ADDIS ABABA UNIVERSITY

COLLEGE OF BUSINESS AND ECONOMICS

Department of Economics

A normal-weighted exponential stochastic frontier model

A thesis submitted in partial fulfilment of the requirements for the award of Master of Science in Economics.

Misgan Desale Nigusie

Supervisor: Adane Tufa (Ph.D)

College of Business and Economics

Department of Economics

June 2022

Addis Ababa, Ethiopia

Declaration

I, Misgan Desale Nigusie, hereby certify that this thesis entitled "A normal-weighted exponential stochastic frontier model" is my original work and has not been presented in any place.

Date …………………   Signature ………………

Certification

This is to certify that the thesis prepared by Misgan Desale Nigusie entitled "A normal-weighted exponential stochastic frontier model" fulfills the requirement of the university and meets the accepted standards with respect to originality and quality.

Advisor

Name: ________________________ Date: _________________ Signature: ______________

Internal Examiner

Name: ________________________ Date: _________________ Signature: ______________

External Examiner

Name: ________________________ Date: _________________ Signature: ______________

Chairperson

Name: ________________________ Date: _________________ Signature: ______________

Acknowledgement

I am thankful to my supervisor, Adane Tuffa (Ph.D), for giving me a freedom in shaping the thesis. I have received a constructive comments and encouragement while defending the proposal, therefore I thank Atnafu Gebremeskel (Ph.D.), Mekonnen Bersisa (Ph.D.), economics (Ph.D.) students and my classmates.

Abstract

This thesis introduces a new stochastic frontier model called a normal-weighted exponential stochastic frontier model. We have derived a closed form log-likelihood function and JLMS inefficiency estimator of a normal-weighted exponential stochastic frontier model. In addition, we have derived the gradient and hessian matrix of a normal-weighted exponential stochastic frontier model. A Monte Carlo (MC) simulation is carried out to verify the correctness of the derivations, of a normal-weighted exponential stochastic frontier model, and to study the finite sample properties of maximum likelihood estimator. Our simulation result shows that a normal-weighted exponential stochastic frontier model performs well compared to a normal-exponential stochastic frontier model. In our simulation result, it shows that as sample size increases the bias and standard errors decreases. Moreover, a real data application is performed, and it is about estimation of carbon efficiency of manufacturing firms in Africa. We have estimated an input requirement production function, using fuel consumption as dependent variable and output and other inputs as independent variables. Our estimated result shows that the estimates of coefficients are the same across models. However, there is differences in carbon efficiency estimates of manufacturing firms. Using a normal-half normal stochastic frontier model, a carbon efficiency of manufacturing firms in Africa gives an estimate ranging between 1.002344 (99.8984%) and 1.002362 (99.8976%). For a normal-exponential stochastic frontier model the range of carbon inefficiency estimates are between 1.074752 (97.5092%) and 1.090364 (96.3126%). Similarly, for a normal-weighted exponential stochastic frontier model the carbon inefficiency estimates are between 1.122895 (95.0907%) and 1.237519 (91.1602%). We have used the carbon efficiency estimates to rank African countries and Egypt is the most carbon efficient country in Africa. We have also run a multiple linear regression on carbon inefficiency estimates to see the determinants. In all three stochastic frontier models: top manager work experience, obstacle to access finance, firm size, export status, and foreign ownership are the key determinants.

Keywords: Inefficiency Estimator, Stochastic Frontier Model, Weighted Exponential Distribution, Carbon efficiency, Manufacturing firms in Africa

Table of Contents

List of Figures

List of Tables

Chapter one: Introduction

1.1. Background of the study

The Stochastic Frontier Analysis (SFA) is a widely used methodology for the measurement of productivity and efficiency of decision making (DM) units, either firms or countries. Using stochastic frontier models, it is possible to estimate the production efficiency, cost efficiency, or profit efficiency of a firm. Kumbhakar and Lovell (2000) have a book-length discussion about the applicability of a stochastic frontier analysis to efficiency measurement of a firm, using the production function, cost function, and profit function. A firm is fully efficient if it can produce the maximum possible output for a given technology and cost, or if it can attain the minimum possible cost for a given level of output and technology. Data Envelopment Analysis (DEA) and Stochastic Frontier Analysis (SFA) are widely used approaches for the efficiency analysis of firms. These approaches, with many extensions, are widely discussed in Coelli, et al. (2005). The general framework of stochastic frontier models for estimating the efficiency of firms was introduced by Aigner, et al (1977) and Meeusen and Broeck (1977).

The basic difference of stochastic frontier models from that of deterministic frontier models is that in stochastic frontier models the deviation from the frontier is assumed to be due to two error terms, which is statistical error and inefficiency score. In the deterministic frontier model, deviation from the maximum possible output is only due to two-sided statistical error. However, in Stochastic Frontier Analysis (SFA) two distributional assumptions are required, for the statistical error and inefficiency score.

Aigner, et al. (1977) used normal distribution for the two-sided statistical error, whereas for the inefficiency part they used normal and exponential distributions. Likewise, Meeusen & Broeck (1977) proposed normal distribution of two-sided statistical error and exponential distribution for the inefficiency part. Any probability density function with the support defined on an interval $(0, \infty)$ can be a candidate model for the inefficiency score, u.

The half normal and exponential distribution of Aigner, et al. (1977) and Meeusen and Broeck (1977) are the most widely used assumptions for the inefficiency part of stochastic frontier models. These models are easily tractable and have closed-form moments, and the log-likelihood function can be expressed in closed form. However, assuming half-normal and exponential distribution for the inefficiency part of the stochastic frontier model has shortcomings (Greene, 1990; Kumbhakar, et al., 2018). Both, exponential and half-normal, distributions have mode at zero value of inefficiency, which means the model assumes the firms to be fully efficient (Kumbhakar, et al., 2018). Moreover, since both exponential and half-normal distributions are governed by a single scale parameter, they are less flexible (Greene, 1990). Another limitation of exponential and half-normal distributions is because of their shape, both distributions are asymmetrically positively skewed, which means the model assumes a very low probability of being inefficient, a priori (Kumbhakar, et al., 2018; Carree, 2002).

Due to the limitations of using exponential and half-normal distribution for the inefficiency part of stochastic frontier models, different one-sided distributions have been proposed. Some of these distributions are, the truncated normal distribution in Stevenson (1980), the gamma distribution was studied in (Stevenson, 1980; Beckers & Hammond, 1987; Greene, 1990; Ritter & Simar, 1997; Greene, 2003). The normal-gamma model is flexible, and it generalizes the normal-exponential stochastic frontier model. However, there is no closed form likelihood function for normal-gamma stochastic frontier model. Stevenson (1980) derived different likelihood functions by restricting the shape parameter of the gamma distribution into positive integers. Stevenson (1980) restricts the shape parameter into only integer values; however, it is desirable to estimate parameters in a continuous form. As a solution Beckers and Hammond (1987) proposed a method estimating the likelihood function by kummer's series using pochhammer's symbols. Greene (1990) derived the likelihood function by estimating the integral function using the quadrature, both the Laguerre quadrature and Newton-Cotes quadrature. Later Ritter and Simar (1997) noticed the identification problem while estimating the quadrature based normal-gamma stochastic fronter likelihood function. Recently, Greene (2003) used the Monte Carlo method with Halton sequences and derived a simulated log-likelihood function of the normal-gamma stochastic frontier model.

Other more flexible inefficiency distributions are used for the Bayesian stochastic frontier model. Some of them are, the gamma distribution in Tsionas (2000), the Weibull distribution in Tsionas (2007), the beta distribution in Tsionas (2012), and the generalized gamma distribution and mixtures of generalized gamma distributions in Griffin and Steel (2008). If one is interested in Bayesian modeling the most flexible distribution is the mixtures of generalized gamma distribution.

In this paper the weighted exponential distribution is introduced as the distribution of the inefficiency score in a stochastic frontier model. So far, I have not seen a stochastic frontier model with the inefficiency part following the weighted exponential distribution. The weighted exponential distribution has both a scale parameter and a shape parameter, like the gamma distribution. But the advantage of using the weighted exponential distribution is that it provides a closed form likelihood function and a closed form function for inefficiency estimator.

1.2. Statement of the problem

So far, in the stochastic frontier literatures, a one parameter inefficiency distributions provide a closed form likelihood function of the stochastic frontier model. A flexible probability density function with one more parameter has been proposed for the inefficiency part of the stochastic frontier model. However, none of these flexible distributions provide a closed form likelihood function and inefficiency estimator. This thesis fills this gab by using a flexible distribution and derives a closed form likelihood function and inefficiency estimator.

The stochastic frontier model was introduced in Aigner, et al. (1977) and Meeusen and Broeck (1977). By assigning different distributional assumption for inefficiency score, many more stochastic frontiers models can be derived. The distributional assumptions, half-normal distribution in (Aigner, et al., 1977) and the exponential distribution in (Aigner, et al., 1977; Meeusen & Broeck, 1977), for the inefficiency score are easily tractable and the most widely used assumptions in stochastic frontier analysis. However, these models are less flexible; their mode is zero i.e., probability of being inefficient is close to zero, the mean and variance of both probability density functions is governed by a single parameter, and their shape is independent from the parameter values (Parmeter & Kumbhakar, 2014).

An exponential distribution and a half-normal distribution are commonly used assumptions for the inefficiency part in stochastic frontier analysis. The advantage of using a half-normal distribution and an exponential distribution is that they provide a closed form likelihood function for a stochastic frontier model. However, there are also limitations of using the exponential distribution and the half-normal distribution; see e.g. Greene (2003) and Stevenson (1980). The first limitation is that both the half-normal distribution and exponential distribution have a mode value of zero. Which means the model assumes a highest probability value for zero value of inefficiency score. Alternative statement is that firms are assumed to be fully efficient a priori. Another limitation is that both probability distributions are governed by a single parameter, a scale parameter. Moreover, both distributions are asymmetrically positively skewed and assumes very low probability of being inefficient Carree (2002). Due to these limitations various probability distributions are assumed for the inefficiency score.

Hajargasht (2015) introduce a Rayleigh distribution into a stochastic frontier model. A Rayleigh distribution is a one-parameter distribution, and it has a non-zero mode. Papadopoulos (2021) present a single parameter generalized exponential distribution and the mode of the distribution is away from zero. These non-zero mode distributions can represent cases where the highest probability is non-zero inefficiency scores. However, since they are governed by a single parameter, they are less flexible. In fact, any probability density function defined on positive value can be a candidate model for the inefficiency score.

In order make the inefficiency score more flexible various two parameter distribution are introduced. The gamma distribution is proposed in (Stevenson (1980), Beckers and Hammond (1987) and Greene (2003)). The Weibull distribution and the beta distribution is introduced in Tsionas (2007) and Tsionas (2012), respectively. However, assuming these flexible distributions for the inefficiency part creates a problem of deriving a closed form likelihood function. As an alternative to the flexibility, the gamma distribution is proposed in (Stevenson, 1980; Beckers & Hammond, 1987; Greene, 2003). However, there is no closed form likelihood function for the normal-gamma stochastic frontier model. Moreover, the normal-gamma stochastic frontier model does not provide a closed form density function for the inefficiency estimates. Greene (2003) has noted the complexity of deriving a closed form likelihood function for gamma-normal stochastic frontier model and proposed simulated maximum likelihood estimation method. The stochastic frontier model with the Weibull inefficiency score is estimated using the Bayesian approach Tsionas (2007). While the beta distribution in Tsionas (2012) is estimated using the Fourier transform method. Some of probability distributions proposed for the inefficiency score in a stochastic frontier modeling are presented in the following table.

Table 1. Probability distributions used in SFMs

Authors	Density function of inefficiency score	Parameters	Likelihood function
Aigner et al. (1977)	Half-normal	One parameter	Closed form
Meeusen and van den Broeck (1977)	Exponential	One parameter	Closed form
Stevenson (1980)	Truncated-normal	Two parameters	Closed form
Hajargasht (2015)	Rayleigh	One parameter	Closed form
Papadopoulos (2020)	Generalized Exponential	One parameter	Closed form
Stevenson (1980) and Greene (2003)	Gamma	Two parameters	No closed form
Tsionas (2007)	Weibull	Two parameters	No closed form
Tsionas (2012)	Beta	Two parameters	No closed form

In this thesis a new probability distribution, a weighted exponential distribution, is introduced for the inefficiency part of a stochastic frontier model. The weighted exponential distribution is a flexible distribution with two parameters, shape parameter and scale parameter. The weighted exponential distribution is flexible as the gamma distribution. However, the advantage of the weighted exponential distribution is that it enables to get a closed-form likelihood function and inefficiency estimator. The weighted exponential distribution contains the one parameter generalized exponential distribution as its special case. Another argument for introducing a new probability distribution is based on the aphorism in statistics saying that all models are wrong, but some are useful. Therefore, it is recommended for the practitioners to have the weighted exponential distribution in their tool kit and verify its usefulness based on the data (Gupta & Kundu, 2009).

1.3. Research questions

The research addresses the following specific questions.

- ✓ Is it possible to have a closed form likelihood function and inefficiency estimator for a normal-weighted exponential stochastic frontier model?
- ✓ Does a Monte Carlo study on normal-weighted exponential distribution give a satisfactory result?
- ✓ What is the carbon efficiency level of manufacturing firms in Africa?

1.4. Objective of the study

1.4.1. General objective

The general objective of the study is to introduce the normal-weighted exponential stochastic frontier model and apply it for estimating carbon efficiency of manufacturing firms in Africa.

1.4.2. Specific objective

Specific objectives of the study are:

- To derive the likelihood function of a normal-weighted exponential stochastic frontier model and its inefficiency estimator.
- To provide a Monte Carlo study of the normal-weighted exponential stochastic frontier model.
- To estimate carbon efficiency of manufacturing firms in Africa.

1.5. Research hypothesis

The first specific objective of the study is to derive a closed form likelihood function for the normal-weighted exponential stochastic frontier model. There is no statistical hypothesis for this specific research question; rather it is a mathematical problem. The second specific objective is to perform a Monte Carlo study on the normal-weighted exponential stochastic frontier model. Again, there is no statistical hypothesis for this second specific objective. Statistical hypothesis is provided only for the third specific objective of estimating carbon efficiency of manufacturing firms in Africa.

Firm's fuel consumption is used to measure the carbon emission, which is the dependent variable. The independent variables are output, labor, and capital.

All independent variables used in fitting the regression model are continuous variables and the dependent variable is also a continuous variable. The independent variables Output, Capital and Intermediate input are hypothesized to affect the dependent variable, fuel consumption, positively. While the independent variable labor is assumed to affect fuel consumption negatively. In general, the null hypothesis is

H_0: The independent variables do not affect the dependent variable, fuel consumption.

And the alternative hypothesis is

H_1: The independent variables significantly affect the dependent variable fuel consumption.

Note that estimating the carbon efficiency of each firm is an integral part of the model. Therefore, the efficiency level of each firm is decomposed from the composite residual of the stochastic frontier model.

After obtaining carbon inefficiency estimates from the above stochastic frontier model, we run a multiple linear regression to see the determinants of carbon inefficiency. Carbon inefficiency is the dependent variable, and the independent variables are top manager work experience, obstacle to financial access, firm size, export, and foreign ownership.

The hypothesis for estimating the determinants of a carbon efficiency of manufacturing firms in Africa is:

H_0: The independent variables do not affect the dependent variable, carbon inefficiency.

And the alternative hypothesis is

H_1: The independent variables significantly affect the dependent variable, carbon inefficiency.

1.6. Significance of the study

The study on a flexible and closed-form likelihood function of the stochastic frontier model helps the applied researchers working on the productivity and efficiency analysis. The stochastic frontier analysis is a powerful tool for analyzing the efficiency of decision-making unit. It can be applied in areas of financial economics, environmental and energy economic, international trade, and many more. The study also includes an estimation of carbon efficiency of manufacturing firms in Africa, and this helps the policy makers on the problem of carbon emission and climate change. It also helps investors, both domestic and abroad, by providing appropriate information about the weakest and strongest side of manufacturing firms in relation to carbon emission. Moreover, it is also possible to rank African countries based on carbon efficiency and this helps the international community to proposed appropriate environmental and energy policy.

1.7. Scope of the study

The study of a flexible and closed-form likelihood functions of the stochastic frontier model; An application to carbon efficiency of manufacturing firms in Africa is restricted on several dimensions. First the study focuses on the classical or frequentist approach of estimating a parametric model. It does not consider the Bayesian approach of estimating a stochastic frontier model. Second, the likelihood function of the normal-weighted exponential stochastic frontier model is derived from the cross-sectional context. It does not include the panel data stochastic frontier model nor the time series approach to the stochastic frontier modeling. Finally, the application is restricted to carbon efficiency of manufacturing firms through their fuel consumption. The study does not concern about the energy efficiency of other sectors.

1.8. Organization of the study

The current chapter, Introduction, starts by presenting the background of the study and proceeds to the statement of the problem and research questions. In addition, chapter 1 discusses the objective of the study and the hypothesis testing. The significance, limitation and scope of the study are also presented in this chapter. Chapter 2 provides a Literature Review; it is subdivided into theoretical literature review, empirical literature review, and conceptual frameworks. The next chapter, Chapter 3, Research Methodology, describes the research strategy and methodology used to address each specific objective. Research findings are presented in chapter 5. In chapter five we have derived a closed form likelihood function of a normal-weighted exponential stochastic frontier model. Furthermore, chapter 5 presents a Monte Carlo (MC) simulation study and a real data applications of a normal-weighted exponential stochastic frontier model. The final chapter, chapter 5, concludes by presenting a summary of research findings and a few recommendations.

Chapter Two: Literature Review

In this chapter we present literature reviews related to the stochastic frontier analysis. It contains three sections which are theoretical literature review, empirical literature review, and conceptual framework. In the first section, theoretical literature reviews, we review literatures about the general framework and procedures for deriving a stochastic frontier model. Next empirical literature review with particular focus on carbon efficiency analysis is reviewed. The final section presents the conceptual framework.

2.1. Theoretical literature review

This section presents a theoretical literature review related to the stochastic frontier analysis. First the general structure of the stochastic frontier model is presented. Then we discussed the problem of having a closed form likelihood function of composite error term when the inefficiency part is assumed to follows a flexible probability density function. Finally, there will be discussions about the JLMS inefficiency estimator, named after Jondrow, et al., (1982).

2.1.1. Stochastic frontier models

The stochastic frontier model is a popular methodology for measuring inefficiency of a firm. A firm is efficient if it produces the maximum possible output for a given cost or achieves the minimum possible cost for a given level output. The stochastic frontier model is the measurement of production function or cost function with symmetric statistical error and inefficiency score. Both Aigner et al. (1977) and Meeusen and van den Broeck (1977) assumed the statistical error to follow the normal distribution with zero mean and constant variance. For the inefficiency score different probability distribution have been proposed.

The production function of a firm is the starting point for the construction of the stochastic frontier models. The production function in its generic form is

$$Y_i = f(\boldsymbol{x_i};\ \theta).$$

where Y_i output of firm i, $\boldsymbol{x_i}$ is vector of variable inputs for firm i, and θ is the unknown vector of parameters. The production function $f(\boldsymbol{x_i};\theta)$. describes the maximum possible output that a firm could achieve for a given level of inputs, $\boldsymbol{x_i}$ and θ represent a vector of inputs and parameters of the production function. The functional form of $f(\boldsymbol{x_i};\ \theta)$ can be either the Cobb-Douglas production function of Cobb and Douglas (1928), the translog production of Christensen et al. (1973), the constant elasticity production function of Arrow et al. (1961), or any other functional form.

It is assumed that the output level Y_i of a firm i may be less than the maximum possible output $f(\boldsymbol{x_i};\ \theta)$, and this is because of two reasons. The first reason is that we may have a data on output less than the maximum possible due to measurement errors. The second reason is that firms produce less than the maximum possible due to their managerial inability or economic inefficiency. Aigner et al. (1977) and Meeusen and van den Broeck (1977) introduced the stochastic frontier models. Thus, the specification of a stochastic frontier model with production function and two multiplicative error terms of the efficiency level U and the statistical error V is

$$Y_i \ = \ f(\boldsymbol{x_i};\ \theta)UV.$$

where $U \in [0,1]$, $V \in [0,\infty]$, $f(\boldsymbol{x_i};\ \theta)$ is the production function. Any deviation from the production function is assumed to be the result of two disturbances, the statistical error V and efficiency level U. The efficiency level U is the ratio of actual output to potential output, that is

$$U \ = \frac{Y_i}{f(\boldsymbol{x_i};\ \theta)V} \ .$$

The potential output $f(\boldsymbol{x_i};\ \theta)V$ is random because of the statistical error V, therefore the model is called the stochastic frontier model. A firm achieves its full efficiency if the actual output Y_i is equal to the stochastic frontier, otherwise it is inefficient.

The multiplicative form of the above stochastic frontier model can be transformed into additive form by using natural logarithm and defining the inefficiency score u implicitly as

$$U = e^{-u}.$$

The efficiency level U is inversely related with u. If the inefficiency score u approaches zero, then $U = e^{-u}$ approaches 1, which means that the firm is producing almost near the frontier, and vice versa.

For the statistical error V we define

$$V = e^{v}.$$

Thus, by logarithmic transformation, the stochastic frontier model in additive form is

$$ln\, Y_i \;=\; ln(f(\boldsymbol{x_i};\, \theta)) \;+\; v \;-\; u$$

where $v \;=\; ln(V\,)$ and $u \;=\; -\,ln(U)$.

Once the functional form of the production function is stated, the next step is to propose distributional assumptions for the inefficiency score u and the statistical error term v.

Assuming independence between u and v, then the density function of the composite error $\varepsilon = v - u$ will be derived using the convolution method. Getting a tractable density function of ε depends on the choice of probability density functions for the statistical error v and inefficiency score u. The distributional assumption for the statistical error v has been relaxed. For example, Wheat et al. (2019) assumes the student t-distribution for the statistical error v. Similarly, various probability distributions have been considered for the inefficiency score. Aigner et al. (1977) used both half normal and exponential distributions for the inefficiency score. Meeusen and van den Broeck (1977) assume only the exponential distribution for the inefficiency score u. Both Meeusen and van den Broeck (1977) and Aigner et al. (1977) assumed the statistical error v to follow a normal distribution with mean zero and constant variance. The exponential probability density function of a random variable u with scale parameter λ is specified as

$$f(u;\, \lambda) \;=\; \lambda\, exp(-\,u\,\lambda\,)$$

where $\lambda \;>\; 0$.

For the half-normal distribution, the probability density function of a random variable v with a scale parameter σ^2 is given by

$$f(v;\ \sigma)\ =\ \frac{2}{\sqrt{2\pi\sigma^2}}\ exp(-\ \frac{v^2}{2\sigma^2}).$$

where $\sigma\ >\ 0$.

The advantage of assuming the exponential distribution and the half-normal distribution is that it provides a closed-form likelihood function for the stochastic frontier model. However, these distributions are governed by a single parameter and are less flexible. Therefore, for the inefficiency score, u, a more flexible probability distributions have been considered.

2.1.2. The distribution of combined error term

The combined error term ε, in case of production frontier, is defined as

$$\varepsilon = v - u,$$

where u is the inefficiency score and v is the statistical error. Having a tractable probability density function of ε depends on the choices of probability distribution made for u and v, and on the assumption whether u and v are independent or not. Assuming independence between the statistical error v and inefficiency score u, the joint distribution of v and u is given by

$$f(u, v) =\ f(u)f(v),$$

where $f(u)$ and $f(v)$ are the probability density function of u and v, respectively. The next step is to substitute one of the random variables from the combined error $\varepsilon = v - u$, and integrate the remaining variable. We have

$$f(\varepsilon) = \int_{-\infty}^{\infty} f(v-\varepsilon)f(v)dv,$$

For some choices of parametric distributions u and v, the integral above can be computed and we have a closed-form distribution function of ε. Both the pioneering articles of Aigner, et al. (1977) and Meeusen and Broeck (1977) and most subsequent applied researchers have assumed the random error v to follow a normal (Gaussian) distribution with zero mean $\mu = 0$ and constant variance σ^2. For u, different probability distributions have been proposed. Only a few probability density functions provide a closed-form probability density function of ε. Assuming an exponential or a half-normal distribution for u provides us with a closed-form probability density function of ε (Aigner, et al., 1977; Meeusen & Broeck, 1977). Recently, Hajargasht (2015) proposed Rayleigh distribution for u and derived a closed form likelihood function for the normal-rayleigh stochastic frontier model. However, when a flexible probability distribution is assumed for u, getting a closed form probability distribution for ε becomes challenging.

Alternatively, recent works have relaxed the assumption that the inefficiency score u is independent of the statistical error v, and used copula methods to derive the probability distribution function of $\varepsilon, f(\varepsilon)$, (Smith, 2008; Gómez-Déniz & Pérez-Rodríguez, 2015). A copula method allows us to estimate a stochastic frontier model with dependent errors. In this paper, we keep the independence assumption.

2.1.3. The likelihood functions

Once the probability density function of ε is derived, then the likelihood function of the stochastic frontier model for N observation is

$$L(\theta \mid \varepsilon) = \prod_{i=1}^{N} f(\varepsilon \mid \theta),$$

where $\varepsilon = v - u = \log(y) - \log[f(x; b)]$.

The vector of parameters θ contains the coeffects of the production function b and parameters from the probability distribution of inefficiency score u and the statistical error v. The production function $f(x; b)$ can be either a Cobb-Douglass production function or another production function that is linear after logarithmic transformation. Maximizing the likelihood function requires the likelihood function to be in a closed form, and this has been a challenge when a flexible distribution is assumed for the inefficiency score u.

A gamma distribution for u was introduced in Stevenson (1980). However, there is no closed-form probability distribution for ε in this case. As a solution, Stevenson (1980) restricted the value of the shape parameter of the gamma distribution to integer values and derived different probability distributions, but this is a very strong restriction. Greene (1990) used Simpson's rule and areas of the trapezoid to approximate the integral but was not satisfied. Later Greene (2003) proposed the inverse transform method.

2.1.4. JLMS inefficiency estimator

JLMS stands for Jondrow, Lovell, Materov, and Schmidt, after their work of (Jondrow, et al., 1982). The primary goal of the stochastic frontier analysis is to have estimates for inefficiency score u. Aigner, et al. (1977) used the mean of u and the maximum likelihood estimators to get the inefficiency estimates of each firm. It is possible to estimate the average inefficiency score $E(\hat{u})$ based on the estimates of the average of composite error $E[\hat{\varepsilon}]$, since $E[u] = E[\varepsilon]$. However, it is also desirable to have inefficiency estimates for each firm and a complete probability distribution for $\hat{u}$. As a solution Jondrow, et al. (1982) proposed a method of deriving the conditional distribution of u from the probability density function of ε. The conditional probability density of function of u, $f(u \mid \varepsilon)$, is derived from the ratio of the joint probability density of u and ε, $f(\varepsilon, u)$, to the marginal probability density of ε, $f(\varepsilon)$.

$$f(u \mid \varepsilon) = \frac{f(\varepsilon, u)}{f(\varepsilon)}.$$

2.2. Empirical literature review

2.2.1. Applications of the stochastic frontier models to carbon efficiency analysis

Since the 1970s oil crisis, monitoring energy efficiency has become an essential goal of economic and energy policies in many countries around the world. This concern grew in the late 1980s because of the increasing awareness of global warming. A key issue in the strategy of the countries that aim to reduce their energy consumption and mitigate their greenhouse gas emissions is the adoption of measures that improve the efficiency of energy use in all economic sectors and especially in those that are energy intensive, as in the case of transport (Llorca, et al., 2017). Energy efficiency of manufacturing industry has been investigated by relating it productivity and efficiency analysis. Energy consumption has been continuously increasing globally, at the same time the concentration of greenhouse gas in the atmosphere has increased.

There are problems of fossil fuel depletion and an increase in fossil fuel price. There is an economic crises and new international energy policies are being proposed for industries/counties to cut energy wastes and inefficiency, and to control their consumption

After the industrial revolution, the global warming has been increasing continuously. Industries have continued to emit more carbon dioxide to the environment than before. There is a global concern about the global warming and monitoring of carbon emission of industries and countries is required. There are two approaches, to use renewable energy or reduce the consumption of fuel energy.

The solution to this global warming is to substitute the renewable energy for the fossil fuel consumption. Another best strategy is to consume these fossil fuels efficiently. Focus has been shifting from fossil energy to clean and renewable energy. Moreover, it is also necessary to use the fossil fuel efficiently. Consumption of fossil fuel accounts for the largest proportion of energy consumption, 80% of the worldwide energy usage (Li & Tao, 2017).

2.2.2. Carbon efficiency analysis

Energy efficiency of a firm or a country is estimated using various methodologies. Carbon efficiency can be analysed using either the Data Envelopment Analysis (DEA) method or the Stochastic Frontier (SFA). The Data Envelopment Analysis (DEA) can be used to evaluate firm's carbon efficiency from production perspective. Furthermore, they have estimated the effect of financial performance on carbon efficiency. The four major evaluation methodologies of energy efficiency are, the stochastic frontier analysis, data envelopment analysis, energy analysis and benchmarking comparison (Li & Tao, 2017). Li and Tao (2017) have reviewed all the four methods and summarized that the SFA approach as solid fundamental work in modeling application. There are three widely used stochastic frontier methodologies for estimating energy efficiency. These are the input demand frontier functions (Llorca, et al., 2017)), Shephard input distance function (Hu & Honma, 2014) and input requirement functions. The Stochastic Frontier Analysis (SFA) can be either input oriented SFA or output oriented SFA. The input oriented SFA measures how much extra input is employed to produce a given level of output. While the output oriented SFA measures how much the output falls below the frontier (Jin & Kim, 2019)

Hu and Honma (2019) used the stochastic frontier analysis to estimate the energy efficiency of industries in 14 developed countries. They have used a panel data for the period of 1995-2005 and 10 industries are included. The countries included in the study are United States, United Kingdom, Sweden, Finland, Germany, Italy, the Netherlands, Portugal, Australia, Austria, Denmark, the Czech Republic, Japan, and South Korea. The industrial sectors included in the study are, the construction industry, the food and tobacco industry, the chemical and petrochemical industry, the iron and steel industry, the machinery industry, the paper industry, the non-metallic minerals industry, the textile and leather industry, the wood industry, the pulp and printing industry, and the transport equipment industry. They used 4 variable inputs (labour, capital, energy, intermediate input) and the variable output is measured using the value added. They used the stochastic frontier distance function in which the production function part is specified as Cobb-Douglass production function. Their regression result shows a decreasing efficiency for the industries, construction, paper, and textile. On the contrary, the industry sector shows an increase in efficiency. The efficiency estimate of more than half industries shows insignificant change. The most efficient performance of industries can be classified into countries. The food industry, the textile industry, and the machinery industry are more efficient in Portugal. The construction industry and the wood industry are better in United Kingdom. While the chemical industries and the paper industries are efficient in Denmark. The rest of industries are distributed as the non-metallic mining in Czech Republic, the transport industry in Italy, and the iron and metal industry in South Korea. Moreover, their result also shows the food industry is the most efficiency with efficiency level of 82.5 percent and the lowest efficiency level is for the wood industry which is 12.7 percent.

2.2.3. Determinants of carbon efficiency

To propose effective policy approach, it is desired to know the determinants of carbon efficacy. Knowing the key determinants of carbon efficiency helps to improve energy efficiency of firms. Studies have been done on the determinants of energy efficiency using both macroeconomic data and microeconomic level data. Li and Tao (2017) used macro level data and identify 5 factors affecting energy efficiency technology. These energy efficiency technologies are GDP, labour, capital investment, structural indicators, energy price, and environmental indicators.

Energy efficiency of countries can be estimated using the energy demand function. Llorca, et al. (2017) used a stochastic frontier analysis for estimating energy efficiency in the transport sector of Latin American and the Caribbean. Llorca, et al. (2017) measured energy efficiency and compared countries in Latin America and the Caribbean. Energy efficiency scores are measured by deviations from the minimum energy consumption predicted by the frontier. They have used a panel data for the 1990-2010 period and 24 countries in Latin America and the Caribbean. The dependent variable is the energy consumption of transport sector. While the dependent variables are GDP, population size, energy price, and other control variables. For the demand function they have used both the Cobb-Douglas and the translog functional forms. The coefficients of the model, in both functional forms, are elasticity. For Cobb-Douglas specification, they have estimated the income and price elasticity to be 0.81 and -0.23, respectively. The coefficient on the population variable is positive and statistically significant.

2.3. Conceptual framework

The study objective is to introduce a weighted exponential distribution as the distribution of the inefficiency score in a stochastic frontier model. First, we will derive a closed form likelihood function for a normal-weighted exponential stochastic frontier model. We also derive the JLMS inefficiency estimator for the proposed estimator. Moreover, we have presented the gradient vector and hessian matrix for the likelihood function of a normal-weighted exponential stochastic frontier model. The second specific objective is to present a Monte Carlo (MC) study for a normal-weighted exponential stochastic frontier model. In doing a Monte Carlo (MC) study, artificial data is generated from a data generating process (DGP) that is normal-weighted exponential stochastic frontier production function. This artificial data is used to estimate the parameters of a normal-weighted exponential stochastic frontier model, and comparison is made with estimates from a normal-exponential stochastic frontier model. The third objective is empirical data application to the normal-weighted exponential stochastic frontier model. And comparison is made with a normal-half normal stochastic frontier model and a normal-exponential stochastic frontier model. The stochastic frontier model is fitted to obtain carbon efficiency estimates of manufacturing firms in Africa. We have compared the carbon efficiency estimates derived from three stochastic frontier models: normal-half normal, normal-exponential, and normal-weighted exponential. Once the carbon efficiency levels are obtained, the multiple linear regression is fitted to determine factors affecting carbon efficiency level of firms. The general framework can be depicted as follows.

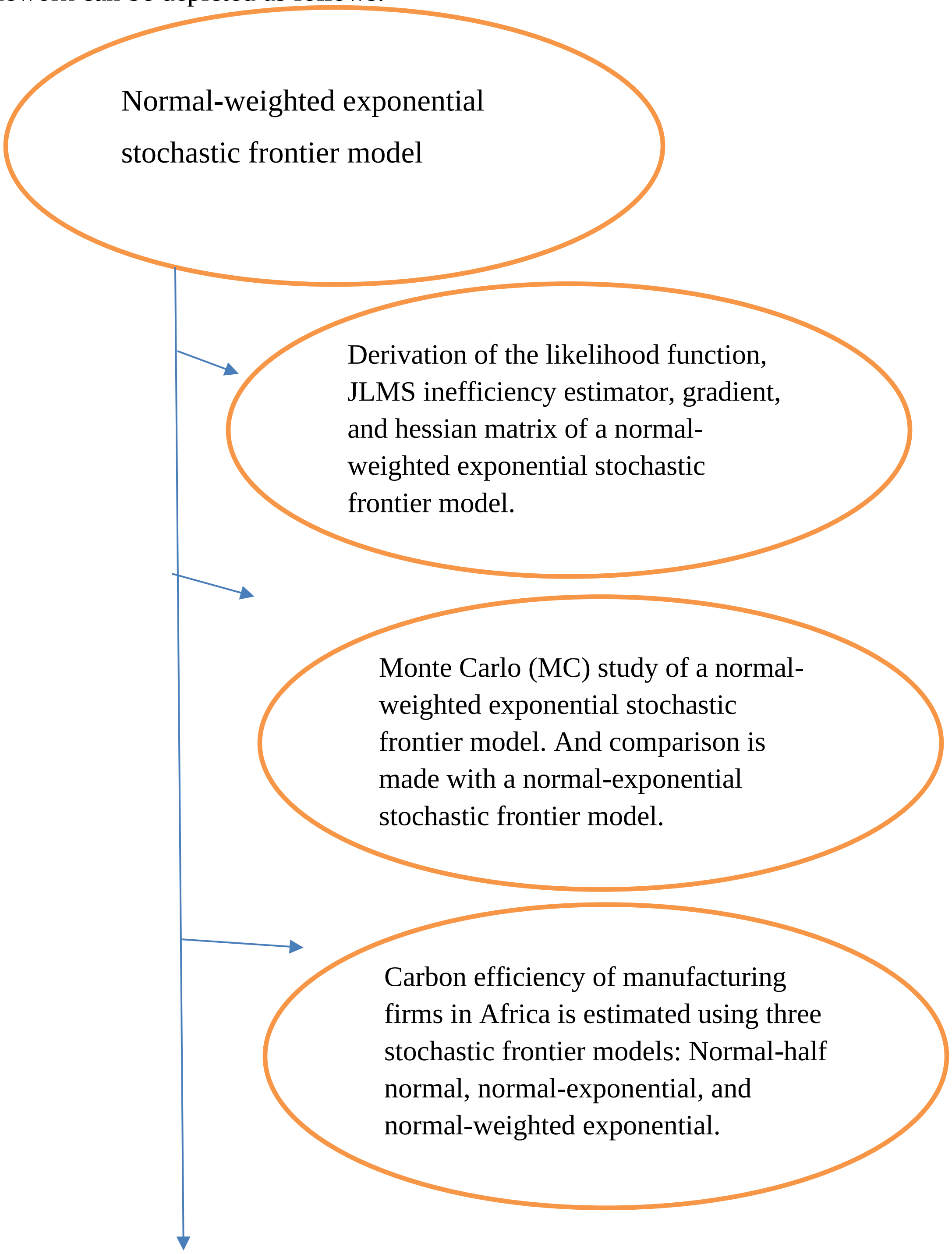


Figure 1. Conceptual Framework

Chapter Three: Methodology

3.1. Area of the study

The study is mainly on econometric tool that used for productivity and efficiency analysis. This study introduces the weighted exponential distribution into a stochastic frontier analysis and derives a closed form likelihood function and JLMS inefficiency estimator. Moreover, a gradients and hessian matrix of a normal-weighted exponential stochastic frontier model are derived. This new stochastic frontier model is studied using Monte Carlo (MC) simulation, and it is compared with a normal-exponential stochastic frontier model. Finally, carbon efficiency of manufacturing firms in Africa is estimated using a normal-weighted exponential stochastic frontier model, and the results are compared with estimates from other stochastic frontier models.

3.2. Source of Data

Of the three specific objectives, the real data is required for the third specific objective, which is to estimate the carbon efficiency of manufacturing firms in Africa. Data on African manufacturing firms is provided by the World Bank Enterprise Survey (WBES) data, and it is accessible online, through the link https://www.enterprisesurveys.org/. In estimating the carbon efficiency of manufacturing firms in Africa, the dependent variable is fuel consumption of firms. And the World Bank's Enterprise Survey (WBES) data set contains information about firms' fuel consumption. Moreover, the WBES data contains information about the level of sales, which is used as proxy for output. Also note that, the sale variable as proxy of output has also been used by Saliola and Seker (2011) study on productivity, using same enterprise survey data. Moreover, the enterprise survey data also includes information about labor, and it is measured by total compensation of workers including wages, salaries, and bonus. Capital is measured by replacement value of Machinery, vehicles, and equipment. Another independent variable is the intermediate good and it is measured by the cost of raw materials and intermediate materials. The WBES survey covers most African countries, however for most countries there is no information about fuel consumption of firms. Therefore, after dropping observations for missing value on fuel consumption, output, labor and capital, and intermediate input, we have used a sample of 1911 firms of 18 African countries.

After obtaining efficiency estimates of each manufacturing firms, we fit a multiple linear regression model to determine factors affecting carbon efficiency of manufacturing firms in Africa. Data on independent variables is needed and the WBES survey data set contains information about manufacturing firms' top manager experience, the degree of obstacle to access finance, firm size, export status, and foreign ownership. Thus, using carbon efficiency as dependent variable multiple linear regression model is estimated.

3.3. Normal-weighted exponential stochastic frontier model

The first specific objective is to derive a closed form likelihood function for a normal-weighted exponential stochastic frontier model. And we also derived a closed form JLMS inefficiency estimator for a normal-weighted exponential stochastic frontier model. Moreover, we also derive the gradient and hessian matrix for a likelihood function of a normal-weighted exponential stochastic frontier model.

The methodology for deriving these mathematical results is to use different algebraic tricks and follow calculus rules. To derive a closed form likelihood function, we need to assume independence between the inefficiency score, u and the statistical error v. Then we use the same procedure as in (Aigner, et al., 1977; Meeusen & Broeck, 1977), which is to evaluate the following integral,

$$f(\varepsilon) = \int_0^\infty f(u) f(\varepsilon + u) du,$$

where $\varepsilon = v - u$ is a composite error, u is a inefficiency score, and v is a statistical error.

To get inefficiency estimator we use (Jondrow, et al., 1982) and it is called a JLMS inefficiency estimator. Jondrow, et al. (1982) proposed a method of deriving the conditional distribution of u from the probability density function of ε. The conditional probability density of function of u, $f(u \mid \varepsilon)$, is derived from the ratio of the joint probability density of u and ε, $f(\varepsilon, u)$, to the marginal probability density of ε, $f(\varepsilon)$. Which is

$$f(u \mid \varepsilon) = \frac{f(\varepsilon, u)}{f(\varepsilon)}.$$

Once the probability density function of ε is derived, then the likelihood function of the stochastic frontier model for N observation is

$$L(\theta \mid \varepsilon) = \prod_{i=1}^{N} f(\varepsilon \mid \theta),$$

where $\varepsilon = v - u = \log(y) - \log[f(x; b)]$.

The vector of parameters θ contains the coeffects of the production function b and parameters from the probability distribution of inefficiency score u and the statistical error v. Then the gradient vector is the first order partial derivate of a likelihood function with respect to each parameter. Finally, the hessian matrix is the second order direct and cross derivatives a likelihood function with respect to parameters.

3.4. Monte Carlo Simulation Study

The second specific objective of the study is to perform Monte Carlo (MC) simulation study on a normal-weighted exponential stochastic frontier model. Monte Carlo (MC) simulation study helps us to confirm that the mathematical derivations, of the first specific objective, are correct. In addition, a Monte Carlo (MC) simulation study used to study the finite sample properties of an estimator.

The idea of Monte Carlo (MC) simulation study is to use a data generating process (DGP) with a particular parameter value and generate artificial data. Then, we use these artificial data and estimate the parameters of the model.

3.5. Carbon Efficiency of Manufacturing Firms in Africa

We estimate the carbon efficiency of manufacturing firms in Africa using a normal-weighted exponential stochastic frontier model. A normal-weighted exponential stochastic frontier model is a newcomer into literatures of efficiency analysis. The performance of a normal-weighted exponential stochastic frontier model is compared with estimates from a normal-half normal and normal-exponential stochastic frontier models. The stochastic fronter model is applicable in many areas of economics. In this study we apply a stochastic frontier model for analyzing a carbon efficiency of manufacturing firms in Africa.

Energy efficiency is defined as the ratio of the minimum feasible use of energy resource to observed use of energy, conditional upon given production technology, level of output and levels of the other inputs (Jiang, et al., 2017). Equivalently, we can define a carbon efficiency as the ration of a minimum feasible use of fuel resource to observed use of energy, conditional a production function, output, and other inputs. Therefore, the stochastic frontier model can be expressed as

$$\begin{aligned}\log(Fuel\ Consumption) &= \alpha + \beta_1 \log(Output) + \beta_2 \log(Labour) + \beta_3 \log(Capital) \\ &+ \beta_4 \log(Intermediate\ input) + v + u,\end{aligned}$$

where v is the statistical error and u is the inefficiency score. The statistical error v is assumed to follow a normal distribution with mean zero and constant variance, i.e., $v \sim \mathcal{N}(0, \sigma^2)$. For the inefficiency score three one-sided probability distributions are consider. These one-sided distributions are, half normal, exponential, and the weighted exponential distributions.

Table 2 provides a description of the hypothetical relationship between the dependent variable and independent variables.

Table 2. Description of variables for a stochastic frontier model

Variables		Descriptions	Expected Signs
The dependent variable: Fuel Consumption		Continuous variable	
Independent variables	Output	Continuous variable	+
	Labor	Continuous variable	-
	Capital	Continuous variable	+
	Intermediate input	Continuous variable	+

Estimating the above stochastic frontier model provides us with two important results: elasticities and carbon inefficiency. The estimates of elasticity represent a percentage change in fuel consumption for a percentage change in the output and other inputs.

After obtaining a carbon inefficiency estimate of each manufacturing firms in Africa, we can further study the determinants of carbon inefficiency. This is carried out by running a multiple linear regression model. The estimated carbon inefficiency is the dependent variable, and independent variables are, top managers work experience, the degree obstacle to access finance, firm size, export status, and foreign ownership of the firm. The regression model is:

$$Carbon\_inefficiency_i = \alpha + \beta_1 Managerial\_experiance_{1i} + \beta_2 Obstacle\_to\_finance_{2i} + \beta_3 Firm\_size_{3i} + \beta_4 Export_{4i} + \beta_5 Foreign_{5i} + \varepsilon_i$$

An experienced top manager is assumed to be more efficient in being more environmentally friendly than a new entrant firm. And this is because an experienced top manager has an advantage of learning from his/her past mistakes and can perform carbon efficient production system. The next independent variable is about the degree of obstacle a firm faces to access finance. Access to finance is a categorical variable. And the classification is as follows:

$$\text{Obstacle to access finance} = \begin{pmatrix} 0 & if\ no\ obstacle \\ 1 & if\ minor\ obstacle \\ 2 & if\ moderate\ obstacle \\ 3 & if\ major\ obstacle \\ 4 & if\ very\ severe\ obstacle \end{pmatrix}$$

The other independent variable is firm size. Furthermore, firm size also affects the efficiency level of firms. Specifically, large firms are assumed to be more efficient than micro and small firms. This can be due to economies of scale or any other factor. Moreover, the kinds of products, the geographic location and the legal status of manufacturing firms are also regressed as an independent variable. Firm size is a categorical variable and classification is based on the number of employees who are actively working on the firm. Size stratification is defined as follows: micro (below 5), small (5 to 19 employees), medium (20 to 99 employees), and large (more than 99 employees). Export is another independent variable because exporter firms are assumed to be carbon efficient. This assumption makes sense because exporter firms are exposed to international competition and most likely to be carbon efficient than those firms producing only for domestic market. The independent variable foreign, indicates whether the firm is owned by foreigner or not. Firms with foreign ownership are assumed to be efficient than those owned by domestic citizens. This is because the foreigner is assumed to have an abroad experience.

Chapter four: Research Findings and Discussion

In this chapter we present the research findings in three sections, corresponding to each of the three specific objectives. In the first section the mathematical derivation of a normal-weighted exponential stochastic frontier model is presented. Next, we present a Monte Carlo (MC) simulation study of a normal-weighted exponential stochastic frontier model. The third section is about an empirical analysis of carbon efficiency of manufacturing firms in Africa.

4.1. Normal-weighted exponential stochastic frontier model

In this chapter, we will assume that u follows a weighted exponential distribution and that v follows a normal distribution. We will call this the "normal-weighted exponential stochastic frontier model". The weighted exponential distribution is introduced in Gupta and Kundu (2009). The advantage of the weighted exponential distribution is that it provides a closed form marginal probability density function of the composite error, $v - u$.

4.1.1. The weighted exponential distribution

As it is shown in Gupta and Kundu (2009), if the inefficiency score u follows the weighted exponential distribution, then its probability density function is given by

$$f(u) = \frac{(\alpha+1)}{\alpha}\lambda e^{-\lambda u}(1 - e^{-\lambda\alpha u}),$$

where $\alpha > 0$ is the shape parameter, and $\lambda > 0$ is the scale parameter.

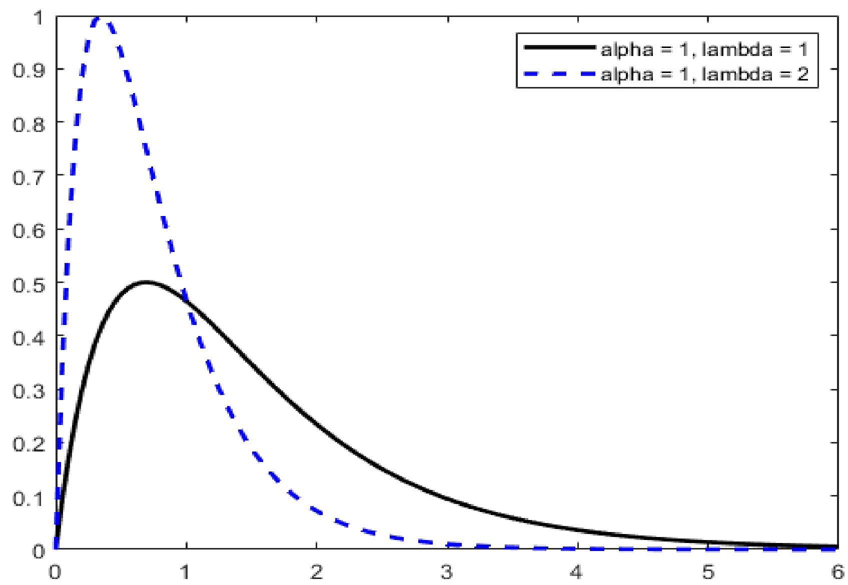


Figure 2. Weighted exponential distribution

If $\alpha = 2$ the probability density function reduces to one parameter generalized exponential distribution. This one parameter generalized exponential distribution was used for the inefficiency part of the stochastic frontier model in Papadopoulos (2021).

4.1.2. Likelihood function of the normal-weighted exponential stochastic frontier model

We will assume that v follows a normal distribution and u follows a weighted exponential distribution, therefore, we will call this the "normal-weighted exponential stochastic frontier model".

The probability density function (pdf) of u is given by

$$f(u) = \frac{(\alpha + 1)}{\alpha} \lambda e^{-\lambda u}\left(1 - e^{-\lambda \alpha u}\right),$$

where $\alpha > 0$ is the shape parameter, and $\lambda > 0$ is the scale parameter.

The statistical error, v, is assumed to be normal with mean 0 and constant variance σ^2, which is

$$f(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v)^2}{2\sigma^2}},$$

where $\sigma^2 > 0$.

Assuming independence between the two random variables and setting $v = \varepsilon - u$, the joint density function is given by

$$f(u, \varepsilon - u) = \frac{(\alpha + 1)}{\alpha} \lambda e^{-\lambda u}(1 - e^{-\lambda \alpha u}) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\varepsilon-u)^2}{2\sigma^2}}.$$

Integrating the above joint density function with respect to inefficiency score u gives us the marginal density function of composite error ε. Which is

$$f(\varepsilon) = \int_0^\infty f(u, \varepsilon - u) du,$$

$$f(\varepsilon) = \frac{\lambda(\alpha + 1)}{\alpha\sqrt{2\pi\sigma^2}} \int_0^\infty e^{-\lambda u}(1 - e^{-\lambda\alpha u}) e^{-\frac{(\varepsilon-u)^2}{2\sigma^2}} du$$

$$f(\varepsilon) = \frac{\lambda(\alpha + 1) e^{-\frac{\varepsilon^2}{2\sigma^2}}}{\alpha\sqrt{2\pi\sigma^2}} \int_0^\infty (1 - e^{-\lambda\alpha u}) e^{-\left(\frac{u^2 - 2u(\varepsilon - \sigma^2\lambda)}{2\sigma^2}\right)} du$$

$$f(\varepsilon) = \frac{\lambda(\alpha + 1) e^{-\frac{\varepsilon^2}{2\sigma^2} + \frac{(\varepsilon-\sigma^2\lambda)^2}{2\sigma^2}}}{\alpha\sqrt{2\pi\sigma^2}} \int_0^\infty (1 - e^{-\lambda\alpha u}) e^{-\left(\frac{u^2 - 2u(\varepsilon - \sigma^2\lambda)}{2\sigma^2}\right) - \frac{(\varepsilon-\sigma^2\lambda)^2}{2\sigma^2}} du$$

$$f(\varepsilon) = \frac{\lambda(\alpha + 1) e^{-\varepsilon\lambda + .5\sigma^2\lambda^2}}{\alpha\sqrt{2\pi\sigma^2}} \int_0^\infty (1 - e^{-\lambda\alpha u}) e^{-\frac{\left(u - (\varepsilon - \sigma^2\lambda)\right)^2}{2\sigma^2}} du$$

$$f(\varepsilon) = \frac{\lambda(\alpha + 1) e^{-\varepsilon\lambda + .5\sigma^2\lambda^2}}{\alpha\sqrt{2\pi\sigma^2}} \left[\int_0^\infty e^{-\frac{\left(u - (\varepsilon - \sigma^2\lambda)\right)^2}{2\sigma^2}} du - \int_0^\infty e^{-\left(\frac{u^2 - 2u\left(\varepsilon - \sigma^2\lambda(\alpha+1)\right) + \left(\varepsilon - \sigma^2\lambda\right)^2}{2\sigma^2}\right)} du \right]$$

$$f(\varepsilon) = \frac{\alpha + 1}{\alpha} \lambda \Phi\left(\frac{(\varepsilon - \sigma^2\lambda)}{\sigma}\right) e^{-\varepsilon\lambda + .5\sigma^2\lambda^2}$$
$$- \frac{\alpha + 1}{\alpha} \lambda \Phi\left(\frac{(\varepsilon - \sigma^2\lambda(\alpha + 1)}{\sigma}\right) e^{-\varepsilon\lambda(\alpha+1) + .5\sigma^2\lambda^2(\alpha+1)^2}.$$

For a stochastic frontier production function, we have

$$f(\varepsilon) = \frac{\alpha+1}{\alpha}\lambda\Phi\left(-\frac{(\varepsilon+\sigma^2\lambda)}{\sigma}\right)e^{\varepsilon\lambda+.5\sigma^2\lambda^2}$$
$$-\frac{\alpha+1}{\alpha}\lambda\Phi\left(-\frac{(\varepsilon+\sigma^2\lambda(\alpha+1))}{\sigma}\right)e^{\varepsilon\lambda(\alpha+1)+.5\sigma^2\lambda^2(\alpha+1)^2}.$$

By rearranging the marginal density function of composite error ε, is given by

$$f(\varepsilon) = \frac{\alpha+1}{\alpha}\lambda e^{\varepsilon\lambda+.5\sigma^2\lambda^2}\left\{\Phi\left(-\frac{(\varepsilon+\sigma^2\lambda)}{\sigma}\right) - \Phi\left(-\frac{(\varepsilon+\sigma^2\lambda(\alpha+1))}{\sigma}\right)e^{\varepsilon\lambda\alpha+.5\sigma^2\lambda^2(\alpha^2+2\alpha)}\right\}.$$

The log-likelihood function for N observation is

$$\log L(\alpha,\lambda,\sigma \mid \varepsilon_i)$$
$$= \sum_{i=1}^{N}\varepsilon_i\lambda + \frac{N\sigma^2\lambda^2}{2} + N\{\log[\lambda]\} + N\{\log[\alpha+1]\} - N\{\log[\alpha]\}$$
$$+ \log\left\{\sum_{i}^{N}\left[\Phi\left(-\frac{(\varepsilon+\sigma^2\lambda)}{\sigma}\right)\right.\right.$$
$$\left.\left. - \Phi\left(-\frac{(\varepsilon+\sigma^2\lambda(\alpha+1))}{\sigma}\right)e^{\varepsilon\lambda\alpha+.5\sigma^2\lambda^2(\alpha^2+2\alpha)}\right]\right\}.$$

4.1.3. JLMS inefficiency estimator

The primary goal of the stochastic frontier analysis is to have estimates of inefficiency for each firm. Jondrow, et al. (1982) proposed the conditional distribution of u and the maximum simulated likelihood estimator. The conditional distribution, $f(u \mid \varepsilon)$, is derived by dividing the joint distribution of the inefficiency score and the composite error, (ε, u), and dividing by the marginal distribution of the composite error ε. The joint distribution of the inefficiency score u and the statistical error ε is given by

$$f(u,\varepsilon-u) = \frac{(\alpha+1)}{\alpha}\lambda e^{-\lambda u}(1-e^{-\lambda\alpha u})\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\varepsilon-u)^2}{2\sigma^2}}.$$

The marginal distribution of ε, is given by

$$f(\varepsilon) = \frac{\alpha+1}{\alpha}\lambda\Phi\left(\frac{(\varepsilon-\sigma^2\lambda)}{\sigma}\right)e^{-\varepsilon\lambda+.5\sigma^2\lambda^2}$$

$$-\frac{\alpha+1}{\alpha}\lambda\Phi\left(\frac{(\varepsilon-\sigma^2\lambda(\alpha+1)}{\sigma}\right)e^{-\varepsilon\lambda(\alpha+1)+.5\sigma^2\lambda^2(\alpha+1)^2}.$$

$$f(\varepsilon) = \frac{\alpha+1}{\alpha}\lambda e^{-\varepsilon\lambda+.5\sigma^2\lambda^2}\left[\Phi\left(\frac{(\varepsilon-\sigma^2\lambda)}{\sigma}\right)-\Phi\left(\frac{(\varepsilon-\sigma^2\lambda(\alpha+1))}{\sigma}\right)e^{-\varepsilon\lambda\alpha+.5\sigma^2\lambda^2(\alpha^2+2\alpha)}\right].$$

The conditional probability distribution of u is given by

$$f(u \mid \varepsilon) = \frac{f(u, \varepsilon-u)}{f(\varepsilon)}$$

$$f(u \mid \varepsilon) = \frac{e^{-\lambda u}(1-e^{-\lambda\alpha u})\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\varepsilon-u)^2}{2\sigma^2}}.}{Ce^{-\varepsilon\lambda+.5\sigma^2\lambda^2}}$$

where $C = \Phi\left(\frac{\mu}{\sigma}\right)-\Phi\left(\frac{\mu_*}{\sigma}\right)e^{-\varepsilon\lambda\alpha+.5\sigma^2\lambda^2(\alpha^2+2\alpha)}$, $\mu = (\varepsilon-\sigma^2\lambda)$, and $\mu_* = \left(\varepsilon-\sigma^2\lambda(\alpha+1)\right)$.

$$f(u \mid \varepsilon) = \frac{\left(1-e^{-\lambda\alpha u}\right)}{C}\ \frac{e^{-\frac{(u-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}.$$

We can see that the distribution of the inefficiency score is a weighted sum of a truncated normal distribution. That is

$$f(u \mid \varepsilon) = W_1\frac{e^{-\frac{(u-\mu)^2}{2\sigma^2}}}{\Phi\left(\frac{\mu}{\sigma}\right)\sqrt{2\pi\sigma^2}} - W_2\frac{e^{-\frac{(u-\mu_*)^2}{2\sigma^2}}}{\Phi\left(\frac{\mu_*}{\sigma}\right)\sqrt{2\pi\sigma^2}}$$

where $W_1 = \frac{\Phi\left(\frac{\mu}{\sigma}\right)}{C}$ and $W_2 = \frac{\Phi\left(\frac{\mu_*}{\sigma}\right)e^{-\varepsilon\lambda\alpha+.5\sigma^2\lambda^2(\alpha^2+2\alpha)}}{C}$ are weights.

We can use the mean of the above inefficacy distribution with maximum likelihood estimator of the parameters, $E[\hat{u} \mid \hat{\varepsilon}]$, as an estimator to each firm, which is commonly called the JLMS inefficacy estimator (Jondrow, et al., 1982). And the mean of the sum of a random variable is the sum of the mean of each random variable.

$$E[\hat{u} \mid \hat{\varepsilon}] = \widehat{W}_1\left(\hat{\mu} + \hat{\sigma}\frac{\phi\left(\frac{\hat{\mu}}{\hat{\sigma}}\right)}{\Phi\left(\frac{\hat{\mu}}{\hat{\sigma}}\right)}\right) - \widehat{W}_2\left(\widehat{\mu_*} + \hat{\sigma}\frac{\phi\left(\frac{\hat{\mu}_*}{\hat{\sigma}}\right)}{\Phi\left(\frac{\hat{\mu}_*}{\hat{\sigma}}\right)}\right).$$

4.1.4. Gradients and hessian matrix

The maximum likelihood estimators are derived by differentiating the loglikelihood function with respect to parameters. Most programing languages including MATLAB have built-in optimization functions. However, it is also desirable to have a gradients and hessian matrix of likelihood function and use it for numerical optimizations. The log-likelihood function of a normal-weighted exponential stochastic frontier model is given by

$$\log L(\alpha, \lambda, \sigma \mid \varepsilon) = \log(\alpha + 1) - \log(\alpha) + \log(\lambda) + \lambda\varepsilon + \frac{\sigma^2\lambda^2}{2} + \log\{\mathrm{K}\},$$

where $\varepsilon = y - X\beta$, $\mathrm{K} = \Phi_1 - \Phi_2 e^C$, $C = \varepsilon\lambda\alpha + .5\sigma^2\lambda^2(\alpha^2 + 2\alpha)$, $\Phi_1 = \Phi\left(-\frac{(\varepsilon+\sigma^2\lambda)}{\sigma}\right)$, and $\Phi_2 = \Phi\left(-\frac{(\varepsilon+\sigma^2\lambda(\alpha+1)}{\sigma}\right)$.

To get the gradients, we need to differentiate the loglikelihood function with respect to each parameter. Therefore, we have

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \alpha} = \frac{1}{\alpha + 1} - \frac{1}{\alpha} + \frac{1}{\mathrm{K}}\frac{\partial \mathrm{K}}{\partial \alpha}.$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \lambda} = \frac{1}{\lambda} + \varepsilon + \lambda\sigma^2 + \frac{1}{\mathrm{K}}\frac{\partial \mathrm{K}}{\partial \lambda}.$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \sigma} = \lambda^2\sigma + \frac{1}{\mathrm{K}}\frac{\partial \mathrm{K}}{\partial \sigma}.$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \beta} = X\lambda + \frac{1}{\mathrm{K}}\frac{\partial \mathrm{K}}{\partial \beta}.$$

The derivative quantities from the above equations are

$$\frac{\partial \mathrm{K}}{\partial \alpha} = -\left(\frac{\partial \mathrm{C}}{\partial \alpha} e^{C} \Phi_2 + \frac{\partial \Phi_2}{\partial \alpha} e^{C}\right).$$

$$\frac{\partial \mathrm{K}}{\partial \lambda} = \frac{\partial \Phi_1}{\partial \lambda} - \left(\frac{\partial \mathrm{C}}{\partial \lambda} e^{C} \Phi_2 + \frac{\partial \Phi_2}{\partial \lambda} e^{C}\right).$$

$$\frac{\partial \mathrm{K}}{\partial \sigma} = \frac{\partial \Phi_1}{\partial \sigma} - \left(\frac{\partial \mathrm{C}}{\partial \sigma} e^{C} \Phi_2 + \frac{\partial \Phi_2}{\partial \sigma} e^{C}\right).$$

$$\frac{\partial \mathrm{K}}{\partial \beta} = \frac{\partial \Phi_1}{\partial \beta} - \left(\frac{\partial \mathrm{C}}{\partial \beta} e^{C} \Phi_2 + \frac{\partial \Phi_2}{\partial \beta} e^{C}\right).$$

Furthermore, we can state the following derivations

$$\frac{\partial \mathrm{C}}{\partial \alpha} = \varepsilon\lambda + \sigma^2\lambda^2\alpha + \sigma^2\lambda^2,$$

$$\frac{\partial \mathrm{C}}{\partial \lambda} = \varepsilon\alpha + \sigma^2\lambda(\alpha^2 + 2\alpha),$$

$$\frac{\partial \mathrm{C}}{\partial \sigma} = \varepsilon\lambda\alpha + \sigma\lambda^2(\alpha^2 + 2\alpha),$$

$$\frac{\partial \mathrm{C}}{\partial \beta} = -X\lambda\alpha.$$

And the derivatives of a standard normal cumulative distribution are given by

$$\frac{\partial \Phi_1}{\partial \lambda} = -\sigma\phi\left(-\frac{(\varepsilon + \sigma^2\lambda)}{\sigma}\right)$$

$$\frac{\partial \Phi_1}{\partial \sigma} = \left(\frac{\varepsilon}{\sigma^2} - \lambda\right)\phi\left(-\frac{(\varepsilon + \sigma^2\lambda)}{\sigma}\right)$$

$$\frac{\partial \Phi_1}{\partial \beta} = \left(\frac{X}{\sigma}\right)\phi\left(-\frac{(\varepsilon + \sigma^2\lambda)}{\sigma}\right)$$

$$\frac{\partial \Phi_2}{\partial \alpha} = -\sigma\lambda\phi\left(-\frac{(\varepsilon + \sigma^2\lambda(\alpha + 1)}{\sigma}\right)$$

$$\frac{\partial \Phi_2}{\partial \lambda} = -\sigma(\alpha + 1)\phi\left(-\frac{(\varepsilon + \sigma^2\lambda(\alpha + 1)}{\sigma}\right)$$

$$\frac{\partial \Phi_2}{\partial \sigma} = \left(\frac{\varepsilon}{\sigma^2} - \lambda(\alpha + 1)\right)\phi\left(-\frac{(\varepsilon + \sigma^2\lambda(\alpha + 1)}{\sigma}\right)$$

$$\frac{\partial \Phi_2}{\partial \beta} = \left(\frac{X}{\sigma}\right)\phi\left(-\frac{(\varepsilon + \sigma^2\lambda(\alpha + 1)}{\sigma}\right)$$

Combining the above derivations, elements of the gradient vector of a normal-weighted exponential stochastic frontier model is given by

$$\frac{\partial \log L(\alpha, \lambda, \sigma, \beta \mid \varepsilon)}{\partial \alpha} = \frac{1}{\alpha + 1} - \frac{1}{\alpha} + \frac{1}{\mathrm{K}}\left(-\left((\varepsilon\lambda + \sigma^2\lambda^2\alpha + \sigma^2\lambda^2)e^C\Phi_2 + -\sigma\lambda\phi_2 e^C\right)\right)$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma, \beta \mid \varepsilon)}{\partial \lambda} = \frac{1}{\lambda} + \varepsilon + \lambda\sigma^2 + \frac{1}{\mathrm{K}}\left(-\sigma\phi_1 - \left(\left(\varepsilon\alpha + \sigma^2\lambda(\alpha^2 + 2\alpha)\right)e^C\Phi_2 + -\sigma(\alpha + 1)\phi_2 e^C\right)\right)$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma, \beta \mid \varepsilon)}{\partial \sigma} = \lambda^2\sigma + \frac{1}{\mathrm{K}}\left(\left(\frac{\varepsilon}{\sigma^2} - \lambda\right)\phi_1 - \left(\varepsilon\lambda\alpha + \sigma\lambda^2(\alpha^2 + 2\alpha)e^C\Phi_2 + \left(\frac{\varepsilon}{\sigma^2} - \lambda(\alpha + 1)\right)\phi_2 e^C\right)\right)$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma, \beta \mid \varepsilon)}{\partial \beta} = X\lambda + \frac{1}{\mathrm{K}}\left(\left(\frac{X}{\sigma}\right)\phi_1 - \left(-X\lambda\alpha e^C\Phi_2 + \left(\frac{X}{\sigma}\right)\phi_2 e^C\right)\right)$$

where $\Phi_1 = \Phi\left(-\frac{(\varepsilon + \sigma^2\lambda)}{\sigma}\right)$, and $\Phi_2 = \Phi\left(-\frac{(\varepsilon + \sigma^2\lambda(\alpha + 1)}{\sigma}\right)$.

The hessian matrix is the second order derivative of the log-likelihood function, and it is given by

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \alpha\alpha} = -\frac{1}{(\alpha + 1)^2} + \frac{1}{\alpha^2} + \frac{\frac{\partial^2 \mathrm{K}}{\partial \alpha\alpha}\mathrm{K} - \left(\frac{\partial \mathrm{K}}{\partial \alpha}\right)^2}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \alpha\lambda} = \frac{\frac{\partial^2 \mathrm{K}}{\partial \alpha\lambda}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \alpha}\frac{\partial \mathrm{K}}{\partial \lambda}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \alpha\sigma} = \frac{\frac{\partial^2 \mathrm{K}}{\partial \alpha\sigma}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \alpha}\frac{\partial \mathrm{K}}{\partial \sigma}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \alpha\beta} = \frac{\frac{\partial^2 \mathrm{K}}{\partial \alpha\beta}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \alpha}\frac{\partial \mathrm{K}}{\partial \beta}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \lambda\lambda} = -\frac{1}{\lambda^2} + \sigma^2 + \frac{\frac{\partial^2 \mathrm{K}}{\partial \lambda\lambda}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \lambda}\frac{\partial \mathrm{K}}{\partial \lambda}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \lambda\sigma} = 2\sigma\lambda + \frac{\frac{\partial^2 \mathrm{K}}{\partial \lambda\sigma}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \lambda}\frac{\partial \mathrm{K}}{\partial \sigma}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \lambda\beta} = -X + \frac{\frac{\partial^2 \mathrm{K}}{\partial \lambda\beta}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \lambda}\frac{\partial \mathrm{K}}{\partial \beta}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \sigma\sigma} = \lambda^2 + \frac{\frac{\partial^2 \mathrm{K}}{\partial \sigma\sigma}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \sigma}\frac{\partial \mathrm{K}}{\partial \sigma}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \sigma\beta} = \frac{\frac{\partial^2 \mathrm{K}}{\partial \sigma\beta}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \sigma}\frac{\partial \mathrm{K}}{\partial \beta}}{\mathrm{K}^2}$$

$$\frac{\partial \log L(\alpha, \lambda, \sigma \mid \varepsilon)}{\partial \beta\beta} = \frac{\frac{\partial^2 \mathrm{K}}{\partial \beta\beta}\mathrm{K} - \frac{\partial \mathrm{K}}{\partial \beta}\frac{\partial \mathrm{K}}{\partial \beta}}{\mathrm{K}^2}$$

Expanding the derivatives within the above equations, we get

$$\frac{\partial^2 \mathrm{K}}{\partial \alpha\alpha} = -\left(\frac{\partial^2 \mathrm{C}}{\partial \alpha\alpha} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial \alpha}\left(\frac{\partial \mathrm{C}}{\partial \alpha} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial \alpha} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial \alpha\alpha} e^C + \frac{\partial \Phi_2}{\partial \alpha}\frac{\partial \mathrm{C}}{\partial \alpha} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial \alpha\lambda} = -\left(\frac{\partial^2 \mathrm{C}}{\partial \alpha\lambda} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial \alpha}\left(\frac{\partial \mathrm{C}}{\partial \lambda} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial \lambda} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial \alpha\lambda} e^C + \frac{\partial \Phi_2}{\partial \alpha}\frac{\partial \mathrm{C}}{\partial \lambda} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\alpha\sigma} = -\left(\frac{\partial^2 \mathrm{C}}{\partial\alpha\sigma} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\alpha}\left(\frac{\partial \mathrm{C}}{\partial\sigma} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\sigma} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\sigma} e^C + \frac{\partial \Phi_2}{\partial\alpha}\frac{\partial \mathrm{C}}{\partial\sigma} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\alpha\beta} = -\left(\frac{\partial^2 \mathrm{C}}{\partial\alpha\beta} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\alpha}\left(\frac{\partial \mathrm{C}}{\partial\beta} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\beta} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\alpha\beta} e^C + \frac{\partial \Phi_2}{\partial\alpha}\frac{\partial \mathrm{C}}{\partial\beta} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\lambda\lambda} = \frac{\partial^2 \Phi_1}{\partial\lambda\lambda} - \left(\frac{\partial^2 \mathrm{C}}{\partial\lambda\lambda} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\lambda}\left(\frac{\partial \mathrm{C}}{\partial\lambda} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\lambda} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\lambda\lambda} e^C + \frac{\partial \Phi_2}{\partial\lambda}\frac{\partial \mathrm{C}}{\partial\lambda} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\lambda\sigma} = \frac{\partial^2 \Phi_1}{\partial\lambda\sigma} - \left(\frac{\partial^2 \mathrm{C}}{\partial\lambda\sigma} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\lambda}\left(\frac{\partial \mathrm{C}}{\partial\sigma} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\sigma} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\lambda\sigma} e^C + \frac{\partial \Phi_2}{\partial\lambda}\frac{\partial \mathrm{C}}{\partial\sigma} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\lambda\sigma} = \frac{\partial^2 \Phi_1}{\partial\lambda\beta} - \left(\frac{\partial^2 \mathrm{C}}{\partial\lambda\beta} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\lambda}\left(\frac{\partial \mathrm{C}}{\partial\beta} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\beta} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\lambda\beta} e^C + \frac{\partial \Phi_2}{\partial\lambda}\frac{\partial \mathrm{C}}{\partial\beta} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\sigma\sigma} = \frac{\partial^2 \Phi_1}{\partial\sigma\sigma} - \left(\frac{\partial^2 \mathrm{C}}{\partial\sigma\sigma} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\sigma}\left(\frac{\partial \mathrm{C}}{\partial\sigma} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\sigma} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\sigma\sigma} e^C + \frac{\partial \Phi_2}{\partial\sigma}\frac{\partial \mathrm{C}}{\partial\sigma} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\sigma\beta} = \frac{\partial^2 \Phi_1}{\partial\sigma\beta} - \left(\frac{\partial^2 \mathrm{C}}{\partial\sigma\beta} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\sigma}\left(\frac{\partial \mathrm{C}}{\partial\beta} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\beta} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\sigma\beta} e^C + \frac{\partial \Phi_2}{\partial\sigma}\frac{\partial \mathrm{C}}{\partial\beta} e^C\right)$$

$$\frac{\partial^2 \mathrm{K}}{\partial\beta\beta} = \frac{\partial^2 \Phi_1}{\partial\beta\beta} - \left(\frac{\partial^2 \mathrm{C}}{\partial\beta\beta} e^C \Phi_2 + \frac{\partial \mathrm{C}}{\partial\beta}\left(\frac{\partial \mathrm{C}}{\partial\beta} e^C \Phi_2 + \frac{\partial \Phi_2}{\partial\beta} e^C\right)\right) - \left(\frac{\partial^2 \Phi_2}{\partial\beta\beta} e^C + \frac{\partial \Phi_2}{\partial\beta}\frac{\partial \mathrm{C}}{\partial\beta} e^C\right).$$

Moreover, we have also the following results

$$\frac{\partial^2 \mathrm{C}}{\partial\alpha\alpha} = \sigma^2\lambda^2$$

$$\frac{\partial^2 \mathrm{C}}{\partial\alpha\lambda} = \varepsilon + 2\sigma^2\lambda\alpha + 2\sigma^2\lambda$$

$$\frac{\partial^2 \mathrm{C}}{\partial\alpha\sigma} = 2\sigma\lambda^2\alpha + 2\sigma\lambda^2$$

$$\frac{\partial^2 \mathrm{C}}{\partial\alpha\beta} = -X\lambda.$$

$$\frac{\partial^2 \mathrm{C}}{\partial\lambda\lambda} = \sigma^2(\alpha^2 + 2\alpha).$$

$$\frac{\partial^2 \mathrm{C}}{\partial\lambda\sigma} = 2\sigma\lambda(\alpha^2 + 2\alpha).$$

$$\frac{\partial^2 \mathrm{C}}{\partial\lambda\beta} = -X\alpha.$$

$$\frac{\partial^2 \mathrm{C}}{\partial\sigma\sigma} = \lambda^2(\alpha^2 + 2\alpha)$$

$$\frac{\partial^2 \mathrm{C}}{\partial\sigma\beta} = -X\lambda\alpha$$

$$\frac{\partial^2 \mathrm{C}}{\partial\beta\beta} = 0$$

Now let us specify the second order derivatives a standard normal cumulative distribution function

$$\frac{\partial^2 \Phi_2}{\partial\alpha\alpha} = -\sigma\lambda\frac{\partial\phi_2}{\partial\alpha}$$

$$\frac{\partial^2 \Phi_2}{\partial\alpha\lambda} = -\left(\sigma\phi_2 + \sigma\lambda\frac{\partial\phi_2}{\partial\lambda}\right)$$

$$\frac{\partial^2 \Phi_2}{\partial\alpha\sigma} = -\left(\lambda\phi_2 + \sigma\lambda\frac{\partial\phi_2}{\partial\sigma}\right)$$

$$\frac{\partial^2 \Phi_2}{\partial\alpha\beta} = -\sigma\lambda\frac{\partial\phi_2}{\partial\beta}$$

$$\frac{\partial^2 \Phi_2}{\partial\lambda\lambda} = -\sigma(\alpha + 1)\frac{\partial\phi_2}{\partial\lambda}$$

$$\frac{\partial^2 \Phi_2}{\partial\lambda\sigma} = -\left((\alpha + 1)\phi_2 + \sigma(\alpha + 1)\frac{\partial\phi_2}{\partial\sigma}\right)$$

$$\frac{\partial^2 \Phi_2}{\partial\lambda\beta} = -\sigma(\alpha+1)\frac{\partial\phi_2}{\partial\beta}$$

$$\frac{\partial^2 \Phi_2}{\partial\sigma\sigma} = \left(\frac{-2\varepsilon}{\sigma^3} + \frac{\varepsilon}{\sigma^2}\frac{\partial\phi_2}{\partial\sigma}\right) - \lambda(\alpha+1)\frac{\partial\phi_2}{\partial\sigma}$$

$$\frac{\partial^2 \Phi_2}{\partial\sigma\beta} = \left(\frac{-X}{\sigma^2} + \frac{\varepsilon}{\sigma^2}\frac{\partial\phi_2}{\partial\beta}\right) - \lambda(\alpha+1)\frac{\partial\phi_2}{\partial\beta}$$

$$\frac{\partial^2 \Phi_2}{\partial\beta\beta} = \left(\frac{X}{\sigma}\right)\frac{\partial\phi_2}{\partial\beta}$$

Moreover, we have

$$\frac{\partial^2 \Phi_1}{\partial\lambda\lambda} = \ -\sigma\frac{\partial\phi_1}{\partial\lambda}$$

$$\frac{\partial^2 \Phi_1}{\partial\lambda\sigma} = -\left(\phi_1 + \sigma\frac{\partial\phi_1}{\partial\sigma}\right)$$

$$\frac{\partial^2 \Phi_1}{\partial\lambda\beta} = \ \sigma\frac{\partial\phi_1}{\partial\beta}$$

$$\frac{\partial^2 \Phi_1}{\partial\sigma\sigma} = \left(\frac{\varepsilon}{\sigma^2} - \lambda\right)\phi_1$$

$$\frac{\partial^2 \Phi_1}{\partial\sigma\sigma} = \left(\frac{-2\varepsilon}{\sigma^3}\phi_1 + \frac{\varepsilon}{\sigma^2}\frac{\partial\phi_1}{\partial\sigma}\right) - \lambda\frac{\partial\phi_1}{\partial\sigma}$$

$$\frac{\partial^2 \Phi_1}{\partial\sigma\beta} = \left(\frac{-X}{\sigma^2}\phi_1 + \frac{\varepsilon}{\sigma^2}\frac{\partial\phi_1}{\partial\beta}\right) - \lambda\frac{\partial\phi_1}{\partial\beta}$$

$$\frac{\partial^2 \Phi_1}{\partial\beta\beta} = \left(\frac{X}{\sigma}\right)\frac{\partial\phi_1}{\partial\beta}$$

Note that the standard normal density function ϕ_1 is given by

$$\phi_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(-\frac{(\varepsilon+\sigma^2\lambda)}{\sigma}\right)^2}$$

Then the derivatives with respect to each parameter are

$$\frac{\partial\phi_1}{\partial\lambda} = -(\varepsilon + \sigma^2\lambda)\phi_1$$

$$\frac{\partial\phi_1}{\partial\sigma} = -\left(-\frac{(\varepsilon + \sigma^2\lambda)}{\sigma}\right)\left(\frac{\varepsilon}{\sigma^2} - \lambda\right)\phi_1$$

$$\frac{\partial\phi_1}{\partial\beta} = -\left(-\frac{(\varepsilon + \sigma^2\lambda)}{\sigma}\right)\frac{X}{\sigma}\phi_1$$

Similarly, the standard normal density function ϕ_2 is given by

$$\phi_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(-\frac{\left(\varepsilon+\sigma^2\lambda(\alpha+1)\right)}{\sigma}\right)^2}$$

$$\frac{\partial\phi_2}{\partial\alpha} = -\left(-\frac{\left(\varepsilon + \sigma^2\lambda(\alpha + 1)\right)}{\sigma}\right)(-\sigma\lambda)\phi_1$$

$$\frac{\partial\phi_2}{\partial\lambda} = -\left(-\frac{\left(\varepsilon + \sigma^2\lambda(\alpha + 1)\right)}{\sigma}\right)\left(-\sigma(\alpha + 1)\right)\phi_1$$

$$\frac{\partial\phi_2}{\partial\sigma} = -\left(-\frac{\left(\varepsilon + \sigma^2\lambda(\alpha + 1)\right)}{\sigma}\right)\left(-\left(\frac{-\varepsilon}{\sigma^2} + \lambda(\alpha + 1)\right)\right)\phi_1$$

$$\frac{\partial\phi_2}{\partial\beta} = -\left(-\frac{\left(\varepsilon + \sigma^2\lambda(\alpha + 1)\right)}{\sigma}\right)\left(\frac{X}{\sigma}\right)\phi_1$$

Substituting the above results gives us an element of a hessian matrix.

4.2. Monte Carlo (MC) simulation study

In this section we use Monte Carlo (MC) study to examine the finite sample properties of the maximum likelihood estimator obtained from a normal-weighted exponential stochastic frontier model. A comparison is made between the maximum likelihood estimator of a normal-exponential stochastic frontier model and the maximum likelihood estimator of a normal-weighted exponential stochastic frontier model.

To simulate artificial data, we have the following data generating process (DGP) of stochastic frontier model,

$$y = Xb + v - u,$$

where X is $N\ by$ 3 matrix of inputs, v is the statistical error and follows the normal distribution. And the inefficiency score, $u,$ follows a weighted exponential distribution.

Artificial data on explanatory variables, $x's$, are derived from a standard uniform distribution, using built in functions in $MATLAB$.

The parameter values, of the coefficients of the production function, needed for generating random output vector y are

$$b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The pseudo-random numbers for v are from a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.25$. i.e., $v \sim N(0,0.25)$. The inefficiency score, u is generated from a weighted exponential distribution. Depending on shape parameter of the weighted exponential distribution, we use two data generating process (DGP).

In the first part we have generated pseudo random numbers from the weighted exponential distribution with shape parameter $\alpha = 1$ and scale parameter $\lambda = 0.5$. i.e., $u \sim WED(1,0.5)$. In the second part of simulation study, we used a weighted exponential distribution with the shape parameter of $\alpha = 0.5$ and scale parameter $\lambda = 0.5$, i.e., $u \sim WED(0.5,0.5)$. There is no built-in function in $MATLAB$ for generating pseudo random numbers from the weighted exponential distribution. However, we can generate two independent random samples from the exponential distributions $u_1 \sim Exp(\lambda)$ and $u_2 \sim Exp(\lambda(\alpha + 1))$, and we add these samples to make them samples from the weighted exponential distribution, that is $u = u_1 + u_2 \sim WED(\alpha, \lambda)$ (Farahani & Khorram, 1994).

In the simulation study, we have used to sample sizes of $N = 500$ and $N = 1000$. And each simulation is iterated two hundred times, (simu=200). Table 3 and Table 4 shows a Monte Carlo (MC) simulation study of the maximum likelihood estimator for a normal-weighted exponential stochastic frontier model and a normal-exponential stochastic frontier model. In the first column, the parameters of composite error term and parameter of production function are listed, and the corresponding true parameter values are in the second column. As the simulation result shows, the maximum likelihood estimates of the normal-weighted exponential stochastic frontier model are not far from the true parameter values.

When the sample size is 500, the average estimate for the shape parameter is 1.1047 and the bias is $(1 - 1.1047 = -0.1047)$. But if we increase the sample size to 1000 the average estimate for the shape parameter becomes 0.9812 and the bias decreases to 0.0188. Similarly, when the sample size increases, the standard error of estimates for the shape parameter decrease from 0.9421 to 0.6235. Because there is no shape parameter in a normal-exponential stochastic frontier model, there is no estimate for the shape parameter of $\alpha = 1$. For the rest of the parameters the average estimate and the standard errors, under the two stochastic frontier models, are reported in the Table 3 below.

The simulation study shows the superior performance of a normal-weighted exponential stochastic frontier model over a normal-exponential stochastic frontier model. A very significant difference is in estimating the intercept of the production function. In our simulation study, the true value of an intercept in the data generating process (DGP) is 3 and the corresponding estimate under a normal-weighted exponential stochastic frontier model is 2.9806, which means the bias is 0.0194. However, under a normal-exponential stochastic frontier model the intercept estimate is 2.3144, and the bias is 0.6856. If we increase the sample size, when $N = 1000$, the bias of estimating the intercept, in a normal-weighted exponential stochastic frontier model, is 0.0049, but for a normal-exponential the bias is 0.6907.

Table 3. Simulation of a normal-weighted exponential stochastic frontier model (When α=1)

Parameters	True Values	N=500				N=1000			
		Weighted Exponential		Exponential		Weighted Exponential		Exponential	
		Est.	Std. Err.	Est.	Std. Err.	Est.	Std. Err.	Est.	Std. Err.
α	1	1.1047	0.9421	-	-	0.9812	0.6235	-	-
λ	0.5	0.5243	0.0795	0.4347	0.0272	0.5203	0.0591	0.4330	0.0193
σ	0.25	0.1918	0.1298	0.5174	0.0854	0.2278	0.0767	0.5270	0.0612
$Cons.$	3	2.9806	0.1758	2.3144	0.1733	2.9951	0.116	2.3093	0.1256
b_1	2	2.0177	0.1943	2.0028	0.1955	2.0039	0.1386	2.0077	0.1462
b_2	1	1.0145	0.1918	1.0095	0.1992	1.0049	0.1262	1.0063	0.1311

***Note: Est. stand for parameter estimates, while Std. Err. is the corresponding standard error

In the second part of our simulation study, we have generated artificial data for the inefficiency error distribution of a weighted exponential distribution with a shape parameter $\alpha = 0.5$. As shown in Table 4, the simulated maximum likelihood estimates of the normal-weighted exponential stochastic frontier are satisfactory. For the sample size of 500, the biased of estimating the shape parameter α is 0.197. When the sample size increases to 1000 the biased of estimating the shape parameter decreases to 0.0146. Similarly, other parameters are estimated with a small bias. When the sample size increased from 500 to 1000 the bias also decreased substantially.

The second simulation study shows the better performance of the normal-weighted exponential stochastic frontier model over the normal-exponential stochastic frontier model. For example, under a normal-weighted exponential stochastic frontier model, the bias in estimating the intercept of the production function is 0.0351. However, under a normal-exponential stochastic frontier model, the bias of estimating the intercept is 0.8263.

Table 4. Simulation of a normal-weighted exponential stochastic frontier model (When α=0.5)

Parameters	True Values	N=500				N=1000			
		Weighted Exponential		Exponential		Weighted Exponential		Exponential	
		Est.	Std. Err.	Est.	Std. Err.	Est.	Std. Err.	Est.	Std. Err.
α	0.5	0.6970	0.6286			0.5146	0.4854		
λ	0.5	0.5069	0.0757	0.4014	0.0257	0.5193	0.0642	0.4001	0.0185
σ	0.25	0.1923	0.1342	0.5976	0.0988	0.2254	0.0855	0.6089	0.0715
$Cons.$	3	2.9649	0.1866	2.1737	0.1991	2.9898	0.1260	2.1666	0.1469
b_1	2	2.0175	0.2136	2.0028	0.2191	2.0032	0.1517	2.0080	0.1658
b_2	1	1.0218	0.2181	1.0088	0.2229	1.0053	0.1385	1.0071	0.1483

4.3. Estimating Carbon Efficiency of Manufacturing Firms in Africa

First, we estimate the stochastic frontier models and get carbon efficiency estimates of each manufacturing firms in Africa. Next, we present a descriptive statistics of carbon efficiency estimates. Moreover, African countries will be ranked based on their carbon efficiency level in their manufacturing sector. The last section presents the determinants of carbon efficiency of manufacturing firms in Africa.

4.3.1. Descriptive Statistics of Variables

After cleaning the missing values on dependent variables and independent variables the sample size reduces to 1911. Descriptive statistics of the variables used in the stochastic input requirement function and in the determinants of efficiency are presented bellow in Table 5. Fuel consumption is the dependent variable, and the independent variables are output, labor, capital, and intermediate input.

Table 5. Descriptive Statistics of Variables for SFMs

Variable	Observation	Mean	Standard Deviation	Minimum	Maximum
Fuel Consumption (in logs)	1,911	12.23132	3.326794	1.609438	24.97176
Output (in logs)	1,911	16.64467	3.210953	8.055158	27.63102
Labor (in logs)	1,911	14.2756	3.389969	.6931472	25.12999
Capital (in logs)	1,911	15.10818	4.123346	0	32.59386
Intermediate (in logs)	1,911	14.87828	3.668987	1.098612	27.01484

4.3.2. Estimating Fuel Consumption Using a Stochastic Frontier Model

In estimating carbon efficiency of manufacturing firms, we have estimated the input requirement function. Fuel consumption is the dependent variable, and the independent variables are firm's output, labor, capital, and intermediate inputs. We have estimated stochastic frontier models and three different probability distributions are assumed for the inefficiency part. As it is shown in the Table 6 below all the input variables and output are statistically significant in explaining variations in fuel consumptions. In all three stochastic frontiers model specifications the estimated coefficients are almost the same and all are statistically significant at 1% level of significance. However, in a normal-half normal stochastic frontier model the intercept and the parameter of the inefficiency distribution are not statistically significant. In the normal-exponential and normal-weighted exponential stochastic frontier models all the estimates are statistically significant. Fuel consumption is positive related with output and other factors of production.

Table 6. Estimation of Fuel Consumption under different SFMs

Variables	Stochastic Frontier Models		
	Half-normal	Exponential	Weighted Exponential
ln_Output	0.239 ***	0.239 ***	0.238 ***
	(0.025)	(0.018)	(0.001)
ln_Labor	0.371 ***	0.371 ***	0.371 ***
	(0.025)	(0.019)	(0.001)
ln_Capital	0.099 ***	0.099 ***	0.099 ***
	(0.015)	(0.010)	(0.002)
ln_Intermediate	0.169 ***	0.169 ***	0.169 ***
	(0.025)	(0.022)	(0.003)
_cons	-1.055	-1.105 ***	-1.229 ***
	(3.140)	(0.034)	(0.002)
σ_v	1.735 ***	1.734 ***	1.729 ***
	(0.028)	(0.025)	(0.0003)
σ_u	0.002		
	(3.927)		
λ		18.623 ***	7.013 ***
		(1.209)	(0.001)
α			1.965 ***
			(0.002)
Log likelihood	3764.4058	3764.4	3753.4
Number of Observation	1,911	1,911	1,911

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

4.3.3. Descriptive Statistics of Carbon inefficiency Estimates

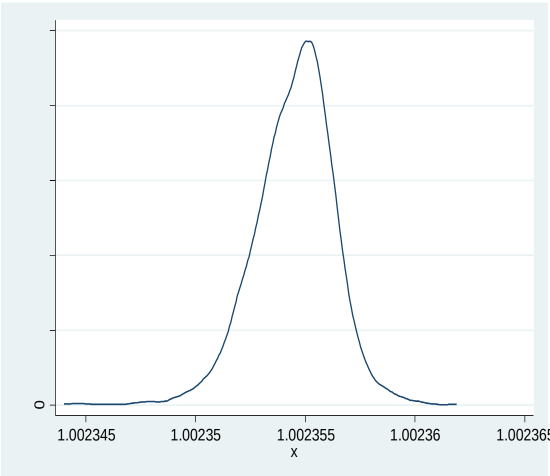
The primary interest in fitting a Stochastic Frontier Model is to obtain the estimates of carbon inefficiency score for each manufacturing firms. Carbon inefficiency is the ratio of actual fuel consumption to the frontier (minimum) fuel consumption. Since the actual fuel consumption is always greater than the optimal fuel consumption the inefficiency estimates are always greater than one. Therefore, if a firm's fuel consumption is equal to the minimum possible frontier fuel consumption level, then the firm is fully efficient. However, if a firm consume higher than the minimum required, by the frontier function, then we have inefficiency. Alternatively, we can use another representation of carbon efficiency. Carbon efficiency level is the exponent of negative of value of the inefficiency score. And the inefficiency scores estimates are the values that we get from a composite error term.

Table 7 shows an average, minimum, and maximum values of carbon inefficiency estimate across different stochastic frontier models. The estimates in the parenthesis are the corresponding carbon efficiency level. All manufacturing firms have an estimate of efficiency level which is close to one, which is a super efficiency estimate. If we estimate using a normal-half normal stochastic frontier model, we get carbon efficiency between 99.897% and 99.898%. Which indicates that all manufacturing firms in Africa are supper efficient in their full consumption. Similarly, if we use a normal-exponential stochastic frontier model, the estimates of carbon efficiency are between 96.312% and 97.509%. For a normal-weighted exponential stochastic frontier model the minimum carbon efficiency estimate is 91.16% and the maximum carbon efficiency estimate is 95.09%.

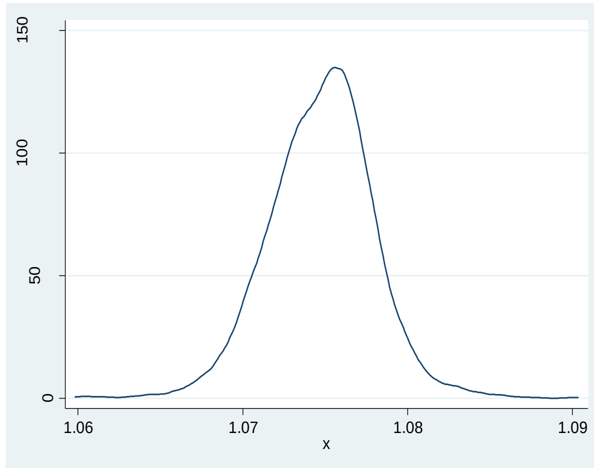
Table 7. Descriptive statistics of carbon inefficiency estimates

	Mean	Minimum	Maximum
Half-normal	1.002354 (0.998979)	1.002344 (0.998984)	1.002362 (0.998976)
Exponential	1.074752 (0.969177)	1.059799 (0.975092)	1.090364 (0.963126)
Weighted Exponential	1.15506 (0.939315)	1.122895 (0.950907)	1.237519 (0.911602)

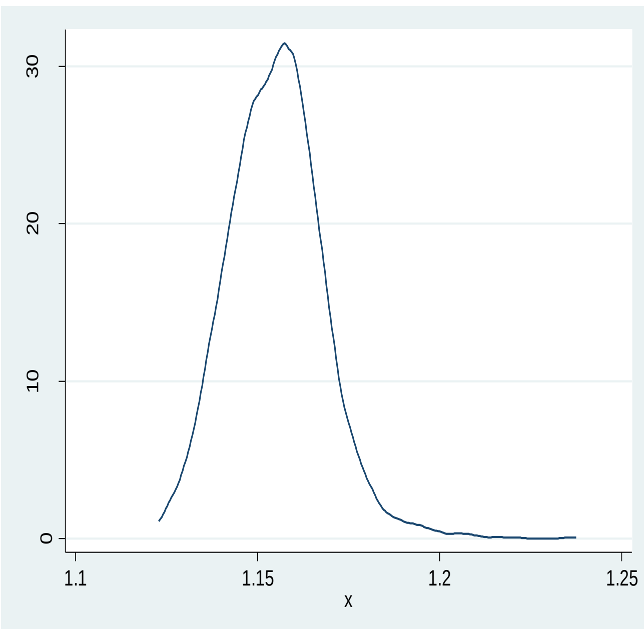
We can plot the carbon inefficiency estimates and see the distribution of the estimates. Figure 3. shows a kernel density of the estimates of efficiency score of manufacturing firms, in Africa.



a) Normal-Half normal



b) Normal-Exponential



c) Normal-Weighted Exponential

Figure 3. Kernel density of inefficiency estimates

4.3.4. Rank of African Countries Based on Carbon Efficiency

There are two ways of interpreting the inefficiency estimates. The most straight forward approach is to measure the proportion of actual fuel consumption to the estimated fuel consumption. Alternatively, it is also possible to transform these estimates to the values ranging between 0 and 1.

In the production frontier function, the efficiency level, $U = \frac{y}{f(x;\theta)e^{v}} = e^{-u}$ is defined as the ratio of actual production, y to the stochastic frontier, $f(x;\theta)e^{v}$. We can extend this relationship of inefficiency score and efficiency level to the cost frontier as well. If the inefficiency score u approaches to zero, then the efficiency level e^{-u} approaches to 1. Which means the firm is producing almost near to the frontier. In general, the relationship between the efficiency level $U\epsilon(0,1)$ and the inefficiency score $u\epsilon(0,\infty)$ can be stated as.

$\lim_{u\to\infty} e^{-u} = 0$ and $\lim_{u\to 0} e^{-u} = 1$

In our study of the carbon efficiency of manufacturing firms in Africa, we have ranked African countries based on their carbon efficiency. We have efficiency estimates under three distribution assumptions for the inefficiency error in the stochastic frontier model. Table 8 shows that in all different model specifications, Egypt is the most carbon efficient country. Egypt has carbon efficiency of 99.89% under the normal-half normal stochastic frontier model, 96.99% under a normal-exponential stochastic frontier model, and 94.21% under a normal-weighted exponential stochastic frontier model. The rank of three most efficiency countries remain the same regardless of different specification of the inefficiency error.

Table 8. Rank of African Countries Based on Carbon Efficiency

	Half-normal		Exponential		Weighted Exponential	
Rank	**Countries**	**Efficiency**	**Countries**	**Efficiency**	**Countries**	**Efficiency**
1	Egypt	0.998979632	Egypt	0.969897048	Egypt	0.942098777
2	Morocco	0.998979447	Morocco	0.969595461	Morocco	0.941066612
3	Ghana	0.998979397	Ghana	0.969546412	Ghana	0.940978019
4	Ethiopia	0.998979333	Ethiopia	0.969430163	Botswana	0.94037868
5	Tunisia	0.99897931	Tunisia	0.969391918	Tunisia	0.940375189
6	Botswana	0.998979285	Botswana	0.969390826	Ethiopia	0.94020189
7	Angola	0.998979177	Burkina	0.969171487	Burkina	0.939469457
8	Burkina	0.998979167	Angola	0.969148905	Angola	0.939353536
9	Burundi	0.998978972	Burundi	0.968828249	Burundi	0.938101772
10	Zambia	0.998978941	Zambia	0.968745477	Tanzania	0.937792434
11	Nigeria	0.998978894	Tanzania	0.968737156	Djibouti	0.937787973
12	Tanzania	0.998978888	Nigeria	0.968716511	Madagascar	0.937497806
13	Madagascar	0.998978866	Djibouti	0.96871447	Nigeria	0.937467785
14	Malawi	0.998978786	Madagascar	0.968671356	Zambia	0.93733418
15	South Sudan	0.998978761	Malawi	0.968537718	Malawi	0.937004853
16	Djibouti	0.998978744	South Sudan	0.96845909	South Sudan	0.936802902
17	Uganda	0.998978699	Uganda	0.968408457	Uganda	0.936035393
18	Mauritania	0.998978628	Mauritania	0.9682597	Mauritania	0.935745438

4.3.5. Determinants of Carbon Efficiency of Manufacturing Firms in Africa

After estimating the carbon efficiency level of each manufacturing firm, it is also possible to estimate the determinants of carbon efficiency of manufacturing firms in Africa. To estimate the determinants of a carbon efficiency of manufacturing firms we have used the Ordinary Least Square (OLS) method. The efficiency estimate derived from the stochastic frontier model is the dependent variable and the independent variables are managerial experience, financial obstacle, firm size, import or export status, and foreign ownership. We have run the regression model on the efficiency estimates derived from three stochastic frontier models. These stochastic frontier models are normal-half normal, normal-exponential, and normal-weighted exponential.

Here we are estimating the determinants of carbon efficiency of manufacturing firms in Africa for two reasons. One is to compare the performance of the normal-weighted stochastic frontier model with other stochastic frontier models. The second reason is to examine the derivers of carbon efficiency of manufacturing firms in Africa. Table 9 shows that most of the variables included the regression equation are statistically significant. Comparing the three stochastic frontier models, most variables are statistically significant and R^2 is higher under the normal-weighted exponential stochastic frontier model. Therefore, of the three stochastic frontier models the normal-weighted exponential stochastic frontier model explains most of the variations in the efficiency level of manufacturing firms in Africa.

There exists a copious of studies in estimating the determinant of efficiency level of manufacturing firms (Smriti & Khan, 2018). Smriti and Khan (2018) found that the firm size, manager's experience, and annual losses due to power outage are important variable in explaining why some firms are more efficient than others. In addition, foreign ownership and exporting status are important variable in explaining the productivity difference in manufacturing sector (Islam & Hyland, 2018).

Even though most of the variables are statistically significant, their marginal effect is small, and all independent variables poorly explain the overall variations in the efficiency level. Managerial experience is a continues variable and it measures the number of years that the CEO or top manager has served the company. As it is shown in the Table 9 in all three stochastic frontier models, managerial experience negatively affects carbon efficiency of the manufacturing firms.

However, the marginal effects are very small, for the normal-half normal stochastic frontier model a one-year increase in managerial experience increases the carbon efficiency by $2.95e-08$. And the marginal effect of managerial experience in normal-exponential and normal-weighted exponential models are 0.0000510 and 0.000218, respectively.

Table 9. Determinants of carbon inefficiency

Independent Variables	Models		
	Half normal	Exponential	Weighted Exponential
Manager Experience	-2.95e-08***	-0.0000510***	-0.000218***
	(3.83e-09)	(0.00000662)	(0.0000272)
Financial Obstacle			
Minor Obstacle	0.000000440***	0.000789***	0.00366***
	(0.000000126)	(0.000218)	(0.000899)
Moderate Obstacle	0.000000218	0.000375	0.00149
	(0.000000125)	(0.000217)	(0.000891)
Major Obstacle	0.000000372**	0.000639**	0.00241**
	(0.000000130)	(0.000225)	(0.000925)
Very Sever Obstacle	0.000000263	0.000441	0.00155
	(0.000000146)	(0.000253)	(0.00104)
Firm Size			
Small	-0.000000767	-0.00138	-0.00560
	(0.000000590)	(0.00102)	(0.00418)
Medium	-0.00000111	-0.00198	-0.00824*

	(0.000000591)	(0.00102)	(0.00418)
Large	-0.00000149*	-0.00261*	-0.0104*
	(0.000000592)	(0.00102)	(0.00419)
Export	8.95e-18*	1.62e-14*	7.14e-14*
	(4.06e-18)	(7.02e-15)	(2.88e-14)
Foreign	3.38e-09*	0.00000598*	0.0000267**
	(1.39e-09)	(0.00000240)	(0.00000990)
_cons	1.002***	1.077***	1.164***
	(0.000000591)	(0.00102)	(0.00419)
N	1759	1759	1745
R^2	0.086	0.086	0.089

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Chapter five: Conclusion and Recommendation

5.1. Conclusion

In this study a weighted exponential distribution is used as the distribution of the inefficiency score in a stochastic frontier model. A weighted exponential distribution is a flexible two parameter distribution, and it is possible to derive a closed form likelihood function and JLMS inefficiency estimator for a normal-weighted exponential stochastic frontier model. Moreover, we have derived the gradient and hessian matrix of the likelihood function of a normal-weighted exponential stochastic frontier model.

A Monte Carlo (MC) simulation study is implemented to examine the finite sample properties of the maximum likelihood estimator of a normal-weighted exponential stochastic frontier model. Pseudo random numbers are generated using built-in functions for generating random numbers in $MATLAB$. A comparison is made with maximum likelihood estimator of a normal-exponential stochastic frontier model. The simulation result shows that, a normal-weighted exponential stochastic frontier model performs well compared to a normal-exponential stochastic frontier model, given the data generating process is a normal-weighted exponential stochastic frontier model. As the sample size increases the bias and the standard errors of the maximum likelihood estimator of a normal-weighted exponential distribution decrease.

To demonstrate the usefulness a normal-weighted exponential stochastic frontier model, we have used a real data application and compared it with other stochastic frontier models. The real data application is on carbon efficiency of manufacturing firm in Africa. Three stochastic frontier models are estimated to get the carbon efficiency estimates of each manufacturing firms in Africa. All the three stochastic frontier models give almost similar estimates for the parameters of a production function. A summary of descriptive statistics of carbon efficiency estimates are discussed. Moreover, African countries will be ranked based on their carbon efficiency level in their manufacturing sector. Our estimation result shows that of the 18 African countries covered in the study, Egypt it top one carbon efficiency country. Moreover, we have also estimated a multiple linear regression model to see the determinants of carbon efficiency of manufacturing firms in Africa. Top managers experience in the firm, the degree of obstacle for financial access, firm size, export, and foreign ownership are important variables explaining variations in carbon efficiency of manufacturing firms in Africa.

5.2. Recommendation

Based our studies we recommend the following, both for researchers and policy makers. The likelihood function of a normal-weighted exponential stochastic frontier model is flexible model, and it provides a closed form solution. Therefore, applied researchers are recommended to use it for their productivity and efficiency analysis. Our derivation of the likelihood function is for cross sectional data. Therefore, a researcher is recommended to extend the model for times series and panel data models.

Researchers interested on Monte Carlo (MC) simulation study can extend the study into times series and panel data cases. Moreover, it is also possible to compare other stochastic frontier models with a normal-weighted exponential stochastic frontier model.

Appendix A: MATLAB Code for Monte Carlo (MC) Simulation Study

A.1. Parameter and their estimate

```
par1=[1,.5,.25,3,2,1];
sim=200; n1=500;n2=1000;
[wed_1,exp_1]=Simu_wed(par1,n1,sim);
[wed_2,exp_2]=Simu_wed(par1,n2,sim);
```

A.1.1. Parameters and normal-weighted exponential estimates for the first simulation

```
[par1' wed_1 wed_2]
```

ans = 6×5

1.0000	1.1047	0.9421	0.9812	0.6235
0.5000	0.5243	0.0795	0.5203	0.0591
0.2500	0.1918	0.1298	0.2278	0.0767
3.0000	2.9806	0.1758	2.9951	0.1160
2.0000	2.0177	0.1943	2.0039	0.1386
1.0000	1.0145	0.1918	1.0049	0.1262

A.1.2. Parameters and normal-exponential estimates for the first simulation

```
[exp_1 exp_2]
```

ans = 5×4

0.4347	0.0272	0.4330	0.0193
0.5174	0.0854	0.5270	0.0612
2.3144	0.1733	2.3093	0.1256
2.0028	0.1955	2.0077	0.1462
1.0095	0.1992	1.0063	0.1311

```
par2=[.5,.5,.25,3,2,1];
[wed_3,exp_3]=Simu_wed(par2,n1,sim);
[wed_4,exp_4]=Simu_wed(par2,n2,sim);
```

A.1.3. Parameters and normal-weighted exponential estimates for the second simulation

```
[par2' wed_3 wed_4]
```

ans = 6×5

0.5000	0.6970	0.6286	0.5146	0.4854
0.5000	0.5069	0.0757	0.5193	0.0642
0.2500	0.1923	0.1342	0.2254	0.0855
3.0000	2.9649	0.1866	2.9898	0.1260
2.0000	2.0175	0.2136	2.0032	0.1517
1.0000	1.0218	0.2181	1.0053	0.1385

A.1.4. Normal-Exponential estimates for the second simulation

```
[exp_3 exp_4]
```

ans = 5×4

0.4014	0.0257	0.4001	0.0185
0.5976	0.0988	0.6089	0.0715
2.1737	0.1991	2.1666	0.1469
2.0028	0.2191	2.0080	0.1658
1.0088	0.2229	1.0071	0.1483

A.2. Simulating from weighted exponential distribution and MLE for normal-wed and normal-exponential models

```
function [WED_estimate,Exp_estimate,Wed,Exp] = Simu_wed(par,n,sim)
rng(123);
a=par(1);
lam=par(2);
s=par(3);
Coef=par(4:end);
x=ones(n,1);x1=rand(n,1);x2=rand(n,1);X=[x x1 x2];
theta0_wed = [1,1,1,1,1,1];       % this sets the initial parameter vector
theta0_exp = [1,1,1,1,1];
wed_sim_estimate=zeros(6,sim);
exp_sim_estimate=zeros(5,sim);
```

```
for i=1:sim
v=normrnd(0,s,n,1);
u=draw_wed(n,lam,a);
y=X*Coef'+v-u;
options = optimoptions(@fminunc,'Algorithm','quasi-
newton','Display','off','MaxFunEvals',10000,'MaxIter',8000,'TolFun',1e-20,'TolX',1e-16);
[MLE_wed,~,~,~] = fminunc(@(par)loglik_wed(par,y,X),theta0_wed,options);
[MLE_exp,~,~,~] = fminunc(@(par)loglik_exp(par,y,X),theta0_exp,options);

wed_sim_estimate(:,i)=MLE_wed';
exp_sim_estimate(:,i)=MLE_exp';
end
Wed = wed_sim_estimate;
Exp = exp_sim_estimate;

WED_estimate=[mean(wed_sim_estimate,2) std(wed_sim_estimate')'];
Exp_estimate = [mean(exp_sim_estimate,2) std(exp_sim_estimate')'];
end
```

A.3. Log-likelihood function of a normal-weighted exponential stochastic frontier model

```
function LL = loglik_wed(par,y,X)
a=par(1);
lam=par(2);
s=par(3);
coef=par(4:end);
if a<=0
   LL=-100000;
elseif lam<=0
   LL=-100000;
elseif s<=0
   LL=-100000;
else
e=y-X*coef';
lik=normal_wed_pdf(e,s,lam,a);
lik=log(lik);
```

```
LL=sum(lik);
end
LL=-LL;
end
```

A.4. Log-likelihood function of a normal-exponential stochastic frontier model

```
function LL = loglik_exp(par,y,X)
lam=par(1);
s=par(2);
coef=par(3:end);
if lam<=0
    LL=-100000;
elseif s<=0
    LL=-100000;
else
e=y-X*coef';
lik=normal_exp_pdf(e,s,lam);
lik=log(lik);
LL=sum(lik);
end
LL=-LL;
end
```

A.5. A normal-weighted exponential probability distribution

```
function f = normal_wed_pdf(e,s,lam,a)
mu1=-(e+s^2*lam);
mu2=-(e+s^2*lam+s^2*lam*a);
f_1 = ((a+1)/a)*lam*exp(lam*e+.5*s^2*lam^2).*normcdf(mu1/s);
f_2 = ((a+1)/a)*lam*exp((a+1)*lam*e+.5*s^2*lam^2*(a+1)^2).*normcdf(mu2/s);
f = f_1-f_2;
end
```

A.6. A normal-exponential probability distribution

```
function f = normal_exp_pdf(e,s,lam)
```

```
mu1=-(e+s^2*lam);
f = lam*exp(lam*e+.5*s^2*lam^2).*normcdf(mu1/s);
end
```

A.7. Drawings from a weighted exponential distribution

```
function draw = draw_wed(n,lam,a)
r1 = exprnd(1/lam,n,1);
r2 = exprnd(1/(lam*(a+1)),n,1);
draw=r1+r2;
end
```

Appendix B: MATLAB Code for a Normal-Weighted Exponential Stochastic Frontier Model

B.1. Load the data

```
Data=xlsread("Data.xls");
y=log(Data(:,1));
N=length(y);
x=ones(N,1);
Output=log(Data(:,2));
Labour=log(Data(:,3));
Capital=log(Data(:,4));
Intermediate=log(Data(:,5));
X=[x Output Labour Capital Intermediate];
```

B.2. Initialization

```
init = [1,1,1,1,0,0,0,0];      % this sets the initial parameter vector
```

B.3. Maximum Likelihood Estimate

```
options = optimoptions(@fminunc,'Display','off');
[MLE,fval,~,~,~,hessian]=fminunc(@(par)loglik(par,y,X),init,options);
```

B.4. Test statistic and p-value

```
se=real(sqrt(diag(inv(hessian))));
t_static=abs(MLE'./se);
df=length(y)-length(MLE);
p_value=2*(1-tcdf(t_static,df));
[MLE' se  p_value]
```

ans = 8×3

```
1.9658    0.0019         0
7.0129    0.0004         0
```

```
   1.7297    0.0003         0
  -1.2278    0.0023         0
   0.2377    0.0014         0
   0.3708    0.0029         0
   0.0990    0.0015         0
   0.1699    0.0026         0
```

```
'Log-Likelihood'
```

ans = 'Log-Likelihood'

```
fval
```

fval = 3.7534e+03

B.5. Efficiency Estimate

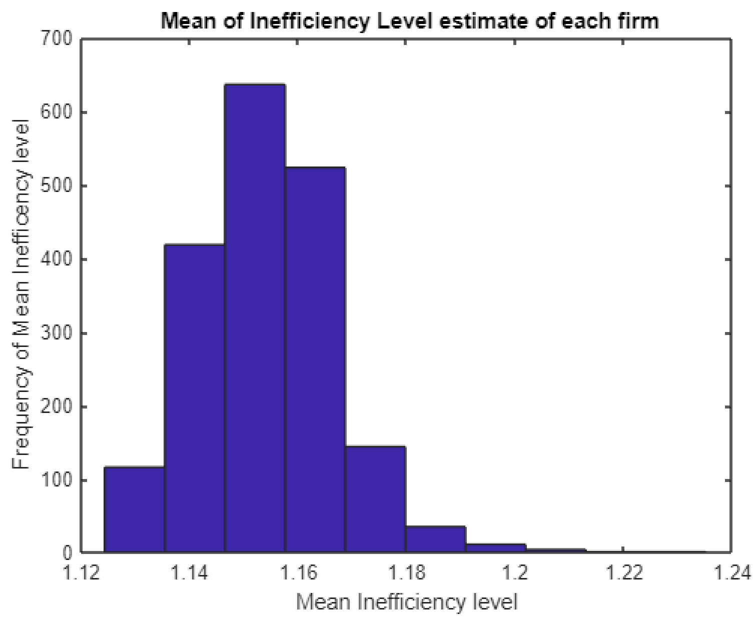
```
res=y-X*MLE(4:end)';
a=MLE(1);
lam=MLE(2);
s=MLE(3);
mu1=res-lam*s^2;
mu2=res-(a+1)*lam*s^2;
EC=-res*lam*a+.5*lam^2*s^2*(a^2+2*a);
C=normcdf(mu1/s)-normcdf(mu2/s).*EC;
W1=normcdf(mu1/s)./C;
W2=normcdf(mu2/s).*EC./C;
Eff_1=mu1+s*(normpdf(mu1/s)./normcdf(mu1/s));
Eff_2=mu2+s*(normpdf(mu2/s)./normcdf(mu2/s));
Eff_mean=W1.*Eff_1-W2.*Eff_2;
U_mean=exp(Eff_mean);
[min(U_mean) max(U_mean)]
```

ans = 1×2

```
    1.1244    1.2351
```

B.6. Graphical display of efficiency estimate

```
hist(U_mean)
U_Range = min(U_mean):0.000001:max(U_mean);
N = hist(U_mean,U_Range);
title(' Mean of Inefficiency Level estimate of each firm')
xlabel('Mean Inefficiency level');
ylabel('Frequency of Mean Inefficency level');
```



B.7. The log-likelihood function

```
function [LL] = loglik(par,y,X)
a=par(1);
lam=par(2);
s=par(3);
if a<=0
   lik=-100000;
elseif lam <= 0
   lik=-100000;
elseif s <= 0
   lik=-100000;
else
coef=par(4:end);
e=y-X*coef';
```

```
mu1=(e-s^2*lam);
mu2=(e-s^2*lam*(1+a));
lik=log(lam)+log(a+1)-log(a)-e*lam+.5*s^2*lam^2+log(normcdf(mu1/s)-...
    exp(-e*lam*a+s^2*lam^2*a+.5*s^2*lam^2*a^2).*normcdf(mu2/s));
end
LL=-sum(lik);
end
```

References

Aigner, D., Lovell, C. K. & Schmidt, P., 1977. Formulation and Estimation of Stochastic Frontier Production Function Model. *Journal of Econometrics,* pp. 21-37.

Arrow, K. J., Chenery, H. B., Minhas, B. S. & Solow, R. M., 1961. Capital-Labor Substitution and Economic Efficiency. *The Review of Economics and Statistics,* 43(3), pp. 225-250.

Beckers, D. E. & Hammond, C. J., 1987. A tractable likelihood function for the normal-gamma stochastic frontier model. *Economics Letters,* Volume 24, pp. 33-38.

Beckers, E. D. & Hammond, E. C., 1987. A Tractable Likelihood Funciton for the Normal-Gamma Stochastic Frontier Model. *Economics Letters,* pp. 33-38.

Carree, M. A., 2002. T echnological inefficiency and the skewness of the error component in stochastic frontier analysis. *Economics Letters,* pp. 101-107.

Christensen, L. R. & Lau, D. W. J. L. J., 1973. Transcendental Logarithmic Production Frontiers. *The Review of Economics and Statistics,* 55(1), pp. 28-45.

Cobb, C. W. & Douglas, P. H., 1928. A Theory of Production. *American Economic Association,* 18(1), pp. 139-165.

Coelli, T. J., Rao, D. P., O'Donnell, C. J. & Battese, G. E., 2005. *An Introduction to Efficiency and Productivity Analysis.*. 2nd ed. USA: Library of Congress Cataloging-in-Publication Data, Springer.

Farahani, Z. S. M. & Khorram, E., 1994. Bayesian Statistical Inference for Weighted Exponential Distribution. *Communications in Statistics—Simulation and Computation,* p. 1362–1384.

Gómez-Déniz, E. & Pérez-Rodríguez, J. V., 2015. Closed-form solution for a bivariate distribution in stochastic frontier models with dependent errors. *Journal of Productivity Analysis,* 43(2), pp. 215-223 .

Greene, W. H., 1990. A gamma-distributed stochastic frontier model. *Journal of Econometrics,* Issue 46, pp. 141-163..

Greene, W. H., 2003. Simulated Likelihood Estimation of the Normal-Gamma Stochastic Frontier Function. *Journal of Productivity Analysis,* 19(2/3), pp. 179-190.

Greene, W. H., 2003. Simulated Likelihood Estimation of the Normal-Gamma Stochastic Frontier Function. *Journal of Productivity Analysis,* Volume 19, pp. 179-190.

Griffin, J. E. & Steel, M. F. J., 2008. Flexible mixture modelling of stochastic frontiers. *Journal of Productivity Analysis*, 29(1), pp. 33-50.

Gupta, R. D. & Kundu, D., 2009. A new class of weighted exponential distributions. *A journal of theoretical and applied statistics* , 43(6), pp. 621-634.

Hajargasht, G., 2015. Stochastic frontiers with a Rayleigh distribution. *Journal of productivity analysis,* pp. 199-208.

Hu, J.-L. & Honma, S., 2014. A Comparative Study of Energy Efficiency of OECD Countries: An Application of the Stochastic Frontier Analysis. *Energy Procedia*, Issue 61, pp. Jin-Li Hu;Satoshi Honmab.

Islam, A. & Hyland, M., 2018. *The Drivers and Impacts of Water Infrastructure Reliability: A Global Analysis of Manufacturing Firms,* s.l.: World Bank.

Jiang, L., Folmer, H., Ji, M. & Tang, J., 2017. Energy efficiency in the Chinese provinces: a fixed effects stochastic frontier spatial Durbin error panel analysis. *Ann Reg Sci*, p. 301–319.

Jin, T. & Kim, J., 2019. A comparative study of energy and carbon efficiency for emerging countries using panel stochastic frontier analysis. *Nature Scientific Reports.*

Jondrow, J., Lovell, C. K., Materov, S. I. & Schmidt, P., 1982. On the estimation of technical inefficiency in the stochastic frontier production function model. *Journal of Econometrics*, Issue 19, pp. 233-238.

Kumbhakar, S. C. & Lovell, C. A. K., 2000. *Stochastic Frontier Analysis.* s.l.:Cambrige University Press.

Kumbhakar, S. C., Parameter, C. F. & Zelenyuk, V., 2018. Stochastic Frontier Analysis: Foundations and Advances. *School of Economics, University of Queensland. Australia*, p. 105.

Li, M.-J. & Tao, W.-Q., 2017. Review of methodologies and polices for evaluation of energy efficiency in high energy-consuming industry. *Applied Energy*, Issue 187, p. 203–215.

Llorca, M., Banos, J., Somoza, J. & Arbue´s, P., 2017. A Stochastic Frontier Analysis Approach for Estimating Energy Demand and Efficiency in the Transport Sector of Latin America and the Caribbean. *The Energy Journal*, 38(5), pp. 153-174.

Meeusen, W. & Broeck, V. D. J., 1977. Efficiency Estimation From Cobb-Douglass Production Function With Composed Error. *International Economic Review*, Volume 18, pp. 435-444.

Papadopoulos, A., 2021. Stochastic frontier models using the Generalized Exponential. *Journal of Productivity Analysis*, p. 15.

Parmeter, C. F. & Kumbhakar, S. C., 2014. *EFFICIENCY ANALYSIS: A PRIMER ON RECENT ADVANCES*. s.l., s.n.

Ritter, C. & Simar, L., 1997. Pitfalls of Normal-Gamma Stochastic Frontier Models. *Journal of Productivity Analysis*, 8(2), pp. 167-182.

Saliola, F. & Seker, M., 2011. Total Factor Productivity Across the Developing World. *Enterprise Surveys, World Bank,* pp. 1-8.

Smith, M. D., 2008. Stochastic frontier models with dependent error components. *The Econometrics Journal*, 11(1), pp. 172-192 .

Smriti, T. N. & Khan, M. H. R., 2018. Efficiency analysis of manufacturing firms using Data Envelopment Analysis (DEA) technique. *Journal of Data Science,* pp. 69-78.

Stevenson, R. E., 1980. Likelihood Function for Generalized Stochastic Frontier Estimation. *Journal of Econometrics*, pp. 57-66.

Stevenson, R. E., 1980. Likelihood Functions for Generalized Stochastic Frontier Estimation. *Journal of Econometrics,* Volume 13, pp. 57-66.

Tsionas, E. G., 2000. Full Likelihood Inference in Normal-Gamma Stochastic Frontier Models. *Journal of Productivity Analysis*, 13(3), pp. 183-205.

Tsionas, E. G., 2007. Efficiency Measurement with the Weibull. *Oxford Bulletin of Economics and Statistics*, 69(5), pp. 693-704.

Tsionas, E. G., 2012. Maximum likelihood estimation of stochastic frontier models by the Fourier transform. *Journal of Econometrics,* Issue 170, pp. 234-248.

Wheat, P., Stead, A. D. & Greene, W. H., 2019. Robust stochastic frontier analysis: a Student’s t-half normal model with application to highway maintenance costs in England. *Journal of Productivity Analysis*, pp. 21-38.