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LEGENDRE DIFFERENTIAL EQUATION AND ITS APPLICATION

BY

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Approval

This thesis has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

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Permission

This is to certify that this thesis is prepared by Teame Asfeha in the Department of Mathematics Addis Ababa University, under my supervision. I here also confirm that the thesis can be submitted for evaluation by examiner and eventual defense.

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Declaration

I declare that this thesis has been composed by me and no part of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other similar title institution of learning. All relevant sources of materials have been duly acknowledged.

TEAME ___________

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Abstract

In this thesis we study various solution techniques of the Legendre differential equations, properties of the Legendre polynomials and the physical applications of Legendre equations and Legendre polynomials to heat conduction and expansion of electromagnetic potential.

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Legendre's equation occur in many areas of applied mathematics, physics and chemistry in physical situation with a spherical geometry such as flow of an ideal fluid past a sphere, the determination of the electric field due to a charged sphere and the determination of the temperature distribution in a sphere given its surface.

Legendre differential equation was introduced by Legendre in the last 18^{th}century and takes the form of

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+k\,(k+1)y\;=0. \tag{1.1}$$

The general solution of the above equation in case where k = 0,1,2,3,… (a positive integer) is given by $y\;=a_1p_k+a_2Q_k$ where a_1 and a_2 are constants, $P_k(\,x)$ is Legendre polynomial (polynomial solution with even exponent) and $Q_k(x)$ is the Legendre polynomial (polynomial solution with odd exponent) . We will obtain explicit representation of these polynomials and discuss their various properties and applications.

1.2 Preliminaries

1.2.1 Definitions

Definition An equation involving independent and dependent variables and the derivatives or differentials of one or more dependent variables with respect to one or more independent variable is called a *differential equation*.

Example 1 i) $\frac{dy}{dx}=4x+5$ is first order ODE.

ii) $\frac{e^y d^2y}{dx^2}+2(\frac{dy}{dx})^2=1$ is second order ODE.

iii) $\frac{\partial^2y}{\partial t^2}-4\frac{\partial^2y}{\partial x^2}=0$ is second order PDE.

iv) $\frac{4d^3y}{dx^3}+sin(x)\,(\frac{dy}{dx})^2+5xy=0$ is third order ODE.

All the above equations are examples of differential equation.

Definition A differential equation which involves derivatives with respect to a single independent variable is known as an *ordinary differential equation* (ODE).

Definition A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a *partial differential equation* (PDE).

Definition The order of the highest order derivative involved in a differential equation is called *the order of the differential equation.*

Definition The degree of a differential equation is the degree of the highest order derivative present in the equation after the differential equation has been made free from the radicals and fractions as far as the derivatives are concerned.

Definition A differential equation in which the dependent variables and all its derivatives present occur in the first degree only and no products of dependent variables and /or derivatives occur is known as a *linear*. It has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots \quad + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)\ . \qquad (1.2)$$

Definition A differential equation which is not linear is said to be a *nonlinear* differential equation.

Definition A homogeneous second order linear differential equation (ODE) has the form

$$y'' + p_1(x)y' + p_2(x)y = 0 \qquad (1.3)$$

Where $p_1(x)$ and $p_2(x)$ are continuous in some interval I.

Definition A solution of a differential equation is a relation between dependent and independent variables not involving the derivatives such that this relation and the derivative obtained from it satisfies the given differential equation.

Definition A solution which contains a number of arbitrary constants equal to the order of the differential equation is called the *general solution* of the differential equation.

Definition A solution obtained from a general solution by giving particular values to the constants is called *a particular solution* of the differential equation.

Theorem: Two solutions y_1 and y_2 of equation (1.3) are linearly independent in the interval I if and only if they satisfy Wronskain property which is defined by

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = y_1(x)\, y'_2(x) - y'_1(x)y_2(x) \neq 0\ .$$

Theorem: If $y_1(x)$ and $y_2(x)$ are solutions of equation (1.3) and a_1 and a_2 are arbitrary constants, then $a_1y_1(x) + a_2y_2(x)$ is also solution of equation(1.3).

Definition Power series is a series of the form

$$\sum_0^\infty a_n(x-x_o)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n x^n + \ldots \qquad (1.4)$$

in which the coefficients a_n (n=0, 1, 2, 3,4...) and the point x_o are independent of x.

The power series centered at $x_0 = 0$ is often referred as $\sum_0^\infty a_n x^n$.

Definition: A Legendre's differential equation is second order ordinary differential equation of the form

$(1-x^2)y'' - 2xy' + k(k+1)y = 0$ Where k is a positive integer.

Theorem: for Legendre differential equation we can find two linearly independent power series basic solution y_1 and y_2 centered at $x = 0$ of (1.1) in the form $y = \sum_0^\infty a_n x^n$

1.2.2 The method of reduction of order

Let $$y'' + Py' + Qy = 0 \qquad (1.5)$$

be a linear second order homogeneous differential equation.

The main idea is reducing equation (1.5) to a linear first order differential equation on the interval I .Let $y_1(x)$ is a known solution of (1.5) on I and $y_1(x) \neq 0$ for every x in the interval I. if we define

$$y(x) = y_1(x)u(x). \qquad (1.6)$$

it follows that

$$\begin{aligned} y' &= uy'_1 + y_1u' \\ y'' &= uy''_1 + y'_1u' + y_1u'' + y'_1u' = uy''_1 + 2y'_1u' + y_1u'' \end{aligned} \qquad (1.7)$$

Substitute (1.7) to (1.6) we get

$$uy''_1 + 2y'_1u' + y_1u'' + p(uy'_1 + y_1u') + Qy_1(x)u = 0 \qquad (1.8)$$

$$u(y''_1 + py'_1 + Qy_1) + u'(2y'_1 + py_1) + y_1u'' = 0$$

$$y_1u'' + (2y'_1 + py_1)\, u' = 0 \qquad (1.9)$$

Let w = u' then $w' = u''$

$$y_1w' + (2y'_1 + py_1)\, w = 0$$

$$y_1\frac{dw}{dx} + (2y'_1 + py_1)\, w = 0$$

$$\frac{dw}{dx} + \left(\frac{2y'_1}{y_1} + p\right) w = 0 \qquad (1.10)$$

$$\frac{dw}{dx} = \left(\frac{2y'_1}{y_1} + p\right) w$$

$$\frac{dw}{w} = -\left(\frac{2y'_1}{y_1} + p\right)dx$$

$$\frac{dw}{w} = -\frac{2y'_1}{y_1}dx - pdx \qquad (1.11)$$

$$\ln|w| = -2\ln(y_1) - \int pdx + c_1$$

$$\ln|w| + 2\ln(y_1) = -\int pdx + c_1$$

$$\ln|wy_1{}^2| = -\int pdx + c_1$$

$$wy_1{}^2 = c_1e^{-\int pdx} \qquad (1.12)$$

We solve the last equation for w, use $w = u'$ and integrate again

$$u'y_1{}^2 = c_1e^{-\int pdx} \qquad (1.13)$$

$$uy_1{}^2 = c_1\int e^{-\int pdx}\,dx + c_2$$

$$u = c_1\int \frac{e^{-\int pdx}}{y_1{}^2}dx + c_2 \qquad (1.14)$$

Since c_1 and c_2 are arbitrary constant we can choose $c_1 = 1$ and $c_2 = 0$

Then (1.14) becomes

$$u = \frac{\int e^{-\int pdx}dx}{y_1{}^2}dx. \qquad (1.15)$$

Substitute (1.15) into (1.6) we get

$y_2 = y_1(x)\frac{\int e^{-\int pdx}dx}{y_1{}^2}dx$ is the second solution of (1.5)

Therefore the general solution of (1.5) is the linear combination $y = c_1y_1 + c_2y_2$ where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I.

1. 2.3 Power series solution

Let the given equation be

$$y'' + p(x)y' + Q(x)y = 0 \qquad (1.16)$$

Theorem: Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation(1.16), we can find two linearly independent solutions in the form of a power series centered at x_0, that is

$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.

For every power series there is a number $R \geq 0$, known as the radius of convergence. A series solution converges at least on some interval defined by $|x - x_0| < R$ otherwise divergent.

Consider $x = 0$ is an ordinary point of (1.16) then based on the above theorem (1.16) has two non-trivial linearly independent power series solutions centered at $x_0 = 0$ of the form

$$y=\sum_{n=0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (1.17)$$

In order to get the coefficient $a_n's$ (1.17) we follow the following procedure.

Differentiating (1.17) twice in succession with respect to 'x'

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad (1.18)$$

putting the values of y, y' and y'' in (1.16) and equating the constant term and the coefficients of various powers of x to zero since it is an identity finally solving equation (1.17),we can obtain the coefficients of (1.17) in terms of a_0 and a_1 where a_0 and a_1 are arbitrary constants. Substituting these coefficients in (1.17) we obtain the required series solution of (1.16) in power of x.

Some of the properties of power series are:

Every power series has an interval of convergence.

Every interval of convergence has a radius of convergence R.

We can determine the radius of convergence by finding the ratio of successive terms of the series

$i.e \ \lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < R.$

A power series convergences for $|x - x_o|$<R and diverges for $|x - x_o|$>R.

When R = 0 the interval of convergence consists of the single number x when R = ∞ the power series convergences for all number x.

If R is not 0 or ∞ then the interval of convergence may include the endpoints $x_0 - R$ and $x_0 + R$.

A power series represents a continuous function within its interval of convergence.

A power series can be differentiated term wise within its interval of convergence.

Two power series with a common interval of convergence can be added term by term.

***Example 2*:** Find the power series solution of the equation $y'' + xy' + y = 0$ in powers of x

Solution*:* Given that

$$y'' + xy' + y = 0 \qquad (1.19)$$

Substitute (1.18) to (1.19) we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad (1.20)$$

$$2a_2 + 6a_3x + 12a_4x^2 + \cdots + x(a_1 + 2a_2x + 3a_3x^2 + \ldots) + a_o + a_1x + a_2x^2 + a_3x^3 = 0 \qquad (1.21)$$

And collect terms

$$2a_2 + a_0 + x(6a_3 + 2a_1) + x^2(12a_4 + 3a_2) + x^3(20a_5 + 4a_3)\cdots = 0 \qquad (1.22)$$

Next we require the coefficient of each power of x^n to vanish giving

$2a_2 + a_0 = 0$ so that $a_2 = \frac{-a_0}{2}$ (1.23)

$6a_3 + 2a_1 = 0$ so that $a_3 = \frac{-a_1}{3}$ (1.24)

$12a_4 + 3a_2 = 0$ so that $a_4 = \frac{(-a_2)}{4}$ (1.25)

$20a_5 + 4a_3 = 0$ so that $a_5 = \frac{(-a_3)}{5}$ (1.26)

eq (1.23) − (1.26) have the recurrence relation

$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$, n =1, 2, 3... (1.27)

or $a_{n+2} = -\frac{a_n}{(n+2)}$ for all $n \geq 1$ (1.28)

The general solution of eq (1.19)

$$y = a_0 f(x) + a_1 g(x) \qquad (1.29)$$

where $f(x) = 1 - \frac{x2}{2} + \frac{1}{2.4}x^4 - \frac{3}{3.4.6}x^6 + \frac{x^8}{2.4.6.8} + \cdots$ (1.30)

$g(x) = x - \frac{x^3}{3} + \frac{1}{3.5}x^5 - \frac{1}{3.5.7}x^7 + \frac{x^9}{3.5.7.9} + \cdots$ (1.31)

$$\text{i.e } y = a_0\left[1 - \frac{x2}{2} + \frac{1}{2.4}x^4 - \frac{3}{3.4.6}x^6 + \frac{x^8}{2.4.6.8} + \cdots\right]$$

$$+ a_1\left[x - \frac{x^3}{3} + \frac{1}{3.5}x^5 - \frac{1}{3.5.7}x^7 + \frac{x^9}{3.5.7.9} + \cdots\right]$$

Chapter 2

Legendre Differential Equation and Its Applications

2.1 Legendre Differential Equation

The Legendre differential equation is the homogeneous second order ordinary differential equation of the form

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \qquad |x| < 1 \tag{2.1}$$

where $\lambda = k(k+1)$, k is a non -negative integer

Or equivalently $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0$ is its self adjoin form.

When we divide (2.1) by $(1 - x^2)$ we get

$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{k(k+1)}{(1-x^2)}y = 0$. We write this equation as

$$y'' + P(x)y' + Q(x)y = 0 \tag{2.2}$$

where $P(x) = \frac{-2x}{1-x^2}$ and $Q(x) = \frac{k(k+1)}{(1-x^2)}$.

both $P(x)$ and $Q(x)$ are analytic everywhere except at the point $x = \pm 1$.

The behavior of the coefficients $P(x)$ and $Q(x)$

a) at $x = 0$, i) $\lim_{x\to 0} p(x) = \lim_{x\to 0}\frac{-2x}{(1-x^2)} = 0$ which is finite

ii) $\lim_{x\to 0} Q(x) = \lim_{x\to 0}\frac{k(k+1)}{(1-x^2)} = k(k+1)$ which is finite

Therefore $x = 0$ is the ordinary point.

b) at $x = \pm 1$, i) $\lim_{x\to 1} p(x) = \lim_{x\to 1}\frac{2x}{(1-x^2)} = \infty$

ii) $\lim_{x\to 1} Q(x) = \lim_{x\to 1} \frac{k(k+1)}{(1 - x^2)} = \infty$

But $\lim_{x\to 1}(x-1)\, p(x) = \lim_{x\to 1}(x-1)\frac{2x}{(1-x^2)} = -1$ finite

and $\lim_{x\to 1}(x-1)^2\, Q(x) = \lim_{x\to 1}(x-1)^2 \frac{k(k+1)}{(1-x^2)} = 0$ finite

Therefore we can conclude that eq (2.1) has singular point at $x = -1$ and at $x = 1$ while $x = 0$ is an ordinary point.

We can find power series solutions centered at $x = 0$ that converges for $|x| < 1$.

Now we construct such series solution.

2.2 Power series solution of Legendre's differential equation

Legendre's differential equation is one of the important equations in mathematical physics. It is usually written in the following form of

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0 \qquad k > 0\,, |x| < 1$$

We can solve it using a power series expansion in the neighborhood of x = 0, which is a regular point for the equation. We assume the solution is form of

$$y = \sum_{n=0}^{\infty} a_n (x-0)^n$$

$$y(x) = \sum_0^{\infty} a_n x^n = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \tag{2.3}$$

Next, we need first and second derivative of expression(2.3) and differentiate it i.e.

$$y' = \sum_{n=1}^{\infty} na_n x^{n-1} \tag{2.4}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \tag{2.5}$$

Now, substituting (2.2) , (2.4) and(2.5) to eqn(2.1). the resulting expression yields

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0 \tag{2.6}$$

The first summation contributes only for $n = 2$ and the second contributes for $n = 1$ then

$$(1 - x^2)\left[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right] - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0 \tag{2.7}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + k(k+1)\sum_{n=0}^{\infty} a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n = 0 \qquad (2.8)$$

In the third summation we can start counting from 0, because terms with n = 0 is null.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + k(k+1)\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2n\sum_{n=0}^{\infty} a_n x^n = 0 \quad (2.9)$$

Degree	1^{st}summation	2^{nd}summation	3^{rd}summation	4^{th}summation
x^0	2(1)a_2	k(k+1)a_0	0	0
x^1	3(2)a_3	k(k+1)a_1	0	-2(1)a_1
x^2	4(3)a_4	k(k+1)a_2	-2(1)a_2	-2(2)a_2
x^3	5(4)a_5	k(k+1)a_3	-3(2)a_3	-2(3)a_3
x^4	6(5)a_6	k(k+1)a_4	-4(3)a_4	-2(4)a_4
. . . x^n	. . . (n+2)(n+1) a_{n+2}	. . . k(k+1)a_n	. . . -n(n-1) a_n	. . . -2(n)a_n

As we absorb the sum of the coefficients of x^n is equal to zero

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}x^n + (k(k+1)-2.n - n(n-1))a_n]\, x^n = 0 \qquad (2.10)$$

$$(n+2)(n+1)a_{n+2} + [k(k+1)-2n - n(n-1)]a_n = 0$$

$$(n+2)(n+1)a_{n+2} + (k^2+k-2n-n^2+n)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = -(k^2 - n^2 + k - n)a_n$$

$$(n+2)(n+1)a_{n+2} = -[(k-n)(k+n) + (k-n)]\, a_n$$

$$(n+2)(n+1)a_{n+2} = -(k-n)(k+n+1)\, a_n$$

$$a_{n+2} = -\frac{(k-n)(k+n+1)}{(n+2)(n+1)} a_n \quad n = 0,1,2.3\ldots \qquad (2.11)$$

is the recurrence relation formula of the Legendre equation.

From (2.11) one can calculate all coefficients a_n, once $a_0$and a_1 are known.

For even coefficients we have:

for $n = 0$ $$a_2 = \frac{-k(k+1)}{2x1} a_0 = \frac{-k(k+1)}{2!} a_0$$

for $n = 2$ $$a_4 = -\frac{(k-2)(k+3)}{4x3} a_2 = -\frac{(k-2)(k+3)}{4x3} \frac{(-k(k+1))}{2x1} a_0$$

$$= \frac{(k+3)(k+1)k(k-2))}{4!} a_0$$

for $n = 4$ $$a_6 = -\frac{(k-4)(k+5))}{6x5} a_4 = -\frac{(k+5)}{6x5} \frac{(k+3)(k+1)k(k-2)(k-4)}{4!} a_0$$

$$= -\frac{(k+5)(k+3)(k+1)k(k-2)(k-4)}{6!} a_0$$

. . .

. . .

$$a_{2m} = (-1)^m \frac{[(k+2m-1)(k+2m-3)\ldots(k+1)k(k-2)\ldots(k-2m+2)}{(2m)!} a_0 \quad \text{with } m = 0,1,2, \qquad (2.12)$$

Similarly for odd coefficients,

for $n = 1$ $$a_3 = -\frac{(k-1)(k+2)}{3.2} a_1 = \frac{-(k+2)(k-1)}{3!} a_1$$

for $n = 3$ $$a_5 = -\frac{(k-3)(k+4)}{5x4} a_3 = \frac{(k+4)(k+2)(k-1)(k-3)}{5!} a_1$$

for $n = 5$ $$a_7 = -\frac{(k-5)(k+6)}{7x6} a_5$$

$$= -\frac{(k-5)(k+6)}{7x6} \frac{(k-3)(k+4)(k-1)(k+2)}{5!} a_1$$

$$= -\frac{(k+6)(k+4)(k+2)(k-1)(k-3)(k-5)}{7!} a_1$$

.. . .

$$a_{2m+1} = \frac{(-1)^m[(k+2m)(k+2m-2)\ldots(k+2)(k-1)(k-3)\ldots(k-2m+1]}{(2m+1)!} a_1 \text{ , } m = 0, 1, 2,\ldots \qquad (2.13)$$

Substituting the above coefficients back in(2.3) the assumed solution we get

$$y(x) \ = \sum_0^{\infty} a_n x^n = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \qquad (2.14)$$

$$y(x) = \ a_0y_1(x) + a_1y_2(x) \qquad (2.15)$$

where $y_1(x) \ = \ a_o + a_2x^2 + a_4x^4 + a_6x^6 + \ldots$

$$y_1(x) = a_0[1 + \sum_{m=0}^{\infty}(-1)^m \ \frac{[(k+2m-1)(k+2m-3)\ldots k(k-2)\ldots(k-2m+2)}{(2m)!} x^{2m} \ ]$$

$$= a_o \ [1 - \frac{(k+1)k}{2!}x^2 + \frac{(k+3)(k+1)k(k-2)}{4!}x^4 - \frac{((k+5)(k+3)(k+1)k(k-2)(k-4)}{6!}x^6 \ldots] \qquad (2.16)$$

$$y_2(x) \ = \ a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \ \ldots$$

$$y_2(x) = a_1[x + \sum_{m=0}^{\infty}(-1)^m \frac{[(k+2m)(k+2m-2)\ldots(k+2)(k-1)(k-3)\ldots(k-2m+1]}{(2m+1)!} x^{2m+1}]$$

$$= a_1[x^1 - \frac{-(k+2)(k-1)}{3!}x^3 + \frac{(k+4)(k+2)(k-1)(k-3)}{5!}x^5 - \frac{(k+6)(k+4)(k+2)(k-1)(k-3)(k-5)}{7!}x^7 \ldots] \ (2.17)$$

$$y(x) = a_0 \left[1 + \sum_{m=0}^{\infty}(-1)^m \ \frac{[(k+2m-1)(k+2m-3)\ldots k(k-2)\ldots(k-2m+2)}{(2m)!} x^{2m} \ \right]$$

$$+ a_1[x + \sum_{m=0}^{\infty}(-1)^m \frac{[(k+2m)(k+2m-2)\ldots(k+2)(k-1)(k-3)\ldots(k-2m+1]}{(2m+1)!} x^{2m+1}]$$

$$y(x) \ = \ a_o[1 - \frac{(k+1)k}{2!}x^2 + \frac{(k+3)(k+1)k(k-2)}{4!}x^4 - \frac{((k+5)(k+3)(k+1)k(k-2)(k-4)}{6!}x^6 \ldots]$$

$$+ \ a_1[x^1 - \frac{(k+2)(k-1)}{3!}x^3 + \frac{(k+4)(k+2)(k-1)(k-3)}{5!}x^5 - \frac{(k+6)(k+4)(k+2)(k-1)(k-3)(k-5)}{7!}x^7 \ldots] \ \ (2.18)$$

is the series solution of Legendre differential where a_0 , a_1 are arbitrary constants and

Given that the coefficients with even index (a_2, a_4, a_6,…) are only dependent on a_0 and those with odd index (a_3, a_5, a_7, . .) are only dependent on a_1.
The two solutions are converges for |x|<1 i. e on the interval $(-1, 1)$ and these series are linearly independent solutions which can be verified by evaluating the Wronskian at the ordinary point, where $y_1(x)$ and $y_2(x)$ are as defined above and $y'_1(x)$ and $y'_2(x)$ and are the first order derivatives of the $y_1(x)$ and $y_2(x)$ respectively. At the ordinary point x = 0.

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

***Remark*:**

A series converges if the ratio of two consecutive terms converges to a number smaller than 1 for n getting bigger and bigger.

- using the ratio test that the series defining y_1 and y_2 converge on the interval $(-1,1)$

$$\lim_{m\to\infty} \left|\frac{a_{2(m+1)}x^{2(m+1)}}{a_{2m}x^{2m}}\right| = \lim_{m\to\infty} \left|\frac{k(k+1)-2m(2m+1)}{(2m+2)(2m+1)} x^2\right| = |x^2| = x^2$$

Thus the power series converges for $|x| < 1$

- every k either y_1 or y_2 is unbounded on $(-1,1)$.That is as $x \to 1$ or as $x \to -1$ one of the following holds ,either $|y_1(x)| \to \infty$ or $|y_2(x)| \to \infty$
- The only case in which Legendre equation has abounded solution on [-1, 1] is when the parameter λ has the form $\lambda=k\ (k+1)$ with k=0 or $k\in \mathbb{Z}^+$.In this case either y_1 or y_2 is polynomial (the series terminate) this case is considered below.

2.3 Legendre Solution (Legendre polynomials)

Definition: The polynomial solution, denoted by $P_k(x)$, of degree k of (2.1) which satisfies $P_k(1)$, = 1 is called the Legendre polynomial of degree k. The Legendre functions (Legendre polynomials) are the solutions of Legendre differential equation with variable coefficients $(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0$ where k = 0 or $k\in \mathbb{Z}^+$

Since the Legendre differential equation is a second order ordinary differential equation it has two linearly independent solutions.

Consider the recurrence relation

$$a_{n+2} = -\frac{(k-n)(k+n+1)a_n}{(n+2)(n+1)} \qquad \forall n = 0,1,2,3,\ldots \tag{2.19}$$

To obtain the expression of the Legendre polynomial we first invert the recurrence relation to re-write it as

$$a_n = -\frac{(n+2)(n+1)}{(k-n)(k+n+1)} a_{n+2}\,. \tag{2.20}$$

The coefficient a_0 and $a_1$are arbitrary so, we choose $a_0 = 1$.

$a_k = \frac{(2k)!}{2^k(k!)^2} = \frac{1.3.5.7...(2k-1)}{k!}$ for k = 1, 2, 3,… the reason for this choice of a_k is to have all Legendre polynomial with $x = 1$ as 1i.e. all of $p_k(x) = 1\ \forall k \in \mathbb{Z}^+$

$$p_n(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.21}$$

for $n = k - 2$, $a_{k-2} = \frac{-k(k-1)}{2(2k-1)} a_k$ substituting the expression a_k

$$a_{k-2} = \frac{-k(k-1)}{2(2k-1)} \frac{(2k)!}{2^k(k!)^2} = \frac{-k(k-1)}{2(2k-1)} \frac{(2k)(2k-1)(2k-2)!}{2^k\, k(k-1)(k-2)!k(k-1)!} = \frac{-(2k-2)!}{2^k(k-2)!(k-1)!}$$

For $n = k - 4$, $a_{k-4} = \frac{-(k-2)(k-3)}{4(2k-3)} a_{k-2}$ substituting the expression of a_{k-2} we get

$$a_{k-4} = \frac{-(k-2)(k-3)}{4(2k-3)} a_{k-2} = \frac{-(k-2)(k-3)}{4(2k-3)} \left[\frac{(2k-2)!}{2^k(k-2)!(k-1)!}\right]$$

$$= \frac{-(k-2)(k-3)}{4(2k-3)} \frac{(2k-2)!}{2^k(k-2)(k-3)(k-4)!(k-1)(k-2)!} = \frac{(2k-4)!}{2^k 2!(k-2)!(k-4)!}$$

Therefore in general when $k - 2m \geq 0$ we can write the coefficient a_{k-2m} as

$$a_{k-2m} = \frac{(-1)^m (2k-2m)!}{2^k m!(k-m)!(k-2m)!} \tag{2.22}$$

Substitute it to $p_n(x) = \sum_{n=0}^{\infty} a_n x^n$

The resulting solution of Legendre's differential equation is called the Legendre polynomial denoted by

$$p_k(x) = \sum_{m=0}^{M} a_{k-2m} x^{k-2m} = \sum_{m=0}^{M} \frac{(-1)^m (2k-2m)!}{2^k m!(k-m)!(k-2m)!} x^{k-2m} \tag{2.23}$$

$$Where\ M = \begin{cases} \frac{k}{2}, if\ k\ is\ even \\ \frac{(k-1),}{2}\ if\ k\ is\ odd \end{cases}$$

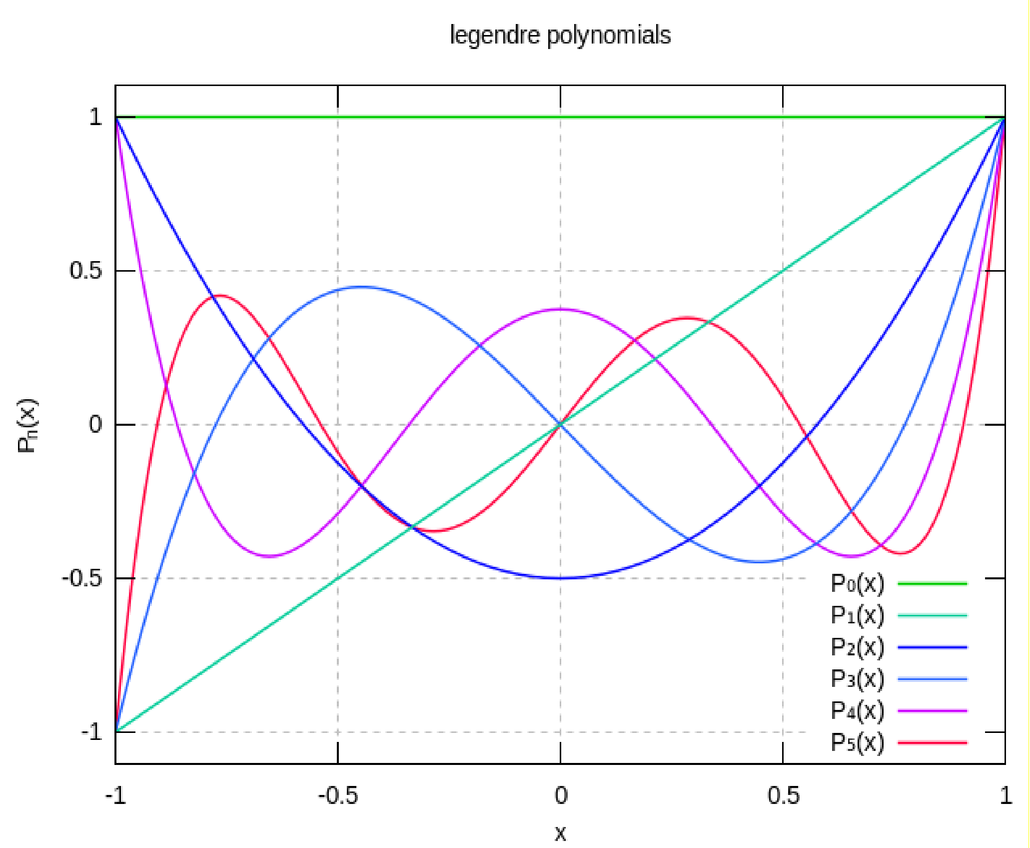
Example 3: Write the first six Legendre polynomials, using formula (2.23).

Solution: The first six Legendre polynomials are:

$p_o(x) = 1$ $\qquad\qquad$ $p_1(x) = x$

$p_2(x) = \frac{1}{2}(3x^2 - 1)$. $\qquad\qquad$ $p_3(x) = \frac{1}{2}(5x^3 - 3x)$

$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ $\qquad\qquad$ $p_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$



Note that: if k is even (respectively odd) then the only powers of x involved in p_k are even (respectively odd) and so an even (respectively odd).

Proposition: If y(x) a bounded solution on the interval $(-1, 1)$ of the Legendre equation (2.15) then there exists a constant c such that $y(x) = cp_k(x)$ where $p_k(x)$ the k^{th} Legendre polynomial.

Theorem: (Rodrigues formula for Legendre‘s polynomials)

Legendre polynomials can be computed iteratively one after the other with the aid of a formula which makes use of repeated derivatives. This formula is known as Rodrigues' formula.

Rodrigues formula of Legendre‘s polynomials is expressed as

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n \tag{2.24}$$

Proof

$$\text{Let } y = (x^2 - 1)^n \tag{2.25}$$

Differentiating (2.25) w. r. t x we get

$$\frac{dy}{dx} = n(x^2 - 1)^{n-1}2x = 2nx(x^2 - 1)^{n-1} \tag{2.26}$$

Multiplying both sides of equation (2.26) by (x^2-1)

$$(x^2-1)\frac{dy}{dx} = 2nx(x^2 - 1)^n \tag{2.27}$$

Using equation (2.25) we rewrite

$$(x^2-1)\frac{dy}{dx} = 2nxy \tag{2.28}$$

By Leibnitz theorem we know that for two functions u and v the nth differential equation is expressed as

$$D^n(vu) = \sum_{k=0}^{n}\binom{n}{k} D^k v D^{n-k} u \tag{2.29}$$

Where D^n is differentiation performed n times .Thus differentiating equation (2.27) $(n + 1)$ times by Leibnitz's theorem {using $v = (x^2-1)$ and $u = \frac{dy}{dx}$}

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + \binom{n+1}{1}(2x)\frac{d^{n+1}y}{dx^{n+1}} + \binom{n+1}{2}(2)\frac{d^n}{dx^n} = 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + \binom{n+1}{2}(1)\frac{d^n y}{dx^n}\right]$$

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + 2(n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + \frac{2(n+1)n}{2}\frac{d^n}{dx^n} = 2n\left[x\frac{d^{n+1}y}{dx^{n+2}} + \frac{(n+1)!1!}{(n+1-1)!1!}\frac{d^n y}{dx^n}\right]$$

$$(x^2 - 1)\frac{d^{n+2}y}{dx^{n+2}} + 2(n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + \frac{2(n+1)n}{2}\frac{d^n}{dx^n} = 2nx\frac{d^{n+1}y}{dx^{n+2}} + 2n(n + 1)\frac{d^n y}{dx^n}$$

$$(x^2-1)\,\frac{d^{n+2}y}{dx^{n+2}}+[(2nx+2x)-2nx\,]\,\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)-2n(n+1)\,\frac{d^n}{dx^n}=0$$

$$(x^2-1)\,\frac{d^{n+2}y}{dx^{n+2}}+2x\,\frac{d^{n+1}y}{dx^{n+1}}-n(n+1)\frac{d^ny}{dx^n}=0$$

$$(1-x^2)\,\frac{d^{n+2}y}{dx^{n+2}}-2x\,\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)\,\frac{d^ny}{dx^n}=0 \tag{2.30}$$

Letting $\emptyset(x)=\frac{d^ny}{dx^n}$ and subsequently $\emptyset'(x)=\frac{d^{n+1}y}{dx^{n+1}}$ and $\emptyset''(x)=\frac{d^{n+2}y}{dx^{n+2}}$ .we write the above equation as

$$(1-x^2)\,\emptyset''(x)-2x\emptyset'(x)+n(n+1)\emptyset(x)=0 \tag{2.31}$$

Which is the Legendre equation with the solution $\emptyset(x)=\frac{d^ny}{dx^n}$ therefore we can relate the solution $\emptyset(x)$ with the Legendre polynomial $p_n(x)$ as a constant c times the $\emptyset(x)$

$$p_n(x)=c\emptyset(x) \tag{2.32}$$

We only have to determine this constant c for which we re-express $y=(x-1)^n(x+1)^n$ and differentiate the above equation n times using the Leibnitz theorem and get

$$\frac{d^ny}{dx^n}=(x-1)^n\,\frac{d^n}{dx^n}(x+1)^n+\binom{n}{1}n(x-1)^{n-1}\,\frac{d^{n-1}}{dx^{n-1}}(x+1)^n+\ldots$$

$$+\binom{n}{n-1}(x+1)^{n-1}\,\frac{d^{n-1}}{dx^{n-1}}(x-1)^n+\binom{n}{n}(x+1)^n\,\frac{d^n}{dx^n}(x-1)^n \tag{2.33}$$

Putting x = 1 on both sides of the above equation

$$\left.\frac{d^ny}{dx^n}\right|_{x=1}=2^n\left.\frac{d^n(x-1)^n}{dx^n}\right|_{x=1}$$

$$=2^n(n)\left.\frac{d^{n-1}(x-1)^{n-1}}{dx^{n-1}}\right|_{x=1}$$

$$=2^n(n(n-1)\left.\frac{d^{n-2}(x-1)^{n-2}}{dx^{n-2}}\right|_{x=1}$$

$$=2^n(n(n-1)(n-2)\left.\frac{d^{n-3}(x-1)^{n-3}}{dx^{n-3}}\right|_{x=1}$$

$$=2^n n!$$

Substituting $x = 1$ in equation (2.33) we see that

$$p_n(1) = c\emptyset(x)|_{x=1} \quad = \quad c\frac{d^n y}{dx^n}\Big|_{x=1} \quad = c2^n n! \tag{2.34}$$

Since we know that for any n when $x = 1$ the Legendre polynomial $p_n(x) = 1$.Therefore the value of the constant c is $c = \frac{p_n(1)}{2^n n!} = \frac{1}{2^n n!}$

Substituting c back in equation (2.32) we get

$$p_n(x) = c\emptyset(x) \quad = \quad \frac{1}{2^n n!}\frac{d^n y}{dx^n} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \blacksquare$$

Example 4: Using Rodrigue's formula compute the first six Legendre polynomials.

Solution: The zeroth-order derivative of a function is simply the function itself.

So,

For $n = 0$ $\quad p_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1$

For $n = 1$ $\quad p_1(x) = \frac{1}{2^1 1!}\frac{d^1}{dx^1}(x^2-1)^1 = x$

For $n = 2$ $\quad p_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2(4)}\frac{d^2}{dx^2}(x^2-1)^2$

$$= \frac{d(}{dx}\frac{x(x^2-1))}{2} = \frac{1}{2}(3x^2-1))$$

For $n = 3$ $\quad p_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{2}(5x^3-3x)$

For $n = 4$ $\quad p_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4 = \frac{1}{8}(35x^4-30x^2+3)$

For $n = 5$ $\quad p_5(x) = \frac{1}{2^5 5!}\frac{d^5}{dx^5}(x^2-1)^5 = \frac{1}{8}(63x^5-70x^3+15x)$

which is the same result obtained in example 3 .

2.4 Properties of Legendre polynomials

In this subtopic we will prove some properties of Legendre functions

2.4.1 Generating function of Legendre's polynomials

Many facts about Legendre functions can be proved by using its generating function. Here we want to determine the generating function.

Definition: The function $\mathrm{G}\,(\mathrm{t},\mathrm{x}) = \frac{1}{\sqrt{1-2\mathrm{xt}+\mathrm{t}^2}}$ is called the generating function for the Legendre polynomials $\mathrm{P_n(x)}$. If we extend $\mathrm{G}\,(\mathrm{t},\mathrm{x})$ as Taylor series in 't' then the coefficient of t^n is the polynomial $\mathrm{P_n(x)}$. It can be shown that for small t i.e.|t|<1,

$$\frac{1}{\sqrt{1-2\mathrm{xt}+\mathrm{t}^2}} = \sum_{\mathrm{n}=0}^{\infty} \mathrm{P_n(x)t^n} \tag{2.35}$$

Proof: Using the binomial theorem which is for $|v| < 1$ and p is any real number, then

$(1 + v)^p = \sum_{n=0}^{\infty}\binom{p}{n} v^n$ where the binomial coefficient $\binom{p}{n}=\frac{p!}{(p-1)!n!}$

$$(1 + \mathrm{x})^{\mathrm{p}} = 1+\mathrm{pv}+\frac{\mathrm{p(p-1)v^2}}{2!}+\frac{\mathrm{p(p-1)(p-3)v^3}}{3!}+\frac{\mathrm{p(p-1)(p-3)\ldots(p-n+1)v^n}}{\mathrm{n}!} + \cdots$$

$$(1 - 2\mathrm{xt} + \mathrm{t}^2)^{-\frac{1}{2}} = [1 - \mathrm{t}(2\mathrm{x} - \mathrm{t})]^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\mathrm{t}(2\mathrm{x} - \mathrm{t})+\frac{1}{2}\frac{3}{4}\mathrm{t}^2(2\mathrm{x} - \mathrm{t})^2+\ldots+\frac{1,3.,,,(2\mathrm{n}-3)}{2,4....(2\mathrm{n}-2)}\mathrm{t}^{\mathrm{n}-1}(2\mathrm{x} - \mathrm{t})^{\mathrm{n}-1}$$

$$+\frac{(1,3....(2\mathrm{n}-1)}{2.4...(2\mathrm{n})}\mathrm{t}^{\mathrm{n}}(2\mathrm{x} - \mathrm{t})^{\mathrm{n}} +\ldots \tag{2.36}$$

Now the coefficient of $\mathrm{t^n}$ in this term

$$\frac{1,3....(2\mathrm{n}-1)}{2.4...(2\mathrm{n})}\mathrm{t}^{\mathrm{n}}(2\mathrm{x} - \mathrm{t})^{\mathrm{n}} = \frac{(1,3....(2\mathrm{n}-1)}{2.4...(2\mathrm{n})}(2\mathrm{x})^{\mathrm{n}}$$

$$= \frac{1,3,5....(2\mathrm{n}-1)2^{\mathrm{n}}\mathrm{x}^{\mathrm{n}}}{(2,1)(2,2(2.3)...(2\mathrm{n})}$$

$$= \frac{1,3,5....(2\mathrm{n}-1)2^{\mathrm{n}}\mathrm{X}^{\mathrm{n}}}{2^{\mathrm{n}}\mathrm{n}!}$$

$$= \frac{1,3,5....(2\mathrm{n}-1)}{\mathrm{n}!}\mathrm{x}^{\mathrm{n}} \tag{2.37}$$

Again the coefficient of $\mathrm{t^n}$ in $n - 1$ term

$$\frac{1,3....(2\mathrm{n}-3)}{2.4...(2\mathrm{n}-2)}\mathrm{t}^{\mathrm{n}-1}(2\mathrm{x} - \mathrm{t})^{\mathrm{n}-1} = \frac{1,3....(2\mathrm{n}-3)}{2.4...(2\mathrm{n}-1)}[-(\mathrm{n} - 1)(2\mathrm{x})^{\mathrm{n}-2}]$$

$$= - \frac{1,3,5....(2\mathrm{n}-3)2^{\mathrm{n}}\mathrm{x}^{\mathrm{n}}}{2^{\mathrm{n}-1}1.2.3.(\mathrm{n}-1)}\frac{2\mathrm{n}-1}{\mathrm{n}}\frac{\mathrm{n}}{2\mathrm{n}-1}[(\mathrm{n}-1)\mathrm{x}2^{\mathrm{n}}\mathrm{x}^{\mathrm{n}-2}]$$

on multiplying and dividing by $\frac{(2\mathrm{n}-1)}{\mathrm{n}}$

$$= \frac{1,3,5....(2n-1)}{n!}\frac{n.(n-1)}{2(2n-1)}x^{n-2} \qquad (2.38)$$

and so on. Using (2.37) and (2.38) we see that the coefficient of t^n in the expansion of

$(1-2xt+t^2)^{-\frac{1}{2}}$ namely (2.36) is given by

$$\frac{1,3...(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2,4.(2n-1)(2n+3)}x^{n-4} - \cdots\right] \qquad (2.39)$$

This we can prove by integrating the above eq (2.39) n times from 0 to x to get

$=\frac{1,3,5...(2n-1)}{(2n)!} = \left[x^n - nx^{2n-2} + \frac{n(n-1)}{2!}x^{2n-4} - \cdots\right]$ which can be expressed as

$\frac{1,3,5....(2n-3)(2n-1)}{2n(n-1)(2n-3)...3.2.1}[(x^2-1)^n]$

Cancelling the odd terms to get

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \qquad \blacksquare$$

Thus the generating function of $p_n(x)$ is $(1-2xt+t^2)^{-\frac{1}{2}}$.

Note we find that $p_1(x), p_2(x), p_3(x)\ldots$will be the coefficient of t, t^2, t^3 in the expansion of $(1-2xt+t^2)^{-\frac{1}{2}}$.

Example 5**:** Show that i) $P_n(1) = 1$

ii) $P_n(-x) = (-1)^n P_n(x)$

Solution: i) Consider the generating function $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$

We substitute $x = 1$ then

$$(1-2t+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(1)t^n$$

$$(1-t)^{-1} = 1+tP_1(1) + t^2P_2(1) + t^3P_3(1) + \ldots + t^nP_n(1)$$

$$\frac{1}{1-t} = 1+t + t^2 + t^3 + \ldots + t^n \text{ (by toyler series expansion)}$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$$

Then equating coefficients of t^n on either side we get

$$P_n(1)=1 \qquad \blacksquare$$

ii) $P_n(-x) = (-1)^n P_n(x)$

Solution: Again use $(1-2t+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n$

We substitute $-x = x$ and then $-t = t$ respectively to get

$$(1+2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} p_n(-x)t^n$$

$(1+2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} p_n(x)(-t)^n$

By equating the two

$$\sum_{n=0}^{\infty} p_n(-x)t^n = \sum_{n=0}^{\infty} p_n(x)(-t)^n$$

$$\Rightarrow p_n(-x)t^n = p_n(x)(-1)^n(t)^n$$

$$\Rightarrow p_n(-x) = (-1)^n p_n(x) \blacksquare$$

2.4.2 Recurrence relation (formulae) for Legendre's polynomials

Proposition: The Legendre polynomials satisfy the following

i) $(n+1)p_{n+1}(x) = (2n+1)x\, p_n(x) - np_{n-1}(x)$ n ≥1

ii) $np_n(x) = x\, p'_n(x) - p'_{n-1}(x)$.

iii) $(2n+1)p_n(x) = p'_{n+1}(x) - p'_{n-1}(x)$

iv) $p'_{n+1}(x) - x\, p'_n(x) = (n+1)p_n(x)$

v) $(1-x^2)p'_n(x) = n(p_{n-1}(x) - x\, p_n(x))$

vi) $(1-x^2)p''_n(x) - 2xp'_n(x)) = -n(n+1)\, p_n(x))$

Proof:

i). $(n+1)p_{n+1}(x) = (2n+1)x\, p_n(x) - np_{n-1}(x)$ n ≥1 (2.40)

Similarly from generating function, we have of

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} p_n(x)t^n \qquad (2.41)$$

Differentiating both sides of (2.41) with respect to 't' we get

$$\frac{-1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty} np_n(x)t^{n-1} \qquad (2.42)$$

Multiplying both sides by $(1-2xt+t^2)$, (2.42) gives

$$(x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} nt^{n-1}p_n(x)$$

Or $(x-t)\sum_{n=0}^{\infty} p_n(x)t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} nt^{n-1}p_n(x)$ by (2.35)

$$x\sum_{n=0}^{\infty} t^n p_n(x) - \sum_{n=0}^{\infty} t^{n+1}p_n(x) = \sum_{n=0}^{\infty} nt^{n-1}p_n(x) - 2x\sum_{n=0}^{\infty} nt^n p_n(x) + \sum_{n=0}^{\infty} nt^{n+1}p_n(x)$$

Equating coefficients of t^n from both sides, we get

$$x\, p_n(x) - p_{n-1}(x) = (n+1)p_{n+1}(x) - 2xnp_n(x) + (n-1)p_{n-1}(x),$$

$$(n+1)p_{n+1}(x) = (2n+1)x\,p_n(x) - np_{n-1}(x) \quad n \geq 1 \blacksquare$$

An alternative form is obtained if $n-1$ be substitute for n as

$$np_n(x) = (2n-1)\,x\,p_{n-1}(x) - (n-1)p_{n-2}(x) \quad n \geq 1. \tag{2.43}$$

ii) $np_n(x) = x\,p'_n(x) - p'_{n-1}(x).$ (2.44)

Generating function, we have of $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} p_n(x)t^n$.

Differentiating both sides of the above equation with respect to 't' we get

$$\frac{-1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty} np_n(x)t^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} np_n(x)t^{n-1} \tag{2.45}$$

Differentiating the generating function w.r.t x

$$\frac{-1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty} np'_n(x)t^{n-1}$$

$$t\,(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} np'_n(x)t^{n-1} \tag{2.46}$$

Where $p'_n(x) = \frac{d^n p_n(x)}{dx}$ let as divided eq(2.45) by (2.46)

we get $\frac{x-t}{t} = \frac{\sum_{n=0}^{\infty} nt^{n-1}p_n(x)}{\sum_{n=0}^{\infty} np'_n(x)t^{n-1}}$ cross multiply to get

$$(x-t)\sum_{n=0}^{\infty} p'_n(x)nt^{n-1} = \sum_{n=0}^{\infty} nt^n\,p_n(x)$$

Equating the coefficient of t^n

$$\Rightarrow np_n(x) = x\,p'_n(x) - p'_{n-1}(x). \blacksquare$$

iii) $(2n+1)p_n(x) = p'_{n+1}(x) - p'_{n-1}(x)$ (2.47)

Consider (2.40) and differentiate with respect to x

$$\frac{d}{dx}[(n+1)p_{n+1}(x) = (2n+1)x\,p_n(x) - np_{n-1}(x)]$$

$$(n+1)p'_{n+1}(x) = (2n+1)(x'p_n(x) + xp'_n(x)) - np'_{n-1}(x)$$

$$(n+1)p'_{n+1}(x) = (2n+1)p_n(x) + (2n+1)xp'_n(x)) - np'_{n-1}(x) \tag{2.48}$$

Substituting the value of $xp'_n(x)$ by $np_n(x) + p'_{n-1}(x)$ from (2.44) We get

$$(n+1)p'_{n+1}(x) = (2n+1)p_n(x) + (2n+1)[np_n(x) + p'_{n-1}(x)] - np'_{n-1}(x) \tag{2.49}$$

$$= (2n+1)p_n(x) + (2n+1)np_n(x) + (2n+1)p'_{n-1}(x)) - np'_{n-1}(x)$$

$$= (2n+1)(n+1)p_n(x) + (2n)p'_{n-1}(x) + p'_{n-1}(x) - np'_{n-1}(x)$$

$$= (2n+1)(n+1)p_n(x) + np'_{n-1}(x) + p'_{n-1}(x)$$

$$(n+1)p'_{n+1}(x) = (2n+1)(n+1)p_n(x) + (n+1)p'_{n-1}(x) \tag{2.50}$$

$$(n+1)p'_{n+1}(x) = (2n+1)(n+1)(p_n(x) + p'_{n-1}(x))$$

$$p'_{n+1}(x) = (2n+1)p_n(x) + p'_{n-1}(x)$$

Therefore $p'_{n+1}(x) - p'_{n-1}(x) = (2n+1)p_n$ ■

An alternative form is obtained if $n-1$ be substitute for n as

$$p'_n(x) - p'_{n-2}(x) = (2n-1)p_{n-1} \tag{2.51}$$

iv. $p'_{n+1}(x) - x\,p'_n(x) = (n+1)p_n(x)$ (2.52)

from eq (2.40)

i.e. $(n+1)p_{n+1}(x) = (2n+1)\,x\,p_n(x) - np_{n-1}(x)$ Differentiating wrt x to get

$$(n+1)p'_{n+1}(x) = (2n+1)p_n(x) + (2n+1)x\,p'_n(x) - np'_{n-1}(x)$$

$$= (2n+1)p_n(x) + (n+1+n)x\,p'_n(x) - np'_{n-1}(x)$$

$$= (2n+1)p_n(x) + (n+1)x\,p'_n(x) + nx\,p'_n(x) - np'_{n-1}(x)$$

Substituting the value of

$x\,p'_n(x) - p'_{n-1}(x) = np_n(x)$ from (2.43) we get

$$(n+1)p'_{n+1}(x) = (2n+1)p_n(x) + (n+1)x\,p'_n(x) + n[np_n(x)]$$

$$= (2n + n2 + 1)p_n(x) + (n+1)x\,p'_n(x)$$

$$= (n+1)^2 p_n(x) + (n+1)x\,p'_n(x) \text{ thus}$$

$$p'_{n+1}(x) = (n+1)^2 p_n(x) + (n+1)x\,p'_n(x)$$

$$= (n+1)p_n(x) + x\,p'_n(x)$$

$p'_{n+1}(x) - x\,p'_n(x) = (n+1)p_n(x)$ ■

An alternative form is obtained if $n-1$ be substitute for n as

$$p'_n(x) - x\,p'_{n-1}(x) = np_{n-1}(x) \tag{2.53}$$

v. $(1-x^2)p'_n(x) = n(p_{n-1}(x) - x\,p_n(x))$ (2.54)

Multiply eq(2.44) by x and then subtract it from eq(2.53) We get

$$n(x\,p_n(x) - p_{n-1}(x)) = (x^2-1)p'_n(x)$$

$(1-x^2)p'_n(x) = n(p_{n-1}(x) - xp_n(x))$. ■

vi) $(1-x^2)p''_n(x) - 2xp'_n(x)) = -n(n+1)\,p_n(x))$

Differentiating both sides of (2.54) with respect to x and use (2.44) we get

$$(1-x^2)p''_n(x) - 2xp'_n(x)) = n(p'_{n-1}(x) - xp'_n(x) - p_n(x))$$

$$= n[\{p'_{n-1}(x) - nxp'_n(x)) - p_n(x)]$$

$$= n(-np_n(x) - p_n(x))$$

$$=-n(n+1)p_n(x)\,.$$

So the $p_n(x)$ in (2.41) satisfies Legendre's equation.

2.4.3 Orthogonality of Legendre's polynomials

The most important property of the Legendre polynomials is the fact that

$$\int_{-1}^{1} p_m(x)p_n(x)dx = \begin{cases} 0, & \text{if} m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases} \tag{2.55}$$

Proof: (i) Since $p_m(x)$and $p_n(x)$ satisfy Legendre's equation we have

$$(1-x^2)p''_m - 2xp'_m + m(m+1)p_m = 0 \tag{2.56}$$

$$(1-x^2)p''_n - 2xp'_n + n(n+1)p_n = 0 \tag{2.57}$$

Multiply the first equation by p_n the second by p_m

We get, $(1-x^2)p_n\, p''_m - 2xp_n\, p'_m + m(m+1)p_n\, p_m = 0$

$$\text{or } \frac{d}{dx}\left[(1-x^2)p_n\, p'_m\right] + m(m+1)p_n\, p_m = 0 \tag{2.58}$$

$$(1-x^2)p_m p''_n - 2xp_m p'_n + n(n+1)p_m p_n = 0$$

$$\text{or } \frac{d}{dx}[\,(1-x^2)p_m p'_n] + n(n+1)p_m p_n = 0 \tag{2.59}$$

subtract (2.59)from (2.58) . We get

$$\frac{d}{dx}[(1-x^2)(p_n p'_m - p'_n p_m)] + (m^2 + m - n^2 - n)p_m p_n = 0$$

$$\frac{d}{dx}[(1-x^2)(p_n p'_m - p'_n p_m)] + (m-n)(n+m+1)p_m p_n = 0$$

$$\frac{d}{dx}[(1-x^2)(p_n p'_m - p'_n p_m)] = (n-m)(n+m+1)p_m p_n$$

Integrating both sides w.r.t x from $-1$to 1 we get

$$(n-m)(n+m+1)\int_{-1}^{1} p_m(x)p_n(x)dx = [(1-x^2)(p_n p'_m - p'_n p_m)]\Big|_{-1}^{1}$$

$$n-m)(n+m+1)\int_{-1}^{1} p_m(x)p_n(x)dx = 0$$

therefore $\int_{-1}^{1} p_m(x)p_n(x)dx = 0$ as $m \neq n$■

(ii) When $m = n$,

Then the required result takes the form

$\int_{-1}^{1}[p_n(x)]^2 dx = \frac{2}{(2n+1)}$ (2.60)

To prove this with start generating function

$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} p_n(x)t^n$ (2.61)

Also (2.50) may be re-written as

$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} p_m(x)t^m$ (2.62)

Multiplying the corresponding sides of (2.61) and (2.62) we get

$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_m(x)\, p_n(x)t^{m+n} = (1-2xt+t^2)^{-1}$ (2.63)

Integrating both sides of the above equation $\mathrm{w.r.t}$ x we get

$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[\int_{-1}^{1} p_m(x)p_n(x)dx]t^{m+n} = \int_{-1}^{1}(1-2xt+t^2)^{-1}dx$ (2.64)

Making use of (2.60), (2.64) reduces to

$$\sum_{n=0}^{\infty}[\int_{-1}^{1}(p_n(x))^2 dx]t^{2n} = \int_{-1}^{1}\frac{1}{1-2xt+t^2}dx$$

$$= \left[\frac{\ln(1+t^2-2xt)}{-2t}\right]_{-1}^{1}$$

$$= -\frac{1}{2t}[\ln(1-t)^2 - \ln(1+t)^2]$$

$$= -(\frac{1}{2t})[2\ln(1-t) - 2\ln(1+t)]$$

$$= (\frac{1}{t})[\ln(1+t) - \ln(1-t)]$$

from the Taylor series for $|x|<1$,

$$\ln(x+1) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} ... +\right) \text{ and}$$

$$\ln(-x+1) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-x)^n}{n}$$

$$= \sum_{n=1}^{\infty}(-1)^{2n-1}\frac{x^n}{n}$$

$$= \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \cdots\right)$$

Hence for $|t|<1$ we have

$$\frac{1}{t}[\ln(1+t) - \ln(1-t)] = \frac{1}{t}(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}t^n - \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}(t)^n$$

$$= \frac{1}{t}\left[\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots +\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots\right)\right]$$

$$= \frac{2}{t}\left(t + \frac{t^3}{3} + \cdots + \frac{t^{2n+1}}{2n+1} + \cdots +\right)$$

$$= 2\left(1 + \frac{t^2}{3} + \cdots + \frac{t^{2n}}{2n+1} + \cdots +\right)$$

$$= 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}$$

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}[p_n(x)]^2 dx\right] t^{2n} = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n} \qquad (2.65)$$

Equating coefficients of t^{2n} from both sides (2.65) gives

$$\int_{-1}^{1}[p_n(x)]^2 dx = \frac{\mathbf{2}}{(\mathbf{2n+1})} \blacksquare$$

2.4.4 Fourier-series expansion of Legendre's polynomials

Any function f which is finite and single-valued in the interval $-1 \le x \le 1$, and which has a finite number of discontinuities within this interval can be expressed as a series of Legendre polynomials. We let

$f(x) = A_0 p_0(x) + A_1 p_1(x) + A_2 p_2(x) + A_3 p_3(x) + \ldots - 1 \le x \le 1$

$$= \sum_{n=0}^{\infty} A_n p_n(x)$$

Multiplying both sides by $p_m(x)dx$ and integrating with respect to x from

$x = -1$ to $x = 1$ gives

$$\int_{-1}^{1} p_m(x) f(x) dx = \int_{-1}^{1} \sum_{n=0}^{\infty} A_n p_n(x)\, p_m(x) dx$$

$$= \sum_{n=0}^{\infty} A_n \int_{-1}^{1} p_m(x)\, p_n(x)\, dx$$

By means of the orthogonality property of the Legendre polynomials we can write

$$\int_{-1}^{1} p_m(x) f(x) dx = A_n \frac{2}{2n+1} \text{ for } m = n$$

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, p_n(x) dx$$

Since $p_n(x)$ is an even function of x when n is even, and an odd function when n is odd, it follows that if f(x) is an even function of x the coefficients A_n will vanish when n is odd;

whereas if f(x) is an odd function of x, the coefficients A_n will vanish when n is even. Thus for an even function f(x) we have

$$A_n = \begin{cases} 0 & \text{if n is odd} \\ (2n+1)\int_0^1 f(x)\,p_n(x)dx & \textit{if n is } \text{even} \end{cases}$$

Whereas for an odd function f(x) we have

$$A_n = \begin{cases} (2n+1)\int_0^1 f(x)\,p_n(x)dx & \text{for n is odd} \\ 0 & \textit{for } \text{n is even} \end{cases}$$

***Example 6*:** find the Legendre series for the polynomial f(x) = $4x+3x^2-5x^3$.

Solution: Since any polynomial of degree n can expressed as a linear combination of Legendre polynomial up to the nth degree using Rodriues formula i. e

$p_0(x) = 1, p_1(x) = x, p_2(x) = \frac{1}{2}(3x^2-1),\ p_3(x) = \frac{1}{2}(5x^3-3x)$ and so on in this case the function is polynomial with degree 3.it will be the combination of Legendre polynomial $p_0(x)$, $p_1(x)$, $p_2(x)$and $p_3(x)$.to compute the coefficient of the linear combination the formula

$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)p_n(x)dx$ can be used four times.

$a_0 = \frac{2.0+1}{2}\int_{-1}^{1} f(x)p_0(x)dx = \frac{1}{2}\int_{-1}^{1}[(4x+3x^2-5x^3)]dx = 1$

$a_1 = \frac{3}{2}\int_{-1}^{1} f(x)p_1(x)dx = \frac{3}{2}\int_{-1}^{1}[(4x+3x^2-5x^3)x]dx = 1$

$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)p_2(x)dx = \frac{5}{2}\int_{-1}^{1}[(4x+3x^2-5x^3)\frac{1}{2}(3x^2-1)]dx = 2$

$a_3 = \frac{7}{2}\int_{-1}^{1} f(x)p_3(x)dx = \frac{7}{2}\int_{-1}^{1}[(4x+3x^2-5x^3)\frac{1}{2}(5x^3-3x)]dx = -2$

Then the expression will be

$4x+3x^2-5x^3 = p_0(x) + p_1(x) + 2p_2(x) - 2p_3(x)$.

***Example 7*:** Expand f(x) in the form $\sum_{n=0}^{\infty} a_n p_n(x)$

Where f(x) = $\begin{cases} 0 \text{ where} -1 < x < 0 \\ 1 \text{ where } 0 < x < 1 \end{cases}$

Solution: Given that f(x) = $\begin{cases} 0 \text{ where} -1 < x < 0 \\ 1 \text{ where } 0 < x < 1 \end{cases}$ (2.66)

We know that f(x) = $\sum_{n=0}^{\infty} a_n p_n(x)$ (2.67)

Where $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)p_n(x)dx$ (2.68)

$= \frac{2n+1}{2}\left[\int_{-1}^{0} f(x)p_n(x)dx + \int_0^1 f(x)p_n(x)dx\right]$

$$a_n = \frac{2n+1}{2}\int_0^1 f(x)p_n(x)dx \text{ by (2.66)}$$

Putting n=0, 1, 2, 3… successively in (2.68) we get

for $n = 0, a_0 = \frac{1}{2}\int_0^1 f(x)p_0(x)dx = \frac{1}{2}\int_0^1 (1)dx = \frac{1}{2}$

for $n = 1, a_1 = \frac{2+1}{2}\int_o^1 f(x)p_1(x)dx = \frac{3}{2}\int_o^1 p_1(x)dx = \frac{3}{2}\int_o^1 xdx = \frac{3}{4}$

for $n = 2, a_2 = \frac{2(2)+1}{2}\int_0^1 f(x)p_2(x)dx = \frac{5}{2}\int_o^1 p_2(x)dx = \frac{5}{2}\int_o^1 \frac{3x^2-1}{2}dx = 0$

for $n = 3, a_3 = \frac{2(3)+1}{2}\int_0^1 f(x)p_3(x)dx = \frac{7}{2}\int_0^1 p_3(x)dx = \frac{7}{2}\int_0^1 \frac{5x^3-3x}{2}dx = \frac{-7}{16}$

for $n = 4, a_4 = \frac{2(4)+1}{2}\int_0^1 f(x)p_4(x)dx = \frac{7}{2}\int_0^1 p_4(x)dx = \frac{9}{2}\int_0^1 \frac{5x^3-3x}{4}dx = \frac{-9}{32}$

and so on .using these values in (2.60) we get

$$f(x) = \frac{1}{2}p_0(x) + \frac{3}{4}p_1(x) - \frac{7}{16}p_3(x) - \frac{9}{32}p_4(x) + \cdots + a_n p_n(x) + \cdots$$

$$\text{Hence } f(x) = \begin{cases} 1 = \frac{1}{2}p_0(x) + \frac{3}{4}p_1(x) - \frac{7}{16}p_3(x) + \cdots & ,0 < x < 1 \\ 0 = \frac{1}{2}p_0(x) + \frac{3}{4}p_1(x) - \frac{7}{16}p_3(x) + \cdots, & -1 < x < 0 \end{cases}$$

2.5 Physical Application of the Legendre Polynomials

In this section we present some examples of Legendre polynomials as they arise in mathematical models of heat conduction in spherical geometries and expansion of electromagnetic potential. In general we will encounter the Legendre equation in situations where we have to solve partial differential equations containing the Laplacian polar coordinates.

2.5.1 Heat conduction

Let’s derive the equation that governs the evaluation of an initial distribution of heat in a solid body with temperature T, density ρ, specific heat capacity c and thermal conductivity k.

The specific heat capacity c, is the amount of heat required to raise the temperature of a unit mass of a substance by one degree.

The thermal conductivity k, of a body appears in Fourier’s law which states that the heat flux per unit area, per unit time $Q = (Q_{x,} Q_{y,} Q_z)$,is related to the temperature gradient ∇T, by the simple linear relationship $Q = -k\nabla T$ .If now consider a small element of our solid body at (x, y , z) with sides of length $\sigma x, \sigma y$ and σz, the temperature change in this element over a time interval σt is determined by the difference between the amount of heat that flow out in which gives

$$\rho c(T(x, y, z, t + \sigma t) - T(x, y, z, t)\sigma x\sigma y\sigma z$$

$$= \{(Q_x(x,y,z,t) - (Q_x(x+\ \sigma x, y, z, t)\}\sigma t\sigma y\sigma z$$
$$+\{Q_{y(}x,y,z,t+\sigma t) - Q_{y(}x,y+\sigma y,z,t)\}\sigma t\sigma x\sigma z$$
$$+\{Q_z(x,y,z,t) - Q_z(x,y+\sigma z,z,t)\}\sigma t\sigma x\sigma y \qquad (2.69)$$

Note that typical term on the right hand side of this for example,

$\{\ (Q_x(x,y,z,t) - (Q_x(x+\ \sigma x,y,z,t)\}\ \sigma t\sigma y\sigma z$ is the amount of heat crossing the x- orientated faces of the element, each with area yσz,during the time interval (t, $\mathrm{t}+\sigma\mathrm{t}$) taking the limit $\sigma t, \sigma x, \sigma y, \sigma z \to 0$.we obtain

$$\rho c\frac{\partial T}{\partial t} = -\{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z}.\} = -\nabla.\mathrm{Q}$$

Substituting in Fourier's law $\mathrm{Q} = -\mathrm{k}\nabla.\mathrm{T}$ given the diffusion equation

$$\frac{\partial T}{\partial t} = k\nabla^2 T \qquad (2.70)$$

Where $\mathrm{K} = \frac{k}{\rho c}$ is called the thermal diffusivity

When the temperature reaches a steady state ($\frac{\partial T}{\partial t}$ = 0) this equation takes the simple form

$$\nabla^2 T = 0 \qquad (2.71)$$

Which is known as Legendre's equation .It must be solved in conjunction with appropriate boundary conditions which drive the temperature gradients to the body.

***Exampl*e 8: Temperature distribution in a sphere**

Lets try to know the steady state temperature inside a sphere of radius a whose upper hemisphere is obtained at a temperature T = T_0 and the lower hemisphere is obtained at a temperature T = $-T_0$ to this end we have Laplace's equation in spherical coordinate subject to the given boundary conditions. Since temperature distribution is in dependent of ϕ we can write

$$T(r,\theta) = \sum_{m=0}^{\infty}(A_m r^m + \frac{B_m}{r^{n+1}}\ )p_m(cos\theta) \qquad (2.72)$$

Since temperature will be finite at the center of the sphere(r = 0), we must have B_m = 0 for all m otherwise solution will diverge. Hence expression in (2.72) reduces to

$$T(r,\theta) = \sum_{m=0}^{\infty} A_m r^m\ \ p_m(cos\theta) \qquad (2.73)$$

In view of the given conditions the temperature distribution on the surface of the sphere can be written as $f(x) = \begin{cases} -T_0 \text{ if} - 1 < x < 0 \\ \ T_0 \ \text{ if } 0 < x < 1 \end{cases}$ (2.74)

Where $x = cos\theta$.Applying the boundary conditions to (2.69) we get

$$f(x) = \sum_{m=0}^{\infty} A_m a^m\ \ p_m(x) \qquad (2.75)$$

To determine the constant $\mathrm{A_m}$ we use the orthogonality relation. To this end we multiply both sides of the equation by $p_n(x)$ and integrate the resulting expression over x in the range -1 to 1.This yields

$$\int_{-1}^{1} p_n(x)\mathrm{f}(x)dx = \sum_{m=0}^{\infty} \mathrm{A_m}\mathrm{r^m} \int_{-1}^{1} p_n(x)p_m(x)\, dx \tag{2.76}$$

Using the orthogonality relation for the Legendre polynomials we get

$$\mathrm{A_n} = \frac{2n+1}{2a^n}\int_{-1}^{1} p_n(x)\mathrm{f}(x)dx = \frac{2n+1}{2a^n}\mathrm{T_0}[\int_{-1}^{0} p_n(x) + \int_{0}^{1} p_n(x)dx\,]$$

$$= \left(\frac{2n+1}{2}\right)(\frac{\mathrm{T_0}}{\boldsymbol{a^n}})[\int_{0}^{1} p_n(x) - \int_{0}^{1} p_n(-x)dx\,] \tag{2.77}$$

Since $p_n(-x) = (-1)^n\, p_n(x)$, the above expression simplifies to

$$\mathrm{A_n} = \begin{cases} (2n+1)(\frac{\mathrm{T_0}}{\boldsymbol{a^n}}) \int_{0}^{1} p_n(x)dx & \text{for n} = \text{odd} \\ 0 & for\ n = even \end{cases} \tag{2.78}$$

From this we can readily write the values of the first few coefficients

$$\mathrm{A_1} = \frac{\mathbf{3}\mathrm{T_0}}{\boldsymbol{a}}\int_{0}^{1} A_n(x)dx = \frac{3\mathrm{T_0}}{a}\int_{0}^{1} xdx = \frac{3\mathrm{T}}{\mathbf{2}a}$$

$$\mathrm{A_3} = \frac{\mathbf{7}\mathrm{T_0}}{\mathbf{2}}\int_{0}^{1} p_3(x)dx = \frac{\mathbf{3}\mathrm{T_0}}{\boldsymbol{a^3}}\int_{0}^{1}\frac{(5x^3-3x)}{2}dx = \frac{\mathbf{-7}\mathrm{T_0}}{8\mathrm{a}^3}$$

Hence the temperature distribution inside the sphere is given by

$$T(r,\theta) = \mathrm{T_0}\,[\frac{3r}{2a}\cos\theta - \frac{7r^3}{16a^3}(5cos^3\theta\text{-}3\cos\theta)+\ldots] \tag{2.79}$$

2.5.2 Expansion of electromagnetic potential

Example 9: **From Electrostatics**

In electrostatics, the potential due to a unit point charge at r = r_0 is

$$\mathrm{V} = \frac{1}{\|\boldsymbol{r}-r_0\|}\,.$$

If this unit charge lies on the *z*-axis, at *x* = *y* = 0, *z* = *a*, this becomes

$$\mathrm{V} = \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}.$$

In terms of spherical polar coordinates, (*r, θ, φ*),

$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta.$

$$x^2 + y^2 + (z-a)^2 = x^2 + y^2 + z^2 - 2az + a^2$$

$$= (\,r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 + (r\cos\theta)^2 - 2a(r\cos\theta) + a^2$$

$$= r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta - 2a(r\cos\theta) + a^2$$

$$= r^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi)\ r^2\cos^2\theta - 2a(r\cos\theta) + a^2$$

$$= r^2(\sin^2\theta + \cos^2\theta) - 2a(r\cos\theta) + a^2$$

$$= r^2 - 2a(r\cos\theta) + a^2$$

$$V = \frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}$$

$$= \frac{1}{a}(1 - 2r\cos\theta\ \frac{a}{r} + \frac{r^2}{a^2})^{\frac{-1}{2}}.$$

there is no dependence upon the azimuthal angle, φ. We can now use the generating function to write this as a power series, $V = \frac{1}{a}\sum_{n=0}^{\infty} p_n(\cos\theta)(\frac{r}{a})^n$

***Example 10*: Electric potential inside a sphere**

Consider a sphere of radius a such that $V(r,\theta)]_{r=a} = V_0 cos^3\theta$ and assume that there are no charge at the origin. Since V must satisfy Laplace's equation and the boundary condition has no φ dependence, the solution will be obtained in terms of Legendre polynomials. The general solution can be written as

$$V(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + \frac{B_n}{r^{n+1}})p_n(\cos\theta) \tag{2.80}$$

From this form of solution we note that V can be finite at the origin only if $B_n = 0$ for all n.

then expression in eq(2.80) reduces to

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n r^n\ \ p_n(\cos\theta) \tag{2.81}$$

On applying the given boundary condition, we have

$$V(r,\theta) = V_0\cos^3\theta = \sum_{n=0}^{\infty} A_n a^n\ \ p_n(\cos\theta) \tag{2.82}$$

To solve the constant A_n we write $\cos^3\theta$ in terms of Legendre polynomials. To this end we recall that $p_3(\cos\theta) = (\frac{(5\cos^3\theta - 3\cos\theta}{2})$ and we can write $\cos^3\theta = \frac{2}{5}p_3(\cos\theta) + \frac{3}{5}\cos\theta$ since $\cos\theta = p_1(\cos\theta)$we find that $\cos^3\theta = \frac{2}{5}p_3(\cos\theta) + \frac{3}{5}p_1\cos\theta$.Integrating this result in (2.81)

we obtain

$$\frac{1}{5}[2V_0 p_3(\cos\theta) + 3V_0 p_1(\cos\theta)] = \sum_{n=0}^{\infty} A_n a^n\ \ p_n(\cos\theta) \tag{2.83}$$

Using the orthogonality property of Legendre polynomials we can easily see that

$$A_1 = \frac{3V_0}{5a} \text{ and } A_3 = \frac{2V_0}{5a^3}.$$

Hence $V(r,\theta) = \frac{3}{5}V_0(\frac{r}{a})p_1(\cos\theta) + \frac{2V_0}{5}(\frac{r}{a})^3 p_3(\cos\theta)$.

***Example 11*:** Consider a charge q located at position R from the origin .We want to compute the potential at some other position r, let the polar angle θ be the angle between r and R.

Solution method 1

Recall that Gauss's law says:

$\nabla^2\Phi(\mathrm{r},\theta,\emptyset,)=\frac{\rho(\mathrm{r},\theta,\emptyset)}{\epsilon_0}$.for all $\mathrm{r}=\mathrm{R}$ the charge density ρ is zero

$$\nabla^2\Phi=\left[\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2\frac{\partial}{\partial \mathrm{r}}\right)+\frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\Phi(\mathrm{r},\theta)=0.$$

We can find separable solutions $\Phi(\mathrm{r},\theta)=\mathrm{R_n}(\mathrm{r})\mathrm{p_n}(\theta)$.

The general solution is given by $\Phi(\mathrm{r},\theta)=\mathrm{R_n}(\mathrm{r})\mathrm{p_n}(\cos\theta)$

where $\mathrm{R_n}(\mathrm{r})=\mathrm{Ar^n}+\frac{\mathrm{B}}{\mathrm{r^{n+1}}}$ and $\mathrm{p_n}(\cos\theta)$ is the $\mathrm{n^{th}}$ Legendre polynomial .

For $\mathrm{R_n}(\mathrm{r})=\mathrm{Ar^n}+\frac{\mathrm{B}}{\mathrm{r^{n+1}}}$ finite solution at r=0 requires B=0.Hence

$\Phi(\mathrm{r},\theta)=\sum_{\mathrm{n}=0}^{\infty}\mathrm{a_n r^n}\,\mathrm{p_n}(\cos\theta)$.

To determine the constants $\mathrm{a_n}$ we need boundary conditions, when $\theta=0$ we must recover the potential of a point charge:

$$\Phi(\mathrm{r},0)=\sum_{\mathrm{n}=0}^{\infty}\mathrm{a_n r^n}=\frac{\mathrm{q}}{4\pi\epsilon_0}\frac{1}{\mathrm{R-r}}$$

$$=\frac{\mathrm{q}}{4\pi\epsilon_0}\left(\frac{1}{\mathrm{R}}+\frac{\mathrm{r}}{\mathrm{R}^2}+\frac{\mathrm{r}^2}{\mathrm{R}^3}+\cdots\right).\text{Therefore}$$

$$\mathrm{a_n}=\frac{\mathrm{q}}{4\pi\epsilon_0}\frac{1}{\mathrm{R^{n+1}}},$$

The full solution is:

$$\Phi(\mathrm{r},\theta)=\frac{\mathrm{q}}{4\pi\epsilon_0}\sum_{\mathrm{n}=0}^{\infty}\frac{\mathrm{r^n}}{\mathrm{R^{n+1}}}\mathrm{p_n}(\cos\theta)$$

Solution method 2

From the potential function $\Phi=\frac{\mathrm{q}}{4\pi\epsilon_{0\mathrm{d}}}$ where $\mathrm{d}=|\mathrm{R}-\mathrm{r}|$.

The law of cosine gives

$$\mathrm{d}=|\mathrm{R}-\mathrm{r}|=\sqrt{\mathrm{R}^2-2\mathrm{Rr}\cos\theta+\mathrm{r}^2}$$

$$=\mathrm{R}\sqrt{1-2\frac{\mathrm{r}}{\mathrm{R}}\cos\theta+(\frac{\mathrm{r}}{\mathrm{R}})^2}.$$

Change of variables $t=\frac{r}{R}$ and $\mathrm{x}=\cos\theta$.Then

$\Phi=\frac{q}{4\pi\epsilon_{0R}}G(x,t)$ where $G(x,t)$ is the generating function. This can be expressed in terms of Legendre polynomials as $\Phi(r,\theta)=\frac{q}{4\pi\epsilon_{0R}}\sum_{n=0}^{\infty}t^n\,p_n(x)$

$$=\frac{q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{r^n}{R^{n+1}}p_n(cos\theta).$$

Summary

A homogeneous second order differential equation of the type

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0$$

where k=0, 1, 2, 3 … (a non- negative integer) is known as the Legendre's differential equation. One of the powerful methods used to solve Legendre differential equation is a power series technique. For the ordinary point x = 0 the series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ yields the recurrence relation / formula

$a_{n+2} = -\frac{(k-n)(k+n+1)\mathbf{a_n}}{(n+2)(n+1)}$**.** The recurrence relation leads to the solution

$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_o + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$

$$y(x) = a_o[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots]$$

$$+a_1[x^1 - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 + \ldots.]$$

The two series in the solution converges for $|x| < 1$ and the solution can be written in the form

$$y(x) = a_o\ y_1(x) + a_1y_2(x)$$

where the two solutions are

$$y_1(x) = [1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots]$$
$$y_2(x) = [x^1 - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 + \ldots]$$

The above two solutions are linearly independent solutions which can be verified by evaluating the Wronskian at the ordinary point, x=0, where $y_1(x)$ and $y_2(x)$ are as defined above and $y'_1(x)$ and $y'_2(x)$ are the first order derivatives of $y_1(x)$ and $y_2(x)$ respectively.

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{x } 1 = 1 \neq 0$$

For k is positive integer we have Legendre polynomial solutions from Legendre differential equation which is denoted by

$$p_k(x) = \sum_{m=0}^{M} \frac{(-1)^m(\mathbf{2}k-\mathbf{2}m)!}{\mathbf{2}^k m!(k-m)!(k-\mathbf{2}m)!}x^{k-\mathbf{2}m} \quad \text{where } M=\frac{k}{2} \text{ or } M = \frac{k-1}{2}.$$

Rodrigues formula of Legendre's polynomials is expressed as

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

The Legendre polynomial $p_n(x)$ *is* the coefficient of t^n in the expansion of $(1 - 2xt + t^2)^{-\frac{1}{2}}$

in ascending powers of t i.e. the generating function is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} p_n(x)t^n$$

$p_n(1) = 1$

$p_n(-x) = (-1)^n p_n(x)$

The orthogonal and normalization properties of Legendre's Polynomials are expressed as

$$\int_{-1}^{1} p_m(x)p_n(x)dx = \begin{cases} 0, & if m \neq n \\ \frac{2}{2n+1}, & if\ m = n \end{cases}$$

Recurrence Formulae for Legendre's Polynomials

- ✓ $(n+1)p_{n+1}(x) = (2n+1)x\, p_n(x) - np_{n-1}(x)$ n ≥1
- ✓ $np_n(x) = x\, p'_n(x) - p'_{n-1}(x)$.
- ✓ $(2n+1)p_n(x) = p'_{n+1}(x) - p'_{n-1}(x)$
- ✓ $p'_{n+1}(x) - x\, p'_n(x) = (n+1)p_n(x)$
- ✓ $(1-x^2)p'_n(x) = n(p_{n-1}(x) - x\, p_n(x))$

Finally

Legendre Polynomials are applied in different area such as in solving Laplace equation in spherical polar coordinates in heat conduction and in the expansion of electromagnetic potential.

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