ADDIS ABABA UNIVERSITY

DEPARTMENT OF MATHEMATICS

On Riccati Equations and Their Analytic Solutions

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Thesis title: "On Riccati Equations and Their Analytic Solutions".

We, the undersigned, here have read and examined a thesis on the "On Riccati Equations and Their Analytic Solutions": done by BEFIKADU AREGAHEGN in partial fulfillment of the requirement for the degree of Master of Science and recommended to the school of graduate studies for acceptance of the thesis.

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ABSTRACT

This thesis, "On Riccati Equations and Their Analytic Solutions," examines the Riccati differential equations, a family of nonlinear equations, and their solutions. The study aims to analyze the forms and properties of Riccati equations and provide analytical solutions. Riccati equations were converted to linear or separable equations by means of analytical substitutions and transformations. Three major classes were examined. By applying integrating factors and validating specific solutions, Type One equations were resolved. Partial fractions were used to implicitly solve Type Two problems by converting them into separable forms. Explicit answers were obtained by reducing Type Three equations to first-order linear forms. The findings demonstrate that analytical techniques offer precise, insightful and effective answers.

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CHAPTER ONE - INTRODUCTION

1.1 Background of the study

Differential equations are mathematical formulations that describe how physical quantities such as those governing motion, population growth, and heat distribution evolve over time.

The Riccati differential equation is one particular type and is expressed as follows:

$$y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)$$

The Riccati equation owes its name to the 18th-century Italian mathematician Jacopo Riccati. Although initially a theoretical pursuit, it has proven indispensable for modeling diverse real-world systems, from fluid dynamics and population growth to quantum mechanics. Consequently, the development of efficient solution methods remains an active area of research.

A recent contribution by Rivera-Oliva , titled "Solving the Riccati equation using the generalized recursive integrating factor method," introduces a novel technique that simplifies the equation prior to applying transformations [8]. Their technique simplifies the equation prior to applying transformations, offering a new pathway towards a solution.

Tumbarell Aranda and Oliveira employed the variation of parameters approach to study population dynamics and generate accurate approximations in their paper "Analytical and numerical solutions of the Riccati equation using the method of variation of parameters"[3].

Despite its broad applicability, finding general analytical solutions for Riccati equations remains a challenging task. Recent African studies have greatly influenced how this topic is addressed. Equatorial Guinean Asumu presented a new type of ordinary Riccati differential equations for which the general solution can be more easily derived as its particular solution is already known [4]. It also enhances analytical approaches to such equations.

Because of its wide application and continuous development of solving methods, this study focuses on Riccati Equations and their Analytic Solutions.

1.2 Statement of the problem

Riccati equations model complex systems in physics and finance. But their nonlinear nature makes them very hard to solve exactly. Only a few special cases have clear answers. For most real-world problems, researchers must use estimates and numerical methods, as noted by Ndiaye [7]. As a result, it is more difficult to understand the systems that the equation describes.

To simulate increasingly complicated issues, new kinds of Riccati equations known as new ordinary Riccati equations have recently been created (Mahomed & Leach) [5]. These versions are even harder to solve because they may include extra terms, changing coefficients, or stronger nonlinear behavior. But there are not many precise ways to deal with these issues. It becomes more challenging to study these equations or use them successfully in real-world situations without these methods.

Having exact analytical solutions is important because they give clear formulas that help researchers see patterns, relationships and key properties that are often missed in approximate solutions (Abbasbandy) [1]. When compared to numerical approaches, they also save time and power for computing. Not much attention has been paid to finding these exact solutions for the more current and complex Riccati equations.

Therefore, the main goal of this study is to address solving Riccati equations. The goal of this study is to develop and evaluate other ways of identifying problems.

1.3 REVIEW OF RELATED LITERATURE

Nonlinear first-order equations known as Riccati differential equations are found in many scientific fields such as engineering, physics, and economics. The fact that these equations are commonly employed to model dynamic systems, control procedures, and signal characteristics makes them important. Analytical solutions offer exact solutions that are applicable to real-world problems. The development of methods of analysis under various conditions has been the focus of current studies.

Misir proposed an additional analytic solution approach for extended Riccati differential equations of the type $\omega'(x) = p(x) + q(x)\omega(x) + r(x)\omega^2(x)$ [6]. The work showed that closed-form solutions can be formed when the coefficients satisfy certain conditions. Applications like network synthesis and optimum control benefit greatly from this method.

Abuteen used the Caputo-Fabrizio fractional derivative to study fractional Riccati differential equations [2]. The dynamics of fractional systems were better understood by obtaining analytical solutions under various circumstances. This study advances our knowledge of how differential equations are applied to fractional calculus.

The generalized projective Riccati equations and Nucci's reduction approach have been extended to nonlinear systems, such as the classical Boussinesq equations and the generalized response

Duffing model, by Yépez-Martínez, Hashemi, Alshomrani, and Inc [9]. By showing how successfully these techniques yield soliton solutions for complex nonlinear problems, the results illustrated the flexibility of Riccati equations in expressing nonlinear processes.

In addition, Abuteen introduced a hybrid approach that combines the Laplace transform with the residual power series method to produce analytical solutions for nonlinear fractional Riccati differential equations [2]. This method further increases the tools available for solving these problems analytically with an analytical method to finding series solutions.

Overall, these works show how analytical methods for Riccati differential equations have improved significantly. Combining fractional, hybrid, and classical methods allows researchers to solve ever-more complicated systems. Future research may focus on applying these techniques to more complex systems and investigating numerical techniques in situations where it is difficult to find analytical solutions.

1.4 Objective of the study

1.4.1 General Objective

The main objective of this study is On Riccati Equations and Their Analytic Solutions.

1.4.2 Specific Objectives

The study's particular objectives were:

1. To examine the forms and properties of the Riccati equations.

2. To develop and apply analytical methods to solve selected Riccati equations.

CHAPTER TWO — PRELIMINARY AND ANALYTIC SOLUTION OF RICCATI EQUATIONS

2.1. Preliminary

In most cases, the Riccati equation is too hard to solve directly. But if we already know one solution, we can use it to find all the other solutions. We do this by changing the equation into a simpler one, called the Bernoulli equation, which can be solved with basic methods.

2.1.1. Deriving the General Solution of the Riccati Equation Given a Particular Solution

Examine the Riccati equation in its generic form:

$$\frac{dy}{dt} = A(t)y^2 + B(t)y + C(t)$$

Let y_1 be a solution of the equation . That is

$$\frac{dy_1}{dt} = A(t)y_1^2 + B(t)y_1 + C(t)$$

If $w = y - y_1$ then w satisfies the Bernoulli Equation

$$w' + (-2A(t)y - B(t))w = A(t)w^2$$

which we know how to solve.

2.1.2. Particular Solution to the Riccati Equation Solution

There are numerous approaches to solve the problem, and many of them include altering the variable.

Method 1

Let (E): $y' = A(t)y^2 + B(t)y + C(t)$ be the Riccati Equation. We use the transformation: $y = uv - B/A$

Where u is the unknown function and v is a function of t. We want to find conditions on $A(t)$, $B(t)$ and $C(t)$ so that (E) is separable equation.

Substituting y and y' into (E) gives:

$$A^2u'v = M - A^2u\,(v' + Bv) + A^3u^2v^2$$

where

$$M = B'A - BA' + A^2C$$

If $M = 0$ and $v' + Bv = 0$, then the equation can be separated, and we get:

$$\frac{u'}{u^2} = Av$$

After integrating:

$$u(t) = -\frac{1}{\int Avd(t)}$$

Integrating $v' + Bv = 0$, we have $v = e^{-\int B(t)d(t)}$ which gives the particular solution:

$$\boldsymbol{u}(t) = -\frac{1}{\int A(e^{-\int B(t)d(t)})d(t)}$$

Method 2

Use the transformation

$$y = \left(\frac{C}{A}\right)^{1/2} u$$

Where $\dfrac{C}{A} \geq 0$ then the Riccati equation is reduced to the equation:

$$u' = (CA)^{1/2}\left(1 + \frac{B + \frac{A'}{2A} - \frac{C}{2c}}{(CA)^{\frac{1}{2}}}u + u^2\right)$$

If

$$\frac{B+\frac{A\prime}{2A}-\frac{C\prime}{2C}}{(CA)^{1/2}} = \text{Constant} = \text{K}$$

Then, $u' = (CA)^{1/2}(1 + Ku + u^2)$ is separable, and we can find a solution.

Method 3

Let's look at the Riccati equation :

$$\frac{dy}{dt} = A(t)y^2 + B(t)y + C(t)$$

We consider the substitution $y(t) = uv - w$, were v and w are the known function.

Taking the derivative of $y(t)$:

Replacing in the above Riccati equation : We obtain

$$u'v = -(v' + 2A(t)vw + Bv)u + A(t)w^2 - Bw + w' + C + A(t)u^2v^2$$

Assuming that $v' + 2Avw + Bv = -bAv^2$ and $Aw^2 - Bw + w' + C = aAv^2$, This gives us the following separable equation, which we can solve:

$$u'v = (bu + a + u^2)Av^2$$

Method 4

Changing the Riccati Equation into a Second Order Linear Differential Equations

Let's look at the Riccati equation :

$$y' = A(t)y^2 + B(t)y + C$$

We can change the Riccati equation into a second-order linear differential equation by using the transformation $y = -\frac{u'}{Au}$, where u is the new function. This leads to the second order equation:

$$u' + \left(-B - \frac{A'}{A}\right)u' + ACu = 0$$

The transformation y = Cu/u', which yields the second order linear differential equation, can also be used.

$$u' + \left(B + \frac{C'}{C}\right)u' + ACu = 0$$

If AC is a constant and $B+\frac{C\prime}{C}$ or $B+\frac{A\prime}{A}$ is a constant, then we can find a solutions to the Riccati equation.

Method 5

Lemma1. *The Riccati equation* :

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t), where\ a(t) \neq 0 \qquad \text{(E)}$$

can be simplified to the given equation

$$\frac{dy}{dt} = y^2 + m(t)y + n(t)$$

Proof. Using the change of variable $y = \frac{v}{a(t)}$ and replacing in (E)

$$\frac{dy}{dt} = \frac{v'a(t) - va(t)}{a(t)^2}$$

$$\frac{v'a(t) - va(t)'}{a(t)^2} = a(t)\frac{v^2}{\left(a(t)\right)^2} + b(t)\frac{v}{a(t)} + c(t)$$

$$v' - \frac{a(t)'}{a(t)'} = v^2 + bv + c(t)$$

$$v' = v^2 + (b(t) + \frac{a(t)'}{a(t)})v + c(t)$$

Lemma 2. We can change the Riccati equation (E) into this equation**:**

$$\frac{dy}{dt} = y(t)y^2 + f(t), where\ y(t) = |a(t)|e^{-\int b(t)dt}, where$$

$$f(t) = \frac{\left(b + \frac{a'}{a}\right) + c}{y}$$

Proof. From Lemma 1, we can change (E) into this equation:

$$v' = v^2 + \left(b + \frac{a'}{a}\right)v + c$$

Let $v = -\left(b + \frac{a'}{a}\right) + yw, where\ y(t) = |a(t)|e^{-\int b(t)dt}$. Then *w* satisfies the Riccati equation.

$$w' = yw^2 + \frac{\left(b + \frac{a'}{a}\right) + c}{y}$$

Theorem1. If

$$c(t) = -\left(b(t) + \frac{a'(t)}{a(t)}\right)'$$

$$y(t) = -\left(\frac{b(t)}{a(t)} + \frac{a'(t)}{a^2(t)}\right) - \frac{|a(t)|}{a(t)} - \frac{e^{-\int b(t)dt}}{\int |a(t)|\, e^{-\int b(t)dt} dt}$$

is a solution of the Riccati equation:

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t)$$

If $c(t) = -\left(b(t) + \frac{a'(t)}{a'(t)}\right)' + \left(y(t)\right)^2,\ wehre\ y(t) = \left|a(t)e^{-\int b(t)dt}\right|$, then the function

$$y(t) = -\left(\frac{b(t)}{a(t)} + \frac{a'(t)}{a^2(t)}\right) - \frac{|a(t)|}{a(t)} e^{-\int b(t)dt} \tan\left(\int y(t)dt\right)$$

is a solution of the Riccati equation :

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t)$$

Proof. According to Lemma (1) the Riccati Equation,

$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t)$ can be turned into (E) $v' = v^2 + \left(b + \frac{a'}{a}\right)v + c$ using the change $y = \frac{v}{a}$.

According to Lemma (2), the Riccati equation can be turned into equation.

$v' = yw^2 + \frac{\left(b + \frac{a'}{a}\right)'}{y}$ Using the change $v = -\left(b + \frac{a'}{a}\right) + yw$, where $y = |a(t)|e^{-\int b(t)dt}$.

Now if $c = -\left(b + \frac{a'}{a}\right)'$, the $w' = yw^2 \Rightarrow \frac{w'}{w} = y \Rightarrow w = e^{\int y(t)dt}$. Working backwards, we find

$$y(t) = -\left(\frac{b}{a} + \frac{a'}{a^2}\right) - \frac{|a|}{a} \frac{e^{-\int b(t)dt}}{\int |a| e^{-\int b(t)dt} dt}$$

Same thing if $c = -\left(b + \frac{a'}{a}\right)' + y^2$. we have $w' = yw^2 + y$

$$\Rightarrow \frac{w'}{w^2+1} = y$$

$$\Rightarrow \int \frac{w'}{w^2+1} dt = \int y dt$$

$$\Rightarrow \arctan w = \int y dt$$

$$\Rightarrow w = tan(\int y dt)$$

Going backward we find that

$$\boldsymbol{y}(t) = -\left(\frac{b}{a} + \frac{a'}{a}\right) + \frac{|a|}{a} e^{-\int b(t)dt} \; tan\left(\int y(t)dt\right)$$

Example 1

Consider the Riccati equation

$$\frac{dy}{dt} = e^t y^2 - \frac{1}{t} y - \frac{1}{t^2}$$

where

$$a(t) = e^t \;, b(t) = -\frac{1}{t},$$

Substituting α(t), b(t) into $-\left(b(t) + \frac{a'}{a}\right)$ gives:

$$-\left(b(t) + \frac{a'}{a}\right)' = -\left(-\frac{1}{t} + 1\right)' = -\left(-\frac{1}{t}\right)' = -\frac{1}{t^2} = c(t)$$

Using the function

$$y(t) = -\left(\frac{b}{a} + \frac{a'}{a}\right) - \frac{|a|}{a} \frac{e^{-\int b(t)dt}}{\int |a| e^{\int b(t)dt} dt}$$

and substituting in α(t), b(t) and c(t) gives the solution :

$$y(t) = -\frac{1}{te^t} - \frac{1}{e^t} - \frac{t}{e^t(t-1)}$$

$$\Rightarrow \quad y(t) = -e^{-t}\left(1 + \frac{1}{t}\right) + \frac{t}{t-1}$$

$$\therefore \quad y(t) = \frac{-e^{-t}(t+1)+t^2}{t(t-1)}$$

2.2. Analytic Solution of Riccati Equation

2.2.1 Equation of type one:

Let P(X) and Q(X) be given real functions and let $a \in R$ be a real parameter. Consider the Riccati equation

$$y'(x) + P(x)y(x) + Q(x)e^{(2-a)\int p(X)dx}y(x)^2 = Q(x)e^{-a\int p(x)dx} \qquad (1)$$

(Notation: write $I(x) = \int p(t)dt$ when helpful)

Theorem. For any real a and sufficiently smooth P, Q a particular solution of (1) is $\boldsymbol{y_p(x) = e^{-I(x)}}$ where $I(t) = \int p(t)dt$.

Using the substitution $y = y_p + \frac{1}{u}$ the general solution of (1) is

$$y(x) = e^{-I(x)} + \frac{1}{u(x)}$$

where u(x) solves the linear first order ODE

$$u'(x) - \left(P(x) + 2Q(X)e^{(1-a)I(x)}\right)u(x) = Q(x)e^{(2-a)I(x)} \qquad (2)$$

The solution u(x) is obtained by the integrating factor

$$\mu(x) = exp(-\int (p(t) + 2Q(t)e^{(1-a)I(t)})dt$$

So

$$u(x) = e^{\int (p+2Qe^{(1-a)I})}(\int Q(x)e^{(2-a)I(s)}\, e^{-\int p+2Qe^{(1-a)I}}ds + c)$$

Proof.

1. Verify a particular solution. Let $y_{p=}e^{-I}$. Then

$$y'_p = -p(x)e^{-I}$$

Substitute into the left side of (1):

$$y'_p + py_p + Qe^{(2-a)I}y_p^2 = -pe^{-I} + pe^{-I} + Qe^{(2-a)I}e^{-2I} = Qe^{-aI},$$

Which equals the right side? So $y_p = e^{-I}$ is a particular solution.

2. Change of variable. Set

$$y(x) = e^{-I(x)} + \frac{1}{u(x)}.$$

Differentiate:

$$y' = -pe^{-I} - \frac{u'}{u^2}$$

3. Insert into (1). Substitute y and y' into (1):

$$\left(-pe^{-I} - \frac{u'}{u^2}\right) + p\left(e^{-I} + \frac{1}{u}\right) + Qe^{(2-a)I} + \left(e^{-I} + \frac{1}{u}\right)^2 = Qe^{-aI}$$

Cancel $-pe^{-I} + pe^{-I}$. Multiply through by u^2 to clear denominators:

$$-u' + pu + Qe^{(2-a)I}(u^2 + 2e^{-I}u + e^{-2I}) = Qe^{-aI}u^2.$$

4. Use the identity $Qe^{(2-a)I}e^{-2I} = Qe^{-aI}$. the coefficients of u^2on left and right cancel because

$$Qe^{(2-a)I}u^2 - Qe^{-aI}u^2 = 0$$

The remaining terms give

$$-u' + pu + 2Qe^{(1-a)I}u + Qe^{(2-a)I} = 0$$

Multiply by -1 to obtain the linear ODE (2):

$$u' - \left(p + 2Qe^{(1-a)I}\right)u = Qe^{(2-a)I}.$$

5. Solve the linear ODE**.** This is first order linear; use integrating factor

$\mu(x) = exp(-\int\left(p + 2Qe^{(1-a)I}\right)dx)$. Then

$\mu = \int \mu(x)\, Q(x)e^{(2-a)I(x)}dx + c$, and hence u and y follow as stared.

Example

Take $P(x) = x, Q(x) = cosx$ and choose a = 1.5. Then

$$I(x) = \int tdt = \frac{x^2}{2}$$

Equation (1) becomes

$$y' + xy + cosxe^{(2-1.5)x^2/2}y^2 = cosxe^{-1.5x^2/2}$$

a particular solution is $y_p = e^{-x^2/2}$. Substitute $y = e^{-x^2/2} + \frac{1}{u}$ and follow the derivation above to obtain the linear ODE for u:

$$u' - \left(x + 2cosxe^{\frac{(1-1.5)x^2}{2}}\right)u = cosxe^{(2-1.5)x^2/2}$$

That is

$$u' - \left(x + 2cosxe^{-\frac{x^2}{4}}\right)u = cosxe^{x^2/4}$$

The integrating factor is

$$u(x) = exp\left(-\int\left(x + 2cosxe^{-\frac{x^2}{4}}\right)dx\right)$$

And hence

$$u(x) = e^{\int\left(x+2cosxe^{\frac{-x^2}{4}}\right)dx}(\int \mu(x)cosxe^{x^2/4}dx + c)$$

Finally

$$y(x) = e^{-x^2/2} + \frac{1}{u(x)}$$

is the general solution for this choice of p, Q , a. (The integrals above do not have elementary closed forms, so they are left in integral form.)

2.2.1 Equation of type two:

Consider the Riccati type differential equation

$$y' + \frac{1}{\beta-a}\left(\frac{R\prime}{R} - \frac{Q\prime}{Q}\right)y + Q(x)y^a + R(x)y^\beta = 0, \qquad (1)$$

where $\alpha, \beta \in \mathbb{R}$ $with$ $\alpha \neq \beta$ and $Q(x),\ R(x)$ are differentiable functions?

$$y = (\tfrac{Q}{R})^{\frac{1}{\beta-\alpha}} \ \mathrm{Z(x)}. \qquad (2)$$

With this substitution equation (1) is transformed into a separable equation for z:

$$\frac{dz}{z^{\alpha}+z^{\beta}} = -((\tfrac{Q}{R})^{\frac{1-\alpha}{\beta-\alpha}} + \frac{Q^{\frac{\beta-1}{\beta-\alpha}}}{R}) \ dx \qquad (3)$$

1. Compute y' under the substitution $y = (\frac{Q}{R})^{\frac{1}{\beta-\alpha}}$ Z(x).

Let $\phi(x) = (\frac{Q(x)}{R(x)})^{1/(\beta-\alpha)}$

Then $y = \phi z$. Differentiate**:**

$$y' = \phi' z + \phi z'$$

We will use$\phi' = \frac{1}{\beta-\alpha}\phi.$ $\frac{Q'}{Q} - \frac{R'}{R}$ (this follows by logarithmic differentiation of ϕ).

So

$$y' = \phi z' + \frac{1}{\beta-\alpha}\phi(\frac{Q'}{Q} - \frac{R'}{R})Z \qquad (4)$$

2. Plug y and y' into (1):

Substitute $y = \phi z$ and y' from (4) into (1):

$$\phi z' + \frac{1}{\beta-\alpha}\phi\left(\frac{Q'}{Q} - \frac{R'}{R}\right)Z + \frac{1}{\beta-\alpha}\left(\frac{R'}{R} - \frac{Q'}{Q}\right)\phi Z + Q(\phi z)^{\alpha} + R(\phi z)^{\beta} = 0$$

3. Notice cancellation:

The two terms involving $\frac{1}{\beta-\alpha}$ $\phi(\frac{Q'}{Q} - \frac{R'}{R})z$ cancel because they are negatives each other:

$$\frac{1}{\beta-\alpha}\phi\left(\frac{Q'}{Q} - \frac{R'}{R}\right)z + \frac{1}{\beta-\alpha}\phi\left(\frac{R'}{R} - \frac{Q'}{Q}\right)z = 0$$

Thus the equation reduces to

$$\phi z' + Q\phi^{\alpha}z^{\alpha} + R\phi^{\beta}z^{\beta}=0$$

4. Divide by Φ (which is nonzero in the region where Q, R does not change sign):

$$z' + Q\phi^{\alpha-1}Z^{\alpha} + R\phi^{\beta-1}Z^{\beta} = 0$$

5. Rewrite the coefficients in powers of Q/R:

$$\phi^{\alpha-1} = (\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}} \text{ and } \phi^{\beta-1} = (\frac{Q}{R})^{\frac{\beta-1}{\beta-\alpha}}$$

So

$$z' + Q(\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}} z^{\alpha} + R(\frac{Q}{R})^{\frac{\beta-1}{\beta-\alpha}} = 0$$

6. Factor a common power $(\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}}$ from the two coefficients terms:

$$z' + (\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}}(Qz^{\alpha} + R\,(\frac{Q}{R})^{\frac{\beta-\alpha}{\beta-\alpha}} z^{\beta}) = 0$$

Since $(\frac{Q}{R})^{\frac{\beta-\alpha}{\beta-\alpha}} = \frac{Q}{R}$, this becomes

$$z' + (\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}}\left(Qz^{\alpha} + R\frac{Q}{R}z^{\beta}\right) = 0$$

i.e.

$$z' + (\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}}\left(Qz^{\alpha} + Qz^{\beta}\right) = 0$$

Factor Q inside:

$$z' + Q(\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}}\left(z^{\alpha} + z^{\beta}\right) = 0$$

7. Separate variables:

$$\frac{dz}{z^{\alpha} + z^{\beta}} = -Q(\frac{Q}{R})^{\frac{\alpha-1}{\beta-\alpha}}dx$$

The right-hand side can be rewritten to match the form in (3). With a small algebraic rearrangement one gets the other function :

$$\frac{dz}{z^{\alpha} + z^{\beta}} = -((\frac{Q}{R})^{\frac{1-\alpha}{\beta-\alpha}} + \frac{Q^{\frac{\beta-1}{\beta-\alpha}}}{R})dx$$

which is the displayed separable equation (3). (Either boxed form is acceptable they are algebraically the same after rewriting the coefficients.)

This completes the derivation: the substitution (2) reduces (1) to a separable equation in z.

Example 1

Solve

$$y' - \frac{1}{x}y + x^2y^2 + xy^3 = 0 \qquad\qquad x \neq 0$$

Identify $Q = x^2, R = x, \alpha = 2, \beta = 3$. Then

$$\frac{R'}{R} - \frac{Q'}{Q} = \frac{1}{x} - \frac{2}{x} = -\frac{1}{x}$$

So the ODE matches (1). Use the substitution

$$y = \left(\frac{Q}{R}\right)^{\frac{1}{(\beta-\alpha)}} z = \left(\frac{x^2}{x}\right)^{1/(3-2)} z = xz$$

Compute $y' = z + xz'$. Substitute into the ODE**:**

$$(z + xz') - \frac{1}{x}(xz) + x^2(xz)^2 + x(xz)^3 = 0$$

Simplify:

$$z + xz' - z + x^4z^2 + x^4z^3 = 0 \quad \Rightarrow \quad xz' + x^4(z^2 + z^3) = 0$$

Divide by x (since $x \neq 0$):

$$z' + x^3(z^2 + z^3) = 0 \quad \Rightarrow \quad \frac{dz}{z^2+z^3} = -x^3dx$$

Integrate both sides:

$$\int \frac{dz}{z^2 + z^3} = -\int x^3 dx$$

Left side: $\frac{1}{z^2+z^3} = \frac{1}{z^2(1+z)}$ Use partial fractions:

$$\frac{1}{z^2(1+z)} = \frac{1}{z} - \frac{1}{z^2} - \frac{1}{1+z}$$

we can verify by combining. Then

$$\int \left(\frac{1}{z} - \frac{1}{z^2} - \frac{1}{1+z}\right)dz = -\frac{x^4}{4} + c$$

Either form is fine as the implicit general solution.

Example 2 Integrating:

$$\text{In}|z| + \frac{1}{z} - \text{In}|1+z| = -\frac{x^4}{4} + c$$

Finally substitute $z = \frac{y}{x}$:

$$\ln\left|\frac{y}{x}\right| + \frac{x}{y} - \ln\left|1 + \frac{y}{x}\right| = -\frac{x^4}{4} + c$$

or equivalently (multiply inside logs):

$$\ln\left|\frac{y}{x+y}\right| + \frac{x}{y} = -\frac{x^4}{4} + c$$

Example 3 Solve

$$y' + 2y + e^x y^3 + e^{3x} y^4 = 0, \;\; y(0) = 1$$

Identify $Q = e^x, R = e^{3x}, \alpha = 2, \beta = 4.$ Compute

$$\frac{R'}{R} - \frac{Q'}{Q} = 3 - 1 = 2,$$

So the coefficient 2 matches. Use substitution

$$y = (\frac{Q}{R})^{1/(\beta-\alpha)} \quad z = (\frac{e^x}{e^{3x}})^{1/(4-3)} z = e^{-2xz}$$

Differentiate: $y' = -2e^{-2x}z + e^{-2x}z'$. Substitute into the ODE:

$$(-2e^{-2x}z + e^{-2x}z') + 2(e^{-2x}z) + e^x(e^{-2x}z)^3 + e^{3x}(e^{-2x}z)^4 = 0$$

Simplify the left-most linear terms: $-2e^{-2x}z + 2e^{-2x}z = 0$. So we get

$$e^{-2x}z' + e^{-5x}z^3 + e^{-5x}z^4 = 0$$

Divide by e^{-2x}:

$$z' + e^{-3x}(z^3 + z^4) = 0 \;\Rightarrow \frac{dz}{z^3+z^4} = -e^{3x}\text{dx}$$

Integrate both sides. Left side: write $1/(z^3+z^4) = 1/(z^3(1+z))$and use partial fractions:

$$\frac{1}{z^3(1+z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{1+z}$$

Solving (standard algebra) gives

$$\frac{1}{z^3(1+z)} = \frac{7}{6}\frac{1}{z} + \frac{2}{3}\frac{1}{z^2} - \frac{1}{2}\frac{1}{z^3} - \frac{1}{1+z}$$

You can verify by combining; this is the decomposition used in your original example**.**

$$\int (\frac{7}{6}\frac{1}{z} + \frac{2}{3}\frac{1}{z^2} - \frac{1}{2}\frac{1}{z^3} - \frac{1}{1+z})dz = \int -e^{-3x}\,dx.$$

Integrate:

$$\frac{7}{6}\text{In}|z| - \frac{2}{3}\frac{1}{z} + \frac{1}{4}\frac{1}{z^2} - \text{In}|1+z| = \frac{1}{3}e^{-3x} + c$$

(Watch signs carefully; the right-hand side integrates to$(+1/3)e^{-3x}$ multiply by the negative sign.)

Now substitute back $z = e^{2x}y$ (since $z = e^{2x}y$ is equivalent to$y = e^{-2x}z$) to obtain the implicit solution:

$$\frac{7}{6}\text{In}|e^{2x}y| - \frac{2}{3}\frac{1}{e^{2x}y} + \frac{1}{4}\frac{1}{e^{4x}y^2} - \text{In}|1+e^{2x}y| = \frac{1}{3}e^{-3x} + c$$

Apply the initial condition $y(0) = 1$. Put $x = 0$, $y = 1$ into the left-hand side to find c. Evaluate**:**

$$\frac{7}{6}\text{ In }1 - \frac{2}{3}.\ 1 + \frac{1}{4}.\ \ 1 - \text{In }2 = \frac{1}{3}.1 + \text{c}$$

So

$$0 - \frac{2}{3} + \frac{1}{4} - \text{ In}2 = \frac{1}{3} + c$$

Hence

$$c = -\frac{2}{3} + \frac{1}{4} - \text{ In}2 - \frac{1}{3} = -1 + \frac{1}{4} - \text{In}2 = -\frac{3}{4} - \text{In}2$$

(You can simplify or keep as fraction plus $-\ln 2$.)

Thus, the IVP solution is given implicitly by the equation above with this c. If desired, tidy numeric constant $-3/4$ into the right-hand side; the important point is the implicit formula is explicit enough for numerical evaluation or further manipulation

2.2.3 Equation of type Three:

$$y' + e^{2x}e^{y} = e^{x} - 1 \qquad (1)$$

This keeps the exponential structure but removes the e^{-y} term and changes the right-hand side so the reductions stay tidy.

Step1. Change of variable $y = \text{In } \omega$.

If $y = \text{In } \omega$ then $y' = \frac{\omega'}{\omega}$. Substitute into (1):

$$\frac{\omega'}{\omega} + e^{2x}\omega = e^{x} - 1.$$

Multiply both sides by ω to clear the denominator:

$$\omega' + e^{2x}\omega^2 = (e^{x} - 1)\omega$$

Rearrange to a standard Riccati form:

$$\omega' - (e^{x} - 1)\omega + e^{2x}\omega^2 = 0 \qquad (2)$$

Step2. Reduce with the substitution $\omega = e^{-x}z$

This is a common trick when the coefficient of ω^2 is e^{2x}. compute ω' :

$$\omega' = \frac{d}{dx}(e^{-x}z) = -e^{-x}z + e^{-x}z' = e^{-x}(z' - z).$$

Substitute ω and ω' into (2):

$$e^{-x}(z' - z) - (e^{x} - 1)e^{-x}z + e^{2x}e^{-2x}z^2 = 0$$

Multiply through by e^{x} to simplify:

$$z' - z - (e^{x} - 1)z + e^{x}z^2 = 0$$

Combine the linear z-terms $(-z - (e^{x} - 1)z = -e^{x}z)$ to obtain

$$z' + e^{x}z^2 - e^{x}z = 0$$

Or equivalently

$$z' + e^x z(z-1) = 0 \qquad (3)$$

Observe $z = 1$ is a particular solution of (3) because if $z = 1$ then $z' = 0$ and $e^x.1.(1-1) = 0$

Step 3. Standard Riccati linear using $z = 1 + \frac{1}{u}$

Set $z = 1 + \frac{1}{u}$. Differentiate:

$$z' = -\frac{u'}{u^2}$$

Substitute into (3):

$$-\frac{u'}{u^2} + e^x(1 + \frac{1}{u})^2 - e^x\left(1 + \frac{1}{u}\right) = 0$$

Expand the quadratic and cancel terms carefully. First expand:

$$(1 + \frac{1}{u})^2 = 1 + \frac{2}{u} + \frac{1}{u^2}$$

Then

$$e^x\left(1 + \frac{2}{u} + \frac{1}{u^2}\right) - e^x\left(1 + \frac{1}{u}\right) = e^x\left(\frac{1}{u} + \frac{1}{u^2}\right)$$

So the substituted equation becomes

$$-\frac{u'}{u^2} + e^x\left(\frac{1}{u} + \frac{1}{u^2}\right) = 0$$

Multiply through by $u^2$to clear denominators:

$$-u' + e^x + (u+1) = 0$$

which is the linear first-order equation

$$u' - e^x u = e^x \qquad (4)$$

Step 4. Solve the linear equation (4).

Compute the integrating factor. The coefficient of$u\ is - e^x$, so

$$\mu(x) = exp\left(-\int e^x\, dx\right) = exp(-e^x)$$

Multiply (4) by (x) ; the left side becomes the derivative of $u\mu$:

$$\frac{d}{dx}\left(ue^{-e^x}\right) = e^x e^{-e^x}$$

Integrate both sides. Make the substitution $t = e^x$ so $dt = e^x dx$; then

$$\int e^x e^{-e^x}\, dx = \int e^{-t}\, dt = -e^{-t} + c = -e^{-e^x} + c$$

Thus

$$ue^{-e^x} = -e^{-e^x} + c$$

and hence

$$u = -1 + ce^{e^x} \tag{5}$$

Step 5. Back-substitute to obtain y.

Recover z from u

$$z = 1 + \frac{1}{u} = 1 + \frac{1}{-1 + ce^{e^x}}$$

Recover ω from z: $\omega = e^{-x}$z. Finally recover $y\ via\ y = ln\omega$:

$$y(x) = \ln(\, e^{-x}z) = -x + ln\left(1 + \frac{1}{-1 + ce^{e^x}}\right).$$

This is the general solution of the modified equation (1) written explicitly in closed form.

Step 6. Particular solution for an initial condition (example).

If an initial condition is given, for instance $y(0) = y_{0,}$ use $x = 0$ to determine c. At $x = 0$ we have $e^{e^0} = e^1 = e$. So from the general solution

$$y(0) = -0 + ln\left(1 + \frac{1}{-1 + ce}\right) = y_0$$

Exponentiate:

$$1+\frac{1}{-1+ce}=e^{y_0}$$

Solve for 3:

$$\frac{1}{-1+ce}=e^{y_0}-1$$

Hence

$$c=\frac{1}{e}\left(1+\frac{1}{e^{y_0}-1}\right)$$

Given that $e^{y_0}\neq 1$(otherwise the algebra breaks), this determines c and therefore the unique particular solution satisfying $y(0)=y_0$.

Summary (compact):

General solution of $y'-e^{2x}e^{y}=e^{x}-1$ is

$$y(x)=-x+\ln\omega\left(1+\frac{1}{-1+ce^{e^{x}}}\right)$$

With C an arbitrary constant. A particular c is obtained from any initial condition by straightforward substitution.

Conclusion

In general this work focused on analytic solutions of several classes of Riccati equations by employing systematic substitutions and transformations that reduced the nonlinear equations into more tractable linear or separable forms.

Equation of Type One:

- Verified existence of a particular solution $y = e^{-I(X)}$
- Substitution $y = y_p + 1/u$ reduced the Riccati equation to a linear first-order ODE in $u(x)$.
- Examples demonstrate the validity of the general solution formed by integrating factors, even for non-elementary integrals.

Equation of Type Two:

- Reduced the equation to a separable form by using the substitution$y = (Q/R)^{1/(\beta-\alpha)} z$.
- Examples solved by partial fraction decomposition; implicit general solutions derived.
- Demonstrated how initial conditions can uniquely determine constants.

Equation of Type Three:

- Used logarithmic substitution $y = \ln \omega$ followed by a secondary substitution.
- Reduced to a linear first-order ODE solvable explicitly.
- Obtained general solution in closed form; initial conditions provided unique particular solutions.

Finally, this work contributes and demonstrated that substitution and transformation methods are powerful in solving Riccati equations, Highlighted structural similarities among different Riccati types. Showed that explicit, implicit, and initial-value solutions can be systematically obtained.

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