THE OUTPUT LIGHT FROM A SUBHARMONIC GENERATOR AND A COHERENTLY DRIVEN CAVITY MODE

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Abstract

Applying the propagator method discussed in J. Math. Phys. 33 (1992) 2179, we calculate the Q function of the cavity coherent light and the signal mode produced by a degenerate subharmonic generator. We then determine the superposition of these two light beams. Employing the input-output relation, we obtain the Q function of the output light. With the aid of the resulting Q function, the squeezing and statistical properties of the output light are analyzed. It is found that the coherently driven cavity mode has no effect on the quadrature variance of the output light. However, it enhances the mean photon number of the output light.

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Chapter 1

Introduction

A considerable attention has been paid to squeezed states of light [1-5], because in these states the quantum noise in one quadrature is below the coherent-state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. This property makes squeezed light to be very useful in the detection of weak signals and in low-noise communications [1, 3, 6, 7].

Various quantum optical processes such as subharmonic [1-9] and second harmonic [1, 3, 6, 7] generation produce squeezed light. A degenerate subharmonic generator, consisting of a nonlinear crystal pumped by coherent light, is a prototype source of a single-mode squeezed light. In this system a pump photon of frequency 2ω is down converted into a pair of highly correlated signal photons each of frequency ω.

Subharmonic generator is one of the most reliable source of squeezed light. A theoretical analysis of the quantum fluctuations and photon statistics of the signal mode produced by a subharmonic generator has been made by a number of authors [1, 2, 5, 6, 7, 9]. A limit of 50 % squeezing of the intracavity signal mode produced by a subharmonic generator has been predicted by a number of authors [1, 2, 5, 6].

In this thesis, the Q function of the signal and the coherently driven cavity mode are derived separately using the propagator method discussed in [10]. We next determine the superposition of these two Q functions [1]. We then obtain the Q function of the output light employing the input-output relation. With the aid of the resulting Q function, we calculate the normally-ordered quadrature variance, the squeezing spectrum, the mean photon number, the normally-ordered variance of the photon number and the photon number distribution.

Chapter 2

The Q Function

Various quantum distribution functions are widely used in quantum optics, because they provide convenient means of describing the quantum properties of light modes. Here we will confine our discussion to the Q function which is best suited to the evaluation of the expectation value of antinormally ordered operators. We proceed to calculate the Q function of a subharmonic generator, a coherently driven cavity mode and the superposition of these two Q functions.

2.1 The Q function of a coherently driven cavity mode

We seek here to obtain the Q function for a cavity mode driven by coherent light and coupled to a vacuum reservoir. The interaction between a cavity mode and a driving coherent light, treated classically, can be described by

$$\hat{H} = i\varepsilon(\hat{a}^{\dagger} - \hat{a}), \tag{2.1.1}$$

where $\hat{a}$ is the annihilation operator for the cavity mode and ε is proportional to the amplitude of the driving light. The master equation for the cavity mode driven by coherent light and coupled to a vacuum reservoir has the form

$$\frac{d\hat{\rho}}{dt} = -\varepsilon(\hat{a}\hat{\rho} - \hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^{\dagger}) + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}). \tag{2.1.2}$$

We next proceed to obtain the Fokker-Planck equation for the Q function. To this end, using the relations

$$[\hat{a}, f(\hat{a}^\dagger, \hat{a})] = \frac{\partial}{\partial \hat{a}^\dagger} f(\hat{a}^\dagger, \hat{a}), \tag{2.1.3}$$

$$[\hat{a}^\dagger, f(\hat{a}^\dagger, \hat{a})] = -\frac{\partial}{\partial \hat{a}} f(\hat{a}^\dagger, \hat{a}), \tag{2.1.4}$$

we find

$$\begin{aligned} \hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger &= [\hat{a}, \hat{\rho}] - [\hat{a}^\dagger, \hat{\rho}] \\ &= \frac{\partial}{\partial \hat{a}^\dagger}\hat{\rho} + \frac{\partial}{\partial \hat{a}}\hat{\rho}. \end{aligned} \tag{2.1.5}$$

Moreover, we see that

$$\begin{aligned} 2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} &= -[\hat{a}^\dagger, \hat{a}\hat{\rho}] + [\hat{a}, \hat{\rho}\hat{a}^\dagger] \\ &= \frac{\partial}{\partial \hat{a}}(\hat{a}\hat{\rho}) + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{\rho}\hat{a}^\dagger). \end{aligned} \tag{2.1.6}$$

We note that

$$[\hat{a}, \hat{\rho}] = \frac{\partial}{\partial \hat{a}^\dagger}\hat{\rho}. \tag{2.1.7}$$

It then follows that

$$\hat{a}\hat{\rho} = \hat{\rho}\hat{a} + \frac{\partial}{\partial \hat{a}^\dagger}\hat{\rho}. \tag{2.1.8}$$

We also find in a similar manner that

$$\hat{\rho}\hat{a}^\dagger = \hat{a}^\dagger\hat{\rho} + \frac{\partial}{\partial \hat{a}}\hat{\rho}. \tag{2.1.9}$$

Upon substituting (2.1.8) and (2.1.9) into (2.1.6), there follows

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} = \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho}) + 2\frac{\partial^2}{\partial \hat{a}\partial \hat{a}^\dagger}\hat{\rho}. \tag{2.1.10}$$

Now on account of (2.1.5) and (2.1.10), the master equation takes the form

$$\frac{d\hat{\rho}}{dt} = -\varepsilon(\frac{\partial}{\partial \hat{a}^\dagger}\hat{\rho} + \frac{\partial}{\partial \hat{a}}\hat{\rho}) + \frac{\kappa}{2}(\frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho}) + 2\frac{\partial^2}{\partial \hat{a}\partial \hat{a}^\dagger}\hat{\rho}), \tag{2.1.11}$$

where $\hat{\rho} = \hat{\rho}(\hat{a}^\dagger, \hat{a}, t)$ is assumed to be in the normal order. Upon replacing $\hat{a}, \hat{a}^\dagger$ and $\hat{\rho}(\hat{a}^\dagger, \hat{a}, t)$ by α, α^* and $Q(\alpha^*, \alpha, t)$, we find the Fokker-Planck equation for the Q function to be

$$\frac{\partial}{\partial t}Q(\alpha, t) = \left[\frac{\partial}{\partial \alpha^*}(\frac{k}{2}\alpha^* - \varepsilon) + \frac{\partial}{\partial \alpha}(\frac{k}{2}\alpha - \varepsilon) + k\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right]Q(\alpha, t). \tag{2.1.12}$$

In terms of Cartesian coordinates, one can write

$$\alpha = x + iy, \tag{2.1.13}$$

$$\frac{\partial}{\partial \alpha} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \tag{2.1.14}$$

$$\frac{\partial^2}{\partial \alpha^2} = \frac{1}{4}(\frac{\partial^2}{\partial x^2} - 2i\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}), \tag{2.1.15}$$

$$\frac{\partial^2}{\partial \alpha^* \partial \alpha} = \frac{1}{4}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}). \tag{2.1.16}$$

Employing (2.1.13), (2.1.14), (2.1.15) and (2.1.16) along with their complex conjugate, the Fokker-Planck equation for the Q function can be put in the form

$$\frac{\partial}{\partial t}Q(x, y, t) = [\frac{\kappa}{4}\frac{\partial^2}{\partial x^2} + \frac{\kappa}{4}\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x}(\frac{\kappa}{2}x - \varepsilon) + \frac{\kappa}{2}\frac{\partial}{\partial y}y]Q(x, y, t). \tag{2.1.17}$$

This differential equation can be solved using the propagator method. We first transform (2.1.17) into a schrodinger-type equation. To this end, upon replacing ($\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x, y,$ $Q(x, y, t)$) by ($i\hat{p}_x, i\hat{p}_y, \hat{x}, \hat{y}, |Q(t)\rangle$), we get

$$i\frac{d}{dt}|Q(t)\rangle = \hat{H}|Q(t)\rangle, \tag{2.1.18}$$

where

$$\hat{H} = -i\frac{\kappa}{4}\hat{p}_x^2 - i\frac{\kappa}{4}\hat{p}_y^2 - \hat{p}_x(\frac{\kappa}{2}\hat{x} - \varepsilon) - \frac{\kappa}{2}\hat{p}_y\hat{y}. \tag{2.1.19}$$

The formal solution of (2.1.18) can be written as

$$|Q(t)\rangle = e^{-i\hat{H}t}|Q(0)\rangle. \tag{2.1.20}$$

Multiplying (2.1.20) by $\langle x, y|$ on the left-hand side and applying the completeness relation for two-dimensional position eigenstates, we have

$$\langle x, y|Q(t)\rangle = \int dx'dy'\langle x, y|e^{-i\hat{H}t}|y', x'\rangle\langle x', y'|Q(0)\rangle \tag{2.1.21}$$

or

$$Q(x, y, t) = \int dx'dy'Q(x, y, t|x', y', 0)Q_0(x', y'), \tag{2.1.22}$$

in which

$$Q(x, y, t) = \langle x, y|Q(t)\rangle, \tag{2.1.23}$$

$$Q(x, y, t|x', y', 0) = \langle x, y|e^{-i\hat{H}t}|y', x'\rangle \tag{2.1.24}$$

is the Q function propagator, and

$$Q_0(x', y') = \langle x', y'|Q(0)\rangle. \tag{2.1.25}$$

According to Fesseha [10], the propagator associated with a quadratic Hamiltonian

$$\hat{H}(\hat{x}, \hat{p}_x) = a_x\hat{p}_x^2 + b_x(t)\hat{p}_x\hat{x} + c_x(t)\hat{x}^2 \tag{2.1.26}$$

is expressible as

$$K(x, t|x', 0) = \big(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\big)^{\frac{1}{2}} exp\big(- \gamma\int_0^t b_x(t')dt' + iS_c\big), \tag{2.1.27}$$

where $S_c = S_c(x', x, t)$ is the classical action, a_x is a constant different from zero and γ is a parameter connected with operator ordering. Comparing (2.1.19) and (2.1.26), we have

$$b_x = -\frac{\kappa}{2}, \tag{2.1.28}$$

$$b_y = -\frac{\kappa}{2} \tag{2.1.29}$$

and recalling that for the antistandard form of ordering $\gamma = \frac{1}{2}$ and employing the two-dimensional form of (2.1.27), the Q function propagator associated with (2.1.19) is expressible as

$$Q(x, y, t|x', y', 0) = \big(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\big)^{\frac{1}{2}}\big(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial y'\partial y}\big)^{\frac{1}{2}} exp(\frac{\kappa}{2}t + iS_c). \tag{2.1.30}$$

The classical Hamlitonian corresponding to (2.1.19) is

$$H = -i\frac{\kappa}{4}p_x^2 - i\frac{\kappa}{4}p_y^2 - p_x(\frac{\kappa}{2}x - \varepsilon) - \frac{\kappa}{2}p_y y. \tag{2.1.31}$$

Applying Hamilton's equation of motion

$$\dot{x} = \frac{\partial H}{\partial p_x} \tag{2.1.32}$$

and

$$\dot{y} = \frac{\partial H}{\partial p_y}, \tag{2.1.33}$$

we arrive at

$$p_x = \frac{2i}{\kappa}\left(\dot{x} + (\frac{\kappa}{2}x - \varepsilon)\right), \tag{2.1.34}$$

$$p_y = \frac{2i}{\kappa}(\dot{y} + \frac{\kappa}{2}y). \tag{2.1.35}$$

Now on account of (2.1.34) and (2.1.35), the classical Hamiltonian takes the form

$$H = \frac{i}{\kappa}\dot{x}^2 - \frac{i}{\kappa}(\frac{\kappa}{2}x - \varepsilon)^2 + \frac{i}{\kappa}\dot{y}^2 - i\frac{\kappa}{4}y^2. \tag{2.1.36}$$

We recall that the Lagrangian is given by

$$L = \sum_j p_j \dot{x}_j - H, \tag{2.1.37}$$

where H is the classical Hamiltonian. We then see that

$$L = \frac{i}{\kappa}\left[\left(\dot{x} + (\frac{\kappa}{2}x - \varepsilon)\right)^2 + (\dot{y} + \frac{\kappa}{2}y)^2\right]. \tag{2.1.38}$$

The Euler-Lagrange equations are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{2.1.39}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0. \tag{2.1.40}$$

It then follows that

$$\ddot{x} - \frac{\kappa^2}{4}x + \frac{\varepsilon\kappa}{2} = 0, \tag{2.1.41}$$

$$\ddot{y} - \frac{\kappa^2}{4} y = 0. \tag{2.1.42}$$

The general solutions of (2.1.41) and (2.1.42) are given by

$$x(t) = c_1 e^{\frac{\kappa}{2} t} + c_2 e^{-\frac{\kappa}{2} t} + \frac{2\varepsilon}{\kappa}, \tag{2.1.43}$$

$$y(t) = c_3 e^{\frac{\kappa}{2} t} + c_4 e^{-\frac{\kappa}{2} t}, \tag{2.1.44}$$

there follows

$$\dot{x} = c_1 \frac{\kappa}{2} e^{\frac{\kappa}{2} t} - c_2 \frac{\kappa}{2} e^{-\frac{\kappa}{2} t}, \tag{2.1.45}$$

$$\dot{y} = c_3 \frac{\kappa}{2} e^{\frac{\kappa}{2} t} - c_4 \frac{\kappa}{2} e^{-\frac{\kappa}{2} t}. \tag{2.1.46}$$

Upon substituting (2.1.43), (2.1.44), (2.1.45) and (2.1.46) into (2.1.38), we find

$$L = \frac{i}{\kappa} \Big[\kappa^2 c_1^2 e^{\kappa t} + \kappa^2 c_3^2 e^{\kappa t} \Big]. \tag{2.1.47}$$

Imposing the boundary conditions

$$x(T) = x'', \tag{2.1.48}$$

$$x(0) = x', \tag{2.1.49}$$

$$y(T) = y'' \tag{2.1.50}$$

and

$$y(0) = y', \tag{2.1.51}$$

one easily obtains

$$c_1 = \frac{x'' - (x' - \frac{2\varepsilon}{\kappa}) e^{-\frac{\kappa}{2} T} - \frac{2\varepsilon}{\kappa}}{e^{\frac{\kappa}{2} T} - e^{-\frac{\kappa}{2} T}}, \tag{2.1.52}$$

$$c_3 = \frac{y'' - y' e^{-\frac{\kappa}{2} T}}{e^{\frac{\kappa}{2} T} - e^{-\frac{\kappa}{2} T}}. \tag{2.1.53}$$

On account of (2.1.52) and (2.1.53), Eq. (2.1.47) can be put in the form

$$L = i\kappa \Big[[\frac{(x'' - \frac{2\varepsilon}{\kappa}) e^{\frac{\kappa}{2} T} - x' + \frac{2\varepsilon}{\kappa}}{e^{\kappa T} - 1}]^2 e^{\kappa t} + (\frac{y'' e^{\frac{\kappa}{2} T} - y'}{e^{\kappa T} - 1})^2 e^{\kappa t} \Big]. \tag{2.1.54}$$

With the aid of (2.1.54) and recalling the relation

$$S_c = \int_0^T L(t) dt, \tag{2.1.55}$$

the classical action can be expressed as

$$S_c = i\kappa[\frac{(x'' - \frac{2\varepsilon}{\kappa})e^{\frac{\kappa}{2}T} - x' + \frac{2\varepsilon}{\kappa}}{e^{\kappa T} - 1}]^2 \int_0^T e^{\kappa t} dt + i\kappa(\frac{y'' e^{\frac{\kappa}{2}T} - y'}{e^{\kappa T} - 1})^2 \int_0^T e^{\kappa t} dt \qquad (2.1.56)$$

After carrying out the integration, we readily obtain

$$S_c = i\frac{[(x'' - \frac{2\varepsilon}{\kappa})e^{\frac{\kappa}{2}T} - x' + \frac{2\varepsilon}{\kappa}]^2}{e^{\kappa T} - 1} + i\frac{(y'' e^{\frac{\kappa}{2}T} - y')^2}{e^{\kappa T} - 1}. \qquad (2.1.57)$$

Upon replacing (x'', y'', T) by (x,y,t), we have

$$S_c = i\frac{[(x - \frac{2\varepsilon}{\kappa})e^{\frac{\kappa}{2}t} - x' + \frac{2\varepsilon}{\kappa}]^2}{e^{\kappa t} - 1} + i\frac{(y e^{\frac{\kappa}{2}t} - y')^2}{e^{\kappa t} - 1}. \qquad (2.1.58)$$

It then follows that

$$\frac{\partial}{\partial x'}\frac{\partial}{\partial x} S_c = -2i\frac{e^{\frac{\kappa}{2}t}}{e^{\kappa t} - 1}, \qquad (2.1.59)$$

$$\frac{\partial}{\partial y'}\frac{\partial}{\partial y} S_c = -2i\frac{e^{\frac{\kappa}{2}t}}{e^{\kappa t} - 1}. \qquad (2.1.60)$$

In view of (2.1.58), (2.1.59) and (2.1.60), Eq. (2.1.30) can be rewritten as

$$Q(x, y, t|x', y', 0) = \frac{1}{\pi}\frac{e^{\kappa t}}{e^{\kappa t} - 1} exp\Bigg[- \frac{[(x - \frac{2\varepsilon}{\kappa})e^{\frac{\kappa}{2}t} - x' + \frac{2\varepsilon}{\kappa}]^2}{e^{\kappa t} - 1} - \frac{(y e^{\frac{\kappa}{2}t} - y')^2}{e^{\kappa t} - 1}\Bigg]. \qquad (2.1.61)$$

We are interested in the Q function satisfying the intial condition

$$Q_o(x', y') = \frac{1}{\pi} exp[-(x' - 0)^2 - (y' - 0)^2]. \qquad (2.1.62)$$

Applying (2.1.22), (2.1.61) and (2.1.62), one gets

$$Q(x, y, t) = \frac{1}{\pi^2}\frac{e^{\kappa t}}{e^{\kappa t} - 1}\int dx' dy' exp\Bigg[- \frac{[x e^{\frac{\kappa}{2}t} - \frac{2\varepsilon}{\kappa}(e^{\frac{\kappa}{2}t} - 1) - x']^2}{e^{\kappa t} - 1} - \frac{(y e^{\frac{\kappa}{2}t} - y')^2}{e^{\kappa t} - 1}\Bigg] exp[-x'^2 - y'^2]. \qquad (2.1.63)$$

Let

$$C = \frac{1}{\pi^2}\frac{e^{\kappa t}}{e^{\kappa t} - 1}, \qquad (2.1.64)$$

$$D = \frac{2\varepsilon}{\kappa}(e^{\frac{\kappa}{2}t} - 1), \qquad (2.1.65)$$

$$E = e^{\frac{\kappa}{2}t}, \qquad (2.1.66)$$

$$F = e^{\kappa t} - 1. \tag{2.1.67}$$

Now Eq. (2.1.63) can be written as

$$Q(x,y,t) = C\int dx'dy' exp[-\frac{(Ex - D - x')^2}{F} - \frac{(Ey - y')^2}{F}]$$
$$\times exp[-x'^2 - y'^2] \tag{2.1.68}$$

or

$$Q(x,y,t) = Cexp\left[\frac{-E^2x^2 + 2DEx - D^2 - E^2y^2}{F}\right]$$
$$\times \int dx' exp\left[-(\frac{1+F}{F})x'^2 + (\frac{2Ex - 2D}{F})x'\right]$$
$$\times \int dy' exp\left[-(\frac{1+F}{F})y'^2 + \frac{2Ey}{F}y'\right]. \tag{2.1.69}$$

Employing the mathematical relation

$$\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, a > 0 \tag{2.1.70}$$

along with (2.1.69), we obtain

$$Q(x,y,t) = \frac{\pi C F}{1+F} exp\left[-\frac{[Ex - D]^2}{1+F} - \frac{(Ey)^2}{1+F}\right]. \tag{2.1.71}$$

On account of (2.1.64), (2.1.65), (2.1.66) and (2.1.67), Eq. (2.1.71) can also be written in the form

$$Q(x,y,t) = \frac{1}{\pi} exp\left[-\frac{[e^{\frac{\kappa}{2}t}x - \frac{2\varepsilon}{\kappa}(e^{\frac{\kappa}{2}t} - 1)]^2}{e^{\kappa t}} - \frac{(e^{\frac{\kappa}{2}t}y)^2}{e^{\kappa t}}\right]. \tag{2.1.72}$$

We note that

$$\alpha = x + iy, \tag{2.1.73}$$

$$\alpha^* = x - iy. \tag{2.1.74}$$

Then we have

$$x = \frac{\alpha + \alpha^*}{2} \tag{2.1.75}$$

and

$$y = \frac{\alpha - \alpha^*}{2i}. \tag{2.1.76}$$

In a similar manner, one can see that

$$x_o = \frac{\alpha_o + \alpha_o^*}{2}, \tag{2.1.77}$$

$$y_o = \frac{\alpha_o - \alpha_o^*}{2i}. \tag{2.1.78}$$

Applying the above relations together with (2.1.72), we arrive at

$$Q(\alpha, t) = \frac{1}{\pi} exp\bigg[- \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa}{2}t})^2 \bigg] \times exp\bigg[- \alpha\alpha^* + [\frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t})]\alpha + [\frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t})]\alpha^* \bigg] \tag{2.1.79}$$

or

$$Q(\alpha, t) = \frac{A(t)}{\pi} exp\bigg(- \alpha\alpha^* + b(t)(\alpha + \alpha^*) \bigg), \tag{2.1.80}$$

where

$$A(t) = exp\bigg[- \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa}{2}t})^2 \bigg], \tag{2.1.81}$$

$$b(t) = \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t}). \tag{2.1.82}$$

2.2 The Q function of a signal mode

We seek here to obtain the Q function of a signal mode produced by a degenerate subharmonic generator. In a degenerate subharmonic generating system, a pump mode of frequency 2ω is down converted into a pair of signal photons each of frequency ω. With the pump mode treated classically, the process of degenerate subharmonic generation is described by the Hamiltonian

$$\hat{H} = \frac{i\varepsilon'}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}), \tag{2.2.1}$$

where ε', considered to be real and constant, is proportional to the amplitude of the pump mode and $\hat{a}$ is the annihilation operator for the signal mode. The master equation for the signal mode is given by

$$\frac{d\hat{\rho}}{dt} = \frac{\varepsilon'}{2}(\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{\rho}) + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}). \tag{2.2.2}$$

The Fokker-Planck equation for the Q function corresponding to (2.2.2) can be constructed by putting all operators in the normal order. To this end, one can find employing (2.1.3) and (2.1.4) that

$$\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{\rho} = \hat{a}[\hat{a},\hat{\rho}] + [\hat{a},\hat{\rho}]\hat{a} - \hat{a}^\dagger[\hat{a}^\dagger,\hat{\rho}] - [\hat{a}^\dagger,\hat{\rho}]\hat{a}^\dagger$$
$$= \hat{a}\frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger} + \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}\hat{a} + \hat{a}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial\hat{\rho}}{\partial\hat{a}}\hat{a}^\dagger. \tag{2.2.3}$$

We note that

$$[\hat{a}, \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}] = \frac{\partial^2\hat{\rho}}{\partial\hat{a}^{\dagger 2}}. \tag{2.2.4}$$

It then follows that

$$\hat{a}\frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger} = \frac{\partial^2\hat{\rho}}{\partial\hat{a}^{\dagger 2}} + \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}\hat{a}. \tag{2.2.5}$$

We alo find in a similar manner

$$\frac{\partial\hat{\rho}}{\partial\hat{a}}\hat{a}^\dagger = \hat{a}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}^2}. \tag{2.2.6}$$

Upon substituting (2.2.5) and (2.2.6) into (2.2.3), there follows

$$\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{\rho} = \frac{\partial^2\hat{\rho}}{\partial\hat{a}^{\dagger 2}} + 2\frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}\hat{a} + 2\hat{a}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}^2}. \tag{2.2.7}$$

Moreover, one can see that

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} = -[\hat{a}^\dagger, \hat{a}\hat{\rho}] + [\hat{a}, \hat{\rho}\hat{a}^\dagger]$$
$$= \frac{\partial}{\partial\hat{a}}(\hat{a}\hat{\rho}) + \frac{\partial}{\partial\hat{a}^\dagger}(\hat{\rho}\hat{a}^\dagger). \tag{2.2.8}$$

Employing the relations

$$[\hat{a}, \hat{\rho}] = \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger} \tag{2.2.9}$$

and

$$[\hat{a}^\dagger, \hat{\rho}] = -\frac{\partial\hat{\rho}}{\partial\hat{a}} \tag{2.2.10}$$

along with (2.2.8), we have

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} = 2\frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{a}^\dagger} + \frac{\partial}{\partial\hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial\hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho}). \tag{2.2.11}$$

Now on account of (2.2.7) and (2.2.11), the master equation can be put in the form

$$\frac{d\hat{\rho}}{dt} = \frac{\varepsilon'}{2}\left(\frac{\partial^2\hat{\rho}}{\partial\hat{a}^{\dagger 2}} + 2\frac{\partial\hat{\rho}}{\partial\hat{a}^{\dagger}}\hat{a} + 2\hat{a}^{\dagger}\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}^2}\right) + \frac{\kappa}{2}\left(2\frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{a}^{\dagger}} + \frac{\partial}{\partial\hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial\hat{a}^{\dagger}}(\hat{a}^{\dagger}\hat{\rho})\right), \tag{2.2.12}$$

where $\hat{\rho} = \hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)$ is assumed to be in the normal order. Upon replacing $\hat{a}, \hat{a}^{\dagger}, \hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)$ by α, α^* and $Q(\alpha^*, \alpha, t)$, we find the Fokker-Planck equation for the Q function to be

$$\frac{\partial Q(\alpha^*, \alpha, t)}{\partial t} = \left[\frac{\varepsilon'}{2}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\alpha^{*2}} + 2\frac{\partial}{\partial\alpha}\alpha^* + 2\frac{\partial}{\partial\alpha^*}\alpha\right)\right.$$
$$\left. + \frac{\kappa}{2}\left(2\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)\right]Q(\alpha^*, \alpha, t). \tag{2.2.13}$$

Applying (2.1.13), (2.1.14), (2.1.15) and (2.1.16) together with their complex conjugate, the Fokker-Plank equation for the Q function take the form

$$\frac{\partial Q(x, y, t)}{\partial t} = \left[\left(\frac{\varepsilon' + \kappa}{4}\right)\frac{\partial^2}{\partial x^2} + \left(\frac{\kappa - \varepsilon'}{4}\right)\frac{\partial^2}{\partial y^2}\right.$$
$$\left. + \left(\frac{\kappa}{2} + \varepsilon'\right)\frac{\partial}{\partial x}x + \left(\frac{\kappa}{2} - \varepsilon'\right)\frac{\partial}{\partial y}y\right]Q(x, y, t). \tag{2.2.14}$$

This differential equation can be solved using the propagator method. We first transform (2.2.14) into a schrodinger-type equation. To this end upon replacing $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x, y, Q(x, y, t))$ by $(i\hat{p}_x, i\hat{p}_y, \hat{x}, \hat{y}, |Q(t)\rangle)$, we then get

$$i\frac{d}{dt}|Q(t)\rangle = \hat{H}|Q(t)\rangle, \tag{2.2.15}$$

where

$$\hat{H} = -i\left(\frac{\varepsilon' + \kappa}{4}\right)\hat{p}_x^2 - i\left(\frac{\kappa - \varepsilon'}{4}\right)\hat{p}_y^2 - \left(\frac{\kappa}{2} + \varepsilon'\right)\hat{p}_x\hat{x} - \left(\frac{\kappa}{2} - \varepsilon'\right)\hat{p}_y\hat{y}. \tag{2.2.16}$$

The formal solution of (2.2.15) can be written as

$$|Q(t)\rangle = e^{-i\hat{H}t}|Q(0)\rangle. \tag{2.2.17}$$

Multiplying (2.2.17) by $\langle x, y|$ on the left-hand side and applying the completeness relation for two-dimensional position eigen states, we can write

$$\langle x, y|Q(t)\rangle = \int dx'dy'\langle x, y|e^{-i\hat{H}t}|y', x'\rangle\langle x', y'|Q(0)\rangle \tag{2.2.18}$$

or

$$Q(x,y,t) = \int dx'dy' Q(x,y,t|x',y',0)Q_0(x',y'), \tag{2.2.19}$$

in which

$$Q(x,y,t) = \langle x,y|Q(t)\rangle, \tag{2.2.20}$$

$$Q(x,y,t|x',y',0) = \langle x,y|e^{-i\hat{H}t}|y',x'\rangle \tag{2.2.21}$$

is the Q function propagator and

$$Q_0(x',y') = \langle x',y'|Q(0)\rangle. \tag{2.2.22}$$

In view of (2.1.26) and (2.1.27) together with (2.2.16), we obtain

$$b_x = -(\frac{\kappa}{2} + \varepsilon'), \tag{2.2.23}$$

$$b_y = -(\frac{\kappa}{2} - \varepsilon') \tag{2.2.24}$$

and recalling that for the antistandard form of ordering $\gamma = \frac{1}{2}$ and employing the two dimensional form of (2.1.27), the Q function propagator associated with (2.2.16) is expressible as

$$Q(x,y,t|x',y',0) = \left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\right)^{\frac{1}{2}}\left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial y'\partial y}\right)^{\frac{1}{2}} exp(\frac{\kappa}{2}t + iS_c). \tag{2.2.25}$$

The classical Hamiltonian corresponding to (2.2.16) is

$$H = -i\left(\frac{\varepsilon' + \kappa}{4}\right)p_x^2 - i\left(\frac{\kappa - \varepsilon'}{4}\right)p_y^2 - \left(\frac{\kappa}{2} + \varepsilon'\right)p_x x - \left(\frac{\kappa}{2} - \varepsilon'\right)p_y y. \tag{2.2.26}$$

Applying Hamilton's equation of motion described by (2.1.32) and (2.1.33), one gets

$$p_x = \frac{2i}{\varepsilon' + \kappa}\left(\dot{x} + (\frac{\kappa}{2} + \varepsilon')x\right) \tag{2.2.27}$$

and

$$p_y = \frac{2i}{\kappa - \varepsilon'}\left(\dot{y} + (\frac{\kappa}{2} - \varepsilon')y\right). \tag{2.2.28}$$

On combining (2.2.26), (2.2.27) and (2.2.28), we have

$$H = \frac{i}{\varepsilon' + \kappa}\left[\dot{x}^2 - (\frac{\kappa}{2} + \varepsilon')^2 x^2\right] + \frac{i}{\kappa - \varepsilon'}\left[\dot{y}^2 - (\frac{\kappa}{2} - \varepsilon')^2 y^2\right]. \tag{2.2.29}$$

On account of (2.1.37) and (2.2.29), one readily obtains

$$L = \frac{i}{\kappa + \varepsilon'}\left[\dot{x} + (\kappa/2 + \varepsilon')x\right]^2 + \frac{i}{\kappa - \varepsilon'}\left[\dot{y} + (\kappa/2 - \varepsilon')y\right]^2. \tag{2.2.30}$$

Employing (2.1.39) and (2.1.40), we arrive at

$$\ddot{x} - (\frac{\kappa}{2} + \varepsilon')^2 x = 0 \tag{2.2.31}$$

and

$$\ddot{y} - (\frac{\kappa}{2} - \varepsilon')^2 y = 0. \tag{2.2.32}$$

The general solutions of the above differential equations are given by

$$x(t) = Ae^{(\frac{\kappa}{2}+\varepsilon')t} + Be^{-(\frac{\kappa}{2}+\varepsilon')t}, \tag{2.2.33}$$

$$y(t) = Ce^{(\frac{\kappa}{2}-\varepsilon')t} + De^{-(\frac{\kappa}{2}-\varepsilon')t}. \tag{2.2.34}$$

It then follows that

$$\dot{x} = A(\frac{\kappa}{2} + \varepsilon')e^{(\frac{\kappa}{2}+\varepsilon')t} - B(\frac{\kappa}{2} + \varepsilon')e^{-(\frac{\kappa}{2}+\varepsilon')t}, \tag{2.2.35}$$

$$\dot{y} = C(\frac{\kappa}{2} - \varepsilon')e^{(\frac{\kappa}{2}-\varepsilon')t} - D(\frac{\kappa}{2} - \varepsilon')e^{-(\frac{\kappa}{2}-\varepsilon')t}. \tag{2.2.36}$$

Upon substituting (2.2.33), (2.2.34), (2.2.35) and (2.2.36) into (2.2.30), we obtain

$$L(t) = \frac{i}{\varepsilon' + \kappa}(\kappa + 2\varepsilon')^2 A^2 e^{(\kappa+2\varepsilon')t} + \frac{i(\kappa - 2\varepsilon')^2}{(\kappa - \varepsilon')}C^2 e^{(\kappa-2\varepsilon')t}. \tag{2.2.37}$$

Applying the the boundary conditions

$$x(T) = x'', \tag{2.2.38}$$

$$x(0) = x', \tag{2.2.39}$$

$$x(T) = y'', \tag{2.2.40}$$

$$y(0) = y', \tag{2.2.41}$$

one easily obtains

$$A = -\left(\frac{x' - x'' e^{(\frac{\kappa}{2}+\varepsilon')T}}{e^{(\kappa+2\varepsilon')T} - 1} \right) \tag{2.2.42}$$

and

$$C = -\left(\frac{y' - y'' e^{(\frac{\kappa}{2}-\varepsilon')T}}{e^{(\kappa-2\varepsilon')T} - 1} \right). \tag{2.2.43}$$

On account of (2.2.42) and (2.2.43), Eq. (2.2.37) can be put in the form

$$L(t) = \frac{i}{\varepsilon' + \kappa} (\kappa + 2\varepsilon')^2 \left(\frac{x' - x'' e^{(\frac{\kappa}{2}+\varepsilon')T}}{e^{(\kappa+2\varepsilon')T} - 1} \right)^2 e^{(\kappa+2\varepsilon')t}$$
$$+ i \frac{(\kappa - 2\varepsilon')^2}{\kappa - \varepsilon'} \left(\frac{y' - y'' e^{(\frac{\kappa}{2}-\varepsilon')T}}{e^{(\kappa-2\varepsilon')T} - 1} \right)^2 e^{(\kappa-2\varepsilon')t}. \tag{2.2.44}$$

In view of (2.1.55) and (2.2.44), the classical action can be expressed as

$$S_c = i \left[\frac{\kappa + 2\varepsilon'}{\kappa + \varepsilon'} \frac{(x' e^{-(\frac{\kappa}{2}+\varepsilon')T} - x'')^2}{1 - e^{-(\kappa+2\varepsilon')T}} + \frac{\kappa - 2\varepsilon'}{\kappa - \varepsilon'} \frac{(y' e^{-(\frac{\kappa}{2}-\varepsilon')T} - y'')^2}{1 - e^{-(\kappa-2\varepsilon')T}} \right]. \tag{2.2.45}$$

Upon replacing (x'', y'', T) by (x,y,t), we have

$$S_c = i \left[\frac{\kappa + 2\varepsilon'}{\kappa + \varepsilon'} \frac{(x' e^{-(\frac{\kappa}{2}+\varepsilon')t} - x)^2}{1 - e^{-(\kappa+2\varepsilon')t}} + \frac{\kappa - 2\varepsilon'}{\kappa - \varepsilon'} \frac{(y' e^{-(\frac{\kappa}{2}-\varepsilon')t} - y)^2}{1 - e^{-(\kappa-2\varepsilon')t}} \right]. \tag{2.2.46}$$

It then follows that

$$\frac{\partial}{\partial x'} \left(\frac{\partial S_c}{\partial x} \right) = \frac{-2i(\kappa + 2\varepsilon') e^{-(\frac{\kappa}{2}+\varepsilon')t}}{(\kappa + \varepsilon')(1 - e^{-(\kappa+2\varepsilon')t})}, \tag{2.2.47}$$

$$\frac{\partial}{\partial y'} \left(\frac{\partial S_c}{\partial y} \right) = \frac{-2i(\kappa - 2\varepsilon') e^{-(\frac{\kappa}{2}-\varepsilon')t}}{(\kappa - \varepsilon')(1 - e^{-(\kappa-2\varepsilon')t})}. \tag{2.2.48}$$

Applying (2.2.46), (2.2.47) and (2.2.48), the Q function propagator can be written as

$$Q(x, y, t | x', y', 0) = \frac{1}{\pi} \sqrt{\frac{\kappa^2 - 4\varepsilon'^2}{(\kappa^2 - \varepsilon'^2)(1 - e^{-(\kappa+2\varepsilon')t})(1 - e^{-(\kappa-2\varepsilon')t})}}$$
$$\times exp \left[- \frac{(\kappa + 2\varepsilon')(x' e^{-(\frac{\kappa}{2}+\varepsilon')t} - x)^2}{(\varepsilon' + \kappa)(1 - e^{-(\kappa+2\varepsilon')t})} - \frac{(\kappa - 2\varepsilon')(y' e^{-(\frac{\kappa}{2}-\varepsilon')t} - y)^2}{(\kappa - \varepsilon')(1 - e^{-(\kappa-2\varepsilon')t})} \right]. \tag{2.2.49}$$

We are interested in the Q function which satisfy the initial condition

$$Q_o(x', y') = \frac{1}{\pi} exp[-(x' - x_o)^2 - (y' - y_o)^2]. \tag{2.2.50}$$

Employing (2.2.19), (2.2.49) and (2.2.50), one can see that

$$Q(x,y,t) = \int dx'dy' \frac{1}{\pi}\sqrt{\frac{\kappa^2 - 4\varepsilon'^2}{(\kappa^2-\varepsilon'^2)(1-e^{-(\kappa-2\varepsilon')t})(1-e^{-(\kappa-2\varepsilon')t})}}$$
$$\times exp\Bigg[-\frac{(\kappa+2\varepsilon')}{\varepsilon'+\kappa}\frac{(x'e^{-(\frac{\kappa}{2}+\varepsilon')t}-x)^2}{(1-e^{-(\kappa+2\varepsilon')t})} - \frac{(\kappa-2\varepsilon')(y'e^{-(\frac{\kappa}{2}-\varepsilon')t}-y)^2}{(\kappa-\varepsilon')(1-e^{-(\kappa-2\varepsilon')t})}\Bigg]$$
$$\times\frac{1}{\pi}exp[-(x'-x_0)^2-(y'-y_0)^2]. \tag{2.2.51}$$

Let

$$A = \kappa + 2\varepsilon', \tag{2.2.52}$$

$$B = e^{-(\frac{\kappa}{2}+\varepsilon')t}, \tag{2.2.53}$$

$$C = (\kappa+\varepsilon')(1-e^{-(\kappa+2\varepsilon')t}), \tag{2.2.54}$$

$$D = \kappa - 2\varepsilon', \tag{2.2.55}$$

$$E = e^{-(\frac{\kappa}{2}-\varepsilon')t}, \tag{2.2.56}$$

$$F = (\kappa-\varepsilon')(1-e^{-(\kappa-2\varepsilon')t}) \tag{2.2.57}$$

and

$$G = \frac{1}{\pi^2}\sqrt{\frac{\kappa^2-4\varepsilon'^2}{(\kappa^2-\varepsilon'^2)(1-e^{-(\kappa+2\varepsilon')t})(1-e^{-(\kappa-2\varepsilon')t})}} \tag{2.2.58}$$

and hence Eq. (2.2.51) takes the form

$$Q(x,y,t) = \int dx'dy'Gexp\Bigg[-\frac{(AB^2x'^2-2ABxx'+Ax^2}{C}$$
$$-\frac{(DE^2y'^2-2DEyy'+Dy^2)}{F}\Bigg]\times exp\Bigg(-x'^2+2x_0x'-x_o^2-y'^2$$
$$+2y_oy'-y_o^2\Bigg) \tag{2.2.59}$$

or

$$Q(x,y,t) = Gexp\Bigg(-\frac{A}{C}x^2-\frac{D}{F}y^2-x_o^2-y_o^2\Bigg)\int dx'exp\Bigg[-\Bigg(\frac{AB^2+C}{C}\Bigg)x'^2$$
$$+\Bigg(\frac{2ABx+2Cx_0}{C}\Bigg)x'\Bigg]\int dy'exp\Bigg[-\Bigg(\frac{DE^2+F}{F}\Bigg)y'^2$$
$$+\Bigg(\frac{2DEy+2Fy_o}{F}\Bigg)y'\Bigg]. \tag{2.2.60}$$

Applying the mathematical relation given by (2.1.70), one obtains

$$Q(x,y,t) = G\pi\sqrt{\frac{CF}{(AB^2+C)(DE^2+F)}}exp\Big[-\frac{A}{AB^2+C}(x-Bx_0)^2 - \frac{D}{DE^2+F}(y-Ey_0)^2\Big]. \tag{2.2.61}$$

On combining (2.2.52), (2.2.53), (2.2.54), (2.2.55), (2.2.56), (2.2.57) and (2.2.58) into (2.2.61), we get

$$Q(x,y,t) = \frac{1}{\pi}\sqrt{\frac{\kappa^2-4\varepsilon'^2}{[\kappa+\varepsilon'(1+e^{-(\kappa+2\varepsilon')t})][\kappa-\varepsilon'(1+e^{-(\kappa-2\varepsilon')t})]}} \times exp\Big[-\frac{(\kappa+2\varepsilon')(x-e^{-(\frac{\kappa}{2}+\varepsilon')t}x_o)^2}{\kappa+\varepsilon'(1+e^{-(\kappa+2\varepsilon')t})} - \frac{(\kappa-2\varepsilon')(y-e^{-(\frac{\kappa}{2}-\varepsilon')t}y_o)^2}{\kappa-\varepsilon'(1+e^{-(\kappa-2\varepsilon')t})}\Big]. \tag{2.2.62}$$

Applying the relation described by (2.1.75), (2.1.76), (2.1.77) and (2.1.78) along with (2.2.62), we find

$$Q(\alpha^*,\alpha,t) = \frac{A'(t)}{\pi}exp\Big[-B'(t)\alpha^*\alpha + \frac{C'(t)}{2}(\alpha^{*2}+\alpha^2) + D'(t)\alpha + D'^*(t)\alpha^*\Big], \tag{2.2.63}$$

where

$$A'(t) = \sqrt{B'^2(t)-C'^2(t)}exp\Big[-\Big(\frac{[B'(t)+C'(t)]e^{-(\kappa-2\varepsilon')t}}{2} + \frac{[B'(t)-C'(t)]e^{-(\kappa+2\varepsilon')t}}{2}\Big)\alpha_o^*\alpha_o + \Big(\frac{[B'(t)+C'(t)]e^{-(\kappa-2\varepsilon')t} - [B'(t)-C'(t)]e^{-(\kappa+2\varepsilon')t}}{4}\Big)(\alpha_o^{*2}+\alpha_o^2)\Big], \tag{2.2.64}$$

$$B'(t) = \frac{1}{2}\Big[\frac{\kappa-2\varepsilon'}{\kappa-\varepsilon'(1+e^{-(\kappa-2\varepsilon')t})} + \frac{\kappa+2\varepsilon'}{\kappa+\varepsilon'(1+e^{-(\kappa+2\varepsilon')t})}\Big], \tag{2.2.65}$$

$$C'(t) = \frac{1}{2}\Big[\frac{\kappa-2\varepsilon'}{\kappa-\varepsilon'(1+e^{-(\kappa-2\varepsilon')t})} - \frac{\kappa+2\varepsilon'}{\kappa+\varepsilon'(1+e^{-(\kappa+2\varepsilon')t})}\Big], \tag{2.2.66}$$

$$D'(t) = \frac{1}{2}\Big[[B'(t)+C'(t)]e^{-(\kappa/2-\varepsilon')t}(\alpha_o^*-\alpha_o) + [B'(t)-C'(t)]e^{-(\kappa/2+\varepsilon')t}(\alpha_o^*+\alpha_o)\Big]. \tag{2.2.67}$$

2.3 The Q function of the superposition of the coherently driven cavity and the signal mode

We proceed to obtain the Q function of the superposition of the light beams produced by the subharmonic generator and the coherently driven cavity mode. The superposition of the light beams from the subharmonic generator and the coherently driven cavity mode in terms of the Q function can be expressed as [1]

$$Q(\alpha^*,\alpha,t) = \frac{1}{\pi}\int d^2\beta d^2\gamma Q(\beta^*,\beta+\frac{\partial}{\partial\beta^*})Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma^*})$$
$$\times exp(-\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \alpha^*\beta + \alpha\beta^* + \alpha^*\gamma + \alpha\gamma^* - \beta^*\gamma - \beta\gamma^*). \quad (2.3.1)$$

On account of Eq. (2.1.80), one can write

$$Q(\beta^*,\beta+\frac{\partial}{\partial\beta^*}) = \frac{A(t)}{\pi}exp(-\beta\beta^* + b(t)\beta + b(t)\beta^*)exp\Big([-\beta^* + b(t)]\frac{\partial}{\partial\beta^*}\Big) \quad (2.3.2)$$

and in view of (2.2.63), we have

$$Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma^*}) = \frac{A'(t)}{\pi}exp\Big[-B'(t)\gamma\gamma^* + D'(t)\gamma + D'^*(t)\gamma^*$$
$$+\frac{C'(t)}{2}(\gamma^{*2}+\gamma^2)\Big]exp\Big[\Big(-B'(t)\gamma^* + D'(t) + C'(t)\gamma\Big)\frac{\partial}{\partial\gamma^*} + \frac{C'(t)}{2}\frac{\partial^2}{\partial\gamma^{*2}}\Big]. (2.3.3)$$

Upon substituting (2.3.2) and (2.3.3) into Eq. (2.3.1), one obtains

$$Q(\alpha^*,\alpha,t) = \frac{1}{\pi}\frac{A(t)}{\pi}A'(t)\int d^2\beta\frac{d^2\gamma}{\pi}exp[-\alpha\alpha^* + \alpha^*\beta + \alpha^*\gamma - \beta\beta^*$$
$$+b(t)\beta + b(t)\beta^* - B'(t)\gamma\gamma^* + D'(t)\gamma + D'^*(t)\gamma^* + \frac{C'(t)}{2}(\gamma^{*2}+\gamma^2)]$$
$$\times exp\Big[\Big(-\beta^* + b(t)\Big)\frac{\partial}{\partial\beta^*}\Big]exp(-\beta^*\beta + \alpha\beta^* - \beta^*\gamma)$$
$$\times exp\Big[\Big(-B'(t)\gamma^* + D'(t) + C'(t)\gamma\Big)\frac{\partial}{\partial\gamma^*} + \frac{C'(t)}{2}\frac{\partial^2}{\partial\gamma^{*2}}\Big]$$
$$\times exp(-\gamma^*\gamma + \alpha\gamma^* - \beta\gamma^*). \quad (2.3.4)$$

On account of the power series expansions

$$exp[(-\beta^* + b(t))\frac{\partial}{\partial\beta^*}] = \sum_i \frac{1}{i!}\Big(-\beta^* + b(t)\Big)^i\frac{\partial^i}{\partial\beta^{*i}}, \quad (2.3.5)$$

$$exp[(-B'(t)\gamma^* + D'(t) + C'(t)\gamma)\frac{\partial}{\partial\gamma^*}] = \sum_j \frac{1}{j!}\Big(- B'(t)\gamma^* + D'(t) + C'(t)\gamma\Big)^j \frac{\partial^j}{\partial\gamma^{*j}}, \quad (2.3.6)$$

$$exp(\frac{C'(t)}{2}\frac{\partial^2}{\partial\gamma^{*2}}) = \sum_k \frac{1}{k!}\Big(\frac{C'(t)}{2}\Big)^k \frac{\partial^{2k}}{\partial\gamma^{*2k}}, \quad (2.3.7)$$

expression (2.3.4) can be put in the form

$$\begin{aligned} Q(\alpha^*, \alpha, t) = & \frac{1}{\pi}\frac{A(t)}{\pi}A'(t)\int d^2\beta\frac{d^2\gamma}{\pi}exp\Big[- \alpha\alpha^* + \alpha^*\beta + \alpha^*\gamma - \beta\beta^* \\ & + b(t)\beta + b(t)\beta^* - B'(t)\gamma\gamma^* + D'(t)\gamma + D'^*(t)\gamma^* + \frac{C'(t)}{2}(\gamma^{*2} + \gamma^2)\Big] \\ & \times\sum_i \frac{1}{i!}\Big(- \beta^* + b(t)\Big)^i \frac{\partial^i}{\partial\beta^{*i}}exp(-\beta^*\beta + \alpha\beta^* - \beta^*\gamma) \\ & \times\sum_{jk} \frac{1}{j!}\frac{1}{k!}\Big(- B'(t)\gamma^* + D'(t) + C'(t)\gamma\Big)^j \Big(\frac{C'(t)}{2}\Big)^k \\ & \times\frac{\partial^{j+2k}}{\partial\gamma^{*j+2k}}exp(-\gamma^*\gamma + \alpha\gamma^* - \beta\gamma^*), \end{aligned} \quad (2.3.8)$$

so that carrying out the differentiation, we arrive at

$$\begin{aligned} Q(\alpha^*, \alpha, t) = & \frac{1}{\pi}\frac{A(t)}{\pi}A'(t)\int d^2\beta\frac{d^2\gamma}{\pi}exp\Big[- \gamma\gamma^* + (-b(t) + \alpha^*)\gamma \\ & + \Big(D'^*(t) - B'(t)\alpha + B'(t)\beta + \alpha - \beta\Big)\gamma^* + \frac{C'(t)}{2}\gamma^{*2}\Big] \\ & \times exp\Big(- \alpha\alpha^* + b(t)\alpha + D'(t)\alpha + \frac{C'(t)}{2}\alpha^2 + \alpha^*\beta - \beta\beta^* + b(t)\beta \\ & + b(t)\beta^* - b(t)\beta - D'(t)\beta - C'(t)\alpha\beta + \frac{C'(t)}{2}\beta^2\Big). \end{aligned} \quad (2.3.9)$$

Employing the relation

$$\begin{aligned} & \int \frac{d^2z}{\pi}exp(-azz^* + bz + cz^* + Az^2 + Bz^{*2}) \\ & = \Big[\frac{1}{a^2 - 4AB}\Big]^{\frac{1}{2}} exp\Big[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\Big], \end{aligned} \quad (2.3.10)$$

Eq. (2.3.9) takes the form

$$Q(\alpha^*,\alpha,t)=\frac{1}{\pi}A(t)A'(t)exp\Bigg[D'(t)\alpha+\frac{C'(t)}{2}\alpha^2-b(t)D'^*(t)$$
$$+b(t)B'(t)\alpha+D'^*(t)\alpha^*-B'(t)\alpha\alpha^*+\frac{C'(t)}{2}b^2(t)-C'(t)b(t)\alpha^*$$
$$+\frac{C'(t)}{2}\alpha^{*2}\Bigg]\int\frac{d^2\beta}{\pi}exp\Bigg[-\beta\beta^*+\beta\Bigg(-D'(t)-C'(t)\alpha-b(t)B'(t)$$
$$+b(t)+B'(t)\alpha^*\Bigg)+b(t)\beta^*+\frac{C'(t)}{2}\beta^2\Bigg]. \quad (2.3.11)$$

Upon carrying out the integration over β, we have

$$Q(\alpha^*,\alpha,t)=\frac{R(t)}{\pi}exp[-B'(t)\alpha\alpha^*+S(t)\alpha+S^*(t)\alpha^*+\frac{C'(t)}{2}(\alpha^{*2}+\alpha^2)], \quad (2.3.12)$$

where

$$R(t)=A(t)A'(t)exp\Bigg[-b(t)D'^*(t)+\frac{C'(t)}{2}b^2(t)-D'(t)b(t)$$
$$-b^2(t)B'(t)+b^2(t)+\frac{C'(t)}{2}b^2(t)\Bigg], \quad (2.3.13)$$

$$S(t)=D'(t)+b(t)B'(t)-C'(t)b(t). \quad (2.3.14)$$

We are interested in the case for which the signal mode is initially in a vacuum state and hence upon setting $\alpha_0=\alpha_0^*=0$, we have

$$Q(\alpha^*,\alpha,t)=\frac{r(t)}{\pi}exp[-B'(t)\alpha\alpha^*+S'(t)(\alpha+\alpha^*)+\frac{C'(t)}{2}(\alpha^{*2}+\alpha^2)], \quad (2.3.15)$$

where

$$r(t)=\sqrt{B'^2(t)-C'^2(t)}exp[b^2(t)(C'(t)-B'(t)+1)], \quad (2.3.16)$$

$$S'(t)=b(t)\Bigg(B'(t)-C'(t)\Bigg), \quad (2.3.17)$$

$$b(t)=\frac{2\varepsilon}{k}(1-e^{-\frac{\kappa}{2}t}). \quad (2.3.18)$$

According to the input-output relation, we have

$$\hat{a}_{out}(t)=\sqrt{\kappa}\hat{a}(t)-\hat{a}_{in}(t). \quad (2.3.19)$$

For a cavity mode coupled to a vacuum reservoir and with the cavity mode represented by c-number variables associated with the normal ordering, one can write

$$\alpha_{out}(t) = \sqrt{\kappa}\alpha(t). \tag{2.3.20}$$

On account of (2.3.12), the Q function of the output light from a subharmonic generator and a coherently driven cavity mode turns out to be

$$Q(\alpha^*, \alpha, t) = \frac{R(t)}{\pi\kappa} exp[-\frac{B'(t)}{\kappa}\alpha\alpha^* + \frac{S(t)}{\sqrt{\kappa}}\alpha + \frac{S^*(t)}{\sqrt{\kappa}}\alpha^* + \frac{C'(t)}{2\kappa}(\alpha^{*2} + \alpha^2)]. \tag{2.3.21}$$

If we consider the case in which the signal mode is initially in a vacuum state, the Q function of the output light takes the form

$$Q(\alpha^*, \alpha, t) = \frac{r(t)}{\pi\kappa} exp[-\frac{B'(t)}{\kappa}\alpha\alpha^* + \frac{S'(t)}{\sqrt{\kappa}}(\alpha + \alpha^*) + \frac{C'(t)}{2\kappa}(\alpha^{*2} + \alpha^2)]. \tag{2.3.22}$$

Chapter 3

Quadrature Squeezing

We now seek to study the quadrature variance and the squeezing spectrum for the output light produced by the subharmonic generator and the coherently driven cavity mode coupled to a vacuum reservoir.

3.1 The normally-ordered quadrature variance

The squeezing properties of a single-mode light are described by two quadrature operators

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger, \tag{3.1.1}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{3.1.2}$$

The normally-ordered quadrature variance is defined by

$$:(\triangle\, a_\pm)^2: \; = \pm\langle :(\hat{a}^\dagger \pm \hat{a})^2: \rangle \mp \langle(\hat{a}^\dagger \pm \hat{a})\rangle^2. \tag{3.1.3}$$

This is expressible in terms of the c-number variables associated with the normal ordering as

$$:(\triangle\, a_\pm)^2: \; = \pm\langle\alpha_\pm^2\rangle \mp \langle\alpha_\pm\rangle^2, \tag{3.1.4}$$

where

$$\alpha_\pm = \alpha^* \pm \alpha. \tag{3.1.5}$$

The expectation value of a c-number variable associated with normal ordering can be written as

$$\langle\alpha^2\rangle = \frac{1}{\pi\kappa}\int d^2\alpha d^2\beta Q(\alpha^*,\beta)exp\bigg[-\frac{\alpha^*\alpha}{\kappa}-\frac{\beta^*\beta}{\kappa}+\frac{\alpha^*\beta}{\kappa}+\frac{\alpha\beta^*}{\kappa}\bigg]\alpha^2. \quad (3.1.6)$$

Upon introducing (2.3.22) into (3.1.6), we obtain

$$\langle\alpha^2\rangle = \frac{rc_1^2}{\pi}\int d^2\alpha\frac{d^2\beta}{\pi}exp[-c_1B'\alpha^*\beta+\sqrt{c_1}S'\beta+\sqrt{c_1}S'\alpha^*$$
$$+\frac{c_1C'}{2}\alpha^{*2}+\frac{c_1C'}{2}\beta^2-c_1\alpha^*\alpha-c_1\beta^*\beta+c_1\alpha^*\beta+c_1\alpha\beta^*]\alpha^2, \quad (3.1.7)$$

where $c_1=\frac{1}{\kappa}$.

One can write the above equation as

$$\langle\alpha^2\rangle = \frac{rc_1^2}{\pi}\frac{\partial}{\partial a}\int d^2\alpha\frac{d^2\beta}{\pi}exp[-c_1\beta^*\beta+\beta(-c_1B'\alpha^*+\sqrt{c_1}S'+c_1\alpha^*)$$
$$+c_1\alpha\beta^*+\frac{c_1C'}{2}\beta^2]\times exp[\sqrt{c_1}S'\alpha^*+a\alpha^2+\frac{c_1C'}{2}\alpha^{*2}-c_1\alpha^*\alpha]|_{a=0}, \quad (3.1.8)$$

so that on carrying out the integration over β employing the relation described by (2.3.10), there follows

$$\langle\alpha^2\rangle = c_1r\frac{\partial}{\partial a}\int\frac{d^2\alpha}{\pi}exp[-c_1B'\alpha\alpha^*+\sqrt{c_1}S'\alpha+\sqrt{c_1}S'\alpha^*$$
$$+(\frac{c_1C'}{2}+a)\alpha^2+\frac{c_1C'}{2}\alpha^{*2}]_{a=0} \quad (3.1.9)$$

and on performing the the integration over α, we have

$$\langle\alpha^2\rangle = rc_1\frac{\partial}{\partial a}\bigg[\frac{1}{c_1^2B'^2-c_1^2C'^2-2c_1C'a}\bigg]^{\frac{1}{2}}$$
$$\times exp\bigg[\frac{c_1^2B'S'^2+c_1^2C'S'^2+c_1aS'^2}{c_1^2B'^2-c_1^2C'^2-2c_1C'a}\bigg]. \quad (3.1.10)$$

On carrying out the differentiation and applying the condition a=0, one readily obtains

$$\langle\alpha^2\rangle = \frac{\kappa C'}{B'^2-C'^2}+\kappa b^2. \quad (3.1.11)$$

We note that

$$\langle\alpha^{*2}\rangle = \frac{\kappa C'}{B'^2-C'^2}+\kappa b^2. \quad (3.1.12)$$

Further more, one can also write

$$\langle\alpha^*\alpha\rangle = \frac{1}{\pi\kappa}\int d^2\alpha d^2\beta Q(\alpha^*,\beta,t) exp[-\frac{\alpha^*\alpha}{\kappa} - \frac{\beta^*\beta}{\kappa} + \frac{\alpha^*\beta}{\kappa} + \frac{\alpha\beta^*}{\kappa}]\beta^*\alpha \qquad (3.1.13)$$

and on substituting (2.3.22), we see that

$$\langle\alpha^*\alpha\rangle = \frac{rc_1^2}{\pi}\int d^2\alpha\frac{d^2\beta}{\pi} \times exp[-c_1B'\alpha^*\beta + \sqrt{c_1}S'\beta + \sqrt{c_1}S'\alpha^*$$
$$+c_1C'/2(\alpha^{*2}+\beta^2) - c_1\alpha^*\alpha - c_1\beta^*\beta + c_1\alpha^*\beta + c_1\alpha\beta^*]\beta^*\alpha. \qquad (3.1.14)$$

It can also be rewritten as

$$\langle\alpha^*\alpha\rangle = \frac{rc_1^2}{\pi}\frac{\partial}{\partial c_1^*}\int d^2\alpha\frac{d^2\beta}{\pi}exp[-c_1\beta\beta^* + \beta(-c_1B'\alpha^* + \sqrt{c_1}S' + c_1\alpha^*)$$
$$+c_1^*\alpha\beta^* + \frac{c_1C'}{2}\beta^2] \times exp[\sqrt{c_1}S'\alpha^* + \frac{c_1C'}{2}\alpha^{*2} - c_1\alpha\alpha^*]_{c_1^*=c_1}. \qquad (3.1.15)$$

Upon integrating over β employing the relation described by (2.3.10), (3.1,15) can be written as

$$\langle\alpha^*\alpha\rangle = rc_1\int\frac{d^2\alpha}{\pi}exp[-(c_1^*B' - c_1^* + c_1)\alpha\alpha^* + \frac{c_1^*}{\sqrt{c_1}}S'\alpha + \sqrt{c_1}S'\alpha^*$$
$$+\frac{c_1^{*2}}{2c_1}C'\alpha^2 + \frac{c_1C'}{2}\alpha^{*2}]_{c_1^*=c_1} \qquad (3.1.16)$$

and taking out the integration over α, one gets

$$\langle\alpha^*\alpha\rangle = rc_1\frac{\partial}{\partial c_1^*}\left[\frac{1}{(c_1^*B' - c_1^* + c_1)^2 - c_1^{*2}C'^2}\right]^{\frac{1}{2}}$$
$$\times exp\left[\frac{c_1^{*2}B'S'^2 - c_1^{*2}S'^2 + c_1c_1^*S'^2 + c_1^{*2}C'S'^2}{(c_1^*B' - c_1^* + c_1)^2 - c_1^{*2}C'^2}\right]_{c_1^*=c_1}. \qquad (3.1.17)$$

Carrying out the differentiation and putting the condition $c_1^* = c_1$, we have

$$\langle\alpha^*\alpha\rangle = \frac{\kappa\Big(B' + C'^2 - B'^2\Big)}{B'^2 - C'^2} + \kappa b^2. \qquad (3.1.18)$$

In the same manner, one can also write

$$\langle\alpha\rangle = \frac{1}{\pi\kappa}\int d^2\alpha d^2\beta Q(\alpha^*,\beta,t)exp[-\frac{\alpha^*\alpha}{\kappa} - \frac{\beta^*\beta}{\kappa} + \frac{\alpha^*\beta}{\kappa} + \frac{\alpha\beta^*}{\kappa}]\alpha \qquad (3.1.19)$$

and on introducing the Q function described by (2.3.22), we get

$$\langle\alpha\rangle = \frac{rc_1^2}{\pi}\int d^2\alpha\frac{d^2\beta}{\pi}exp[-c_1B'\alpha^*\beta + \sqrt{c_1}S'\beta$$
$$+\sqrt{c_1}S'\alpha^* + \frac{c_1C'}{2}(\alpha^{*2}+\beta^2) - c_1\alpha^*\alpha - c_1\beta^*\beta + c_1\alpha^*\beta + c_1\alpha\beta^*]\alpha. \quad (3.1.20)$$

The above equation can also be rewritten as

$$\langle\alpha\rangle = \frac{rc_1^2}{\pi}\frac{\partial}{\partial d}\int d^2\alpha\frac{d^2\beta}{\pi}exp[-c_1\beta^*\beta + \beta(c_1\alpha^* + \sqrt{c_1}S' - c_1B'\alpha^*)$$
$$+c_1\alpha\beta^* + \frac{c_1C'}{2}\beta^2] \times exp[\sqrt{c_1}S'\alpha^* + \frac{c_1C'}{2}\alpha^{*2} + d\alpha - c_1\alpha\alpha^*]|_{d=0}. \quad (3.1.21)$$

Upon carrying out the integration over β, one can write

$$\langle\alpha\rangle = rc_1\frac{\partial}{\partial d}\int\frac{d^2\alpha}{\pi}exp[-c_1B'\alpha\alpha^* + (\sqrt{c_1}S' + d)\alpha + \sqrt{c_1}S'\alpha^*$$
$$+\frac{c_1C'}{2}\alpha^2 + \frac{c_1C'}{2}\alpha^{*2}]_{d=0} \quad (3.1.22)$$

and on performing the integration over α, there follows

$$\langle\alpha\rangle = rc_1\frac{\partial}{\partial d}\left[\frac{1}{c_1^2B'^2 - c_1^2C'^2}\right]^{\frac{1}{2}}$$
$$\times exp\left[\frac{c_1^2B'S'^2 + c_1\sqrt{c_1}B'dS' + c_1^2C'S'^2 + c_1\sqrt{c_1}dC'S' + \frac{c_1C'}{2}d^2}{c_1^2B'^2 - c_1^2C'^2}\right]_{d=0}. \quad (3.1.23)$$

Carrying out the differentiation and applying the condition d=0, one obtains

$$\langle\alpha\rangle = \sqrt{\kappa}b. \quad (3.1.24)$$

We note that

$$\langle\alpha^*\rangle = \sqrt{\kappa}b. \quad (3.1.25)$$

Upon substituting (3.1.11), (3.1.12), (3.1.18), (3.1.24) and (3.1.25) into (3.1.4), we have

$$:(\triangle\, a_+)^2: \; = \frac{2\kappa C'}{B'^2 - C'^2} + \frac{2\kappa(B' + C'^2 - B'^2)}{B'^2 - C'^2} \quad (3.1.26)$$

and

$$: (\triangle\ a_-)^2 : \ = \frac{2\kappa(B' + C'^2 - B'^2)}{B'^2 - C'^2} - \frac{2\kappa C'}{B'^2 - C'^2}. \tag{3.1.27}$$

Upon substituting the explicit form of B' and C', (3.1.26) and (3.1.27) can be put in the form

$$: (\triangle\ a_+)^2 : \ = \frac{2\kappa\varepsilon'(-1 + e^{-(\kappa+2\varepsilon')t})}{\kappa + 2\varepsilon'} \tag{3.1.28}$$

and

$$: (\triangle\ a_-)^2 : \ = \frac{2\kappa\varepsilon'(1 - e^{-(\kappa-2\varepsilon')t})}{\kappa - 2\varepsilon'}, \tag{3.1.29}$$

so that at steady state, we obtain

$$: (\triangle\ a_+)^2 : \ = \frac{-2\kappa\varepsilon'}{\kappa + 2\varepsilon'}, \tag{3.1.30}$$

$$: (\triangle\ a_-)^2 : \ = \frac{2\kappa\varepsilon'}{\kappa - 2\varepsilon'}. \tag{3.1.31}$$

One can observe that the coherent driving light has no effect on the quadrature variances of the output light.

At threshold, we see that

$$: (\triangle\ a_+)^2 : \ = -\frac{1}{2}\kappa \tag{3.1.32}$$

and

$$: (\triangle\ a_-)^2 : \ = \infty. \tag{3.1.33}$$

At and below threshold, the variances of the quadrature operators indicate that the fluctuations in the plus quadrature are below the vacuum level with enhanced fluctuations in the minus quadrature. This verify that the output mode is in a squeezed state. We also note that at steady state and at threshold, there is a 40 % squeezing of the output light for $\kappa = 0.8$.

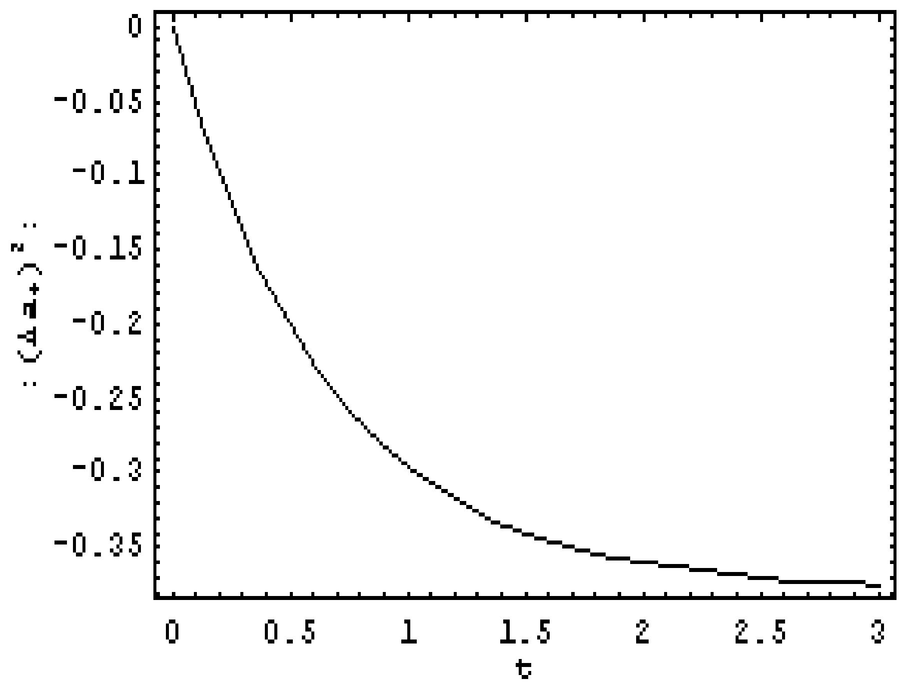

Figure 3.1: Plots of the quadrature variance : $(\Delta a_+)^2$: versus t for $\dfrac{\varepsilon'}{\kappa} = 0.45$.

Fig. 3.1 indicates that the squeezing increases with time and is maximum at steady state.

3.2 Squeezing spectrum

The squeezing spectrum of a single-mode light is expressible in terms of c-number variables associated with the normal ordering as

$$S_{\pm}^{out}(\omega) = \pm \frac{\kappa}{\pi} Re \int_0^{\infty} d\tau \langle \alpha_{\pm}(t), \alpha_{\pm}(t+\tau) \rangle_{ss} e^{i(\omega-\omega_o)\tau}, \tag{3.2.1}$$

where ω_o is the central frequency of the single-mode light and

$$\alpha_{\pm} = \alpha^*(t) \pm \alpha(t). \tag{3.2.2}$$

Applying (3.2.1) and (3.2.2) along with the relation

$$\langle U, V \rangle = \langle UV \rangle - \langle U \rangle \langle V \rangle, \tag{3.2.3}$$

we get

$$S_{\pm}^{out}(\omega) = \pm\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$+\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha^*(t)\alpha(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$+\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha(t)\alpha^*(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$\pm\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha(t)\alpha(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$\mp\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha^*(t+\tau)\rangle_{ss}\langle\alpha^*(t)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$-\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha^*(t)\rangle_{ss}\langle\alpha(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$-\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha^*(t+\tau)\rangle_{ss}\langle\alpha(t)\rangle_{ss}e^{i(\omega-\omega_o)\tau}$$
$$\mp\frac{\kappa}{\pi}Re\int_0^{\infty} d\tau\langle\alpha(t)\rangle_{ss}\langle\alpha(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}. \tag{3.2.4}$$

A two-time correlation function is expressible in the schrodinger picture as

$$\langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss} = Tr[\hat{a}^\dagger(o)\hat{a}^\dagger(\tau)\rho(t)]. \tag{3.2.5}$$

Upon expanding the density operator in the normal order, we have

$$\rho(t) = \sum_{lm} C_{lm}(t)\hat{a}^{\dagger l}\hat{a}^m. \tag{3.2.6}$$

Employing (3.2.5), (3.2.6) and using the completeness relation

$$I = \int\frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|, \tag{3.2.7}$$

one obtains

$$\langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss} = \int\frac{d^2\alpha}{\pi}\sum_{lm} C_{lm}(t)Tr[\hat{a}^\dagger(o)\hat{a}^\dagger(\tau)|\alpha\rangle\langle\alpha|\hat{a}^{\dagger l}\hat{a}^m]$$
$$= \int\frac{d^2\alpha}{\pi}\sum_{lm} C_{lm}(t)\alpha^{*l}Tr[\hat{a}^\dagger(0)\hat{a}^\dagger(\tau)|\alpha\rangle\langle\alpha|\hat{a}^m]. \tag{3.2.8}$$

Now applying the identity

$$|\alpha\rangle\langle\alpha|\hat{a}^n = \left(\alpha+\frac{\partial}{\partial\alpha^*}\right)^n(|\alpha\rangle\langle\alpha|), \tag{3.2.9}$$

we have

$$\langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss} = \int \frac{d^2\alpha}{\pi}\sum_{lm} C_{lm}(t)\alpha^{*l}\left(\alpha + \frac{\partial}{\partial\alpha^*}\right)^m Tr[\hat{a}^\dagger(0)\hat{a}^\dagger(\tau)|\alpha\rangle\langle\alpha|]. \quad (3.2.10)$$

or

$$\langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss} = \int d^2\alpha Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t) Tr[\hat{a}^\dagger(0)\hat{a}^\dagger(\tau)|\alpha\rangle\langle\alpha|]. \quad (3.2.11)$$

We note that

$$Tr[\hat{a}^\dagger(0)\hat{a}^\dagger(\tau)|\alpha\rangle\langle\alpha|] = \alpha^* Tr[\hat{a}^\dagger(\tau)\rho(0)] = Tr[\hat{a}^\dagger(0)\rho(\tau)], \quad (3.2.12)$$

where

$$\rho(0) = |\alpha\rangle\langle\alpha|. \quad (3.2.13)$$

Thus, one can write

$$Tr[\hat{a}^\dagger(0)\hat{a}^\dagger(\tau)|\alpha\rangle\langle\alpha|] = \alpha^* \int d^2\lambda Q(\lambda^*, \lambda, \tau)\lambda^*, \quad (3.2.14)$$

with

$$Q(\lambda^*, \lambda, 0) = \frac{1}{\pi} exp[-|\lambda - \alpha|^2]. \quad (3.2.15)$$

On combining (3.2.11) and (3.2.14), one can arrive at

$$\langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss} = \int d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha^* Q(\lambda^*, \lambda, \tau)\lambda^*. \quad (3.2.16)$$

This represents an expression for a two-time correlation function in terms of the Q function. In the same manner, one also finds

$$\langle\alpha^*(t)\alpha(t+\tau)\rangle_{ss} = \int d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha^* Q(\lambda^*, \lambda, \tau)\lambda, \quad (3.2.17)$$

$$\langle\alpha(t)\alpha^*(t+\tau)\rangle_{ss} = \int d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha Q(\lambda^*, \lambda, \tau)\lambda^*, \quad (3.2.18)$$

$$\langle\alpha(t)\alpha(t+\tau)\rangle_{ss} = \int d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha Q(\lambda^*, \lambda, \tau)\lambda, \quad (3.2.19)$$

$$\langle \alpha(t+\tau)\rangle_{ss} = \int d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}, t) Q(\lambda^*, \lambda, \tau)\lambda, \tag{3.2.20}$$

$$\langle \alpha^*(t+\tau)\rangle_{ss} = \int d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}, t) Q(\lambda^*, \lambda, \tau)\lambda^*. \tag{3.2.21}$$

On account of (3.2.16–21), Eq. (3.2.4) can be put in the form

$$S_{\pm}^{out}(\omega) = \frac{\kappa}{\pi} Re \int_0^\infty d\tau d^2\alpha d^2\lambda Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}, t)(\alpha \pm \alpha^*)$$

$$\times Q(\lambda^*, \lambda, \tau)(\lambda^* \pm \lambda) e^{i(\omega - \omega_o)\tau} \mp \frac{\kappa}{\pi} Re \int_0^\infty d\tau d^2\alpha d^2\lambda \tag{3.2.22}$$

$$\times Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}, t) Q(\lambda^*, \lambda, \tau)(\lambda^* \pm \lambda)(\langle \alpha^*\rangle \pm \langle \alpha \rangle) e^{i(\omega-\omega_o)\tau}. \tag{3.2.23}$$

Now replacing $(\alpha, \alpha^*, \alpha_o, \alpha_o^*, t)$ by $(\lambda, \lambda^*, \alpha, \alpha^*, \tau)$ in the Q function (2.3.12), we have

$$Q(\lambda^*, \lambda, \tau) = \frac{R''(\tau)}{\pi} exp[-B''(\tau)\lambda\lambda^* + S''(\tau)\lambda + S''^*(\tau)\lambda^* + \frac{C''(\tau)}{2}(\lambda^{*2} + \lambda^2)], \tag{3.2.24}$$

where B'', C'', R'' and S'' are described by the expression (2.2.65), (2.2.66), (2.3.13) and (2.3.14) with $(\alpha_o, \alpha_o^*, t)$ replaced by (α, α^*, τ).

It then follows that

$$\int d^2\lambda(\lambda^* \pm \lambda) Q(\lambda^*, \lambda, \tau) = R''(\tau) \int \frac{d^2\lambda}{\pi} (\lambda^* \pm \lambda) \times exp(-B''(\tau)\lambda\lambda^*$$

$$+ S''(\tau)\lambda + S''^*(\tau)\lambda^* + \frac{C''(\tau)}{2}(\lambda^{*2} + \lambda^2)]. \tag{3.2.25}$$

One can also write the above equation in the form

$$\int d^2\lambda(\lambda^* \pm \lambda) Q(\lambda^*, \lambda, \tau) = \left(\frac{\partial}{\partial S''^*} \pm \frac{\partial}{\partial S''}\right) R''(\tau) \int \frac{d^2\lambda}{\pi} exp(-B''(\tau)\lambda\lambda^*$$

$$+ S''(\tau)\lambda + S''^*(\tau)\lambda^* + \frac{C''(\tau)}{2}(\lambda^{*2} + \lambda^2)] \tag{3.2.26}$$

and on performing the integration over λ, there follows

$$\int d^2\lambda(\lambda^* \pm \lambda) Q(\lambda^*, \lambda, \tau) = \left(\frac{\partial}{\partial S''^*} \pm \frac{\partial}{\partial S''}\right) \frac{R''(\tau)}{\sqrt{B''^2 - C''^2}}$$

$$exp\left[\frac{B''S''S''^* + \frac{C''(\tau)}{2} S''^{*2} + \frac{C''(\tau)}{2} S''^2}{B''^2 - C''^2}\right], \tag{3.2.27}$$

so that on carrying out the differentiation, we get

$$\int d^2\lambda(\lambda^* \pm \lambda)Q(\lambda^*, \lambda, \tau) = \frac{(B'' \pm C'')(S'' \pm S''^*)}{(B''^2 - C''^2)}. \tag{3.2.28}$$

Upon substituting the explicit form of B'', C'', S'' and S''^* into (3.2.28), we have

$$\int d^2\lambda(\lambda^* + \lambda)Q(\lambda^*, \lambda, \tau) = e^{-(\kappa/2+\varepsilon')\tau}(\alpha^* + \alpha) + \frac{4\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}\tau}) \tag{3.2.29}$$

and

$$\int d^2\lambda(\lambda^* - \lambda)Q(\lambda^*, \lambda, \tau) = e^{-(\kappa/2-\varepsilon')\tau}(\alpha^* - \alpha). \tag{3.2.30}$$

On combining (3.2.23), (3.2.29) and (3.2.30), we see that

$$\begin{aligned} S_{\pm}^{out}(\omega) &= \frac{\kappa}{\pi} Re \int_0^{\infty} d\tau d^2\alpha e^{-(\kappa/2\pm\varepsilon')\tau} Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t) \\ &(\pm\alpha^{*2} \pm \alpha^2 + 2\alpha\alpha^*)e^{i(\omega-\omega_o)\tau} \mp \frac{\kappa}{\pi} Re \int_0^{\infty} d\tau d^2\alpha e^{-(\kappa/2\pm\varepsilon')\tau} \\ &Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)(\alpha^* \pm \alpha)(\langle\alpha^* \pm \alpha\rangle)e^{i(\omega-\omega_o)\tau}. \end{aligned} \tag{3.2.31}$$

Moreover, the expectation value of $\hat{A}$ can be expressed as

$$\langle\hat{A}\rangle = Tr[\rho(t)\hat{A}(0)]. \tag{3.2.32}$$

Upon expanding the density operator in the normal order and applying the completeness relation for coherent state, we have

$$\langle\hat{A}\rangle = \int \frac{d^2\alpha}{\pi} \sum_{lm} C_{lm}(t)\alpha^{*l} Tr[|\alpha\rangle\langle\alpha|\hat{a}^m A_n(\hat{a}^\dagger, \hat{a})]. \tag{3.2.33}$$

Applying the identity described by (3.2.9), we find

$$\langle\hat{A}\rangle = \int d^2\alpha Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)A_n(\alpha^*, \alpha), \tag{3.2.34}$$

where $A_n(\alpha^*, \alpha)$ is the c-number variables corresponding to $A(\hat{a}^\dagger, \hat{a})$ in the normal order.

Now with the aid of (3.2.34), Eq. (3.2.31) can be expressed as

$$S_{\pm}^{out}(\omega) = \frac{\kappa}{\pi} Re \int_0^{\infty} d\tau e^{-(\kappa/2 \pm \varepsilon')\tau} \langle (\pm \hat{a}^2 \pm \hat{a}^{\dagger 2} + 2\hat{a}^{\dagger}\hat{a}) \rangle e^{i(\omega-\omega_o)\tau}$$
$$\mp \frac{\kappa}{\pi} Re \int_0^{\infty} d\tau e^{-(\kappa/2 \pm \varepsilon')\tau} (\langle \hat{a}^{\dagger} \pm \hat{a} \rangle)^2 e^{i(\omega-\omega_o)\tau}. \tag{3.2.35}$$

Employing the Q function (2.3.15), one can write

$$\langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle = r(t) \int \frac{d^2\alpha}{\pi} (\alpha^2 + \alpha^{*2}) \times exp(-B'(t)\alpha\alpha^* + S'(t)(\alpha + \alpha^*)$$
$$+ \frac{C'(t)}{2}(\alpha^2 + \alpha^{*2})]. \tag{3.2.36}$$

The above equation can also be put in the form

$$\langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle = 2\frac{\partial}{\partial C'} r(t) \int \frac{d^2\alpha}{\pi} \times exp(-B'(t)\alpha\alpha^* + S'(t)(\alpha + \alpha^*)$$
$$+ \frac{C'(t)}{2}(\alpha^2 + \alpha^{*2})] \tag{3.2.37}$$

and hence carrying out the integration over α and differentiating with respect to C', we get

$$\langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle = \frac{2C'}{B'^2 - C'^2} + 2b^2(t). \tag{3.2.38}$$

Furthermore, one can write

$$\langle \hat{a} + \hat{a}^{\dagger} \rangle = r(t) \int \frac{d^2\alpha}{\pi} (\alpha + \alpha^*) \times exp(-B'(t)\alpha\alpha^* + S'(t)(\alpha + \alpha^*)$$
$$+ \frac{C'(t)}{2}(\alpha^2 + \alpha^{*2})]. \tag{3.2.39}$$

The above equation can also be expressed as

$$\langle \hat{a} + \hat{a}^{\dagger} \rangle = \frac{\partial}{\partial S'} r(t) \int \frac{d^2\alpha}{\pi} \times exp(-B'(t)\alpha\alpha^* + S'(t)(\alpha + \alpha^*)$$
$$+ \frac{C'(t)}{2}(\alpha^2 + \alpha^{*2})], \tag{3.2.40}$$

there follows

$$\langle \hat{a} + \hat{a}^{\dagger} \rangle = 2b(t). \tag{3.2.41}$$

We note that

$$\langle \hat{a} - \hat{a}^{\dagger} \rangle = 0. \tag{3.2.42}$$

In the same manner, we have

$$\langle \hat{a}^{\dagger}\hat{a} \rangle = r(t) \int \frac{d^2\alpha}{\pi} (\alpha\alpha^* - 1) \times exp(-B'(t)\alpha\alpha^* + S'(t)(\alpha + \alpha^*)$$
$$+ \frac{C'(t)}{2}(\alpha^2 + \alpha^{*2})]. \tag{3.2.43}$$

One can rewrite the above equation in the form

$$\langle \hat{a}^{\dagger}\hat{a} \rangle = -\frac{\partial}{\partial B'} r(t) \int \frac{d^2\alpha}{\pi} \times exp(-B'(t)\alpha\alpha^* + S'(t)(\alpha + \alpha^*)$$
$$+ \frac{C'(t)}{2}(\alpha^2 + \alpha^{*2})] - 1, \tag{3.2.44}$$

so that performing the integration employing the relation given by (2.3.10) and carryingout the differentiation, we find

$$\langle \hat{a}^{\dagger}\hat{a} \rangle = \frac{B'}{B'^2 - C'^2} + b^2(t) - 1. \tag{3.2.45}$$

Upon substituting (3.2.38), (3.2.41), (3.2.42) and (3.2.45) into (3.2.35), one obtains

$$S_{\pm}^{out}(\omega) = \pm \frac{1}{\pi} : (\triangle a_{\pm})^2 : Re \int_0^{\infty} d\tau e^{-(\kappa/2 \pm \varepsilon')\tau} e^{i(\omega - \omega_o)\tau}. \tag{3.2.46}$$

Upon carrying out the integration, the squeezing spectrum turns out to be of the form

$$S_{\pm}^{out}(\omega) = \pm : (\triangle a_{\pm})^2 : \left[\frac{(\kappa \pm 2\varepsilon')/2\pi}{[\frac{1}{2}(\kappa \pm 2\varepsilon')]^2 + (\omega - \omega_o)^2} \right]. \tag{3.2.47}$$

This represents the squeezing spectrum of the output light produced by the degenerate subharmonic generator.

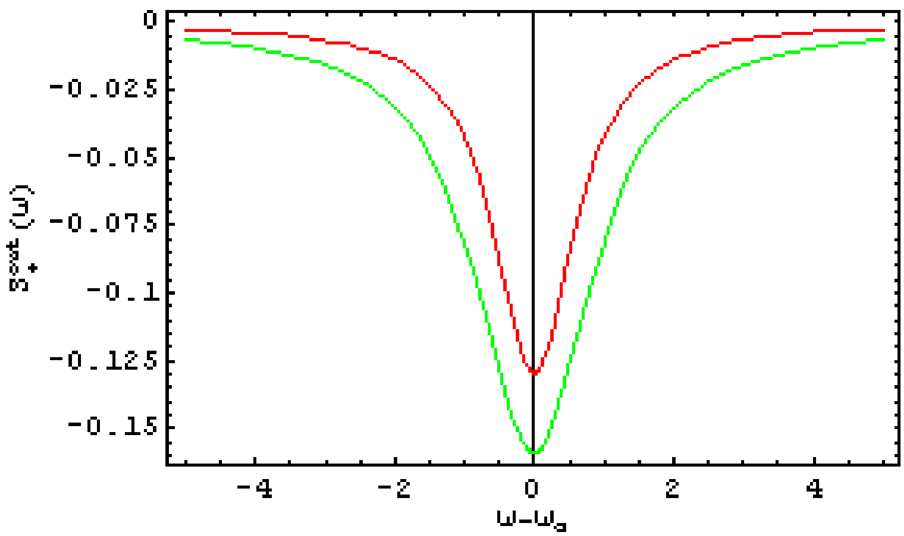

Figure 3.2: Plots of the squeezing spectrum $S_+^{out}(\omega)$ versus ω for $\dfrac{\varepsilon'}{\kappa} = 0.2$ (red curve) and $\dfrac{\varepsilon'}{\kappa} = 0.5$ (green curve).

Fig. 3.2 indicates that the squeezing spectrum $S_+^{out}(\omega)$ of the output mode produced by the degenerate subharmonic generator operating at and below threshold. We notice from Fig. 3.2 that the minimum value of the squeezing spectrum occurs at $\omega = \omega_0$. Moreover, we also observe that the value of the squeezing spectrum increases with increasing or decreasing ω.

Now on integrating both sides of (3.2.47) over ω, we have

$$\int_{-\infty}^{\infty} S_\pm^{out}(\omega) d\omega =: (\triangle a_\pm)^2 : . \tag{3.2.48}$$

On the basis of this relation, we note that $S_\pm^{out}(\omega)d\omega$ is the normally-ordered quadrature variance of the output light in the interval between ω and $\omega + d\omega$ [1].

Chapter 4

Photon Statistics

The statistical properties of a light beam is described in terms of the mean photon number, the variance of the photon number and the photon number distribution. Here we wish to calculate the photon number distribution, the mean and variance of the photon number for the output mode employing the Q function.

4.1 The mean photon number

The mean photon number is expressible in terms of the Q function and c-number variables associated with normal ordering as

$$\bar{n} = \frac{1}{\pi\kappa}\int d^2\alpha d^2\beta Q(\alpha^*, \beta) exp[-\frac{\alpha^*\alpha}{\kappa} - \frac{\beta^*\beta}{\kappa} + \frac{\alpha^*\beta}{\kappa} + \frac{\alpha\beta^*}{\kappa}]\beta^*\alpha \tag{4.1.1}$$

and upon substituting (2.3.22) into (4.1.1) and letting $\frac{1}{\kappa} = c_1$, we find

$$\bar{n} = \frac{r(t)c_1^2}{\pi}\int d^2\alpha \frac{d^2\beta}{\pi} exp[-c_1 B'\alpha^*\beta + \sqrt{c_1}S'\beta + \sqrt{c_1}S'\alpha^*$$
$$+\frac{c_1 C'}{2}(\alpha^{*2} + \beta^2) - c_1\alpha^*\alpha - c_1\beta^*\beta + c_1\alpha^*\beta + c_1\alpha\beta^*]\alpha\beta^*. \tag{4.1.2}$$

One can write the above equation in the form

$$\bar{n} = \frac{r(t)c_1^2}{\pi}\frac{\partial}{\partial c_1^*}\int d^2\alpha \frac{d^2\beta}{\pi} exp[-c_1\beta\beta^* + \beta(-c_1 B'\alpha^* + \sqrt{c_1}S' + c_1\alpha^*) + c_1^*\alpha\beta^*$$
$$+\frac{c_1 C'}{2}\beta^2] exp[\sqrt{c_1}S'\alpha^* + \frac{c_1 C'}{2}\alpha^{*2} - c_1\alpha\alpha^*]_{c_1^*=c_1}, \tag{4.1.3}$$

so that carrying out the integration over β employing the relation given by (2.3.10), we obtain

$$\bar{n} = c_1 r(t)\frac{\partial}{\partial c_1^*}\int \frac{d^2\alpha}{\pi} exp[-(c_1 + c_1^* B' - c_1^*)\alpha\alpha^* + \frac{\sqrt{c_1}}{c_1}c_1^* S'\alpha + \sqrt{c_1}S'\alpha^*$$
$$+\frac{C'}{2c_1}c_1^{*2}\alpha^2 + \frac{c_1 C'}{2}\alpha^{*2}]_{c_1^*=c_1}. \quad (4.1.4)$$

Upon performing the integration over α, there follows

$$\bar{n} = c_1 r(t)\frac{\partial}{\partial c_1^*}\left[\frac{1}{[c_1 + c_1^* B' - c_1^*]^2 - c_1^{*2}C'^2}\right]^{\frac{1}{2}}$$
$$\times exp\left[\frac{c_1 c_1^* S'^2 + c_1^{*2}B'S'^2 - c_1^{*2}S'^2 + c_1^{*2}C'S'^2}{[c_1 + c_1^* B' - c_1^*]^2 - c_1^{*2}C'^2}\right]_{c_1^*=c_1} \quad (4.1.5)$$

and carrying out the differentiation and putting the condition $c_1^* = c_1$, we readily obtain

$$\bar{n} = \frac{\kappa(B' - B'^2 + C'^2)}{B'^2 - C'^2} + \kappa b^2. \quad (4.1.6)$$

Upon substituting the explicit form of B', C' and b, we get

$$\bar{n} = \frac{4\kappa\varepsilon'^2 + \kappa^2\varepsilon'(e^{-(\kappa+2\varepsilon')t} - e^{-(\kappa-2\varepsilon')t}) - 2\kappa\varepsilon'^2(e^{-(\kappa+2\varepsilon')t} + e^{-(\kappa-2\varepsilon')t})}{2(\kappa^2 - 4\varepsilon'^2)}$$
$$+\frac{4\varepsilon^2}{\kappa}(1 - e^{-\frac{\kappa}{2}t})^2. \quad (4.1.7)$$

At steady state the above equation reduces to

$$\bar{n} = \frac{2\kappa\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2} + \frac{4\varepsilon^2}{\kappa}. \quad (4.1.8)$$

In the absence of the coherently driving light $\varepsilon = 0$, the above equation reduces to

$$\bar{n} = \frac{2\kappa\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}, \quad (4.1.9)$$

which is the mean photon number of the output light from a subharmonic generator.

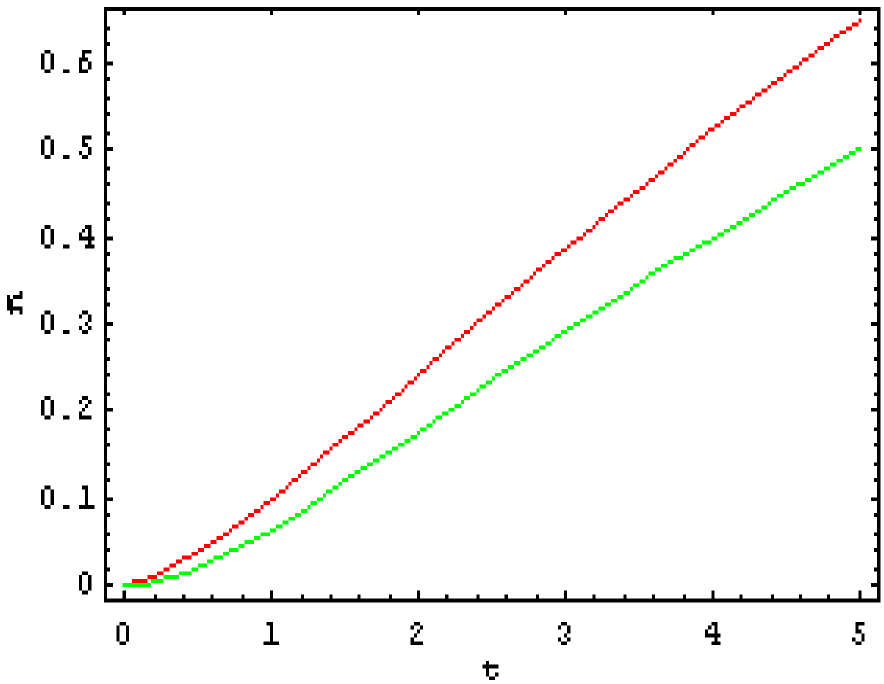

Figure 4.1: Plots of the mean photon number (Eq. (4.1.7)) vs t for κ=0.8, $\dfrac{\varepsilon'}{\kappa} = 0.45$ and ε =0.3 in the presence of the coherent driving light (red curve) and in the abscence of the coherent driving light (green curve)

Fig. 4.1 shows that the coherent driving light results in an increase in the mean photon number.

4.2 The normally-ordered variance of the photon number

Next we wish to calculate the normally-ordered variance of the photon number for the output mode. The normally-ordered variance of the photon number is expressible as

$$:(\triangle n)^2: \; = \langle(\hat{a}^{\dagger 2}\hat{a}^2)\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2. \tag{4.2.1}$$

In terms of c-number variables associated with the normal ordering, Eq. (4.2.1) can be written as

$$:(\triangle n)^2:\ = \langle(\alpha^{*2}\alpha^2)\rangle - \langle\alpha^*\alpha\rangle^2. \tag{4.2.2}$$

The expectation value of the c-number variables associated with the normal ordering can be put in terms of the Q function as

$$\langle(\alpha^{*2}\alpha^2)\rangle = \frac{1}{\pi\kappa}\int d^2\alpha d^2\beta Q(\alpha^*,\beta) exp[-\frac{\alpha^*\alpha}{\kappa} - \frac{\beta^*\beta}{\kappa} + \frac{\alpha^*\beta}{\kappa} + \frac{\alpha\beta^*}{\kappa}]\beta^{*2}\alpha^2. \tag{4.2.3}$$

On account of (2.3.22), one can write

$$\langle(\alpha^{*2}\alpha^2)\rangle = \frac{r(t)c_1^2}{\pi}\int d^2\alpha\frac{d^2\beta}{\pi} exp[-c_1\beta\beta^* + \beta(c_1\alpha^* + \sqrt{c_1}S' - c_1B'\alpha^*) + c_1^*\alpha\beta^*$$
$$+\frac{c_1C'}{2}\beta^2] exp[-c_1\alpha\alpha^* + \frac{c_1C'}{2}\alpha^{*2} + \sqrt{c_1}S'\alpha^*]\beta^{*2}\alpha^2. \tag{4.2.4}$$

One can also put the above equation in the form

$$\langle(\alpha^{*2}\alpha^2)\rangle = \frac{r(t)c_1^2}{\pi}\frac{\partial^2}{\partial c_1^{*2}}\int d^2\alpha\frac{d^2\beta}{\pi} exp[-c_1\beta\beta^* + \beta(c_1\alpha^* + \sqrt{c_1}S' - c_1B'\alpha^*)$$
$$+c_1^*\alpha\beta^* + \frac{c_1C'}{2}\beta^2] exp[-c_1\alpha\alpha^* + \frac{c_1C'}{2}\alpha^{*2} + \sqrt{c_1}S'\alpha^*]_{c_1^*=c_1} \tag{4.2.5}$$

and applying the mathematical relation (2.3.10), we get

$$\langle(\alpha^{*2}\alpha^2)\rangle = r(t)c_1\frac{\partial^2}{\partial c_1^{*2}}\int\frac{d^2\alpha}{\pi} exp[-(c_1 + c_1^*B' - c_1^*)\alpha\alpha^* + \frac{c_1^*}{\sqrt{c_1}}S'\alpha$$
$$+\sqrt{c_1}S'\alpha^* + \frac{c_1^{*2}}{2c_1}C'\alpha^2 + \frac{c_1}{2}C'\alpha^{*2}]_{c_1^*=c_1}. \tag{4.2.6}$$

Upon carrying out the integration over α, we have

$$\langle(\alpha^{*2}\alpha^2)\rangle = r(t)c_1\frac{\partial^2}{\partial c_1^{*2}} exp\left[\frac{1}{[c_1 + c_1^*B' - c_1^*]^2 - c_1^{*2}C'^2}\right]^{\frac{1}{2}}$$
$$\times exp\left[\frac{c_1c_1^*S'^2 + c_1^{*2}B'S'^2 - c_1^{*2}S'^2 + c_1^{*2}C'S'^2}{[c_1 + c_1^*B' - c_1^*]^2 - c_1^{*2}C'^2}\right]_{c_1^*=c_1}. \tag{4.2.7}$$

Performing the differentiation and applying the condition $c_1^* = c_1$, we get

$$\langle(\alpha^{*2}\alpha^2)\rangle = \frac{3\kappa^2(B'^2 - B' - C'^2)^2}{(B'^2 - C'^2)^2} + \frac{\kappa^2(2B' + C'^2 - 1 - B'^2)}{B'^2 - C'^2}$$
$$+\frac{6\kappa^2b^2(B' - B'^2 + C'^2)}{B'^2 - C'^2} + (\kappa b^2)^2 + \frac{2\kappa^2b^2}{B' + C'}(B' + C' - 1). \tag{4.2.8}$$

On account of (4.1.6), we have

$$\langle\alpha^*\alpha\rangle^2 = \frac{\kappa^2(B' + C'^2 - B'^2)^2}{(B'^2 - C'^2)^2} + \frac{2\kappa^2 b^2(B' + C'^2 - B'^2)}{B'^2 - C'^2} + (\kappa b^2)^2. \tag{4.2.9}$$

On combining (4.2.8) and (4.2.9), the normally-ordered variance of the photon number takes the form

$$\begin{aligned} :(\triangle n)^2: \; &= \frac{2\kappa^2(B'^2 - B' - C'^2)^2}{(B'^2 - C'^2)^2} + \frac{\kappa^2(2B' + C'^2 - 1 - B'^2)}{B'^2 - C'^2} \\ &+ \frac{4\kappa^2 b^2(B' - B'^2 + C'^2)}{B'^2 - C'^2} + \frac{2\kappa^2 b^2}{B' + C'}(B' + C' - 1) \end{aligned} \tag{4.2.10}$$

and upon introducing the explicit form of B', C' and b, we get

$$\begin{aligned} :(\triangle n)^2: \; &= 2\kappa^2\left[\frac{4\varepsilon'^2 + \kappa\varepsilon'(e^{-(\kappa+2\varepsilon')t} - e^{-(\kappa-2\varepsilon')t}) - 2\varepsilon'^2(e^{-(\kappa+2\varepsilon')t} + e^{-(\kappa-2\varepsilon')t})}{2(\kappa^2 - 4\varepsilon'^2)}\right]^2 \\ &+\kappa^2\Bigg[\frac{(\kappa - 2\varepsilon')(\kappa + \varepsilon' + \varepsilon' e^{-(\kappa+2\varepsilon')t}) + (\kappa + 2\varepsilon')(\kappa - \varepsilon' - \varepsilon' e^{-(\kappa-2\varepsilon')t})}{\kappa^2 - 4\varepsilon'^2} \\ &- \frac{(\kappa - \varepsilon' - \varepsilon' e^{-(\kappa-2\varepsilon')t})(\kappa + \varepsilon' + \varepsilon' e^{-(\kappa+2\varepsilon')t})}{\kappa^2 - 4\varepsilon'^2} - 1\Bigg] + 16\varepsilon^2(1 - e^{-\frac{\kappa}{2}t})^2 \\ &\times\left[\frac{4\varepsilon'^2 + \kappa\varepsilon'(e^{-(\kappa+2\varepsilon')t} - e^{-(\kappa-2\varepsilon')t}) - 2\varepsilon'^2(e^{-(\kappa+2\varepsilon')t} + e^{-(\kappa-2\varepsilon')t})}{2(\kappa^2 - 4\varepsilon'^2)}\right] \\ &+ \frac{8\varepsilon^2\varepsilon'(1 - e^{-\frac{\kappa}{2}t})^2(e^{-(\kappa-2\varepsilon')t} - 1)}{\kappa - 2\varepsilon'}, \end{aligned} \tag{4.2.11}$$

so that at steady state, we obtain

$$:(\triangle n)^2: \; = \frac{\kappa^2\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}\left(\frac{\kappa^2 + 4\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}\right) + 32\varepsilon^2\frac{\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2} - \frac{8\varepsilon^2\varepsilon'(\kappa + \varepsilon')}{\kappa^2 - \kappa\varepsilon' - 2\varepsilon'^2}. \tag{4.2.12}$$

In the absence of the driving coherent light, we have

$$:(\triangle n)^2: \; = \frac{\kappa^2\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}\left(\frac{\kappa^2 + 4\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}\right). \tag{4.2.13}$$

This represents the variance of the photon number of the output light of a subharmonic generator. According to Berihu Teklu [7], the variance of the photon number of the intracavity of the subharmonic generator is found to be

$$:(\triangle n)^2: \; = \frac{\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}\left(\frac{\kappa^2 + 4\varepsilon'^2}{\kappa^2 - 4\varepsilon'^2}\right). \tag{4.2.14}$$

From the above two expressions, one can conclude that the variance of the photon number of the output light from the subharmonic generator is κ^2 times that of the intracavity.

4.3 The photon number distribution

We now seek to study the photon number distribution of the output light employing the Q function (2.3.22). The photon number distribution of the output mode is expressible in terms of the Q function as [1]

$$P(n,t) = \frac{\pi\kappa^n}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left[Q(\alpha^*,\alpha,t)e^{\frac{\alpha^*\alpha}{\kappa}}\right]_{\alpha=\alpha^*=0}. \tag{4.3.1}$$

Upon substituting (2.3.22) into (4.3.1), we have

$$P(n,t) = \frac{r(t)\kappa^n}{\kappa n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left[exp[(\frac{1}{\kappa}-\frac{B'}{\kappa})\alpha\alpha^* + \frac{S'}{\sqrt{\kappa}}(\alpha+\alpha^*) + \frac{C'}{2\kappa}(\alpha^{*2}+\alpha^2)\right]_{\alpha=\alpha^*=0} \tag{4.3.2}$$

On account of the power series expansion, one can write

$$exp[(\frac{1}{\kappa}-\frac{B'}{\kappa})\alpha\alpha^*] = \sum_i \frac{1}{i!}\left(\frac{1}{\kappa}-\frac{B'}{\kappa}\right)^i \alpha^i\alpha^{*i}, \tag{4.3.3}$$

$$exp\left(\frac{S'}{\sqrt{\kappa}}\alpha\right) = \sum_j \frac{1}{j!}\left(\frac{S'}{\sqrt{\kappa}}\right)^j \alpha^j, \tag{4.3.4}$$

$$exp\left(\frac{S'}{\sqrt{\kappa}}\alpha^*\right) = \sum_l \frac{1}{l!}\left(\frac{S'}{\sqrt{\kappa}}\right)^l \alpha^{*l}, \tag{4.3.5}$$

$$exp\left(\frac{C'}{2\kappa}\alpha^{*2}\right) = \sum_m \frac{1}{m!}\left(\frac{C'}{2\kappa}\right)^m \alpha^{*2m}, \tag{4.3.6}$$

$$exp\left(\frac{C'}{2\kappa}\alpha^2\right) = \sum_r \frac{1}{r!}\left(\frac{C'}{2\kappa}\right)^r \alpha^{2r} \tag{4.3.7}$$

and hence the photon number distribution takes the form

$$P(n,t) = \frac{r(t)\kappa^n}{\kappa n!}\sum_{ijlmr}\frac{1}{2^{m+r}}\frac{1}{\kappa^{i+\frac{1}{2}j+\frac{1}{2}l+m+r}}\frac{(1-B')^i S'^j S'^l C'^{m+r}}{i!j!l!m!r!}$$
$$\times\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}[\alpha^{i+j+2r}\alpha^{*(i+l+2m)}]_{\alpha=\alpha^*=0}. \tag{4.3.8}$$

Applying the relation

$$\frac{\partial}{\partial x^n}x^m = \frac{m!}{(m-n)!}x^{m-n}, \tag{4.3.9}$$

(4.3.8) can be written as

$$P(n,t) = \frac{r(t)\kappa^n}{\kappa n!}\sum_{ijlmr}\frac{1}{2^{m+r}}\frac{1}{\kappa^{i+\frac{1}{2}j+\frac{1}{2}l+m+r}}\frac{(1-B')^i S'^j S'^l C'^{m+r}}{i!j!l!m!r!}$$
$$\times\frac{(i+j+2r)!}{(i+j+2r-n)!}\alpha^{i+j+2r-n}\frac{(i+l+2m)!}{(i+l+2m-n)!}\alpha^{*(i+l+2m-n)}|_{\alpha=\alpha^*=0}. \tag{4.3.10}$$

Now applying the condition $\alpha = \alpha^* = 0$, one finds

$$P(n,t) = \frac{r(t)\kappa^n}{\kappa n!}\sum_{ijlmr}\frac{1}{2^{m+r}}\frac{1}{\kappa^{i+\frac{1}{2}j+\frac{1}{2}l+m+r}}\frac{(1-B')^i S'^j S'^l C'^{m+r}}{i!j!l!m!r!}$$
$$\times\frac{(i+j+2r)!}{(i+j+2r-n)!}\delta_{i+j+2r,n}\frac{(i+l+2m)!}{(i+l+2m-n)!}\delta_{i+l+2m,n}. \tag{4.3.11}$$

Hence in view of the property of the kronecker delta, we see that

$$j = n - i - 2r, \tag{4.3.12}$$

$$l = n - i - 2m. \tag{4.3.13}$$

On introducing (4.3.12) and (4.3.13) into (4.3.11), one obtains

$$P(n,t) = \frac{r(t)}{\kappa n!}\sum_{imr}\frac{(n!)^2}{2^{m+r}}\frac{(1-B')^i S'^{2(n-i-r-m)} C'^{m+r}}{i!(n-i-2r)!(n-i-2m)!m!r!}. \tag{4.3.14}$$

This represents the photon number distribution of the output light from the subharmonic generator and the coherently driven cavity mode.

If we consider the case in which the driving light is not present ($\varepsilon = 0$) and employing (2.3.16), (2.3.17), (2.3.18) and (4.3.10), we get

$$P(n,t) = \frac{\sqrt{B'^2 - C'^2}\kappa^n}{\kappa n!}\sum_{imr}\frac{1}{2^{m+r}}\frac{1}{\kappa^{i+m+r}}\frac{(1-B')^i C'^{m+r}}{i!m!r!}$$
$$\times\frac{(i+2r)!}{(i+2r-n)!}\delta_{i+2r,n}\frac{(i+2m)}{(i+2m-n)!}\delta_{i+2m,n}. \tag{4.3.15}$$

Applying the property of the kronecker delta, we have

$$m = r, \tag{4.3.16}$$

$$i = n - 2m. \tag{4.3.17}$$

Hence the photon number distribution takes the form

$$P(n,t) = \frac{\sqrt{B'^2 - C'^2}}{\kappa n!} \sum_{m=0}^{[n]} \frac{(n!)^2}{2^{2m}} \frac{(1-B')^{n-2m} C'^{2m}}{(n-2m)!(m!)^2}, \tag{4.3.18}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n.

This expression indicates that there is a finite probability of observing odd number of output signal photons. This is due to the fact that, although the signal photons are generated in pairs, there is also the possibility for an odd number of photons to pass through the port mirror.

Comparing the results given by (4.3.18) with the intracavity photon number distribution of a subharmonic generator found by B. Daniel and K. Fesseha [2], one can observe that the photon number distribution of the output light from the subharmonic generator is $\frac{1}{\kappa}$ of the intracavity.

Chapter 5

Conclusion

In this thesis we have studied the squeezing and statistical properties of the output light from a subharmonic generator and a coherently driven cavity mode. We have obtained the Q function of the signal mode and the coherently driven cavity mode employing the propagator method [10]. Then we have obtained the superposition of these two light beams. Applying the input-output relation, we have determined the Q function of the output light.

Employing the resulting Q function, we have calculated the normally-ordered quadrature variance and the squeezing spectrum. We have found that the normally-ordered quadrature variance of the output light depends on the cavity damping constant and the amplitude of the pump mode. At steady state and at threshold, there is a 40 % squeezing of the output mode for $\kappa = 0.8$. We have also observed that the squeezing spectrum increases with increasing or decreasing ω. In addition, we have noticed that the squeezing spectrum is just the quadrature variance of the given components of the output light. Further more, the driving coherent light has no effect on the normally-ordered quadrature variance and squeezing spectrum of the output mode.

Moreover, we have determined with the aid of the Q function of the output light, the mean photon number, the normally-ordered variance of the photon number and the photon number distribution. The presence of the coherent driving light results in an increase in the mean photon number of the output light. Finally, from the photon number distribution of the output light from a subharmonic generator, we have seen that there is a possibility of finding odd number of output photons. This is due to the fact that, although the signal photons are generated in pairs, there is a possibility for an odd number of photons to leave the cavity through the port mirror.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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