College of natural and Computational Science

Department of Mathematics

Heat equation and its application in homogeneous Dirichelt boundary condition

By

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A Thesis submitted to the college of Natural and Computational sciences of Addis Ababa University in partial fulfillment of the requirements for degree of master of science in mathematics .

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Addis Ababa University

Department of Mathematics

The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled on **heat equation and its application on homogeneous Dirichelt boundary condition** by Bezuneh Mekonnen in partial fulfillment of the requirements for the degree of masters of Science.

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Dedication

This thesis is dedicated to my beloved mother Bekelech Wiziro

Notation

∂ = Partial derivative

∇^2 = Laplacian operator

Ø = Flow rate of heat energy

c = Specific heat capacity

ρ = Mass density

∂Q = Boundary of Q

$\|.\|$ = Norm

$\propto$ = Directly proportional

e = Heat energy density

c^k = k times differentiable function

k_o = Thermal conductivity

$\hat{n}$ = Unit normal vector

∇ = Gradient operator

Abstract

In this paper, heat equation is derived using Fourier's law of heat conduction and conservation of energy. In addition to this two laws Greens and divergence Theorem are used to change or transform line integral in to surface integral and volume integral in to surface integral respectively while deriving heat equation. The solution of heat equation is also investigated using separation of variables by considering homogenous Dirchelt boundary condition. The application of heat equation is also included.

Introduction

The heat equation is an important partial differential equation which describes the distribution of heat or Variation in temperature in a given region over time. The heat equation has the general form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

For a function $u(x, y, z, t)$ of three spatial variables x, y, z and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} = k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

Or equivalently

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

Where k is a constant

The heat equation predicts that if a hot body is placed in a box of cold water, the temperature of the body will decrease, and eventually (after infinite time, and subject to no external heat sources) the temperature in the box will equalize.

The heat equation is derived from Fourier's law and conservation of energy. By Fourier's law, the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface,

$$\emptyset = -\, \mathrm{k}\, \nabla u$$

Where, k is the thermal conductivity and u is the temperature.

In the absence of work done, a change in internal energy per unit volume in the material ΔQ is proportional to the change in temperature. That is,

$$\Delta \mathrm{Q} = c\rho\Delta \mathrm{u}$$

Where c is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at temperature zero, this can be rewritten as

$$Q = c\rho u$$

One of the methods used to solve heat equation is using separation of variables. To solve heat equation by separation of variables boundary and initial conditions are needed.

Unit one

Preliminaries

1.1 Greens Theorem in a plane

Let R be a closed bounded region in the X-Y plane whose boundary C is a smooth curve. Let M(X, Y) and N(X, Y) be functions that are continuous and have continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ everywhere in some domain containing R. Then

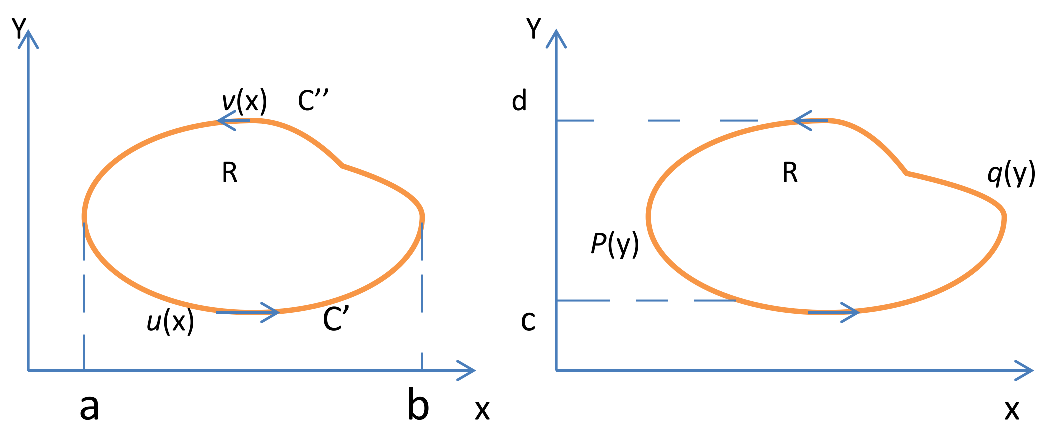
$$\iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint\limits_C (Mdx + Ndy)$$

Or in vector form

$$\iint\limits_R cur\boldsymbol{F} dxdy = \oint\limits_C \boldsymbol{F}.\boldsymbol{n} dr$$

Where $\boldsymbol{F}(x, y) = M(x, y)i + N(x, y)j$ and n is a unit normal vector to C

<u>Proof</u>: Consider special regions below



$a \leq x \leq b$ and $u(x) \leq Y \leq v(x)$ $\qquad$ $c \leq Y \leq d$ and $p(y) \leq X \leq q(y)$

$$\iint\limits_R \frac{\partial M}{\partial y} dxdy = \int_a^b [\int_{u(x)}^{v(x)} \frac{\partial M}{\partial y} dy] dx$$

$$= \int_a^b [\, M(x, v(x)) - M(x, u(x))] dx$$

$$= \int_a^b M(x, v(x)) dx - \int_a^b M(x, u(x))\, dx$$

$$= - \int_b^a M(x, v(x)) dx - \int_a^b M(x, u(x))\, dx$$

$$= - \int_{C''} M(x, y) dx - \int_{C'} M(x, y)\, dx$$

$$= - \oint_C M(x, y)\, dx$$

Therefore,

$$- \iint_R \frac{\partial M}{\partial y} dxdy = \oint_C M(x, y)\, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (1)$$

Next, $\iint_R \frac{\partial N}{\partial y} dxdy = \int_C^d [\int_{p(y)}^{q(y)} \frac{\partial N}{\partial y} dx] dy$

$$= \int_C^d [\, N(q(y), y) - N(p(y), y)] dy$$

$$= \int_C^d N(q(y), y) dy - \int_C^d N(p(y), y) dy$$

$$= \int_C^d N(q(y), y) dy + \int_d^c N(p(y), y) dy$$

$$= \oint_C N(x, y) dy$$

$$\therefore \iint_R \frac{\partial N}{\partial y} dxdy = \oint_C N(x, y) dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Adding equation (1) and (2) we get

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C M(x,y)\,dx + \oint_C N(x,y)dy$$

$$\Rightarrow \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C (Mdx + Ndy)$$

$$\Rightarrow \iint_R cur\boldsymbol{F} dxdy = \oint_C \boldsymbol{F}.\boldsymbol{n} dr$$

1.2 Divergence Theorem of Gauss

Let Q C R^3 be a region bounded by a closed surface ∂Q and let **n** the unit outward normal vector to ∂Q. If **F** is a vector function that has continuous partial derivative in Q, then

$$\iint_{\partial Q} \boldsymbol{F}.\boldsymbol{n} ds = \iiint_Q \nabla.\boldsymbol{F} dv$$

Where,

$$\boldsymbol{F}(x,y,z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$$

Proof:

$$\iint_{\partial Q} \boldsymbol{F}.\boldsymbol{n} ds = \iint_{\partial Q} M(x,y,z)\boldsymbol{i}.\boldsymbol{n} ds + \iint_{\partial Q} N(x,y,z)\boldsymbol{j}.\boldsymbol{n} ds + \iint_{\partial Q} P(x,y,z)\boldsymbol{k}.\boldsymbol{n} ds \ldots .(3)$$

And $\iiint_Q \nabla.\boldsymbol{F} dv = \iiint_Q \frac{\partial \mathrm{M}}{\partial \mathrm{x}} dv + \iiint_Q \frac{\partial \mathrm{N}}{\partial \mathrm{x}} dv + \iiint_Q \frac{\partial \mathrm{P}}{\partial \mathrm{Z}} dv \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ldots \ldots \ldots .$ (4)

Equating (3) and (4) we get

$$\iint_{\partial Q} M(x,y,z)\boldsymbol{i}.\boldsymbol{n} ds + \iint_{\partial Q} N(x,y,z)\boldsymbol{j}.\boldsymbol{n} ds + \iint_{\partial Q} P(x,y,z)\boldsymbol{k}.\boldsymbol{n} ds$$
$$= \iiint_Q \frac{\partial \mathrm{M}}{\partial \mathrm{x}} dv + \iiint_Q \frac{\partial \mathrm{N}}{\partial \mathrm{y}} dv + \iiint_Q \frac{\partial \mathrm{p}}{\partial \mathrm{z}} dv$$

Thus the Theorem will be proved if we can show

$$\iint_{\partial Q} M(x,y,z)\boldsymbol{i}.\boldsymbol{n} ds = \iiint_Q \frac{\partial \mathrm{M}}{\partial \mathrm{x}} dv$$

$$\iint_{\partial Q} N(x,y,z)\boldsymbol{j}.\boldsymbol{n}ds = \iiint_{Q} \frac{\partial \mathrm{N}}{\partial \mathrm{y}} dv$$

$$\iint_{\partial Q} P(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds = \iiint_{Q} \frac{\partial \mathrm{p}}{\partial \mathrm{z}} dv$$

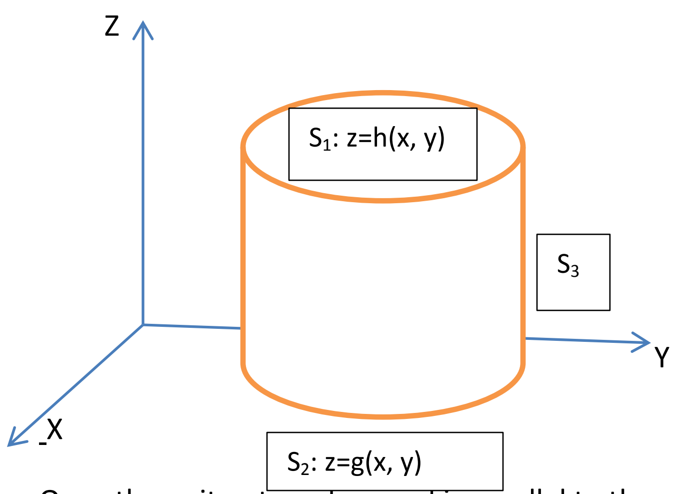
Since all the proofs are similar we will focus on only

$$\iint_{\partial Q} P(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds = \iiint_{Q} \frac{\partial \mathrm{p}}{\partial \mathrm{z}} dv$$

Suppose the region Q can be described as

$Q = \left\{\{(x,y,z)\}\middle| g(x,y) \le Z \le h(x,y), for\ (x,y) \in R \right\}$ where R is a region in the xy-plane

Think of Q as being bounded by three surfaces s_1(top), s_2(bottom), and s_3(side)



On s_3 the unit outward normal is parallel to the xy-plane and thus

$$\iint_{\partial Q} p(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds = \iint_{\partial Q} 0ds = 0$$

Now we can calculate the surface integral over s_1

$$s_1 = \{(x,y,z)|z - h(x,y) = 0 for\ (x,y) \in R\}$$

The unit outward normal **n** is given by

$$\boldsymbol{n} = \frac{\nabla(z - h(x,y))}{\|\nabla(z - h(x,y))\|} = \frac{-h_x(x,y)i - h_y(x,y)j + k}{\sqrt{[-h_x(x,y)]^2 + [-h_y(x,y)]^2 + 1}}$$

And

$$\boldsymbol{k}.\boldsymbol{n} = \frac{1}{\sqrt{[h_x(x,y)]^2 + [h_y(x,y)]^2 + 1}}$$

$$\iint\limits_{s_1} p(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds = \iint\limits_{s_1} \frac{p(x,y,z)}{\sqrt{[-h_x(x,y)]^2 + [-h_y(x,y)]^2 + 1}}ds = \iint\limits_{R} P(x,y,h(x,y))dA$$

Similarly, the surface integral over s_2 is

$$\iint\limits_{s_2} p(x,y,z)k.nds = -\iint\limits_{R} P(x,y,g(x,y))dA$$

Finally,

$$\iint\limits_{\partial Q} p(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds = \iint\limits_{s_1} p(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds + \iint\limits_{s_2} p(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds + \iint\limits_{s_3} p(x,y,z)\boldsymbol{k}.\boldsymbol{n}ds$$

$$= \iint\limits_{R} P(x,y,h(x,y))dA - \iint\limits_{R} P(x,y,g(x,y))dA$$

$$= \iint\limits_{R} [P\big(x,y,h(x,y)\big) - P\big(x,y,g(x,y)\big)]dA$$

$$= \iint_R \int\limits_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}dzdA = \iiint\limits_{Q} \frac{\partial P}{\partial z}dv$$

Hence,

$$\iint\limits_{\partial Q} \boldsymbol{F}.\boldsymbol{n}ds = \iiint\limits_{Q} \nabla.\boldsymbol{F}dv$$

1.3 Fourier series representation of function

A Fourier series representation of a function $f(x)$ over the interval $-a \leq x \leq a$ is an expression of the form

$$f(x) = a_o + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{a})b_n \sin(\frac{n\pi x}{b})$$

Where,

$$a_o = \frac{1}{a}\int_0^a f(x)dx$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos(\frac{n\pi x}{a})dx$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin(\frac{n\pi x}{a})\,dx$$

If the Fourier series representation of $f(x)$ contains only sine Fourier series on the interval $-a \le x \le a$ $(i.e. f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{a})$, then the Fourier coefficient b_n is given by

$$b_n = \frac{2}{a}\int_0^a f(x)\sin(\frac{n\pi x}{a})\,dx$$

For two dimensional case, a function $f(x,y)$ containing only double sine Fourier series for $-a \le x \le a \; and -b \le y \le b$ $(i.e. f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b}))$ the coefficient b_{mn} is given by

$$b_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})$$

For three dimensional case, a $f(x,y,z)$ containing only triple sine Fourier series for

$$-a \le x \le a\,, -b \le y \le b\,, and\; -c \le z \le c$$

$(i.e. f(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} b_{mnl} \sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\sin(\frac{l\pi z}{c}))$. The coefficient $b_{mnl}$is given by

$$b_{mnl} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\sin(\frac{l\pi z}{c})$$

Unit Two

Derivation of heat equation

To derive heat equation two concepts are needed:

1. Conservation of energy
2. Fourier's law of heat conduction

1. Conservation of energy

It states that the rate of change of heat energy between two ends is equal to the sum of rate of heat energy flowing through ends plus rate of heat energy generated inside the segment of the rod.

2. Fourier's law of heat conduction

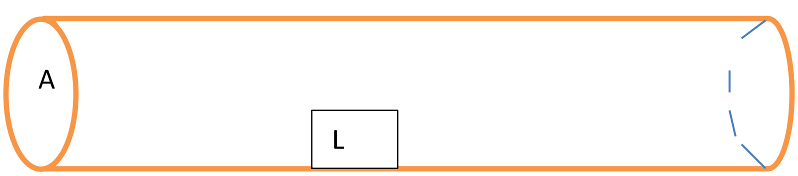
It states that the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface. If the temperature is constant at different positions, then no heat energy flows (i.e. $\emptyset = 0$). If there are temperature differences at different positions then heat energy flows from hot to cold. The greater the difference in temperature, the greater the flux or heat energy flow (i.e. $\emptyset \propto -\frac{\partial u}{\partial x}$).

Heat flow also depends on the type of material

$\emptyset(\mathrm{x,t}) = -k_o \frac{\partial}{\partial x}\mathrm{u(x,t)}$, where k_o is thermal conductivity of the material

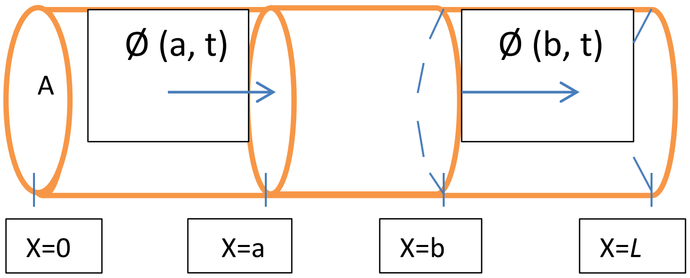
2.1 One dimensional heat equation derivation

Consider a rod of length L and cross sectional area A



To understand the distribution of heat energy in the rod consider the heat energy density

$\mathrm{e} = \mathrm{e(x,t)}$, i.e. the heat energy per unit length at position x and time t. Assume that e depends only on x and t .This means the rod is insulated (except possibly at the ends). The heat flux or the amount of heat energy per unit flowing through a unit cross sectional area at x is denoted by Ø (x, t).



Consider possible heat source Q(x, t), i.e. The amount of heat energy per unit volume generated per unit time.

Using conservation of energy:

Rate of change heat energy between x=a and x=b

=

Rate of heat energy flowing through ends .... (5)

+

Rate of heat energy generated inside segment of the rod

Total heat energy between x=a and x=b is: $A\int_a^b e(x,t)dx$ so that the change of heat energy between x=a and x=b is: $\frac{d}{dt}\left[A\int_a^b e(x,t)dx\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$

The rate of heat energy flowing through ends is: $A\,\text{Ø}\,(\mathrm{a,t}) - A\,\text{Ø}\,(\mathrm{b,t}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(7)$

And the rate of heat generated inside the rod is: $A\int_a^b Q(x,t)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$

Substituting 6, 7, 8 in to (1) we get:

$$\frac{d}{dt}\left[\int_a^b e(x,t)dx\right] = \text{Ø}\,(\mathrm{a,t}) - \text{Ø}\,(\mathrm{b,t}) + \int_a^b Q(x,t)dx$$

Provided that e is continuous and a, b are constants with respect to t, then

$$\frac{d}{dt}\int_a^b e(x,t)dx = \int_a^b \frac{\partial}{\partial t} e(x,t)dx$$

And by the Fundamental Theorem of Calculus provided that $\emptyset \in C^1$

$$\emptyset\,(\mathrm{a,t}) - \emptyset\,(\mathrm{b,t}) = \int_b^a \frac{\partial}{\partial t}\emptyset(x,t)dx = -\int_a^b \frac{\partial}{\partial x}\emptyset(x,t)dx$$

So conservation of energy becomes

$$\int_a^b \frac{\partial}{\partial t}e(x,t)dx = -\int_a^b \frac{\partial}{\partial x}\emptyset(x,t)dx + \int_a^b Q(x,t)dx$$

Or

$$\int_a^b [\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}\,\emptyset(x,t) - Q(x,t)]dx = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(9)$$

Since, $\int_a^b [\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}\,\emptyset(x,t) - Q(x,t)]dx = 0$ holds for arbitrary a and b we have the following

$$\frac{\partial}{\partial t}\,e(x,t) + \frac{\partial}{\partial x}\,\emptyset(x,t) - Q(x,t) = 0$$

$$\frac{\partial}{\partial t}\,e(x,t) = -\frac{\partial}{\partial x}\,\emptyset(x,t) + Q(x,t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(10)$$

Energy density e(x, t) is given by the formula:

$e(x,t) = c(x).\rho(\mathrm{x}).\mathrm{u(x,t)}$, where $c(x)$ = specific heat

$\mathrm{u(x,t)}$ = temperature

$\rho(\mathrm{x})$ = mass density and

Substituting the value of e(x, t) in equation (10) we get

$$\frac{\partial}{\partial t}\,[c(x).\rho(\mathrm{x}).\mathrm{u(x,t)}] = -\frac{\partial}{\partial x}\,\emptyset(x,t) + Q(x,t)$$

$$c(x).\rho(\mathrm{x})\frac{\partial}{\partial t}\,\mathrm{u(x,t)} = -\frac{\partial}{\partial x}\,\emptyset(x,t) + Q(x,t)\,, but\;\emptyset(x,t) = -\,k_o(\mathrm{x})\,\frac{\partial}{\partial x}\,\mathrm{u(x,t)}$$

$$c(x).\rho(\mathrm{x})\frac{\partial}{\partial t}\,\mathrm{u(x,t)} = \frac{\partial}{\partial x}\left[k_o(\mathrm{x})\frac{\partial}{\partial x}\,\mathrm{u(x,t)}\right] + \mathrm{Q(x,t)}$$

$$c.\rho\frac{\partial}{\partial t}\,\mathrm{u(x,t)} = k_o\frac{\partial^2}{\partial x^2}u(x,t) + Q(x,t)$$

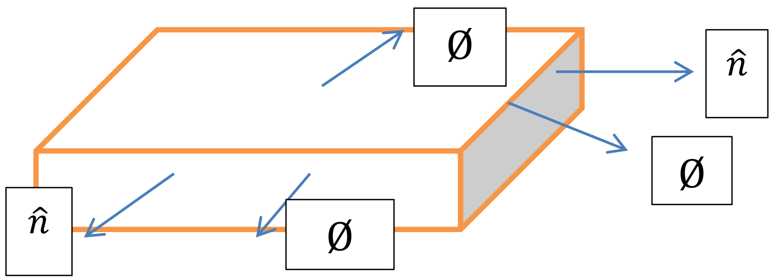
$\frac{\partial}{\partial t}$ u(x,t) $= k\frac{\partial^2}{\partial x^2}u(x,t) + q(x,t)$, where $k = \frac{k_o}{c\rho}$ (k is called thermal diffusivity) and

$q(x,t) = \frac{Q(x,t)}{c\rho}$.

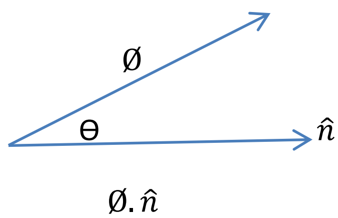
If no source of heat is present, then the one dimensional heat equation becomes

$\frac{\partial}{\partial t}$ u(x,t) $= k\frac{\partial^2}{\partial x^2}u(x,t)$ Or $u_t = ku_{xx}$

2.2 Two dimensional heat equation derivation



Considering only the normal component of the heat flux, i.e. taking the component of Ø along $\hat{n}$



$Proj_{\hat{n}}^{Ø} = \|Ø\| \cos\Theta$ and $\cos\Theta = \frac{Ø.\hat{n}}{\|Ø\|\,\|\hat{n}\|}$ hence, projection of Ø along $\hat{n}$ is equal to Ø. $\hat{n}$

The vector form of Greens Theorem that uses divergence is given by:

$\oint_c \vec{F}.\vec{n}.ds = \iint_D div\, \boldsymbol{F}\, dA$, where $div\, F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$

The total heat energy on the surface is equal to:

$\iint_D e(x,y,t)dA = \iint_D c(x,y)\rho(x,\ y).u(x,y,t)dA$ and the rate of change of heat energy is $\frac{d}{dt}\iint_D c(x,y)\rho(x,\ y)u(x,y,t)dA$

Similarly, the rate of heat energy generated inside the region is: $\iint_D Q(x,y,t)dA$

Using only the normal component of the heat flux we get the rate of heat energy flowing through the boundary line: $-\oint Ø(x,y,t).\hat{n}(x,y)ds$

Substituting the quantities in the conservation of heat energy equation (1) we get the following

$$\frac{d}{dt}\iint_D c\rho\mathrm{u}(\mathrm{x},\mathrm{y},\mathrm{t})\mathrm{dA} = \iint_D Q(x,y,t)dA - \oint \emptyset(\mathrm{x},\mathrm{y},\mathrm{t}).\check{\mathrm{n}}(\mathrm{x},\mathrm{y})\mathrm{ds} \ldots\ldots\ldots\ldots (11)$$

In order to get the heat equation in PDE form we need to convert the line integral in to area integral using Greens Theorem

$$\oint_c \emptyset\ .\hat{n}\ ds = \iint_D \nabla.\emptyset\ \ dA$$

Equation (11) becomes

$$\iint_D c\rho\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t})\mathrm{dA} = \iint_D Q(x,y,t)dA - \iint_D \nabla.\emptyset\ \ dA$$

$$\iint_D c\rho\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t})\mathrm{dA} - \iint_D Q(x,y,t)dA + \iint_D \nabla.\emptyset\ \ dA = 0$$

$$\iint_D [c\rho\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t}) - Q(x,y,t) + \nabla.\emptyset]\mathrm{dA} = 0$$

Since the above equation holds for arbitrary D, then we get

$$c\rho\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t}) = Q(x,y,t) - \nabla.\emptyset \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (12)$$

Using Fourier's law of heat conduction in 2D, the heat flux Ø is proportional to the temperature gradient $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$

$$\emptyset(\mathrm{x},\mathrm{y},\mathrm{t}) = -k_o\nabla u(x,y,t)$$

$$\emptyset(\mathrm{x},\mathrm{y},\mathrm{t}) = -k_o(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

Hence equation (12) becomes

$$c\rho\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t}) = Q(x,y,t) + \left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)k_o(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

$$c\rho\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t}) = Q(x,y,t) + k_o(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

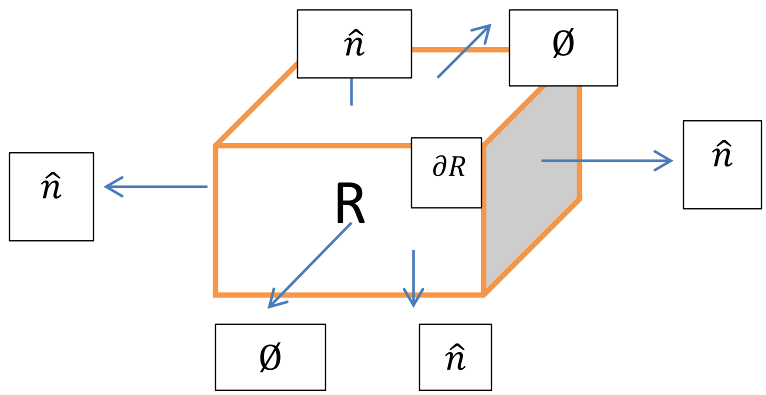
$$\frac{\partial}{\partial \mathrm{t}}u(\mathrm{x},\mathrm{y},\mathrm{t}) = \frac{Q}{c\rho} + \frac{k_o}{c\rho}(u_{xx} + u_{yy})$$

$$u_t = q + k(u_{xx} + u_{yy}),\, where \quad q = \frac{Q}{c\rho} \; and \; k = \frac{k_o}{c\rho}$$

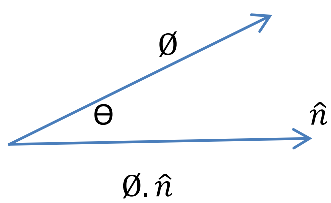
If no source of heat is present, then the two dimensional heat equation becomes

$$u_t = k(u_{xx} + u_{yy})$$

2.3 Three dimensional heat equation derivation



Considering only the normal component the flux, i.e. taking the component of Ø along $\hat{n}$



$Proj_{\hat{n}}^{\emptyset} = \|\emptyset\| \cos\Theta$ and $\cos\Theta = \frac{\emptyset . \hat{n}}{\|\emptyset\|\|\hat{n}\|}$. Hence, projection of Ø along $\hat{n}$ is equal to Ø. $\hat{n}$

The total heat energy on the region R is equal to:

$\iiint_R e(x,y,z,t)dv = \iiint_R c(x,y,z)\rho(\mathrm{x,y,z})\mathrm{u}(\mathrm{x,y,z,t})\mathrm{dv}$

And the rate of change heat energy is $\frac{d}{dt}\iiint_R c(x,y,z)\rho(\mathrm{x,y,z})\mathrm{u}(\mathrm{x,y,z,t})\mathrm{dv}$

Similarly, the rate of heat energy generated inside the region is: $\iiint_R Q(x,y,t)dv$

Using only the normal component of the heat flux, we get the rate of heat energy flowing through boundary surface: $-\iint_{\partial R} \emptyset\,(x,y,z,t).\hat{n}\,(\mathrm{x,y,z})\mathrm{ds}$

Substituting the quantities in the conservation of heat energy equation (1) we get the following

$$\frac{d}{dt}\iiint_R c\rho u \mathrm{dv} = \iiint_R Q dv - \iint_{\partial R} \emptyset . \hat{n}\; \mathrm{ds} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(13)$$

In order to get the heat equation in PDE form we need to convert the surface integral in to volume integral using the divergence theorem i.e.

$$\iint_{\partial R} \emptyset . \hat{n}\, \mathrm{ds} = \iiint_R \nabla . \emptyset \mathrm{dv}$$

Hence, equation (13) becomes

$$\iiint_R c\rho \frac{\partial}{\partial \mathrm{t}} \mathrm{udv} = \iiint_R Q dv - \iiint_R \nabla . \emptyset \mathrm{dv}$$

$$\iiint_R [c\rho \frac{\partial}{\partial \mathrm{t}} \mathrm{u} - Q + \nabla . \emptyset] \mathrm{dv} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Since equation 10 holds for arbitrary R we get

$$c\rho \frac{\partial}{\partial \mathrm{t}} \mathrm{u} - Q + \nabla . \emptyset = 0$$

$$c\rho \frac{\partial}{\partial \mathrm{t}} \mathrm{u} = Q - \nabla . \emptyset \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Now using Fourier's law of heat conduction in its 3D form, the flux Ø is proportional to the temperature gradient $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$

$$\emptyset(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t}) = -k_o \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Substituting equation 16 in to equation 15 we get

$$c\rho \frac{\partial}{\partial \mathrm{t}} \mathrm{u} = Q + \nabla . (k_o \nabla u)$$

$$c\rho \frac{\partial}{\partial \mathrm{t}} \mathrm{u} = Q + k_o \nabla^2 u$$

$$\frac{\partial}{\partial \mathrm{t}} \mathrm{u} = \frac{Q}{c\rho} + \frac{k_o}{c\rho} \nabla^2 u$$

$$\frac{\partial}{\partial \mathrm{t}} \mathrm{u} = q + \mathrm{k}\, \nabla^2 u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Where, $q = \frac{Q}{c\rho}$, $\mathrm{k} = \frac{k_o}{c\rho}$ and $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

If there is no heat source heat equation (17) becomes $\mathrm{u}_t = k(u_{xx} + u_{yy} + u_{zz})$

Unit Three

Solution of heat equation using separation of variables

To solve heat equation by separation of variables boundary conditions and initial conditions are needed and the following steps are followed

Step 1: In the first step, we need to find all solutions $\mathrm{u(x,t)}, \mathrm{u(x,y,t)}$ and $\mathrm{u(x,y,z,t)}$ that are of the special form

$\mathrm{u(x,t) = X(x)T\,(t)}$, for one dimensional heat equation.

$\mathrm{u(x,y,t) = X(x)Y(y)T\,(t)}$, for two dimensional heat equation and

$\mathrm{u(x,y,z,t) = X(x)Y(y)Z\,(z)T\,(t)}$, for three dimensional heat equation such that $\mathrm{X(x)}$ depends only on X, $\mathrm{Y(y)}$ depends only on y, $\mathrm{Z(z)}$ depends only on z and $\mathrm{T(t)}$ depends only on t. If we find a bunch of solutions $\mathrm{X_i(x)Y_i(y)Z_i(z)T_i(t)}$ of this form, then since heat equation is a linear equation $\sum_i \mathrm{a_i\, X_i(x)Y_i(y)Z_i(z)T_i(t)}$ is also a solution for any choice of the constant $\mathrm{a_i}$.

Step 2: we impose the boundary conditions in to the solution form

Step 3: we impose the initial condition to get the general solution of heat equation

3.1 solution of one dimensional heat equation for homogeneous Dirichelt boundary condition

$\frac{\partial}{\partial t}\,\mathrm{u(x,t)} = k\frac{\partial^2}{\partial x^2}u(x,t)$

$\mathrm{u(x,t)}$ Denotes the temperature at a position x and time t in a long, thin rod of length l that runs from x=0 to x= l

Suppose the temperature at the ends of the rod is held fixed at 0 (boundary condition)

$$\mathrm{u(0,t)} = 0 \; for\ all \;\; t \ge 0$$

$$\mathrm{u}(l,\mathrm{t}) = 0 \; for\ all \;\; t \ge 0$$

Assume that we know the temperature throughout the rod at time $t = 0$. So there is some given function $f(x)$ such that the initial condition

$$\mathrm{u(x,0)} = f(x) \; for \; all \; 0 \leq \; x \leq l \; is \; satisfied$$

The problem is to determine $\mathrm{u(x,t)}$ for all x and t. $\mathrm{u(x,t)}$ can be solved using method of separation.

We seek a separated solution of the form $\mathrm{u(x,t)} = \mathrm{X(x)T\,(t)}$. In this case

$$\frac{\partial}{\partial t}\,\mathrm{u(x,t)} = XT'$$

$$\frac{\partial^2}{\partial x^2} u(x,t) = X''T$$

$$XT' = kX''T$$

$$\frac{X''}{X} = \frac{T'}{kT}$$

The left hand side is independent of t and the right hand side is independent of x. The two sides are equal, so both sides must be independent of both x and t and hence equal to some constant, say $-\lambda$. So we have

$\frac{X''}{X} = -\lambda$ and $\frac{T'}{kT} = -\lambda$

Together with the boundary conditions we obtain the system

$X'' + \lambda X = 0$ and $T' + k\lambda T = 0$, for $\mathrm{X(0)} = \mathrm{X}(l\,) = 0$

<u>Case 1</u>: For $\lambda = 0$,

$$X'' = 0$$

$$\Rightarrow X(x) = Cx + D$$

$\mathrm{X\,(0)} = 0 \Longrightarrow \mathrm{X\,(0)} = \mathrm{C\,(0)} + \mathrm{D} = 0 \Longrightarrow \mathrm{D} = 0$ and

$\mathrm{X}\,(l) = 0 \Longrightarrow \mathrm{C}\,(l) = 0 \Longrightarrow \mathrm{C} = 0$

$\Longrightarrow X(\,x) = 0$

$\Longrightarrow \mathrm{u(x,t)} = X.T = 0$ This is a trivial solution

<u>Case 2</u>: For $\lambda > 0$, $\lambda = \beta^2$

$$X'' + \beta^2 X = 0$$

$$\Longrightarrow X = C\cos\beta x + D\sin\beta x$$

$$\Longrightarrow X(0) = C\cos 0 + D\sin 0$$

$$\Longrightarrow C = 0$$

And $X(l) = D\sin\beta l = 0$

$\Longrightarrow \beta l = n\pi \Longrightarrow \beta = \frac{n\pi}{l}$, for $n = 1, 2, 3 \ldots..$

Therefore, $\lambda_n = (\frac{n\pi}{l})^2$ and $X_n(x) = D_n \sin(\frac{n\pi}{l})x$

<u>Case 3</u>: For $\lambda < 0$, $\lambda = -\beta^2, \beta \neq 0$

$$X'' - \beta^2 X = 0$$

$$\Longrightarrow X(x) = C_1 e^{\beta x} + C_2 e^{-\beta x}$$

$$X(0) = 0 \Longrightarrow 0 = C_1 e^0 + C_2 e^0$$

$$\Longrightarrow C_1 + C_2 = 0$$

$$\Longrightarrow C_1 = -C_2$$

$$X(l) = 0 \Longrightarrow 0 = -C_2 e^{\beta l} + C_2 e^{-\beta l}$$

$$\Longrightarrow C_2\left(-e^{\beta l} + e^{-\beta l}\right) = 0$$

Since, $\beta \neq 0, -e^{\beta l} + e^{-\beta l} \neq 0$, hence $C_2 = 0$ and $C_1 = 0$

Therefore, $(x) = 0$. This is a trivial solution.

The solution of the above equation is given by

$$X_n(x) = D_n \sin(\frac{n\pi}{l})x \ , n = 1, 2, 3, \ldots \ldots \ldots \ldots$$

And

$$T' + k\lambda T = 0$$

$$\Longrightarrow T' = -k\lambda T$$

$$\Longrightarrow \frac{T'}{T} = -k\lambda$$

$$\Longrightarrow (lnT)' = -k\lambda dt$$

$$\Longrightarrow \int (lnT)'dt = \int (-k)\lambda dt$$

$$\Longrightarrow lnT = -k\lambda T + lnC$$

$$\Longrightarrow T = Ce^{-k\lambda t}$$

$$T_n = C_n e^{-k\lambda t}$$

Therefore, the general solution of one dimensional heat equation for homogeneous Dirichelt boundary condition is

$$\begin{aligned} \mathrm{u}_n(\mathrm{x},\mathrm{t}) &= X_n(x).T_n(t) \\ &= D_n \sin(\frac{n\pi}{l})x\,.\,C_n e^{-k(\frac{n\pi}{l})^2 t} \\ &= A_n \sin(\frac{n\pi}{l})x\,.\,e^{-k(\frac{n\pi}{l})^2 t} \end{aligned}$$

$$\therefore \quad \mathrm{u}(\mathrm{x},\mathrm{t}) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{l})x\,.\,e^{-k(\frac{n\pi}{l})^2 t}$$

Using the initial condition, $\mathrm{u}(\mathrm{x},0) = f(x)$ and expressing $f(x)$ in Fourier sine series,

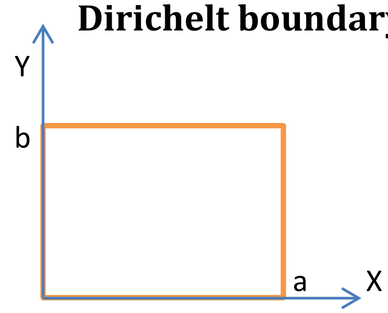
$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{l})x$$

From Fourier series of $f(x)$ and $f\ is\ a\ C^1$ function, we have

$A_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi}{l})x\, f(x)dx \quad , n = 1, 2, 3 \ldots$

Hence, $\mathrm{u}(\mathrm{x},\mathrm{t}) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{l})x\,.\,e^{-k(\frac{n\pi}{l})^2 t}$, where $A_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi}{l})x\, f(x)dx$ and $n = 1,2 \ldots$

3.2 solution of two dimensional heat equation for homogeneous Dirichelt boundary condition



$u_t = k(u_{xx} + u_{yy})$ for $0 < x < a$, and $0 < y < b$

$u(0, y, t) = u(a, y, t) = 0$, $for\ 0 \leq y \leq b, and\ t \geq 0$

$u(x, 0, t) = u(x, b, t) = 0$, $for\ 0 \leq x \leq a, and\ t \geq 0$

Physically, the above boundary conditions correspond to holding the temperature along the edges of the plate to be 0.

The way the plate is heated initially is given by the initial condition

$$u(x, y, 0) = f(x, y)\ , (x, y) \in R\ , where\ R = [0, a] \times [0, b]$$

So, we seek a separated solution of the form

$$u(x, y, t) = X(x)\, Y(y)\, T(t)$$

$$u_t = XYT',\ u_{xx} = X''YT\ and\ u_{yy} = XY''T$$

Inserting the above value in two dimensional heat equation we get

$$XYT' = k(X''YT + XY''T)$$

$$\Rightarrow \quad \frac{XYT'}{XYT} = k(\frac{X''YT}{XYT} + \frac{XY''T}{XYT})$$

$$\Rightarrow \quad \frac{T'}{T} = k\frac{X''}{X} + k\frac{Y''}{Y}$$

The left hand side $\frac{T'}{T}$ is independent of $x\ and\ y$ and the right hand side $\frac{X''}{X}$ is independent of $y\ and\ t$. The second part of the right hand side $\frac{Y''}{Y}$ is independent of $x\ and\ t$.Hence,

$$k\frac{X''}{X} = k_1 \qquad \Longrightarrow \frac{X''}{X} = \frac{k_1}{k} = B \ , \qquad k_1 \ is \ constant.$$

$$\Longrightarrow \qquad X'' - BX = 0$$

And

$$k\frac{Y''}{Y} = k_2 \ \ \Rightarrow \frac{Y''}{Y} = \frac{k_2}{k} = C \ , \qquad k_2 \ is \ \ constant$$

$$\Longrightarrow Y'' - CY = 0$$

And

$$\frac{T'}{T} = k_1 + k_2 = Bk + Ck$$

$$\Longrightarrow T' - k(B + C)T = 0$$

The solution for $X'' - BX = 0$, for $B < 0 \ and \ B = -\beta^2$

$$X'' - (-\beta^2)X = 0$$

$$\Longrightarrow X'' + \beta^2 X = 0$$

The ODE has solution of the form $X = e^{\lambda x}$ and substituting in to the above equation, we get the characteristic equation $\lambda^2 + \beta^2 = 0$ and $\lambda = \pm\beta i$. Hence,

$$X_1(x) \ \ = e^{i\beta x} = \ \cos\beta x + i\sin\beta x \ \ and$$

$$X_2(x) \ \ = \ e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

Let $\widehat{X_1} = \frac{X_1+X_2}{2} = \cos\beta x \ and \ \widehat{X_2} = \frac{X_1+X_2}{2i} = \sin\beta x$

$$X(x) = C_1\cos\beta x + C_2\sin\beta x \ \ is \ the \ general \ solution \ of \ X'' + \beta^2 X = 0$$

$$X(0) = 0 \ \Rightarrow C_1 = 0 \ and \ X(a) = 0 \ \Rightarrow \ C_2\sin\beta a = 0$$

$$C_2\sin\beta a = 0 \Rightarrow \beta a = n\pi \ \Rightarrow \beta = \frac{n\pi}{a} \ for \ n = 1, 2, 3 \ldots$$

Therefore, $B_n = \ -(\frac{n\pi}{a})^2 \ and \ the \ general \ solution \ of X'' + \beta^2 X = 0 \ \ is$

$$X_n(x) = C_n \sin(\frac{n\pi}{a}x)\ for\ n = 1, 2, 3, \ldots$$

Similarly, the solution for $Y'' - CY = 0\ \ and\ C = \ -\gamma^2\ is$

$$Y_m(y) = D_m \sin(\frac{m\pi}{b}y)\ for\ m = 1, 2, 3 \ldots \ldots \ldots$$

The solution for $T' - k(B + C)T = 0$

$$T' - k(-\beta^2 - \gamma^2)T = 0$$

$$\Rightarrow T' + k(\beta^2 + \gamma^2)T = 0$$

$$\Rightarrow \frac{T'}{T} = \ -k(\beta^2 + \gamma^2)$$

$$\Rightarrow (lnT)' = -k(\beta^2 + \gamma^2)$$

$$\Rightarrow T(t) = e^{-k(\beta^2+\gamma^2)t}$$

$$\Longrightarrow\ T_{mn}(t) = e^{-\lambda_{mn}^2 t}\ , where\ \lambda_{mn} = \pi\sqrt{k}\ \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

Assembling these results, we find that for any pair $m, n\ \geq 1$, we have

$$u_{mn}(x, y, t) = X_n(x)Y_m(y)T_{mn}(t)$$

$$u_{mn}(x, y, t) = C_n \sin(\frac{n\pi}{a}x)D_m \sin(\frac{m\pi}{b}y)e^{-\lambda_{mn}^2 t}$$

$$u_{mn}(x, y, t) = A_{mn} \sin(\frac{n\pi}{a}x) \sin(\frac{m\pi}{b}y)e^{-\lambda_{mn}^2 t}\ \ , where\ C_n\ D_m = A_{mn}$$

For any choice of constants A_{mn}, by the Principle of Superposition, we have the general solution

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(\frac{n\pi}{a}x) \sin(\frac{m\pi}{b}y)e^{-\lambda_{mn}^2 t}$$

The coefficients A_{mn} can be determined using the initial condition

$u(x, y, 0) = f(x, y)$

$$f(x, y) = u(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(\frac{n\pi}{a}x) \sin(\frac{m\pi}{b}y)$$

This is just the double Fourier series for $f(x,y)$ and f $is\ a\ C^2$ function then we have

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)dydx$$

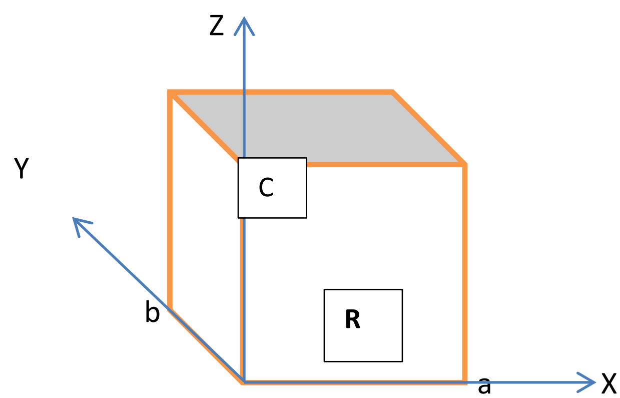
Therefore, the solution to heat equation in two dimensions with homogeneous Dirichlet boundary condition and initial condition is given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)e^{-\lambda_{mn}^2 t}$$

Where,

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)dydx\ \ and\ \lambda_{mn} = \pi\sqrt{k}\ \sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$$

3.3 solution of three dimensional heat equation for homogeneous Dirichelt boundary condition



$u_t = k(u_{xx} + u_{yy} + u_{zz})$ for $0 < x < a$, $0 < y < b$ and $0 < z < c$

$u(0,y,z,t) = u(a,y,z,t) = 0$, $for\ 0 \le y \le b, 0 < z < c$, and $t \ge 0$

$u(x,0,z,t) = u(x,b,z,t) = 0$, $for\ 0 \le x \le a, 0 < z < c\ and\ t \ge 0$

$u(x,y\ 0,t) = u(x,y,c,t) = 0$, $for\ 0 \le x \le a, 0 \le y \le b\ \ and\ t \ge 0$

Physically, the above boundary condition corresponds to holding the temperature along the boundary of the region R to be 0.

The way the region is heated initially is given by the initial condition

$$u(x,y,z,0) = f(x,y,z)\ , where\ (x,y,z) \in R$$

So, we seek a separated solution of the form

$$\mathrm{u(x,y,z,t)\ =\ X(x)\ Y(y)Z(z)\ T\ (t)}$$

$$u_t = XYZT', \qquad u_{xx} = X''YZT\ , \qquad u_{yy} = XY''ZT\ and\ u_{zz} = XYZ''T$$

Inserting the above value in three dimensional heat equation, we get

$$XYZT' = k(X''YZT + XY''ZT + XYZ''T\ )$$

$$\Rightarrow \frac{XYZT'}{XYZT} = k(\frac{X''YZT}{XYZT} + \frac{XY''ZT}{XYZT} + \frac{XYZ''T}{XYZT})$$

$$\Rightarrow \frac{T'}{T} = k\frac{X''}{X} + k\frac{Y''}{Y} + k\frac{Z''}{Z}$$

The left hand side $\frac{T'}{T}$ is independent of $x, y\ and\ z$.

And the right hand side:

$\frac{X''}{X}$ is independent of $y, z\ and\ t$.

$\frac{Y''}{Y}$ is independent of $x, z\ \ and\ t$.

$\frac{Z''}{Z}$ is independent of $x, y\ \ and\ t$.

Hence,

$$k\frac{X''}{X} = k_1\ \Rightarrow \frac{X''}{X} = \frac{k_1}{k} = B$$

$$\Rightarrow X'' - BX = 0$$

And

$$k\frac{Y''}{Y} = k_2\ \ \Rightarrow \frac{Y''}{Y} = \frac{k_2}{k} = C$$

$$\Rightarrow Y'' - CY = 0$$

And

$$k\frac{Z''}{Z} = k_3 \quad \Longrightarrow \frac{Z''}{Z} = \frac{k_3}{k} = D$$

$$\Longrightarrow Z'' - DZ = 0$$

And

$$\frac{T'}{T} = k_1 + k_2 + k_3 = Bk + Ck + Dk$$

$$\Longrightarrow T' - k(B + C + D)T = 0$$

The solution for $X'' - BX = 0$, for $B < 0 \; and \; B = -\beta^2$

$$X'' - (-\beta^2)X = 0$$

$$\Longrightarrow X'' + \beta^2 X = 0$$

The ODE has solution of the form $X = e^{\lambda x}$ and substituting in to the above equation we get the characteristic equation $\lambda^2 + \beta^2 = 0$ and $\lambda = \pm\beta i$. Hence,

$$X_1(x) = e^{i\beta x} = \cos\beta x + i\sin\beta x \;\; and$$

$$X_2(x) = e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

Put $\widehat{X_1} = \frac{X_1 + X_2}{2} = \cos\beta x \; and \; \widehat{X_2} = \frac{X_1 + X_2}{2i} = \sin\beta x$

$$X(x) = C_1\cos\beta x + C_2\sin\beta x \;\; is \; the \; general \; solution \; of \; X'' + \beta^2 X = 0$$

$$X(0) = 0 \; \Rightarrow C_1 = 0 \; and \; X(a) = 0 \; \Rightarrow \; C_2\sin\beta a = 0$$

$$C_2\sin\beta a = 0 \Rightarrow \beta a = n\pi \; \Rightarrow \beta = \frac{n\pi}{a} \; for \; n = 1, 2, 3, \ldots$$

Therefore, $B_n = -(\frac{n\pi}{a})^2 \; and \; the \; general \; solution \; of X'' + \beta^2 X = 0 \;\; is$

$$X_n(x) = C_n\sin(\frac{n\pi}{a}x) \; for \; n = 1, 2, 3, \ldots$$

Similarly, the solution for $Y'' - CY = 0 \;\; and \; C = -\gamma^2 \; is$

$$Y_m(y) = D_m\sin(\frac{m\pi}{b}y) \; for \; m = 1, 2, 3 \ldots,$$

And the solution for $Z'' - DZ = 0, D = -\alpha^2$ is

$$Z_l(z) = E_l \sin(\frac{l\pi}{c} z) \; for \; l = 1, 2, 3, \ldots$$

The solution for $T' - k(B + C + D)T = 0$

$$T' - k(-\beta^2 - \gamma^2 - \alpha^2)T = 0$$

$$\Rightarrow T' + k(\beta^2 + \gamma^2 + \alpha^2)T = 0$$

$$\Rightarrow \frac{T'}{T} = -k(\beta^2 + \gamma^2 + \alpha^2)$$

$$\Rightarrow (lnT)' = -k(\beta^2 + \gamma^2 + \alpha^2)$$

$$\Rightarrow \quad T(t) = e^{-k(\beta^2+\gamma^2+\alpha^2)t}$$

$$\Rightarrow T_{mnl}(t) = e^{-\lambda_{mnl}^2 t}, \qquad where \; \lambda_{mnl} = \pi\sqrt{k}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}}$$

Assembling these results, we find that for any pair $m, n, l \geq 1$, we have

$$u_{mnl}(x, y, z, t) = X_n(x)Y_m(y)Z_l(z)T_{mnl}(t)$$

$$u_{mnl}(x, y, z, t) = C_n \sin(\frac{n\pi}{a} x)D_m \sin(\frac{m\pi}{b} y)E_l \sin(\frac{l\pi}{c} z)e^{-\lambda_{mnl}^2 t}$$

$$u_{mnl}(x, y, z, t) = A_{mnl} \sin(\frac{n\pi}{a} x) \sin(\frac{m\pi}{b} y) \sin(\frac{l\pi}{c} z)e^{-\lambda_{mn}^2 t}$$

$$where, C_n \, D_m E_l = A_{mnl}$$

For any choice of constants A_{mnl}, by the Principle of Superposition, we have the general solution

$$u(x, y, z, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{mnl} \sin(\frac{n\pi}{a} x) \sin(\frac{m\pi}{b} y) \sin(\frac{l\pi}{c} z)e^{-\lambda_{mn}^2 t}$$

The coefficients A_{mnl} can be determined using the initial condition

$u(x, y, z, 0) = f(x, y, z)$

$$f(x,y,z) = u(x,y,z,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{mnl}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(\frac{l\pi}{c}z)$$

This is just the triple Fourier series for $f(x,y,z)$ and $f\ is\ a\ C^3$ function then we have

$$A_{mnl} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(\frac{l\pi}{c}z)$$

Therefore, the solution to heat equation in three dimensions with homogeneous Dirichlet boundary condition and initial condition is given by

$$u(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{mnl}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(\frac{l\pi}{c}z)e^{-\lambda_{mn}^2 t}$$

Where,

$$A_{mnl} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c f(x,y,z)\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(\frac{l\pi}{c}z)$$

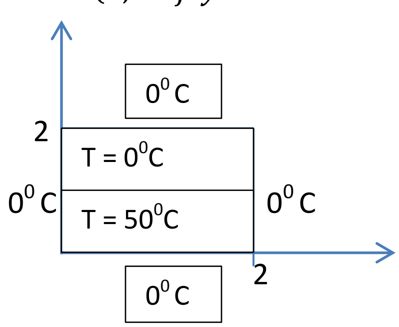
And

$$\lambda_{mn} = \pi\sqrt{k}\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{l^2}{c^2}}$$

3.4 Application of heat equation

A 2x2 square plate with $k = \frac{1}{3}$ is heated in such a way that the temperature in the lower half is 50^0 C, while the temperature in the upper half is 0^0 C. After that, it is insulated laterally and the temperature at its edges is held at 0^0 C. Find an expression that gives the temperature in the plate for$t > 0$.

$$f(x,y) = \begin{cases} 50\ , if\ y \le 1 \\ 0, \quad if\ y > 1 \end{cases}$$



The coefficients in the solution is determined by

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)dydx$$

$$= \frac{4}{2\times 2}\int_0^2\int_0^2 f(x,y)\sin(\frac{n\pi}{2}x)\sin(\frac{m\pi}{2}y)dydx$$

$$= \int_0^2\int_0^1 f(x,y)\sin(\frac{n\pi}{2}x)\sin(\frac{m\pi}{2}y)dydx + \int_0^2\int_1^2 f(x,y)\sin(\frac{n\pi}{2}x)\sin(\frac{m\pi}{2}y)dydx$$

$$= \int_0^2\int_0^1 50\ \sin(\frac{n\pi}{2}x)\sin(\frac{m\pi}{2}y)dydx$$

$$= \int_0^2 50\ \sin(\frac{n\pi}{2}x)dx\int_0^1 \sin(\frac{m\pi}{2}y)dy$$

$$= 50\ \frac{2}{n\pi}(-\cos(\frac{n\pi}{2}))\Big|_0^2 . \frac{2}{m\pi}\ (-\cos(\frac{n\pi}{2}))\Big|_0^1$$

$$= \frac{100}{n\pi}[-\cos n\pi + \cos 0].\frac{2}{m\pi}\left[-\cos\frac{m\pi}{2} + \cos 0\right]$$

$$= \frac{200}{\pi^2}\frac{(1-\cos n\pi)}{n}\frac{(1-\cos\frac{m\pi}{2})}{m}$$

$$= \frac{200}{\pi^2}\frac{(1+(-1)^{n+1})}{n}\frac{(1-\cos\frac{m\pi}{2})}{m}$$

The temperature throughout the plate at any time t is given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)e^{-\lambda_{mn}^2 t}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{200}{\pi^2}\frac{(1+(-1)^{n+1})}{n}\frac{(1-\cos\frac{m\pi}{2})}{m}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)e^{-(\pi\sqrt{k}\ \sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}})^2 t}$$

$$= \frac{200}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1+(-1)^{n+1})}{n}\frac{(1-\cos\frac{m\pi}{2})}{m}\sin(\frac{n\pi}{2}x)\sin(\frac{m\pi}{2}y)e^{\frac{-\pi^2\sqrt{n^2+m^2}}{36}t}$$

3.5 Fundamental solution of heat equation

A fundamental solution, also called a heat kernel, is a solution of the heat equation corresponding to the initial condition of an initial point source of heat at a known position. These can be used to find a general solution of the heat equation over certain domains; in one variable, the Green's function is a solution of the initial value problem

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t), & -\infty < x < \infty, 0 < t < \infty \\ u(x,0) = \delta(x) \end{cases}$$

Where δ is the Dirac delta function. The solution to this problem is the fundamental solution.

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{x^2}{4kt})$$

One can obtain the general solution of the one variable heat equation with initial condition

$u(x,0) = g(x)$ for $-\infty < x < \infty$ and $0 < t < \infty$ by applying a convolution:

$$u(x,t) = \int \Phi(x-y,t)g(y)dy$$

In several spatial variables, the fundamental solution solves the analogous problem

$$\begin{cases} u_t(x,t) = k\sum_{i=1}^{n} u_{x_i x_i}(x,t), & -\infty < x_i < \infty, i = 1, \dots, n \textit{ and } 0 < t < \infty \\ u(x,0) = \delta(x) \end{cases}$$

The n-variable fundamental solution is the product of the fundamental solutions in each variable; i.e.

$$\Phi(X,t) = \Phi(x_1,t)\Phi(x_2,t) \dots \Phi(x_n,t) = \frac{1}{\sqrt{(4\pi kt)^n}} \exp(-\frac{x^2}{4kt})$$

The general solution of the heat equation on $\mathbb{R}^n$ is then obtained by a convolution, so that to solve the initial value problem with $u(x,0) = g(x)$, one has

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)d^n y$$

The general problem on a domain Ω in $\mathbb{R}^n$ is

$$\begin{cases} u_t(X,t) = k\sum_{i=1}^{n} u_{x_i x_i}(X,t), \quad X\epsilon\,\Omega\; 0 < t < \infty \\ u(X,0) = \delta(X) \end{cases}$$

With Dirichlet boundary data. A Green's function always exists, but unless the domain Ω can be readily decomposed into one-variable problems , it may not be possible to write it down explicitly. The method of images provides one additional technique for obtaining Green's functions for non-trivial domains. Some Green's function solutions in 1D. A variety of elementary Green's function solutions in one-dimension are recorded here. In some of these, the spatial domain is the entire real line$(-\infty,\infty)$. In others, it is the semi-infinite interval $(0,\infty)$ with Dirichlet boundary conditions.

Initial value problem on $(-\infty,\infty)$.

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t), \qquad -\infty < x < \infty\,, 0 < t < \infty \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} exp(-\frac{(x-y)^2}{4kt})g(y)dy$$

Initial value problem on $(0,\infty)$ with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t), \qquad 0 \le x < \infty\,, 0 < t < \infty \\ u(x,0) = g(x) \\ u(0,t) = 0 \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{0}^{\infty} (exp\left(-\frac{(x-y)^2}{4kt}\right) - exp\left(-\frac{(x+y)^2}{4kt}\right)g(y)dy)$$

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