College of Natural Science
Department of Mathematics

Graduate Project Report on The Method of Characteristics and Classical Solutions Of First Order PDEs

Submitted in partial fulfilment of the requirements for the Degree of Master of Science in Mathematics

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Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **The Method of Characteristics and Classical Solutions of first order PDEs** by Surafel Abiy in partial fulfillment of the requirements for the degree of master of Science.

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Introduction

One of the main results of the classical theory of first-order partial differential equations (PDEs) is the characteristic method which asserts that under certain assumptions the Cauchy problem can be reduced to the corresponding characteristic system of ordinary differential equations (ODEs). To illustrate this, let us consider the Cauchy problem for the nonviscid Burger equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, t > 0, x\epsilon R \tag{1}$$

$$u(0, x) = h(x), x\epsilon R \tag{2}$$

We try to reduce the problem $(1)-(2)$ to an ODE along some curve $x = x(t)$. More precisely, let us find $x = x(t)$ such that

$$\frac{d}{dt}u(t, x(t)) = \frac{\partial u}{\partial t}(t, x(t)) + u(t, x(t))\frac{\partial u}{\partial x}(t, x(t))$$

By the chain rule, we may simply require $\frac{dx}{dt} = u$, and so the characteristics $x = x(t)$ can be defined by

$$\frac{dx}{dt} = u(t, x) \tag{3}$$

Along each characteristic $x = x(t)$ we have $du/dt = 0$, i.e., $u = u(t, x(t))$ takes a constant value and then the characteristic must be a straight line with slope given by(3).

Thus, by the initial data (2), the characteristic passing through any given point $(0, s)$ on the $x-$axis is

$$x = s + h(s)t \tag{4}$$

on which u has the constant value:

$$u = h(s) \tag{5}$$

Hence, if the C^1-norm of $h = h(s)$ is bounded, then, by means of the implicit function theorem and (4), we can get

$$s = s(t, x) \tag{6}$$

for small values of t. Substituting (6) into (5) gives the classical solution (C^1- solution)

$$u = h(s(t,x)) \tag{7}$$

to our Cauchy problem $(1) - (2)$. However, in general, this solution exists only locally in time. In fact, if $h = h(s)$ is not a nondecreasing function of s, there exist two points $(0, s_1)$and$(0, s_2)$ on the x-axis such that

$$s_1 < s_2 \quad and \quad h(s_1) > h(s_2) \tag{8}$$

Then the characteristic curves beginning from $(0, s_1)$ and $(0, s_2)$ will intersect at time

$$t = \frac{s_2 - s_1}{h(s_1) - h(s_2)}$$

Since the solution $u = u(t,x)$ is constant along each of the two curves but has different values $h(s_1)$ and $h(s_2)$, respectively, at the intersection point, the value of the classical solution cannot be uniquely determined. Hence, in this case the Cauchy problem $(1) - (2)$ never admits a global classical solution on $\{t \geq 0\}$; in fact, the classical solution will blow up in a finite time no matter how smooth and small the initial data $h = h(s)$ are. On the other hand, if $h = h(s)$ is a nondecreasing function of s, then the characteristics emanating from distinct points $(0, s_1)$ and $(0, s_2)$ on the $x - axis$ will not intersect, and thus the solution $u = u(t,x)$ will exist globally for $t \geq 0$. The previous example shows, generally speaking, that for (first-order) nonlinear partial differential equations or systems, classical solutions to the Cauchy problem exist only locally in time, while singularities may occur in a finite time, even if the initial data are sufficiently smooth and small.

Chapter 1

The method of characterstic,Local Theory

Partial differential equations of first-order have been studied from various points of view: for example, classical mechanics, variational method, geometrical optics, etc. In this chapter we will always suppose that the equations and solutions are real valued. The classical method to solve the equations is the characteristic method. As this is the fundamental tool in the subsequent discussions, we will give here a brief explanation of the method.

1.1 Characterstic curves and system of ODES

First we consider a quasi-linear partial differential equation of first-order :

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} a_i(t,x,u)\frac{\partial u}{\partial x_i} = a_0(t,x,u) \quad in \quad \cdot U, \tag{1.1}$$

$$u(0,x) = \phi(x) \quad on \quad u_0 = \{x\varepsilon R^n : (0,x)\varepsilon U\} \tag{1.2}$$

where U is an open neighborhood of $(t,x) = (0,0)$. Let V be an open neighborhood of $\{(0,x,\phi(x)) : x\epsilon U_0\}$ in R^{n+2}. Assume that $a_i = a_i(t,x,u) \quad (i = 0,1,...,n)$ and $\phi = \phi(x)$ are of class C^1 in V and U_0, respectively. A function is said to be of class C^k if it is $k-$times continuously differentiable, and $C^k(U)$ is the family of functions being of class C^k in U. A C^k-function means that it is a function of class C^k Characteristic curves of $(1.1)-(1.2)$ are defined by solution curves of the following system of ordinary differential equations:

$$\begin{cases} \frac{dx}{dt} = a_i(t,x,v) & (i = 1,2,...,n) \\ \frac{dv}{dt} = a_0(t,x,v) \end{cases} \tag{1.3}$$

In accordance with (1.2), the initial condition for (1.3) is given by

$$x_i(0) = y_i(i = 1, 2, ..., n), v(0) = \phi(y) \tag{1.4}$$

The ordinary differential equations in (1.3) are called the "characteristic equations" for (1.1), where we use $v = v(t, y)$ instead of $u = u(t, x)$ to avoid confusion. In the following discussions, $u = u(t, x)$ is a solution of (1.1) and $v = v(t, y)$ is a solution of $(1.3) - (1.4)$ which is equal to the value of $u = u(t, x)$ restricted on the corresponding solution curve $x = x(t, y)$ of $(1.3) - (1.4)$. As $a_i = a_i(t, x, v)(i = 0, 1, ..., n)$ and $\phi = \phi(x)$ are of class C^1 in V and U_0, respectively, the Cauchy problem $(1.3) - (1.4)$ has a system of solutions $x_i = x_i(t, y)(i = 1, 2, ..., n)$ and $v = v(t, y)$ which are of class C^1 in a neighborhood of $\{(0, y) : y\epsilon U_0\}$. Let us fix our notations on derivatives of functions. A vector x is n-dimensional column vector, i.e.,$x = (x_1, x_2, ..., x_n)$ Therefore $dx/dt = (\frac{dx_1}{dt}, \frac{dx_2}{dt}..., \frac{dx_n}{dt})^t$. On the other hand, given any real-valued function $\phi = \phi(x)$, we write grad $\phi(x) = \frac{\partial\phi}{\partial x} = \phi^1(x) = (\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}..., \frac{\partial\phi}{\partial x_n})$. For an n-vector valued function $x = x(y)$ of an n-vector y, we define its Jacobi matrix and Jacobian, respectively, by

$$(\frac{\partial x_i}{\partial y_j})_{i,j=1,2,...,n} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

and

$$\frac{D_x}{D_y}(y) = det(\frac{\partial x_i}{\partial y_j})_{i,j=1,2,...,n}$$

We will sometimes write the Jacobi matrix simply by $\frac{\partial x_i}{\partial y_j}$ Since $x(O, y) = y$, we see that $\frac{D_x}{D_y}(t, y) = 1$ for $t = 0, y\epsilon U_0$.

1.2 Existence of solution of the cauchy problem

In a neighborhood of $\{(0, y) : y\epsilon U_0\}$, as $\frac{D_x}{D_y}(t, y)$ does not vanish, we can uniquely solve the equation $x = x(t, y)$ with respect to y and write the solution by $y = y(t, x)$. Putting $u(t, x) = v(t, y(t, x))$, we will prove that $u = u(t, x)$ satisfies $(1.1) - (1.2)$ in a neighborhood of the origin.

Theorem 1.2.1. *The Cauchy problem* $(1.1) - (1.2)$ *has a unique solution of class* C^1 *in a neighborhood of the origin.*

Proof. We use the notations introduced in the above. The following discussions are true only in the definition domain of $y = y(t,x)$. This domain is a neighborhood of the origin, where the Jacobian $\frac{D_x}{D_y}(t,y)$ does not vanish. As $x = x(t, y(t,x))$, we have

$$(\frac{\partial x_i}{\partial y_j})_{i,j=1,2,...n}.(\frac{\partial y_i}{\partial x_j})_{i,j=1,2,...,n} = I \qquad (Identity \quad matrix) \tag{1.5}$$

and

$$\frac{\partial x}{\partial t} + (\frac{\partial x_i}{\partial y_j})_{i,j=1,2,...n}.\frac{\partial y}{\partial t} = 0 \tag{1.6}$$

As $u(t,x) = v(t, y(t,x))$,we have

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}(t,y) + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial t}(t,x) \tag{1.7}$$

By (1.7),using (1.5) and (1.6), we get

$$\frac{\partial u}{\partial t}(t,x) = a_0(t,x,u(t,x)) - \frac{\partial v}{\partial y}.(\frac{\partial y_i}{\partial x_j})_{1\leq i,j \ n}\frac{\partial x}{\partial t}$$

$$= a_0(t,x,u(t,x)) - \frac{\partial u}{\partial x}(t,x).\frac{\partial x}{\partial t}$$

$$= a_0(t,x,u(t,x)) - \sum_{i=1}^{n} a_i(t,x,u).\frac{\partial u}{\partial x_i}$$

As $u(0,x) = v(0, y(0,x)) = \phi(x)$, we see that $u = u(t,x)$ satisfies the Cauchy problem $(1.1) - (1.2)$ in the aforementioned neighborhood of the origin.

1.3 Uniqueness of solution of the Cauchy problem

We will show the uniqueness of solutions. Let $u = u(t,x)$ be any solution of class C^1 of$(1.1) - (1.2)$, and put $\kappa(t,y) = u(t, x(t,y))$ where $x = x(t,y)$ and $v = v(t,y)$ are the solutions of $(1.3) - (1.4)$. Then the difference $\omega(t,y) = \kappa(t,y) - v(t,y)$ satisfies the following Cauchy problem:

$$\begin{cases} \frac{d}{dt}\omega(t,y) = \sum_{j=1}^{n}(a_j(t,x,v) - a_j(t,x,\kappa))\frac{\partial u}{\partial x_j} + (a_0(t,x,\kappa) - a_0(t,x,v)) \\ \omega(0,y) = 0 \end{cases}$$

As the right-hand side of this differential equation can be estimated by $M \mid \kappa - v \mid = M \mid \omega \mid$, we get $\omega(t,y) \equiv 0$ i.e $\kappa(t,y) \equiv v(t,y)$ for any (t,y) in a neighborhood of the origin. This means that the solution of C^1-class is unique along the curves $x = x(t, y)$. That is to say, as long as the Jacobian $(\frac{D_x}{D_y})(t, y)$ does not vanish, the solution of $(1.1) - (1.2)$ is unique in the C^1-space.

1.4 c^2-solution of the Cauchy problem

Next we consider the Cauchy problem for general partial differential equations of first-order,

$$\frac{\partial u}{\partial t} + f(t, x, u, \frac{\partial u}{\partial x}) = 0 \qquad in \qquad U \tag{1.8}$$

$$u(0, x) = \phi(x) \quad on \quad U_0 = \{x \epsilon R^n : (0, x) \epsilon U\} \tag{1.9}$$

where U is an open neighborhood of the origin. Let V be an open neighborhood of $\{(0, x, \phi(x), \phi'(x)) : x\epsilon U_0\}$ in $R \times R^n \times R \times R^n$.Assume that $f = f(t, x, u, p)$ and $\phi = \phi(x)$ are of class C^2 in V and U_0, respectively.
Characteristic strips for $(1.8) - (1.9)$ are defined as solution curves of the following system of ordinary differential equations:

$$\begin{cases} \frac{dx_i}{dt} = \frac{\partial f}{\partial p_i}(t, x, v, p)(i = 1, 2, ..., n), \\ \frac{dv}{dt} = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i}(t, x, v, p) - f(t, x, v, p), \\ \frac{dp_i}{dt} = \frac{\partial f}{\partial x_i}(t, x, v, p) - p_i \frac{\partial f}{\partial u}(t, x, v, p)(i = 1, 2..., n), \end{cases} \tag{1.10}$$

with

$$x_i(0) = y_i, v(0) = \phi(y), p_i(0) = \phi'(y)(i = 1, 2, ..., n) \tag{1.11}$$

We remark that system (1.10) is called the "characteristic system of differential equations," or simply "characteristic equations," for equation (1.8). As f and ϕ are of class C^2, the Cauchy problem $(1.10) - (1.11)$ has uniquely the solutions $x = x(t, y), v = v(t, y)$ and $p = p(t, y)$ in a neighborhood of $t = 0$. Moreover, they are of class C^1 with respect to (t, y). As $x(0, y) = y$, we have $\frac{Dx}{Dy}(0, y) = 1$ for any $y\epsilon U_0$. Therefore, there exists an open neighborhood W of $\{(0, y) : y\epsilon U_0\}$ where the Jacobian $\frac{Dx}{Dy}(t, y)$ does not vanish and the equation $x = x(t, y)$ can be uniquely solved with respect to y. Denote the solution by $y = y(t, x)$, and put $u(t, x) = v(t, y(t, x))$. We will prove that $u = u(t, x)$ is of class C^2 ,and that it satisfies the Cauchy problem $(1.8) - (1.9)$. For this aim, we prepare some lemmas.

Lemma 1.4.1. *For all* (t, y) *in the existence domain of solutions to* $(1.10) - (1.11)$*, we have*

$$\frac{\partial v}{\partial y}(t, y) = p(t, y) \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix} \tag{1.12}$$

where $p(t, y) = (p_1(t, y), p_2(t, y), ..., p_n(t, y))$

proof,we put

$$z(t, y) = \frac{\partial v}{\partial y}(t, y) = -p(t, y) \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial [] x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

using(1.10),we have

$$\begin{cases} \frac{d}{dt} z(t, Y) = -\frac{\partial f}{\partial u}(t, x(t, y), v(t, y), p(t, y)) z(t, y) \\ z(0, y) = 0 \end{cases}$$

As this is a linear ordinary differential equation concerning $z = z(t, y)$, we get $z(t, y) \equiv 0$

Remark. Fix any $y \epsilon U_0$, and let $J \subset R$ be an interval around 0 on which the solutions $x = x(t, y), v = v(t, y)$ and $p = p(t, y)$ of the characteristic equations $(1.10) - (1.11)$ exist. Then (1.12) is true for each $t \in J$ even if the Jacobian may vanish at (t, y).

Recall that in W we have $(\frac{Dx}{Dy})(t, y) \neq 0$, and we can uniquely solve the equation $x = x(t, y)$ with respect to y. The solution has been denoted by $y = y(t, x)$ and used to define $u(t, x) = v(t, y(t, x))$.

Corollary 1.4.1. *In the domain of* $y = y(t, x)$*, we have*

$$\frac{\partial u}{\partial x_i}(t, x) = p_i(t, y(t, x))(i = 1, 2, ..., n) \tag{1.13}$$

Using Lemma 1.4.1 and its corollary, we get the following

Theorem 1.4.1. *Suppose* $f \in C^2$ *and* $\phi \in C^2$*. Then the Cauchy problem* $(1.8) - (1.9)$ *has a unique solution of class* C^2 *in a neighborhood of the origin.*

Proof. We first prove the existence of solutions. Let $u(t,x) = v(t, y(t,x))$ as in the above notations. By Lemma 1.4.1 and Corollary 1.4.1, using (1.10), we have

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial v}{\partial y}(t, y(t,x)) + \frac{\partial v}{\partial t}(t, y(t,x))\frac{\partial y}{\partial t}(t,x)$$

$$= \sum_{i=1}^{n} p_i(t, y(t,x))\frac{\partial f}{\partial p_i} - f(t,x,v,p) - p(t,y(t,x))(\frac{\partial x_i}{y_j})_{1\leq i,j\leq n}.\frac{\partial y}{\partial t}$$

$$= \sum_{i=1}^{n} p_i(t, y(t,x))\frac{\partial f}{\partial p_i} - f(t,x,v,p) - p(t,y(t,x)).\frac{\partial y}{\partial t}$$

$$= -f(t,x,v(t,y(t,x)),p(t,y(t,x)))$$

$$= -f(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x))$$

Moreover, by $(1.11), u(0,x) = v(0,y(0,x)) = \phi(x)$. It follows that $u = u(t,x)$ is a solution of $(1.8) - (1.9)$. As $p_i = p_i(t,y)(i = 1,2,...,n)$ are of class C^1 in W and $y = y(t,x)$ is of class C^1 in its definition domain, we see by (1.13) that $u = u(t,x)$ is of class C^2 in the definition domain of $y = y(t,x)$. This domain is actually a neighborhood of the origin.

Finally, we give the sketch of a proof of the uniqueness of solutions. Let $u = u(t,x)$ be a solution of class C^2 of $(1.8) - (1.9)$, and $z = z(t,y) = (z_1(t,y),...,z_n(t,y))$ be a solution of

$$\begin{cases} \frac{dz_i}{dt} = \frac{\partial f}{\partial p_i}(t,z,u(t,z),\frac{\partial u}{\partial x}(t,z))(i = 1,2,...,n) \\ z_i(0) = y_i \end{cases}$$

We put $\omega(t,y) = u(t,z(t,y))$ and $q(t,y) = \frac{\partial u}{\partial x}(t,z(t,y))$, then we get

$$\begin{cases} \frac{d\omega}{dt} = -f(t,z,\omega,q) + \sum_{i=1}^{n} q_{i(t,y)}\frac{\partial f}{\partial p_i}(t,z,\omega,q), \\ \frac{dq_i}{dt} = -\frac{\partial f}{\partial x_i}(t,z,\omega,q) - q_i(t,y)\frac{\partial f}{\partial u}(t,z,\omega,q) \qquad (i = 1,2,...,n) \end{cases}$$

with $\omega(0,y) = \phi(y)$ and $q(0,y) = \phi^{'}(y)$. Hence

$$(z,\omega,q) = (z(t,y),\omega(t,y),q(t,y))$$

satisfies the Cauchy problem $(1.10) - (1.11)$. By the uniqueness of solutions of ordinary differential equations, we have

$$x(t,y) = z(t,y) \qquad v(t,y) = \omega(t,y) \qquad p(t,y) = q(t,y)$$

where $(x,v,p) = (x(t,y),v(t,y),p(t,y))$ is the solution of $(1.10)-(1.11)$ which has already appeared in Lemma 1.4.1. This says that the solution of class C^2 is unique along the curves $x = x(t,y)$. Therefore, as long as the Jacobian $(\frac{D_x}{D_y})(t,y)$ does not vanish, the solution of class C^2 of $(1.8) - (1.9)$ is unique.

Chapter 2

Classical Solutions and Global Existence

As we have shown in Chapter 1, the Cauchy problems $(1.1)-(1.2)$ and $(1.8){-}(1.9)$ have locally classical solutions. It is well-known that the solutions may generally have singularities in finite time even for smooth initial data.

2.1 Classical solution of a conservation Law

For some equations of the conservation law, the life spans of classical solutions have been exactly calculated, The principal aim of this chapter is to determine the life spans of classical solutions of general partial differential equations of first-order. Moreover, using the results on the life spans of classical solutions, we will give necessary and sufficient conditions which guarantee the global existence of classical solutions for (1.1)-(1.2) and (1.8)- (1.9). As an example, we consider a simple equation of the conservation law as follows:

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}=0 \qquad in\{t>0, x\in R^n\}, \tag{2.1}$$

$$u(0,x)=\phi(x) \qquad on\{t=0, x\in R^n\}, \tag{2.2}$$

where $a_i = a_i(u)$ $(i = 1, 2, ..., n)$and $\phi = \phi(y)$ are of class c^1 in R and R^n respectively. If $n = 1$ and $a_1(u) = u$ the characteristic curves which are the solutions of equations $(1.3)-(1.4)$ are given by

$$x = y + ta(\phi(y)) \qquad v(t, y) = \phi(y) \tag{2.3}$$

where $(u) = (a_1(u), a_2(u), ..., a_n(u))$.Then the Jacobian of the mapping $x = x(t, y)$ is given by $\frac{D_x}{D_y}(t, y) = 1 + t\lambda(y)$ with $\lambda(y) = \sum_{i=1}^{n} a_i^{'}(\phi(y))(\frac{\partial\phi}{\partial y_i}(y)$

Obviously, the Jacobian $(\frac{D_x}{D_y})(t, y)$ does not vanish in a neighborhood of $(t, y) = (0, 0)$.Therefore, as we have shown in section 1.1, the Cauchy problem (2.1) − (2.2) has a unique C^1-solution $u = u(t, r)$ in a neighborhood of the origin. Since $u(t, x) = \phi(y(t, x))$ where $y = y(t, x)$ is the solution of the equation $x = x(t, y)$, we have

$$(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}) = \frac{1}{1 + t\lambda(y)}(\frac{\partial \phi}{\partial y_1}, ..., \frac{\partial \phi}{\partial y_n}) \mid_{y=y(t,x)} \tag{2.4}$$

Assume $\lambda(y^0) < 0$.Then the Jacobian $(Dx/Dy)(t_0, y_0) = 0$ for $t^0 = 1/(\lambda(y^0))$. We see by (2.4) that, when (t, x) goes to (t^0, x^0) along the curve $x = x(t, y^0)$ where $x^0 = x(t^0, y^0)$ at least one of the first derivatives of $u = u(t, x)$ tends to infinity. Therefore, if the Jacobian vanishes somewhere, then the Cauchy problem (2.1)(2.2) can not admit a global solution of class C^1. In section 2.2 and 2.3, we will give similar results on the life spans of classical solutions for general partial differential equations of first-order. What we would like to remark here is to point out that there exist some differences between quasi-linear equations and general partial differential equations. See Theorem 2.2.1 and Theorem 2.2.2. In section 2.4, we will consider the global existence of classical solutions. These existence results are corollaries of theorems given in section 2.2 and 2.3.

2.2 Life spans of classical solutions

Let us consider a quasi-linear partial differential equation of first-order :

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} a_i(t, x, u)\frac{\partial u}{\partial x_i} = a_0(t, x, u) \quad in \quad \{t > 0, x \in R^n\} \tag{2.5}$$

$$u(0, x) = \phi(x) \quad on \quad \{t = 0, x \in R^n\} \tag{2.6}$$

where $a_i = a_i(t, x, u)(i = 0, 1, ..., n)$ and $\phi = \phi(y)$ are of class C^1 in $R \times R^n \times R \quad and \quad R^n$, respectively. The characteristic equations for (2.5)−(2.6) are written by

$$\begin{cases} \frac{dx_i}{dt} = a_i(t, x, v) \\ \frac{dv}{dt} = a_0(t, x, v) \end{cases} \tag{2.7}$$

with the initial conditions

$$x_i(0) = y_i(i = 1, 2, ..., n), v(0) = \phi(y) \tag{2.8}$$

We write the solutions of (2.7) − (2.8) by $x = x(t, y)$ and $v = v(t, y)$. Then $v = v(t, y)$ means the value of a solution of (2.5) restricted on the curve

$x = x(t,y)$. Here we assume the following condition:
(A_1) The Cauchy problem $(2.7) - (2.8)$ has a unique global solution $x = x(t,y), v = v(t,y)$ on $t \geq 0$ for any $y \in R^n$ It is not easy to write down sufficient conditions which guarantee the assumption (A_1). When we assume (A_1), we get a smooth mapping $x = x(t,y)$ from R^n to R^n for each $t \geq 0$. The life span of classical solutions of $(2.5) - (2.6)$ is determined by the following:

Theorem 2.2.1. *Under Condition (A_1), suppose that $(D_x/D_y)(t^0, y^0) = 0$ and $(D_x/D_y)(t, y^0) \neq 0$ for $t < t^0$. Then at least one of the first derivatives of the solution $u = u(t,x)$ tends to infinity when t goes to t_0 along the curve $x = x(t, y^0)$.*

Proof
Let us put $L = \{(t,y^0) : 0 \leq t < t^0\}$ and $C_0 = \{(t,x) : x = x(t,y^0), 0 \leq t < t^0\}$. By the assumption and by the theorem of inverse functions, we can get an open neighborhood ν of L so that the Jacobian $(D_x/D_y)(t,y)$ does not vanish on $V = \nu \cap \{0 \leq t < t^0\}$ and that the equation $x = x(t,y)$ can be uniquely solved with respect to y for any (t,x) in $U = (t,x) : x = x(t,y), (t,y) \in V$. Let us write the solution by $y = y(t,x)$ and define $\{u(t,x) = v(t, y(t,x))\}$. We already proved in section 1.1 that $u = u(t,x)$ is a C^1-solution of $(2.5)-(2.6)$ in the domain U. The uniqueness of C^1- solution of $(2.5) - (2.6)$ is assured by Theorem 1.1. Here we have,

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial v}{y_k} \frac{\partial y_k}{x_i} \quad in \quad U, \tag{2.9}$$

$$\frac{\partial v}{\partial y_j} = \sum_{k=1}^{n} \frac{\partial u}{x_k} \frac{\partial x_k}{y_j} \quad in \quad V$$

On the other hand, from (2.7), $\partial x_i/\partial y_j$ *and* $\partial v/\partial y_j$ $(i,j = 1,2,...,n)$ satisfy the following system of linear equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial x_i}{\partial y_j}\right) = \sum_{k=1}^{n} \frac{\partial a_i}{\partial x_k} \frac{\partial x_k}{\partial y_j} + \frac{\partial a_i}{\partial v} \frac{\partial v}{\partial y_j} \\ \frac{d}{dt}\left(\frac{\partial v}{\partial y_j}\right) = \sum_{k=1}^{n} \frac{\partial a_0}{\partial x_k} \frac{\partial x_k}{\partial y_j} + \frac{\partial a_0}{\partial v} \frac{\partial v}{\partial y_j} \end{cases} \tag{2.10}$$

with the initial data

$$\frac{\partial x_i}{\partial y_j}(0) = \begin{cases} 1, & if \quad i = j, \\ 0, & if \quad i \neq j, \end{cases} \quad and \qquad \frac{\partial v}{\partial y_j}(0) = \frac{\partial \phi}{\partial y_j}(y)$$

By the linearity of (2.10) we have

$$rank\begin{pmatrix} \frac{\partial x_1}{\partial y}(t) \\ ... \\ ... \\ ... \\ \frac{\partial x_n}{\partial y}(t) \\ \frac{\partial v}{\partial y}(t) \end{pmatrix} = rank\begin{pmatrix} \frac{\partial x_1}{\partial y}(0) \\ ... \\ ... \\ ... \\ \frac{\partial x_n}{\partial y}(0) \\ \frac{\partial v}{\partial y}(0) \end{pmatrix} = n, \forall t \geq 0 \tag{2.11}$$

where we write $\partial x_i/\partial y = (\partial x_i/\partial y_1, ..., \partial x_i/\partial y_n)$ and $\partial v \partial y = (\partial v/\partial y_1, ..., \partial v/\partial y_n)$. Assume that all the components of $\partial u/\partial x$ remain bounded along the curve C_0,then we can pick up a sequence $\{t^m\}_m \subset [0, t^0)$ and a $c = (c_1, c_2, ...c_n) \in R^n$ such that

$$1) \lim_{m\to\infty} t^m = t^0, and$$

$$2) \lim_{m\to\infty} \frac{\partial u}{\partial x}(t^m, x(t^m, y^0)) = c$$

From (2.9) we get

$$\frac{\partial v}{\partial y}(t^0, y^0) = \sum_{k=1}^{n} c_k \frac{\partial x_k}{\partial y}(t^0, y^0) \tag{2.12}$$

As $(D_x/D_y)(t^0, y^0) = 0$, (2.12) contradicts (2.11). This means that at least one component of $(\partial u/\partial x)(t, x(t, y^0))$ tends to infinity when $t \to t^0$ along C_0. Next we consider general partial differential equations of first-order,

$$\frac{\partial u}{\partial t} + f(t, x, u, \frac{\partial u}{\partial x}) = 0, in \qquad \{t > 0, x \in R^n\}, \tag{2.13}$$

$$u(0, x) = \phi(x), on \qquad \{t = 0, x \in R^n\}, \tag{2.14}$$

where $f = f(t, x, u, p)$ and $\phi = \phi(y)$ are of class C^2 in $R \times R^n \times R \times R^n$ and R^n, respectively. The characteristic equations for $(2.13) - (2.14)$ are written by

$$\begin{cases} \frac{dx_i}{dt} = \frac{\partial f}{\partial p_i}(t, x, v, p) & (i = 1, 2, ..., n) \\ \frac{dv}{dt} = \sum_{j=1}^{n} p_j \frac{\partial f}{\partial p_j}(t, x, v, p) - f(t, x, v, p) \\ \frac{dp_i}{dt} = -\frac{\partial f}{\partial x_i}(t, x, v, p) - \frac{\partial f}{\partial u}(t, x, v, p) & (i = 1, 2, ..., n) \end{cases} \tag{2.15}$$

with

$$x_i(0) = y_i, v(0) = \phi(y), p_i(0) = \frac{\partial \phi}{\partial y_i}, \qquad (i = 1, 2, ..., n) \tag{2.16}$$

We assume here the following condition:
(A_2) The Cauchy problem $(2.15) - (2.16)$ has a unique global solution on $\{t \geq 0\}$ for any $y \in R^n$.
We denote the solution of $(2.15) - (2.16)$ by $x = x(t, y)$, $v = v(t, y)$, and $p = p(t, y)$. As $(D_x/D_y)(O, y) = 1$ for all $y \in R^n$, the Jacobian does not vanish in a neighborhood of {t = 0}. Therefore we can uniquely solve the equation $x = x(t, y)$ with respect to y, and denote it by $y = y(t, x)$. Define u(t,x)=v(t,y(t,x)) Then $u = u(t, x)$ is a C^2-solution of $(2.13) - (2.14)$ in a neighborhood of $t = 0$. Moreover, we can see by Theorem 1.4 that there does not exist another C^1-solution of $(2.13) - (2.14)$ in a neighborhood of $\{t = 0\}$. When we extend this solution for large t, we get the following.

Theorem 2.2.2. *Under Condition (A_2), suppose that $(D_x/D_y)(t^0, y^0) = 0$ and $(D_x/D_y)(t, y^0) \neq \quad 0 for \quad t < t^0. Then \sum_{i,j=1}^{n} |\frac{\partial^2 u}{\partial x_i \partial x_j}(t, x)|$ tends to infinity when t goes to t^0 along the curve $x = x(t, y^0)$*

Proof
. As the proof is similar to that of Theorem 2.2.1, we use the same notations introduced there. Put $L = \{(t, y^0) : 0 \leq t < t^0\}$ Next we choose an open neighborhood ν of L so that the Jacobian $(D_x/D_y)(t, y)$ does not vanish on $V = \nu \cap \{0 \leq t < t^0\}$, and that the equation $x = x(t, y)$ can be uniquely solved with respect to y for any (t, x) in $U = \{(t, z) : x = x(t, y), (t, y) \in V\}$ (theorem of inverse functions). We write the solution by $y = y(t, x)$. Here we define $u(t, x) = v(t, y(t, x))$. Then $u = u(t, x)$ is a C^2-solution of $(2.13) - (2.14)$ in the domain U.
Our aim is to show that, when t goes to t^0 along the curve $x = x(t, y^0)$, at least one of $\{\partial u/\partial x_i, \partial^2 u/\partial x_i \partial x_j : i, j = 1, 2, ..., n\}$ tends to infinity.
Differentiating (2.15) with respect to y_j, $(j = 1, 2, ..., n)$, we get a system of linear ordinary differential equations of first-order concerning
$\partial x_i/\partial y_j, \partial v/\partial y_j, \partial p_i/\partial y_j : i, j = 1, 2, ..., n$ like (2.10). As this system of equations is linear, we have,

$$rank \begin{pmatrix} \partial x_1/\partial y(t) \\ ... \\ \partial x_n/\partial y(t) \\ \partial v/\partial y(t) \\ \partial p_1/\partial y(t) \\ ... \\ \partial p_n/\partial y(t) \end{pmatrix} = rank \begin{pmatrix} \partial x_1/\partial y(0) \\ ... \\ \partial x_n/\partial y(0) \\ \partial v/\partial y(t) \\ \partial p_1/\partial y(0) \\ ... \\ \partial p_n/\partial y(0) \end{pmatrix} = n \quad for \quad any \quad t \geq 0 \tag{2.17}$$

since (2.17) is true at $(t, y) = (t_0, y_0)$, we can choose n vectors in
$\{\partial x_1/\partial y, ...\partial x_n/\partial y, \partial v/\partial y, \partial p_1/\partial y, ...\partial p_n/\partial y\}$ such that they are linearly in-

dependent at $(t,y)=(t_0,y_0)$. We denote them by $a_i(t,y)=(\partial/\partial y)b_i(t,y)(i=1,2,...,n)$, where $b_i(t,y)$ is one of $\{x_k(t,y), v(t,y), p_k(t,y) : k=1,2,...,n\}$. As $(D_x/D_y)(t^0,y^0)=0$, we can find $v(t,y)$ or some $p_k(t,y)$ in $\{b_1(t,y),...,b_n(t,y)\}$. Here we recall Corollary 1.4.1 :

$$\frac{\partial u}{\partial x_j}(x,t)=p_j(t,y(t,x)) \qquad in \quad U \qquad (j=1,2,...,n) \tag{2.18}$$

On the other hand,we get in U,

$$[\frac{\partial}{\partial x_j}b_i(t,y(t,x))]_{1\le i,j\le n}=\begin{pmatrix} a_1(t,y)\\ a_2(t,y)\\ ...\\ ...\\ a_n(t,y)\end{pmatrix}[\frac{\partial y_i}{\partial x_j}(t,x)]_{1\le i,j\le n}$$

$$=\begin{pmatrix} a_1(t,y)\\ a_2(t,y)\\ ...\\ ...\\ a_n(t,y)\end{pmatrix}([\frac{\partial x_i}{\partial y_j}(t,y)]_{1\le i,j\le n})^{-1}$$

Since $\{a_1(t,y), a_2(t,y),...,a_n(t,y)\}$ are linearly independent in a neighborhood of (t^0,y^0) and $(D_x/D_y)(t^0,y^0)=0$,it follows that at least one component of the matrix $[\partial/\partial x_j b_i(t,y(t,x))]_{1\le i,j\le n}$ tends to infinity when t goes to t^0 along the curve $x=x(t,y^0)$. But we see by (A_2) and (2.18) that

$$\frac{\partial}{\partial x_j}v(t,y(t,x))|_{x=x(t,y^0)}=\frac{\partial u}{\partial x_j}(t,x(t,y^0))=p_j(t,y^0),(j=1,2,...,n)$$

remain bounded (though the Jacobian vanishes); therefore, some $p_k(t,y)$ must be contained in $\{b_1(t,y), b_2(t,y),...b_n(t,y)\}$ Hence we get Theorem 2.2.2.

Theorem 2.2.2 says that, if the Jacobian vanishes at a point (t^0,y^0), then the second derivatives of classical solutions blow up at time $t=t^0$. But this does not prevent the existence of C^1-solutions even if the Jacobian may vanish. To understand this situation, we need to know the behavior of characteristic curves in a neighborhood of the point where the Jacobian vanishes. After having studied this subject, we will again consider the extension of classical solutions in the next Chapter.

2.3 Global existence of classical solutions

As we have shown in section 1.1, the Cauchy problem for a partial differential equation of first-order has locally a classical solution. We have also seen in

section 2.2 that the solution may generally have singularities in finite time even for smooth initial data. But in some cases the Cauchy problem admits a global classical solution. In this section, we will give necessary and sufficient conditions which guarantee the global existence of classical solutions.
Now we review the results on the diffeomorphism of the Euclidean space R^n. Let H be a C^1-mapping from R^n to R^n defined on the whole space. Then the mapping H is said to be proper if $H^{-1}\{K\}$is compact whenever K is compact. We have:

Lemma 2.3.1. *The mapping H is a diffeomorphssm from R^n to R^n if and only if H is proper and the Jacobian of H does not vanish anywhere on the whole space.*

First we consider the Cauchy problem for a quasi-linear equation, say Problem $(2.5) - (2.6)$. We assume the following conditions. (Here and subsequently,
$|x|$denotes the Euclidean norm of $x \in R^n$)
(A_3) For any $y \in R^n$ the Cauchy problem $(2.7) - (2.8)$ has always a unique global C^1-solution $x = x(t, y), v = v(t, y) \quad$ on $t \geq 0$.
(A_4) On the solution curves of $(2.7) - (2.8)$, it holds that

$$| a_i(t, x(t, y)), v(t, y)) | \leq M(1+ | x(t, y) |)$$

from any $t \in [0, T], y \in R^n$ and $i = 1, 2, ..., n$ where the constant M depends only on T. When Conditions (A_3) and (A_4) are satisfied, we can define, for any $t \geq 0$,a C-mapping H_t from R^n to R^n by $x = x(t, y) = H_t(y)$. Then we get:

Lemma 2.3.2. *Under the assumptions (A_3) and (A_4), the mapping is proper for any $t \geq 0$.*

proof. As we have

$$\frac{d}{dt} | x(t, y) |= \frac{1}{| x(t, y) |} \sum x_i(t, y) a_i(t, x(t, y), v(t, y))$$

$$\geq -nM(1+ | x(t, y) |),$$

we get $| x(t, y) | \geq (1+ | y |)e^{-nMt} - 1$.
Therefore $| x(t, y) | \to \infty \quad as \mid y \mid \to \infty$.This means that H_t is proper.
Using lemma 2.3.1,we have have the following lemma

Lemma 2.3.3. *For any fixed $t \geq 0$, the mapping H_t is a diffeomorphism from R^n to R^n if and only if its Jacobian does not vanish at any point $y \in R^n$.*

By means of Lemma 2.3.3, we obtain the following.

Theorem 2.3.1. *under the assumptions (A_3) and (A_4), the Cauchy problem* $(2.5) - (2.6)$ *has a global* C^1 *-solution uniquely on the domain* $D = \{t \geq 0, x \in R^n\}$ *if and only if the Jacobian* $(D_x/D_y)(t, y)$ *of the mapping* H_1 *does not vanish for any* $t \geq 0$ *and* $y \in R^n$.

proof.
When $(D_x/D_y)(t, y) \neq 0$ for any $y \in R^n$ and $t \geq 0$, we can see from Lemma 2.5 that H_1, is a diffeomorphism from R^n to R^n for any $t \geq 0$. Hence the inverse function $y = y(t, x)$ of $x = H_1(y)$ is a function of class C^1 defined on D and $u = u(t, x) = v(t, y(t, x))$ is the unique global C^1-solution of $(2.5)-(2.6)$. (For the uniqueness, follow the proof of Theorem 1.1)
Next, for the necessity, suppose that $(D_x/D_y)(t', y^0) = 0$ for some $t' \in (0, \infty)$ and $y^0 \in R^n$.As $(D_x/D_y)(0, y) = 1$ for all $y \in R^n$, we can get a unique C^1-solution of $(2.5) - (2.6)$ in a neighborhood of $\{t = 0\}$. Then we see by Theorem 2.2.1 that this classical solution blows up at time $t^0 = \inf\{t > 0 : D_x/D_y(t, y^0) = 0\}$.(Notice that $0 < t^0 < t'$.) Next we consider the Cauchy problem $(2.13)-(2.14)$ for general partial differential equations of first order. We assume the following hypotheses:
(A_5) For any $y \in R^n$, the Cauchy problem $(2.15)-(2.16)$ has always a unique global C^1-solution $x = x(t, y), v = v(t, y), p = p(t, y)$ on the half space $t \geq 0$.
(A_6) On the solution curves of$(2.15) - (2.16)$, it holds that

$$| \frac{\partial f}{\partial p_i}(t, x(t, y), v(t, y), p(t, y) | \leq M(1+ | x(t, y) |)$$

for any $t \in [0, T], y \in R^n$ and $i = 1, 2, ..., n$ where the constant M depends only on T.
For any $t \geq 0$, we define a C^1 -mapping H_1 from R^n to R_n by $x = x(t, y) = H_1(y)$. Then Conditions (A_5) and (A_6) guarantee that the mapping H_1 is proper. Therefore, Lemma 2.5 is also true for this H_1

Theorem 2.3.2. *Under the assumptions* (A_5) *and* (A_6), *the Cauchy problem* $(2.13) - (2.14)$ *has uniquely a global* C^2*-solutzon on the domain* $D = \{t \geq 0, x \in R^n\}$ *if and only if the Jacobian of the mapping* $x = H_1(y)$ *does not vanish for any* $t \geq 0$ *and* $y \in R^n$

proof. When $(D_x/D_y)(t, y) \neq 0$ for any $t \geq 0$ and $y \in R^n$, we can uniquely solve the equation $x = H_1(y)$ with respect to y for any $(t, x) \in D$. Let $y = y(t, x)$ be the inverse function. A global C^2 -solution of $(2.13)-(2.14)$ is given by $u = u(t, x) = v(t, y(t, x))$. It can also be deduced from the method of proof used in Theorem 1.4.1. The necessity of the above condition comes from Theorem2.2.1.

Chapter 3

Extension of classical solutions and sufficient condition for characteristic collision

This chapter is continued from Theorem 2.2 in section 2.2. Let us rewrite the equation which we will again consider here:

$$\frac{\partial u}{\partial t} + f(t, x, u, \frac{\partial u}{\partial x}) = 0, in\{t > 0, x \in R^n\}, \tag{3.1}$$

$$u(0, x) = \phi(x), on\{t = 0, x \in R^n\} \tag{3.2}$$

Where $f = f(t, x, u, p)$ and $\phi = \phi(x)$ are of class C^2 in $R \times R^n \times R \times R^n$ and R^n respectively. The characteristic equations corresponding to $(3.1) - (3.2)$ are given by $(2.15) - (2.16)$. Let $x = x(t, y), v = v(t, y)$, and $p = p(t, y)$ be solutions of $(2.15) - (2.16)$. We consider the Cauchy problem $(3.1) - (3.2)$ in the following situation:
(I) $(D_x/D_y)(t^0, y^0) = 0$, and (II) $(D_x/D_y)(t, y^0) \neq 0$ for $t < t^0$
. We put $x = x(t^0, y^0)$. Theorem 2.2.1 says that, when (t, x) goes to (t^0, x^0) along the curve $x = x(t, y^0)$, one of the second derivatives of the solution $u = u(t, x)$ of $(3.1) - (3.2)$ tends to infinity. But this does not prevent the existence of C^1-solution in a neighborhood of the point (t^0, x^0). Our problem is to see whether or not we can extend the classical solution $u = u(t, x)$ beyond the time t^0. On the other hand, we will show later that, if the characteristic curves meet in a neighborhood of (t^0, x^0), then the Cauchy problem $(3.1) - (3.2)$ cannot admit a classical solution there. Therefore, it is necessary for us to consider whether or not the characteristic curves meet in a neighborhood of (t^0, x^0), i.e., whether or not there exist two points y_1 and y_2 $(y_1 \neq y_2)$ satisfying $x(t, y_1) = x(t, y_2)$ for some t. In section 3.1,

we will give two examples in which characteristic curves do not meet though the Jacobian vanishes. In section 3.2, we will consider the case where we can extend classical solutions of (3.1) – (3.2) beyond a point where the Jacobian vanishes. In section 3.3 and section 3.4, we will give sufficient conditions so that the characteristic curves meet in a neighborhood of the point (t^0, x^0).

3.1 Illustration

Example 3.1.1. *We consider the Cauchy problem for a quasi-linear partial differential equation*

$$\begin{cases} \frac{\partial u}{\partial t} + a(t,u)\frac{\partial u}{\partial x} = u, & in \quad \{t > 0, x \in R^1\} \\ u(0,x) = x, & on \quad \{t = 0, x \in R^1\} \end{cases} \tag{3.3}$$

where $a(t,u) = \alpha'(t)e^{-t}u + \beta'(t)e^{-3t}u^3$ *and the two functions* $\alpha = \alpha(t), \beta = \beta(t)$ *satisfy the following conditions:*
1)$\alpha = \alpha(t)$ *and* $\beta = \beta(t)$ *are in* $C^1(R^1)$
2) $\alpha(t) \geq 0$ *for each* t, $\alpha(0) = 1$ *,and* $\alpha(t) = 0$ *for all* $t \geq K =$ *constant* > 0.
3) $\beta(t) \geq 0$ *for each* t, $\beta(0) = 0$.
4)$\alpha(t) + \beta(t) \neq 0$ *for all* $t \in R^1$
Then the characteristic curves for (3.3) *are written by*

$$x = x(t,y) = \alpha(t)y + \beta(t)y^3 \qquad and \qquad v = v(t,y) = e^t y \tag{3.4}$$

from which we easily see that

$$\frac{D_x}{D_y}(t,y) = \alpha(t) + 3\beta(t)y^2 \qquad and \qquad \frac{D_x}{D_y}(t,0) = 0, for \quad all \quad t \geq K$$

But we can also see from (3.4) *that the characteristic curves* $x = x(t,y)$ *do not meet for all* $t \geq 0$. *In this case the solution* $u = u(t,x)$ *is represented as*

$$u = \beta(t^{-\frac{1}{3}})e^t x^{\frac{1}{3}} \qquad for \quad t \geq K$$

This representation says that the solution contains algebraic singularity at $x = 0$, *and that the singularity of shock type does not appear though the Jacobian vanishes.*

Example 3.1.2. *We consider the following Cauchy problem:*

$$\begin{cases} \frac{\partial u}{\partial t} + f(t,u,\frac{\partial u}{\partial x}) = 0 & in \quad \{t > 0, x \in R^1\} \\ u(0,x) = \frac{1}{2}x^2 & on \quad \{t = 0, x \in R^1\} \end{cases} \tag{3.5}$$

where

$$f(t,u,p)=\frac{1}{2}\alpha'(t)e^{-t}p^2+\beta'(t)e^{-3t}p^4-u$$

and the functions $\alpha=\alpha(t),\beta=\beta(t)$ *are the same functions that we introduced in Example* 3.1.1. *This example is not of quasi-linear type, and it satisfies Conditions* (A_5) *and* (A_6) *given in section* 2.4. *The characteristic curves for* (3.5) *are written as*

$$x=x(t,y)=\alpha(t)y+4\beta(t)y^3 \qquad \textit{and} \qquad v=v(t,y)=\frac{1}{2}\alpha(t)e^ty^2+3\beta(t)e^tp^4$$

Therefore, the Jacobian $(D_x/D_y)(t,y)=\alpha(t)+12\beta(t)y^2$ *vanishes on* $L=\{(t,y):t\geq K andy=O\}.But$ $x=x(t,y)$ *is a bijective mapping defined in a neighborhood of* $y=0$ *for each* $t\geq K$. *In a neighborhood of* L, *the solution* $u=u(t,x)$ *is written by*

$$u(t,x)=const.\beta(t)^{-\frac{1}{3}}e^tx^{\frac{4}{3}} \qquad for \quad t\geq K$$

This says that the solution $u=u(t,x)$ *is of class* C^1, *but not of class* C^2, *in a neighborhood of* L.

3.2 Extension of classical solutions

As we have shown in section 3.1, there exists the case where the characteristic curves do not meet in a neighborhood of points where the Jacobian vanishes. In this case we can uniquely extend classical solutions even if the Jacobian may vanish. This is the problem which we would like to prove in this section. Now let us make clear the situation under which we consider the Cauchy problem (3.1) − (3.2).
We always assume Condition (A_5) (Chapter 2) which assures the global existence of characteristic curves. Let $x=x(t,y),v=v(t,y)$ and $p=p(t,y)$ be the solutions of (2.15) − (2.16), and define a mapping H from R^{n+1} to R^{n+1} by $H(t,y)=(t,x(t,y))$. Suppose:
(i) $(D_x/D_y)(t^0,y^0)=0$,
(ii)$(D_x/D_y)(t,y^0)\neq 0$ for $t^0<t$
In this section we consider the case where the characteristic curves do not meet. Therefore, furthermore, we assume the following condition:
(A_7) The mapping H is bijective from a neighborhood of (t^0,y^0) to another one of (t^0,z^0) where $x^0=x(t^0,y^0)$.
By Condition (A_7), we can uniquely solve the equation $x=x(t,y)$ with respect to y, and denote it by $y=y(t,x)$ (for (t,x) in a neighborhood of

$(t^0, x^0), (t, y)$ in a neighborhood of (t^0, y^0)). The function $y = y(t, x)$ is obviously continuous, though it may not be differentiable. Then we get the following.

Theorem 3.2.1. *Under the hypothesis (A_5), suppose $(i) - (ii)$ and (A_7). Then the solution $u = u(t, x)$ remains a C^1-solution of* (3.1) *in a neighborhood of (t^0, x^0), though it is not of class C^2.*

Proof.
Let V and U be open neighborhoods of (t^0, y^0) and (t^0, x^0), respectively, such that the mapping H is bijective from V to U. Consider $S = \{(t, y) \in V : (D_x/D_y)(t, y) = 0)\}$ and $H(S) = \{H(t, y) : (t, y) \in S\}$. But the Lebesgue measure of $H(S)$ is zero. Therefore $U \backslash H(S)$ is dense in U. We see that $u = u(t, x)$ is of class C^2 in the domain $U \backslash H(S)$. The reason is as follows. For any $(t\prime, x\prime) \in U \backslash H(S)$, there exists uniquely a point $(t', y') \in V$ satisfying $D_x/D_y(t', y') \neq 0$ and $x' = x(t', y')$. The inverse function $y = y(t, x)$ is of class C^1 in a neighborhood of (t', x') which is contained in $U \backslash H(S)$. Here we recall Lemma 1.4.1 and Corollary 1.4.1 in section 1.1, and we see that $u = u(t, x)$ is a function of class C^2 satisfying the equation (3.1) in the neighborhood of (t', x'). Next we show that $u = u(t, x)$ is continuously differentiable in the domain U. We pick up arbitrarily a point (t'^0, x'^0) in $H(S)$. Then we can choose a sequence $\{(t^m, x^m)\}_m$ of points in $U \backslash H(S)$ such that, when m tends to infinity, (t^m, x^m) is convergent to (t'^0, x'^0). As the mapping H is bijective from V to U, there exists a unique point $(t^m, y^m) \in V \backslash S$ satisfying $H(t^m, y^m) = (t^m, x^m)$ for each m. Since $y = y(t, x)$ is continuous in U, it follows that $y^m = y(t^m, x^m)$ is convergent to $y'^0 = (y(t'^0, x'^0)$ when m tends to infinity. As $u = u(t, x)$ is continuously differentiable at (t^m, x^m), we have by (1.13)

$$p(t^m, y^m) = \frac{\partial u}{\partial x}(t^m, x^m)$$

Because $p = p(t, y)$ is continuous on the whole half space $\{t \geq 0, y \in R^n\}$, we get

$$\lim_{m \to \infty} \frac{\partial u}{\partial x}(t^m, x^m) = \lim_{m \to \infty} p(t^m, y^m) = p(t'^0, y'^0)$$

The derivative $\partial u/\partial x$ of $u = u(t, x)$ can thereby be continuously extended (in such a way that $(\partial u/\partial x)(t, x) = p(t, y(t, x)))$ over U. That is to say, $u = u(t, x)$ is of class C^1 in the domain U.

3.3 Sufficient conditions for collision of characteristic curves

3.3.1 Characteristic curves for quasi-linear PDE of first order

For quasi-linear equations of first-order, Theorem 2.2.1 says that, if the Jacobian vanishes somewhere, the classical solutions blow up there. Therefore we cannot extend the classical solutions beyond the time t^0 when the Jacobian vanishes. This obliges us to treat weak solutions for $t > t^0$. The typical singularity of weak solutions is "shock.", the shock appears by the collision of characteristic curves. Therefore we try here to give sufficient conditions so that the characteristic curves meet after the Jacobian vanishes.
In this section we consider quasi-linear equations of first-order in one space dimension,

$$\frac{\partial u}{\partial t} + a_1(t,x,u)\frac{\partial u}{\partial x} = a_0(t,x,u) \quad in \quad t > 0, x \in R^1, \tag{3.6}$$

$$u(0,x) = \phi(x), on, t = 0, x \in R^1, \tag{3.7}$$

where $a_i = a_i(t,x,u)(i = 0,1)$ and $\phi = \phi(y)$ are of class C^1 in R^3 and R^1, respectively. The characteristic curves for $(3.6)-(3.7)$ are defined as solution curves of $(2.7) - (2.8)$ for $n = 1$. Here, as in Chapter 2, we again assume Condition $(A.I)$ which assures the global existence of characteristic curves. Let $x = x(t,y)$ and $v = v(t,y)$ be the solutions of $(2.7) - (2.8)$ for $n = 1$.

Theorem 3.3.1. *Under Condition $(A.I)$, we assume:*

$$(i) \qquad \frac{\partial x}{\partial y}(t^0, y^0),$$

and

$$(ii) \qquad \frac{\partial x}{\partial y}(t, y^0) \neq 0 \qquad for \quad t < t^0$$

If $(\partial a_1/\partial u)(t^0, x^0, v^0) \neq 0$ where $x^0 = x(t^0, y^0)$ and $v^0 = v(t^0, y^0)$, then the characteristic curves meet in a neighborhood of (t^0, x^0).

Proof.
This theorem follows immediately if the monotonicity of $x = x(t,y)$ with respect to y is violated. First we prove that $(\partial x/\partial y)(t, y^0)$ is negative for $t > t^0$ with $t - t^0$ small. Actually, differentiating (2.7) with respect to y, we get the system of ordinary differential equations (2.10) for $n = 1$.Then (2.10)

is linear with respect to $\partial x/\partial y$ *and* $\partial v/\partial y$. As the initial data are not zero, we get

$$(\frac{\partial x}{\partial y}(t,y), \frac{\partial v}{\partial y}(t,y)) \neq 0 \quad for \quad any \quad t \geq 0, \quad y \in R^1$$

Hence we have $(\partial v/\partial y)(t^0, y^0) \neq 0$. By the assumption, we get

$$\frac{d}{dt}(\frac{\partial x}{\partial y}) \mid_{(t,y)=(t^0,y^0)} = \frac{\partial a_1}{\partial u}(t^0, x^0, v^0)\frac{\partial v}{\partial y}(t^0, y^0) \neq 0$$

Moreover, as $(\partial x/\partial y)(t^0, y^0) = 0$ and $\partial x/\partial y)(t, y^0) > 0 \quad for \quad t < t^0$, we get

$$\frac{d}{dt}(\frac{\partial x}{\partial y}) \mid_{(t,y)=(t^0,y^0)} \leq 0, \qquad i.e \quad \frac{d}{dt}(\frac{\partial x}{\partial y}) \mid_{(t,y)=(t^0,y^0)} < 0$$

Therefore we get $(\partial x/\partial y)(t, y^0) < 0$ for $t > t^0$ with $t - t^0$ small. That is to say, $(\partial x/\partial y)(t, y^0)$ changes its sign at $t = t^0$. We now define a function $h = h(y)$ by setting $h(y) = \inf\{t : (\partial x/\partial y)(t, y) = 0\}$ for each y. Then $h = h(y)$ is of class C^1 in a neighborhood of $y = y^0$.In fact, as we have $(\partial x/\partial y)(t^0, y^0) = 0$ and $(\partial^2 x/\partial t\partial y)(t^0, y^0) < 0$, we can uniquely solve the equation $(\partial x/\partial y)(t, y) = 0$ with respect to y in a neighborhood of (t^0, y^0). Let us restrict our following discussions into a small neighborhood of (t^0, x^0, v^0) only. We see by the same reasoning as in the above that, as a function of t, $(\partial x/\partial y)(t, y)$ changes its sign at $t = h(y)$ for each y.
Next, we will show that $x = x(t, y)$ is not monotone with respect to y. Pick up a point $y' \neq y^0$. If $h(y') < t^0$, then we get

$$\frac{\partial x}{\partial y}(t, y') < 0 \qquad and \qquad \frac{\partial x}{\partial y}(t, y^0) > 0$$

for $t \in (h(y'), t^0)$. As this means that $x = x(t, y)$ is not monotone with respect to y, the characteristic curves meet in a neighborhood of (t^0, x^0). When $h = h(y)$ is constant in a neighborhood of $y = y^0$,it follows that $(\partial x/\partial y)(t^0, y) \equiv 0$ in a neighborhood of $y = y^0$, i.e., $x = x(t^0, y)$ is constant there.This says that the characteristic curves meet at the point (t^0, x^0). When there exists a point $y = y'$ such that $h(y') > t^0$, we can similarly see that $x = x(t, y)$ is not monotone with respect to y in a neighborhood of $y = y^0$ for each $t \in (t^0, h(y'))$. Hence the characteristic curves $x = x(t, y)$ meet in a neighborhood of the point (t^0, x^0).

3.3.2 Characteristic curves for general PDE of first order

In this section we consider the same problem as in section 3.3.1 for general partial differential equations of first-order in one space dimension as follows:

$$\frac{\partial u}{\partial t} + f(t, x, u, \frac{\partial u}{\partial x}) = 0, in\{t > 0, x \in R^1\} \tag{3.8}$$

$$u(0, x) = \phi(x), on\{t = 0, x \in R^1\}, \tag{3.9}$$

where $f = f(t, x, u, p)$ and $\phi = \phi(x)$ are of class C^2 in R^4 and R^1, respectively. The characteristic curves for $(3.8) - (3.9)$ are obtained as solution curves of $(2.15) - (2.16)$ for $n = 1$. Here we assume Condition (A_3) (Chapter 2) which assures the global existence of characteristic curves. Let $x = x(t, y), v = v(t, y)$,and $p = p(t, y)$ be the solutions of $(2.15) - (2.16)$ for $n = 1$.

Theorem 3.3.2. *Under Condition* (A_3)*, we assume:*

$$(i) \quad \frac{\partial x}{\partial y}(t^0, y^0),$$

and

$$(ii) \quad (i)\frac{\partial x}{\partial y}(t, y^0) \neq 0 \qquad for \quad t < t^0$$

If $(\partial^2 f/\partial p^2)(t^0, x^0, v^0, p^0) \neq 0$ *where* $x^0 = x(t^0, y^0), \quad v^0 = v(t^0, y^0)$ *and* $p^0 = p(t^0, y^0)$*, then the characteristic curves meet in a neighborhood of* (t^0, x^0)*.*

Proof.

As in the proof of Theorem 3.3.1, we will show that $x = x(t, y)$ is not monotone with respect to y. When the Jacobian $(\partial x/\partial y)(t, y) \neq 0$, we can solve the equation $x = x(t, y)$ with respect to y, and denote the solution by $y = y(t, x)$. Then the solution of $(3.8) - (3.9)$ is given by $u(t, x) = v(t, y(t, x))$ and

$$p(t, y(t, x)) = \frac{\partial u}{\partial x}(t, x) = \frac{\partial v}{\partial y}(t, y(t, x))\frac{\partial y}{\partial x}(t, x)$$
$$= \frac{\partial v}{\partial y}(t, y(t, x)).(\frac{\partial x}{\partial y}(t, y))^{-1} |_{y=y(t,x)}$$

When a point (t, x) goes to (t^0, x^0) along the curve $\{(t, x) : x = x(t, y^0)\}$, $\partial x/\partial y(t, y(t, x))$ tends to zero and $p(t, y(t, x))$ remains bounded. Therefore it must be true that $(\partial v/\partial y)(t^0, y^0) = 0$ Differentiating the system (2.15) with respect to y, we get a system of linear ordinary differential equations for $\partial x/\partial y, \partial v/\partial y$ and $\partial p/\partial y$ just like (2.10). As the initial data

$$(\partial x/\partial y, \partial v/\partial y, \partial p/\partial y)(0, y) = (1, \phi'(y), \phi''(y))$$

are not zero, it follows that $(\partial x/\partial y, \partial v/\partial y, \partial p/\partial y)(t, y) \neq (0, 0, 0)$ for any $t \geq 0$. Hence $(\partial p/\partial y)(t^0, y^0) \neq 0$. We can see that $(\partial x/\partial y)(t, y)$ satisfies the following differential equation:

$$\frac{d}{dt}(\frac{\partial x}{\partial y}) = \frac{\partial^2 f}{\partial x \partial p}\frac{\partial x}{\partial y} + \frac{\partial^2 f}{\partial u \partial p}\frac{\partial v}{\partial y} + \frac{\partial^2 f}{\partial p^2}\frac{\partial p}{\partial y}$$

Using the assumptions and the above results, we get

$$\frac{d}{dt}(\frac{\partial x}{\partial y}) \mid_{(t,y)=(t^0,y^0)} = \frac{\partial^2 f}{\partial p^2}(t^0, x^0, v^0, p^0)\frac{\partial p}{\partial y}(t^0, y^0) \neq 0$$

Moreover, as $(\partial x/\partial y)(t^0, y^0) = 0$ and $(\partial x/\partial y)(t, y^0) > 0$ for $t < t^0$, it must be true that

$$\frac{d}{dt}(\frac{\partial x}{\partial y}) \mid_{(t,y)=(t^0,y^0)} \leq 0 \quad i.e \quad \frac{d}{dt}(\frac{\partial x}{\partial y}) \mid_{(t,y)=(t^0,y^0)} < 0$$

Hence$(\partial x/\partial y)(t, y^0)$ changes its sign at $t = t^0$. Here we introduce a function $h = h(y)$,just as in the proof of Theorem 3.3.1, by setting $h(y) = \inf\{t : (\partial x/\partial y(t, y) = 0\}$ for each y. We consider the problem in a small neighborhood of (t^0, x^0, v^0, p^0) where $\partial^2 f/\partial p^2 \neq 0$. We pick up a point $y'(y' \neq y^0)$. By the same reasoning as in the above, we see that $(\partial x/\partial y)(t, y')$ changes its sign at $t = h(y')$. If $h(y') > t^0$,we have

$$\frac{\partial x}{\partial y}(t, y') > 0 \qquad and \qquad \frac{\partial x}{\partial y}(t, y^0) < 0$$

for $t \in (t, h(y'))$. As this means that $x = x(t, y)$ is not monotone with respect to y, the characteristic curves meet in a neighborhood of (t^0, x^0). In the case where $h(y') < t^0$ or $h = h(y)$ is constant, we can similarly prove that the characteristic curves meet in a neighborhood of (t^0, x^0).

Summary

The method of characteristics is a powerful method that allows one to reduce any first-order linear PDE to an ODE, which can be subsequently solved using ODE techniques and it can be generalized to quasilinear equations as well. the principal results of this paper are: 1) The Cauchy problem

$$\frac{\partial u}{\partial t} + f(t, x, u, \frac{\partial u}{\partial x}) = 0, in\{t > 0, x \in R^n\},$$

$$u(0, x) = \phi(x), on\{t = 0, x \in R^n\}$$

has locally a unique C^2-solution.
2) If the Jacobian $(Dx/Dy)(t, y)$ of the mapping $x = x(t, y)$ vanishes somewhere, it is impossible to extend the C^2-solution beyond a point where the Jacobian vanishes.
3) Suppose that the characteristic curves do not meet in a neighborhood of the point where the Jacobian vanishes. Then the solution keeps being of class C^1, but not of class C^2, in the neighborhood of the point.

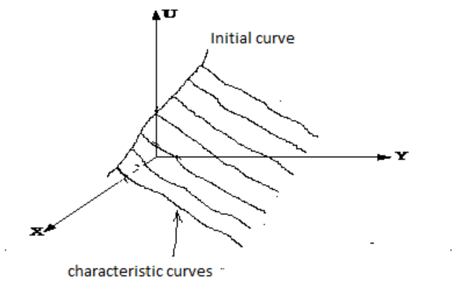
Appendix

Origin of the method of characteristics

We solve first-order PDEs by the method of characteristics. This method was developed in the middle of the nineteenth century by Hamilton. Hamilton investigated the propagation of light. He sought to derive the rules governing this propagation from a purely geometric theory, akin to Educlidean geometry. Hamilton was well aware of the wave theory of light, which was proposed by the Dutch physicist Christian Huygens $(1629-1695)$ and advanced early in the nineteenth century by the English scientist Thomas Young $(1773-1829)$ and the French physicist Augustin Fresnel $(1788-1829)$. Yet, he chose to base his theory on the principle of least time that was proposed in 1657 by the French scientist (and lawyer!) Pierre de Fermat $(1601-1665)$. Fermat proposed a unified principle, according to which light rays travel from a point A to a point B in an orbit that takes the least amount of time. Hamilton showed that principle can serve as a foundation of a dynamical theory of rays. He thus derived an axiomatic theory that provided equations of motion for light rays. The main building block in the theory is a function that completely characterizes any given optical medium. Hamilton called it the characteristic function. He showed that Femats principle implies that the characteristic function must satisfy a certain first-order nonlinear PDE. Hamiltons characteristic function and characteristic equation are now called the Eikonical function and Eikonical equation after the Greek word $\epsilon\iota\kappa\omega\nu$ (or $\epsilon\iota\kappa o\nu$) which means an image.

Hamilton discovered that the Eikonical equation can be solved by integrating it along special curves that he called characteristics. Furthermore, he showed that in a uniform medium, these curves are exactly the straight light rays whose existence has been assumed since ancient times. In 1911 it was shown by the German physicists Arnold sommerfeld $(1868-1951)$ and Carl Runge $(1856-1927)$ that the Eikonical equation, proposed by Hamilton from his geometric theory, can be derived as a small wavelength limit of the wave equation. Notice that although the Eikonical equation is of first order, it is

in fact fully nonlinear and not quasi linear.
The characteristics method is based on knitting the solution surface with a one-parameter family of curves that intersect a given curve in space.



Sketch of method of characteristics.

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