SCHOOL OF GRADUATE STUDIES

FACULTY OF COMPUTER AND MATHEMATICAL SCIENCES

DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT ON

MATRIX-TREE THEOREM AND ITS APPLICATION

Compiled by: Abayneh Fentie

This project is submitted to Addis Ababa University, Faculty of Computer and Mathematical Sciences, Department of Mathematics in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Advisor: Dr.Yirgalem Tsegaye

Addis Ababa, Ethiopia

June, 2011

DECLARATION

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Abayneh Fentie

Signature: -

PERMISSION

This is to certify that this project is compiled by Abayneh Fentie in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr.Yirgalem Tsegaye

Signature:

ACKNOWLEDGEMENT

First, I want to praise my GOD, the Almighty GOD, who has passed me many unspeakable situations throughout my life from the beginning until this time and about my future.

Next, I would like to present my gratitude and grateful respect to my advisor Dr. Yirgalem Tsegaye for her valuable enlightening discussions, suggestions, comments and her generous hospitality in preparing this Project and her critical reading of the paper and providing me constructive comments, assistance, material support and encouragement. Her guidance is not only for this paper, but for my future life too.

Next, I would like to thank Professor Melkamu Zeleke for providing all the necessary reading material support for the project.

Finally, I would like to thank for the department of Mathematics for providing all the necessary departmental facilities.

List of Graph theoretic symbols of the project

G - Graphs

T - Trees

$H \subseteq G$ - Sub graph

$\tau(G)$ - spanning tree of a graph

$V(G)$ - Vertex set of a graph

$E(G)$ - Edge set of a graph

$G - e$ - Deletion of edge e

$G.e$ - Contraction of an edge e

$\omega(G)$ - Number of components of G

O_n - Empty graph

C_n - Cycle graph

K_n - Complete graph

$K_{m,n}$ - Complete bipartite graph

H_n - Hyper cube graphs

deg (v) - Degree of v in G

$G_1 + G_2$ - The disjoint union of graphs G_1 *and* G_2

$D(G)$ - Degree matrix of a graph

$A(G)$ - Adjacency matrix of a graph

$L(G)$ - Laplacian matrix of a graph

$N(G)$ - Incidence matrix of a graph

$M(G)$- Reduced incidence matrix of a graph

SUMMARY OF THE PROJECT

Counting the number of spanning trees of a graph is one of the fundamental problems in enumerative combinatorics. Spanning trees have enormous interesting applications in telecommunication, computer science, and so on.

In this project, we will see various approaches of counting spanning trees of a graph by introducing some mathematical method like deletion-contraction method, direct counting, conditioning, deletion, Inclusion-exclusion, recurrence relation, cryptographic method, and we arrive at the conclusion that the number of spanning trees of $K_{2,n}$ is $n2^{n-1}$ and we have also counted some small graphs and in the process we generalize the number of spanning trees of a cycle graph C_n is n.

Then the less illuminating, but the most efficient and a generalized ways of counting spanning trees of a graph is matrix-tree theorem or Kirchhoff's matrix-tree theorem.

In this paper, we are mainly interested to prove different versions of matrix- tree theorem by involve matrix-theory and some combinatorial approaches of counting spanning trees are introduced that helps as to prove matrix- tree theorem.

We arrive at a conclusion that the number of spanning trees of a graph K_n is n^{n-2} and $K_{m,n}$ is $m^{n-1}n^{m-1}$ which are known as Cayley's theorem and Scoin's theorem respectively. So in general we can say that matrix-tree theorem is a fundamental tool of calculating the number of spanning trees of a graph G.

LIST OF TABLE OF CONTENTES

INTRODUCTION

A graph G=(V,E) is can be defined informally as a non empty set of vertices or nodes V , with or without edges or arcs E. Enumerating objects, graphs, sets are the main concern of enumerative part of combinatorics.

A tree is a special type of graph in which there is no cycle in it. In this project, we are interested in analyzing various approaches for the enumeration of spanning trees. Spanning tree of a graph is a sub graph of G which is a tree that contains all vertices of the graph G.

It has many economical applications in the real world, for instance in telecommunication, computer science, and so on. Therefore counting the number of spanning trees is a basic problem in combinatorics.

In the first chapter of this paper, we start by doing introductions to some basic concepts in graph theory such as graph, connectedness of a graph, spanning trees and existence of spanning trees.

In the second chapter, we enumerate spanning trees of a complete bipartite graph $K_{2,n}$ by introducing some techniques of counting such as deletion-contraction method, direct counting, conditioning, deletion, inclusion-exclusion, recursion relation, and cryptographic approach and matrix algebra. Then we proceed to count spanning trees of a graph G, Combinatoricaly and algebraically using the matrix-tree theorem.

Finally, we will see the application of matrix-tree theorem by visiting a certain application problem, cayley's theorem and scions theorem which is the generalization of any complete graph and complete bipartite graph G respectively.

CHAPTER ONE: PRELIMINARIES

1.1 Introductory concepts

1.1.1 Graphs and Some examples

A graph G is a triple $(V(G), E(G), \varphi_G)$ consisting of a vertex set V(G) , an edge set E(G), and a function φ_G that associates with each edge in E, unordered pair of vertices (not necessary distinct) called its **endpoint.** We can represent a graph on a paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints.

Example let $G = (V\,(G), E\,(G), \varphi_G)$ where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$,

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ And φ_G is defined by

$$\varphi_G(e_1) = v_1v_2, \qquad \varphi_G(e_5) = v_2v_4$$

$$\varphi_G(e_2) = v_2v_3 \qquad \varphi_G(e_6) = v_4v_5$$

$$\varphi_G(e_3) = v_3v_3 \qquad \varphi_G(e_7) = v_2v_5$$

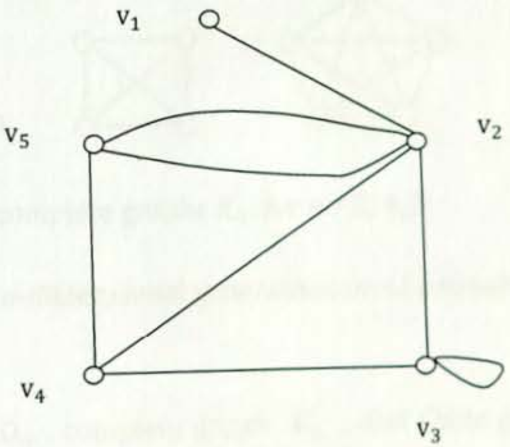


Figure 1.1 pictorial representation of a graph G

DEFINITIONS:

- **A loop** is an edge corresponding to pairs of the type (v,v) for all v in V.
- **Multiple edges** are edges having the same pair of end points.
- **A simple graph** is a graph that has no loops or multiple edges.
- **Empty graph** is a graph with no edges.

A Graph G is **finite** if both its vertex set and edge sets are finite, Infinite, otherwise. A finite graph with just one vertex is called **trivial graph** and all the other graphs are termed as **non-trivial graphs.**

Order, size and degree of a graph: The order and the size of the graph G is denoted by |G| and ||G|| which is the cardinality of its vertex set and edge set respectively. Where |G| = |V| and ||G|| =|E|. For instance in the figure 1.1 Order of G , |G| =5 , Size of G , ||G|| =8 and The Degree of G; Deg(v_1)=1 deg(v_2)=5; deg(v_3)=4 and deg(v_4)= deg(V_5)= 3

1.1.2 REGULAR GRAPHS AND THEIR PROPERTIES

- A graph G is **regular**, if every vertex has the same degree. That is A graph G is said to be regular of degree r (r-regular), if deg (v) = r for all vertices v in G.
- **A complete graph (cliques)**: A complete graph on n vertices denoted as K_n which is a simple graph whose vertices are pair wise adjacent. For instance K_3, K_4 *and* K_5 graph are shown in the fig 1.2

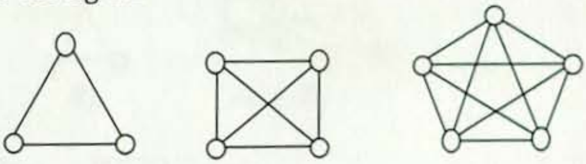


Figure 1.2 complete graphs K_n, for n= 3, 4, 5

- **Hypercube graph** H_n is n-dimensional generalization of a square (n=2) and A cube (n=3).

REMARK: Empty graph O_n , complete graph K_n , and Cycle graph C_n are all regular graphs of order n. and Hypercube graph H_n is regular graph of order 2^n.

- **A complete bipartite graphs (bi cliques):** A Graph is complete bipartite graph if the vertices can be partitioned in to two sets **X** and **Y** such that every vertex in **X** joined to every vertex in **Y** in such away that no vertex is adjacent in the same partite sets.

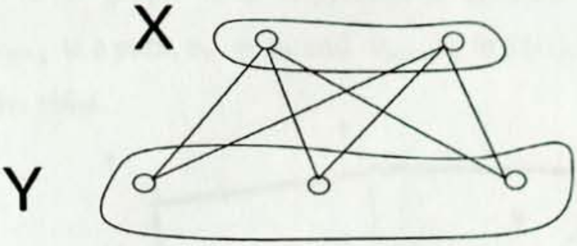


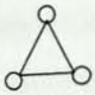
Figure 1.3 The complete bipartite graph $K_{2,3}$.

- **Complimentary graphs:**

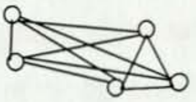
A graph $\bar{G} = (V.\bar{E})$ is called complimentary to a graph $G = (V, E)$, if $\bar{E} = \binom{V}{2} - E$. in other word e is an edge of $\bar{G}$ if and only if e is not an edge of G. Here the complementary graph $\bar{O}_n = K_n$ *and* $\overline{K_m + K_n} = K_{m,n}$.

To verify this fact, clearly $\bar{O}_n = K_n$,but we need to show the second one. Consider for intance K_2 *and* K_3.

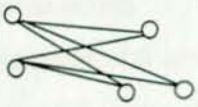
K_2



K_3



$K_2 + K_3$



$\overline{K_2 + K_3} = K_{2,3}$.

Therefore $\overline{K_m + K_n} = K_{m,n}$

1.1.3. PATHS AND CONNECTEDNESS OF A GRAPH

- A path in a graph is a sequence of distinct vertices v_1 , v_2 , … , v_n such that $v_i\, v_j$ in E(G) for i,= 1,2 ,…,n-1.and j=1,2,…, n. The length of the path is the number of edges on the path.
- A cycle in a graph is a sequence of vertices v_1, v_2 , … , v_n such that v_1 , v_2 , … , v_{n-1} is a path, $v_1 = v_n$ and $v_{n-1}\, v_n$ in $E(G)$. for example of a Path: abcdefg and Cycle: abha.

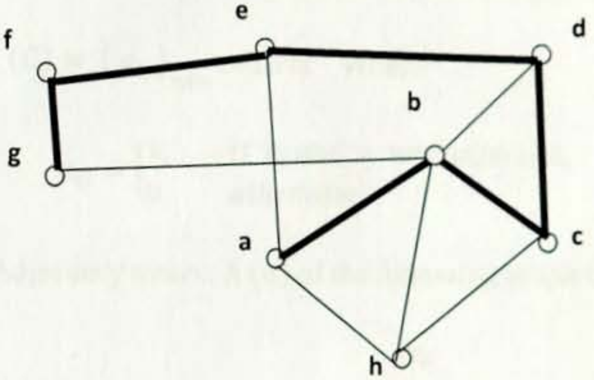


Figure1.4 path and cycle of graph

- **Connectedness of a graph:** A non empty graph G is said to be connected, if any two of its vertices are linked by a path in G; otherwise it is disconnected.
- A maximal connected sub graph of G is said to be a **component** of G and we denote the number of components of G by $\omega(G)$. In the figure 1.5 note that number of component of G $\omega(G) = 1$ and that of H $\omega(H) = 3$

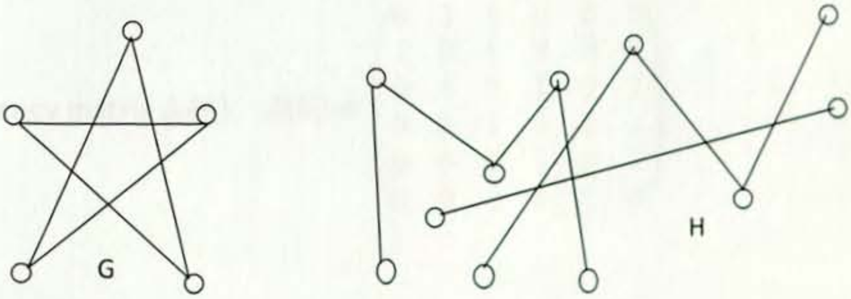


Figure 1.5 connectivity of graph

- ***Sub graphs*:** A Graph H is a sub graph of a graph G if V(H)⊆ V(G) , E(H) ⊆ E(G) And φ_H is a restriction of φ_G to H .In this case we write H⊆ G, and we say that G contains H. for instance each component of H is a sub graph of G.

1.1.4. ADJACENCY MATRIX AND INCIDENCE MATRIX OF A GRAPH

i. Adjacency matrix A(G)

Definition: Let G be a graph with vertices $v_1 \quad v_2 \; \dots \; v_n$. The Adjacency matrix of G is

$$A\,(G) = \left(a_{ij}\right)_{nxn} \; matrix \quad \text{Where}$$

$$a_{ij} = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent. \\ 0, & otherwise \end{cases}$$

Example 1 Determine Adjacency matrix A (G) of the following graph G.

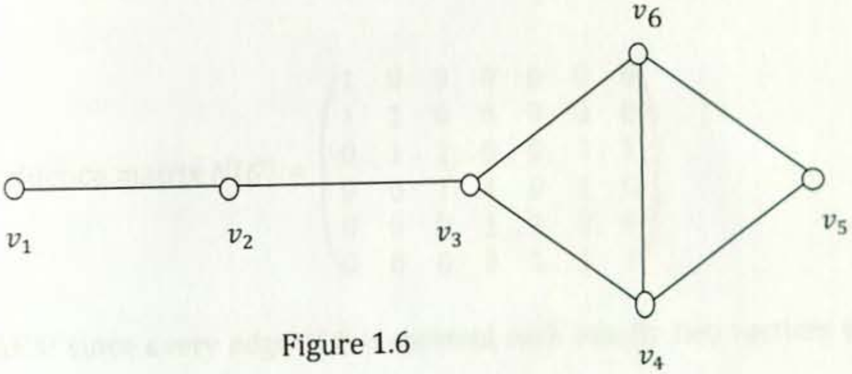


Figure 1.6

The Adjacency matrix A (G), $A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$

ii. Incidence matrix N(G)

Definition: Let G be a graph with n vertices $(v_1, v_2, \dots\ v_n)$ and k edges $(e_1, e_2, \dots \dots\ e_k)$. The incidence matrix of G is $N(G) = (\ n_{ij}\ )_{nxk}$ matrix where

$$n_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } e_j \text{ are incident} \\ 0 & \text{otherwise} \end{cases}$$

Example 2 Determine the incidence matrix $N(G)$ of the following graph

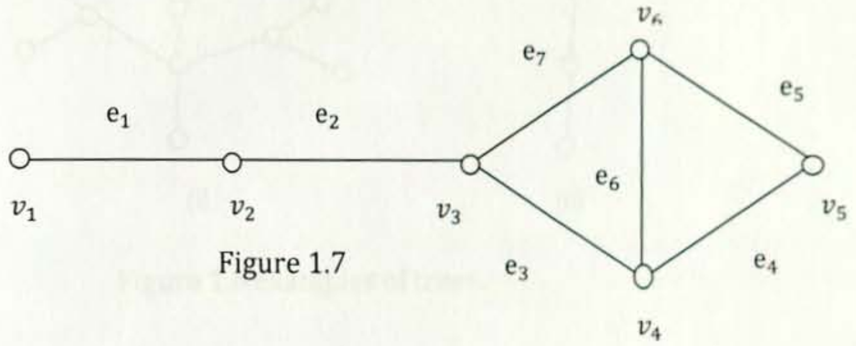


Figure 1.7

The incidence matrix $N(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$

REMARK: since every edge of G is incident with exactly two vertices of G, Each column of N(G) contains two 1's and n-2 zeros.

1.2. TREES

1.2.1 Definition and some properties of a tree.

Definition: A **tree** is a connected graph that has no cycle. A set of trees is called a **forest**. In a tree a vertex of degree one is called a **leaf**. Examples of trees are

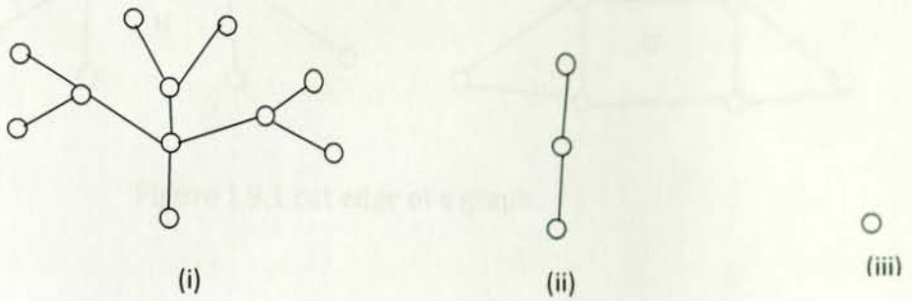


Figure 1.8 examples of trees

Some properties of a tree

- Any two vertices of a tree are connected by a unique path.
- Any tree of order n has n-1 edges. That is $|E| = |V| - 1$
- Any tree of order n has at least two leaves, for n≥ 2

1.2.2 SPANNING TREES

A Spanning tree T of a graph G is a spanning sub graph of G which is tree.

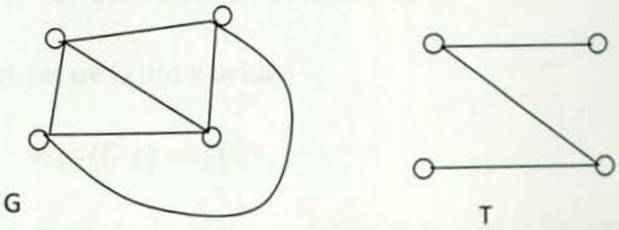


Figure 1.9. Graph G and one of its spanning trees.

1.2.3 Existence of spanning trees of a graph

Do all Graphs have spanning trees?

- **Cut- edge:** An edge e of a graph G is said to be a **cut edge** or **bridge**, if $\omega(G-e) > \omega(G)$ where $\omega(G)$ is the number of components of G.

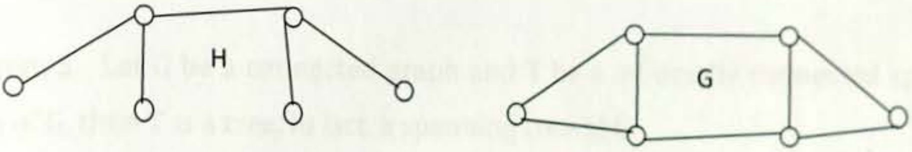


Figure 1.9.1 cut edge of a graph

Lemma 1 an edge e of a connected graph G is a bridge if and only if e does not lie on any cycle of a Graph G.

Proof (⟹) suppose e=uv is a bridge of G.

Suppose that e= uv lies on cycle of a graph G,

Then removing e= uv leaves u and v in the same component of G-e.

$\Rightarrow \omega(G-e) = \omega(G)$

⇒e is not a bridge.

(⟸) suppose e= uv does not lie on any of a cycle.

Suppose that e= uv is not a bridge

$\Rightarrow \omega(G-e) = \omega(G)$

There is a path P joining u and v and P+e is a cycle .This implies e lies on a cycle which is contradiction. Thus e lies on a cycle. Hence the proof is complete, and therefore an edge is a bridge if and only if it doesn't lie on a cycle.

Theorem 2 A connected graphs G is a tree if and only if every edge e is a bridge.

Proof ($\Rightarrow$) let e be any edge of a tree G.

$\Rightarrow$ e does not lie on a cycle. Therefore

$\Rightarrow$ e is a bridge. By lemma1.

Theorem 3 Let G be a connected graph and T be a minimally connected spanning sub graph of G, then T is a tree, in fact, a spanning tree of G.

Proof By Hypothesis $\omega(T) = 1$ and $\omega(T\text{-}e) > 1$ for any edge e of T.

It follows that each edge of T is a bridge. Therefore by theorem 2 T is a tree. Hence the theorem is proved. Now we have seen that spanning tree exists in connected graph G. so how can we count all spanning trees for a given connected graph G.

Counting the number of spanning trees is an important problem in enumerative combinatorics. In the next part, we develop a technique how to count the number of spanning trees of a graph G and we drive a simple formula that helps us to count exact number of spanning trees of some special graphs such as cycle graphs, complete graphs, complete bipartite graphs, hypercube graphs and some other graph G.

CHAPTER TWO: ENUMRATION OF SPANNING TREES OF A GRAPH

2.1. DELETION-CONTRACTION FORMULA.

This is a simple and elegant recursive formula for the number of spanning trees of a graph. It involves the operation of deletion and contraction of an edge. For instance

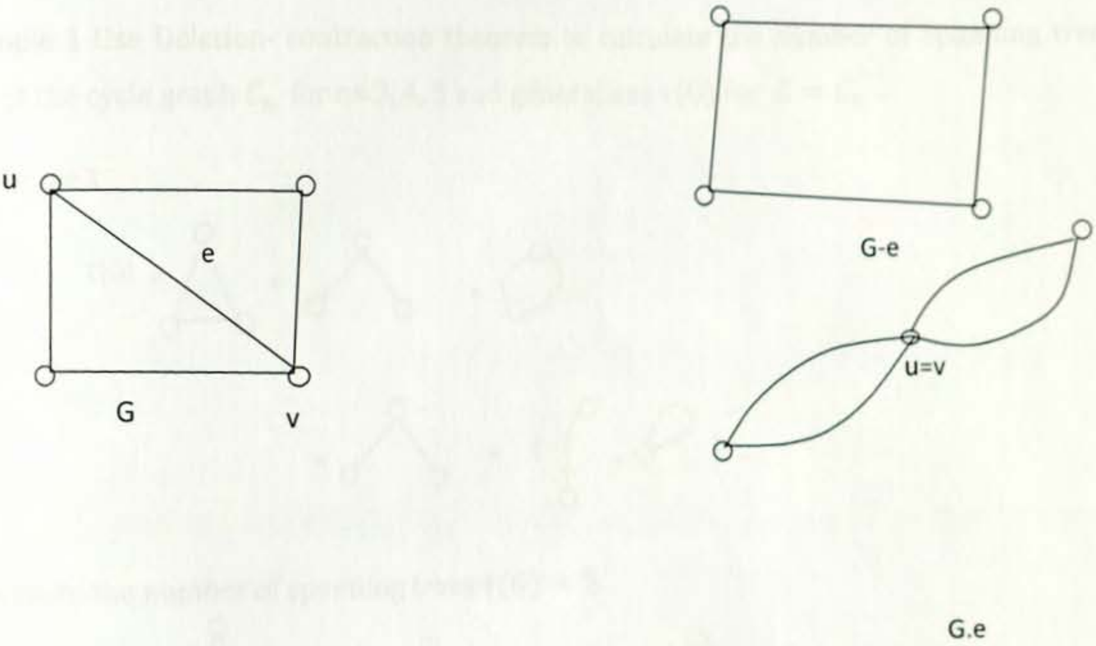


Figure 2.1 Deletion & contracting of an edge in agape G.

The contraction G.e is obtained from G-e by identifying u and v ("fusing" the two vertices together).for contraction to make sense, we usually require that e not be a loop.

REMARK: If e is a link of G, then

- $V(G.e) = V(G) - 1$
- $E(G.e) = E(G) - 1 \text{ and } K(G.e) = K(G)$

Therefore, if T is a tree so too is T.e

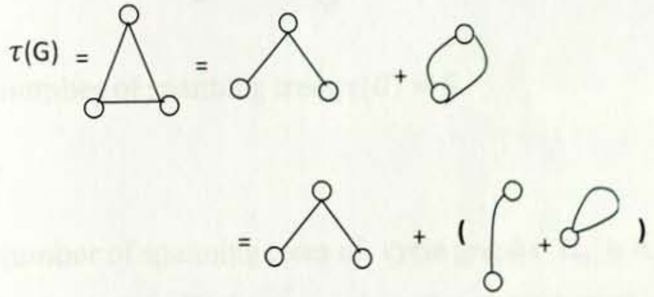
Theorem 2.1 let $\tau(G)$ be the number of spanning trees in G. If e is a link of G,

Then $\tau(G) = \tau(G - e) + \tau(G.e)$

Proof: since every spanning tree of G that do not contain e is also spanning tree of $G-e$, and conversely $\tau(G-e)$ is the number of spanning trees of G that do not contain e .Now to each spanning tree T of G that contain e, there corresponds a spanning tree $T.e$ of $G.e$, this correspondence is a *bijection*.Therefore $\tau(G.e)$ is precisely the number of spanning trees of G that contain e. It follows that $\tau(G) = \tau(G-e)+\tau(G.e)$

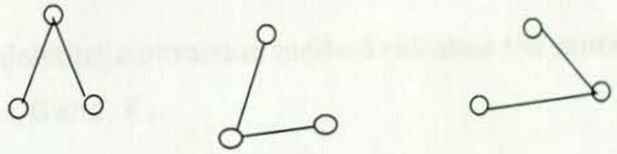
Example 1 Use Deletion- contraction theorem to calculate the number of Spanning trees τ(G) of the cycle graph C_n for n=3, 4, 5 and generalizes τ(G) for $G = C_n$.

- for n=3

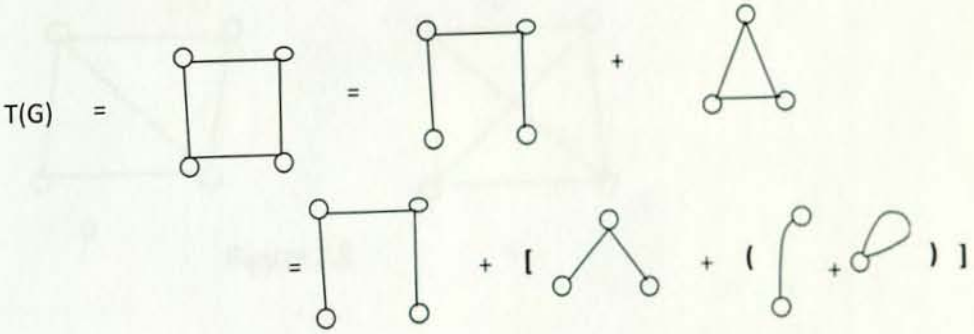


Therefore the number of spanning trees $\tau(G) = 3$.

Thus are

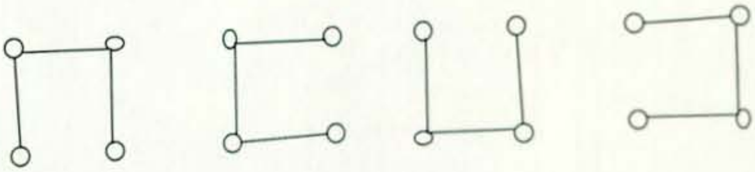


For n=4

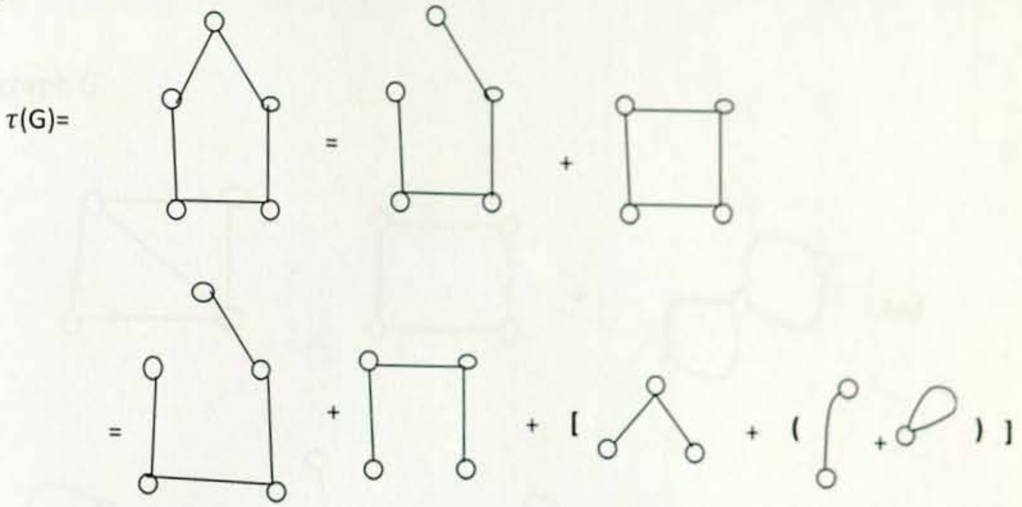


Therefore the number of spanning trees $\tau(G) = 4$.

Thus are



- For n=5

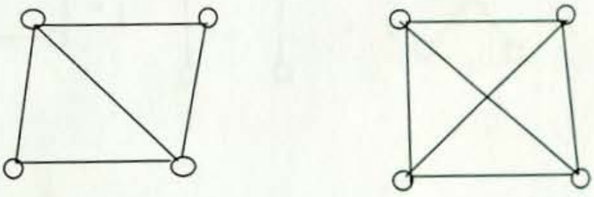


Then the number of spanning trees $\tau(G) = 5$.

REMARK:

- The number of spanning trees of cycle graphs C_n is n. That is $\tau(G) = n$ for $G = C_n$.
- A loop does not affect the number of spanning trees, so we can delete or ignore the loops in a graph and determine the number of spanning trees.

Example 2 Use deletion contraction method calculate the number of spanning trees of the following graphs, G and K_4.



G

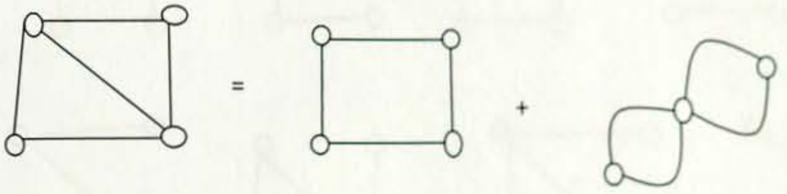
Figure 2.2

K_4

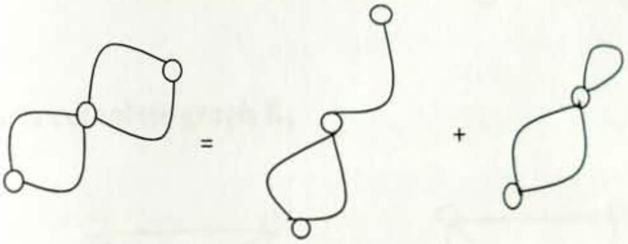
Solutions

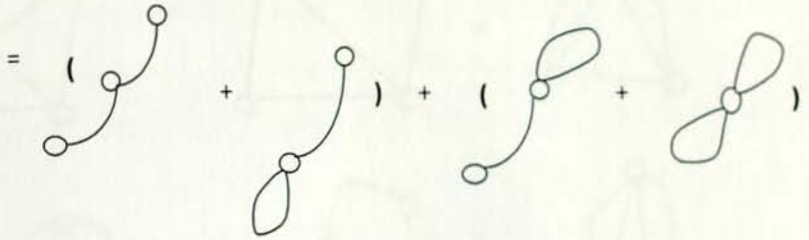
- For a graph G.

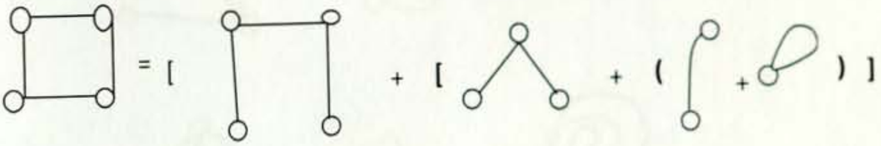
τ (G)=



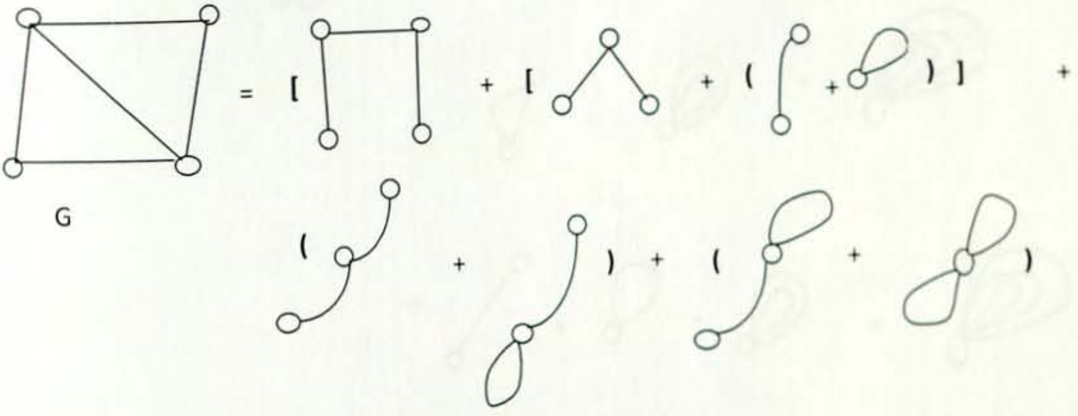
And



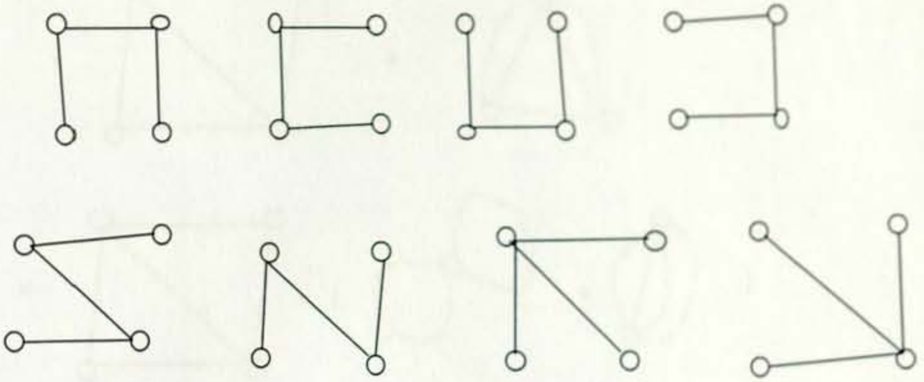




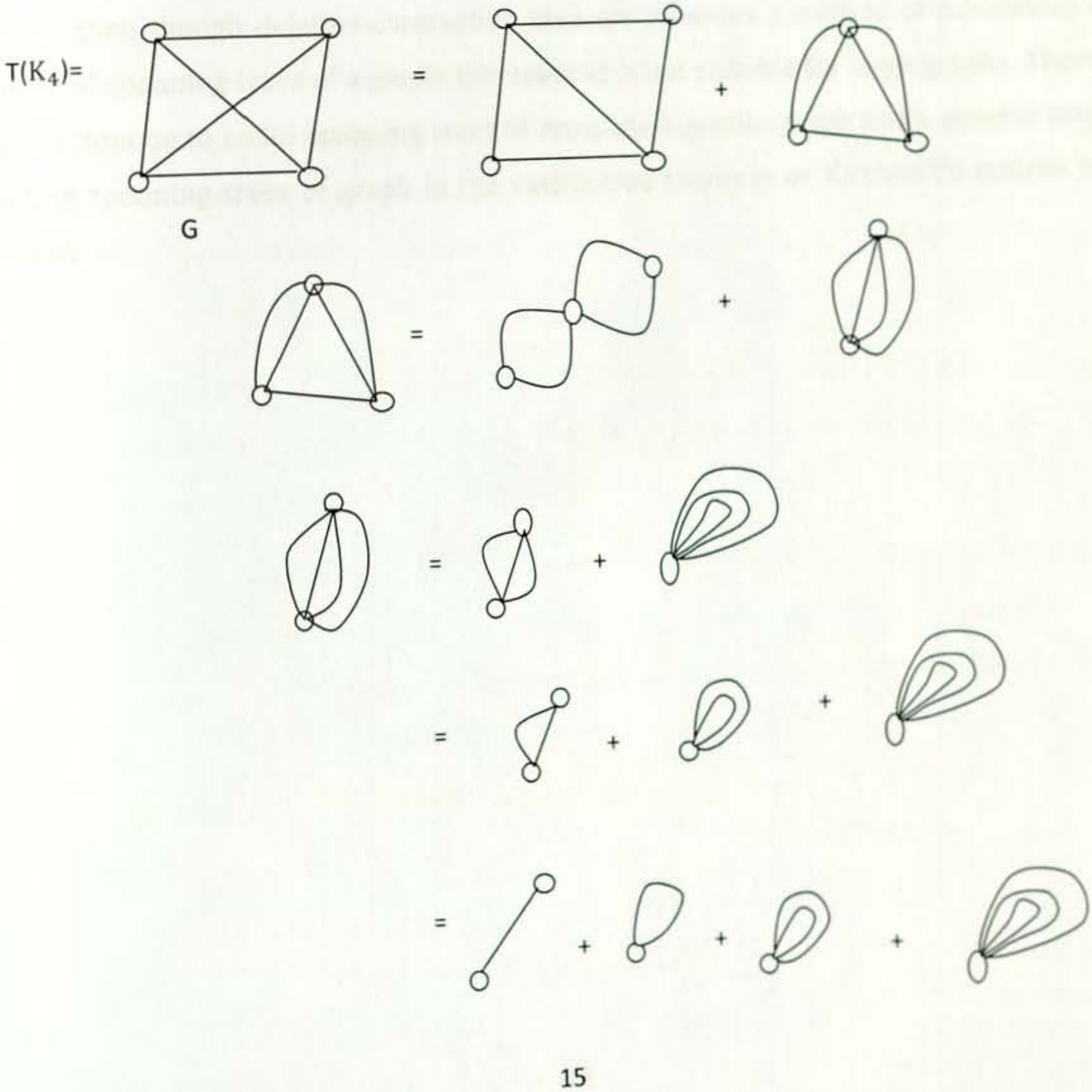
Therefore



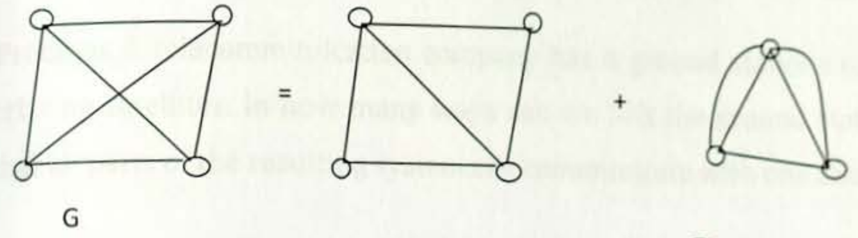
Then $\tau(G) = 8$, and the spanning trees of G are

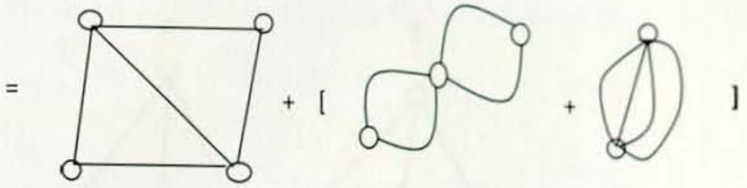


- For a complete graph K_4



Therefore,





Now the number of spanning trees of K_4 is $T(G) = 8 + [4 + 4] = 16$. Thus are

Even though deletion-contraction theorem provides a method of calculating the number of spanning trees of a graph, this method is not suitable for large graphs. There is also a technique to count spanning trees of complete bipartite graph and a general way of counting spanning trees of graph is the matrix-tree theorem or Kirchhoff's matrix- tree theorem

2.2. SOME APPROACHS IN COUNTING SPANNING TREES OF $K_{2,n}$

Problem. A telecommunication company has n ground stations to be linked with its two orbiting satellites. In how many ways can we link the ground stations to the satellites so that all parts of the resulting system can communicate with one another efficiently?

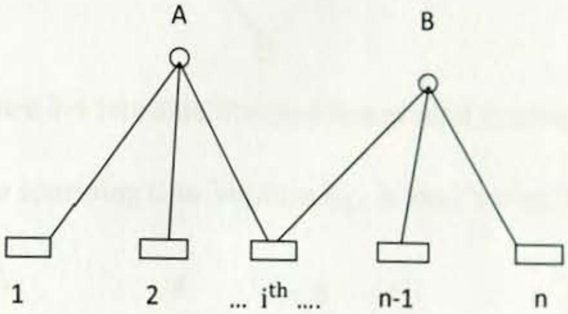


Figure 2.3 A possible configuration of two satellites and n ground station.

Figure 2.1 shows one possibility. Every pair of ground stations can communicate (via either one or two intermediate satellite connections): the two satellites A and B can also communicate with each other via a ground station. This is to be achieved, moreover, using the smallest possible number of links.

To abstract this problem, we see that a candidate solution is undirected graph on n+2 vertices, such that all vertices are accessible (either directly or indirectly) from one another using edges (proposed communication links).

It is a simple requirement that the candidate graph G be connected: every pair of distinct vertices of G is joined by some path, and for the sake of efficiency; no unnecessary edges should be present. That is G should be without cycles. A connected graph without cycles is called a tree, and our problem requires a tree that includes all vertices - spanning tree. So the original problem can now be restated as find the number τ_n of spanning trees in $K_{2,n}$.

We know that $K_{2,n}$ has n+2 vertices and 2n edges. So a spanning tree T of $K_{2,n}$ has n+2 vertices and n+ 1 edge.

Let us look for the simplest possible cases

- $For\ n = 1,$

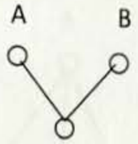


Figure 2.4 two satellites and one ground station

We have only one spanning tree .because $K_{2,1}$ is itself a tree. Therefore $t_1 = 1$

- $For\ n = 2,$

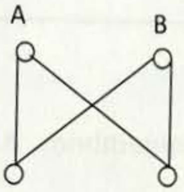


Figure 2.5 two satellites and two ground station

We have four spanning trees. Because removing any one of the four edges of $K_{2,2}$, we get spanning tree. Therefore $t_n = 4$.

- $For\ n = 3$, And above t_n gets more complicated to find. There are other seven different methods or ways to solve this problem for arbitrary n.

  i. **DIRECT COUNTING:** We know that a complete bipartite graph $K_{2,n}$ has n+2 vertices and 2n edges.

  -Both satellites A and B can be connected by unique path to any one of the i ground stations. For $i = 1, 2, \dots, n$. This can be done in *n different ways.*

  -Either A or B must be connected to the remaining n-1 ground stations. This can be done in *2 different ways for each ground stations.* So there are $\underbrace{2.2.2 \dots 2}_{n-1\ times} = 2^{n-1}$ different ways. Then by *multiplication rule*, the number of spanning trees of $K_{2,n}$ $t_n = n.2^{n-1}$.

ii. **CONDITIONING METHOD** : In this method , we condition on the number k of edges in a tree T that emanate from A. then clearly n+1-k edges emanate from B. because spanning tree of $K_{2,n}$ has n+1 edges.

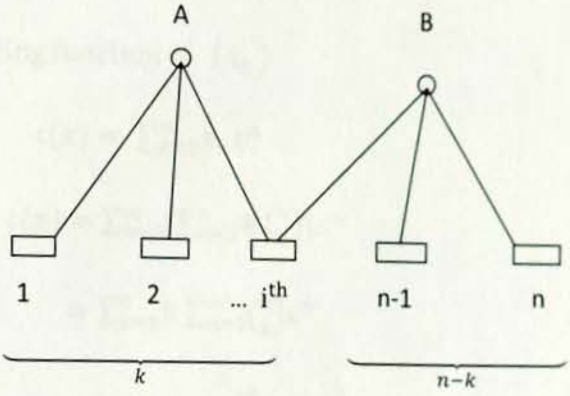


Figure 2.6 conditioning on the k edges from A

As we observed from the direct counting method, precisely one vertex i is joined directly to both A and B. The number of ways of selecting k vertices that are joined to A is $\binom{n}{k}$.Since for each such selection there are k choices for special vertex i , then by multiplication rule.

There are $t_k = k\binom{n}{k}$ ways to construct a spanning tree with k tree edges emanating from *A*. Since the situations for different k are mutually exclusive, the number of spanning trees constructed in this way by direct counting taking k vertices from A and the remaining .

Hence $t_n = \sum_{k=1}^{n} k\binom{n}{k}$ and it remains to show that $t_n = n2^{n-1}$

Method 1 Using binomial identity. We know that $\sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n$

By differentiating with respect to x, we get $\sum_{k=1}^{n} k\binom{n}{k}x^{k-1} = n(1+x)^{n-1}$

Now set x=1, we have $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

Therefore, $t_n = n2^{n-1}$

Method 2 using snake oil method:

Given that $t_n = \sum_{k=1}^{n} \mathrm{k}\binom{n}{k} = \sum_{k=0}^{n} \mathrm{k}\binom{n}{k}$.

We want to show that $t_n = \sum_{k=0}^{n} \mathrm{k}\binom{n}{k} = n2^{n-1}$

Let t(x) be the generating function of $\{ t_n \}$

$$t(x) = \sum_{n=0}^{\infty} t_n x^n$$

Then $t(x) = \sum_{n=0}^{\infty}[\sum_{k=0}^{n} k\binom{n}{k}]x^n$

$$= \sum_{k=0}^{n} k \sum_{n=0}^{\infty}\binom{n}{k}x^n$$

$$= \sum_{k=0}^{n} k \,.\, \frac{x^k}{(1-x)^{k+1}} \qquad \text{Book keeper identity}$$

$$= \sum_{k=0}^{n} k\left(\frac{x}{1-x}\right)^k . \frac{1}{1-x}$$

$$= \sum_{k=0}^{n} k\left(\frac{x}{1-x}\right)^{k-1} \frac{x}{(1-x)^2}$$

$$= x\sum_{k=0}^{n} \frac{d}{dx}\left(\frac{x}{1-x}\right)^k$$

$$\Rightarrow t(x) = x\frac{d}{dx}\sum_{k=0}^{n}\left(\frac{x}{1-x}\right)^k = x\frac{d}{dx}\left(\frac{1}{1-\frac{x}{1-x}}\right) = x\frac{d}{dx}\left(\frac{1-x}{1-2x}\right) = \frac{x}{(1-2x)^2}$$

$$\Rightarrow t(x) = \frac{1}{2}\frac{x}{(1-2x)^2} = \frac{1}{2}\sum_{n=0}^{\infty} n(2x)^n = \sum_{n=0}^{\infty} n2^{n-1}x^n.$$

Hence, $t_n = [x^n]t(x) = n2^{n-1}$

iii. **DELETION METHOD**: Instead of considering different ways to select the n+1 tree edges out of the 2n edges of $K_{2,n}$, we can focus on the $2n-(n+1)=n-1$ *edges to be deleted.* The vital criterion is that *no two deleted edges can be incident with the same ground station.* Because to avoid isolation of any ground station of $K_{2,n}$.

-The first deleted edge, incident with ground station i_1, can be chosen out of the *2n edges* of the complete bipartite graph$K_{2,n}$.

-The second deleted edge, incident with ground station i_2, can be chosen out of the *2n-2 edges of* $K_{2,n}$ as it must avoid isolating vertex i_1.

- The third deleted edge incident with ground station i_3, can be chosen out of 2n-*4 edge of* $K_{2,n}$ as it must avoid, isolating of either vertex i_1, or i_2.

- In general, the (j+1) $^{\text{sty}}$ deleted edge chosen from any 2n-2j edges.

Then by multiplication rule, the total choice of possibilities of these deleted edges *from* $K_{2,n}$ is $(2n).(2n-2).(2n-4)\dots(2n-2j)=\prod_{j=0}^{n-2}2n-2j$, $j=0,1,2,\dots,n-2$.

However we have *seriously over counted,* since selecting in order edges e_1 $e_2,\dots,e_{n-1}$ gives the same end result as selecting those edges in different order. Indeed every permutation of e_1 $e_2,\dots,e_{n-1}$ gives the same result so the above product needs to be divided by (n-1)!

Therefore the number of spanning trees obtained by deleting n-1 edges $t_n=\frac{\prod_{j=0}^{n-2}2n-2j}{(n-1)!}$

$$\Rightarrow t_n=\frac{1}{(n-1)!}\ \underbrace{(2n)(2n-2)(2n-4)(2n-6),\dots,(2n-2(n-2))}_{n-1\ terms}$$

$$\Rightarrow t_n=\frac{1}{(n-1)!}2^{n-1}n(n-1)(n-2)(n-3),\dots,(2).$$

$$\Rightarrow t_n=\frac{1}{(n-1)!}2^{n-1}n! \qquad \text{Since } n!=n(n-1)! \text{ then}$$

$$\Rightarrow t_n=n2^{n-1}.$$

iv. **INCLUSION-EXCLUSION METHOD:** Let us count items by developing successive overestimates and underestimates. Here we are interested in selecting $n+1$ edge from $2n$ edges of $\mathrm{K}_{2,n}$ to form spanning tree. So there are $N = \binom{2n}{n+1}$ selections in all, but some of them isolate aground station.

Let $\mathrm{E_i}$ to be the set of selections of $n+1$ edge in which ground station i is isolated. $i = 1,2,\dots,n$. Note that in any selections of $n+1$ edges, vertices A and B can not be isolated. Clearly $\overline{\mathrm{E}}_\mathrm{i}$ is the set of selections of n+ 1 edge in which any ground station I is not isolated for $i = 1,2,\dots,n$.

We are interested on the number of n+1 edge selections that do not isolate any ground station. $\mathrm{t_n} = |\bar{E}_1 \cap \bar{E}_2 \cap \dots \cap \bar{E}_n|$ And this quantity can be found by introducing inclusion-exclusion principle. We know that from set theory, $|\bar{E}_1 \cap \bar{E}_2 \cap \dots \cap \bar{E}_n| = N - |E_1 \cup E_2 \cup ,\dots, \cup E_n|$ and

$$|E_1 \cup E_2 \cup \dots \cup E_n|$$

$$= \sum_{i=1}^{n}|E_i| - \sum_{i<j}|E_i \cap E_j| + \sum_{i<j<k}|E_i \cap E_j \cap E_k| - \sum_{i<j<k<l}|E_i \cap E_j \cap E_k \cap E_l| + \dots + (-1)^n|E_1 \cap E_2 \cap \dots \cap E_n|$$

Then $\quad t_n = |\bar{E}_1 \cap \bar{E}_2 \cap \dots \cap \bar{E}_n|$

$$= N - \sum_{i=1}^{n}|E_i| + \sum_{i<j}|E_i \cap E_j| - \dots + (-1)^n|E_1 \cap E_2 \cap \dots \cap E_n|. \quad (1)$$

Since none of the n+ 1 selected edge can be incident with i. That is the vertex i is not incident with all the n+1 selected edges.

So we choose here $|\mathrm{E_i}| = \binom{2n-2}{n+1}$ for each i. *

Then let us find $|E_i \cap E_j|$. Since the vertices i and j vertices are not incident with all of the n+ 1 selected edges of $\mathrm{K}_{2,n}$.

So we choose $|E_i \cap E_j| = \binom{2n-4}{n+1}$ for each i and j. **

Similarly, that is since none of the n+ 1 selected edge can be incident with i, j, and k vertices. We have $\left|E_i \cap E_j \cap E_k\right| = \binom{2n-6}{n+1}$ for each i, j, k.

To calculate $|E_1 \cap E_2 \cap \ldots \cap E_k|$.In this case we have k isolated vertex or ground station and these vertices has 2k edges that are not incident with them.

So we choose $|E_1 \cap E_2 \cap \ldots \cap E_k| = \binom{2n-2k}{n+1}$ for each *i*, *j*, … *k*. ***

Substitution of those equations stars to the equation (1) yields

$$t_n = \binom{2n}{n+1} - \sum_{i=1}^{n}\binom{2n-2}{n+1} + \sum_{i<j}\binom{2n-4}{n+1} - \cdots$$

$$= 1.\binom{2n}{n+1} - n.\binom{2n-2}{n+1} + \frac{n(n-1)}{2}\binom{2n-4}{n+1} + \cdots$$

$$= \binom{n}{0}\binom{2n}{n+1} - \binom{n}{1}\binom{2n-2}{n+1} + \binom{n}{2}\binom{2n-4}{n+1} - \cdots + (-1)^k\binom{n}{k}\binom{2n-2k}{n+1} + \cdots$$

Therefore $t_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{k}\binom{2n-2k}{n+1}$

Now we need to show that $t_n = n2^{n-1}$

Let $t(x) = \sum_{n=0}^{\infty} t_n x^n$ *be the* ***generating function*** *for* $\{t_n\}_{n=0}^{\infty}$.

Then $t_n = [x^n]t(x)$ *Then **apply snake oil method***

$$\text{Now } t(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n+1}x^n$$

$$= \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(-1)^k\binom{n}{k}\sum_{n=0}^{\infty}\binom{2n-2k}{n+1}x^n$$

$$= \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(-1)^k\binom{n}{k}x^{-n+2k}\sum_{n=0}^{\infty}\binom{2n-2k}{n+1}x^n x^{n-2k}$$

$$= \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(-1)^k\binom{n}{k}x^{-n}x^{2k}\sum_{n=0}^{\infty}\binom{2n-2k}{n+1}x^{2n-2k}$$

$$= \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(-1)^k\binom{n}{k}x^{-n}x^{2k} \cdot \frac{x^{n+1}}{(1-x)^{n+2}} \qquad \text{(Book keeper's identity)}$$

$$= \frac{x}{(1-x)^{n+2}} \cdot \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{k} (-x^2)^k$$

$$= \frac{x}{(1-x)^{n+2}} . (1-x^2)^n \qquad \text{Binomial identity}$$

This simplifies $t(x) = \frac{x}{(1-x)^2} . (1+x)^n$

$$= \sum_{n=0}^{\infty} nx^n . \sum_{k=1}^{\infty} \binom{n}{k} x^k$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=1}^{\infty} n \binom{n}{k-1} x^{k-1} \right] x^n$$

Then $t_n = [x^n] t(x) = \sum_{k=1}^{\infty} n \binom{n}{k-1} x^{k-1}$

but $\sum_{k=1}^{\infty} n \binom{n}{k-1} x^{k-1} = n(1+x)^{n-1}$

set $x = 1$, *we have* $t_n = n2^{n-1}$

v. **RECURRENCE RELATION METHOD:** Many counting problems can be reduced to similar problems of smaller size. This leads to recurrence relation that can be solved to produce the general answer. Specifically a spanning tree involving stations $2, 3, \ldots, n$ can be extended to include station 1 in two ways.

- First we add a single edge from 1 to either A or B, so the t_{n-1} spanning trees for 2, ...n give rise to $2t_{n-1}$ spanning trees for 1,2,...,n.

-Alternatively, there are 2^{n-1}spanning trees in which station 1 is joined to both A and B. The following recurrence relation, with initial condition $t_0 = 0$, results.

$$t_n = 2t_{n-1} + 2^{n-1}, \text{ for } n \geq 1$$

We need to show that $t_n = n2^{n-1}$for $n \geq 1$

Proof: Using generating functions.

Insert summation on both sides that runs from 1 up to ∞ to the equation or (multiply x power of the largest index in the equation, then insert summation that runs 1 to ∞).

$$\sum_{N=1}^{\infty} t_n x^n = \sum_{n=1}^{\infty} (2t_{n-1} + 2^{n-1})\, x^n$$

$$\sum_{n=1}^{\infty} t_n x^n = \sum_{n=1}^{\infty} (2t_{n-1}) x^n + \sum_{n=1}^{\infty} (2^{n-1})\, x^n \quad \text{Distribute the summation}$$

$$\sum_{n=1}^{\infty} t_n x^n = 2x \sum_{n=1}^{\infty} (t_{n-1}) x^{n-1} + x \sum_{n=1}^{\infty} (2^{n-1})\, x^{n-1} \quad \text{Make similar index.}$$

$$\sum_{N=1}^{\infty} t_n x^n = 2x \sum_{n=1}^{\infty} (t_{n-1}) x^{n-1} + x \sum_{n=1}^{\infty} (2x)^{n-1}$$

$$\sum_{n=1}^{\infty} t_n x^n = 2x \sum_{n=1}^{\infty} (t_{n-1}) x^{n-1} + x \sum_{n=1}^{\infty} (2x)^{n-1}$$

$$\sum_{n=1}^{\infty} t_n x^n = 2x \sum_{n=1}^{\infty} (t_{n-1}) x^{n-1} + x \,.\, \frac{1}{1-2x} \quad \text{Geometric sum identity}$$

Let $t(x) = \sum_{n=0}^{\infty} t_n x^n = t_o + t_1 x + t_2 x^2 + t_3 x^3 + \cdots$

Then $t(x) - t_o = 2x \,.\, t(x) + \frac{x}{1-2x}$

Since $t_o = 0$, we have $t(x) = \frac{x}{(1-2x)^2}$ and $t(x)$ can be written as $t(x) = \frac{1}{2} \cdot \frac{2x}{(1-2x)^2}$.

This implies $t(x) = \frac{1}{2} \sum_{n=0}^{\infty} n(2x)^n$ Generating function identity.

Therefore $t(x) = \sum_{n=0}^{\infty} n 2^{n-1} x^n$.

Hence, the number of spanning tree of $K_{2,n}$ is $t_n = [x^n] t(x) = n 2^{n-1}$ for n≥1

vi. **CRYPTOGRAPHIC APPROACH METHOD:** We can sometimes solve problem by recognizing it as disguised ("cryptomorphic") version of more familiar (or smilingly more tractable) problem. In this case we set up a 1-1 correspondence between a spanning trees T of $K_{2,n}$ and edges of the hypercube H_n. The hypercube is an n- dimensional generalization of the square ($n = 2$) and the cube ($n = 3$); H_n has 2^n vertices, each of degree n. therefore the total degree of H_n is $\sum_{v \in v(H_n)} \deg (v_i) = n2^n$.

To provide 1-1 correspondence between spanning trees T of $K_{2,n}$ and edges of Hypercube H_n .consider spanning tree T $K_{2,n}$. We construct an n- vector $\vec{x} = (x_1\, x_2\, , \ldots , x_n)$ from T in the following way. For each ground station i,

$$\text{we assign} \quad x_i = \begin{cases} 0 & if\ i\ is\ connected\ only\ to\ A \\ 1 & if\ i\ is\ connected\ only\ to\ B \\ * & if\ i\ is\ conneted\ to\ both\ A\ and\ B \end{cases}$$

Since there is unique ground station i, that is connected to A and B. Therefore H_n contains exactly one *.

Example consider A particular spanning tree T of $K_{2,5}$ giving $\vec{x}$= (0, 1,* , 0 , 1) .

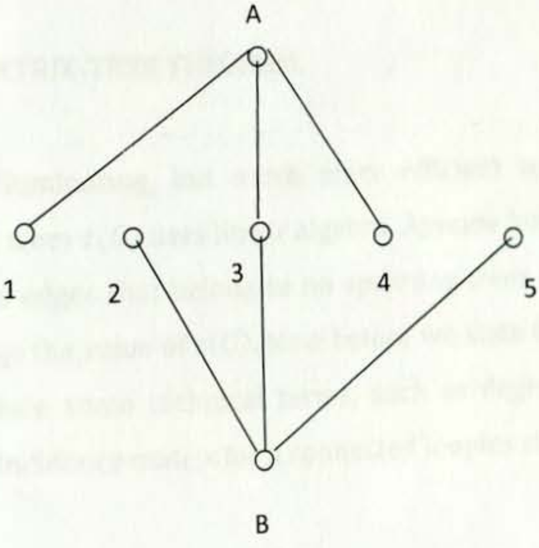


Figure 2.7 spanning tree T and its corresponding vector $\vec{x}$

Now we observe that $\vec{x}$= (0, 1,*, 0, 1) corresponds to a unique edge of H_5; namely, the edge connecting (0,1, 0,0,1) and(0,1,1,0,1).

In general the constructed vector, and thus the spanning tree of $K_{2,n}$, uniquely corresponds to the edge of H_n. Our equivalent problem now becomes to count the edge of the hypercube H_n .we know from first theorem of Graph theory (handshaking lemma or degree sum formula) that:

$$\sum_{v_i \epsilon H_n} d(v_i) = 2|E| .$$

This implies that $|E| = \frac{1}{2}\sum_{v_i \epsilon H_n} d(v_i)$ and we have seen the total degree sum of the hyper cube H_n is $n2^n$. Hence $t_n = \frac{1}{2}\, n\, 2^n = n2^{n-1}$.

vii. Matrix algebra method

This method will be clear to the reader if we discuss matrix- tree theorem first. So we will discuss matrix-tree theorem in the next section and we will solve soon. This can be taken as one application of matrix- tree theorem or sometimes we call it Kirchhoff matrix-tree theorem.

2.3 MATRIX-TREE THEOREM.

A less illuminating, but much more efficient way to calculate the number of spanning trees $\tau(G)$ uses linear algebra. Assume for the moment that G has no loops. Loops are edges that belong to no spanning trees of G, so deleting all the loops do not change the value of $\tau(G)$. Now before we state the matrix-tree theorem, we need to introduce some technical terms, such as degree matrix, laplacian matrix, and reduced incidence matrix for a connected looples simple graph G.

2.3.1. LAPLACIAN AND DEGREE MATRIX OF A GRAPH

i. **Degree matrix D(G)**

Definition: Let G be a graph with Vertices $v_1, v_2, \dots, v_n$. The Degree matrix of G is

$D(G) = (d_{ij})_{nxn}$ *matrix* , Where

$$d_{ij} = \begin{cases} \deg(v_i) & if\ i = j \\ 0 & otherwise \end{cases}$$

ii. **Laplacian matrix L(G)**

Definition: Let G be a graph of order n. Let D (G) and A (G) be and their corresponding degree matrix and adjacency matrix of G. A Laplacian matrix of G, denoted by $L(G)$ is a matrix obtained by taking the difference of the adjacency matrix from the corresponding degree matrix:$L(G) = D(G) - A(G)$.

Example 1 Determine the Degree matrix D(G) and the laplacian matrix L(G) of the following graph G.

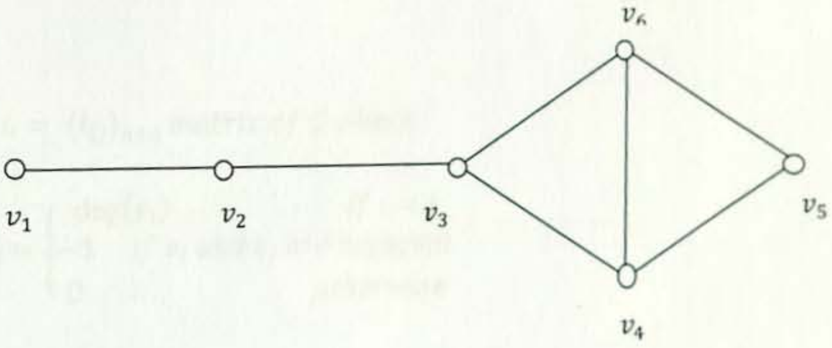


Figure 2.8

SOLUTION: a degree matrix D(G) and an adjacency matrix A(G) corresponding to the order $v_1, v_2, \dots v_6$ of the graph G are

$$D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{And} \quad A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Then the laplacian matrix is

$$L(G) = D(G) - A(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$L(G) \quad = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

REMARK: $D - A = L = (l_{ij})_{nxn}$ *matrix of G* where

$$l_{ij} = \begin{cases} \deg(v_i) & if\ i = j \\ -1 & if\ v_i\ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases}$$

iii. **Reduced incidence matrix M(G)**

Definition: A reduced incidence matrix M(G) of a connected graph G is a matrix which can be obtained by deleting the n^{th} row of the incidence matrix N (G).

Example 2 Determine the reduced incidence matrix $M(G)$ of the following graph

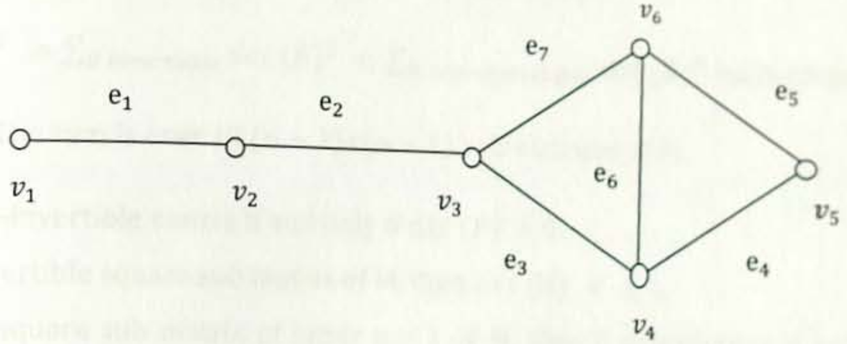


Figure 2.9

SOLUTION: First we calculate the incidence matrix N(G), Then the reduced incidence matrix M(G)

$$N(G)=\begin{pmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\0&1&1&0&0&0&1\\0&0&1&1&0&1&0\\0&0&0&1&1&0&0\\0&0&0&0&1&1&1\end{pmatrix},\quad \mathrm{M(G)}=\begin{pmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\0&1&1&0&0&0&1\\0&0&1&1&0&1&0\\0&0&0&1&1&0&0\end{pmatrix}$$

REMARK: Reduced incidence matrix $M = (n_{ij})_{(n-1)xk}$ *matrix of N.*

2.3.2 ALGEBRAIC APPROACH TO PROVE MATRIX- TREE THEOREM

- **Some basic properties of determinants**

1. Let M is nxk matrix , where $n \leq k$.Then

$$\det(\mathrm{M}.\mathrm{M}^{\mathrm{T}}) = \sum \det(\mathrm{B}.\mathrm{B}^{\mathrm{T}})$$

Where the sum is over all $(\mathrm{n}-1)\mathrm{X}(\mathrm{n}-1)$sub matrices B of M and B^{T} of M^{T}.

Where B is made out of M in the same way as B^{T} is made out of M^{T}.

That is the number of columns deleted from M to get B are the same as the number of rows deleted from M^{T} to get B^{T}. This formula is called *Cauchy- Binet formula.*

2. $det\ (B.B^T) = det\ (B).\det\ (B^T)$-product property of determinant

3. $det\ (B^T) = det\ (B)$ –transpose property of determinant

4. $\sum det\ (B)^2 = \sum_{B\ invertable} det\ (B)^2 + \sum_{B\ non-invertable} det\ (B)^2$ –sum property of det.

   The sum is over all $(n-1)x(n-1)$ sub matrices of M.

5. B is non-invertible matrix if and only if $det\ (B) = 0$.
6. If B is invertible square sub matrix of M, then $det\ (B) = \pm 1$.
7. If B is a square sub matrix of order $n-1$ of M, then B is invertible if and only if the edges corresponding to the column of B determine spanning sub tree of G.
8. If the row sums and column sums of a matrix are all 0, then the cofactors all have the same value.

THEOREM 2.2 (MATRIX-TREE THEOREM): If M is a reduced incidence matrix of a connected graph G, then the number of spanning trees of G is $\tau(G) = det\ (M.M^T)$

Proof

$det\ (M.M^T) = \sum det\ (B.B^T)$- *by cauchy binet formula*

the sum is over all $(n-1)x(n-1)$sub matririces B of M.

$= \sum det\ (B).det\ (B)^T$ - *product property of detrminants*

$= \sum det\ (B)^2$ *, since* $det\ (B^T) = det\ (B)$

$= \sum_{B\ invertable} det\ (B)^2 + \sum_{B\ non-invertable} \det\ (B)^2$ *by proerty 4*

$= \sum_{B\ invertable} det\ (B)^2$ *by property 5*

$= \sum_{B\ \mathrm{invertable}} 1$ by property 6

= *the number of* $(n-1) \times (n-1)$ *invertable sub matrices of B of M.*

$= t(G)$ *by property 7,*

Hence $t(G) = det\ (M.M^T)$.

Example 1 Use reduced incidence matrix M to find the number spanning trees of the graph G.

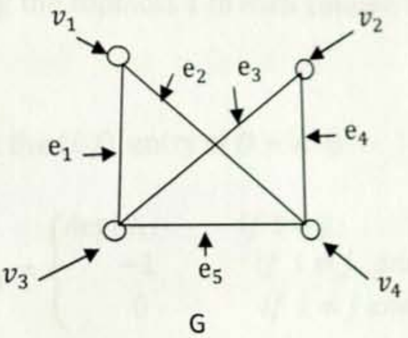


Figure 2.9.1

SOLUTION:

The incidence matrix of G is

$$N(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

Then the reduced incidence matrix of G is

$$\mathrm{M(G)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad .$$

The number of spanning trees of G is given by the formula

$$\tau(G) = det(M.M^T)$$

$$= \det\left[\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= det\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = 8.$$

So the number of spanning tree of G is 8.

Lemma 2.3 Let N be nxk incidence matrix of a connected graph G. Let M be the nxk matrix that results from changing the topmost 1 in each column to -1, then $MM^T = D - A$. where M^T is the transpose of M.

Proof First, notice that the (i,j) entry of $D - A$ is

$$[D-A]_{ij} = \begin{cases} \deg(v_i) & if\ i = j \\ -1 & if\ i \neq j\ \ and\ v_i v_j \in E(G) \\ 0 & if\ i \neq j\ and v_i v_j \notin E(G) \end{cases}$$

Now from matrix multiplication, the (i, j) entry of MM^T is the dot product of the ith row of M and the jth column of M^T. That is

$$[MM^T]_{ij} = ([M]_{i1}\ \ [M]_{i2}\ \ \dots\ \ [M]_{ik}).\big([M^T]_{1j}\ \ [M^T]_{2j}\ \ \dots\ \ [M^T]_{kj}\big)$$

$$= ([M]_{i1}\ \ [M]_{i2}\ \ \dots\ \ [M]_{ik}).\big([M]_{j1}\ \ [M]_{j2}\ \ \dots\ \ [M]_{jk}\big)$$

$$= [M]_{i1}[M]_{j1} + \ [M]_{i2}\ [M]_{j2} + \dots + \ [M]_{ik}\ [M]_{jk}$$

$$= \sum_{r=1}^{n} [M]_{ir}[M]_{jr}$$

- If $i = j$, then this sum counts one for every non zero row entry in the ith row. That is it counts the degree of v_i.
- If $i \neq j\ and\ v_i v_j \notin E(G)$, the there is no column of M in which the ith row and the jth row entries are non zero. Hence the value of the sum in this case is 0.
- If $i \neq j\ and\ v_i v_j \in E(G)$, then the only column in which both the ith row and the jth row entries are non zero is the column that represents the $edge\ v_i v_j$. Since one of these entries is 1 and the other is -1. The value of the sum is -1.We have shown that the (i,j) entry of MM^T is the same as the (i,j) entry of $D - A$ .Hence it is proved.

Let H be a sub graph of G with n vertices and n-1 edges let p be an arbitrary integer between 1 and, n, and let M be the $(n-1) \times (n-1)$ sub matrix of M formed by all rows of M except row p and columns that correspond to the edges in H.

Lemma 2.4 If H is a tree, then $|\det(M')| = 1$.Otherwise, $|\det(M')| = 0$.

First suppose that H is not a tree since H has n vertices and n-1 edges, H must be disconnected. LetH_1, be a connected component of H that does not contain the vertex V_P

Let M″ be the $|V(H_1) \times (n-1)|$sub matrix of M′ formed by eliminate all rows other than the ones corresponding to the vertices of H_1. each column of M″contains exactly two non-zero entries: 1 and -1. Therefore the sum of all of rows of vectors of M″is a zero vector, so the rows of M″are linearly dependent. Since these rows are also rows of M′: we see that $det\,(M') = 0$.

Now suppose that H is a tree. Choose some leaf of H that is not in V_P calling it u_1 let us also say that, e_1 is the edge of H that is incident with u_1. In a tree H-U_1, choose u_2 to be some leaf other than V_P. Let e_2 be the edge of H-u_1 incident with u_2. Keep removing leaves in this fashion until V_P is the only vertex left. Having established the list of Vertices $u_1, u_2, \ldots, u_{n-1}$,. We now create a new$(n-1)\ x\ (n-1)$, matrix M* by rearranging the rows of M′ in the following way: the i^{th} row of M* will be the row of M′ that correspond to the vertex u_i. i=1,2,...,n-1. An important property of matrix M* is that it is lower triangular because for each i, vertex u_i is not incident with any of $e_{i+1}, e_{i+2}, \ldots, e_{n-1}$.

Thus the determinant of M* is equal to the product of the main diagonal entries, which are either 1 or -1, since every u_i is incident with e_i.Thus $|\det(M*)| = 1$ and $|\det(M')| = 1$.This proves the lemma.

Let M be nxn matrix. The (i, j) cofactor of M is defined to be $(-1)^{i+j} det\ (M(i/j))$. Where $M(i/j)$ represents the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of M.

THEOREM 2.5 (MATRIX-TREE THEOREM): If G is a connected labeled graph with Degree matrix D and Adjacency matrix A, then the number of unique spanning trees of G is equal to the absolute value of any cofactor of the Laplacian matrix $D - A$.

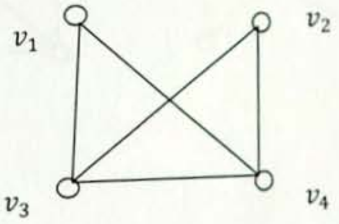
Proof: We know that $D - A = MM^T$ *by lemma* 2.3 . now we are ready to investigate the cofactor of $D - A = MM^T$. It is a fact from matrix theory that if the row sums and column sums of a matrix are all 0, then the cofactors all have the same value. Since the matrix MM^T satisfies this condition, we need to consider only one of its cofactors. We might as well choose i and j such that i+j is even. Let us choose $i = 1$ and $j = 1$ so the (1,1) cofactor of $D - A$ is

$$= det\ ((D-A)\ (1/1)) = det\ (MM^T(1/1))$$

$= det\ (M_1 {M^T}_1)$ Where M_1 is a matrix obtained by deleting the first row of $D - A$ At this point we make use of the Cauchy- Binet formula, which says that the determinant above is equal to the sum of the determinants of $(n-1) \times (n-1)$ sub matrices of M_1, by lemma 2.4. We have already seen that any $(n-1) \times (n-1)$ sum matrices that corresponds to a spanning tree of G will contribute 1 to the sum, while all others contribute 0. This shows that the value of $det((D-A)(1/1)) = det\ (MM^T)\ (1/1)$ is precisely the number of spanning trees of G by theorem 2.2, hence proved.

Example 1 Consider the Graph from figure 2.9.1, what is the number of spanning trees of G?

SOLUTION:



The degree matrix D(G) and the Adjacency matrix A(G) are

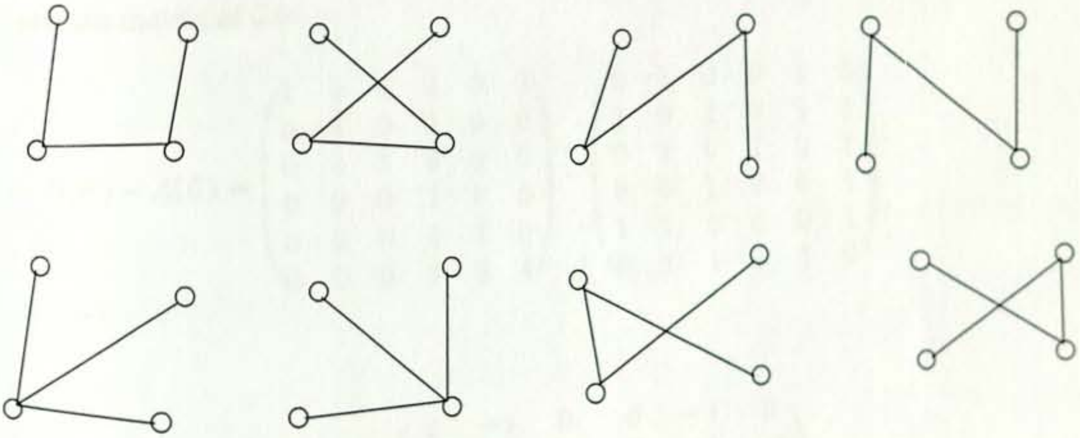
$$D(G) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \text{, and } \; A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\text{And so } D - A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix},$$

The $(3, 3)$ cofactor of $D - A$ is

$$\tau(G) = det\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} = 8.$$

It is true that G has 8 spanning trees;



Example 2 Find the spanning trees of the following graph

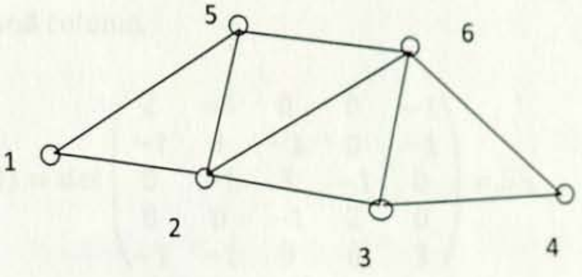


Figure 2.9.2

SOLUTION:

The degree matrix and the adjacency matrix of G are

$$D(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The laplacian matrix of G is

$$L(G) = D(G) - A(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$L(G) = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 3 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

Then $\tau(G)$ is the absolute value of any cofactor of $L(G)$.We could find the cofactor could be by removing the last row and column.

$$\tau(G) = det\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 3 \end{pmatrix} = 55.$$

THEOREM 2.6 (MATRIX-TREE-THEOREM) Let L be the laplacian matrix of a graph G. The number of spanning tress of G is $\tau(G) = \frac{1}{n}\lambda_1\lambda_2 \dots \lambda_{n-1}$ for $\lambda_n = 0, and$ where $\lambda_1\lambda_2 \dots \lambda_{n-1}$ are non zero Eigen values of the laplacian matrix $L(G)$.

Proof: Given undirected graph G, Let D be the directed graph with edges (i,j) and (d_{ij}) for every edge of G .We first observe that there is a bijection between the set of oriented spanning trees of D rooted at r and the set of spanning trees of G .We can take any oriented spanning tree of D rooted at r and get spanning tree of G by disregarding the root and the orientation of the edges. For any spanning tree T of G, we get an oriented spanning tree of D by oriented edges along the unique path from each vertex to r. such path exists because t is connected and unique because T has no cycle. Then $n\tau(G) = \sum_{r=1}^{n} \tau(D,r)$. Let L be the laplacian matrix of D. then the characteristic polynomial of L

$P(t) = \det(tI - L)$. *it is true that* : $\sum_{r=1}^{n} detL_r = (-1)^{n-1}[t]P(t)$

Where $[t]P(t)$ the coefficient of t in $P(t)$, so we have that Where *the* λ_i 's are Eigen values of L and $\lambda_n = 0$, $n\,\tau(G) = \sum_{r=1}^{n} detL_r = (-1)^{n-1}\,[t]P(t)$

$$= (-1)^{n-1}[t]\prod_{i=1}^{n}(t - \lambda_i)$$

$$= (-1)^{n-1}(-1)^{n-1}\lambda_1\lambda_2 \dots \lambda_{n-1}$$

$$= \lambda_1\lambda_2 \dots \lambda_{n-1}$$

$$Therefore\ \tau(G) = \frac{1}{n}\lambda_1\lambda_2 \dots \lambda_{n-1}$$

2.3.3 COMBINATORAL APPROACH TO PROVE MATRX- TREE THEOREM.

In this approach we count spanning trees by formulating a generating function polynomial and we try to relate the matrix-tree theorem.

- **Enumeration of trees by degree of each vertex.** Let us now try to count all trees on a vertex set [n] that have a given degree sequence. The degree sequences of a tree on a vertex set [n] is the sequence $(d_1\ d_2\ ...d_n)$ where d_i is the degree sequence of vertex i. for i=1,2,..,n.

Now for n≤3, we have the following table

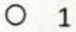
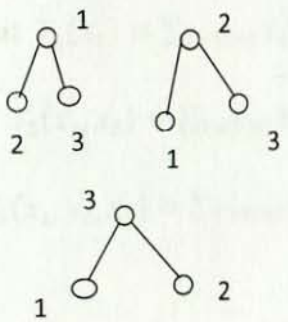
For natural number [n]	Diagram	Degree sequence the spanning trees S	The # of degree sequence	The # of trees
$n = 1$	 $deg(1) = 0$	(1)	1	1
$n = 2$	 deg(1)=deg(2)=1	(1,1)	1	1
$n = 3$		(2,1,1),(1,2,1),(1,1,2)	3	3

Table 2.1 Labeled tree with degree sequence S.

For $n = 4$, the situation is more complicated while there are 16 trees on the vertex set [4], there are only 10 degree sequences.

Example1: Indeed, all degree sequences must consist of either a 3 or three 1 s (four possibilities), or two 2 s and two 1 s (six possibilities) .Different trees may have the same degree sequence. For example in the following graphs show, both trees have the same degree sequence (2, 2, 1, 1,).

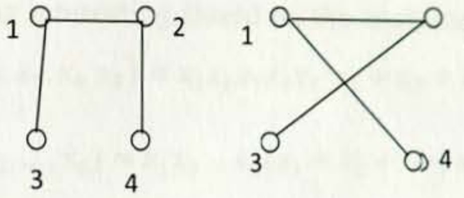


Figure 2.9.3 different trees of the same degree sequences

A more convenient way to count objects according to several parameters is by introducing a generating function in several variables. It is natural to start with a generating function n variables, the i^{th} of which is responsible for the degree d_i of the vertex labeled i in the tree under consideration of our trees into account. More precisely, we can define the generating function as $T_n(x_1, x_2, \dots, x_n) = \sum_{T\, \epsilon T(n)} x_1^{d_1}\ x_2^{d_2} \dots x_n^{d_n}$

Where $T(n)$ the set of all trees on a vertex is set [n], and $(d_1\,, d_2, \dots, d_n)$ is the degree sequence of T in $T(n)$. The examples that we computed for small values of n in the above table, show that $T_1(x_1) = \sum_{T\, \epsilon T(n)} x_1^{d_1} = x_1^{\,0} = 1$

$$T_2(x_1, x_2) = \sum_{T\, \epsilon T(n)} x_1^{d_1}\ x_2^{d_2} = x_1^{\,1} x_2^{\,1} = x_1 x_2$$

$$T_3(x_1, x_2, x_3) = \sum_{T\, \epsilon T(n)} x_1^{d_1}\ x_2^{d_2} x_3^{d_3} = x_1^{\,2}\ x_2^{\,1} x_3^{\,1} + x_1^{\,1} x_2^{\,2} x_3^{\,1} + x_1^{\,1} x_2^{\,1} x_3^{\,2}$$

$$= x_1 x_2 x_3 (x_1 + x_2 + x_3)$$

$$T_4(x_1, x_2, x_3, x_4) = \sum_{T\, \epsilon T(n)} x_1^{d_1}\ x_2^{d_2} x_3^{d_3} x_4^{d_4} = x_1 x_2 x_3 x_4 (x_1 + x_2 + x_3 + x_4)^2$$

The function $T_4(x_1, x_2, x_3, x_4)$ has 16 summands. In order to find some pattern in the sequence of functions T_n, we look at T_2 and T_3 and see that they are all divisible by

- Tree type (1) can be labeled in *5 different ways,* and they correspond to the 5 monomials that look like $x_1{}^4x_2x_3x_4x_5$. That is monomial of degree 5 in which one variable has exponent 4, and the rest have exponents 1.
- Tree type(2) can be labeled in 5.4.3 = *60 different ways*, corresponding to the 60 monomials of degree 5 in which one variable is exponent 3, and one is exponent 2 .

  -Finally tree type (3) can be labeled in $\frac{5!}{2}$ = 5.4.3 = 60 different ways, corresponding to the 60 monomial of degree 5 in which there are three Variable of exponent 2. It is a direct consequence of the multinomial theorem that these *125 monomial* together form $x_1x_2x_3x_4x_5(x_1+x_2+x_3+x_4+x_5)^3$ Confirming our conjecture. Let us see the following general theorem.

Theorem 2.7 Let $n \geq 2$, then the number of labeled trees on a vertex set [n] having degree sequence $(d_1\ d_2\ \dots\ d_n)$ is equal to the coefficient $\binom{n-2}{d_1\ d_2 \dots d_n}$ of $x_1{}^{d_1}x_2{}^{d_2}\dots x_n{}^{d_n}$ in the expansion of $x_1x_2\dots x_n(x_1+x_2+\cdots+x_n)^{n-2}$

Or equivalently $\quad T_n(x_1,x_2,\dots,x_n) = x_1x_2\dots x_n(x_1+x_2+\cdots+x_n)^{n-2}$.

Where $T_n(x_1,x_2,\dots,x_n) = \sum_{T \in T(n)} x_1{}^{d_1}\ x_2{}^{d_2}\dots x_n{}^{d_n}$. This interesting theorem counts the number of labeled trees on n vertices. (The proof is found from bona book).

Remark: The number of labeled trees on n vertices is equal to $T_n = \binom{n-2}{d_1\ d_2 \dots d_n}$.

1. ***Enumeration of spanning trees:*** In this combinatorial approach, we will compute the number $\tau(G)$ of spanning trees. We construct the polynomial f_G for a graph G. Let $G = (V,E)$ be a graph on the set of vertices V with set of edges $E \subset \binom{V}{k} := \{e \subset V : |e| = 2\}$. We associate the variable x_i with each vertex i in V. For each trees T on the set of Vertices V, $|V| \geq 2$. we define ***a* monomial of variable** x_i:

$$m(T) = \prod_{i \in V} x_i{}^{dT(i)-1} \qquad (1)$$

Where $dT(i)$ denotes degree of the vertex i in the tree T. that is the number of edges of T incident to the vertex i.

For a graph G, we construct the polynomial t_G of variables x_i:$t_G := \sum_T m(T)$ where the sum is over all spanning trees T in the graph G. The polynomial t_G in case of the complete graph K_n first was considered by A .cayley, Who found the exact formula $\tau(K_n) = n^{n-2}$ and the polynomial $t_n(G)$ for arbitrary graph was defined by A. Renyi.

Let 0 be not the member of V and $\overline{V} = V \cup \{0\}$, for a graph G on set V, we define the extended graph $\bar{G}$ on the set $\overline{V}$ be agraph obtained from G by adding edges $\{0, i\}$ for all vertices i in V. *Let the variable x be associated with added vertex 0.* For non empty graph G, we construct another polynomial f_G of variables x and x_i, i in V.

$$f_G := t_{\overline{G}} \tag{2}$$

Let $V = \{1,2, \dots, n\}$and $f_G = f_G(x; x_1, \dots, x_n)$ the monomials of the polynomial f_G, which do not include x, correspond to spanning trees of a graph G such that the degree of the vertex 0 is equal to 1. Hence the vertex 0 is connected by an edge with certain vertex i of G.

$$t_G(x_1, \dots, x_n).(x_1 + \dots + x_n) = f_G(0; x_1, \dots, x_n) \tag{3}$$

2. Reciprocity theorem for a polynomial f_G

Theorem 2.8: Let G be a graph on the set of vertices, $V = \{1,2, \dots, n)$.

Then $f_{\overline{G}}(x; x_1, \dots, x_n) = (-1)^{n-1}. f_G(-x - x_1 - \dots - x_n; x_1, \dots, x_n)$ (4)

If $x_i = 1$, this formula was found by S.D Bedrosian and A.k Kelmans.

3. **Computation of spanning trees for certain graphs:** Let G_1, G_2 be two graphs on disjoint sets of vertices .Let $G_1 + G_2$ denote disjoint union of the graphs. We associated variables $y_1, y_2, \dots, y_m$ to the vertices of G_1 and $z_1, z_2, \dots, z_n$ to the vertices of G_2. Then, $f_{G_1+G_2}(x; y_1,, \dots, y_m, z_1, \dots, z_n) = x.\ f_{G_1}(x; y_1, \dots, y_m).\ f_{G_2}(x; z_1, \dots, z_n)$ (5) Indeed, every spanning tree T in the graph $\overline{G_1 + G_2}$ splits in to two spanning trees T_1 and T_2 in the graph $\overline{G_1}$ and $\overline{G_2}$ respectively. And $dT(o) - 1 = (dT_1(0) - 1) + (dT_2(0) - 1) + 1$. Hence the factor x in the right part occurs.

Example1 let O_n denote the empty graph on the set $[n] = \{1,2 \dots n\}$. $f_{O_1} = 1$ Consequently applying definition of equation (5), we get $f_{O_n} = x^{n-1}$. For $\mathrm{K_n}$ on the set {1, 2,..., n} By reciprocity theorem: $\mathrm{f_{K_n} = f_{\bar{O}_n}} = (-1)^{\mathrm{n}-1}(-x - x_1 - \cdots - x_{\mathrm{n}})^{\mathrm{n}-1}$

$$= (-1)^{\mathrm{n}-1}(-1)^{\mathrm{n}-1}(x + x_1 + \cdots + x_{\mathrm{n}})^{\mathrm{n}-1} = (x + x_1 + \cdots + x_{\mathrm{n}})^{\mathrm{n}-1} \quad (6)$$

Example 2 let $\mathrm{K_{m,n}}$ be the full bipartite graph, variables $y_1, y_2, \dots, y_m$ correspond to m vertices of the first part. And variables $z_1, z_2, \dots, z_n$ correspond to be vertices of the second part, $\mathrm{f_{K_m+K_n}} = x.\, \mathrm{f_{K_m}}(x; y_1, \dots, y_m).\, \mathrm{f_{K_n}}(x; z_1, \dots, z_n)$ by definition of equation (5)

$$= x.(x + y_1 + \cdots + y_m)^{m-1}.(x + z_1 + \cdots + z_n)^{n-1} \quad \text{By equation (6)}$$

Now, $\mathrm{F_{K_{m,n}} = f_{\overline{K_m+K_n}}}$ since $\mathrm{K_{m,n} = \overline{K_m + K_n}}$

$$\mathrm{F_{\overline{K_m+K_n}}}(x; y_1, \dots, y_m,\ z_1, \dots, z_n)$$

$$= (-1)^{\mathrm{m+n}-1}.\mathrm{f_{K_m+K_n}}[-x - \mathrm{y_1} - \cdots - \mathrm{y_m} - \mathrm{z_1} - \cdots - \mathrm{z_n}; \mathrm{y_1}, \dots, \mathrm{y_m}, \mathrm{z_1}, \dots, \mathrm{z_n}], \text{ By equation (4)}$$

$$= (x + y_1 + \cdots + y_m + z_1 + \cdots + z_n).(x + z_1 + \cdots + z_n)^{m-1}(x + y_1 + \cdots + y_m)^{n-1}$$

We can find the corresponding formula for $\mathrm{t_G}$ by equation (3)

$$\mathrm{t_{K_n}} = (x_1 + \dots + x_{\mathrm{n}})^{\mathrm{n}-2} \quad (7)$$

$$\mathrm{t_{K_{m,n}}} = (\mathrm{y_1 + y_2 + \cdots + y_m})^{\mathrm{n}-1}(z_1 + z_2 + \cdots + z_n)^{m-1} \quad (8)$$

Then Let $t(G)$ denote the number of spanning trees of a graph G.

Substituting $x_1 = x_2 = \cdots = x_n = 1$, we have the following

$$t(K_n) = n^{n-2} \quad (9)$$

$$t(K_{m,n}) = m^{n-1} n^{m-1} \quad (10)$$

4. Generalization to oriented nets:

An oriented net (or simply net) on the set of vertices V is defined by the set of conductivities g_{ij} is in ℝ assigned to every ordered pair of vertices i, j in V. We associate a net with each graph as $g_{ij} = g_{ji} = \begin{cases} 1\ , & \text{if ij is an edge of the graph } \overline{\mathrm{G}} \\ 0\ , & \text{otherwise} \end{cases}$

We will consider only nets $G = (g_{ij}),\ i, j$ in V, without loops. That is $g_{ij} = 0$ for all i inV. Let T be a tree on the set of vertices $\bar{V} = V \cup \{0\}$.Let us orient the tree T from the root in the vertex 0.The multiplicity of T in the net G is $K_G(T) = \prod_{(i,j)} g_{ij}$, where the product is over all ordered pairs (i, j) in $V \times V$ such that (i, j) is an edge of T. If T consist only of edges $(0, i)$ and does not contain any edge (i, j) in VxV, we assume that $\mathrm{K_G(T)} = 1$. If G is a net associated with a graph,

Then $\mathrm{K_G(T)} = \begin{cases} 1, \text{if T is a spanning tree of the graph } \overline{\mathrm{G}} \\ 0, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{otherwise} \end{cases}$

One can define a polynomial $\mathrm{f_G}$ for a net G, $\mathrm{f_G(T)} = \sum_{\mathrm{T}} \mathrm{K_G(T)}.\mathrm{m(T)}$. The sum is over all trees on the set of vertices $\bar{V}$ and $m(T)$ is defined by equation (1)

Example Let $v = \{1,2\}, g_{12} = \alpha, g_{21} = \beta$. then$f_G = x + \alpha x_1 + \beta x_2$.

A net $\bar{G} = (\overline{g_{ij}})$ on the same set V is called complimentary to the net G= (g_{ij}) on the set V if $\bar{g}_{ij} = \begin{cases} 1 - g_{ij} & , for\ all\ i \neq j \\ g_{ij} = 0 & , \ for\ j = j \end{cases}$

It is clear that in the case when the net G corresponds to a graph the concepts complementary net and complimentary graph coincide. The formula in equation (4) remains true for nets.

Theorem 2.9 Let G be an oriented net on the set of vertices {1, 2,...,n),then $f_{\bar{G}}(x; x_1, \dots, x_n) = (-1)^{n-1} f_G(-x - x_1 - \dots - x_n; x_1, \dots, x_n)$.For instance a net from the above example we get $f_{\bar{G}}(x; x_1, x_2) = x + (1-\alpha)x_1 + (1-\beta)x_2 = (-1)[(-x - x_1 - x_2) + \alpha x_1 + \beta x_2]$

$= (-1)^{2-1} f_G(-x - x_1 - x_2; x_1, x_2)$

Matrix -tree theorem: One can express the polynomial f_G as determinant of a matrix. In the beginning we formulate tutte's generalization of matrix-tree theorem. Now let $z_{ij}, i,j\ \ in\ \bar{V} = V \cup \{0\}$, be the collection of commutative variables, we assume that $z_{ij} = 0\ for\ \ i\ \ in\ \bar{V}$.With any tree T on the set of vertices $\bar{V}$, we associate a monomial $M(T)$ of variables z_{ij}. Let us orient the tree T from the root in vertex 0 and assume that $M(T) := \prod_{(i,j)} z_{ij}$ Where the product is over all pairs (i, j) in $\bar{V}x\bar{V}$ which are oriented edges of T ,where the sum is over all trees on the set of vertices $\bar{V}$.Without loss of generality we can assume that $V = \{1,2 \dots n\}$. In this case $F_n := F_{\{1,2,\dots,n\}}$ is a polynomial of z_{ij} , $0 \leq i,j \leq n$. Let Kirchhoff's matrix be nxn matrix A= $(a_{ij}), i,j\ \ in\ \ \{1,2,\dots,n\}$ where

$$a_{ij} = \begin{cases} \sum_{l=0}^{n} z_{lj} & , \ if\ i = j \\ -z_{ij} & , \ if\ i \neq j \end{cases} \qquad (11)$$

Theorem 2.9.1 $F_n = \det A$ .Let now G be a net on a set of vertices V with conductivities g_{ij}.

$$Assume\ that\ \ z_{ij} = x_i g_{ij}, i,j\ \ in\ V, \quad z_{0i} = x\ \ i\ \ in\, V. \qquad (12)$$

It is clear that, $K_G(T).m(T) = M(T)$, where m(T) is defined as monomial of variables x_i in equation (1) and $K_G(T)$ is also defined as multiplicity of T in the net G in equation (2) respectively. Substituting (12) into (11) we get the following Corollary.

Corollary 2.9.2 $x.\ f_G(x; x_1, \dots, x_n) = detB$ where B=$(b_{ij}), 1 \leq i,j \leq n, is\ n\ x\ n\ \ matrix.$ where $b_{ij} = \begin{cases} x + \sum_{l=1}^{n} x_l g_i, & i = j \\ -x_i g_{ij} & , \ i \neq j \end{cases}$. Counting spanning trees using this way anybody can deal further. Let us see some application of matrix tree theorem in the next.

CHAPTER THREE: APPLICATION OF MATRIX-TREE THEOREM

3.1 COUNTING THE SPANNING TREES OF GRAPH

3.1.1 ***COMPLETE GRAPH K_n*** : It seems difficult to count the spanning tree of complete graph of any order n. but it is possible to calculate the exact number of spanning tree of such graph using matrix- tree theorem. This provides a nice formula that helps us to count the complete graph of any order, which is known as cayley's formula. So let us state and proof cayley's theorem.

THEOREM 3.1 (CAYLEY'S THEOREM): The exact number of spanning trees of complete graph of order n is equal to $\mathrm{n}^{\mathrm{n}-2}$.

Proof We know that a complete graph of order n is denoted by $\mathrm{K_n}$ and each vertices of this graph has degree n-1.so the degree matrix D and the adjacency matrix A is

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n-1 \end{pmatrix}_{nxn} \quad \text{And}$$

$$A(K_n) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{nxn}$$

Then the laplacian matrix $L(K_n) = D(K_n) - A(K_n)$

$$L(K_n) = \begin{pmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n-1 \end{pmatrix}_{nxn} - \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}_{nxn}$$

$$L(K_n) = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{pmatrix}_{nxn}$$

Now by matrix- tree theorem, the number of spanning trees of K_n is $T(K_n)$ which is the absolute value of any cofactor of the laplacian matrix $L(K_n)$.so let us consider the(1,1)cofactor of $L(K_n)$.

$$\tau(K_n) = \det\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{pmatrix}_{n-1xn-1}$$

Step1 Add all rows to the first row to get $(n-1+(-1)(n-2)=1$

$$\tau(K_n) = \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{pmatrix}_{n-1xn-1}$$

Step2 Now add the first row to all other rows to make lower triangular.

Then we get $\tau(K_n) = \det\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}_{(n-1)x(n-1)}$

Using the first column we can find the determinant of the above matrix as follows

$$\tau(K_n) = 1.\begin{vmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{vmatrix}_{(n-2)x(n-2)}$$

$$= \underbrace{n.n.n\dots n}_{(n-2)times} = n^{n-2}\ .$$

$$Therefore\quad \tau(K_n) = n^{n-2}\ .$$

3.1.2 COMPLETE BIPARTITE GRAPH $K_{m,n}$

3.1.3 Application Problem. Let us solve one interesting problem that are stated and verified on page 17 using matrix-tree theorem. Find the number spanning trees τ_n of acomplet bipartite graph in $K_{2,n}$

Matrix tree theorem counts the number of spanning trees of any undirected graph G on n vertices by evaluating the absolute value of any cofactor of the laplacian matrix of G.

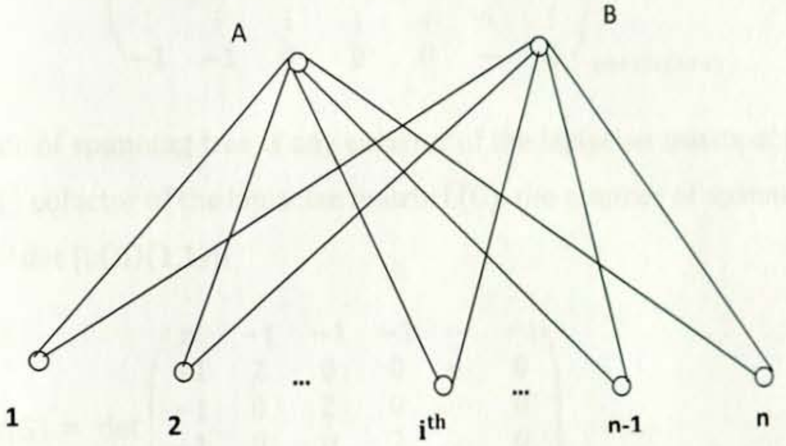


Figure 3.2 A possible configuration of $K_{2,n}$.

Let $G = K_{2,n}$ and if we order the vertices as A, B, 1, 2,..., n. Then the degree matrix D and the adjacency matrix A are

$$D(G) = \begin{pmatrix} n & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & n & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}_{(n+2)x(n+2)} \quad and$$

$$A(G) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n+2)x(n+2)} \quad \text{, Then}$$

$$L(G) = \begin{pmatrix} n & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & n & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}_{(n+2)x(n+2)} - \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n+2)x(n+2)}$$

$$L(G) = \begin{pmatrix} n & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & n & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & 2 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & 2 & 0 & \cdots & 0 \\ -1 & -1 & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}_{(n+2)x(n+2)}$$

Now the number of spanning tree is any cofactor of the laplacian matrix of $L(G)$. Consider the $(\mathrm{i},\mathrm{j}) = (1,1)$ cofactor of the laplacian matrix L(G). the number of spanning trees of G is $\tau(\mathrm{G}) = |(-1)^{1+1}\det\,[\mathrm{L}(\mathrm{G})(1,1)]|$

$$\tau(\mathrm{G}) = \det \begin{pmatrix} n & -1 & -1 & -1 & \cdots & -1 \\ -1 & 2 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}_{(n+1)x(n+1)}$$

This matrix Obtained by deleting the first row and column of $L(G)$. Now using first row, we can evaluate the determinant of the above matrix

$$\tau(G) = \begin{vmatrix} n & -1 & -1 & -1 & \cdots & -1 \\ -1 & 2 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 2 \end{vmatrix}_{(n+1)x(n+1)}$$

$$\tau(G) = n\begin{vmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{vmatrix}_{nxn} + 1\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{vmatrix}_{nxn} - 1\begin{vmatrix} -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{vmatrix}_{nxn}$$

$$-1\begin{vmatrix} -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 2 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{vmatrix}_{nxn} + \cdots + \begin{vmatrix} -1 & 2 & 0 & 0 & \cdots \\ -1 & 0 & 2 & 0 & \cdots \\ -1 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ -1 & 0 & 0 & 0 & \cdots \end{vmatrix}_{nxn}$$

Then,

$\tau(G) =$

$$n2\begin{vmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{vmatrix}_{(n-1)x(n-1)} - 1\begin{vmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{vmatrix}_{(n-1)x(n-1)} - \begin{vmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{vmatrix}_{(n-1)x(n-1)}$$

$$-\begin{vmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{vmatrix}_{(n-1)x(n-1)} + \cdots - 1\begin{vmatrix} 2 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}_{(n-1)x(n-1)}$$

$$= 2n2^{n-1} \underbrace{-2^{n-1} - 2^{n-1} - 2^{n-1} - \cdots - 2^{n-1}}_{n\ terms}$$

$$\tau(G) = 2n2^{n-1} - n2^{n-1} = n2^{n-1}.$$

3.1.4 **SOINS THEOREM:** This theorem gives us the generalized ways of counting spanning tree problem for any complete bipartite graph of any order. So it is efficient method of calculating spanning trees of such graph.

THEOREM 3.2 the exact number of spanning trees of a complete bipartite graph $K_{m,n}$ is equal to $m^{n-1}n^{m-1}$.

Proof Let $G=K_{m,n}$ be a comlet bipartite graph.

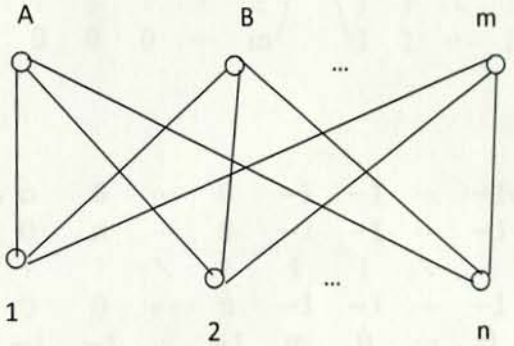


Figure 3.3

let us order the vertices as A,B,C,...,m,1,2,3,...,n. then the degree matrix D and the adjacency matrix A of G are

$$D(G)=\begin{array}{c}A\\B\\\vdots\\m\\1\\2\\\vdots\\n\end{array}\left(\begin{array}{cccccccc}n&0&\cdots&0&0&0&\cdots&0\\0&n&\cdots&0&0&0&\cdots&0\\\vdots&\vdots&\ddots&0&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&n&0&0&\cdots&0\\0&0&\cdots&0&m&0&\cdots&0\\0&0&\cdots&0&0&m&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&0&0&\cdots&m\end{array}\right)_{(m+n)x(m+n)}\qquad \text{And}$$

$$A(G)=\begin{array}{c}A\\B\\\vdots\\m\\1\\2\\\vdots\\n\end{array}\left(\begin{array}{cccccccc}0&0&\cdots&0&1&1&\cdots&1\\0&0&\cdots&0&1&1&\cdots&1\\\vdots&\vdots&\ddots&0&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&1&1&\cdots&1\\1&1&\cdots&1&0&0&\cdots&0\\1&1&\cdots&1&0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1&0&0&\cdots&0\end{array}\right)_{(m+n)x(m+n)}$$

Then the laplacian matrix is $L(G) = D(G) - A(G)$

$$= \begin{pmatrix} n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & n & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & m \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\text{Hence,}\quad L(G) = \begin{pmatrix} n & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & n & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & m & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & m \end{pmatrix}_{(m+n)x(m+n)}$$

Now the number of spanning trees of G is any cofactor of $L(G)$.

Consider the (1,1) cofactor of $L(G)$.

$$\tau(G) = \det\begin{pmatrix} n & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & n & -1 & -1 & \cdots & -1 \\ -1 & \cdots & -1 & m & 0 & \cdots & 0 \\ -1 & \cdots & -1 & 0 & m & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & 0 & 0 & \cdots & m \end{pmatrix}_{(m+n-1)x(m+n-1)}$$

In this matrix the first m rows look "similar", then the last n-1 look "similar". The same is true for columns. To compute this determinant,

Step1 Add all rows to the first one to get

$$\tau(\mathrm{G}) = \det\begin{pmatrix} 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{n} & -1 & -1 & \cdots & -1 \\ -1 & \cdots & -1 & \mathrm{m} & 0 & \cdots & 0 \\ -1 & \cdots & -1 & 0 & \mathrm{m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & 0 & 0 & \cdots & \mathrm{m} \end{pmatrix}_{(\mathrm{m+n-1})\mathrm{x}(\mathrm{m+n-1})}$$

Then

Step2 Add the first row to each of the last n-1 rows, to get triangular matrix

$$\tau(\mathrm{G}) = \det\begin{pmatrix} 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{n} & -1 & -1 & \cdots & -1 \\ 0 & \cdots & 0 & \mathrm{m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \mathrm{m} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \mathrm{m} \end{pmatrix}_{(\mathrm{m+n-1})\mathrm{x}(\mathrm{m+n-1})} = \mathrm{n}^{\mathrm{m}-1}\mathrm{m}^{\mathrm{n}-1}.$$

Therefore the number of spanning trees $\tau(K_{m,n}) = n^{m-1}m^{n-1}$.

REFERENCES

1. Graph theory with application, J.A Bondy and U.S.R. Murty, 1976, the macmillan press Ltd.
2. Combinatorics and Graph theory, John M.Harris, Jeffy L.Hirst.Michael J.Mossinghoff, 2000 Springs-verland, New York Inc.
3. A Walk though combinatorics,bona second edition,2006 Springs-verland, New York Inc.
4. Spanning trees, Douglas R.Shier, Mathematical Association of America. http://www.jstor.org/stable/2690815
5. Matrix tree theorem, Konstantin Karagatsos, May 19, 2000.
6. Matrix Tree Theorem, Janneke van den Boomen, June19, 2007.Radboud Universiteit Nijmegen.
7. Algebraic combinatorics, David Witmer, April 21, 2011. Lionel Levine
8. Enumeration of spanning trees of a graph, Igor Pak Alexander postnikove, November 1, 1994.Harvard University, Massachusetts Institute of Technology.