

ADDIS ABABA UNIVERSITY
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A STUDY OF PERIODIC SOLUTION FOR A CLASS OF SECOND
ORDER DIFFERENTIAL EQUATIONS

BY
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Permission

This is to certify that this thesis is prepared by Asfaw Dillo in the department Of Mathematics, Addis Ababa University, under my supervision. I here also confirm that the thesis can be submitted for evaluation by examiners and eventual defense.

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Abstract

*In this paper we give sufficient conditions ensuring that the periodic solutions for a class of second order differential Equations and positive periodic solutions of second-order differential equations ; system of periodic solutions for autonomous and nonlinear differential equations, each of which has at least one periodic solution. This paper consists three parts. In the **first part** of this paper, we study periodic solutions for second order differential equations are provided. The method of proof relies on the Schauder's fixed point theorem. In the **second part**, we have obtained existence results for single and multiple positive periodic solutions of second order differential equation. The results are based on a fixed Point theorem for cones. In the third part, periodic solutions for nonlinear differential equations and autonomous systems are considered.*

Finally, in each part of the paper, prototypical examples are provided to illustrate the main results of the paper and conclusion is drawn as well.

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Chapter one

1. Introduction and Preliminaries

1.1 Introduction

The study of periodic solutions for second order differential Equations and periodic solutions of ordinary differential Equations has been a significant part of mathematics for several centuries. The earliest motivation for the study was classical mechanics, especially celestial mechanics. For example, the Equations arising in Celestial mechanics are autonomous. The existence and multiplicity of periodic solutions are important aspect in differential equations qualitative analysis. The existence problems for periodic solutions and periodic boundary value problems for first-order and second-order ordinary differential equations have attracted the attention of many authors. Many theorems and methods of nonlinear functional analysis, as one can see [1,3,4,5,6,8,9,16,17,18], have been applied to these problems. These theorems and methods are mainly the upper and lower solution methods and the monotone iterative technique see [20, 26, 30], the continuation method of topological degree (see [21, 27, 28, 31]), fixed point theorem and the variational method and critical point theory (see [22, 24, 25, 29]).

In the **first part** of this paper, we investigate the *existence of periodic solutions* of the following differential equation

$$-x''(t) + a(t)x'(t) = g(t, x(t)) - f(t, x(t), x'(t)) \quad (1.1)$$

Where $a(t)$ is a continuous ω – periodic function, $g(t, u), f(t, u, v)$ are ω – periodic in t and $\omega > 0$.

Eq. (1.1) includes many important models, for example,

$$x''(t) + \mu \sin x(t) = h(t) \quad (1.2)$$

$$x''(t) + f(x'(t)) + g(x(t)) = e(t) \quad (1.3)$$

$$x''(t) + cx'(t) + \mu \sin x(t) = h(t) \quad (1.4)$$

$$x''(t) + f(x(t)) x'(t) + g(x(t)) = e(t) \quad (1.5)$$

$$x''(t) = -X^{-\lambda(T)} \quad , \lambda > 0 \quad (1.6)$$

where λ is constant.

The above equations arise in many fields, such as, physics, mechanics and engineering. (See [2, 10, 11, 13, 14, 15])

One of the main purposes of this paper is to discuss the existence of periodic solutions of equation (1.1) by means of Schauder's fixed point theorem. The method of proof is in a simple idea and is composed of two steps: the first step is to *transform the original equation in to a first order integro-differential equation through a linear integral operator* and the second step is an *application of the Schauder's fixed point theorem*. The existence of single periodic solution for (1.1) has been established under suitable behavior of g and f on some closed sets. So some information on the location of periodic solution is also obtained, leading to multiplicity results. The results are new for (1.2) – (1.5). See (Corollary 2.2, corollary 2.3, theorem 2.3).

In the **second part** of this work, we consider the existence of *positive periodic solutions* of second-order differential equation

$$x''(t)+a(t)x'(t)+f(t, x(t)) = 0 \tag{1.7}$$

where $a(t) \in C(\mathbb{R}, \mathbb{R})$, $f(t, u) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $a(t+\omega) = a(t)$, $f(t+\omega, u) = f(t, u)$.

Li[29] has discussed the existence of positive periodic solutions of differential equation.

$$x''(t)+g(t, x(t)) = 0 \tag{1.8}$$

where $g(t, u) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g(t+\omega, u) = g(t, u)$. By using the theory of the fixed point index for cones, we have obtained the optimal conditions for the non-linear term g so that equation

(1.8) has a positive periodic solution. But this method is not valid for equation (1.7) with the term x' . By using a new fixed point theorem for cones, we obtain existence results for single and multiple positive periodic solutions of Equation (1.7) which extend the results given in [29].

In this work, in the **third part**, we consider the *periodic solutions for autonomous system* of the form.

$$x' = f(x)$$

Where the domain of f_i is an open set D in \mathbb{R}^n and f_i satisfies a local Lipschitz condition and *periodic solutions for nonlinear differential equations* of the form

$$x' = f(t, x)$$

where f has domain $\mathbb{R} \times \mathbb{R}^n$ and f has continuous first derivative in all variables at each point of $\mathbb{R} \times \mathbb{R}^n$.

1.2 PRELIMINARIES

1.2.1. DEFINITIONS AND CONCEPTS

Definition: Let $\{f_n\}$ be a sequence of functions defined on a set E . $\{f_n\}$ is said to be uniformly bounded on E if there exists a number M such that

$$|f_n(x)| < M \text{ for all } x \in E, n = 1, 2, 3, \dots$$

Example: Let $f_n(x) = \sin nx$, $0 \leq x \leq 2\pi$, $n = 1, 2, 3, \dots$

Consider that for each $x \in E = [0, 2\pi]$ and $n \in \mathbb{N}$,

$$|f_n(x)| = |\sin nx| \leq 1$$

Therefore, $\{f_n\}$ are uniformly bounded sequence of continuous functions on compact set $[0, 2\pi]$.

Definition: A family F of real - valued functions f defined on a set E of real numbers is said to be *equicontinuous* on E if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f \in F, x, y \in E, \text{ and } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Theorem: If $\{f_n\}$ is a sequence of continuous functions on $[a, b]$, and suppose $\{f_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ is equicontinuous on $[a, b]$.

Proof: Assume that $f_n \rightarrow f$ uniformly on $[a, b]$. Let $\varepsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_N(y)| < \varepsilon/3 \text{ for all } x \in [a, b] \text{ and for all } n \geq N$$

Since continuous functions are uniformly continuous on $[a, b]$, there is a $\delta > 0$ such that $1 \leq i \leq N$ and

$$|x - y| < \delta \Rightarrow |f_i(x) - f_i(y)| < \varepsilon/3$$

Now if $n > N$ and $|x - y| < \delta$, it follows that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Therefore, $\{f_n\}$ is equicontinuous on $[a, b]$.

The proof is complete

Definition: (Cauchy Sequence)

A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists $n(\varepsilon)$ such that for all $m, n \geq n(\varepsilon)$ we have

$$\|x_n - x_m\| \leq \varepsilon$$

Theorem: Let X be a real vector space. A norm on X is a map $\|\cdot\|: X \rightarrow [0, \infty]$ satisfying the following requirements

- (i) $\|x\| = 0$ if and only if $x = 0$
- (ii) $\|\alpha x\| \leq |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$ and for all $x \in X$

$$(iii) \quad \|x + y\| \leq \|x\| + \|y\| \text{ for } x, y \in X$$

The pair $(X, \|\cdot\|)$ is called a normed vector space.

Definition: An ordered field F is said to be complete (or to have the completeness property) if and only if every non-empty subset of F that has an upper bound has a least upper bound in F .

Definition: A normed Space is complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ converges, i.e. there exists $x \in X$ with

$$\|x_n - x\| \rightarrow 0, \text{ as } n \rightarrow \infty$$

We write $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition: (Banach Space): A Banach space is a vector space X over the field \mathbb{R} of real numbers, or over the field \mathbb{C} of complex numbers, which is equipped with a norm and which is complete with respect to that norm, that is to say, for every Cauchy sequence $\{x_n\}$ in X , there exists an element x in X such that

$$\lim_{n \rightarrow \infty} x_n = x \text{ or equivalently: } \lim_{n \rightarrow \infty} \|x_n - x\| = 0$$

Definition: (Convex Set)

A convex set is a region such that, for every pair of points within the region, every point on the straight line segment that joins the pair of points is also within the region.

Theorem: (Fixed point theorem)

A fixed point of a mapping $K: C \subseteq X \rightarrow C$ is an element $x \in C$ such that

$$K(x) = x.$$

Theorem (ASCOLI- ARZELA THEOREM): Let $\{f_n\}$ be an equicontinuous sequence of functions on $[a, b]$, and suppose there exists $M > 0$ such that for all n and for all $x \in [a, b]$,

$$|f_n(x)| < M$$

Then $\{f_n\}$ contains a subsequence which is uniformly convergent on $[a, b]$

Definition (The concept of convergence):

(1) Given a normed vector space X , we say that a sequence of vectors $\{f_n\}$ converges to a vector f if

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0$$

(2) A mapping $F: X \rightarrow Y$ between two normed spaces is called continuous if

$$\lim_{n \rightarrow \infty} f_n = f \Rightarrow \lim_{n \rightarrow \infty} F(f_n) = F(f)$$

Brouwer Fixed point Theorem: If $X = [a, b] \subseteq \mathbb{R}$ and $f: X \rightarrow X$ is continuous then f has a fixed point. i.e. a fixed point exists provided X is a compact and convex subset of \mathbb{R}^N .

Schauder Fixed point theorem: If D is a non-empty, convex and compact subset of Banach space B and $T: D \rightarrow D$ a continuous function, then T has a fixed point in D , that is, there exists an $x \in D$ such that $T(x) = x$.

Remark: Schauder's Fixed point Theorem is the infinite-dimensional analog of the Brouwer fixed point Theorem.

I.e. Schauder Fixed point Theorem is the extension of Brouwer fixed point Theorem.

Definition (The Lipschitz condition): Let f be a function defined for (x, y) in a set S . We say f satisfies a Lipschitz condition on S in y if there exists a constant K such that

$$|f(x, y_1) - f(x, y_2)| < K |y_1 - y_2|$$

For all $(x, y_1), (x, y_2)$ in S . The constant K is called a Lipschitz constant.

1.2.2 PERIODIC SOLUTIONS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

The main purpose of this section is to discuss the existence of periodic solutions of equation (1.1) by means of Schauder's fixed point theorem. There are two steps to be considered. The first step is to transform the original equation in to a first order integro-differential equation through a linear integral operator and the second step is an application of the Schauder's fixed point theorem.

Let $X = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t + \omega) = x(t) \text{ for all } t \in \mathbb{R}\}$ with the norm $\|x\| = \max_{t \in [0, \omega]} |x(t)|$

Clearly, X is a Banach space. $\lim_{n \rightarrow \infty} \max_{t \in [0, \omega]} |x_n(t) - x(t)| = 0$

Proof: To show X is a Banach space, we have to show that $\{x_n(t)\}$ converges to $x(t)$, $\{x_n\}$ is a Cauchy sequence and x is continuous.

A sequence of functions $\{x_n(t)\}$ converges to $x(t)$ if and only if

$$\lim_{n \rightarrow \infty} \|x_n - x\| = \lim_{n \rightarrow \infty} \max_{t \in [0, \omega]} |x_n(t) - x(t)| = 0$$

That is, in the language of real analysis x_n converges uniformly to x .

For any fixed $t \in [0, \omega]$, $x_n(t)$ is a Cauchy sequence of real numbers. By completeness of \mathbb{R} , there is a limit $x(t)$ for each t . Thus, we get a limiting function $x(t)$. Moreover, letting $m \rightarrow \infty$ in

$$|x_n(t) - x_m(t)| \leq \epsilon \text{ for all } n, m > N_\epsilon, t \in [0, \omega]$$

we see,

$$|x_n(t) - x(t)| \leq \epsilon \text{ for all } n > N_\epsilon, t \in [0, \omega],$$

That is, $x_n(t)$ converges uniformly to $x(t)$.

From real analysis the uniform limit of continuous functions is again continuous.

Fix $t \in [0, \omega]$ and $\epsilon > 0$

We need to find a δ such that

$$|t - s| < \delta \Rightarrow |x(t) - x(s)| < \epsilon$$

Pick n so that

$$\|x_n - x\| < \epsilon/3$$

And so that

$$|t - s| < \delta \Rightarrow |x_n(t) - x_n(s)| < \epsilon/3$$

Then

$$|t - s| < \delta \Rightarrow |x(t) - x(s)| \leq |x(t) - x_n(t)| + |x_n(t) - x_n(s)| + |x_n(s) - x(s)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

as required.

Hence $x(t) \in X$ and thus, every Cauchy sequence in X converges.

That is, X is a Banach space.

The proof is complete.

Let $p, q \in X$ and consider the following two differential equations

$$x'(t) = -p(t)x(t) + q(t), \quad (1.9)$$

$$x'(t) = p(t)x(t) - q(t), \quad (1.10)$$

Lemma 1.1: Assume that $\int_0^\omega p(t)dt \neq 0$, then eq. (1.2.21) has a unique ω -periodic solution

$$\tilde{x}(t) = \int_t^{t+\omega} \frac{\exp(\int_t^s p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds$$

and (1.2.22) has a unique ω -periodic solution

$$\bar{x}(t) = \int_t^{t+\omega} \frac{\exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds$$

Proof: Obviously, the periodic solution of (1.9) is unique if $\int_0^\omega p(t)dt \neq 0$, and we show that

$\bar{x}(t)$ is the periodic solution of (1.9). Differentiating $\tilde{x}(t)$, we obtain that

$$\begin{aligned} \tilde{x}(t) &= \frac{\exp(\int_t^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(t+\omega) - \frac{1}{\exp(\int_0^\omega p(r)dr)-1} q(t) \\ &\quad - \int_t^{t+\omega} \frac{p(t) \exp(\int_t^s p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\ &= q(t) - p(t) \bar{x}(t) \end{aligned}$$

And

$$\begin{aligned} \tilde{x}(t + \omega) &= \int_{t+\omega}^{t+2\omega} \frac{\exp(\int_{t+\omega}^s p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\ &= \int_t^{t+\omega} \frac{\exp(\int_{t+\omega}^{u+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(u+\omega) d(u+\omega) \\ &= \int_t^{t+\omega} \frac{\exp(\int_t^u p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(u) du = \bar{x}(t) \end{aligned}$$

Hence, $\tilde{x}(t)$ is unique ω -periodic solution of (1.9).

The proof is complete

Proof of Equation (1.10): Assume that $\int_0^\omega p(t)dt \neq 0$ by lemma 1.1. Then equation (1.10) has a unique ω -period solution

$$\bar{x}(t) = \int_t^{t+\omega} \frac{\exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds$$

And we show that $\bar{x}(t)$ is the periodic solution of Equation (1.10). Differentiating $\bar{x}(t)$, we obtain that

$$\begin{aligned} \bar{x}(t) &= \frac{1}{\exp(\int_0^\omega p(r)dr)-1} q(t+\omega) - \frac{\exp(\int_t^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(t) + \\ &\quad \int_t^{t+\omega} \frac{p(t+\omega) \exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(t) \\ &= -q(t) + p(t) x(t) \\ &= p(t) x(t) - q(t) \end{aligned}$$

And

$$\begin{aligned} \bar{x}(t+\omega) &= \int_{t+\omega}^{t+2\omega} \frac{\exp(\int_s^{t+2\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\ &= \int_t^{t+\omega} \frac{\exp(\int_s^{t+2\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\ &= \int_t^{t+\omega} \frac{\exp(\int_{u+\omega}^{t+2\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(u+\omega) d(u+\omega) \\ &= \int_t^{t+\omega} \frac{\exp(\int_u^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(u)ds \\ &= \bar{x}(t) \end{aligned}$$

Hence, $\bar{x}(t)$ is a unique ω -periodic solution of Equation (1.10)

The proof is complete

The following well-known Schauder's fixed point theorem is crucial in our arguments.

Lemma 1.2. Let X be a Banach space with $D \subset X$ closed and convex. Assume that

$T: D \rightarrow D$ is a completely continuous operator, then T has a fixed point in D .

Define an operator J on X by

$$(Ju)(t) = \int_t^{t+\omega} \frac{e^{(s-t)p}}{e^{p\omega}-1} u(s)ds, \quad u \in X,$$

where $p > 0$ is a constant which is determined later. For any $u \in X$, $Ju \in X \cap C^1(\mathbb{R})$ and

$$(Ju)'(t) = -p(Ju)(t) + u(t). \quad (1.11)$$

If $u \in X \cap C^1(\mathbb{R})$, then $Ju \in X \cap C^2(\mathbb{R})$ and

$$(Ju)''(t) = -p(Ju)'(t) + u'(t) = P^2(Ju)(t) - pu(t) + u'(t). \quad (1.12)$$

We transform (1.1) to

$$P^2(Ju)(t) - pu(t) + u'(t) - a(t)(-p(Ju)(t) + u(t)) = f(t, (Ju)(t), u(t) - p(Ju)(t)) - g(t, (Ju)(t)),$$

that is,

$$u'(t) = (p+a(t))u(t) - [p^2Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - pJu)]. \quad (1.13)$$

By lemma 1.1, we obtain that if u is a periodic solution of (1.13), u satisfies

$$u(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(s)ds$$

If $\int_0^\omega (p+a(r))dr \neq 0$, where

$$(Hu)(s) = P^2(Ju)(s) + Pa(s)(Ju)(s) + g(s, (Ju)(s)) - f(t, (Ju)(s), u(s) - p(Ju)(s)).$$

Proof: To show u is a periodic solution of equation (1.13), we have the following:

If $\int_0^\omega (p+a(r))dr \neq 0$, Equation (1.13) has a unique solution u .

$$u(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(s)ds$$

Differentiating we have,

$$\begin{aligned} u(t) &= \frac{\exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(s) \int_t^{t+\omega} + \int_t^{t+\omega} \frac{(p+a(t)) \exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(s)ds \\ &= \frac{\exp \int_{t+\omega}^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(t+\omega) - \frac{\exp \int_t^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(t) + (p+a(t))u(t) \\ &= \frac{1}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(t) - \frac{\exp \int_t^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(t) + (p+a(t))u(t) \\ &= -(Hu)(t) \frac{\exp \int_t^{t+\omega} (p+a(r))dr-1}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(t) dt + (p+a(t))u(t) \\ &= -(Hu)(t) + (p+a(t))u(t) \\ &= (p+a(t))u(t) - (Hu)(t) \end{aligned}$$

Thus, u is a periodic solution of equation (1.13)

the proof is complete

In order to put more emphasis on the above facts, we summarize them in the following lemma.

Lemma 1.3: Define an operator T on X by

$$(Tu)(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr-1)} (Hu)(s)ds \quad (1.14)$$

Here $\int_0^\omega (p+a(r))dr \neq 0$. Then the fixed point u of T on X is the periodic solution of (1.13)

and Ju is the periodic solution of (1.1).

Proof: Since $Tu = u$ and

$$\begin{aligned}
(Tu)'(t) &= \frac{\exp \int_t^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr)-1}(Hu)(t+\omega) - \frac{\exp \int_t^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr)-1}(Hu)(t) + \\
&\quad \int_t^{t+\omega} \frac{(p+a(t))\exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr)-1}(Hu)(s)ds \\
&= (Hu)(t) \left(\frac{1}{\exp(\int_0^\omega (p+a(r))dr)-1} - \frac{\exp \int_t^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr)-1} \right) + \\
&\quad (p+a(t))u(t) \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p+a(r))dr}{\exp(\int_0^\omega (p+a(r))dr)-1}(Hu)(s)ds \\
&= -(Hu)(t) + (p+a(t))(Tu)(t) \\
&= (p+a(t))(Tu)(t) - (Hu)(t) \\
(Tu)'(t) &= (p+a(t))(Tu)(t) - [p^2Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - PJu)]
\end{aligned}$$

Where $(Hu)(t) = p^2Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - PJu)$.

Thus, we obtain that u is the periodic solution of (1.13)

The proof is complete

In order to prove that Ju is the periodic solution of (1.1), we only show that Ju satisfies (1.1). From (1.11) – (1.13), this result follows immediately.

1.2.3 SOME PREPARATIONS FOR POSITIVE PERIODIC

SOLUTIONS

The main purpose of this section is to obtain the existence results for single and multiple positive periodic solutions of Eq. (1.7) by using a fixed point theorem for cones. By using the theory of the fixed point index for cones we have obtained the optimal conditions for the nonlinear term g so that Eq. (1.8) has a positive periodic solution.

Let

$$C_\omega = \{u \in C(\mathbb{R}, \mathbb{R}) : u(t + \omega) = u(t)\}, \quad C_\omega^1 = \{u \in C^1(\mathbb{R}, \mathbb{R}) : u(t + \omega) = u(t)\}$$

$$M_1 = \{u \in C_\omega^1 : \int_0^\omega u(t) > 0\}, \quad M_2 = \{u \in C_\omega : \int_0^\omega u(t) > 0\}, \quad M_3 = \{u \in C_\omega : u(t) > 0\}.$$

C_ω is a Banach space, equipped with the norm $\|u\| = \max_{0 \leq t \leq \omega} |u(t)|$

For convenience, we list the following assumptions: let $\alpha \in M_1, \beta \in M_2, r_1 \in M_3, v \geq 0$ and $t \in [0, \omega]$

$$(H_1) \quad a = \alpha + \beta, \quad r_1 = \alpha \beta + \alpha', \quad F_1 = (\alpha \beta + \alpha') v - f(t, v) \geq 0.$$

$$(H_2) \quad a = \alpha - \beta, \quad r_2 = \alpha \beta - \alpha', \quad F_2 = (\alpha \beta - \alpha') v + f(t, v) \geq 0$$

$$(H_3) \quad a = -\alpha - \beta, \quad r_3 = \alpha \beta - \alpha', \quad F_3 = (\alpha \beta - \alpha') v - f(t, v) \geq 0$$

$$(H_4) \quad a = -\alpha + \beta, \quad r_4 = \alpha \beta + \alpha', \quad F_4 = (\alpha \beta + \alpha') v + f(t, v) \geq 0$$

Let $p, q \in C_\omega$ and consider the following two differential equations:

$$u'(t) = -p(t)u(t) + q(t). \quad (1.15)$$

$$u'(t) = p(t)u(t) - q(t). \quad (1.16)$$

Lemma 1. 4: Assume that $p \in M_2$, then eq.(1.15) has a unique ω -periodic solution

$$\tilde{u}(t) = \int_t^{t+\omega} \frac{\exp \int_t^s p(r) dr}{\exp \left(\int_0^\omega p(r) dr \right) - 1} q(s) ds$$

and eq. (1.16) has a unique ω -periodic solution

$$\bar{u}(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} p(r) dr}{\exp \left(\int_0^\omega p(r) dr \right) - 1} q(s) ds$$

Proof: To prove Equation (1.15). Assume that $\int_0^\omega p(t) dt \neq 0$, then the periodic solution of (1.15) is unique and we show that $\tilde{u}(t)$ is the periodic solution of (1.15). Differentiating $\tilde{u}(t)$, we obtain that

$$\begin{aligned} \tilde{u}(t) &= \frac{\exp \int_t^s p(r) dr}{\exp \left(\int_0^\omega p(r) dr \right) - 1} q(t + \omega) - \frac{1}{\exp \left(\int_0^\omega p(r) dr \right) - 1} q(t) - \int_t^{t+\omega} \frac{p(t) \exp \int_t^s p(r) dr}{\exp \left(\int_0^\omega p(r) dr \right) - 1} q(s) ds \\ &= q(t) - P(t) u(t) \end{aligned}$$

And

$$\begin{aligned}
\tilde{u}(t + \omega) &= \int_{t+\omega}^{t+2\omega} \frac{\exp \int_{t+\omega}^s p(r)dr}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\
&= \int_t^{t+\omega} \frac{\exp \int_{t+\omega}^{u+\omega} p(r)dr}{\exp(\int_0^\omega p(r)dr)-1} q(u+\omega)d(u + \omega) \\
&= \int_t^{t+\omega} \frac{\exp \int_t^u p(r)dr}{\exp(\int_0^\omega p(r)dr)-1} q(u)du \\
&= \tilde{u}(t)
\end{aligned}$$

Hence, $\tilde{u}(t)$ is a unique ω -periodic solution of Equation (1.15).

The proof is complete

In similar way we can show that Eq. (1.16) has a unique ω -periodic solution

Consider Eq. (1.16). The periodic solution of (1.16) is unique if $\int_0^\omega P(t)dt \neq 0$, and we show that $\tilde{u}(t)$ is the periodic solution of (1.16). Differentiating $\tilde{u}(t)$, we obtain that

$$\begin{aligned}
\bar{u}'(t) &= \left(\int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} p(r)dr}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \right)', \\
&= \frac{1}{\exp(\int_0^\omega p(r)dr)-1} - \frac{\exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(t) + \int_t^{t+\omega} \frac{p(t)\exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\
&= -q(t) + P(t) u(t) \\
&= P(t) u(t) - q(t)
\end{aligned}$$

And

$$\begin{aligned}
\bar{u}(t + \omega) &= \int_{t+\omega}^{t+2\omega} \frac{\exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(s)ds \\
&= \int_t^{t+\omega} \frac{\exp(\int_{u+\omega}^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(u+\omega)d(u + \omega) \\
&= \int_t^{t+\omega} \frac{\exp(\int_u^{t+\omega} p(r)dr)}{\exp(\int_0^\omega p(r)dr)-1} q(u)du \\
&= \bar{u}(t)
\end{aligned}$$

Hence, $\bar{u}(t)$ is a unique ω -periodic solution of Equation (1.16).

The proof is complete

Put

$$\begin{aligned}
A_\eta(t, s) &= \frac{\exp \int_t^s \eta(r)dr}{\exp(\int_0^\omega \eta(r)dr)-1}, \quad \widetilde{A}\eta(t, s) = \frac{\exp \int_s^{t+\omega} \eta(r)dr}{\exp(\int_0^\omega \eta(r)dr)-1} \\
G_\alpha^1(t, s) &= G_\alpha^2(t, s) = A_\alpha(t, s), \quad G_\alpha^3(t, s) = G_\alpha^4(t, s) = \widetilde{A}_\alpha(t, s), \\
G_\beta^1(t, s) &= G_\beta^4(t, s) = A_\beta(t, s), \quad G_\beta^2(t, s) = G_\beta^3(t, s) = \widetilde{A}_\beta(t, s),
\end{aligned}$$

$$m = \frac{1}{\exp(\int_0^\omega \alpha(r) dr) - 1}, \quad M = \frac{\exp(\int_0^\omega \alpha(r) dr)}{\exp(\int_0^\omega \alpha(r) dr) - 1}$$

Next, we define a mapping $T^i: C_\omega \rightarrow C_\omega$ by

$$(T^i x)(t) = \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) F_i(u, x(u)) du ds, \quad i=1,2,3,4. \quad (1.17)$$

And a cone K in C_ω by

$$K = \{u \in C_\omega : u(t) \geq \delta \|u\|, \forall t \in \mathbb{R}\}$$

Where $\delta = m/M \in (0, 1)$.

Lemma 1.15: Assume that (H_i) ($i = 1, 2, 3, 4$) is satisfied, then the fixed point of F^i is a solution of Eq. (1.7).

Lemma 1.16. Assume that (H_i) ($i = 1, 2, 3, 4$) is satisfied, then $T^i(K) \subset K$ and $T^i: K \rightarrow K$ is completely continuous.

Proof: Let us consider the (H_1) case is satisfied. If (H_1) holds and $x \in K$, then $F_1(u, x(u)) \in C_\omega$. It follows from $\beta \in M_2$ and Lemma 1. 4 that $\int_s^{s+\omega} G_\beta^1(s, u) F_1(u, x(u)) du \in C_\omega$. Further we have $(T^1 x)(t) \in C_\omega$.

On the other hand, we have

$$\begin{aligned} \|T^1 x\| &\leq M \int_t^{t+\omega} \int_s^{s+\omega} G_\beta^1(s, u) F_1(u, x(u)) du ds, \\ (T^1 x)(t) &\geq m \int_t^{t+\omega} \int_s^{s+\omega} G_\beta^1(s, u) F_1(u, x(u)) du ds, \end{aligned}$$

Hence

$$(T^1 x)(t) \geq \frac{m}{M} \|T^1 x\|,$$

Which implies $T^1 x \in K$. Thus $T^1(K) \subset K$.

Obviously, $T^1: K \rightarrow K$ is continuous. Let $D \subset K$ be a bounded set. For every $x \in D$, it is easy to see that there exists a constant $d > 0$ such that $\|T^1 x\| \leq d$. And

$$(T^1 x)'(t) = -\alpha(t)(T^1 x)(t) + \int_t^{t+\omega} G_\beta^1(t, u) F_1(u, x(u)) du.$$

It follows that $\{(T^1 x)' | x \in D\}$ is a bounded set. Consequently, $T^1(D)$ is an equicontinuous and bounded family of functions.

Thus, by the Arzela-Ascoli theorem, $T^1: K \rightarrow K$ is completely continuous. The proof is complete

To conclude this section, we state a new fixed point theorem for cones which will be needed in this work.

Lemma 1.7: Let X be a Banach space and K be a cone in X . Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and suppose that

$$\emptyset: K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

is a completely continuous operator such that

- (i) $\|\emptyset, u\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and there exists $\Psi \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\Psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$, or
- (ii) $\|\emptyset, u\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$ and there exists $\Psi \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\Psi$ for $x \in K \cap \partial\Omega_1$ and $\lambda > 0$.

Then \emptyset has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$

CHAPTER TWO

2. RESULTS

2.1 PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER

DIFFERENTIAL EQUATIONS

The following theorems are the main results of this section.

Theorem 2.1: Assume that there exist constants $m < M, p > 0$ such that

(H₁) $g \in C(R \times [m, M], R)$ and $p(p + a(t))u + g(t, u)$ is non-decreasing in $u \in [m, M], t \in R$

(H₂) $f(t, u, v) \in C(R \times [m, M] \times [p(m - M), p(M - m)], R)$ and

$$g(t, M) \leq f(t, u, v) \leq g(t, m)$$

for any $(t, u, v) \in R \times [m, M] \times [p(m - M), p(M - m)]$.

Then (1.1) has at least one ω -periodic solution x with $m \leq x \leq M$.

Proof: Let $\Omega = \{x \in X: mp \leq x(t) \leq pM \text{ for } t \in [0, \omega]\}$, then Ω is a closed and convex set in X .

For any $u \in \Omega$, we compute to obtain that $m \leq Ju \leq M$ and $pm - pM \leq u - pJu \leq pM - pm$.

Moreover, according to (H₁), we have

$$(p^2 + pa(t))m + g(t, m) \leq (p^2 + pa(t))Ju + g(t, Ju) \leq (p^2 + pa(t))M + g(t, M) \text{ for } u \in \Omega. \quad (2.1)$$

Using (H₂), we obtain that for any $u \in \Omega$,

$$\begin{aligned} (Hu)(t) &= p^2(Ju)(t) + pa(t)(Ju)t + g(t, (Ju)(t)) - f(t, (Ju)(t), u(t) - p(Ju)(t)) \\ &\leq (p^2 + pa(t))M + g(t, M) - g(t, M) \\ &= (p^2 + pa(t))M, \\ (Hu)(t) &= p^2(Ju)(t) + pa(t)(Ju)t + g(t, (Ju)(t)) - f(t, (Ju)(t), u(t) - p(Ju)(t)) \\ &\geq (p^2 + pa(t))m + g(t, m) - g(t, m) \\ &= (p^2 + pa(t))m, \end{aligned}$$

Which imply that $p+a(t) \geq 0$ for all $t \in R$. If $p+a(t) \equiv 0$, according to (H₁) and (H₂), we easily to check that $g(t, u) \equiv f(t, u, v)$ for any $(t, u, v) \in R \times [m, M] \times [p(m - M), p(M - m)]$. Thus any constant $C \in [m, M]$ is the periodic solution of (1.1). We assume that

$p+a(t) \geq 0$ and $p+a(t) \not\equiv 0$. The operator T is well defined. Now we show that T satisfies all conditions of Lemma 1. 2. Noting that

$$\int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (p+a(s))ds \equiv 1,$$

$$\frac{\exp \int_s^{t+\omega} (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} > 0, \text{ for } t \leq s \leq t+\omega,$$

we obtain that for any $u \in \Omega$,

$$\begin{aligned} (Tu)(t) &= \int_t^{t+\omega} \frac{\exp \int_t^s (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (Hu)(s)ds \\ &\leq \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (p^2+pa(t))Mds = pM, \end{aligned}$$

$$\begin{aligned} (Tu)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (Hu)(s)ds \\ &\geq \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (p^2+pa(t))mds = pm, \end{aligned}$$

which imply that $T(\Omega) \subset \Omega$

Next, we show that $T: \Omega \rightarrow \Omega$ is completely continuous. Obviously $T(\Omega)$ is a uniformly bounded set and T is continuous on Ω , so it suffices to show $T(\Omega)$ is equicontinuous by Ascoli-Arzela theorem. For any $u \in \Omega$, we have

$$(Tu)'(t) = (p+a(t))(Tu)(t) - [p^2Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - pJu)]$$

Since $T(\Omega)$ is bounded and f, g, a are continuous, there exists $\mathfrak{Q} > 0$ such that

$$|(Tu)'(t)| \leq \mathfrak{Q}, u \in \Omega$$

Which implies that $T(\Omega)$ is equicontinuous. So T is a completely continuous operator

On Ω . By lemma 1.2, there exists $u \in \Omega$ with $Tu = u$. Moreover, $Ju \in [m, M]$ is the periodic solution of (1.1).

The proof is complete

Theorem 2.2: Assume that there exist constants $m < M, p > 0$ such that

(H₃) $g \in C(R \times [m, M], R)$, and $p(p+a(t))u + g(t, u)$ is non-increasing in $u \in [m, M]$.

(H₄) $f(t, u, v) \in C(R \times [m, M] \times [p(m - M), p(M - m)])$ and

$$g(t, m) \leq f(t, u, v) \leq g(t, M)$$

for any $(t, u, v) \in R \times [m, M] \times [p(m - M), p(M - m)]$. Then (1.1) has at least one ω -periodic solution x with $m \leq x \leq M$.

Proof: Let $\beta = \{x \in X: mp \leq x(t) \leq Mp \text{ for } t \in [0, \omega]\}$, then β is a closed and convex set in X .

For any $u \in \beta$, we compute to obtain that $m \leq Ju \leq M$ and $pm - pM \leq U - pJu \leq pM - pm$.

Moreover, according to (H_3) , we have

$$(p^2+pa(t))M + g(t, M) \leq (p^2+pa(t))Ju + g(t, Ju) \leq (p^2+pa(t))m+g(t, m)$$

Using (H_4) , we obtain that for any $u \in \beta$

$$\begin{aligned} (Hu)(t) &= p^2(Ju)(t)+pa(t)(Ju)(t)+g(t, (Ju)(t)) - f(t, (Ju)(t), u(t) - p(Ju)(t)). \\ &\leq (p^2+pa(t))m+g(t, m) - g(t, m) \end{aligned}$$

$$= (p^2+pa(t))m$$

$$(Hu)(t) = p^2(Ju)(t)+pa(t)(Ju)(t)+g(t, (Ju)(t)) - f(t, (Ju)(t), u(t) - p(Ju)(t)).$$

$$\geq (p^2+pa(t))M +g(t, M) - g(t, M)$$

$$= (p^2+pa(t))M$$

Which imply that $p+a(t) \leq 0$ for all $t \in R$. If $p+a(t) \equiv 0$, according to (H_3) and (H_4) . We easily to check that $g(t, u) \equiv f(t, u, v)$ for any $(t, u, v) \in (R \times [m, M] \times p(m - m), p(M - m))$.

Thus, any constant $C \in [m, M]$ is the periodic solution of (1. 1). We assume that $p+a(t) \leq 0$ and

$P+a(t) \neq 0$. The operator T is well defined. Now we show that T satisfies all conditions of

lemma 1. 2. By theorem 2.1 we obtain that for any $u \in \beta$,

$$\begin{aligned} (Tu)(t) &= \int_t^{t+\omega} \frac{\exp \int_t^{t+\omega} (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (Hu)(s)ds \\ &\geq \int_t^{t+\omega} \frac{\exp \int_t^s (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (p^2+pa(t))m ds = pm, \end{aligned}$$

$$\begin{aligned} (Tu)(t) &= \int_t^{t+\omega} \frac{\exp \int_t^s (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (Hu)(s) ds \\ &\leq \int_t^{t+\omega} \frac{\exp \int_t^s (P+a(r))dr}{\exp \int_0^\omega (P+a(r))dr-1} (p^2+pa(t))M ds = pM, \end{aligned}$$

Which imply that $T(\beta) \subseteq \beta$

Next, we show that $T: \beta \rightarrow \beta$ is completely continuous. Clearly, $T(\beta)$ is a uniformly bounded

set and T is continuous on β . So it suffices to show that $T(\beta)$ is equicontinuous by Ascoli -

Arzela theorem. For any $u \in \beta$, we have

$$(Tu)'(t) = (p+a(t))(Tu)(t) - [p^2Ju + pa(t)Ju+g(t, Ju)- f(t, Ju, u - pJu)]$$

Since $T(\beta)$ is bounded and f, g, a are continuous, there exists $\mathfrak{Q} > 0$ such that

$$| (Tu)'(t) | \leq \mathfrak{Q}, u \in \beta$$

Which implies that $T(\beta)$ is equicontinuous. So T is completely continuous operator on Ω . By Lemma 1.2, there exists $u \in \beta$ with $Tu = u$. Moreover, $Ju \in [m, M]$ is the periodic solution of (1.1). The proof is complete

In Theorem 2.2, if $g_u(t, u)$ is continuous on $[0, \omega] \times [m, M]$, (H_1) and (H_2) are fulfilled for any sufficiently large $p > 0$.

Corollary 2.1: Let $f(t, u, v) \equiv f(t, u)$. Assume that there exist constants $m < M$ such that $g_u(t, u)$, $f(t, u) \in C(R \times [m, M], R)$ and

$$g(t, M) \leq f(t, u) \leq g(t, m)$$

for any $(t, u) \in R \times [m, M]$. Then (1.1) has at least one ω -periodic solution x with $m \leq x \leq M$.

Remark 2.1: Let $c, \mu > 0$ and h be ω -periodic continuous function. In [12], by using critical point theorem we have proved that (1.3) has at least two periodic solutions if $\|h\| < \mu$ and one periodic solutions if $\|h\| = \mu$.

Corollary 2.2: Assume that c, μ are constants and h is ω -periodic continuous function with $\|h\| \leq 1$. Then

- i. Eq. (1.2) has at least one ω -periodic solution
- ii. Further suppose that $c \geq 2\sqrt{|\mu|}$ and $h \not\equiv \pm\mu$, then Eq. (1.3) has at least two ω -periodic solutions.

Proof of (i): To Show Eq. (1.2) has at least one ω -periodic solution.

Suppose that $c \equiv 0$ and $h \not\equiv \pm\mu$ if $\mu = 0$, then $h \equiv 0$. Then any constant K is the periodic solution of (1.2). Now we assume that $\mu \neq 0$. Let $a(t) = 0$, $g(u) = \mu \sin u$ and $f(t, u, v) = h(t)$. Put $p_1 = |\mu| + 1$. Then $p_1(p_1 - 0)u + g(u) = p_1^2 u + g(u)$ is increasing in \mathfrak{R} and (H_1) is fulfilled. If $\mu > 0$, choosing $m_1 = 0.5\pi$, $M_1 = 1.5\pi$; if $\mu < 0$, choosing $m_1 = 1.5\pi$, $M_1 = 2.5\pi$, we obtain that

$$g(M_1) \leq h(t) \leq g(m_1), \forall t \in R$$

that is, for $\mu > 0$

$$\begin{aligned} \mu \sin(1.5\pi) \leq h(t) \leq \mu \sin(0.5\pi) \\ -\mu \leq h(t) \leq \mu \end{aligned}$$

And, for $\mu < 0$

$$\begin{aligned}\mu \sin(2.5\pi) &\leq h(t) \leq \sin(1.5\pi) \\ \cdot(1) &\leq h(t) \leq \cdot(-1) \\ \mu &\leq h(t) \leq -\mu\end{aligned}$$

Thus, (H₂) is fulfilled.

Hence, (1.2) has at least one periodic solution x_1 with $m_1 \leq x_1 \leq M_1$.

Further suppose that $c \equiv 0$ and $h \not\equiv \pm\mu$. Put $p_2 = \frac{1}{|\mu|} + 1$. Then $p_2(p_2 - 0)u + g(u)$ which is $p_2^2 u + g(u)$ is non-increasing in R and (H₃) is fulfilled. If $\mu > 0$, choosing $m_2 = -0.5\pi$, $M_2 = 0.5\pi$; if $\mu < 0$, choosing $m_2 = 0.5\pi$, $M_2 = 1.5\pi$ we obtain that,

$$g(m_2) \leq h(t) \leq g(M_2), \forall t \in R.$$

that is, for $\mu > 0$

$$\begin{aligned}\mu \sin(1.5\pi) &\leq h(t) \leq \mu \sin(0.5\pi) \\ (-1) &\leq h(t) \leq (1) \\ -\mu &\leq h(t) \leq \mu\end{aligned}$$

And, for $\mu < 0$

$$\begin{aligned}\mu \sin(0.5\pi) &\leq h(t) \leq \mu \sin(1.5\pi) \\ \lambda \cdot(1) &\leq h(t) \leq \lambda \cdot(-1) \\ \mu &\leq h(t) \leq -\mu\end{aligned}$$

Thus, (H₄) is fulfilled.

Hence, (1.2) has at least one ω -periodic solution x_2 with $m_2 \leq x_2 \leq M_2$.

Since $h \not\equiv \pm\mu$, $x_i \not\equiv m_j$, $x_i \not\equiv M_j$ ($i, j = 1, 2$), $x_1 \not\equiv x_2$.

The proof is complete

Proof of (ii): Suppose that $c \geq 2\sqrt{|\mu|}$ and $h \not\equiv \pm\mu$, Then Eq. (1.3) has at least two ω -periodic solutions. If $\mu = 0$, then $h \equiv 0$ and any constant K is the periodic solution of (1.3). Now, we assume that $\mu \neq 0$. Let $a(t) = -c$, $g(u) = \mu \sin u$ and $f(t, u, v) = h(t)$. Put $p_1 = (|c|+1)(|\mu|+1)$ then $p_1(p_1 - c)u + g(u)$ is increasing in R and (H₁) is fulfilled. If $\mu > 0$, choosing $m_1 = 0.5\pi$, $M_1 = 1.5\pi$; if $\mu < 0$, choosing $m_1 = 1.5\pi$, $M_1 = 2.5\pi$. By (H₂) we obtain that.

$$g(M_1) \leq h(t) \leq g(m_1), \forall t \in R.$$

That is, for $\mu > 0$

$$\begin{aligned}\mu \sin (1.5\pi) &\leq h(t) \leq \mu \sin (0.5\pi) \\ (-1) &\leq h(t) \leq (1) \\ -\mu &\leq h(t) \leq \mu\end{aligned}$$

And, for $\mu < 0$

$$\begin{aligned}\mu \sin (2.5\pi) &\leq h(t) \leq \mu \sin (1.5\pi) \\ \mu \cdot 1 &\leq h(t) \leq \mu \cdot (-1) \\ &\leq h(t) \leq -\end{aligned}$$

Hence, (1.3) has at least one periodic solution x_1 with $m_1 \leq x_1 \leq M_1$. The proof is complete

Further suppose that $c \geq 2\sqrt{|\mu|}$ and $h \neq \pm\mu$. Put $p_2 = c/2$, then $p_2(p_2 - c)u + g(u)$ is non-increasing in R and (H_3) is fulfilled. If $\mu > 0$, choosing $m_2 = -0.5\pi$, $M_2 = 0.5\pi$; if $\mu < 0$, choosing $m_2 = 0.5\pi$, $M_2 = 1.5\pi$. By (H_2) we obtain that.

$$g(m_2) \leq h(t) \leq g(M_2), \forall t \in R.$$

That is, for $\mu > 0$

$$\begin{aligned}\mu \sin (-0.5\pi) &\leq h(t) \leq \mu \sin (0.5\pi) \\ (-1) &\leq h(t) \leq (1) \\ -\mu &\leq h(t) \leq \mu\end{aligned}$$

and, for $\mu < 0$

$$\begin{aligned}\mu \sin (0.5\pi) &\leq h(t) \leq \mu \sin (1.5\pi) \\ \mu \cdot (1) &\leq h(t) \leq \mu \cdot (-1) \\ \mu &\leq h(t) \leq -\mu\end{aligned}$$

Hence, (1.3) has at least one periodic solution x_2 with $m_2 \leq x_2 \leq M_2$.

Assume that e is ω -periodic continuous function, for (1.5), we have the following result.

Theorem 2.3: Assume that there exist constants $m < M$ such that $g, f \in C'([m, M], R)$ and

$$g(M) \leq e(t) \leq g(m)$$

$\forall t \in [0, \omega]$, then (1.5) has at least one ω -periodic solution x with $m \leq x \leq M$.

Proof: Define the function

$$\Gamma(t, v) = pv + \frac{g(v) - e(t)}{p - f(v)}, \quad t \in R, v \in [m, M],$$

Where $p > 0$ is sufficiently large which is determined later. By computation, we have

$$\Gamma V(t, v) = p + \frac{g'(v)}{p-f(v)} + \frac{f'(v)(g(v)-e(t))}{(p-f(v))^2}.$$

Let

$$P = \max_{m \leq u \leq M} |f(u)| + \max_{m \leq u \leq M} |g'(u)| + \max_{m \leq u \leq M} |f'(u)| \times (\max_{m \leq u \leq M} |g'(u)| + \max_{t \in R} |e(t)|) + 2$$

Then $\Gamma_u(t, v) > 0$ for $t \in R$, $v \in [m, M]$, and Γ is non-decreasing in $v \in [m, M]$ for any fixed $t \in R$.

Hence,

$$Pm + \frac{g(m)-e(t)}{p-f(m)} \leq \Gamma(t, v) \leq pM + \frac{g(M)-e(t)}{p-f(M)}, \quad t \in R, v \in [m, M].$$

Similar to Eq. (1.13), using the operator Ju , we transform (1.5) to the equation

$$u'(t) = (p - f(Ju))u(t) - [p^2 Ju + g(Ju) - e(t) - p(Ju)f(Ju)]. \quad (2.2)$$

Define an operator K on the closed, convex set Ω by

$$\begin{aligned} (Ku)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (Lu)(s) ds, \\ (Lu)(t) &= p^2 Ju(t) + g(Ju(t)) - e(t) - pJu(t)f(Ju(t)), \\ \Omega &= \{x \in X: mp \leq x(t) \leq pM \text{ for } t \in [0, \omega]\}. \end{aligned}$$

Hence, the fact that K has fixed point on Ω implies that Eq. (1.5) has the periodic solution.

For any $u \in \Omega$, $m \leq Ju \leq M$, $p > f(Ju)$ and

$$\begin{aligned} (Ku)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p - f(Ju(s))) \Gamma(s, (Ju(s))) ds \\ &\leq \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p - f(Ju(s))) \left(pM + \frac{g(M)-e(t)}{p-f(M)} \right) ds \\ &\leq pM \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p - f(Ju(s))) ds \\ &= pM, \\ (Ku)(t) &\geq \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p - f(Ju(s))) \left(pm + \frac{g(m)-e(s)}{p-f(m)} \right) ds \\ &\geq pm \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p - f(Ju(s))) ds \\ &= pm. \end{aligned}$$

Hence, $K(\Omega) \subseteq \Omega$

That is,

For any $u \in \Omega$, $m \leq Ju \leq M$, $p > f(Ju)$ and

$$\begin{aligned}
(Ku)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) \Gamma(s, Ju(s)) ds \\
&\leq \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) \left(pM + \frac{g(M)-e(t)}{p-f(M)} \right) ds \\
&\leq pM \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) ds + \\
&\quad \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) \left(\frac{g(M)-e(t)}{p-f(M)} \right) ds \\
&= pM \\
(Ku)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) \left(pm + \frac{g(m)-e(t)}{p-f(m)} \right) ds \\
&\geq \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) pm ds + \\
&\quad \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) \left(\frac{g(m)-e(t)}{p-f(m)} \right) ds \\
&\geq pm \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (p-f(Ju(s))) pm ds \\
&= pm
\end{aligned}$$

For $p > 0$ is sufficiently large.

Hence, $K(\Omega) \subset \Omega$

Next we show that $K: \Omega \rightarrow \Omega$ is completely continuous. Since Ω is closed and convex set in a

Banach space, Ω is a uniformly bounded set. Since $K(\Omega) \subset \Omega$, $K(\Omega)$ is also a uniformly

bounded set and so K is continuous on Ω . So it suffices to show $K(\Omega)$ is equicontinuous by

Ascoli-Arzela theorem. For any $u \in \Omega$, we have

$$(Ku)(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr-1} (Lu)(s) ds$$

And hence,

$$(Ku)'(t) = (p-f(Ju))u(t) - [p^2 Ju + g(Ju) - e(t) - p(Ju) f(Ju)]$$

Where

$$\begin{aligned}
(Lu)(s) &= p^2(Ju)(s) + g((Ju)e(s)) - e(s) - p(Ju)(s) f((Ju)(s)) \\
(Ku)'(t) &= \frac{\exp \int_t^{t+\omega} (p-f(Ju))dr}{\exp(\int_0^\omega (p-f(Ju))dr)-1} (Lu)(t + \omega) - \frac{\exp \int_t^{t+\omega} (p-f(Ju))dr}{\exp(\int_0^\omega (p-f(Ju))dr)-1} (Lu)(t) + \\
&\quad \int_t^{t+\omega} \frac{(p-f(t))\exp \int_s^{t+\omega} (p-f(Ju))dr}{\exp \int_0^\omega (p-f(Ju))dr)-1} (Lu)(s)ds \\
&= \frac{1}{\exp(\int_0^\omega (p-f(Ju))dr)-1} (Lu)(t+\omega) - \frac{\exp \int_t^{t+\omega} (p-f(Ju))dr}{\exp(\int_0^\omega (p-f(Ju))dr)-1} (Lu)(t) + (p-f(t))(Ku)(t) \\
&= -(Lu)(t) + (p-f(t))(Ku)(t)
\end{aligned}$$

And hence

$$(ku)'(t) = (p-f(Ju))u(t) - [p^2Ju + g(Ju) - e(t) - p(Ju) f(Ju)]$$

Since $K(\Omega)$ is bounded and f, g, e are continuous, there exist $\mathfrak{Q} > 0$ such that

$$| (Ku)'(t) | \leq \mathfrak{Q}, u \in \Omega.$$

Which implies that $K(\Omega)$ is equicontinuous. So K is a completely continuous operator on Ω . By lemma 1.2, There exists $u \in \Omega$ with $Ku=u$. Moreover, $Ju \in [m, M]$ is the periodic solution of (1. 5).

Since $h \neq \pm\mu, x_i \neq m_j, x_i \neq M_j(i, j = 1..2), x_1 \neq x_2$.

The proof is complete

Corollary 2.3: Let e be ω -periodic continuous function and f a bounded continuous function on R . Further suppose that there exist constants $m < M$ such that $g \in C^1([m, M], R)$ and

$$g(M) + \alpha \leq e(t) \leq g(m) + \beta, t \in [0, \omega],$$

where $\alpha = \sup_{u \in R} f(u)$ and $\beta = \inf_{u \in R} f(u)$. Then (1.4) has at least one ω - periodic solutions x with $m \leq x \leq M$

Remark 2.2: Using the **same idea**, for the following differential equation

$$x''(t) + f(t, x(t))x'(t) + g(t, x(t)) = e(t), \quad (2.3)$$

we have the following result;

Corollary 2.4: Assume that e is ω -periodic continuous function and there exist constants $m < M$ such that $g_u(t, u), f_u(t, u) \in C(R \times [m, M], R)$,

$$g(t, M) \leq e(t) \leq g(t, m) \text{ for } \forall t \in [0, \omega],$$

Then Eq. (2.3) has at least one ω -periodic solution x with $m \leq x \leq M$

2.1.1 Prototypical examples

In this section, three examples are provided to high light our results obtained in “periodic solution for second order differential equations”

Example2.1: Consider the differential equation with singularity

$$x''(t) = \frac{G(t)}{x^\mu(t)} - \frac{H(t)}{x^\lambda(t)} + F(t), \quad (2.4)$$

where G, H, F are ω -periodic continuous functions, $\mu, \lambda > 0$.

In recent paper [7], R. Hakl and P J. Torres discussed the existence of positive periodic solution of Eq.(2.4) when $G, H \in L^+$ and $F \in L$, where L is the Banach space of ω -periodic Lebesgue integrable function and

$L^+ = \{p \in L: p \geq 0 \text{ for a.e. } t \in [0, \omega]\}$. By using *method of upper and lower functions*, we have established several existence results. By corollary2.1, we obtain the following new result.

Proposition2.1: Assume that there exist $0 < m < M$ such that for $\forall t \in [0, \omega]$,

$$m^{\lambda-\mu}G^+(t) + \frac{m^\lambda}{M^\mu}G^-(t) + m^\lambda F(t) \leq H(t) \leq M^{\lambda-\mu}G^+(t) + \frac{M^\lambda}{m^\mu}G^-(t) + M^\lambda F(t), \quad (2.5)$$

Here, $G^+(t) = \max \{G(t), 0\}$ and $G^-(t) = \min \{G(t), 0\}$. Then Eq.(2.4) has at least one positive ω -periodic solutions.

Proof: By the condition Eq. (2.5), we have

$$\frac{G^+(t)}{m^\mu} + \frac{G^-(t)}{M^\mu} + F(t) \leq \frac{H(t)}{m^\lambda}, \frac{H(t)}{M^\lambda} \leq \frac{G^+(t)}{M^\mu} + \frac{G^-(t)}{m^\mu} + F(t)$$

On the other hand, for any $u \in [m, M]$,

$$\frac{G^+(t)}{M^\mu} + \frac{G^-(t)}{m^\mu} \leq \frac{G(t)}{u^\mu} \leq \frac{G^+(t)}{m^\mu} + \frac{G^-(t)}{M^\mu}$$

Hence for any $u \in [m, M]$, the inequality

$$\frac{H(t)}{M^\lambda} \leq \frac{G(t)}{u^\mu} + F(t) \leq \frac{H(t)}{m^\lambda}$$

holds. By Corollary 2.1, Eq. (2.4) has at least one positive ω -periodic solution.

The condition Eq. (2.5) admits that G changes sign, which is different with those of [7]. For example, using Eq. (2.5), one can check that the differential equation

$$x''(t) = \frac{1+2\sin t}{x^{0.8}(t)} - \frac{2}{x(t)} + 15 \quad (2.6)$$

has one 2π -periodic solution x with $0.1 \leq x \leq 10$.

Check: Given, $m = 0.1, M = 10, \mu = 0.8, \lambda = 1$

$$\mathbf{H(t) = 2, F(t) = 15, G(t) = 1 + 2\sin t}$$

$$\mathbf{Let t = \pi/2 \in [0, 2\pi]}$$

Substituting the given information in the inequality $\frac{H(t)}{M^\lambda} \leq \frac{G(t)}{u^\mu} + F(t) \leq \frac{H(t)}{m^\lambda}$,

Gives $0.2 \leq 17 \leq 20$, holds true.

When $G \equiv 0$, Eq. (2.1,12) reduces to

$$m^\lambda F(t) \leq H(t) \leq M^\lambda F(t) \text{ for } t \in [0, \omega], \quad (2.7)$$

which guarantee that the equation

$$x''(t) = -\frac{H(t)}{x^\lambda(t)} + F(t) \quad (2.8)$$

has at least one positive ω -periodic solution. The new condition Eq. (2.7) in which H is possibly zero on the set of a positive measure gives partial answer to the open problem Eq. (2.5) in [7].

Example 2.2: Consider the differential equation

$$x''(t) + \frac{1}{8}(x'(t))^2 - x^2(t) = \sin t - 1. \quad (2.9)$$

We claim that Eq. (2.9) has at least two 2π -periodic solutions. In fact,

$$g(u) = -u^2, f(t, u, v) = \sin t - 1 - v^2/8.$$

Put $p = 2, m = 0, M = 2$, then for $(t, u, v) \in [0, 2\pi] \times [0, 2] \times [-4, 4]$,

$$g(M) = -4 \leq f(t, u, v) \leq 0 = g(m)$$

By Theorem 2.1, Eq. (2.9) has at least one 2π -periodic solution x_1 with $0 \leq x_1 \leq 2$.

Similarly, one can easily check that Eq.(2.9) has one 2π -periodic solution x_2 with $-2 \leq x_2 \leq 0$.

Proof: To show, Eq. (2.9) has at least one 2π – periodic solution x_2 with $-2 \leq x_2 \leq 0$, we have

$$g(m) = -4 \leq f(t, u, v) \leq 0 = g(M)$$

Hence, Bly theorem (2.1), Eq. (2.9) has at least one 2π - periodic solution x_2 with $-2 \leq x_2 \leq 0$.

Example 2.3: Consider the differential equation

$$-x''(t) + a(t)x'(t) = x^\alpha(t) \sin x(t) - f(t, x(t)), \quad (2.10)$$

Where $\alpha > 0, a$ is a continuous ω -periodic function, $f(t, u): R \times R \rightarrow R$ is continuous and ω -periodic in t .

We claim that Eq. (2.10) has infinitely many periodic solutions if the following condition is fulfilled:

$$\lim_{u \rightarrow +\infty} \frac{|f(t,u)|}{u^\alpha} < 1 \quad (2.11)$$

uniformly with respect to $t \in \mathbb{R}$.

In fact, there exist constant $\mathcal{G} > 0$ such that

$$|f(t, u)| \leq u^\alpha, \quad u \geq \mathcal{G}.$$

Choose integer n such that $2\pi n > \mathcal{G}$. Let $m = 2\pi n + 0.5\pi$ and $M = 2\pi n + 1.5\pi$, then

$$g(u) = u^\alpha \sin u \in C^1([m, M], \mathbb{R})$$

and

$$g(M) = -M^\alpha \leq f(t, u) \leq m^\alpha = g(m)$$

for $(t, u) \in \mathbb{R} \times [m, M]$.

By Corollary 2.1, Eq. (2.10) has at least one ω -periodic solution x with $m \leq x \leq M$. Since n is arbitrary sufficiently large integer, Eq. (2.10) has infinitely many periodic solution.

2.2 Positive Periodic Solutions of Second Order Differential Equations

The following are the main results of this section.

Put

$$\begin{aligned}\emptyset_i(s) &= \min \left\{ \frac{F_i(t,u)}{r_i(t)u}, t \in [0, \omega], u \in [\delta s, s] \right\}, \\ \varphi_i(s) &= \max \left\{ \frac{F_i(t,u)}{r_i(t)u}, t \in [0, \omega], u \in [\delta s, s] \right\}, \\ \emptyset_i^0 &= \lim_{s \rightarrow 0^+} \emptyset_i(s), \quad \emptyset_i^\infty = \lim_{s \rightarrow \infty} \emptyset_i(s), \\ \varphi_i^0 &= \lim_{s \rightarrow 0^+} \varphi_i(s), \quad \varphi_i^\infty = \lim_{s \rightarrow \infty} \varphi_i(s),\end{aligned}$$

Theorem 2.4. Assume that (H_i) holds, and there exist two positive constants a, b : $a \neq b$ such that $\varphi_i(a) \leq 1$ and $\emptyset_i(b) \geq 1$; then Eq. (1.7) has at least a positive periodic solution $x \in K$ with $\min\{a, b\} \leq \|x\| \leq \max\{a, b\}$.

Proof: Without loss of generality, we assume that $a < b$. Let $\Omega_1 = \{x \in K, \|x\| < a\}$ and

$\Omega_2 = \{x \in K, \|x\| < b\}$ by $\varphi_i(a) \leq 1$ and $\emptyset_i(b) \geq 1$, we have

$$\begin{aligned}F_i(t, x) &\leq r_i(t)x, \quad \forall 0 \leq t \leq \omega, \delta a \leq x \leq a, \\ F_i(t, x) &\geq r_i(t)x, \quad \forall 0 \leq t \leq \omega, \delta b \leq x \leq b,\end{aligned}$$

Next, we show that:

- (i) $\|T^1 x\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$.
- (ii) There exists $\Psi \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\Psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$,

To justify (i): Let $x \in K \cap \partial\Omega_1$ then $\|x\| = a$, $\delta a \leq x(t) \leq a$. So we have that

$$\begin{aligned}(T^i x)(t) &= \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) F_i(u, x(u)) du ds \\ &\leq \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) r_i(u) x(u) du ds, \\ &\leq \|x\| \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) r_i(u) du ds, \\ &= \|x\|\end{aligned}$$

And this implies $\|T^i x\| \leq \|x\|$, $x \in K \cap \partial\Omega_1$.

The justification is completed

Next, **to justify (ii):** Let $\Psi \equiv 1$. Assume that there exists $x \in K \cap \partial\Omega_2$, and $\lambda > 0$ such that

$$x = T^1 x + \lambda\Psi. \text{ By } \varphi_i(a) \leq 1 \text{ and } \emptyset_i(b) \geq 1, \text{ we have } \delta b \leq x \leq b.$$

Let $\mathbf{X} = \min\{x(t), t \in [0, \omega]\}$, then we have

$$\begin{aligned}
\mathbf{X} &= x(t) \text{ for some } t \in [0, \omega]. \text{ Thus it follows that} \\
x(t) &= (\mathbb{T}^i x)(t) + \lambda \\
&= \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) F_i(u, x(u)) du ds + \lambda \\
&\geq \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) r_i(u) x(u) du ds + \lambda \\
&\geq \mathbf{X} \int_t^{t+\omega} G_\alpha^i(t, s) \int_s^{s+\omega} G_\beta^i(s, u) r_i(u) du ds + \lambda \\
&= \mathbf{X} + \lambda
\end{aligned}$$

And this implies $\mathbf{X} > \mathbf{X}$, a contradiction.

By Lemma 1.15, it follows that F^i has a fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Furthermore,

$a \leq \|x\| \leq b$ and $x(t) \geq \delta b > 0$, which means that $x(t)$ is an ω -periodic positive solution of (1.7).

Corollary 2.5. Assume that (H_i) holds. If one of the following conditions holds:

- (i) $\varphi_i^0 < 1, \varphi_i^\infty > 1,$
- (ii) $\varphi_i^0 > 1, \varphi_i^\infty < 1,$

Then Eq. (1.7) has at least a positive ω -periodic solution

Remark: Put $\alpha = \beta = c$ (c is a positive constant) and $i = 2$ or $i = 4$; Corollary 2.5 is Theorem 1.1 in [11].

Theorem 2.5. Assume that (H_i) is satisfied, and there exist $N + 1$ positive constants

$$P_1 < P_2 < \dots < P_N < P_{N+1}$$

such that one of the following conditions is satisfied:

- (i) $\varphi_i(p_{2k-1}) < 1, k = 1, 2, \dots, [(N+2)/2], \varphi_i(p_{2k}) > 1, k = 1, 2, \dots, [(N+1)/2],$
- (ii) $\varphi_i(p_{2k-1}) > 1, k = 1, 2, \dots, [(N+2)/2], \varphi_i(p_{2k}) < 1, k = 1, 2, \dots, [(N+1)/2],$

where $[d]$ denotes the integer part of d . then, Eq. (1.7) has at least N positive periodic solutions $x_j \in K, j = 1, 2, \dots, N$ with $p_j < \|x_j\| < p_{j+1}$.

Proof of the case (i): Since $\varphi, \varphi: (0, \infty) \rightarrow (0, \infty)$ are continuous, it follows

that there exist $a_k, b_k: p_k < a_k < b_k < p_{k+1}, k = 1, 2, \dots, N$ such that

$$\begin{aligned}\varphi_i(a_{2k-1}) \leq 1, \varphi_i(b_{2k-1}) \geq 1, k = 1, 2, \dots [(N+1)/2], \\ \varphi_i(a_{2k}) \geq 1, \varphi_i(b_{2k}) \leq 1, k = 1, 2, \dots [(N+1)/2].\end{aligned}$$

It follows from Theorem 2.4 that Eq. (1.7) has at least a periodic solution $x_j \in K$ with $a_j \leq \|x_j\| \leq b_j$ for the pairs $(a_j, b_j), j = 1, 2, \dots, N$. The proof is complete

Proof of the case (ii). Since (H_i) is satisfied by assumption and

$\varphi_i: (0, \infty) \rightarrow (0, \infty)$ and $\varphi_i: (0, \infty) \rightarrow (0, \infty)$ are continuous, it follows that, there exist,

$a_k, b_k: p_k < a_k < b_k < p_{k+1}, k = 1, 2, \dots [(k+1)/2]$ such that

$$\begin{aligned}\varphi_i(b_{2k-1}) \geq 1, \varphi_i(a_{2k-1}) \leq 1, k = 1, 2, \dots [(N+1)/2]. \\ \varphi_i(b_{2k}) \leq 1, \varphi_i(a_{2k}) \geq 1, k = 1, 2, \dots [(N+1)/2],\end{aligned}$$

It follows from Theorem 2.21 that Equation (1.7) has at least a periodic solution

$x_j \in K$ with

$$a_j \leq \|x_j\| \leq b_j$$

for the pair $(a_j, b_j), j = 1, 2, \dots, N$. The proof is complete

Corollary 2.6. Assume that (H_i) is satisfied, and

$$\varphi_i^0 < 1, \varphi_i^\infty < 1.$$

Further there is a positive constant b such that $\varphi_i(b) > 1$. Then Eq. (1.7) has at least two positive ω -periodic solutions x_1 and x_2 such that

$$0 < \|x_1\| < b < \|x_2\|.$$

Corollary 2.7. Assume that (H_i) is satisfied, and

$$\varphi_i^0 > 1, \varphi_i^\infty > 1.$$

Further there is a positive constant a such that $\varphi_i(a) < 1$. Then Eq. (1.7) has at least two positive ω -periodic solutions x_1 and x_2 such that

$$0 < \|x_1\| < a < \|x_2\|.$$

2.2.1 Prototypical Examples

In this section, we apply the main results obtained in previous sections to several examples.

Example 2.4. Consider the differential Equation

$$x''(t) + x'(t) \sin t + e^{-x(t)} - (2 + \sin t)x(t) = 0 \quad (2.12)$$

Let $a(t) = \sin t$. By (H_4) , $S \sin t = -\alpha + \beta$ which implies that $\alpha(t) = 2$ and $\beta(t) = 2 + \sin t$. Then

$$F_4(t, u) = e^{-u} + (2 + \sin t)u \text{ and } r_4 = 2(2 + \sin t) > 0.$$

Since

$$\begin{aligned} \varphi_4^0 &= \lim_{u \rightarrow +0} \max \left\{ \frac{F_4(t, u)}{r_4(t)u}, t \in [0, \omega], u \in [\delta s, s] \right\}, \\ &= \lim_{u \rightarrow +0} \max \left\{ \frac{e^{-u} + (2 + \sin t)u}{2(2 + \sin t)u} \right\} \\ &= \lim_{u \rightarrow +0} \max \left\{ \frac{1}{2ue^u(2 + \sin t)} \right\} \\ &= \infty \end{aligned}$$

And

$$\begin{aligned} \varphi_4^\infty &= \lim_{u \rightarrow \infty} \min \left\{ \frac{F_4(t, u)}{r_4(t)u}, t \in [0, \omega], u \in [\delta s, s] \right\}, \\ &= \lim_{u \rightarrow \infty} \min \left\{ \frac{e^{-u} + (2 + \sin t)u}{2(2 + \sin t)u} \right\} \\ &= \lim_{u \rightarrow \infty} \min \left\{ \frac{1}{2} \right\} \\ &= \frac{1}{2} \end{aligned}$$

Thus, from corollary 2.21, Eq. (2.12.) has at least a positive 2π -periodic solution

Example 2.5. Consider the differential equation

$$x''(t) + x^2(t) - 3x(t) + \frac{1}{5} \sin^2 4\pi t + 1 = 0 \quad (2.13)$$

Eq. (2.13) has at least two positive periodic solutions.

Proof: Let $\alpha(t) = \beta(t) = 1$, then $\delta^{-\frac{1}{2}}$ by lemma 1. 4, and by (H_2) .we have the following

$$\begin{aligned} r_2 &= \alpha \beta - \alpha' \\ &= 1 \times 1 + 0 \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} F_2(t, u) &= (\alpha \beta - \alpha')u + f(t, u) \\ &= (1 \times 1 - 0)u + f(t, u) \\ &= u + f(t, u) \\ &= u + u^2 + 1 - 3u + \frac{1}{5} \sin^2 4\pi t \text{ from eq. (2.13)} \\ &= u^2 + 1 - 2u + \frac{1}{5} \sin^2 4\pi t \end{aligned}$$

Since

$$\begin{aligned} \Phi_2(s) &= \min \left\{ \frac{F_2(t, u)}{r_2(t)u}, t \in [0, \omega], u \in [\delta s, s] \right\}, \\ &= \min \left\{ \frac{u^2 + 1 - 2u + \frac{1}{5} \sin^2 4\pi t}{1 \cdot u} \right\} \\ &= \min \left\{ u + \frac{1}{u} - 2 + \frac{\sin^2 4\pi t}{5u} \right\} \end{aligned}$$

Thus,

$$\begin{aligned} \Phi_2^0 &= \lim_{u \rightarrow +0} \min \left\{ u + \frac{1}{u} - 2 + \frac{\sin^2 4\pi t}{5u} \right\} \\ &= \infty \end{aligned}$$

And

$$\begin{aligned}\Phi_2^\infty &= \lim_{u \rightarrow \infty} \min \left\{ u + \frac{1}{u} - 2 + \frac{\sin^2 4\pi t}{5u} \right\} \\ &= \infty\end{aligned}$$

Thus,

$$\Phi_2^0 = \infty \text{ and } \Phi_2^0 = \infty$$

Let $a = \frac{5}{2}$, then for $\delta a \leq u \leq a$,

$$F_2(t, u) = u^2 + 1 - 2u + \frac{1}{5} \sin^2 4\pi t < u.$$

By corollary 2.3, Equation (2.13) has at least two positive periodic solutions.

Example 2.6. Consider the differential equation

$$x''(t) + 2x'(t) + \sin x(t) - \frac{1}{100}(1 + \sin 200\pi t) = 0 \quad (2.14)$$

Put $\alpha = \beta = 1$, then $\delta = e^{-\frac{1}{100}}$. By (H_1) , we have

$$\begin{aligned}F_1(t, u) &= (\alpha \beta + \alpha^1)u - f(t, u) \\ &= (1 \times 1 + 0)u - \left(\sin u - \frac{1}{100} (1 + \sin 200\pi t) \right) \\ &= u - \sin u + \frac{1}{100} (1 + \sin 200\pi t)\end{aligned}$$

Let

$$p_{2k-1} = 2k\pi + \frac{2\pi}{3}, p_{2k} = 2(k+1)\pi - \frac{2\pi}{3}, k = 1, 2, 3, \dots$$

If $\delta p_{2k-1} \leq u \leq p_{2k-1}$, then $\varphi_1(p_{2k-1}) < 1$

If $\delta p_{2k} \leq u \leq p_{2k}$, then $\varphi_1(p_{2k}) > 1$.

By theorem 2.5, Equation (2.14) has infinite positive periodic solutions.

2.3 PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL

EQUATIONS

In this section we give sufficient conditions ensuring that the system of nonlinear differential equations has at least one periodic solution.

Let $f(t, x)$ be a continuous on $[0, T] \times R^n$ with its range contained in R^n . We give sufficient conditions in order that the system of differential equations

$$x' = f(t, x) \quad (2.15)$$

has at least one periodic solution $x(t)$, i.e a solution $x(t)$ such that $x(0) = x(T)$. We say that $f(t, x)$ is of type K(after Kamke [32]) in a set $D \in R^n$ if for all $x, y \in D$,

$x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, with $x \leq y$ and $x_i = y_i$, it holds that $f_i(t, x) \leq f_i(t, y)$ where $f = (f_1, \dots, f_n)$, $i = 1, \dots, n$

The **main results** of this section is as follows:

Theorem 2. 6:- Let $\alpha, \beta: [0, T] \rightarrow R^n$ be continuous functions such that

$$\begin{aligned} \alpha(t) \leq \beta(t), 0 \leq t \leq T, \alpha(t) \leq \alpha(T), \beta(0) \geq \beta(T) \text{ and } D_-\alpha(t) \leq f(t, \alpha(t)) \\ D^-\beta(t) \geq f(t, \beta(t)), 0 < t \leq T \end{aligned} \quad (2.16)$$

Let f be of type k on the set $\{x: \alpha(t) \leq x \leq \beta(t), 0 \leq t \leq T\}$. Then there exists a periodic solution $x(t)$ of Eq. (2.31) with

$$\alpha(t) \leq x(t) \leq \beta(t), 0 \leq t \leq T.$$

By D_- and D^- we denote the lower left and upper left differential operators, respectively.

In proving theorem 2.6, we proceed as follows:

We first establish a special case where f is assumed to be Lipschitz continuous with respect to x and the inequalities in (2.16) are assumed to be strict. Then we use approximation arguments to obtain the more general result from the special one. Before proving theorem (2.6), we consider the following special case.

Theorem 2.7: Let $f: [0, T] \times R^n \rightarrow R^n$ be continuous and satisfy a Lipschitz condition with respect to x on compact subsets of $[0, T] \times R^n$. Further, let $\alpha, \beta: [0, T] \rightarrow R^n$ be continuous functions such that

$$\alpha(t) < \beta(t), \alpha(0) \leq \alpha(T), \beta(0) \geq \beta(T)$$

and

$$D_-\alpha(t) \leq f(t, \alpha(t)) \text{ and } D^-\beta(t) \geq f(t, \beta(t)), 0 < t \leq T \quad (2.17)$$

If, in addition, f is a type of K on $\{x: \alpha(t) \leq x \leq \beta(t), 0 \leq t \leq T\}$, Then there exists a periodic solution $x(t)$ of (2.15) such that

$$\alpha(t) \leq x(t) \leq \beta(t), 0 \leq t \leq T \quad (2.18)$$

Proof: Let $[\alpha, \beta] = \{(t, x): \alpha(t) \leq x(t) \leq \beta(t), 0 \leq t \leq T\}$, Then $[\alpha, \beta]$ is a compact subset of $[0, T] \times R^n$. Since f satisfies a Lipschitz condition with respect to x on this set, the initial value problem.

$$x' = f(t, x), \quad x(0) = x_0 \quad (2.19)$$

has a unique solution $x(t; x_0)$ for every x_0 with $\alpha(0) \leq x_0 \leq \beta(0)$. It follows from [31] that $f(t; x_0)$ exists on $[0, T]$ and that

$$\alpha(t) \leq x(t, x_0) \leq \beta(t), 0 \leq t \leq T \quad (2.20)$$

Define mapping

$$S = \{x_0: \alpha(0) \leq x_0 \leq \beta(0)\} \rightarrow \{z: \alpha(T) \leq z \leq \beta(T)\} \subseteq \{x_0: \alpha(0) \leq x_0 \leq \beta(0)\} \quad (2.21)$$

By $S(x_0) = X(T, x_0)$. Then S is continuous. It follows from Brouwer's fixed point theorem that S has a fixed point.

With the help of theorem 2.32 we will prove now theorem 2.31.

Proof of theorem 2.6: Let α and β as in theorem 2.31. Define the function $F(t, x)$ in the following way. For each $t \in [0, T]$ and $i = 1, \dots, n$, let

$$F_i(t, x) = \begin{cases} f_i(t, \bar{x}) - \frac{x_i - \beta_i(t)}{1 + |x_i|} & , \text{if } x_i > \beta_i(t) \\ f_i(t, \bar{x}) & \text{if } \alpha_i(t) \leq x_i \leq \beta_i(t) \\ f_i(t, \bar{x}) - \frac{x_i - \alpha_i(t)}{1 + |x_i|} & , \text{if } x_i < \alpha_i(t) \end{cases} \quad (2.22)$$

Where $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$ and

$$\bar{x}_i = \begin{cases} \beta_i(t), & \text{if } x_i > \beta_i(t) \\ x_i, & \text{if } \alpha_i(t) \leq x_i \leq \beta_i(t) \\ \alpha_i(t), & \text{if } x_i < \alpha_i(t) \end{cases} \quad (2.23)$$

The function $F(t, x)$, so defined, is continuous and bounded on $[0, T] \times R^n$ and furthermore is of type K everywhere. Also a function $x(t)$

$$\alpha(t) \leq x(t) \leq \beta(t), 0 \leq t \leq T$$

is a solution of Eq. (2.15) if and only if it is a solution of

$$x' = F(t, x) \quad (2.24)$$

Let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ be a constant vector with $\epsilon_i > 0$, $i = 1, \dots, n$

Let $A(t) = \alpha(t) - \epsilon$, $B(t) = \beta(t) + \epsilon$, then

by $F_i(t, x) = f_i(t, x) - \frac{x_i - \alpha_i(t)}{1 + |x_i|}$, we have

$$F_i(t, x) = f_i(t, x) - \frac{A_i - A_i(t) - \epsilon}{1 + |A_i|}$$

$$F_i(t, A(t)) = f_i(t, \alpha(t)) + \frac{\epsilon_i}{1 + |A_i(t)|} > f_i(t, \alpha(t))$$

and hence.

$$D_- A(t) = D_- \alpha(t) \leq f(t, \alpha(t)) < F(t, A(t)).$$

which implies that

$$D_- A(t) < F(t, A(t))$$

Similarly, by $F_i(t, x) = f_i(t, x) - \frac{x_i - \beta_i(t)}{1 + |x_i|}$, we have

$$F_i(t, B(t)) = f_i(t, \beta_i(t)) - \frac{B_i - (B_i(t) - \epsilon)}{1 + |B_i|}$$

$$F_i(t, x) = f_i(t, \beta_i(t)) - \frac{\epsilon}{1 + |A_i|} \leq f_i(t, \beta_i(t))$$

And hence

$$D^- B(t) = D^- \beta(t) \geq f(t, \beta(t)) > F_i(t, B(t))$$

This implies that

$$D^- B(t) > F(t, B(t)).$$

Consider now the function $F(t, x)$ on the set

$$\mathfrak{A} = \{(t, x) : A(t) \leq x \leq B(t), 0 \leq t \leq T\}$$

and let $\delta > 0$ be such that

$$D_- A_i(t) - F_i(t, A(t)) < -\delta < 0 < \delta < D^- B_i(t) - F_i(t, B(t)), i = 1, \dots, n.$$

Using an approximation argument [31], we may find a function $G(t, x)$ which satisfies a

Lipschitz condition with respect to x , is of type K and is such that

$$|G_i(t, x) - F_i(t, x)| < \delta/2,$$

$i = 1, \dots, n$ for all $(t, x) \in \mathfrak{A}$. This in turn implies that

$$D_- A(t) < G(t, A(t)), D^- B(t) > G(t, B(t)),$$

Hence by Theorem 2.7, the equation $x' = G(t, x)$ has a periodic solution $x(t)$ such that

$$A(t) \leq x(t) \leq B(t)$$

Using the Ascoli–Arzela Theorem (i.e. pick a monotone decreasing sequence of δ 's converging to 0, etc.)

We conclude that Equation (2.40) has a periodic solution $x(t)$ such that

$$A(t) \leq x(t) \leq B(t).$$

This in turn is true for every positive vector ϵ . Using Ascoli-Arzela theorem once more,

We conclude that there exists a periodic solution $x(t)$ of equation (2.24) such that

$$\alpha(t) \leq x(t) \leq \beta(t).$$

By the remark at the beginning of the proof we conclude that $x(t)$ is a periodic solution of Eq.(2.15).

The proof is complete

2.4 PERIODIC SOLUTION FOR NONLINEAR SECOND ORDER AUTONOMOUS DIFFERENTIAL EQUATION

In this section, we shall consider the second-order scalar differential equation of the form

$$x'' + f(x, x') = 0$$

where, f is continuous and smooth enough to ensure the existence and uniqueness of solution with any set of initial data. It will be shown if some conditions are satisfied, then there exists at least one nontrivial periodic solution.

Let us consider the following second-order differential equation.

$$x'' + f(x, x') = 0 \tag{2.25}$$

where f is a continuous real-valued function with domain \mathbb{R}^2 and is smooth enough to ensure the existence and uniqueness of solution with any set of initial data. Under the above assumptions, we shall establish the following theorem.

Theorem 2.8 Let us assume that there exist constants $\alpha, \beta, a > 0$ and $c_2 > 0$ such that

- i) $\alpha \leq \beta$
- ii) $f(\alpha, 0) \leq 0 \leq f(\beta, 0)$
- iii) $c_2 > d = \max \{ |\alpha|, |\beta| \}$
- iv) where $3M < a(c_2 - d)$

$$M = \text{Max} \{ |ax - f(x, x')| : (x, x') \in A \}$$

And

$$A = \{(x, x') \in \mathbb{R}^2 : |x| \leq 2c_2, |x'| \leq 2c_2\sqrt{a}\}$$

Then there exists at least one $\omega \in \left(\frac{\pi}{2\sqrt{a}}, \frac{3\pi}{2\sqrt{a}}\right)$ such that (2.25) has a nontrivial solution satisfying the following boundary conditions

$$x(0) = x(\omega). \tag{2.26}$$

$$x'(0) = x'(\omega). \tag{2.26a}$$

Proof: For each $c_1, \alpha \leq c_1 \leq \beta$, let us define the function $x(t) = x(t, c_1)$ as the solution of the following integral equation.

$$x(t) = c_1 \cos a^{1/2} t + c_2' \sin a^{1/2} t + F(t, x(t), x'(t)) \tag{2.27}$$

where

$$F(t, x(t), x'(t)) = a^{-1/2} \int_0^t \sin a^{1/2} (t-s) \{ax(s) - f(x(s), x'(s))\} ds$$

and

$$c_2' = c_2 \cdot \text{sign } c_1$$

One can easily verify that $x(t)$ satisfies equation(2.25), and furthermore $x(t)$ is not identically constant, since the velocity at the time $t = 0$ is equal to $c_2' \sqrt{a}$ which is not zero.

Now, let

$$B = \left\{ x(t) \in C^1 \left[0, \frac{3\pi}{2\sqrt{a}} \right] : |x(t)| \leq 2C_2, |x'(t)| \leq 2C_2 \sqrt{a} \right\}$$

And define the operator U on B by

$$Ux(t) = c_1 \cos a^{1/2} t + c_2' \sin a^{1/2} t + f(t, x(t), x'(t))$$

Then we have the following estimations

$$|Ux(t)| \leq |c_1| + |c_2| + 3Ma^{-1}.$$

And

$$|(Ux)'(t)| \leq a^{1/2} (|c_1| + |c_2| + 3Ma^{-1}).$$

Hence, by virtue of (iv), U maps B continuously in to itself and it follows from Schauder's fixed point theorem that Equation (2.26) has a solution in B . Now, since for any $x \in B$,

$|F(t, x, x')| \leq \frac{3M}{a}$, one obtains.

$$\left[c_1 - c_2' - F\left(\frac{\pi}{2\sqrt{a}}, x\left(\frac{\pi}{2\sqrt{a}}\right), x'\left(\frac{\pi}{2\sqrt{a}}\right)\right) \right] x \left[c_1 + c_2' - F\left(\frac{3\pi}{2\sqrt{a}}, x\left(\frac{3\pi}{2\sqrt{a}}\right), x'\left(\frac{3\pi}{2\sqrt{a}}\right)\right) \right] < 0$$

Which leads to the existence of an $\omega \in \left(\frac{\pi}{2\sqrt{a}}, \frac{3\pi}{2\sqrt{a}}\right)$ such that

$$c_1(1 - \cos a^{1/2} \omega) - c_2 \sin a^{1/2} \omega = F(\omega, x(\omega), x'(\omega))$$

or equation (2.26).

Now, for each $c_1, \alpha \leq c_1 \leq \beta$, using *Arzela – Ascoli theorem*, one can easily verify that this solution is a C^1 - function on $[0, \omega] \times [\alpha, \beta]$. Furthermore we claim that

$$x(t, \beta) \geq \beta \text{ and } x(t, \alpha) \leq \alpha \text{ for } 0 \leq t \leq \omega \quad (2.28)$$

In fact, let $\epsilon > 0$ be given and define

$$F_{\epsilon}(x, x') = \begin{cases} f(x, x') & \text{if } x \geq \beta \\ f(\beta, x') - \epsilon(x - \beta) & \text{if } x < \beta \end{cases}$$

Taking M, c_1, c_2 and $a > 0$ as above, we repeat the first part of the proof for the modified equation.

$$x'' + F_\epsilon(x, x') = 0 \quad (2.29)$$

It follows that there exists $\omega_\epsilon \in \left(\frac{\pi}{2\sqrt{a}}, \frac{3\pi}{2\sqrt{a}}\right)$ such that equation (2.29) possesses a nontrivial solution, say $x_\epsilon(t; c_1)$ such that

$$x_\epsilon(0; c_1) = x_\epsilon(\omega_\epsilon; c_1) = c_1$$

Now, we would like to show that

$$x_\epsilon(t, \beta) \geq \beta \text{ for } 0 \leq t \leq \omega_\epsilon \quad (2.30)$$

Suppose equation (2.30) is not true, i.e. There exists $t_1 \in [0, \omega_\epsilon]$ such that

$$x_\epsilon(t_1; \beta) - \beta < 0$$

Since,

$$x_\epsilon(0; \beta) = x_\epsilon(\omega_\epsilon; \beta) = \beta,$$

it follows that $x_\epsilon(t; \beta) - \beta$ must have a negative minimum at $t_0 \in [0, \omega_\epsilon]$, and hence

$$x'_\epsilon(t_0, \beta) = 0$$

But

$$x''_\epsilon(t_0, \beta) = -f(\beta, 0) + \epsilon(x_\epsilon(t_0, \beta) - \beta) < 0$$

Which contradicts the fact that the solution has a minimum at t_0 . Hence Equation (2.30) is true.

With a similar argument we can prove that there exists $x_\epsilon(t, \alpha)$ solution of the other modified equation

$$x'' + G_\epsilon(x, x') = 0,$$

such that

$$x_\epsilon(t, \alpha) \leq \alpha$$

Now, from the definition of $F_\epsilon(x, x')$, we observe that ϵ may be chosen such that

$\omega_\epsilon \rightarrow \omega$ as $\epsilon \rightarrow 0$ and that $x(t; \beta)$ and $x(t; \alpha)$ may be obtained as the limits of these solutions satisfying Equation (2.28).

Finally, define the continuous function g on $[\alpha, \beta]$ by

$$g(c) = x'(0, c) - x'(\omega, c)$$

we conclude by Equation (2.44) that

$$g(\alpha) \leq 0 \leq g(\beta).$$

Hence there must be a constant \bar{C} , $\bar{C} \in [\alpha, \beta]$ such that

$$g(\bar{C}) = 0.$$

This completes the proof of the theorem, by remarking that the solution $x(t; \bar{C})$ satisfies the periodic boundary conditions(2.26) and (2.26a).

2.5 Conclusions

- The conclusion of theorem 2.1 and theorem 2.2 for finding the existence of periodic solutions of (1.1) and theorem 2.3 for finding the existence of periodic solutions of (1.5) implies that a solution of the differential equation has a motion that starts and ends on the boundary of the compact set (closed and bounded set), which shows that there exists a periodic solution.
- The upper and lower solutions method and the monotone iterative technique, the continuation method of topological degree, fixed point theorems, the variational method and critical point theory are established for finding the existence problems for periodic solutions and periodic boundary value problem for second order differential equations, a set of easily verifiable sufficient conditions are obtained for the existence of at least one positive ω – periodic solutions.
- Some new sufficient conditions are established ensuring existence and uniqueness of periodic solution for a general class of autonomous systems are obtained.
- In proving theorem 2.6, theorem 2.7 has been established as a special case where the given function f is assumed to be Lipschitz continuous with respect to x and the inequalities in (2.16) are assumed to be strict to obtain the more general result from the special one so that sufficient conditions are given ensuring that the system of nonlinear differential equations has at least one periodic solution.

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