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NUMERICAL METHODS FOR SOLVING SYSTEM OF HYPERBOLIC UNCOUPLED PDEs

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Abstract

This thesis concentrates on numerical methods for solving **hyperbolic uncoupled PDEs systems with two independent variables (space and time)**and whose model problem is $v_t + Av_x = \bar{0}$ for which A is assumed to be a **diagonalized matrix**;discusses the **consistency,stability and convergence** based on the **sup-norm**, $l_{2,\Delta x}$ and **discrete Fourier series methods** on the difference equations; determine the stability and convergence region of the difference equations so that the solution of the **numerical difference equation is optimal**.The given difference equation is analysed on different time and space schemes to find the nature of the difference equation and approximate solutions with the given Initial Boundary Value Problem by taking sample schemes such as FTFS, BTFS AND CTFS show that the schemes have similar precision and accuracy in a stability region with the smallest grid size.

key words:-Numerical methods, Hyperbolic uncoupled PDEs, Model problem, Diagonalized matrix, Consistency, Stability, Convergency, Sup-norm, $\ell_{2,\Delta x}$Norm, Discrete Fourier Transform,Difference Equation, Difference scheme, Initial Boundary Value Problem(IBVP), precision and accuracy

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Contents

Chapter 1

INTRODUCTION

When we talk about partial differential equation, we mean equation where the solution is a function(or a vector function) of at least two variables,which are called **independent variables**.The equation describes a relation involving the solution and its partial derivatives.But the specification of a mathematical model in applications involve much more than just this relation.First of all, the model is associated to ***geometry***.This means that we specify a domain in the space of the independent variables where the difference should be satisfied.This domain can be finite or infinite.In two dimensions, such domain can be a subset of the plane,but also the surface of a cylinder or a sphere.[1]In this context,we consider numerical methods for solving system of hyperbolic uncoupled PDEs. Particularly a first order linear PDE with two independent variables time and space by taking its model problem as the form of $v_t + Av_x = \bar{0}$.
The world is defined by structure in space and time, and it is forever changing in complex ways that can not be solved exactly.Therefore, the numerical solution of partial differential equation leads to some of the most important,and computationally intensive tasks in all of numerical analysis such as

- forecasting the weather
- simulation of natural processes,such as:-
  - chemical processes
  - fluid mechanics
  - structural dynamics
  - quantum physical process
  - electromagnetism
- finance e.t.c

The problem of stability is pervasive in the numerical solution of partial differential equation. In the absence of computational experience, one would be hardly likely to guess that instability was an issue at all,yet it is a dominant consideration, in almost every computations.The relationship between stability and convergence was hinted at by **RICHARD COURANT, KURT FRIEDREICH** and **HANS LEWY** of the university of Göttingen in Germany,published a famous paper entitled "ON THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS" in the 1920's and identified more clearly by **VON NEUMANN** in the 1940's and brought in to organized form by **LAX** and **RICHTMYER** in the 1950's-**LAX EQUIVALENCE THEOREM**.[3]

LAX EQUIVALANCE THEOREM 1. *A consistent finite difference scheme for a PDE for which the Initial Value Problem is well posed and said to be convergent if and only if it is stable[15]*

The brief of this thesis is organised as follows:- In the second chapter, the concept of different difference schemes,convergence, local-global truncation error, consistency and stability are investigated for difference schemes by applying sup-norm,$l_{2\Delta x}$ norm and discrete Fourier transform as well as its amplification matrix or radius of spectrum are introduced.**The purpose of this chapter is to put the pavements which enable us to find the best approximated solution.**

Chapter 3 discusses the basic concepts of first order linear PDE in connection with the formation of its matrices coefficients particularly for hyperbolic uncoupled PDEs cases and focuses on Initial Value Problem and/or Initial Boundary Value Problem.**The basic purpose of this chapter is to transform the uncoupled hyperbolic PDE in to its couple form**.

In chapter 4,evaluation of first order linear PDE's numerically taken place,stability as well as the convergence region of difference schemes are reviewed , identified and determined by using different methods such as ℓ_∞ , $\ell_{2,\Delta x}$ and discrete Fourier transform.**The purposes of this chapter is to determine the stability region of difference schemes by using ℓ_∞ , $\ell_{2,\Delta x}$ and discrete Fourier transform**

In the next chapter,chapter 5, we discuss the application of sup-norm,$l_{2,\Delta x}$ and discrete Fourier transform in connection with the Initial Value Problem and Initial Boundary Value Problem by using MATLAB to analyse applicability of those stabile regions obtained in the previous chapter.**The basic purpose of this chapter is to evaluate the sufficient and necessary condition or stability regions by using MATLAB**

In chapter six,the dispersion relation of each difference are explained.**The basic purpose of this part is to assess the dispersion relation,phase and group velocity of difference schemes**. In chapter seven, we compare the results obtained and the figure out the best approximation methods and each numerical methods is solved by using MATLAB.After that,in chapter eight the result of numerical methods' precision and accuracy of different difference schemes explained. Additionally,in this thesis,equations and formulas are benefited from books [2],[5],[8],[9]

Chapter 2

CONCEPTS OF LOCAL-GLOBAL TRUNCATION ERROR , CONVERGENCE , CONSISTENCY AND STABILITY

In this part,we will review some of the concepts we will need for the remaining chapters.

2.1 AIM OF THE THESIS

The aim of this study is to compute the approximated solution of hyperbolic uncoupled PDEs.

2.1.1 GOAL AND OBJECTIVE

The goal is finding **necessary and sufficient region** for a given difference scheme (DS) to be an optimal approximation to some continuous problem. By optimal, we mean that the solution of the continuous problem and that the error between the approximation and exact solutions should be 'smallest' when measured in some norms.

2.1.2 APPROACHES APPLIED IN THIS STUDY

The approaches are taken in the spaces of SUP-NORM and $\ell_{2,\Delta x}$.In addition to this, **discrete Fourier transform** is applied for the analysis of the stability region to which the sufficient and necessary conditions are determined and satisfied.

2.2 TAYLOR SERIES EXPANSIONS FOR THE FUNCTION OF ONE AND TWO VARIABLES

We will first consider a Taylor series equation with remainder for the function $f(x)$.We assume the derivatives of $\frac{d^n f}{dx^n}$ are continuous in the interval $a - r < x < a + r$ for $0 \leq n \leq k+1$, $r > 0$.Then for each x in this interval there is at least one point η contained in the interval $a < \eta < x$ or $x < \eta < a$ such that

$$f(x) = \sum_{n=0}^{k} \frac{(x-a)^n f^{(n)}(a)}{n!} + \frac{(x-a)^{k+1} f^{(k+1)}(\eta)}{(k+1)!} \tag{2.1}$$

Here, $f^n(x) = \frac{d^n f}{dx^n}$ denotes the n^{th} derivatives of $f(x)$. We assume that we have obtained the following approximate solutions by using Taylor series (from a difference scheme u_k^n that we shall refer to as where **n corresponds to the time step and k to the spatial mesh step) to this problem**, u_k^n which is defined on a grid (with grid spacing Δx and Δt) and satisfies the initial condition $u_k^0 = f(k\Delta x)$,$k \in \mathbb{R}$.
A similar formula exists for two variables differentiated at (x_0, t_0) for first order PDE with two independent variables x and t, Taylor series becomes:-

$$u(x,t) = u(x_0, t_0) + (x - x_0)u_x + (t - t_0)u_t + \mathcal{O}(\Delta x^2, \Delta t^2) \tag{2.2}$$

Let $\mathbf{v}$ denote the exact solution of to our *Initial Value Problem.*We then have the following formulas for $\mathbf{v}_t$ and $\mathbf{v}_x$

TYPES OF DIFFERENCE SCHEME	FORMULATION	ORDER
BACKWARD TIME	$\mathbf{v}_t = \frac{\mathbf{u}_k^{n-1}-\mathbf{u}_k^n}{\Delta t}$	$\mathcal{O}(\Delta t)$
FORWARD TIME	$\mathbf{v}_t = \frac{\mathbf{u}_k^{n+1}-\mathbf{u}_k^n}{\Delta t}$	$\mathcal{O}(\Delta t)$
FIRST ORDER CENTER TIME	$\mathbf{v}_t = \frac{\mathbf{u}_k^{n+1}-\mathbf{u}_k^{n-1}}{2\Delta t}$	$\mathcal{O}(\Delta t^2)$
BACKWARD SPACE	$\mathbf{v}_x = \frac{\mathbf{u}_{k-1}^{n}-\mathbf{u}_k^n}{\Delta x}$	$\mathcal{O}(\Delta x)$
FORWARD SPACE	$\mathbf{v}_x = \frac{\mathbf{u}_{k+1}^{n}-\mathbf{u}_k^n}{\Delta x}$	$\mathcal{O}(\Delta x)$
FIRST ORDER CENTER SPACE	$\mathbf{v}_x = \frac{\mathbf{u}_{k+1}^{n}-\mathbf{u}_{k-1}^n}{2\Delta x}$	$\mathcal{O}(\Delta x^2)$

Table 2.1: DIFFERENCE SCHEMES IN TIME AND SPACE

2.3 NORMS AND RELATED IDEAS

One of the most fundamental properties of any object be its mathematical or physical, of its size.Ofcourse,**in numerical analysis we are always concerned with the size of the error of any particular numerical approximation or computational procedure**.There is a general mathematical object called **Norm** by which we can assign a number corresponding to the size of the various mathematical entities.

DEFINITION 1. *Let V be a (finite or infinite dimensional) vector space, and let $\|.\| : \boldsymbol{V} \to \mathbb{R}^+ \bigcup\{0\}$ with the following properties*

* $\|\boldsymbol{v}\| \geq 0$ *for all* $\boldsymbol{v} \in \boldsymbol{V}$ *with* $\|\boldsymbol{v}\| = 0 \Leftrightarrow \boldsymbol{v} = \bar{0}$
* $\|\lambda\boldsymbol{v}\| = |\lambda|\|\boldsymbol{v}\|, \forall \boldsymbol{v} \in \boldsymbol{V}, \lambda \in \mathbb{R}$
* $\|v + w\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|, \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$

Then $\|.\|$ is called a norm for $\boldsymbol{V}$.[8]

Note that we can take **V** to be a space of vectors,function or even operators,and the above properties apply.It is important to observe that for a given space **V** there are many different mappings $\|.\|$ having the properties required mentioned above.But for our case we use some of them as mentioned below.If $\mathbf{v}\in\mathbf{V}$ is an infinite dimensional space of vectors with elements $\mathbf{v} = [..., v_{-1}, v_0, v_1, ...]^T$,then a familiar measure of the size of $\mathbf{v}$ is its Euclidean length whose mesh length along x-axis is Δx

$$\|\mathbf{v}\|_{2,\Delta x} = (\sum_{j=-\infty}^{\infty} |\mathbf{v}_j|^2)^{\frac{1}{2}} \sqrt{\Delta x} \tag{2.3}$$

is known as $\ell_{2,\Delta x}$ norm satisfies conditions listed under definition 1. Another useful norm often encountered in practice is the ℓ_∞ called **sup-norm or infinity norm** defined as

$$\|\mathbf{v}\|_\infty = \sup_{j\in\mathbb{R}} |\mathbf{v}_j| \tag{2.4}$$

Therefore,ℓ_∞ norm represent the absolute value of the supremium or largest component in the vector $\mathbf{v}$.[1] [2]

2.4 TRUNCATION AND ROUND OFF ERROR

In this section,we will see that some errors that can be produced by using numerical method.When any numerical methods applied to the system errors occurs in two different forms[2][1][11][13]

* *Truncation error*
* *Round off error*

2.4.1 TRUNCATION ERROR

It happens when approximations are made to estimate the solution of given or obtained system.This error can be categorized as *LOCAL or GLOBAL* truncated error.

* Local truncated error denoted by τ_k^n and introduced the local error at $(x_k, t_n) = (k\Delta x, n\Delta t)$ for each k and n.It appears when a numerical method is applied to solve the system.To get the local truncated error,take the difference of the expand Taylor series and numerical solution for each k and n and the remaining term is called ,as shown below, the local truncated error.

$$\|\tau_k^n\| = \mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q) \tag{2.5}$$

  If p and q are larger,then the method is more accurate.

* Global truncated error :- is denoted by e_k^n which is defined as the difference of analytical and numerical solution. A difference scheme approximating a function v and numerical solution u at time t in a given space of partitions Δx_j , as (n+1)$\times\Delta t \to t$,

$$e_k^n = \|u^n - v^n\| \tag{2.6}$$

  Global truncation error is caused by the total accumulation of each local errors occurred at each mesh point for each iteration. Let $u = [..., u_{-1}, u_0, u_n, ...]^T$ be the solution of difference scheme (assume that no round off errors occurred),and $v = [..., v_{-1}, v_0, v_1, ...]^T$ be the exact solution at grid or lattice point.The global error vector is defined as $\mathbf{e} = \mathbf{v} - \mathbf{u}$. Naturally,we can use different norms such as for ℓ_∞ and $\ell_{\Delta x,2}$ respectively as follows:-

$$\|\mathbf{e}\|_\infty = \sup_{i\in\mathbb{Z}} |v_i - u_i| \tag{2.7}$$

$$\|\mathbf{e}\|_{2,\Delta x} = (\sum_{i=-\infty}^{\infty} |v_i - u_i|^2)^{\frac{1}{2}} \sqrt{\Delta x} \tag{2.8}$$

Consistency is used to show the accuracy of the method to be implemented. Consistency condition is satisfied by

$$\mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q) \to 0 \tag{2.9}$$

as $\Delta x, \Delta t \to 0$.Thus,the speed of the of the convergence of approximated solution is u to the exact solution v is related with the speed of the convergence of condition 2.9.When step sizes or grid sizes $\Delta x, \Delta t$ will be chosen small,the intervals will increase which provides us closer to zero.[2] For convenience, we denote the vector of difference equation solution values u_k^n by $\mathbf{u}^n$ and the vector of solution values of the PDE evaluated at the grid points $v(k\Delta x, n\Delta t)$ by $\mathbf{v}^n$

DEFINITION 2. *A difference scheme u approximating the partial differential equation v is a convergent scheme at a time t if ,as $(n+1)\Delta t \to t$, $\| \boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1} \| \to 0$ as Δx,$\Delta t \to 0$. [13]*

REMARK 1. *It should be noted that the norm used above was not specified.This was done in this manner because in different situations, different norms will be used.At other times we will see that the natural norm to choose will be a variation of the ℓ_∞ norm , the $\ell_{2,\Delta x}$ norm.*

2.4.2 ROUND OFF ERROR

It originates due to the operations taken by a computer that performs on limited number of decimals of digits.After calculating the approximation of the value, the result is dropped by some values called Round off error

2.5 THE CONCEPTS OF DIFFERENCE METHOD

It is extremely important for a user of difference techniques to understand precisely

1. *what type of convergence their scheme has*
2. *what kinds of assumptions are made to get this convergence* and
3. *how this convergence affects their accuracy*

The most common approach to approach to convergence of difference equations is through the concepts of *consistency,stability* and the *Lax theorem*.The *Lax theorem* allows us to prove convergence of a difference scheme by showing that the scheme is both consistent and stable.

2.5.1 CONSISTENCY,STABILITY AND CONVERGENCE

Here,we present some basic descriptive terms.By **consistency** we mean that the difference approximation *converges to* the PDE as discretization step size approaches zero,and by **stability** we mean that the solution to the difference equation does not increase with time at faster rate than does the solution to the difference equation.**Convergence** implies that the *solutions* to the difference equation approach those of the PDE,as discretization grid size are refined.If this does not occur,the associated numerical approximation are useless.The solution is u_k^n is stabile if the error remains bounded as n increases.Moreover,the difference scheme is stabile if it gives stabile solution for any Initial Condition.Stability of a scheme is a property of great importance.Thus,conditions for stability should be known,In this thesis,we use the following methods for the explanation of stability and its region.

* $\ell_{2,\Delta x}$ Norm
* ℓ_∞ Norm
* Discrete Fourier transform or Von Neumann's condition.

For difference schemes' solutions to certain PDE is what is really needed which can be made to approximate the solution of the PDE to any desired accuracy. Thus, we want some sort of *convergence of the solution of the the finite difference equation* to the solution of the PDE. At certain times we want to discuss convergence in terms of how fast the solution of the difference equation converges to the solution of the PDE.For this purpose we define convergence of order (p, q) as follows[13]

DEFINITION 3. *A difference scheme u approximating the PDE v is a convergent scheme of order (p, q) if for any t, as $(n + 1)\Delta t$ converges to t,*

$$\|u_k^n - v_k^n\| = \mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q). \tag{2.10}$$

[9] as Δx and Δt converges to zero

For the largest value of p and q,is known as the *order or order of convergence of method*.Convergence occurs for the given PDE then approximated solution u tends or closer to analytic solution v as the mesh or grid sizes $\Delta x, \Delta t$ goes to zero.This describes that convergence in the limits as $\Delta x, \Delta t \to 0$ ($n \to \infty, n\Delta t = t - t_0, k\Delta x = x - x_0$).It can be represented as:-

$$\lim_{(\Delta x,\Delta t)\to(0,0)} u_k^n \to (0, 0) \tag{2.11}$$

When we use the "$\mathcal{O}$" notation, we must always remember that there is a constant involved,i.e, 2.10 is really a short hand notation for there exists a constant C such that $\|u_k^n - v_k^n\| \le C(\Delta x^p + \Delta t^q)$ in this case , the constant C will depend on t.

2.5.2 STABILITY ANALYSIS BY USING $\ell_{2,\Delta x}$, ℓ_∞ AND DISCRETE FOURIER TRANSFORM

In the previous section,we explained how important stability is for proving convergence of difference scheme.This is done largely by introducing tools that can be used to prove stability of Difference Scheme, such as discrete Fourier transform[2][5]. One property of Fourier transform that we do not use directly in solving problems analytically by Fourier transforms, but is very important theoretically is Parseval's Identity,that is, [2][5],[9],[12]

$$\|v\|_2 = \|\mathbf{v}\|_2$$

We can use essentially the same approach to analyze the stability of difference schemes for initial Value Problem.We suppose that we are given a vector in l_2, $\mathbf{u} = (..., u_{-1}, u_0, u_1, ...)$ and define the discrete Fourier transform of $\mathbf{u}$ as follows [2][5]

DEFINITION 4. *The discrete Fourier transform of $u \in L_2$ is the function $\boldsymbol{u} \in l_2[-\pi, \pi]$ defined by*

$$\boldsymbol{u}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} u_m \tag{2.12}$$

for $\xi \in [-\pi, \pi]$

We note that the L_2 vectors that will be using later $l_{2,\Delta x}$ vectors and will be the solutions to our Difference scheme at time step n,u^n

PROPOSITION 1. *If $u \in L_2$ and u is the discrete Fourier transform of $\boldsymbol{u}$ then*

$$u_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} \boldsymbol{u}(\xi) d\xi \tag{2.13}$$

PROPOSITION 2. *If $u \in L_2$ and u is the discrete Fourier transform of u then*

$$\|\boldsymbol{u}\|_2 = \|u\|_2 \tag{2.14}$$

where the first norm is the L_2 and on $[-\pi, \pi]$ and the second norm is the l_2 norm [2][5],[9],[12]

Proof. Proposition 2 can be proved formally by the following calculation [2][5],[9],[12]

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ik\xi} \mathbf{u}(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ik\xi} \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} u_m d\xi$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} u_m \int_{-\pi}^{\pi} e^{i(k-m)\xi} d\xi$$

$$= \frac{1}{2\pi} \sum_{m=-\infty, m\neq k}^{\infty} u_m [\frac{e^{i(k-m)\xi}}{i(k-m)}]_{-\pi}^{\pi} + \frac{1}{2\pi} u_k \int_{-\pi}^{\pi} d\xi$$

$$= \frac{1}{2\pi} \sum_{m=-\infty, k\neq m}^{\infty} u_m (\frac{1}{i(k-m)})[e^{i(k-m)\pi} - e^{-i(k-m)\pi}] + u_k$$

$$= u_k$$

□

The **discrete Fourier transform** is one of the basic tools that we will use in our stability analysis.In the definition of stability,the inequality that was required interms of the energy norm was of the form

$$\|\mathbf{u}^{n+1}\|_{2,\Delta x} \leq \kappa e^{\beta(n+1)\Delta t} \|\mathbf{u}^n\|_{2,\Delta x} \tag{2.15}$$

but since

$$\|\mathbf{u}\|_{2,\Delta x} = \sqrt{\Delta x} \|\mathbf{u}\|_2$$
$$= \sqrt{\Delta x} \|\mathbf{u}\|_2$$

If we can find a κ and β to satisfy

$$\|\mathbf{u}^{n+1}\|_2 \leq \kappa e^{\beta(n+1)\Delta t} \|\mathbf{u}^0\|_2 \tag{2.16}$$

then the same κ and β will also satisfy 2.15.when inequality 2.16 holds,we say that the sequence $\{\mathbf{u}^n\}$ is stabile in the transform space L_2.If we can find κ and β for 2.16, *there is no need to return to the original equation.*

2.6 CONCEPTS OF DIAGONALIZED MATRIX

To make this thesis reasonably self contained, we include here a brief review of some basic definitions from the linear Algebra perspective which is more convenient for this thesis.

DEFINITION 5. *A square matrix $\boldsymbol{A}$ is said to be diagonalizable if $\boldsymbol{A}$ is similar to diagonal matrix,i.e there exists an invertible matrix $\boldsymbol{S}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{SDS}^{-1}$*

In this case,we can say that S diagonalize A and S is called model matrix for A. [12]

2.7 SYMBOL OR AMPLIFICATION MATRIX

The definition of consistency,stability and convergence are the same as they were in the scalar case.The only modification that must be made is that we must use a different norm.Then using the vector norm, we see that the Lax theorem also applies to systems of equations.Thus , as before ,**the job of proving**

convergence is replaced by showing consistency and stability. .Again, the consistency is relatively easy to show and we proceed to discuss methods for showing that difference schemes for systems of PDE are stable.We consider now stability for an invertible matrix A and generalize the Von Neumann condition to the system.We present the procedure how to determine the amplification matrix or radius of the spectrum by using $\ell_{2,\Delta x}$,l_∞ spaces and discrete Fourier transform. To be the **symbol or amplification matrix** of the difference scheme.We see that

$$\hat{\mathbf{u}}^{n+1}(\xi) = \mathbf{G}(\xi)\hat{\mathbf{u}}^n \tag{2.17}$$

$$= \mathbf{G}^{n+1}(\xi)\hat{\mathbf{u}}^0 \tag{2.18}$$

So, the growth of $\hat{\mathbf{u}}^n$ $(\mathbf{u}^n)$ and ,hence, the stability of the scheme, depends on the growth of the amplification matrix raised to the n^{th} power,$\mathbf{G}^n$

THEOREM 1. *Difference scheme is stable with respect to the $l_{2,\Delta x}$ norm if and only if there exists a positive constants Δx_0 and Δt_0 and non negative constants κ and β so that*

$$\|\boldsymbol{G}^n(\xi)\|_2 \leq \kappa e^{\beta t} \tag{2.19}$$

for $0 < \Delta x \leq \Delta x_0, 0 < \Delta t \leq \Delta t_0, t = n\Delta t$ and all $\xi \in [-\pi, \pi]$

Suppose that the amplification matrix **G** associated with different scheme satisfies the **Von Neumann** condition.Then

1. if **G** is Hermitian,the scheme is stable
2. if there exists a matrix **S** such that $\|\mathbf{S}\|_2 \leq \mathbf{C}_1$, $\|\mathbf{S}^{-1}\|_2 \leq \mathbf{C}_2$, $\mathbf{C}_1, \mathbf{C}_2$ are constants and $\mathbf{SGS}^{-1}$ is Hermitian, the scheme is stable.

Proof. 1.Since **G** is Hermitian, $\sigma(\mathbf{G}(\xi)) = \|\mathbf{G}(\xi)\|_2$.Then since $\sigma(\mathbf{G}(\xi) \leq 1 + \mathbf{C}\Delta t$,

$$\|\mathbf{G}(\xi)\|_2 \leq 1 + \mathbf{C}\Delta t \leq e^{\mathbf{C}\Delta t}$$

Then $\|\mathbf{G}^n\|_2(\xi) \leq \|\mathbf{G}\|_2^n \leq e^{\mathbf{C}\Delta t}$ so the scheme is stable by theorem 1
2. Let $\mathbf{H}$=$\mathbf{SGS}^{-1}$.Then

$$\begin{aligned}
\|\mathbf{G}^n(\xi)\|_2 &= \|\mathbf{S}^{-1}\mathbf{H}^n\mathbf{S}\|_2 \\
&\leq \|\mathbf{S}^{-1}\|_2\|\mathbf{H}^n\|_2\|\mathbf{S}\|_2 \\
&\leq \mathbf{C}_1\mathbf{C}_2\|\mathbf{H}^n\|_2 \\
&\leq \mathbf{C}_1\mathbf{C}_2\|\mathbf{H}\|_2^n \\
&= \mathbf{C}_1\mathbf{C}_2[\sigma(\mathbf{H})]^n \\
&= \mathbf{C}_1\mathbf{C}_2[\sigma(\mathbf{G})]^n \\
&\leq \mathbf{C}_1\mathbf{C}_2(1 + \mathbf{C}\Delta t)^n \\
&\leq \mathbf{C}_1\mathbf{C}_2 e^{\mathbf{Cn}\Delta t}
\end{aligned}$$

So again by theorem 1, the scheme is stable.[1] □

Chapter 3

FIRST ORDER UNCOUPLED LINEAR PDE

3.1 LINEAR SYSTEM OF FIRST ORDER HYPERBOLIC UNCOUPLED PDE

A first order linear PDE in two independent variables x and t has the form

$$A(t,x)u_x + B(t,x)u_t + C(t,x)u = f(t,x) \tag{3.1}$$

where A,B,C $\in \mathbb{R}^{k\times k}$; u_t , u_x , u and f $\in \mathbb{R}^k$ are continuously differentiable function of partial derivatives in some region $\mathcal{R}$.For system of hyperbolic equations in one space variable with variable coefficients,we require uniform diagonalizability

It is said to be coupled hyperbolic if A is diagonalizable with the real eigenvalues and there exist an invertible matrix $S(x,t)$ such that

$$A(t,x) = S(t,x)D_1(t,x)S^{-1}(t,x)$$

$$D_1(t,x) = diag(\mu_{i,j}(t,x))$$

for $j = 1, 2, ..., k$
By applying change of variables,assume that $u = Sv$ and then by using product rule of derivatives we obtain the following result

$$u_t = Sv_t + S_tv$$

$$u_x = Sv_x + S_xv$$

Then substituting this result in to equation 3.1 yields

$$A(Sv_x + S_xv) + I(Sv_t + S_tv) + CSv = f$$

and gives

$$D_1 v_x + I v_t + C v + (S^{-1} S_x + S^{-1} S_t) v = f \tag{3.2}$$

which becomes $(D_1 S v_x + S v_t + C S v) + (A S_x + I S_t) v = f$ where $D_1 = S^{-1} A S$ is a diagonal matrix.If S is an invertible matrix of constants equation 3.2 becomes

$$D_1 v_x + I v_t + C v = S^{-1} f \tag{3.3}$$

Now the equations 3.2 and 3.3 are not any longer a decoupled system

DEFINITION 6. *The system*

$$I u_t + A(t,x) u_x + C(t,x) u = F(t,x) \tag{3.4}$$

where I is an identity matrix, with

$$u(0,x) = u_0(x) \tag{3.5}$$

is hyperbolic if there is a matrix function S(t,x) such that

$$S(t,x) A(t,x) S^{-1}(t,x) = \Lambda(t,x) = \begin{bmatrix} \mu_1 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & 0 & \cdots & 0 \\ 0 & 0 & \mu_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_k \end{bmatrix}_{k \times k}$$

is diagonal with real eigenvalues and the matrix norms of S(t,x) and S^{-1}(t,x) are bounded in x and t, for $x \in \mathbb{R}$,$t \geq 0$

3.2 MODEL PROBLEM OF DECOUPLED HYPERBOLIC PDEs

We consider a system of PDE of first order and semi-linear with two independent variables x and t,and k independent variables that taken as a model problem of equation 3.1 assumed to be

$$I u_t + A u_x = \bar{0} \tag{3.6}$$

To begin on operating on 3.6 we need to assume that A and I be constant and identity matrices respectively .From the definition of being diagonalized, we know that if A is diagonalizable and has real eigenvalues ,the decoupled hyperbolic PDE of 3.6 can be changed in to its coupled form as follows. Moreover,there exist an invertible matrix S such that A can be factorized as $A = S^{-1} D_1 S$ and $D_2 = S$ where $D_1 = diag(\mu_j)$ and for each $j = 1, 2, ..., k$. As we have done above by applying variable transformation

$$u = Sv \Leftrightarrow v = S^{-1} u$$

$$u_t = Sv_t$$
$$u_x = Sv_x$$

where S is a constant matrix and substituting this result in to 3.6 gives us

$$ISv_t + ASv_x = Sv_t + (SDS^{-1})S = S(v_t + Dv_x) = \bar{0}$$

we have therefore decoupled 3.6 in to

$$Iv_t + Dv_x = \bar{0} \Leftrightarrow v_{jt} + \mu_j v_{jx} = \bar{0} \tag{3.7}$$

For each equation we have a characteristics equation

$$\frac{dx}{dt} = \mu_j \tag{3.8}$$

for each $j = 1, 2, ..., k$ and therefore we can observe that equation 3.8 has a total of k characteristic equation which results k families of characteristics

$$x(t) = \mu_j t + const \tag{3.9}$$

for each $j = 1, 2, ..., k$

3.3 INITIAL BOUNDARY VALUE PROBLEM

Initial condition specify the unknown function, and possibly its derivatives, at an initial time. Various equation requires a different number of initial condition depending on the order of the time derivative(s). The advection equation $v_t + Av_x = 0$ for example requires only one initial condition since it has only first derivatives in time.Thus,an initial boundary value problem is given as

$$Iu_t + Au_x = \bar{0}, c \in \mathbb{R}, t \geq 0 \tag{3.10}$$

$$u(0, x) = f(x), x \in \mathbb{R} \tag{3.11}$$

$R = \{(x, t) : x \geq 0, t \geq 0\}$

$$v_{jt} - \mu_j v_{jx} = 0, x \in \mathbb{R}, t > 0, j = 1, 2, ..., K \tag{3.12}$$

We know that the equation of the form of 3.12 have a solution of the form

$$v_j(x, t) = v_{0j}(x + \mu_j t) \tag{3.13}$$

where $v_0 = [v_{01}, v_{02}, ..., v_{0K}]^T$ denotes the Initial condition for the system 3.6 (and is given interms of the initial condition for the primitive variables v_0 by $v_0 = Sv_0$).The solution 3.13 can be recoupled to provide a solution to the original Initial value Problem 3.6 along with $v(x, 0) = v_0(x)$ as

$$v(t, x) = \mathbf{S}^{-1}v = \mathbf{S}^{-1} \begin{bmatrix} v_{01}(x + \mu_j t) \\ v_{02}(x + \mu_j t) \\ \vdots \\ v_{0K}(x + \mu_j t) \end{bmatrix} \tag{3.14}$$

we note in solution 3.14 that they are coupled, the solution consists of K waves, propagating with the speed $-\mu_j, j = 1, 2, ..., K$. As in the case of scalar hyperbolic PDE, the phase velocities for solution of systems of hyperbolic PDE must also be approximated. The difference is that we have K phase velocities to approximate and that each component of the solution will involve sums of these propagating wave when they are coupled together as in 3.14

Chapter 4

EVALUATION OF NUMERICAL DIFFERENCE EQUATION FOR FIRST ORDER LINEAR PDE

To explain the numerical schemes for solving hyperbolic equations, we use the model problem

$$v_t + Av_x = \bar{0} \tag{4.1}$$

and we have nine possibilities to determine the stability and convergence of the given difference equation as shown below

$t \times s$	FS	BS	CS
FT	FTFS	FTBS	FTCS
BT	BTFS	BTBS	BTCS
CT	CTFS	CTBS	CTCS

Table 4.1: MAXIMUM POSSIBILITIES TO ASSESS THE STABILITY OF $v_t + Av_x = \bar{0}$

where F=Forward,B=Backward,C=first order Center,T=time and S=Space

4.1 STABILITY AND CONVERGENCE ANALYSIS BASED ON $\ell_{2,\Delta x}$ AND ℓ_∞

The following table shows the difference equation for each difference schemes for which D is assumed to be the diagonal matrix of A whose entries are the eigenvalues of A.

DIFF. SCHEME	DIFF. EQUATION	ORDER
FTFS	$u_k^{n+1} = (I + RA)u_k^n - RAu_{k+1}^n$	$\mathcal{O}(\Delta t^2 + \Delta t\Delta x)$
BTFS	$u_k^{n-1} = (I + RA)u_k^n - RAu_{k+1}^n$	$\mathcal{O}(\Delta t^2 + \Delta t\Delta x)$
CTFS	$u_k^{n+1} = u_k^{n-1} - 2RA(u_{k+1}^n - u_k^n)$	$\mathcal{O}(\Delta t^3 + \Delta t\Delta x)$
FTBS	$u_k^{n+1} = (I + RA)u_k^n - RAu_{k-1}^n$	$\mathcal{O}(\Delta t^2 + \Delta t\Delta x)$
BTBS	$u_k^{n-1} = (I + RA)u_k^n - RAu_{k-1}^n$	$\mathcal{O}(\Delta t^2 + \Delta t\Delta x)$
CTBS	$u_k^{n+1} = u_k^{n-1} - 2RA(u_{k-1}^n - u_k^n)$	$\mathcal{O}(\Delta t^3 + \Delta t\Delta x)$
FTCS	$u_k^{n+1} = u_k^n - \frac{RA}{2}(u_{k+1}^n - u_{k-1}^n)$	$\mathcal{O}(\Delta t^2 + \Delta t\Delta x^2)$
BTCS	$u_k^{n-1} = u_k^n - \frac{RA}{2}(u_{k+1}^n - u_{k-1}^n)$	$\mathcal{O}(\Delta t^2 + \Delta t\Delta x^2)$
CTCS	$u_k^{n+1} = u_k^{n-1} - RA(u_{k+1}^n - u_{k-1}^n)$	$\mathcal{O}(\Delta t^3 + \Delta t\Delta x^2)$

Table 4.2: THE DIFFERENCE EQUATIONS OF DIFFERENCE SCHEMES SHOWN ON TABLE 4.1 ABOVE
where $R = \frac{\Delta t}{\Delta x}$ called mesh ratio, I is an identity matrix and D is a diagonal matrix

From table 4.2,its clear that the following difference schemes equations' right sides are identical and hence, have similar behaviour in ℓ_∞ and $\ell_{2,\Delta x}$ spaces over k

1. FTFS and FTBS
2. BTFS and BTBS
3. CTFS,CTBS and CTCS
4. FTCS and BTCS

Moreover, the inverse of amplification matrix or spectrum radius of FTFS ,FTBS and FTCS is amplification matrix of BTFS ,BTBS and BTCS respectively.In addition to this, CTFS,CTBS and CTCS have similar behaviour. Now, lets solve the difference scheme in ℓ_∞ and $\ell_{2,\Delta x}$ spaces and then determine the stability and convergence of the scheme

- ASSESSMENT OF FTFS IN ℓ_∞ SPACE
  By using ℓ_∞ over k FTFS difference scheme can be written as follows:-

  STABILITY ANALYSIS 1.

$$\sum_{k=-\infty}^{\infty} |u_k^{n+1}|^2 = \sum_{k=-\infty}^{\infty} |(I + RA)u_k^n - RAu_{k+1}^n|^2$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |u_k^{n+1}|^2 \leq \sum_{k=-\infty}^{\infty} (|I+RA|^2|u_k^n|^2+2|I+RA||RA||u_k^n||u_{k+1}^n|+|RA|^2|u_{k+1}^n|^2)$$

Since geometric progression of $2|I+RA||RA||u_k^n||u_{k+1}^n|$ *is faster than its arithmetic progression,it can be replaced as* $2|I+RA||RA||u_k^n|^2$ *Thus,*

$$\sum_{k=-\infty}^{\infty} |u_k^{n+1}|^2 \leq \sum_{k=-\infty}^{\infty} (|I+RA|+|RA|)^2|u_k^n|^2$$

Therefore,its sup-norm over k is

$$\Rightarrow \|u^{n+1}\|_\infty \leq [|I+RA|+|RA|]\|u^n\|_\infty$$

And its iterative result and amplification matrix (whose diagonal entity of diagonalized matrix is the diagonal matrix of D such that each entities are the eigenvalues of A denoted by μ_j*) are respectively:-*

$$\|u^{n+1}\|_\infty \leq [|I+RD|+|RD|]^n\|u^0\|_\infty$$

and

$$\rho_j = |1+R\mu_j|+|R\mu_j|$$

where $G = [|I+RD|+|RD|]^n$ *and* $G \in \mathbb{R}^{k\times k}$ *is a matrix often called Amplification matrix,*$\beta \in \mathbb{R}^k$ *is an arbitrary frequency and* $u^0 \in \mathbb{R}^k$ *an arbitrary vector.Since the necessary condition for stability is then* $\rho(\xi) \leq 1$ *and thus FTFS is said to be stable if* $|\rho_j| \leq 1$*.Since A is diagonalizable,we can write* $A = SDS^{-1}$, $D = diag(\mu_1, \mu_2, ..., \mu_k)$ *then the matrix S diagonalize both A and G. So the eigenvalues of G are on the mid diagonal factor, they are* $\rho_j = [|1+R\mu_j|+|R\mu_j|]$*.Now, by choosing the value of* $\beta = 0$ *and* $\kappa = 1$*,then fix the value of* $R\mu_j \leq 0$*,Thus,*

$$\Rightarrow ||1+R\mu_j|+|R\mu_j|| \leq 1$$

Since $R\mu_j \leq 0$ *,*$|R\mu_j| = -R\mu_j$ *which yields*

$$-1 \leq |1+R\mu_j|-R\mu_j \leq 1 \Rightarrow R\mu_j - 1 \leq |1+R\mu_j| \leq 1+R\mu_j$$

Here,the left side of the inequality is true and hence satisfied.Thus, solving the right hand side of the inequality yields $R\mu_j \in [-1,\infty)$*.Now from the result obtained and 2.15,* $R\mu_j \geq -1$ *and the fixed value* $R\mu_j \leq 0$ *,* $R\mu_j \in [-1,0]$ *and therefore the scheme is stable and by using the definition of Lax equivalence ,it is also convergent.*

- ASSESSMENT OF FTFS IN $\ell_{2,\Delta x}$ SPCACE
  By using $\ell_{2,\Delta x}$ over k FTFS difference scheme can be written as follows:-

  STABILITY ANALYSIS 2.

$$\sum_{k=-\infty}^{\infty} |u_k^{n+1}|^2\sqrt{\Delta x} = \sum_{k=-\infty}^{\infty} |(I+RA)u_k^n - RAu_{k+1}^n|^2\sqrt{\Delta x}$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |u_k^{n+1}|^2\sqrt{\Delta x} \leq \sum_{k=-\infty}^{\infty} (|I+RA|^2|u_k^n|^2\sqrt{\Delta x}+2|I+RA||RA||u_k^n||u_{k+1}^n|\sqrt{\Delta x}+|RA|^2|u_{k+1}^n|^2\sqrt{\Delta x})$$

Since geometric progression of $2|I+RA||RA||u_k^n||u_{k+1}^n|$ *is faster than its arithmetic progression,it can be replaced as* $2|I+RA||RA||u_k^n|^2$ *Thus,*

$$\sum_{k=-\infty}^{\infty} |u_k^{n+1}|^2\sqrt{\Delta x} \leq (|I+RA|+|RA|)^2|u_k^n|^2\sqrt{\Delta x}$$

Therefore,its $\ell_{2,\Delta x}$ *over* k *is*

$$\Rightarrow \|u^{n+1}\|_{2,\Delta x} \leq [|I+RA|+|RA|]\|u^n\|_{2,\Delta x}$$

And its iterative result and amplification matrix whose diagonal entity is μ_j *are respectively as done above:-*

$$\|u^{n+1}\|_{2,\Delta x} \leq [|I+RD|+|RD|]^n\|u^0\|_{2,\Delta x}$$

and

$$\rho_j = |1+R\mu_j|+|R\mu_j|$$

Hence, the amplification matrix's result is the same with the spectrum radius's result which we obtain above. Therefore, the scheme is stable and by using the definition of Lax equivalence ,it is also convergent.

The following tables show the general properties of the rest of the difference schemes in ℓ_∞ and $\ell_{2,\Delta x}$

DIFFERENCE SCHEME	ℓ_∞ AND $\ell_{2,\Delta x}$ SCHEME	ITS ITERATIVE SCHEME
FTFS AND FTBS	$\|u^{n+1}\| \leq (\|I+RD\|+\|RD\|)\|u^n\|$	$\|u^{n+1}\| \leq (\|I+RD\|+\|RD\|)^n\|u^0\|$
BTFS AND BTBS	$\|u^{n-1}\| \leq (\|I+RD\|+\|RD\|)\|u^n\|$	$\|u^0\| \leq (\|I+RD\|+\|RD\|)^n\|u^n\|$
CTFS AND CTBS	$\|u^{n+1}\| \leq (2I+2\|2DR\|)\|u^n\|$	$\|u^{n+1}\| \leq (2I+2\|2RD\|)^n\|u^0\|$
FTCS	$\|u^{n+1}\| \leq (2I+\|RD\|)\|u^n\|$	$\|u^{n+1}\| \leq (2I+\|RD\|)^n\|u^0\|$
BTCS	$\|u^{n-1}\| \leq (2I+\|RD\|)\|u^n\|$	$\|u^0\| \leq (2I+\|RD\|)^n\|u^n\|$
CTCS	$\|u^{n+1}\| \leq (2I+2\|RD\|)\|u^n\|$	$\|u^{n+1}\| \leq (2I+2\|RD\|)^n\|u^0\|$

Table 4.3: THE SCHEME,DIFFERENCE EQUATION AND ITS ITERATIVE IN THE ℓ_∞ AND $\ell_{2,\Delta x}$
where $\|.\|$ implies $\|.\|_\infty$ and $\|.\|_{2,\Delta x}$

DIFFERENCE SCHEME	SPECTRAL RADIUS	STABILITY REGION	STABILITY
FTFS AND FTBS	$\|1+R\mu_j\|+\|R\mu_j\|$	$R\mu_j \in [-1,0]$	STABILE
BTFS AND BTBS	$\frac{1}{\|1+R\mu_j\|+\|R\mu_j\|}$	$R\mu_j \in (-\infty,-1]$	STABILE
CTFS AND CTBS	$(2\|1+2R\mu_j\|)$	$R\mu_j \in [\frac{-3}{4},\frac{-1}{4}]$	STABILE
FTCS	$2(1+\|\frac{R\mu_j}{2}\|$	$R\mu_j \in [1,3]$	STABILE
BTCS	$\frac{1}{2(1+\|\frac{R\mu_j}{2}\|)}$	$R\mu_j \in (-\infty,1]$	STABILE
CTCS	$2+2\|R\mu_j\|$	$R\mu_j \in [\frac{1}{2},\frac{3}{2}]$	STABILE

Table 4.4: THE SPECTRUM RADIUS AND STABILITY REGION OF DIFFERENCE SCHEMES

4.2 STABILITY AND CONVERGENCE ANALYSIS BASED ON DISCRETE FOURIER TRANSFORM

Now lets evaluate the difference equations considered above by using discrete Fourier transform.For a better clarity let see the tables given below

DIFFERENCE SCHEME	DISCRETE FOURIER TRANSFORM
FTFS	$\hat{u}^{n+1} = [I + RD - R(\cos\xi)D + iR(\cos\xi)D]\hat{u}^n$
FTBS	$\hat{u}^{n+1} = [I + RD - RD\cos\xi - iRD\sin\xi]\hat{u}^n$
FTCS	$\hat{u}^{n+1} = [I + [2R - 2R(\cos\xi - i\sin\xi)D]\hat{u}^n$
BTFS	$[I + RD - RD(\cos\xi - i\sin\xi)]\hat{u}^n = \hat{u}^{n-1}$
BTBS	$[I + RD - RD(\cos\xi + i\sin\xi)]\hat{u}^n = \hat{u}^{n-1}$
BTCS	$[I + (iR\sin\xi)D]\hat{u}^n = \hat{u}^{n-1}$
CTFS	$\hat{u}^{n+1} = \hat{u}^{n-1} - [2R\cos\xi - 2iR\sin\xi - 2R]D\hat{u}^n$
CTBS	$\hat{u}^{n+1} = \hat{u}^{n-1} + [2R(I - \cos\xi - i\sin\xi)D]\hat{u}^n$
CTCS	$\hat{u}^{n+1} = \hat{u}^{n-1} + (2iR\sin\xi)D\hat{u}^n$

Table 4.5: THE RELATIONSHIPS BETWEEN DIFFERENCE SCHEMES AND THEIR RESPECTIVE DISCRETE FOURIER TRANSFORM
where $R = \frac{\Delta t}{\Delta x}$

STABILITY ANALYSIS 3. *Hence,by using discrete fourier transform on FTFS we can determine the following of its property.The difference scheme of FTFS is*

$$u_k^{n+1} = (I + RA)u_k^n - RA(u_{k+1}^n)$$

since 3.1.5 yields, $\hat{u}^n = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{im\xi} u_m^n$,

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{ik\xi} u_k^{n+1} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{ik\xi}((I + RA)u_k^n - RA(u_{k+1}^n)))$$

$$\Rightarrow \hat{u}^{n+1} = (I + RA)\hat{u}^n + RA(\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{ik\xi} u_{k+1}^n)$$

By using change of variables $k \pm 1 = m \Rightarrow k = m \mp 1, e^{ik\xi} u_{k\pm1}^n$ *becomes* $e^{i(m\mp1)\xi} u_m^n = e^{\mp i\xi} \hat{u}_n$*,*

$$\Rightarrow \hat{u}^{n+1} = (I + RA)\hat{u}^n - RA(\cos\xi - i\sin\xi)\hat{u}^n$$

$$\Rightarrow \hat{u}^{n+1} = (I + RA)\hat{u}^n - RA(\cos\xi - i\sin\xi)D\hat{u}^n$$

$$\Rightarrow \|\hat{u}^{n+1}\| = [(I + RA) - RA(\cos\xi - i\sin\xi)]\|\hat{u}^n\|$$

And therefore, its iterative form of 2.17 and 2.18 is

$$\Rightarrow \|\hat{u}^{n+1}\| = [(I + RA) - RA(\cos\xi - i\sin\xi)]^n\|\hat{u}^0\|$$

which yield the amplification matrix of the difference scheme is

$$\rho_j^2(\xi) = 1 + 4R\mu_j + 4(R\mu_j)^2)\sin^2\frac{\xi}{2}$$

Since the difference scheme is said to be stabile if $\rho(\xi)_j^2 \leq 1$ *by using 2.15, which yields the stability region of*

$$\Rightarrow R\mu_j \in [-1, 0]$$

Therefore, its stabile.

Similarly others are done and its results are shown below on the following tables.

DIFFERENCE SCHEME	DISCRETE FOURIER TRANSFORM'S ITERATION
FTFS	$\hat{u}^{n+1} = [I + RD - R(\cos\xi)D + iR(\sin\xi)D]^n \hat{u}^0$
FTBS	$\hat{u}^{n+1} = [I + RD - R(\cos\xi)D - iR(\sin\xi)D]^n \hat{u}^0$
FTCS	$\hat{u}^{n+1} = [I + [2R - 2R(\cos\xi - i\sin\xi)D]^n \hat{u}^0$
BTFS	$[I + RD - R(\cos\xi - i\sin\xi)D]^n \hat{u}^n = \hat{u}^0$
BTBS	$[I + RD - R(\cos\xi + i\sin\xi)D]^n \hat{u}^n = \hat{u}^0$
BTCS	$[I + iR(\sin\xi)D]^n \hat{u}^n = \hat{u}^0$
CTFS	$\|\hat{u}^{n+1}\| \leq \|I - [2R(\cos\xi - 2i\sin\xi)D - 2RD]\|^n \|\hat{u}^0\|$
CTBS	$\|\hat{u}^{n+1}\| \leq \|I + 2R(I - \cos\xi - i\sin\xi)D\|\|^n \hat{u}^0\|$
CTCS	$\|\hat{u}^{n+1}\| \leq \|I + 2iR(\sin\xi)D\|\|^0 \hat{u}^0\|$

Table 4.6: THE RELATIONSHIPS BETWEEN DIFFERENCE SCHEMES AND THEIR RESPECTIVE ITERATED DISCRETE FOURIER TRANSFORM
where $R = \frac{\Delta t}{\Delta x}$

DIFF. SCHEME	AMPLIFICATION MATRIX	STABILITY REGION	STABILITY
FTFS	$\rho_j^2(\xi) = (1 + 4R\mu_j \sin^2\frac{\xi}{2} + 4(R\mu_j)^2 \sin^2\frac{\xi}{2})$	$R\mu_j \in [-1, 0]$	STABILE
FTBS	$\rho_j^2(\xi) = (1 + 4R\mu_j \sin^2\frac{\xi}{2} + 4(R\mu_j)^2 \sin^2\frac{\xi}{2}) \sin^2\frac{\xi}{2}$	$R\mu_j \in [-1, 0]$	STABILE
FTCS	$\rho_j^2(\xi) \leq 1 + 4(R\mu_j)\sin^2\frac{\xi}{2} + 4(1 + \sin^2\frac{\xi}{2})(R\mu_j)^2$	$R\mu_j \in [\frac{-4\sin^2\frac{\xi}{2}}{4+4\sin^2\frac{\xi}{2}}, 0]$	STABILE
BTFS	$\rho_j^2(\xi) \leq \frac{1}{[1+4R\mu_j \sin^2\frac{\xi}{2} - 4(R\mu_j)^2 \sin^2\frac{\xi}{2}]}$	$R\mu_j \in \mathbb{R}/(0, 1)$	STABILE
BTBS	$\rho_j^2 = \frac{1}{1+4R\mu_j \sin^2\frac{\xi}{2} + 4(R\mu j)^2 \sin^2\frac{\xi}{2}}$	$R\mu_j \in \mathbb{R}/(-1, 0)$	STABILE
BTCS	$\rho_j^2 = \frac{1}{1+(R\mu_j)^2 \sin^2\xi}$	$R\mu_j \in \mathbb{R}$	STABILE
CTFS	$\rho_j^2 \leq 1 - 8R\mu_j \cos^2\frac{\xi}{2} + 16(R\mu_j)^2 \cos^2\frac{\xi}{2}$	$R\mu_j \in [0, \frac{1}{2}]$	STABILE
CTBS	$\rho_j^2 \leq 1 + 8R\mu_j \sin^2\frac{\xi}{2} + 16(R\mu_j)^2 \sin^2\frac{\xi}{2}$	$R\mu_j \in [\frac{-1}{2}, 0]$	STABILE
CTCS	$\rho_j^2 \leq 1 + (R\mu_j)^2 \sin^2\xi$	$R\mu_j \in \{0\}$	STABILE

Table 4.7: THE RELATION AMONG DIFFERENCE SCHEME WITH RESPECT TO THEIR AMPLIFICATION MATRIX, STABILITY AND ITS REGION BY USING DISCRETE FOURIER TRANSFORM

Chapter 5

DISPERSION RELATION

Any time dependent scalar linear PDEs with constant coefficients on unbounded space domain admits plane wave solution

$$u(x,t) = e^{i(\xi x + wt)}, \xi \in \mathbb{R} \tag{5.1}$$

where ξ is the wave number and w is the frequency and the relation $w = w(\xi)$ is known as the dispersion relation.Now lets turn to discrete difference scheme formulas :

$$u_k^n = e^{i(\xi k \Delta x + wn\Delta t)} \tag{5.2}$$

Thus, for FTFS scheme we have

$$\frac{e^{iw\Delta t} - 1}{\Delta t} + \mu_j(\frac{e^{i\xi\Delta x} - 1}{\Delta x}) = 0 \tag{5.3}$$

which yields

$$\cos w\Delta t + i \sin w\Delta t = (1 + \frac{\mu_j \Delta t}{\Delta x}) - \mu_j \frac{\Delta t}{\Delta x}(\cos \xi\Delta x + i \sin \xi \Delta x)$$

And therefore,

$$\cos w\Delta t = (1 + \mu_j \frac{\Delta t}{\Delta x}) - \mu_j \frac{\Delta t}{\Delta x} \cos \xi \Delta x$$

and

$$\sin w\Delta t = -\mu_j \frac{\Delta t}{\Delta x} \sin \xi \Delta x$$

If we assume that $1 + \frac{\mu_j \Delta t}{\Delta x} = 0$, then

* phase velocity $= \frac{w(\xi)}{\xi}$
* group velocity $= \frac{dw(\xi)}{d\xi}$
* Dispersion relation: $\sin w\Delta t = \sin \xi \Delta x$

Here,suppose that a PDE or finite difference formula admits a solution of $e^{i(wt+\xi x)}$.Any individual "wave crest" of this wave,i.e a point moving in such away that the quantity inside the parenthesis has aconstant value (phase) moves at the velocity called phase velocity and its derivative w.r.t ξ is known as group velocity Therefore,the following table shows dispersion relation,phase and group velocity for the rest of difference schemes

DIFF.SCHEME	IF $\frac{\mu_j \Delta t}{\Delta x} \in$	PHASE VELOCITY	GROUP VELOCITY
FTFS	$\{-1\}$	$\frac{w(\xi)}{\xi}$	$\frac{dw(\xi)}{d\xi}$
BTFS	$\{-1\}$	$\frac{-w(\xi)}{\xi}$	$\frac{-dw(\xi)}{d\xi}$
CTFS	$\mathbb{R}$	$\frac{w(\xi)}{\xi}$	$\frac{dw(\xi)}{d\xi}$
FTBS	$\{-1\}$	$\frac{-w(\xi)}{\xi}$	$\frac{-dw(\xi)}{d\xi}$
BTBS	$\{1\}$	$\frac{-w(\xi)}{\xi}$	$\frac{-dw(\xi)}{d\xi}$
CTBS	$\mathbb{R}$	$\frac{w(\xi)}{\xi}$	$\frac{dw(\xi)}{d\xi}$
FTCS	$\mathbb{R}$	$\frac{w(\xi)}{\xi}$	$\frac{dw(\xi)}{d\xi}$
BTCS	$\mathbb{R}$	$\frac{w(\xi)}{\xi}$	$\frac{dw(\xi)}{d\xi}$
CTCS	$\{-1\}$	$\frac{w(\xi)}{\xi}$	$\frac{dw(\xi)}{d\xi}$

Table 5.1: DISPERSION RELATION,PHASE AND GROUP VELOCITY OF DIFFERENCE SCHEME

Chapter 6

STABILITY ANALYSIS BY USING MATLAB

In this chapter, we are going to elaborate the previous analysis of different difference schemes by using 3D and 2D graphics drawn with the aid of MATLAB which is performed on an example below.As we have seen on table 4.7,we can classify the difference scheme in to three families that have similar properties:-

1. FTFS,FTBS AND FTCS
2. BTFS,BTBS AND BTCS
3. CTFS,CTBS AND CTCS

Thus,lets examine the precision and accuracy of the following sample difference schemes(FTFS,BTFS AND CTFS) from the above in connection with the initial condition given below

6.1 MATLAB ANALYSIS IN THE STABILITY REGIONS OF DISCRETE FOURIER TRANSFORM

Let A be given as follows

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}_{3\times 3}$$

whose eigenvalues are $\mu(j) = \{2, 2, 8\}$ for $j = 1, 2, 3$,eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Moreover,its diagonal matrix D and invertible matrix S are $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ and

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ \frac{-1}{\sqrt{2}} & \frac{-\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \end{bmatrix}$ respectively.By using $v_t + Dv_x = \bar{0}$'s differences schemes,with initial condition $u(x,0) = \cos\pi(x)^2$ and from the coupled system

$$v_{0j}(x,0) = S \times u_{0j}(x,0)$$

formula and using MATLAB ,we used result's on table 4.7 provided that mesh length of space and time are each 100,x$\in [\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}]$ and $t \in [0, 0.0009]$.In the following tables:-

* table 6.1 shows comparative **analytic solutions** of FTFS,BTFS and CTFS w.r.t to the **coupled equation** 3.14 together with the initial conditions given above

* table 6.2 shows comparative **numerical solutions** of FTFS,BTFS and CTFS w.r.t to the **uncoupled equation** 3.10 together with the initial conditions given above

* comparative **global truncated error w.r.t sup-norm or ℓ_∞** of FTFS, BTFS and CTFS w.r.t to the **uncoupled equation's numerical solutions of 3.10, 2.7 and 2.4 obtained and drawn in table 6.3** together with the initial conditions given above

* comparative **global truncated error w.r.t $\ell_{2,Deltax}$ norm** of FTFS, BTFS and CTFS w.r.t to the **uncoupled equation's numerical solutions of 3.10, 2.3 and 2.8 obtained and drawn in table 6.4** together with the initial conditions given above

* comparative **Taylor series solution of 2.2** for FTFS, BTFS and CTFS w.r.t to the **uncoupled equation's solutions of 3.10 obtained and drawn in table 6.5** together with the initial conditions given above

* comparative **Local Truncated Error of 2.6** for FTFS,BTFS and CTFS w.r.t to the **uncoupled equation's solutions of 3.10 obtained and drawn in table 6.6** together with the initial conditions given above

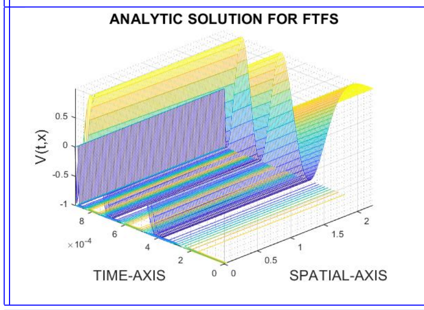
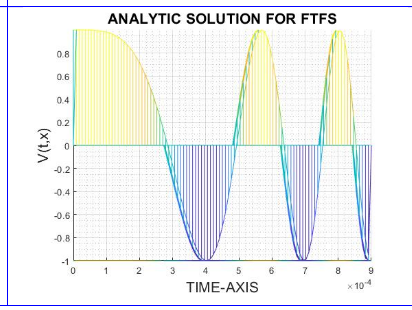
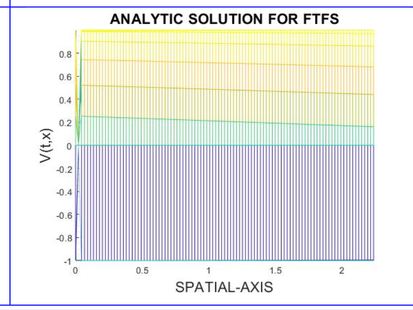
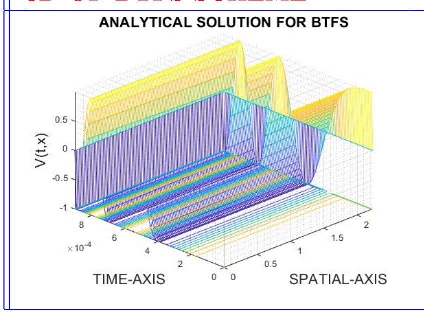
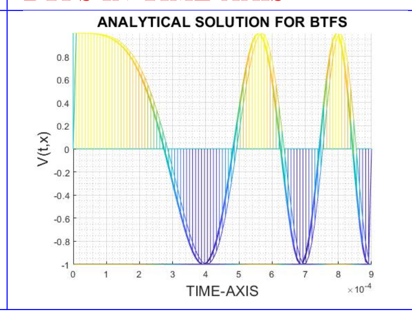
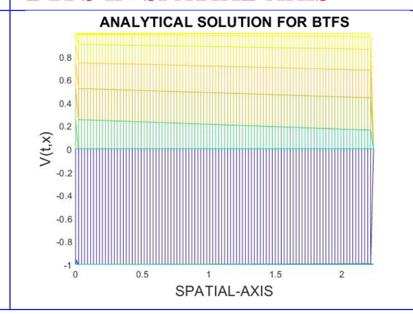
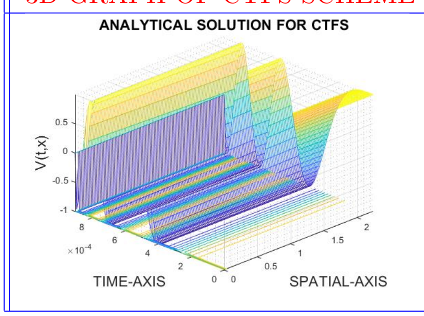
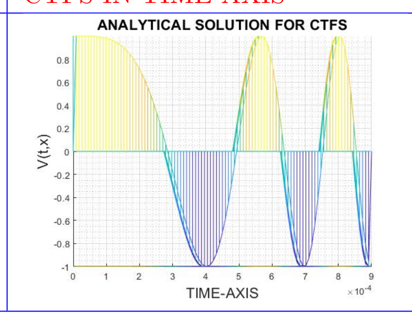
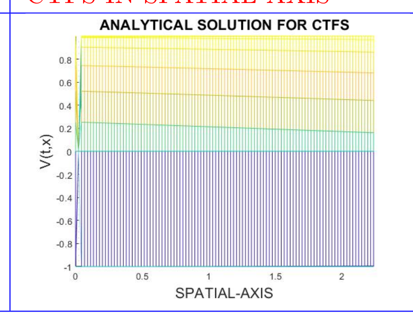
3D OF FTFS SCHEME	FTFS W.R.T TIME	FTFS W.R.T SPATIAL
		
3D OF BTFS SCHEME	BTFS IN TIME AXIS	BTFS IN SPATIAL AXIS
		
3D GRAPH OF CTFS SCHEME	CTFS IN TIME AXIS	CTFS IN SPATIAL AXIS
		

Table 6.1: ANALYTIC SOLUTION FOR SAMPLE SCHEMES

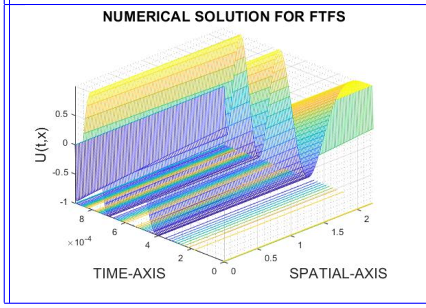
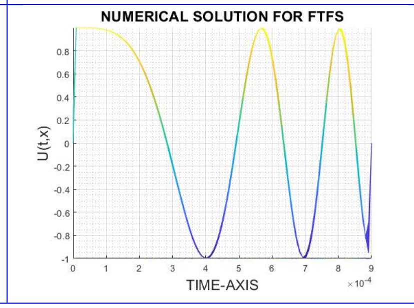
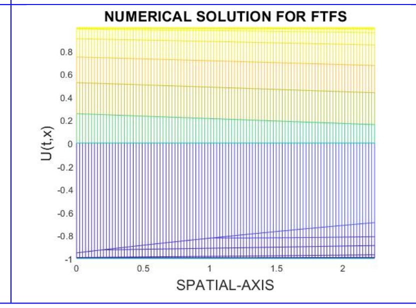
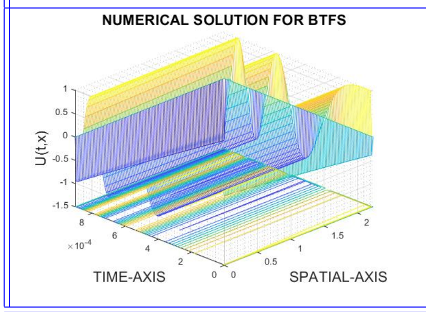
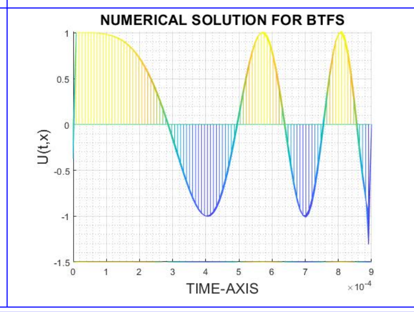
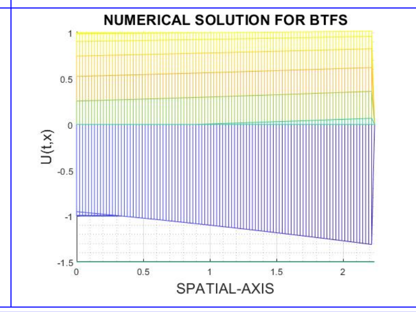
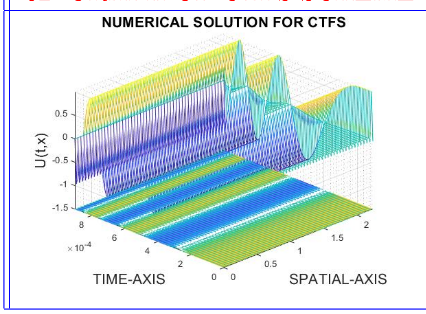
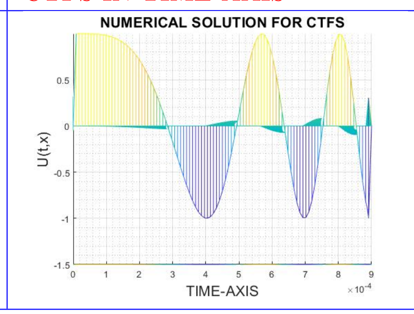
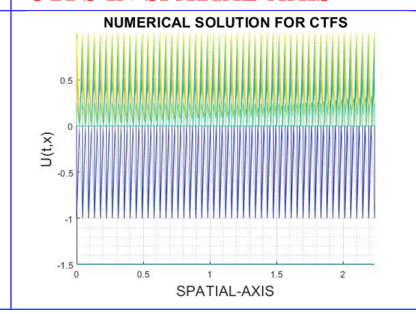
3D OF FTFS SCHEME	FTFS W.R.T TIME	FTFS W.R.T SPATIAL
		
3D OF BTFS SCHEME	BTFS IN TIME AXIS	BTFS IN SPATIAL AXIS
		
3D GRAPH OF CTFS SCHEME	CTFS IN TIME AXIS	CTFS IN SPATIAL AXIS
		

Table 6.2: NUMERICAL DIFFERENCE SOLUTION FOR SAMPLE SCHEMES

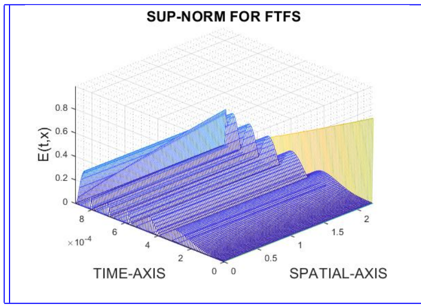
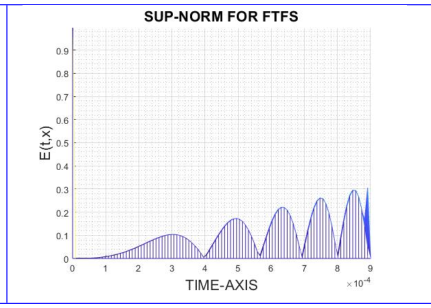
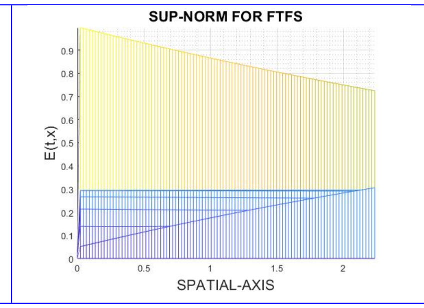
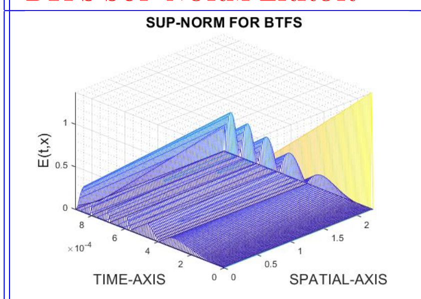
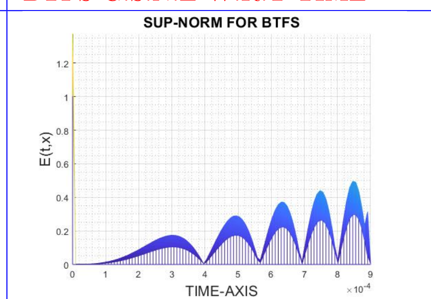
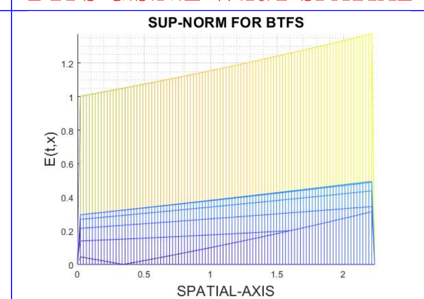
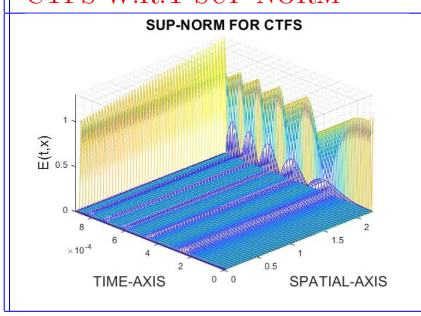
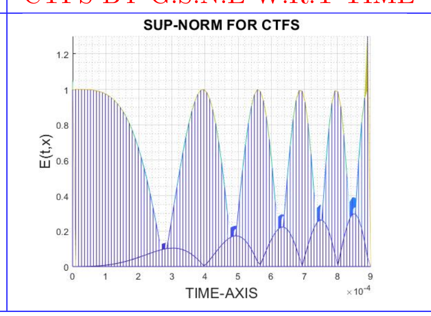
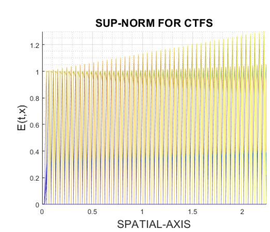
FTFS G.S.N.E	FTFS G.S.N.E IN TIME AXIS	FTFS G.S.N.E FOR SPATIAL
		
BTFS SUP-NORM ERROR	BTFS G.S.N.E W.R.T TIME	BTFS G.S.N.E W.R.T SPATIAL
		
CTFS W.R.T SUP-NORM	CTFS BY G.S.N.E W.R.T TIME	CTFS G.S.N.E W.R.T SPATIAL
		

Table 6.3: GLOBAL TRUNCATED ERROR BASED ON SUP-NORM (G.S.N.E) FOR SAMPLE SCHEMES

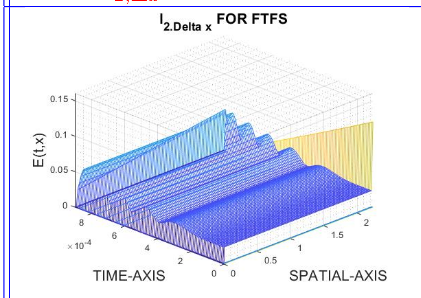
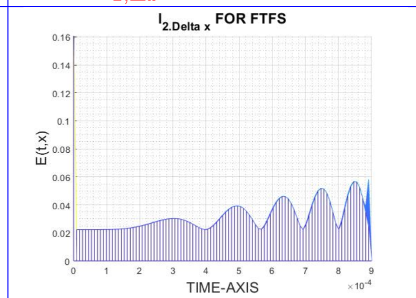
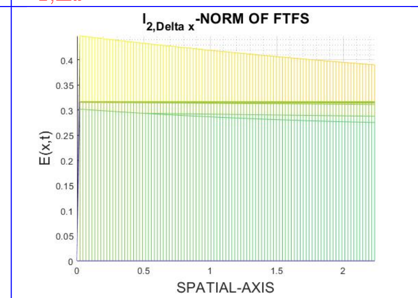
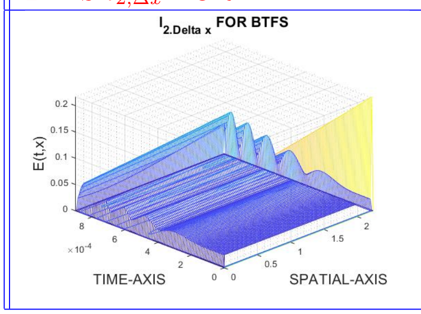
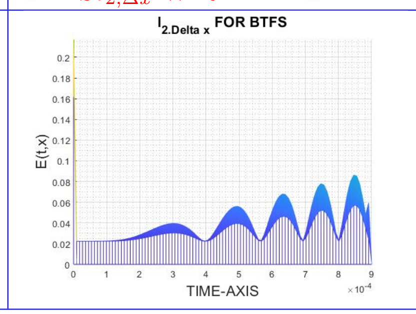
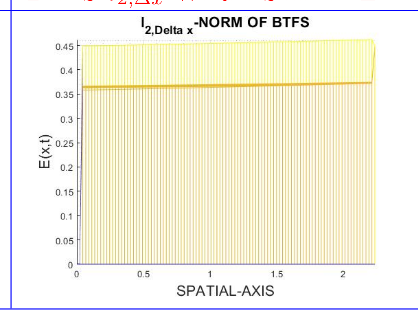
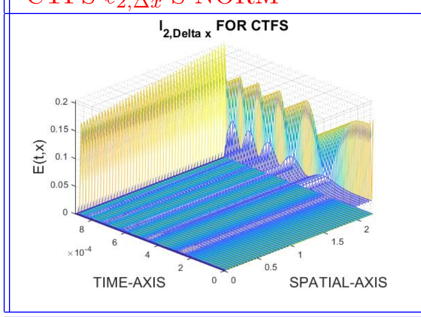
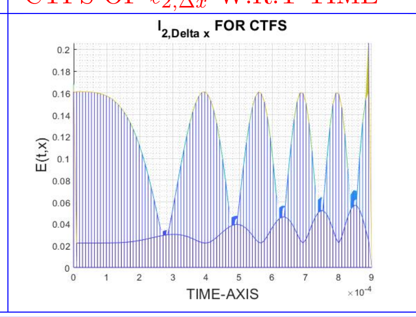
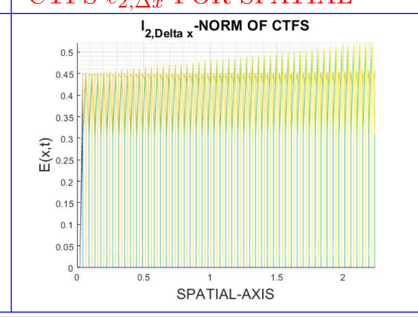
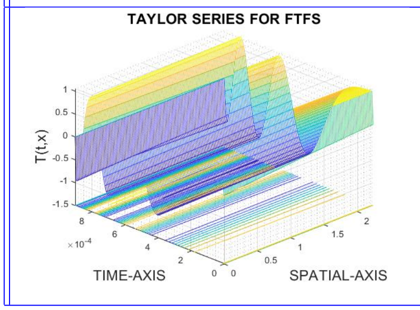
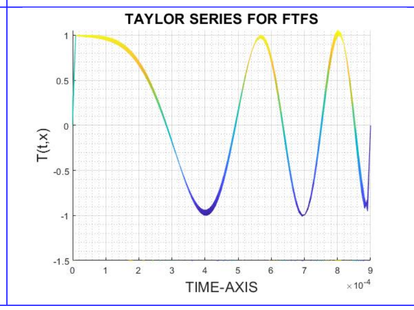
FTFS $\ell_{2,\Delta x}$ NORM	FTFS $\ell_{2,\Delta x}$ W.R.T TIME	$\ell_{2,\Delta x}$ W.R.T SPATIAL
		
BTFS $\ell_{2,\Delta x}$ NORM	BTFS$\ell_{2,\Delta x}$ W.R.T TIME	BTFS $\ell_{2,\Delta x}$ W.R.T SPATIAL
		
CTFS $\ell_{2,\Delta x}$'S NORM	CTFS OF $\ell_{2,\Delta x}$ W.R.T TIME	CTFS $\ell_{2,\Delta x}$ FOR SPATIAL
		

Table 6.4: GLOBAL ERROR OF SAMPLE SCHEMES BASED ON $\ell_{2,\Delta x}$

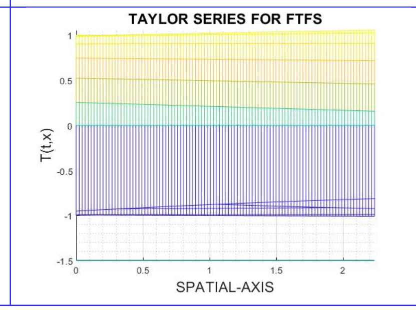
FTFS TAYLOR SERIES



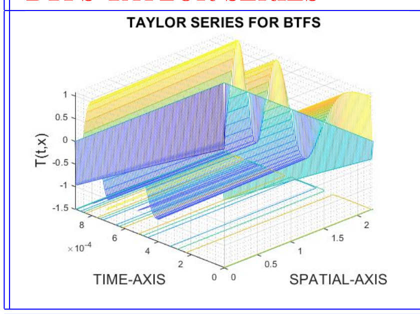
FTFS T.S. W.R.T TIME



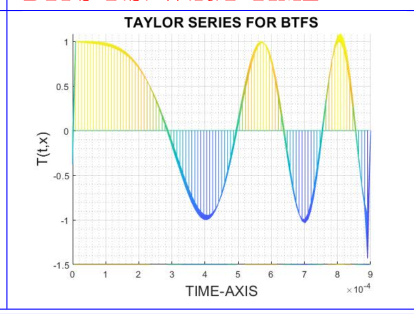
FTFS T.S. W.R.T SPATIAL



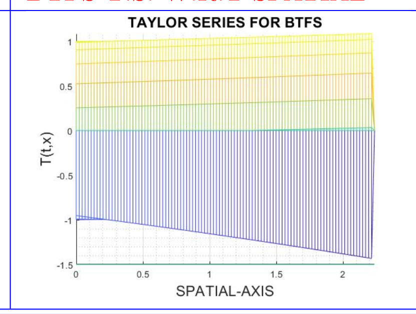
BTFS TAYLOR SERIES



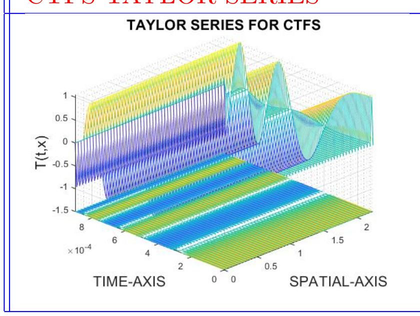
BTFS T.S. W.R.T TIME



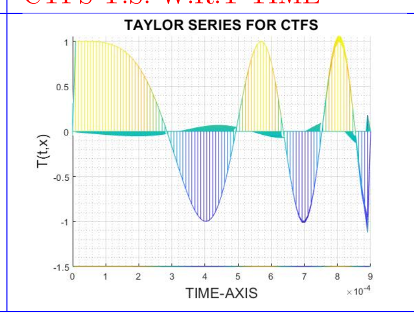
BTFS T.S. W.R.T SPATIAL



CTFS TAYLOR SERIES



CTFS T.S. W.R.T TIME



CTFS T.S. W.R.T SPATIAL

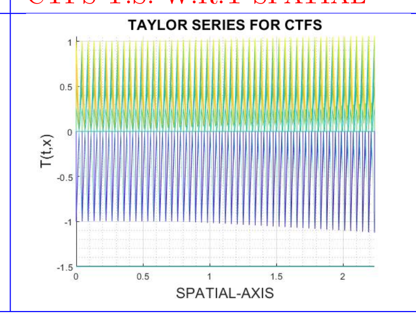


Table 6.5: GRAPHS OF TAYLOR SERIES(T.S) OF SAMPLE DIFFERENCE SCHEMES

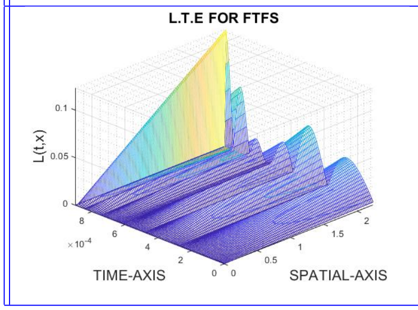
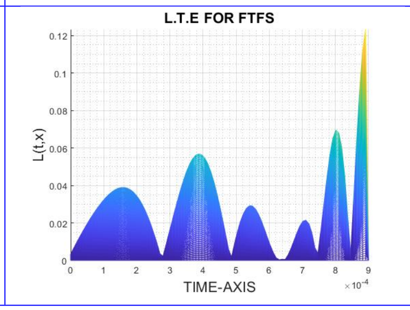
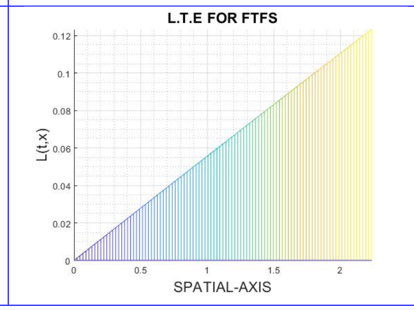
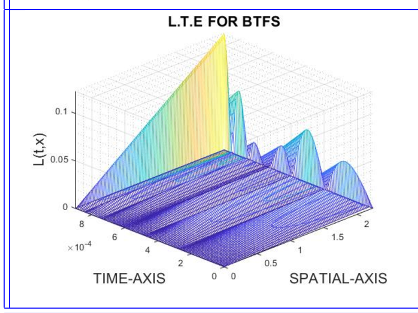
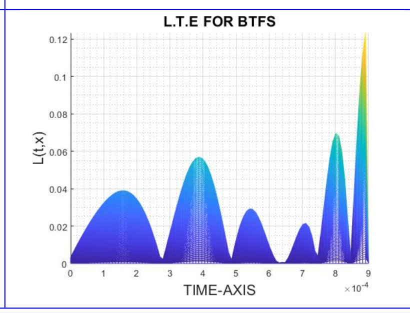
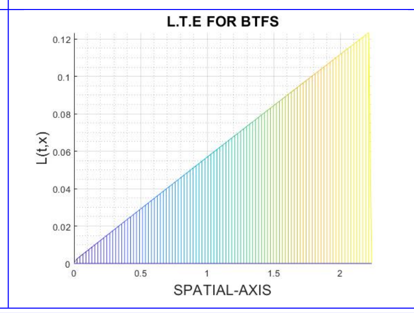
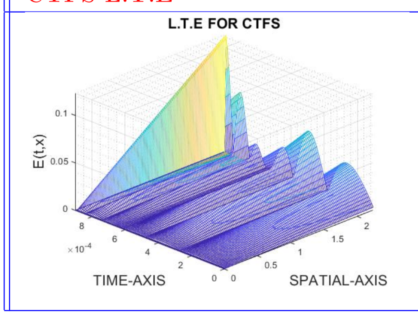
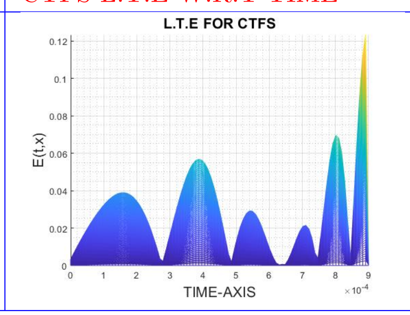
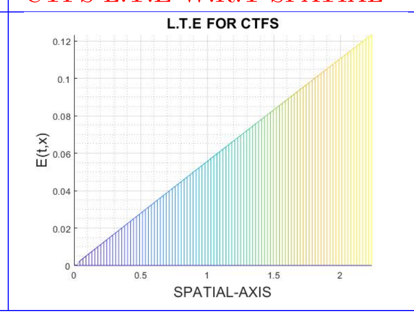
FTFS L.T.E	FTFS L.T.E W.R.T TIME	FTFS L.T.E W.R.T SPATIAL
		
BTFS L.T.E	BTFS L.T.E W.R.T TIME	BTFS L.T.E W.R.T SPATIAL
		
CTFS L.T.E	CTFS L.T.E W.R.T TIME	CTFS L.T.E W.R.T SPATIAL
		

Table 6.6: LOCAL TRUNCATED ERRORS(L.T.E) OF SAMPLE DIFFERENCE SCHEMES

Chapter 7

ERROR ANALYSIS FROM THE MATLAB RESULTS

7.1 METHODS ANALYSIS RESULT

While solving or finding the stability region for the sufficient and necessary conditions to satisfy the CFL we have applied three methods namely ℓ_∞, $\ell_{2,\Delta x}$ and Discrete Fourier Transform.As their results are seen on tables 4.4 and 4.7,the best approach to determine the stability region is Discrete Fourier Transform Method or Von Neumann. In addition to this,we have applied these results on a model problem with a given initial condition which results different graphs drawn by a MATLAB.

- CTFS , CTBS , CTCS
- FTFS , FTBS , FTCS
- BTFS , BTBS , BTCS

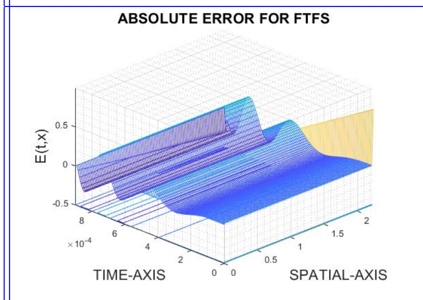
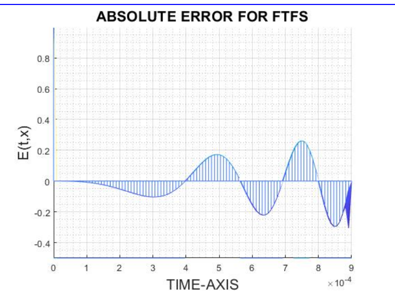
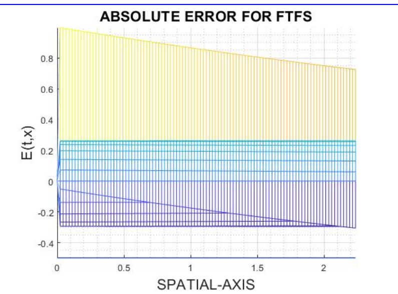
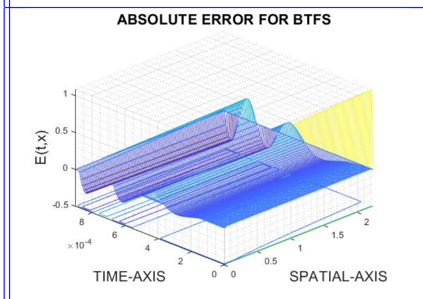
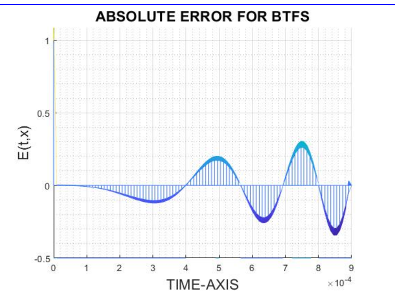
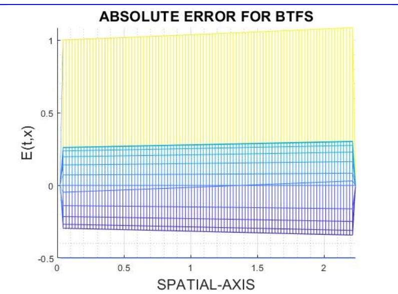
Thus,to compare the over all impact of error committed(absolute and relative errors) on a numerical solution w.r.t analytic solution, its enough to compare the following graphs resulted from the following formulations:-

* ABSOLUTE ERROR(A.E):-difference of Analytic and Numerical values as shown below in table 7.1,i.e,

$$A.E = v_k^n - u_k^n$$

* RELATIVE ERROR(R.E):-obtained from ABSOLUTE ERROR/Analytic value as shown below in 7.2

$$R.E = \frac{v_k^n - u_k^n}{v_k^n}$$

<table>
<tr><td>A.E W.R.T FTFS</td><td>A.E FTFS W.R.T TIME</td><td>A.E FTFS W.R.T SPATIAL</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>A.E W.R.T BTFS</td><td>A.E BTFS W.R.T TIME</td><td>A.E BTFS W.R.T SPATIAL</td></tr>
<tr><td>
</td><td>
</td><td>
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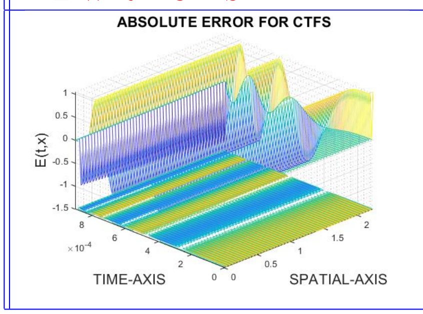
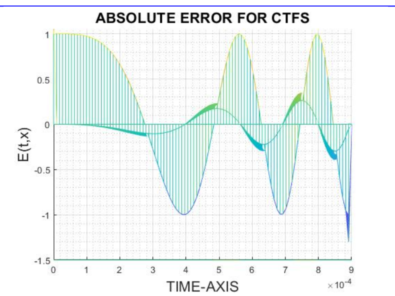
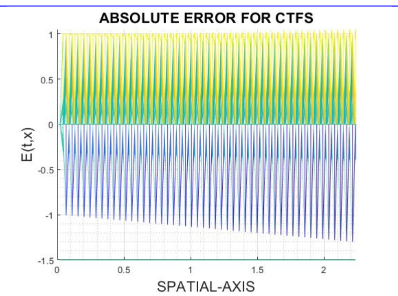
<tr><td>A.E W.R.T CTFS</td><td>A.E CTFS W.R.T TIME</td><td>A.E CTFS W.R.T SPATIAL</td></tr>
<tr><td>
</td><td>
</td><td>
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Table 7.1: ABSOLUTE ERRORS(A.E) OF FTFS,BTFS AND CTFS SCHEMES.

<table>
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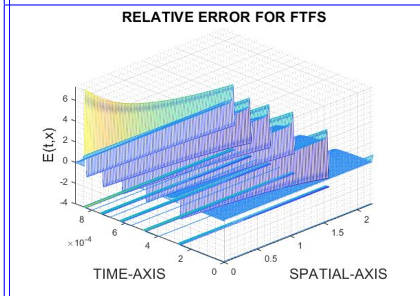
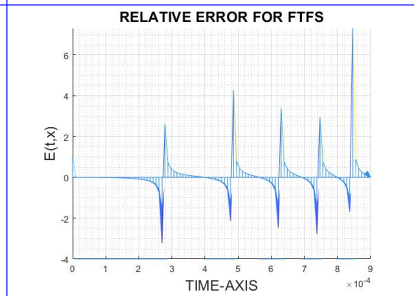
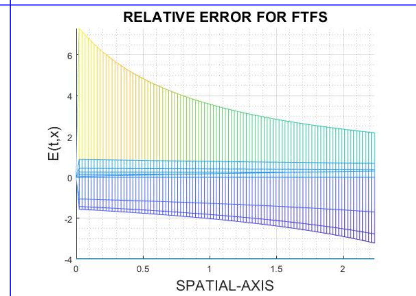
<tr><td>
</td><td>
</td><td>
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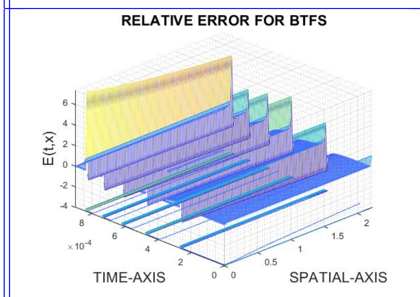
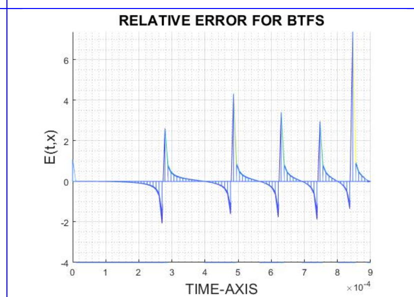
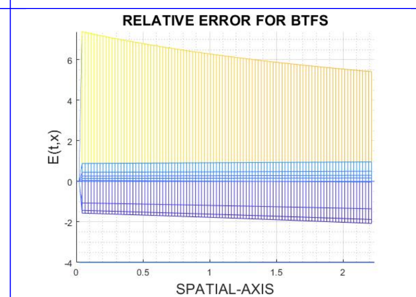
<tr><td>
</td><td>
</td><td>
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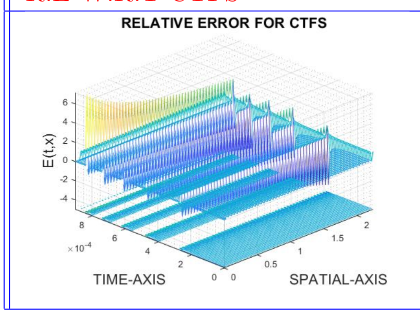
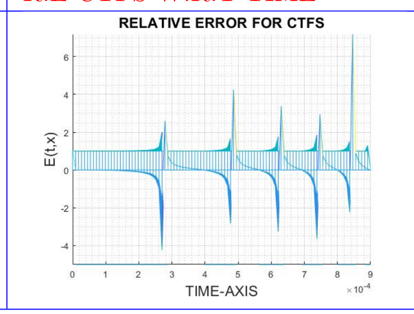
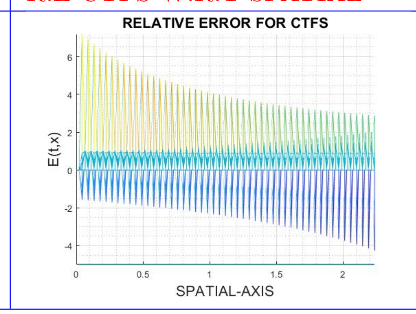
<tr><td>
</td><td>
</td><td>
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Table 7.2: RELATIVE ERRORS(R.E) OF FTFS,BTFS AND CTFS SCHEMES.

Chapter 8

CONCLUSION

We examined errors in our difference schemes' model problem to know the reliability that a given Numerical Analysis' scheme has with in a stability region; determined the errors in the difference schemes;run replicated schemes (schemes about the same size that are carried through an analysis in exactly the same way).By definition we know that precision is the closeness of data to other data that have been obtained in exactly the same way.High precision schemes have small standard deviation.

Accuracy is the closeness of a result to its true or accepted value, it determines how much error (that can be Absolute or Relative error) in the method.Since consistency shows how much the given difference scheme is accurate,table 6.6 shows that each of which have the same accuracy.

Moreover,with in stability regions , from 7.1 and 7.2 as well as from tables 6.3 and 6.4 show that families of FTFS,i.e FTBS and FTCS are the **best** for **numerical approximation** and secondly families of BTFS,i.e BTBS and BTCS are **better** for **approximation** and families of CTFS,i.e CTBS and CTCS are also **good**.

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