ADDIS ABABA UNIVERSITY

DEPARTMENT OF MATHEMATICS

A GRADUARTE SEMINAR REPORT

ON

GENERATING FUNCTIONS AND SOME OF THEIR APPLICATIONS

BY

YOHANNES TADESSE

Advisor

Dr. Demissu Gemeda

July 2004

ADDIS ABABA UNIVERSITY

DEPARTMENT OF MATHEMATICS

A GRADUARTE SEMINAR REPORT

ON

GENERATING FUNCTIONS AND SOME OF THEIR APPLICATIONS

BY

YOHANNES TADESSE

Advisor

Dr. Demissu Gemeda

July 2004

Preface

This seminar report is about generating functions and some of their uses in discrete mathematics. It is a report prepared for the course Math 702 during an M. Sc. study at Addis Ababa University. Generating functions are a bridge between discrete mathematics on one hand and continuous analysis on the other hand. It is possible to study them solely as tools for solving discrete problems. In this report we will see how generating functions are useful to describe different sequences in a meaningful way.

The report has two chapters. Chapter one is mainly about different versions of generating functions. We start the chapter with definitions of polynomials over a ring and then go through formal power series, formal power series generating functions, exponential generating functions and Dirichelet generating functions with some useful properties and examples. We have tried to prove some propositions and list references where any one could find the proofs for those unproved propositions. The second chapter is about applications of generating functions. Even though it is impossible to list all applications of generating functions, due to the vast application it has, we have tried to show how it is useful to describe a sequence analytically. We can apply them to find terms of a sequence having a certain property. As a result, the chapter has four sections: Linear Recurrence, Fibonacci Numbers, Partition and The Hilbert Function.

Finally we have tried to list some important notes in the Glossary to help understand how some operations are applied throughout the material. Moreover, we have tried to list possible Mathematica Functions with illustrations that could be applied in finding the problem in question into Discrete Math package.

It is pleasure to make many acknowledges. First, I would like to express my appreciation to my advisor Dr. Demissu Gemeda for helping me to locate the necessary literatures, critically reading the material and giving constructive suggestions. I am gratefully thankful to Dr Adinew Alemayehu for his invaluable support and fatherly advice. I am also giving my thanks to my colleague Tilahun Abebaw for his support and comments about this report. I want to thank Ato Abate Tibebu and Ato Binyam for letting me to use the computer lab with out any restrictions and also for letting me to print this report. Without them this report would not be to where it is now.

I would also like to thank all my family and my uncle Ato Getu Moges and his family for their commitments to be accessible in my everyday life activities. I have been very fortunate indeed to enjoy such a pleasant informal relations with the academic stuff of AAU, Dep't of Mathematics, my best friends in and out campus. To all those and others who have helped me, I thank you all.

Yohannes Tadesse

July 2004

To all who owe me than I could possibly give back!

Table of Contents

Contents Pages

Chapter One

Generating Functions

1.1 Rings of Polynomials

Definition 1.1.1

Let R be a ring. A ***polynomial*** f in indeterminate x with coefficients in R is an infinite formal sum $\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, where $a_i \in R$ and $a_i = 0$ for all but a finite number of values of i.

The a_i's are called the ***coefficient of*** x^i in f. If for some $i_0 > 0$ it is true that $a_{i_0} \neq 0$, the largest such value of i_0 is the ***degree*** of f and the coefficient a_{i_0} is called the ***leading coefficient*** of f. If no such $i > 0$ exists then f is called a polynomial of degree zero.

Example 1.1.2

The function $f(x) = 3x^4 + 2x^3 - 5x + 4$ is a polynomial with coefficients in the ring Z of integer. It is a polynomial of degree four with leading coefficient 3.

Example 1.1.3

The function $f(x) = x^3 + ix^2 - (2+3i)x + (5-7i)$ is a third-degree polynomial in the ring $Z+iZ$ **of Gaussian Integers**.

Polynomial Operations

To say that two polynomials are equal is to say that the coefficients of the same power of x are equal. Addition and Multiplication of polynomials in a ring R is defined in a formal way; if

$$f(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+a_3x^3+\dots \text{ and}$$

$$g(x)=\sum_{n=0}^{\infty}b_nx^n=b_0+b_1x+b_2x^2+b_3x^3+\dots \text{are polynomials,}$$

then for polynomial addition we have

$$(f+g)(x)=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+\dots, \text{ where } c_n=a_n+b_n$$

and for polynomial multiplication we have

$$(fg)(x)=\sum_{n=0}^{\infty}d_nx^n=d_0+d_1x+d_2x^2+d_3x^3+\dots \text{ where } d_n=\sum_{i=0}^{n}a_ib_{n-i}\ .$$

It is clear that again c_i and d_i are 0 for all but a finite number of values of i, so the definition makes sense.

Proposition 1.1.4

The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative [respectively an integral domain], then so is $R[x]$. If 1 is unity in R, then 1 is also unity for $R[x]$.

Proof: [8] page 121-124

1.2 Formal Power Series

Definition 1.2.1

Let R be a ring and $\{a_n\}_{n=0}^{\infty}$ be a sequence in R. A ***formal power series*** for the sequence $\{a_n\}_{n=0}^{\infty}$ is an expression of the form $a_0+a_1x+a_2x^2+a_3x^3+\dots$. The sequence $\{a_n\}_{n=0}^{\infty}$ is called the ***sequence of coefficients***.

Example 1.2.2

The formal power series for the sequence $\{n\}_{n=0}^{\infty}$ in the ring Z of integers is $0+1x+2x^2+3x^3+...$.

Example 1.2.3

The formal power series for the constant sequence $\{1\}_{n=0}^{\infty}$ is $1+1x+1x^2+1x^3+...$.

Example 1.2.4

The formal power series for the finite sequence $\{1,3,5,7\}$ is the polynomial $1+3x+5x^2+7x^3$.

We usually denote the formal power series $\sum_{n=0}^{\infty} a_n x^n$ for the sequence $\{a_n\}_{n=0}^{\infty}$ by letters f, g, h etc.

To say that two formal power series are equal is to say that their sequences of coefficient are the same.

Formal Power Series Operations

We can do certain kinds of operations with formal power series. Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ be two formal power series, then

i. we can add them by usual addition rule of polynomials

   i.e. $(f+g)(x)=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$ where $c_n = a_n + b_n$

   and

ii. we can multiply them by usual multiplication rule of polynomials

   i.e. $(fg)(x)=\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + ...$ where $d_n = \sum_{i=0}^{n} a_i b_{n-i}$.

This product rule accounts for the wide applicability of series methods in combinatorial problems. This is because frequently we can construct all a_n of the object of type n in some family by choosing an object of type k and an object of type $n - k$ and stitching them together to make the object of type n. The number of ways doing that will be $a_k a_{n-k}$, and if we sum over k we find that the product of two formal power series is directly related to the problem that we are studying.

Proposition 1.2.5

Let R be a ring and denoted by $R[[x]]$, the set of all formal power series with sequence of coefficients in R.

i. $R[[x]]$ is a ring with addition and multiplication defined.

ii. R and $R[x]$ are both subrings of $R[[x]]$.

iii. If R is commutative [respectively a ring with unity or an integral domain], then so is $R[[x]]$.

Proof: [9] page 123 or [4] page 154

Proposition 1.2.6

Let R be a unitary ring and $f(x) = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$.

i. f is a unit if and only if its constant term a_0 is a unit in R.

ii. If a_0 is irreducible in R, then f is irreducible in $R[[x]]$.

Proof: [4] page 155

Proposition 1.2.7

If R is a division ring, then units in $R[[x]]$ are precisely those power series with non-zero constant term. The principal ideal $\langle x \rangle$ consists of the non-units in $R[[x]]$ and is the unique maximal ideal of $R[[x]]$. Thus, if R is a field, $R[[x]]$ is local ring.

Proof: [4] page 155 or [3] page 420

So far we have introduced some important notions and properties of a formal power series with coefficients in any general ring. From here onwards and through out this report we consider only a sequence of numbers as we want to apply the concept to combinatorial problems. Thus by a sequence we mean a sequence of numbers.

We may define the composition of two formal power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ by $f(g(x))=\sum_{n=0}^{\infty} a_n [g(x)]^n$.

If the series $g(x)$ has a non-zero constant term b_0, then every term of the series $f(g(x))=\sum_{n=0}^{\infty} a_n [g(x)]^n$ may contribute to the coefficient of each power of x. On the other hand, if $b_0=0$, then we will be able to compute the coefficient of x^k for some k from the first $k+1$ terms of the series shown.

Notice that every single term $a_n [g(x)]^n = a_n \left[\sum_{n=0}^{\infty} b_n x^n\right]^n$ with $n>k$ will contain only powers of x greater that k, and therefore we won't need to look at those terms to find the coefficient of x^k. Thus if $b_0=0$ then the computation of each one of the coefficients of the series $f(g(x))$ is a finite process, and therefore all those coefficients are well defined, and so is the series. If $b_0 \neq 0$, the computation is infinite process unless f is a polynomial. Thus the composition $f(g(x))$ of two formal series f & g is defined if and only if $b_0 = 0$ or f is a polynomial.

Definition 1.2.8

The ***inverse of a series*** f, if it exists, is a series g such that $f(g(x)) = g(f(x)) = x.$

Proposition 1.2.9

The inverse of a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ exists if and only if the constant term a_o is zero and the coefficient a_1 of x is non-zero.

Proof: Assume that the formal power series $f(x) = \sum_{n=r}^{\infty} a_n x^n$ has inverse $g(x) = \sum_{n=s}^{\infty} b_n x^n$ where $r, s \geq 0$. The constant terms a_o and b_o are zero as the composition $f(g(x)) = g(f(x))$ is defined. Now $f(g(x)) = x = a_r b_s^r x^{rs} + \dots$, where $rs = 1$. Thus $r = 1, s = 1$, which means a_o is zero and the coefficient a_1 of x is non-zero.

The derivative of a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$. Differentiation follows usual rule of calculus, such as the sum, product, and quotient rules.

Proposition 1.2.10

If $f'=0$, then f is a constant.

Proof: Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$.

Then, $f'=0$ means that f' is identical to the formal power series 0.

$\Rightarrow a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots=0$.

$\Rightarrow$ For all $n \in N$, we have $na_n=0$

$\Rightarrow$ For all $n \in N$, we have $a_n=0$

$\Rightarrow f=a_0$ is a constant.

Proposition 1.2.11

If $f'=f$ then $f=ce^x$, c is a constant.

Proof: Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$. Now, $f'=f$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$\Rightarrow$ For all $n \geq 0$, we have $(n+1)a_{n+1}=a_n$.

$\Rightarrow a_{n+1}=\frac{a_n}{n+1}$, whence by induction on n, $a_n=\frac{a_0}{n!}$, for all $n \geq 0$.

$\Rightarrow f(x)=\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_o \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$.

1.3 Generating Functions.

Definition 1.3.1

Given a sequence of numbers (finite or infinite) $a_0, a_1, a_2, a_3, \dots$, we define the formal power series

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

to be the ***generating function*** of the sequence $a_0, a_1, a_2, a_3, \dots$ and we write $G \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$

Example 1.3.2

The generating function for the finite sequence $\left\{\binom{n}{k}\right\}_{k=0}^{n}$, where n is a positive integer is

$$G(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n = (1+x)^n .$$

Thus, we write $(1+x)^n \xleftrightarrow{ops} \binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$.

Example 1.3.3

The generating function for the infinite sequence $\left\{\frac{1}{n!}\right\}_{n\ge 0}$ is

$$G(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots = e^x .$$

Thus, we write $e^x \xleftrightarrow{ops} 1, 1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}, \dots$

Example 1.3.4

The generating function for the sequence with recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 0$, $a_1 = 1$ is $G(x) = \frac{x}{1-5x+6x^2}$.

Thus, we write $\frac{x}{1-5x+6x^2} \xleftrightarrow{ops} \{0, 1, 5 \dots, a_n = 5a_{n-1} - 6a_{n-2}\}_{n=2}^{\infty}$

Example 1.3.5

The formal power series generating function for the constant sequence $\{1\}_{n=0}^{\infty}$ is $G(x) = 1 + x + x^2 + x^3 + x^4 + ... = \frac{1}{1-x}$.

Thus, $\frac{1}{1-x} \xleftrightarrow{ops} \{1\}_{n=0}^{\infty}$

The symbol $G \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$ means that the series G is the *formal (ordinary) power series generating function* for the sequence $\{a_n\}_{n=0}^{\infty}$. Thus $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

Proposition 1.3.6

If $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$ and $H(x) \xleftrightarrow{ops} \{b_n\}_{n=0}^{\infty}$, then

i. $G(x) + H(x) \xleftrightarrow{ops} \{a_n + b_n\}_{n=0}^{\infty}$

ii. $cG(x) \xleftrightarrow{ops} \{ca_n\}_{n=0}^{\infty}$, where c is a fixed number.

iii. $G(x)H(x) \xleftrightarrow{ops} \{d_n\}_{n=0}^{\infty}$, where $d_n = \sum_{k=0}^{n} a_k b_{n-k}$

Proof: Immediate consequence of rules of formal power series operation.

Proposition 1.3.7

If $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, then

i. $xG(x) \xleftrightarrow{ops} 0, a_0, a_1, a_2, a_3, ...$

ii. $x^k G(x) \xleftrightarrow{ops} 0, 0, ..., 0, a_0, a_1, a_2, a_3, ...$ *(k-zeros first)*, $k \in N$.

Proof:

i) Let $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$. That is $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\Rightarrow xG(x) = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n x^{n+1} = 0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

$$\Rightarrow xG(x) \xleftrightarrow{ops} 0, a_0, a_1, a_2, a_3, ...$$

ii) Follows by induction on k.

Proposition 1.3.8

If $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, then

i. $\dfrac{G(x)-a_0}{x} \xleftrightarrow{ops} \{a_{n+1}\}_{n=0}^{\infty}$

ii. $\dfrac{G(x)-a_0-a_1x-a_2x^2-\ldots-a_{k-1}x^{k-1}}{x^k} \xleftrightarrow{ops} \{a_{n+k}\}_{n=0}^{\infty}$

Proof:

i) Let $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$. That is $G(x)=\sum_{n=0}^{\infty} a_n x^n$.

$$\Rightarrow G(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots$$

$$\Rightarrow \frac{G(x)-a_0}{x}=a_1+a_2x+a_3x^2+a_4x^3+\ldots$$

$$\Rightarrow \frac{G(x)-a_0}{x} \xleftrightarrow{ops} \{a_{n+1}\}_{n=0}^{\infty}$$

(ii) Follows by induction on k.

Proposition 1.3.9

If $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, then $\dfrac{d}{dx}(G(x)) \xleftrightarrow{ops} a_1, 2a_2, 3a_3, 4a_4, \ldots$

Proof: Follows from the definition of derivative of formal power series.

Proposition 1.3.10

If $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, and P is any polynomial in n, then $P(x\dfrac{d}{dx})(G(x)) \xleftrightarrow{ops} \{P(n)a_n\}_{n=0}^{\infty}$.

Proof: Let $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, that is $G(x)=\sum_{n=0}^{\infty} a_n x^n$.

Then the formal power series $x\dfrac{d}{dx}(G(x))=\sum_{n=0}^{\infty} na_n x^n$ generates the sequence $\{na_n\}_{n=0}^{\infty}$.

The formal power series $x^2 \frac{d^2}{dx^2}(G(x)) = \sum_{n=0}^{\infty} n^2 a_n x^n$ generates the sequence $\{n^2 a_n\}_{n=0}^{\infty}$. In general, for any (fixed) positive integer k the formal power series $x^k \frac{d}{dx^k}^k (G(x)) = \sum_{n=0}^{\infty} n^k a_n x^n$ generates the sequence $\{n^k a_n\}_{n=0}^{\infty}$. Now let $P(n) = \sum_{m=0}^{m} p_k n^k = p_m n^m + p_{m-1} n^{m-1} + \ldots + p_1 n + p_0$ be the polynomial in n. Then we have

$$P\left(x\frac{d}{dx}\right)G(x) = \left\{ p_m \left(x\frac{d}{dx}\right)^m + p_{m-1}\left(x\frac{d}{dx}\right)^{m-1} + \ldots + p_1\left(x\frac{d}{dx}\right) + p_0 \right\} G(x)$$

$$\Rightarrow P\left(x\frac{d}{dx}\right)G(x) = p_m\left(x\frac{d}{dx}\right)^m G(x) + p_{m-1}\left(x\frac{d}{dx}\right)^{m-1} G(x) + \ldots + p_1\left(x\frac{d}{dx}\right)G(x) + p_0 G(x)$$

$$\Rightarrow P\left(x\frac{d}{dx}\right)G(x) = p_m\left(x^m \frac{d^m G(x)}{dx^m}\right) + p_{m-1}\left(x^{m-1}\frac{d^{m-1}G(x)}{dx^{m-1}}\right) + \ldots + p_1\left(x\frac{dG(x)}{dx}\right) + p_0 G(x)$$

$$\Rightarrow P\left(x\frac{d}{dx}\right)G(x) = p_m \sum_{n=0}^{\infty} n^m a_n x^n + p_{m-1}\sum_{n=0}^{\infty} n^{m-1} a_n x^n + \ldots + p_1 \sum_{n=0}^{\infty} n a_n x^n + p_0 \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow P\left(x\frac{d}{dx}\right)G(x) = \{p_m n^m + p_{m-1} n^{m-1} + \ldots + p_1 n + p_0\}\sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow P\left(x\frac{d}{dx}\right)G(x) = P(n)G(x)$$

Therefore we have $P(x\frac{d}{dx})(G(x)) \xleftrightarrow{ops} \{P(n)a_n\}_{n=0}^{\infty}$.

Remark: We denote $x^k \frac{d}{dx^k}^k (G(x))$ by $\left(x\frac{d}{dx}\right)^k (G(x))$ in the above proposition.

Proposition 1.3.11

If $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, then $\dfrac{G(x)}{1-x} \xleftrightarrow{ops} \left\{\sum_{k=0}^{n} a_k\right\}$.

Proof: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

Now $\dfrac{G(x)}{1-x} = G(x)\dfrac{1}{1-x} = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$ (*)

For any integer $n \geq 0$, the coefficient of x^n in the product series (*) is obtained by taking the product of the coefficient of x^k in the series of *G(x)* and the coefficient of x^{n-k} in the series of $\dfrac{1}{1-x}$ $(0 \leq k \leq n)$. Thus we obtain

$$\frac{G(x)}{1-x} = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \ldots$$

Hence, $\dfrac{G(x)}{1-x} \xleftrightarrow{ops} \left\{\sum_{k=0}^{n} a_k\right\}$.

Thus very roughly multiplication of *G(x)* by $\dfrac{1}{1-x}$ acts as an operator taking a sequence $a_0, a_1, a_2, a_3, \ldots$ to the sequence of its partial sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots$.

More precisely $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$, if and only if $\dfrac{G(x)}{1-x} \xleftrightarrow{ops} \left\{\sum_{k=0}^{n} a_k\right\}$.

Here is the modest example.

$1,1,0,0,0,0,\ldots \xleftrightarrow{ops} 1+x$ so by Proposition 2.3.11, we have

$$1, 1+1, 1+1+0, 1+1+0+0, 1+1+0+0+0, \ldots = 1,2,2,2,2,2,\ldots \xleftrightarrow{ops} \frac{1+x}{1-x}.$$

A second application of the Proposition yields that

$$1,1+2,1+2+2,1+2+2+2,1+2+2+2+2,\ldots = 1,3,5,7,9,11,\ldots \xleftrightarrow{ops} \frac{1+x}{(1-x)^2}.$$

Thus we have chanced upon the generating function for odd numbers. One more application of the proposition gives

$$1,1+3,1+3+5,1+3+5+7,1+3+5+7+9,\ldots = 1,4,,9,16,25,\ldots \xleftrightarrow{ops} \frac{1+x}{(1-x)^3}.$$

It looks like we have found the generating function for the sequence of squares. To make this totally rigorous we would need to show that $\sum_{k=1}^{n} 2k-1 = n^2$ and this is demonstrated by a straightforward proof using induction.

Thus we have shown that $1,4,,9,16,25,\ldots = \sum_{n=0}^{\infty} n^2 x^n \xleftrightarrow{ops} \frac{1+x}{(1-x)^3}$.

1.4 Exponential Generating Functions

Definition 1.4.1

If $a_0, a_1, a_2, a_3, \ldots$ is a sequence, then the corresponding ***exponential generating function*** is the formal power series $G(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and we write $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$.

Example 1.4.2

The exponential generating function for the constant sequence $\{1\}_{n=0}^{\infty}$ is $G(x)=\frac{1}{0!}+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+...=e^x$.

Thus $e^x \xleftrightarrow{egf} \{1\}_{n=0}^{\infty}$.

Example 1.4.3

The exponential generating function for the sequence $\{n!\}_{n=0}^{\infty}$ is

$$G(x)=0!\frac{1}{0!}+1!\frac{1}{1!}x+2!\frac{1}{2!}x^2+3!\frac{1}{3!}x^3+4!\frac{1}{4!}x^4+...=1+x+x^2+x^3+...=\frac{1}{1-x}.$$

Thus $\frac{1}{1-x} \xleftrightarrow{egf} \{n!\}_{n=0}^{\infty}$.

Example 1.4.4 **(Circular Permutations)**

If we want to arrange n distinct elements in a circle there are $(n-1)!$ number of ways to arrange them. The exponential generating function for circular permutation is $C(x)=\sum_{n=1}^{\infty}(n-1)!\frac{x^n}{n!}=\sum_{n=1}^{\infty}\frac{x^n}{n}=\ln\left(\frac{1}{1-x}\right)$.

Thus $\ln\left(\frac{1}{1-x}\right) \xleftrightarrow{egf} \{(n-1)!\}_{n=1}^{\infty}$.

Remark: Note that $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$ is equivalent to

$$G(x) \xleftrightarrow{ops} \left\{\frac{a_n}{n!}\right\}_{n=0}^{\infty}.$$

Proposition 1.4.5

If $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$ and $H(x) \xleftrightarrow{egf} \{b_n\}_{n=0}^{\infty}$, then

i. $G(x)+H(x) \xleftrightarrow{egf} \{a_n + b_n\}_{n=0}^{\infty}$

ii. $cG(x) \xleftrightarrow{egf} \{ca_n\}_{n=0}^{\infty}$, where c is a fixed number.

Proof: They are immediate consequences of definition 1.4.1 and the above remark.

Proposition 1.4.6

If $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$, then

i. $\frac{d}{dx}(G(x)) \xleftrightarrow{egf} \{a_{n+1}\}_{n=0}^{\infty}$

ii. for any integer $k \geq 0$ $\frac{d^k}{dx^k}(G(x)) \xleftrightarrow{egf} \{a_{n+k}\}_{n=0}^{\infty}$.

Proof: i. Let $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$. Then $G(x) \xleftrightarrow{ops} \left\{\frac{a_n}{n!}\right\}_{n=0}^{\infty}$.

$$\Rightarrow G(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \qquad \Rightarrow \frac{d}{dx}(G(x)) = \sum_{n=0}^{\infty} \frac{na_n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1}$$

$$\Rightarrow \frac{d}{dx}(G(x)) \xleftrightarrow{ops} \left\{\frac{a_{n+1}}{n!}\right\}_{n=0}^{\infty} \Rightarrow \frac{d}{dx}(G(x)) \xleftrightarrow{egf} \{a_{n+1}\}_{n=0}^{\infty}.$$

ii. Follows by induction on k.

Proposition 1.4.7

If $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$ and P is a given polynomial in n, then

$P(x\frac{d}{dx})(G(x)) \xleftrightarrow{egf} \{P(n)a_n\}_{n=0}^{\infty}$.

Proof: Let $G(x) \xleftrightarrow{egf} \{a_n\}_{n=0}^{\infty}$ and $P(n)$ be a given polynomial in n.

That is $G(x)=\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$. Then the formal power series $x\frac{d}{dx}(G(x))=\sum_{n=0}^{\infty}\frac{na_n}{n!}x^n$ generates the sequence $\{na_n\}_{n=0}^{\infty}$. The formal power series $x^2\frac{d^2}{dx^2}(G(x))=\sum_{n=0}^{\infty}\frac{n^2a_n}{n!}x^n$ generates the sequence $\{n^2a_n\}_{n=0}^{\infty}$. In general, for any (fixed) positive integer k the formal power series $x^k\frac{d}{dx^k}^k(G(x))=\sum_{n=0}^{\infty}\frac{n^ka_n}{n!}x^n$ generates the sequence $\{n^ka_n\}_{n=0}^{\infty}$. Therefore, for the given polynomial *P*, $P(x\frac{d}{dx})(G(x))\xleftrightarrow{egf}\{P(n)a_n\}_{n=0}^{\infty}$ follows from Proposition 1.4.1 and properties of differentiation.

Proposition 1.4.8

If $G(x)\xleftrightarrow{egf}\{a_n\}_{n=0}^{\infty}$ & $H(x)\xleftrightarrow{egf}\{b_n\}_{n=0}^{\infty}$, then

$$G(x)H(x)\xleftrightarrow{egf}\left\{\sum_{k=0}^{n}\binom{n}{k}a_nb_{n-k}\right\}_{n=0}^{\infty}.$$

Proof: Let $A(x)=G(x)H(x)\xleftrightarrow{egf}\{d_n\}_{n=0}^{\infty}$.

Now observe that $G(x)$, $H(x)$ and $A(x)$ are formal power series generating functions for $\left\{\frac{a_n}{n!}\right\}$, $\left\{\frac{b_n}{n!}\right\}$ and $\left\{\frac{c_n}{n!}\right\}$ respectively and

$$\frac{c_n}{n!}=\sum_{k=0}^{n}\frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!}.$$

$$\Rightarrow\ c_n=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}a_kb_{n-k}=\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}.$$

$$\Rightarrow\ A(x)=G(x)H(x)\xleftrightarrow{egf}\left\{\sum_{k=0}^{n}\binom{n}{k}a_nb_{n-k}\right\}_{n=0}^{\infty}.$$

1.5 Dirichlet Series Generating Function

Definition 1.5.1

Given a sequence $\{a_n\}_{n=1}^{\infty}$; we say that a formal series $F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}=a_1+\frac{a_2}{2^s}+\frac{a_3}{3^s}+\frac{a_4}{4^s}+\ldots$ is the ***Dirichlet Series Generating Function*** of the sequence and we write $F(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$.

Proposition 1.5.2

If $G(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$ and $G(s)\xleftrightarrow{Dir}\{b_n\}_{n=1}^{\infty}$, then

i. $G(s)+H(s)\xleftrightarrow{Dir}\{a_n+b_n\}_{n=1}^{\infty}$

ii. $cG(s)\xleftrightarrow{Dir}\{ca_n\}_{n=1}^{\infty}$, where c is a fixed number.

Proof: They are immediate consequences of definition 1.5.1.

Proposition 1.5.3

If $G(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$, then

i. $\frac{d}{dx}\left(G(s)\right)\xleftrightarrow{Dir}\{-(\ln n)a_n\}_{n=1}^{\infty}$

ii. for any integer $k\geq 0$ $\frac{d^k}{dx^k}\left(G(s)\right)\xleftrightarrow{Dir}\{(-1)^k(\ln n)^k a_n\}_{n=1}^{\infty}$.

Proof: i. Let $G(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$. i.e. $G(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}=\sum_{n=1}^{\infty}a_n n^{-s}$

$$\Rightarrow \quad \frac{d}{dx}(G(s))=\sum_{n=1}^{\infty}-(\ln n)a_n n^{-s}=\sum_{n=1}^{\infty}\frac{-(\ln n)\,a_n}{n^s}$$

$$\Rightarrow \quad \frac{d}{dx}(G(s))\xleftrightarrow{Dir}\{-(\ln n)a_n\}_{n=1}^{\infty}.$$

iii. Follows by induction on k.

The importance of Dirichlet series stems directly from their multiplication rule. Suppose $F(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$ and $G(s)\xleftrightarrow{Dir}\{b_n\}_{n=1}^{\infty}$. The question is what sequence is generated by $F(s)G(s)$?

Proposition 1.5.4

If $F(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$ and $G(s)\xleftrightarrow{Dir}\{b_n\}_{n=1}^{\infty}$, then

$$F(s)G(s)\xleftrightarrow{Dir}\left\{\sum_{d|n} a_d b_{\frac{n}{d}}\right\}_{n=1}^{\infty}.$$

Proof: Given $F(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty}$ and $G(s)\xleftrightarrow{Dir}\{b_n\}_{n=1}^{\infty}$, implies $F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$, $G(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$. Consider the product $F(s)G(s)$.

$$F(s)G(s)=\left(a_1+\frac{a_2}{2^s}+\frac{a_3}{3^s}+\frac{a_4}{4^s}+\dots\right)\left(b_1+\frac{b_2}{2^s}+\frac{b_3}{3^s}+\frac{b_4}{4^s}+\dots\right)$$

$$\Rightarrow F(s)G(s)=(a_1b_1)+(a_1b_2+a_2b_1)2^{-s}+(a_1b_3+a_3b_1)3^{-s}+(a_1b_4+a_2b_2+a_4b_1)4^{-s}+\dots$$

In the product FG, the coefficient of n^{-s} is the sum of all products of *a's* & *b's* where the product of their subscript is n it is $\sum_{rs=n} a_r b_s$. Now if $rs=n$ then r & s are divisors of n, so the above sum can also be written as $\sum_{d|n} a_d b_{\frac{n}{d}}$.

Therefore we have the proposition.

What happens to the sequence generated by the *kth* power of a Dirichlet series? Let's work it out as follows:

$$F(s)\xleftrightarrow{Dir}\{a_n\}_{n=1}^{\infty} \qquad \Rightarrow F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

$$\Rightarrow \qquad F(s)^k=\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)^k=\sum_{n\geq 1} a_{n_1}a_{n_2}a_{n_3}\dots a_{n_k}(n_1n_2n_3\dots n_k)^{-s}$$

$$=\sum_{n\geq 1} n^{-s}\left\{\sum_{n_1 n_2 n_3 \ldots n_k} a_{n_1} a_{n_2} a_{n_3} \ldots a_{n_k}\right\}$$

This shows the following proposition

Proposition 1.5.5

If $F(s) \xleftrightarrow{Dir} \{a_n\}_{n=1}^{\infty}$ then $F(s)^k \xleftrightarrow{Dir}$ a sequence, whose nth member is the sum, extended over all ordered factorization of n into k factors, of the products of the members of the sequence whose subscripts are the factors in that factorization.

What series does F generates the sequence of all 1's: $\{1\}_{n=1}^{\infty}$? When we asked that question in the cases of the formal power series generating function or the exponential generating function, the answers turned out to be the functions $\frac{1}{1-x}$ for *ops* and e^x for *egf*.

In the present case, the formal Dirichlet series whose coefficients are all 1's is not related to any simple function of analysis, it is a new creature, and it gets a new name: the ***Reimman zeta function***.

That is the Dirichlet series $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ and it is one of the most important functions on analysis.

Now from proposition 1.5.4, the coefficient of n^{-s} in the product $\zeta(s)^2$ is $\sum_{d|n} 1.1 = d(n)$, where $d(n)$ is the number of divisors of the integer n.

Likewise from proposition 1.5.5, $\zeta(s)^k$ generates the number of ordered factorization of n.

Chapter 2

Some Applications of Generating Functions

Introduction

Generating function is a bridge between discrete mathematics, on one hand, and continuous analysis (particularly complex variable theory) on the other. It is a close line on which we hang-up a sequence of numbers for display. Suppose we have a problem whose answer is a sequence of numbers $a_0, a_1, a_2, a_3, \ldots$. We want to know what the sequence is. Generating functions helps us to find the sequence. But this may not be as such simple to have, since some sequences are complicated. Although giving a simple formula for the numbers of the sequence may be out of the question in some cases, we might be able to give a simple formula for the sum of a power series, whose coefficients are the sequence we are looking for.

We can apply generating functions to

a. find an exact formula for the sequence of numbers;
b. find a recurrence formula;
c. find averages and other statistical properties;
d. find asymptotic formula for sequences; and
e. prove different identities.

2.1 Linear Recursion

Definition 2.1.1

A mathematical relationship expressing the nth term a_n of a sequence $\{a_n\}_{n=o}^{\infty}$ as some combination of a_i with $i < n$, when formulated as an equation to be solved recurrence relation is known as ***recurrence equation*** or a ***difference equation***.

A recurrence equation is the discrete analogy of a differential equation. It involves an integer function a_n in a form like $a_n - a_{n-1} = p(n)$, where p is some integer function. The above equation is the direct analogy of the first order differential equation $a'(x) = p(x)$.

Example 2.1.2

The equation $a_n = 2a_{n-1}$, $a_0 = 1$, is a recurrence equation of a sequence $\{a_n\}_{n=o}^{\infty}$ in which the *n*th term of the sequence is obtained by doubling the (*n-1*)th term. From this recurrence equation we can formulate a first order linear differential equation $f' - 2f = 0$.

Example 2.1.3

The equation $a_n = 5a_{n-1} - 4a_{n-2}$, $a_0 = 1$, $a_1 = 2$ is a recurrence equation of a sequence $\{a_n\}_{n=o}^{\infty}$ in which the *n*th term of the sequence is obtained by multiplying the (*n-1*)th term by 5 and then subtracting 4 multiple of the (*n-2*)th term . From this recurrence equation we can formulate a second order linear differential equation $f'' - 5f' + 4f = 0$.

Definition 2.1.4

A ***recursive sequence*** $\{a_n\}_{n=1}^{\infty}$ also known as ***recurrence sequence***, is a sequence of numbers indexed by non-negative integers n and generated by solving a recurrence equation. The terms of the sequence can be denoted symbolically in a number of different notations, such as a_n, $a(n), a[n]$, where a is a symbol representing the sequence.

Example 2.1.5

The recurrence equation $a_n = 2a_{n-1}$, $a_0 = 1$ generates a recurrence sequence $a_n = 2^n$.

Example 2.1.6

The recurrence equation $a_n = 4a_{n-1} - 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$ generates the recurrence sequence $a_n = \frac{1}{2}(3^n + 1)$.

Definition 2.1.7

A ***linear recurrence equation*** is a recurrence equation on a sequence of numbers $\{a_n\}_{n=1}^{\infty}$ expressing a_n as a first-degree polynomial in a_i with $i < n$.

For example, $x_n = C_1x_{n-1} + C_2x_{n-2} + C_3x_{n-3} + ... + C_nx_0$ where C_i's are constant coefficients is a linear recurrence equation.

Our aim is to find the generating function for the sequence with a given linear recurrence equation. And from the generating function we will find an explicit formula for terms of the sequence as a function of n only.

First Let's see the following examples.

Example 2.1.8

Find the explicit formula for the sequence $\{a_n\}_{n=0}^{\infty}$ defined by the recurrence equation $a_n = 5a_{n-1} - 6a_{n-2}$, with $a_0 = 0$, $a_1 = 1$.

Solution:

***Method** I: Using Formal Power Series Generating Functions*

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$.

Multiply the given recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ by x^n and then sum it over $n \geq 2$. That is $a_n x^n = 5a_{n-1}x^n - 6a_{n-2}x^n$

$$\Rightarrow \qquad \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} \left(5a_{n-1}x^n - 6a_{n-2}x^n\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - (a_0 + a_1 x) = 5x\sum_{n=0}^{\infty} a_n x^n - (5xa_0) - 6x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow G(x) - x = 5xG(x) - 6x^2 G(x)$$

$$\Rightarrow G(x) = \frac{x}{1 - 5x + 6x^2}$$

Thus $G(x) = \dfrac{x}{1 - 5x + 6x^2}$ is the generating function for the sequence with recurrence $a_n = 5a_{n-1} - 6a_{n-2}$.

To find the formula for a_n we write *G(x)* as sum of simple partial fractions. [see the Glossary]

$$G(x) = \frac{x}{1-5x+6x^2} = \frac{1}{1-3x} - \frac{1}{1-2x}.$$

Each of these two fractions is the sum of an infinite geometric series. So we may write

$$G(x) = \left(1+3x+3^2x^2+3^3x^3+\ldots+3^nx^n+\ldots\right) - \left(1+2x+2^2x^2+2^3x^3+\ldots+2^nx^n+\ldots\right)$$

$$G(x) = 0+(3-2)x+(3^2-2^2)x^2+\ldots+\left(3^n-2^n\right)x^n+\ldots$$

Thus $a_n = 3^n - 2^n$.

Therefore $\frac{x}{1-5x+6x^2} \xleftrightarrow{ops} \left\{3^n-2^n\right\}_{n=0}^{\infty}$.

***Method** II: Using Exponential Generating Functions*

Let $G(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \xleftrightarrow{egf} \left\{a_n\right\}_{n=0}^{\infty}$.

We multiply the recurrence equation given by $\frac{x^{n-2}}{(n-2)!}$ and then sum it over $2 \le n < \infty$ as we have values of a_0 and a_1.

$$\Rightarrow \sum_{n=2}^{\infty} \frac{a_n x^{n-2}}{(n-2)!} = \sum_{n=2}^{\infty}\left(\frac{5a_{n-1}-6a_{n-2}}{(n-2)!}\right)x^{n-2} = 5\sum_{n=2}^{\infty}\frac{a_{n-1}x^{n-2}}{(n-2)!} - 6\sum_{n=2}^{\infty}\frac{a_{n-2}x^{n-2}}{(n-2)!}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{a_n (n(n-1))x^{n-2}}{n!} = 5\sum_{n=2}^{\infty}\frac{a_{n-1}(n-1)x^{n-2}}{(n-1)!} - 6\sum_{n=2}^{\infty}\frac{a_{n-2}x^{n-2}}{(n-2)!}$$

$\Rightarrow G''(x) = 5G'(x) - 6G(x)$ Which is a second order ordinary linear differential equation subject to the initial conditions $a_0 = 0$, $a_1 = 1$.

At the corresponding stage in the solution for the formal power series version, we had an equation to solve for G that didn't involve any derivatives. We solved it and then had to deal with a partial fraction expansion in order to find an exact formula for the *nth* term of the sequence. In this version, we solve the differential equation through the method of solving such equations.

We assume that a solution of this differential equation looks $G(x) = e^{kx}$ for some constant k.

Thus $G'(x) = ke^{kx}$ and $G''(x) = k^2 e^{kx}$.

Substituting these in to $G''(x) = 5G'(x) - 6G(x)$ we get

$$k^2 e^{kx} - 5ke^{kx} + 6e^{kx} = 0 \quad \Leftrightarrow \quad e^{kx}(k^2 - 5k + 6) = 0 \quad \Leftrightarrow \quad k^2 - 5k + 6 = 0$$

$$\Leftrightarrow \quad (k-3)(k-2) = 0 \quad \Leftrightarrow \quad k = 3, \quad k = 2$$

Thus the general solution is $G(x) = C_1 e^{3x} + C_2 e^{2x}$ with the initial conditions $a_0 = 0$, $a_1 = 1$.

Now $G(x) = C_1 e^{3x} + C_2 e^{2x} = C_1 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + C_2 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$

$$= (C_1 + C_2) + (3C_1 + 2C_2)x + \frac{1}{2!}(3^2 C_1 + 2^2 C_2)x^2 + \ldots + \frac{1}{n!}(3^n C_1 + 2^n C_2)x^n \ldots$$

$\Rightarrow a_0 = C_1 + C_2$, $a_1 = 3C_1 + 2C_2$, and $a_n = 3^n C_1 + 2^n C_2$.

$\Rightarrow C_1 = -C_2 = 1$, from $a_0 = 0$, $a_1 = 1$.

$\Rightarrow a_n = 3^n - 2^n$ as required.

The Method

Given: a linear recurrence formula that is to be solved by the method of generating function.

1. Make sure that the set of values of the free variables (say n) for which the recurrence equation is clearly delineated.

2. Give a name to the generating function that you look for, and write out that function in terms of the unknown sequence (e.g., call it $G(x)$, and define it to be $\sum_{n=0}^{\infty} a_n x^n$)

3. Multiply both sides of the recurrence equation by x^n (or $\frac{x^n}{n!}$ depending on the version of generating function you want to apply) and sum it over all values of n for which the recurrence holds.

4. Express both sides of the resulting equation explicitly in terms of your generating function $G(x)$.

5. Solve the resulting equation for the unknown generating function $G(x)$ and its derivatives.

6. If you want an exact formula for the sequence that is defined by the given recurrence equation, then attempt to get such a formula by expanding $G(x)$ into a power series by any method you can think of. In particular, if $G(x)$ is a rational function then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

Example 2.1.9

A certain sequence of numbers $a_0, a_1, a_2, \ldots$ satisfies the condition $a_{n+1} = 2a_n + n$, $(n \geq 0; a_0 = 1)$. Find the sequence.

Solution: Let $G(x) \xleftrightarrow{ops} \{a_n\}_{n=0}^{\infty}$.

Multiplying the recurrence $a_{n+1} = 2a_n + n$ by x^n and sum it over $0 \leq n < \infty$. We have, $\sum_{n=0}^{\infty} a_{n+1} x^n = 2\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} n x^n$

$$\Rightarrow \frac{G(x) - a_0}{x} = 2G(x) + \frac{x}{(1-x)^2} \qquad (a_0 = 1)$$

$$\Rightarrow G(x) = \frac{1 - 2x + 2x^2}{(1-x)^2(1-2x)}$$

$$\Rightarrow G(x) = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}$$

$$\Rightarrow G(x) = -\sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=0}^{\infty} 2^n x^n$$

$$\Rightarrow G(x) = \sum_{n=0}^{\infty}(2^{n+1} - (n+1))x^n$$

$$\Rightarrow a_n = 2^{n+1} - n - 1 \qquad (n = 0, 1, 2, 3, \ldots)$$

Example 2.1.10

How many regions are formed in the plane by n lines if no two of the lines are parallel and no three of the lines meet in a common point?

Solution: A look at small cases yields that zero lines give one region, one line gives two regions, two lines give four regions, and three lines give seven regions. Let r_n be the number of regions formed by n lines so that $r_0 = 1$, $r_1 = 2$, $r_2 = 4$, $r_3 = 7$.

Thus adding a new $(n+1)^{th}$ line keeps the r_n previous regions but also a new line is subdivided into $n+1$ line segments by the first n lines. Each of these line segment form a new region and the recursion equation established as $r_{n+1} = r_n + n + 1$, $n \geq 0$.

Now let $R(x) \xleftrightarrow{ops} \{r_n\}_{n=0}^{\infty}$, and we multiply the recursion by x^n to get $r_{n+1}x^n = r_n x^n + (n+1)x^n$. Next sum starting from $n=0$ to obtain

$$\sum_{n=0}^{\infty} r_{n+1}x^n = \sum_{n=0}^{\infty} r_n x^n + \sum_{n=0}^{\infty} (n+1)x^n$$

$$\Rightarrow \frac{R(x) - r_0}{x} = R(x) + \frac{1}{(1-x)^2}, \text{ but } r_0 = 1$$

$$\Rightarrow R(x) - 1 = xR(x) + \frac{x}{(1-x)^2} \text{ or } R(x)(1-x) = 1 + \frac{x}{(1-x)^2}$$

$$\Rightarrow R(x) = \frac{1}{1-x} - \frac{x}{(1-x)^3}$$

$\Rightarrow R(x)=\sum_{n=0}^{\infty}1.x^n+\sum_{n=0}^{\infty}\binom{n+1}{2}x^n$ and we now can dispense with generating functions since we have

$$r_n=1+\binom{n+1}{2}=1+\frac{(n+1)n}{2}=\frac{n^2+n+2}{2}.$$

Example 2.1.11

Show that the number of ways of selecting r elements from an n-set where order does not count and repetition is allowed is $\binom{n+r-1}{r}$.

Proof:

Let $f(n,r)$ denote the number of ways of select r things from the set of n distinct objects $\{x_1,x_2,x_3,...,x_n\}$, where we allow repetition but order does not matter.

Either x_1 is selected at least once or it is not.

If it is selected then we must choose r-1 things from the set $\{x_2,x_3,...,x_n\}$, to fill out the sample and this can be done in $f(n,r-1)$ ways. If we never choose x_1 then we draw the whole sample from the n-1 set $\{x_2,x_3,...,x_n\}$ and this can be done in $f(n-1,r)$ different ways. This demonstrates a double index recurrence relation.

$f(n,r)=f(n-1,r)+f(n,r-1)$

Now let $G_n(x)=1+f(n,1)x+f(n,2)x^2+f(n,3)x^3+...$ be the ordinary power series generating function for the sequence $\{f(n,r)\}_{r=0}^{\infty}$ for a fixed number n.

Now we have

$$xG_n(x) + G_{n-1}(x) = x + f(n,1)x^2 + f(n,2)x^3 + \ldots + 1 + f(n-1,1)x + f(n-1,2)x^2 + f(n-1,3)x^3 + \ldots$$

$$= \; 1 + (f(n-1,1)+1)x + (f(n,1)+f(n-1,2))x^2 + (f(n,2)+f(n-1,3))x^3 + \ldots$$

$$= 1 + f(n,1)x + f(n,2)x^2 + f(n,3)x^3 + f(n,4)x^4 \ldots$$

$$= G_n(x)$$

Thus we have $G_n(x) = \dfrac{1}{1-x} G_{n-1}(x)$

This also applies to $G_{n-1}(x)$ and we obtain $G_{n-1}(x) = \dfrac{1}{1-x} G_{n-2}(x)$.

And also $G_{n-2}(x) = \dfrac{1}{1-x} G_{n-3}(x)$

This process ends at $G_1(x) = 1 + f(1,1)x + f(1,2)x^2 + f(1,3)x^3 + \ldots$

$$= 1 + x + x^2 + x^3 + x^4 + \ldots$$

$$= \frac{1}{1-x}$$

Therefore $G_n(x) = \dfrac{1}{(1-x)^n}$

By the Binomial theorem for negative expansion $G_n(x)$ could be expanded as

$$G_n(x) = (1-x)^{-n} = 1 + (-n)(-x) + \frac{(-n)(-n-1)}{2!}(-x^2) + \frac{(-n)(-n-1)(-n-2)}{3!}(-x^3) + \ldots$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r \ldots$$

Thus $f(n,r) = \dfrac{n(n+1)(n+2)\ldots(n+r-1)}{r!} = \dfrac{n(n-1)(n-2)\ldots(3)(2)(1)}{((n+r-1)-r)!\,r!}$

$$f(n,r) = \binom{n+r-1}{r}.$$

Example 2.1.12

Suppose that a telephone exchange has n subscribers and let T_n be the number of ways that the n subscribers can be connected up at any moment. At any time a subscriber can either be talking to another subscriber or not using the telephone. No conference calls are allowed and all calls are with in the exchange. Find the recurrence equation and the exponential generating function for the sequence $\{T_n\}_{n=1}^{\infty}$.

Solution: Let $G(z) \xleftrightarrow{egf} \{T_n\}_{n=1}^{\infty}$

It is easy to see that $T_0 = 0$ $T_1 = 1$, $T_2 = 2$, $T_3 = 4$, $T_4 = 10$.

What is the effect of adding one more subscriber? If the new subscriber is not talking on the phone there are T_n possibilities. If the new subscriber is on the phone it's to one of the n other subscribers, excluding these two the remaining $(n\text{-}1)$ subscribers can be connected in T_{n-1} ways. These give us the recursion $T_{n+1} = T_n + nT_{n-1}$.

Multiplying this recursion equation by $\frac{z^n}{n!}$ and sum it over $n \geq 0$ we obtains $T_{n+1}\frac{z^n}{n!} = T_n\frac{z^n}{n!} + nT_{n-1}\frac{z^n}{n!}$.

$$\Rightarrow \quad \sum_{n=0}^{\infty} T_{n+1}\frac{z^n}{n!} = \sum_{n=0}^{\infty}[T_n + nT_{n-1}]\frac{z^n}{n!} = \sum_{n=0}^{\infty} T_n\frac{z^n}{n!} + \sum_{n=0}^{\infty} nT_n\frac{z^n}{n!}$$

$$\Rightarrow \quad G'(z) = G(z) + z\sum_{n=0}^{\infty} T_{n-1}\frac{z^{n-1}}{(n-1)!} = G(z) + zG(z)$$

$$\Rightarrow \quad G'(z) = (1-z)G(z)$$

$$\Rightarrow \quad \frac{dG}{G} = (1-z)dz$$

$$\Rightarrow \quad \ln G(z) = z - \frac{z^2}{2} \qquad \Rightarrow \qquad G(z) = e^{z-\frac{z^2}{2}}$$

2.2 Fibonacci Numbers

Eight centuries ago Fibonacci introduced his famous sequence 1,1,2,3,5,8,13,21,34,55,89,144…. Its original appearance dates back to 1202 with the publication of his book "***Liber Abacia***". This book has an important place in the history of ideas since it was the first book in Europe to use Arabic rather than Roman numerals. This change spread rather slowly because the printing press had not yet come along and copies had to be done by hand.

Here is the original problem usually called "***The Rabbit Problem***". Start with a pair of newborn rabbits that will, starting at two months and every month thereafter, beget a pair of bunnies. The same reproduction applies to all offspring and no rabbits are ever die. How many pairs of rabbits are there at the n*th* month?

We call the number of rabbit alive at month n F_n, the ***nth Fibonacci number***. F_n is formed by starting with the F_{n-1} pairs of rabbits alive last month and adding the rabbits that can only come from the F_{n-2} pairs alive two months ago.

$$F_n = F_{n-1} + F_{n-2} \qquad F_1 = F_2 = 1, \text{ and by convention } F_0 = 0.$$

This is a linear recurrence equation and thus we may apply the above method to prove the following proposition.

Theorem 2.2.1

The nth Fibonacci number is $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

Proof:

***Method** I: Using formal power series generating functions*

Let $G(x) \xleftrightarrow{ops} \{F_n\}_{n=0}^{\infty}$, i.e. $G(x) = \sum_{n=0}^{\infty} F_n x^n$.

then, $$xG(x) = x\sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_n x^{n+1}$$

and $$x^2G(x) = x^2\sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_n x^{n+2}$$

Subtracting the last two equations from the first two yields

$$G(x) - xG(x) - x^2G(x) = F_0 + (F_0 - F_1)x + (F_2 - F_1 - F_0)x^2 + (F_3 - F_2 - F_1)x^3 + \ldots$$

$$= 0 + 1x + 0x^2 + 0x^3 + \ldots$$

$= x$ since the coefficients of all the other powers of x are 0.

Solving this equation for $G(x)$ we find,

$$G(x) = \frac{x}{1 - x - x^2} .$$

Now by the method of partial fractions we can write $G(x)$ as

$$G(x)=\frac{1}{\sqrt{5}}\left[\frac{1}{1-\left(\frac{1+\sqrt{5}}{2}\right)x}-\frac{1}{1-\left(\frac{1-\sqrt{5}}{2}\right)x}\right]$$

Let us use the notations $a:=\dfrac{1+\sqrt{5}}{2}$, $b:=\dfrac{1-\sqrt{5}}{2}$, then

$$\sqrt{5}G(x)=\left(\frac{1}{1-ax}-\frac{1}{1-bx}\right)$$

$$\sqrt{5}G(x)=\left(1+ax+a^2x^2+a^3x^3+\ldots+a^nx^n+\ldots\right)-\left(1+bx+b^2x^2+b^3x^3+\ldots+b^nx^n+\ldots\right)$$

$$=(1-1)+(a-b)x+(a^2-b^2)x^2+\ldots(a^n-b^n)x^n+\ldots$$

Thus $$F_n=\frac{a^n-b^n}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Method II: *Using exponential generating function*

Let $G(x)\xleftrightarrow{egf}\{F_n\}_{n=0}^{\infty}$, i.e. $G(x)=\sum_{n=0}^{\infty}\frac{F_n}{n!}x^n$.

Multiply the recurrence relation $F_n = F_{n-1} + F_{n-2}$ by $\dfrac{x^{n-2}}{(n-2)!}$ and add it over $2\le n<\infty$, we have

$$F_n\frac{x^{n-2}}{(n-2)!}=F_{n-1}\frac{x^{n-2}}{(n-2)!}+F_{n-2}\frac{x^{n-2}}{(n-2)!}$$

$$\Rightarrow\ \sum_{n=2}^{\infty}F_n\frac{x^{n-2}}{(n-2)!}=\sum_{n=2}^{\infty}F_{n-1}\frac{x^{n-2}}{(n-2)!}+\sum_{n=2}^{\infty}F_{n-2}\frac{x^{n-2}}{(n-2)!}$$

$$\Rightarrow \sum_{n=2}^{\infty} F_n \frac{n(n-1)x^{n-2}}{n!} = \sum_{n=2}^{\infty} F_{n-1} \frac{(n-1)x^{n-2}}{(n-1)!} + \sum_{n=2}^{\infty} F_{n-2} \frac{x^{n-2}}{(n-2)!}$$

$\Rightarrow$ $G''(x) = G'(x) + G(x)$, $F_1 = F_2 = 1$, and by convention $F_0=0$, which is a second order linear differential equation with initial conditions. We solve the differential equation, getting

$$G(x) = C_1 e^{ax} + C_2 e^{bx}, \qquad \left(a = \frac{1+\sqrt{5}}{2}, \quad b = \frac{1-\sqrt{5}}{2} \right)$$

where C_1 and C_2 can be determined by the initial conditions (which haven't been used yet!) $F_1 = F_2 = 1$. After applying these two conditions, we find that $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$, from which the exponential generating function of Fibonacci sequence is $G(x) = C_1 e^{ax} + C_2 e^{bx} = \frac{1}{\sqrt{5}}\left(e^{ax} - e^{bx}\right)$. And we found that

$$F_n = \frac{a^n - b^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Probably the original point of this was to give some practice in addition. However an amazing amount of mathematics relates to these numbers including the number of ancestors of a male bee, the arrangements of rings on a pineapple, analysis of algorithms, electrical circuit theory, the number of ways you can walk on a ladder if you take the rungs one or two at a time, and so on.

Example 2.2.2

How many ways can a man climb up a ladder of n rungs if at each step he can climb either one or two rungs?

Solution:

Let the numbers of ways he can climb the ladder be A_n.

For his first step he can climb one rung or two rungs. If he climbs one rung the number of ways he can finish the step is A_{n-1}. If he climbs two rungs the number of ways he can finish the step is A_{n-2}. By the rule of sum, $A_n = A_{n-1} + A_{n-2}$. This sequence grows like Fibonacci sequence but it starts at different point since $A_1 = 1$ and $A_2 = 2$. Hence $A_n = F_{n+1}$.

We may also have the following immediate consequences of the recurrence equation that defines the n^{th} Fibonacci numbers.

How fast is the F_n growing? Let's look at the ratios ${F_{n+1}}/{F_n}$.

${F_1}/{F_0} = 1$, ${F_2}/{F_1} = 2$, ${F_3}/{F_2} = 1.5$, ${F_4}/{F_3} = 1.66666... = 1.\bar{6}$ and so on.

Assume that $\lim\limits_{n\to\infty} {F_{n+1}}/{F_n} = L$, if it exists.

Rewrite $F_{n+1} = F_n + F_{n-1}$ as $\dfrac{F_{n+1}}{F_n} = 1 + \dfrac{F_{n-1}}{F_n} = 1 + \dfrac{1}{\dfrac{F_n}{F_{n-1}}}$.

Now take the limit as $n \to \infty$.

$$L = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 + \frac{1}{\lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}} = 1 + \frac{1}{L}$$

Solving yields $L^2 - L - 1 = 0$ which implies $L = \frac{1+\sqrt{5}}{2} \approx 1.61803398875$, which implies that this L is the famous "***golden ratio***" of art and geometry.

Property 2.2.3

The sum of the first n Fibonacci numbers is one less than the *(n+2)nd*, i.e. $\sum_{k=1}^{n} F_k = F_{n+2} - 1$.

Proof: We start by writing

$F_1 = F_3 - F_2$, $F_2 = F_4 - F_3$, $F_3 = F_5 - F_4$

$F_4 = F_6 - F_5 \ldots$ $F_{n-1} = F_{n+1} - F_n$ $F_n = F_{n+2} - F_{n+1}$

Adding these gives the result.

Property 2.2.4

1. $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$
2. $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$

Proof: 1. $F_1 = F_2$, $F_3 = F_4 - F_2$, $F_5 = F_6 - F_4$

$F_7 = F_8 - F_6$, ... $F_{2n-1} = F_{2n} - F_{2n-2}$

Adding these gives the result.

2. $\sum_{k=1}^{n} F_{2k} = \sum_{k=1}^{n} F_k - \sum_{k=1}^{n} F_{2k-1} = F_{2n+2} - 1 - F_{2n} = F_{2n+2} - F_{2n} - 1 = F_{2n+1} - 1$

Property 2.2.5

$$\sum_{k=1}^{n} F_k^{\,2} = F_n F_{n+1}$$

Proof: It is immediate consequence $F_k^2 = F_k(F_{k+1} - F_{k-1})$ $(1 \le k \le n-1)$.

2.3 Partition

The technique of generating functions was greatly extended by Euler in 1748 in his "***Introductio in Analysin Infinitornum***", where he uses them to attack the problem of Partition.

Definition 2.3.1

$P(n)$ sometimes also denoted $p(n)$, gives the number of ways of writing the integer n as a sum of positive integers, where order of addends is not considered significant. By convention partitions are usually ordered from largest to smallest.

For example 4 = 4 = 3 + 1= 2 + 2

= 2 + 1 + 1= 1 + 1 + 1 + 1

Thus $P(4)=5$. Note that $P(1)=1$, $P(2)=2$, $P(3)=3$.

We want to apply Generating functions to approximate $P(n)$.

Proposition 2.3.2

The coefficient of x^m in the series expansion of $\frac{1}{(1-x^a)(1-x^b)(1-x^c)...}$ is the number of ways of writing m as a sum of *a's, b's, c's*, ...

Proof: We note that

$$\frac{1}{(1-x^a)(1-x^b)(1-x^c)...} = \left(1+x^a+x^{2a}+x^{3a}+...\right)\left(1+x^b+x^{2b}+x^{3b}+...\right)\left(1+x^c+x^{2c}+...\right)..$$

If x^m is formed, for instance, from the product x^{3a}, x^{2b}, x^{4c}, ... then $m = a+a+a+b+b+c+c+c+c+...$.

The term x^m arises exactly as often as m can be written as the sum of *a's, b's, c's*, ... We have the proposition.

Definition 2.3.3

Let $P_k(n)$ be the number of partitions of n into the integers *1, 2, ...,* k with repetition allowed.

Then we have the following immediate consequences of Proposition 2.3.2.

Corollary 2.3.4

The generating function for $P_k(n)$ is

$$P_k(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)...(1-x^k)}$$

Corollary 2.3.5

The generating function for *P(n)* is $P(x) = \dfrac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$.

Theorem 2.3.6

For all n: $P(n) < e^{3\sqrt{n}}$.

Proof: Let $G(x) \xleftrightarrow{ops} \{P(n)\}_{n=0}^{\infty}$

By Corollary 2.3.5, $G(x) = \sum_{n=0}^{\infty} P(n)x^n = \dfrac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$ then

$$\ln G(x) = \ln\left[\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}\right] = -\ln(1-x) - \ln(1-x^2) - \ln(1-x^3) - \ldots$$

Let us recall that the Taylor series for logarithms is $-\ln(1-y) = \sum_{n=1}^{\infty} \dfrac{y^n}{n}$.

So $$\ln G(x) = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) + \left(x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \ldots\right) + \left(x^3 + \frac{x^6}{2} + \frac{x^9}{3} + \ldots\right) + \ldots$$

$$= \left(x + x^2 + x^3 + \ldots\right) + \left(\frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{2} + \ldots\right) + \left(\frac{x^3}{3} + \frac{x^6}{3} + \frac{x^9}{3} + \ldots\right) + \ldots$$

$$= \left(\frac{x}{1-x}\right) + \frac{1}{2}\left(\frac{x^2}{1-x^2}\right) + \frac{1}{3}\left(\frac{x^3}{1-x^3}\right) + \ldots$$

Now let's study the expression $\dfrac{x^n}{1-x^n}$

If we restrict our attention to values of x between 0 and 1 we know that $x^{n-1} < x^{n-2} < x^{n-3} < \ldots < x^2 < x < 1$.

The average of this set of numbers is bigger that the smallest of them.

In other words, $x^{n-1} < \dfrac{1+x+x^2+\ldots+x^{n-1}}{n}$ or $\dfrac{x^{n-1}}{1+x+x^2+\ldots+x^{n-1}} < \dfrac{1}{n}$

Now $\dfrac{x^n}{1-x^n} = \dfrac{x}{1-x}\dfrac{x^{n-1}}{1+x+x^2+\ldots+x^{n-1}} < \dfrac{1}{n}\left(\dfrac{x}{1-x}\right)$

So $\ln G(x) < \dfrac{x}{1-x} + \left(\dfrac{1}{2}\right)^2\left(\dfrac{x}{1-x}\right) + \left(\dfrac{1}{3}\right)^2\left(\dfrac{x}{1-x}\right) + \ldots + \left(\dfrac{1}{n}\right)^2\left(\dfrac{x}{1-x}\right) + \ldots$

$$\Rightarrow \ln G(x) < \left(\frac{x}{1-x}\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots\right)$$

Recall from calculus that $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots < 1 + \displaystyle\int_1^\infty \frac{1}{x^2}dx = 2$

Therefore $\ln G(x) < \dfrac{2x}{1-x}$

Now $G(x) = \displaystyle\sum_{n=0}^{\infty} P(n)x^n > P(n)x^n$ for all n.

Thus $\ln P(n) < \ln G(x) - n\ln x < \dfrac{2x}{1-x} - n\ln x$

Now for $y > 1$, $\ln y < y - 1$ (look at the graphs)

So $-\ln x = \ln\dfrac{1}{x} < \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$.

Our inequality now becomes $\ln P(n) < 2\left(\frac{x}{1-x}\right) + n\left(\frac{1-x}{x}\right)$

If we let $x = \frac{\sqrt{n}}{1+\sqrt{n}}$ we find $\ln P(n) < 3\sqrt{n}$ which is the desired result.

More on Partition

Definition 2.3.7

1 Let $P_o(n)$ be the number of partitions of n into odd integers with repetition allowed.

2 Let $P_d(n)$ be the number of partitions of n into distinct parts.

3 Let $P_t(n)$ be the number of partitions of n into distinct powers of 2, i.e. 1, 2, 4, 8, 16, …

Property 2.3.8

The generating function for $P_o(n)$ is $\frac{1}{(1-x)(1-x^3((1-x^5)\ldots}$

Property 2.3.9

The coefficient of x^m in the series expansion of $(1+x^a)(1+x^b)(1+x^c)\ldots$ is the number of ways of writing m as a sum of a's, b's, c's e.t.c… at most once each.

Thus the generating function for $P_d(n)$ is $(1+x)(1+x^2)(1+x^3)\ldots$

And the generating function for $P_t(n)$ is

$(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots$

Proposition 2.3.10

For all n: $P_0(n) = P_d(n)$.

Proof:

We start with the equation $1 + x^k = \frac{1 - x^{2k}}{1 - x^k}$ for all $k = 1, 2, 3, \ldots$

Taking the product of the above equation over k gives

$$(1+x)(1+x^2)(1+x^3)\ldots = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\ldots = \frac{1}{(1+x)(1+x^3)(1+x^5)}\ldots$$

Thus $P_0(n) = P_d(n)$.

2.4 The Hilbert Function

2.4.1 Noetherian & Artinian Rings, Modules and Modules of finite lengths.

Definition 2.4.1.1

Assume R is a commutative unitary ring and M is an R-Module.

a. By ***ascending chain (ac)*** of submodules of M, we mean a sequence $\{M_n\}_{n\geq 1}$ of submodules of M such that

$$M_1 \subset M_2 \subset M_3 \subset M_4 \subset \ldots$$

b. By a ***descending chain (dc)*** of submodules of M, we mean a sequence $\{M_n\}_{n\geq 1}$ of submodules of M such that

$$M_1 \supset M_2 \supset M_3 \supset M_4 \supset \ldots$$

c. A module M is said to satisfy ***ascending chain condition (acc)*** if every ac of submodules of M is stationary after a finite steps, i.e. if $M_1 \subset M_2 \subset M_3 \subset M_4 \subset \ldots$ is any ac of *M*, then there is a positive integer k such that $M_n = M_k, \forall n \geq k.$

d. A module *M* is said to satisfy ***descending chain condition (dcc)*** if every dc of submodules of *M* is stationary after a finite steps i.e. if $M_1 \supset M_2 \supset M_3 \supset M_4 \supset \ldots$ is any dc of *M*, then there is a positive integer k such that $M_n = M_k, \forall n \geq k.$

Theorem 2.4.1.2

The following three conditions are equivalent on an *R*-module *M*.

i) *M* satisfies acc for submodules.

ii) Every non-empty set of submodules of *M* contains a maximal element under inclusion.

iii) Every submodule of *M* is finitely generated.

Proof: [1] page 74 and 75.

Definition 2.4.1.3

An *R*-module *M* is said to be ***Noetherian*** if it satisfies any one (and hence all) of the conditions of theorem 2.4.1.2.

Definition 2.4.1.4

A ring *R* is said to be ***Noetherian*** if it is noetherian when it is regarded as an *R*-module over itself.

Example 2.4.1.5

Any field K is noetherian as the only ascending chain of ideals of K is (0) $\subset K$.

Example 2.4.1.6

Any principal ideal domain is noetherian since every ideal is finitely generated. In particular the ring Z of integers is noetherian.

Example 2.4.1.7

The ring $R = K[x_1, x_2, x_3, ..., x_n, ...]$ is not noetherian since the sequence $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \ldots$, of ideals of R is an ac which is not stationary.

Properties of noetherian modules.

Proposition 2.4.1.8

If $O \to N \xrightarrow{f} M \xrightarrow{g} L \to O$ be a short exact sequence of R-modules, then M is noetherian if and only if N and L are.

Proof: [1] page 75

Proposition 2.4.1.9

Every submodule of a noetherian module is noetherian.

Proof: [1] page 75

Proposition 2.4.1.10

If N is a sub module of M, then M is noetherian if and only if N and M/N are.

Proof: Follows from proposition 2.4.1.8 by considering the short exact sequence $O \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to O$.

Proposition 2.4.1.11 *(Generalized Hilbert Basis Theorem)*

If R is noetherian ring, then $R[x_1, x_2, x_3, ..., x_n]$ is also noetherian.

Proof: [1] page 81

Proposition 2.4.1.12

If R is noetherian ring and I is an ideal in R, then R/I is also noetherian.

Proof: R/I is finitely generated since both R and I are. Moreover, if the maximal set of linearly independent generators of R has cardinality n and that of generators of I is k $(k \leq n)$ then the maximal set of linearly independent generators of R/I has cardinality n-k.

Theorem 2.4.1.13

The following statements are equivalent for an R-module M.

i) M satisfies dcc for submodules.

ii) Every non-empty collection of submodules of M has a minimal element under inclusion.

Proof: [4] page 373

Definition 2.4.1.14

An R-module M is said to be ***Artinian*** if it satisfies any one (and hence both) of the conditions of theorem 2.4.1.13.

Definition 2.4.1.15

A ring R is said to be ***Artinian*** if it is artinian when it is regarded as an R- module over itself.

Example 2.4.1.16

Any field K is artinian since the only descending chain of ideals of K is $K \supset (0)$.

Example 2.4.1.17

The ring Z of integers is not artinian for if $a \in Z$, $a \neq 0$, $(a) \supset (a^2) \supset (a^3) \supset (a^4) \supset \ldots$ does not satisfy dcc.

Definition 2.4.1.18

Let N be a submodule of an R-module M. A ***composition series from M to N*** is a chain $M = M_0 \supset M_1 \supset M_2 \supset M_3 \supset M_4 \supset \ldots \supset M_n = N$, where the chain factors M_{i-1}/M_i have no proper submodule,

i.e., each M_{i-1}/M_i is simple R-modules.

A composition series from M to (0) is called a ***composition series of M***. Given an R- module M, a composition series may not always exist. If it exists the inclusion in the chain are all strict and moreover there is no submodule strictly between M_{i-1} and M_i. The following theorem gives the conditions for the existence of a composition series for an R-module.

Theorem 2.4.1.29

An R-module M has a composition series if and only if it satisfies acc & dcc.

Proof: [1] page 77 or [4] page 376

Definition 2.4.1.20

Let M be an R-module. We define the ***length*** $l(M)$ ***of composition series of M*** by

$$l(M)=\begin{cases} 0 & \text{if } M=(0) \\ n & \text{if } M\neq(0)\text{, and there is a composition series } M=M_0\supset M_1\supset M_2\supset M_3\supset M_4\supset\ldots\supset M_n=(0) \text{ of } M \\ \infty & \text{if } M\neq(0) \text{ and there is no composition series for } M. \end{cases}$$

Theorem 2.4.1.21

Suppose an R-module M has a composition series of length n. Then every composition series of M has length n, and every chain in M can be extended to a composition series.

Proof: [1] page 77

Theorem 2.4.1.22

If $O\to M_0\xrightarrow{f_1}M_1\xrightarrow{f_2}M_2\xrightarrow{f_3}M_3\ldots M_{s-1}\xrightarrow{f_s}M_s\to O$ is an exact sequence of R-modules of finite length, and $l(M_i)$ is the length of M_i, then $\sum_{i=0}^{s}(-1)^i l(M_i)=0$.

Proof: We shall consider the following cases on s.

Case1: If $s = 0$, then the exact sequence is $O\to M_0\to O$ and hence $M_0=(0)$. Therefore the result holds.

Case 2: If $s = 1$, then the exact sequence is $O \to M_0 \xrightarrow{f_1} M_1 \to O$ and hence $M_0 \cong M_1$. This implies $l(M_0) = l(M_1)$. Therefore the result holds.

Case 3: If $s = 2$, then the exact sequence is $O \to M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to O$.

By the first isomorphism theorem on f_2, we have $M_2 \cong {M_1}/{M_0}$.

$$\Rightarrow \; l(M_2) = l\left({M_1}/{M_0}\right) = l(M_1) - l(M_0).$$

$$\Rightarrow l(M_2) - l(M_1) + l(M_0) = 0$$

Case 4: Now suppose that $s > 2$. We embed the given sequence in a sequence of homomorphism

$$\xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} f_i(M_{i-1}) \longrightarrow M_i \xrightarrow{f_{i+1}} f_{i+1}(M_i) \to M_{i+1} \qquad (1 \le i \le s-1).$$

Then we have exact sequences

$$O \to M_0 \xrightarrow{f_0} f_1(M_0) \to O$$

$$O \to f_i(M_{i-1}) \longrightarrow M_i \xrightarrow{f_{i+1}} f_{i+1}(M_i) \to O$$

$$O \to f_{i+1}(M_i) \longrightarrow M_{i+1} \xrightarrow{f_{i+2}} f_{i+2}(M_{i+1}) \to O$$

$$O \to f_{i+2}(M_{i+1}) \longrightarrow M_{i+2} \xrightarrow{f_{i+3}} f_{i+3}(M_{i+2}) \to O \qquad (1 \le i \le s-1)$$

$$O \to f_s(M_{s-1}) \longrightarrow M_s \to O$$

Which gives the relation

$$l(M_0) = l(f_1(M_0))$$

$$l(M_i) = l(f_i(M_{i-1})) + l(f_{i+1}(M_i))$$

$$l(M_{i+1}) = l(f_{i+1}(M_i)) + l(f_{i+2}(M_{i+1}))$$

$$l(M_{i+2}) = l(f_{i+2}(M_{i+2})) + l(f_{i+3}(M_{i+2})) \qquad (1 \le i \le s-1)$$

$$l(M_s) = l(f_s(M_{s-1}))$$

If we take the alternating sum of those equations, we obtain the result.

That is, $\sum_{i=0}^{s}(-1)^i l(M_i) = 0$.

2.4.2 Graded Ring and Module

Definition *2.4.2.1*

A ***graded ring*** is a ring R together with a family $\{R_i\}_{i=0}^{\infty}$ of subgroups of the additive group R such that

a) $R = \bigoplus_{i=0}^{\infty} R_i$ and

b) $R_i R_j \subseteq R_{i+j}$ for all $i, j = 0,1,2,3,...$

By way of explanation, it needs to be stated that $R_i R_j$ means the set of all elements which can be expanded in the form $\sum_{k=1}^{n} r_k s_k$ where $r_k \in R_i$ and $s_k \in R_j$ $k = 1, 2, 3, ..., n$. Suppose that we have a grading on R by $\{R_i\}_{i=0}^{\infty}$. Each element r of R has a unique representation in the form $r = \sum_{k=0}^{\infty} r_k$ where $r_k \in R_k$ and the sum contains only a finite number of non - zero terms. The elements of R_k are said to be homogeneous elements of degree k and r_k is called homogeneous component of r of degree k.

Thus

a) 0 is homogeneous of degree k for any $k \geq 0$.

b) If x is homogeneous of degree k and y is homogeneous of degree m, then xy is homogeneous of degree $k+m$.

c) We can consider any ring R graded by taking the sequence of subgroups $\{R_i\}_{i=0}^{\infty}$, where $R_0 = R$, and $R_i = (0)$, $i = 1, 2, 3, \ldots$

Properties of graded rings

Proposition 2.4.2.2

Let R be a graded ring. Then the unity element is a homogeneous element of degree zero

Proof: [5] page 114

Proposition 2.4.2.3

R_0 is a subring of R.

Proof: [5] page 114

Proposition 2.4.2.4

Each R_n is an R_0-module.

Proposition 2.4.2.5

$R_+ = \bigoplus_{i=1}^{\infty} R_i$ is an ideal in R.

Proposition 2.4.2.6

R is algebra over R_0.

Definition 2.4.2.7

Let R be a graded ring, by the subgroups $\{R_i\}_{i=0}^{\infty}$. A ***graded R-module*** is an R-module M together with a family $\{M_i\}_{i=0}^{\infty}$ of subgroups of the additive group M such that

a) $M = \bigoplus_{i=0}^{\infty} M_i$ and

b) $R_i M_j \subseteq M_{i+j}$ for all $i, j = 0,1,2,3,...$

In this context $R_i M_j$ means the set of all elements, which can be expressed in the form $\sum_{k=1}^{n} r_k m_k$ where $r_k \in R_i$ and $m_k \in M_j$, $k = 1, 2, 3, ..., n$.

Let M be a graded R-module by $\{M_i\}_{i=0}^{\infty}$ over a graded ring R by $\{R_i\}_{i=0}^{\infty}$. Each element m of M has unique representation in the form of $m = \sum_{k=0}^{\infty} m_k$ where $m_k \in M_k$ and the sum contains only a finite number of non - zero terms. The elements of M_k are called homogeneous of degree k and m_k is called homogeneous component of m of degree k. If x is homogeneous element of R_k and y is homogeneous element of M_n then xy is homogeneous element of M_{k+n}. Moreover any R- module can be graded.

Theorem 2.4.2.8

Each M_i is an R_0-module.

Theorem 2.4.2.9

If N is a submodule of the graded R-module $M = \overset{\infty}{\underset{i=0}{\oplus}} M_i$, then the following statements are equivalent.

i. $N = \overset{\infty}{\underset{i=0}{\oplus}} (N \cap M_i)$

ii. If $y \in N$, then all homogeneous components of y belong to N.

iii. N can be generated as an R-module by homogeneous elements.

Proof: [5] page 115

Definition 2.4.2.10

A submodule N of a graded module $M = \overset{\infty}{\underset{i=0}{\oplus}} M_i$, which satisfies any one (and hence all) of the equivalent conditions of theorem 2.4.2.9 is called ***homogeneous submodule*** of M.

If we put $N_i = N \cap M_i$, then $\{N_i\}_{i=0}^{\infty}$ is grading of N. Assuming that N is a homogeneous submodule of M, let $\phi : M \to M/N$ be a natural mapping. Now both M and M/N can be regarded as R_0-modules and hence ϕ is R_0-homomorphism.
Put $\phi(M_i) = (M/N)_i \quad i = 0,1,2,3,...$
Then $\{(M/N)_i\}_{i=0}^{\infty}$ is grading of M/N.
But $(M/N)_i \cong M_i/N_i \cong (M_i + N)/N$.

Theorem 2.4.2.11

If R is a graded commutative ring, then the following statements are equivalent.

i. R is noetherian ring.

ii. R_0 is noetherian and R is finitely generated as an R_0-algebra.

If the above conditions hold and M is finitely generated graded R-module, and then every M_i is finitely generated.

Proof: (i)$\Rightarrow$(ii); Suppose R is noetherian.

$$R = R_0 \oplus R_+ \qquad \Rightarrow \qquad {(R_0 + R_+)}/{R_+} \cong {R_0}/{R_0 \cap R}$$

$$\Rightarrow \; R = R_0 + R_+ \text{ and } R_0 \cap R_+ = (0).$$

$$\Rightarrow R_0 \cong {R}/{R_+}.$$

Since R_+ is a submodule of a noetherian R-module R, it is noetherian and hence R_0 is noetherian.

Moreover, the ideal R_+ is finitely generated as an R-module by, say, $x_1, x_2, x_3, \ldots, x_s$. Let's replace each x_i by its homogeneous parts or without any loss of generality, let's assume that each x_i is homogeneous elements of R of degrees $k_1, k_2, k_3, \ldots, k_s$ (all >0) respectively. We want to claim that $\{x_1, x_2, x_3, \ldots, x_s\}$ generates R as an R_0-algebra. It suffices to show that each homogeneous element can be written as a polynomial in the x_i with coefficients in R_0. We proceed by induction on homogeneity.

The result is clear if the degree is 0, since every homogeneous element of degree zero is in R_0.

Now suppose that u is a homogeneous element of degree $n>0$. We can write $u=\sum_{i=1}^{s}u_i x_i$, where $u_i \in R$. Equating homogeneous parts, we may assume that u_i is homogeneous of degree $n - deg\ x_i < n$, since $x_i \in R_+$. Then u_i is a polynomial in x_j with coefficients in R_0 and hence the same is true for $u=\sum_{i=1}^{s}u_i x_i$.

Therefore R is finitely generated R_0-algebra.

($\Leftarrow$) Suppose R_0 is noetherian and R is finitely generated as an R_0-algebra by, say, $y_1, y_2, y_3, ..., y_n$. Then $R= R_0[y_1, y_2, y_3, ..., y_n]$. Hence the result follows by Hilbert Generalized Basis Theorem.

Now suppose that R is noetherian and M is finitely generated as graded R-module. We may choose a set of generators $\{u_1, u_2, u_3, ...u_r\}$ for M that are homogeneous. If the x 's are chosen as before, then it is readily seen that every element of M_n is an R_0-linear combination of elements $y_i u_i$, where y_i is a monomial in x 's and deg $y_i + deg\ u_i = n$. Since the number of these elements is finite, M_n is finitely generated.

2.4.3 The Hilbert Function

Let R be a graded noetherian ring and M be a graded R-module. Suppose

i. R_0 is artinian (and as well noetherian)

ii. M is finitely generated as R_0-module and hence is both noetherian and artinian.

Hence every M_n has a composition series as an R_0-module and therefore we have a uniquely determined non-negative integer $l(M_n)$, the length of the composition series of M_n.

Let the generating function for the sequence $\{l(M_n)\}_{n=0}^{\infty}$, be denoted by $P(M,t)$.

Definition 2.4.3.1

The generating function $P(M,t)=\sum_{n=0}^{\infty} l(M_n)t^n$ is called ***the Hilbert Function*** for the graded module M.

Our main objective is to characterize the behavior of $P(M,t)$ by applying the ordinary power series generating functions and the theory we developed so far. We state this as a theorem as follows.

Theorem 2.4.3.2

If R is generated by homogeneous elements $x_1, x_2, x_3, \ldots, x_m$, where $\deg x_i = e_i > 0$, then $P(M, t)$ is a rational function of the form

$$P(M,t)=\frac{f(t)}{\prod_{i=1}^{m}\left(1-t^{e_i}\right)}, \text{ where } f(t)\in Z[t].$$

Proof: We use induction on the number of homogeneous generators.

If $m = 0$, then $R = R_0$ and M is finitely generated as R_0-module. Thus, $M_n = (0)$ only for sufficiently large n.

Assume that $M_n = (0)$ for all $n>k$. We then have $l(M_n)=0$ for all $n>k$.

Hence $P(M,t)=\sum_{n=0}^{k} l(M_n)t^n$ which is a rational function.

Assume the result holds if R has $m - 1$ homogeneous generators. We define an R-module endomorphism $\phi_n : M_n \to M_{n+e_m}$ by $\phi_n(a)=x_m a$ for each $n=0,1,2,3,4,\ldots$

Hence $K_n = Ker\phi_n$ and $C_{n+e_m} = Co\ker\phi_n$, for each $n=0,1,2,3,4,\ldots$, are homogeneous submodules.

We have $K = \bigoplus_{n=0}^{\infty} K_n$ and $C = \bigoplus_{n=0}^{\infty} C_n$ are graded modules by theorems 2.4.2.9 and 2.4.2.11.

Moreover we have the exact sequence

$O \to K_n \longrightarrow M_n \xrightarrow{\phi_n} M_{n+e_m} \longrightarrow C_{n+e_m} \to O$. Hence by theorem 2.4.1.22, we have

$$l(K_n)-l(M_n)+l(M_{n+e_m})-l(C_{n+e_m})=0$$

$$\Rightarrow \quad l(M_n)-l(M_{n+e_m})=l(K_n)-l(C_{n+e_m})$$

If we multiply this relation by t^{n+e_m}, rearrange it and then take the sum over n, we obtain

$$\Rightarrow \quad l(M_n)t^{n+e_m} - l(M_{n+e_m})t^{n+e_m} = l(K_n)t^{n+e_m} - l(C_{n+e_m})t^{n+e_m}.$$

$$\Rightarrow \quad l(M_n)t^n t^{e_m} - l(M_{n+e_m})t^{n+e_m} = l(K_n)t^n t^{e_m} - l(C_{n+e_m})t^{n+e_m}.$$

$$\Rightarrow \sum_{n=0}^{\infty} l(M_n)t^n t^{e_m} - \sum_{n=0}^{\infty} l(M_{n+e_m})t^{n+e_m} = \sum_{n=0}^{\infty} l(K_n)t^n t^{e_m} - \sum_{n=0}^{\infty} l(C_{n+e_m})t^{n+e_m}.$$

$$\Rightarrow P(M,t)t^{e_m} - \sum_{n=0}^{\infty} l(M_{n+e_m})t^{n+e_m} = P(K,t)t^{e_m} - \sum_{n=0}^{\infty} l(C_{n+e_m})t^{n+e_m}.$$

$$\Rightarrow \quad P(M,t)t^{e_m} + \sum_{k=0}^{e_m-1} l(M_k)t^k - \sum_{k=0}^{e_m-1} l(M_k)t^k - \sum_{n=0}^{\infty} l(M_{n+e_m})t^{n+e_m} =$$

$$= P(K,t)t^{e_m} + \sum_{k=0}^{e_m-1} l(C_k)t^k - \sum_{k=0}^{e_m-1} l(C_k)t^k - \sum_{n=0}^{\infty} l(C_{n+e_m})t^{n+e_m}.$$

$$\Rightarrow P(M,t)t^{e_m} - P(M,t) + \sum_{k=0}^{e_m-1} l(M_k)t^k = P(K,t)t^{e_m} - P(C,t) + \sum_{k=0}^{e_m-1} l(C_k)t^k.$$

$$\Rightarrow P(M,t)(1-t^{e_m}) = P(C,t) - P(K,t)t^{e_m} + \sum_{k=0}^{e_m-1} l(M_k)t^k - \sum_{k=0}^{e_m-1} l(C_k)t^k.$$

$$P(M,t)(1-t^{e_m}) = P(C,t) - P(K,t)t^{e_m} + g(t) \qquad (*)$$

where $g(t) = \sum_{k=0}^{e_m-1} l(M_k)t^k - \sum_{k=0}^{e_m-1} l(C_k)t^k. \in Z[t]$

But we can see from the R-module homomorphism $\phi : M \to M$, given by $\phi(a) = x_m a$ that $x_m K = (0)$ and $x_m C = (0)$.

Hence C and K are $R/x_m R$-modules. Since $R/x_m R$ is a graded ring with m-1 homogeneous generators of degree $e_1, e_2, e_3, \ldots, e_{m-1}$ and since C and K are graded modules of this ring, by induction assumption we have

$$P(K,t) = \frac{f_1(t)}{\prod_{i=1}^{m-1}(1-t^{e_i})}, \text{ where } f_1(t) \in Z[t]$$

and

$$P(C,t) = \frac{f_2(t)}{\prod_{i=1}^{m-1}(1-t^{e_i})}, \text{ where } f_2(t) \in Z[t],$$

Finally by substituting the above two equations in (*), we've

$$P(M,t)(1-t^{e_m}) = \frac{f_2(t)}{\prod_{i=1}^{m-1}(1-t^{e_i})} - \frac{f_1(t)}{\prod_{i=1}^{m-1}(1-t^{e_i})}t^{e_m} + g(t)$$

Therefore,

$$P(M,t)=\frac{f_2(t)-f_1(t)t^{e_m}+g(t)\prod_{i=1}^{m-1}\left(1-t^{e_i}\right)}{\prod_{i=1}^{m}\left(1-t^{e_i}\right)}$$

$$P(M,t)=\frac{f(t)}{\prod_{i=1}^{m}\left(1-t^{e_i}\right)}, \text{ where. } f(t)=f_2(t)-f_1(t)t^{e_m}+g(t)\prod_{i=1}^{m-1}\left(1-t^{e_i}\right)\in Z[t]$$

The most important case of the foregoing result is that in which the generators could be chosen in R_1, that is each $e_i=1$.
This is the case if $R=R_0[x_1,x_2,x_3,...,x_m]$, where

i. R_0 is artinian

ii. The x_i 's are indeterminate

iii. The grading is usual (R_n is homogeneous elements of degree n)

Then, $P(M,t)=\dfrac{f(t)}{(1-t)^m}$

GLOSSARY

1. We may need to revise techniques of writing a rational function as a sum of simple partial fractions from calculus. Here are some suggested books
   a. Calculus with Analytic Geometry, Ellis
   b. Calculus with Analytic Geometry, Johnson
   c. Calculus with Analytic Geometry, Edwards
2. Generating functions giving the first few powers of the non-negative integers are given in the following table.

n^p	$f(x)$	Series
1	$\frac{x}{1-x}$	$\sum_{n=1}^{\infty} x^n$
n	$\frac{x}{(1-x)^2}$	$\sum_{n=1}^{\infty} nx^n$
n^2	$\frac{x(x+1)}{(1-x)^3}$	$\sum_{n=1}^{\infty} n^2x^n$
n^3	$\frac{x(x^2+4x+1)}{(1-x)^4}$	$\sum_{n=1}^{\infty} n^3x^n$
n^4	$\frac{x(x+1)(x^2+10x+1)}{(1-x)^5}$	$\sum_{n=1}^{\infty} n^4x^n$

3. The Mathematica 4.0 (Mathematica 5.0) Computer programs add on Package *DiscreteMath`RSolve` (which can be loaded with the command <<DiscreteMath`Rsolve`) has the following lists of functions that could help us to find the problem in question.*
   a. RSolve[equn,a[n],n]-solve the recurrence equation for a[n].

Example: In[1]:=RSolve[a[n+1]=2a[n],a[n],n]

Out[1]={{a[n]→2^nc[1]}}

The solution to this recurrence equation is an exponential. The initial value of the solution sequence is left unspecified. The constant c[1] may be specified using the option RSolveConstants.

b. RSolve[equn,a,n]-solve the recurrence equation for the function a.

c. RSolve[{equn1,equn2,equn3,…},{a1,a2,a3,…},n]- solve list of recurrence equations.

d. RSolve[equn,the initial conditions, a[n],n]-solve the recurrence equation for a[n] subjected to the initial conditions given.

Examples: In[2]:=RSolve[a[n+1]=2a[n],a[0]=5,a[n],n]

Out[2]={{a[n]→5(2^n)}}

This is a linear equation with variable coefficients.

In[3]:=RSolve[{a[0]=a[1]=2,(n+1)(n+2)-2(n+1)a[n+1]-3a[n]=0,a[n],n]

$$\text{Out [3]} = a[n] \to \left\{\left\{\frac{(-1)^n}{n!} + \frac{3^n}{n!}\right\}\right\}$$

e. PowerSum[expr,{x,n}]-gives the formal power series generating function for the expr in terms of x, where expr is treated as a sequence in n and the sum runs from zero to infinity.

Example: In[4]:=PowerSum[n^2,{x,n}]

$$\text{Out [4]} = -\frac{x(1+x)}{(-1+x)^3}$$

f. PowerSum[expr,{x,n,n_0}]-gives the formal power sum generating function for expr in terms of x, where expr is treated as a sequence in n and the sum runs from n_0.

g. CoefficientList[Normal[Series[expr,{x,k1,k2}]],x]- gives the coefficient of the series expansion of expr (a function in x) from k_1 to to k_2 inclusive.

Example: Ln[5]:=CoefficientList [Normal[Series [$\frac{1}{1-x-x^2}$, {x,3,10}]],x]

Out[5]= {2,3,5,8,13,21,34,55}

h. ExponentialPowerSum[expr,{x,n}]- gives the exponential generating function for expr, where the sum runs from zero to infinity.

Example: In[6]:=ExponentiaLPowerSum[n^2,{x,n}]

Out[6]= $e^x x(1+x)$

i. ExponentialPowerSum[expr,{x,n,n_0}]- gives the exponential generating function for expr, where the sum runs from n_0 to infinity.

j. GeneratingFunction[equn,a[n],n,x]-gives the ordinary generating function for the solutions a[n] to the recurrence equation in terms of x.

Example: In[7]:=GeneratingFunction[a[n]=a[n-1] +a[n-2],a[0]=a[1]=1,a[n],n,x]

Out[7]= $\left\{\left\{-\frac{1}{-1+x+x^2}\right\}\right\}$

k. GeneratingFunction[{equn1,equn2,equn3,…},{a1[n],a2[n],a3[n],…},n,x]-give the ordinary generating function for the solutions of the system of equations given.

l. ExponentialGeneratingFunction[equn,a[n],n,x]-gives the exponential generating function for the solutions a[n] to the recurrence equation in terms of x.

m. ExponentialGeneratingFunction[{equn1,equn2,equn3,...},{a1[n],a2[n],a3[n],...},n,x]-give the exponential generating function for the solutions of the system of equations given.

n. SeriesTerm[expr,{x,x_0,n}-gives the nth coefficient in the power series expression of expr about the point x=x_0.

4. Fibonacci numbers are implemented in Mathematica as Fibonacci[n].

5. Some useful power series

a. $\frac{1}{1-x} = \sum_{n\geq 0} x^n$

b. $\ln\left(\frac{1}{1-x}\right) = \sum_{n\geq 0} \frac{x^n}{n}$

c. $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$

d. $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

e. $\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{n} x^n$, k is fixed non-negative integer.

References

[1]. I.G. Macdonald, ***Introduction to Commutative Algebra***, 1969, Addison-Wesley Pub. Com.

[2]. Hideyuki Matsumura, ***Commutative Ring Theory***, 1980, Cambridge University Press, London.

[3]. Nathan Jacobson, ***Basic Algebra II***, 1980, Hindustan Publishing Company, India.

[4]. Thomas W. Hungerford, ***Algebra, A Graduate Text in Mathematics***,1974, Springer, New York.

[5]. D.G. Northcott, F.R.S., ***Lesson on Rings, Modules, and Multiplicities***, 1968, Cambridge University Press, London

[6]. Hilbert S. Wilf, ***Generetingfuntionology,*** 1990, Haracourt Brace Jovanovich publisher, San Diego

[7]. Daniel A. Cohn, ***Basic Techniques of Combinatorial Theory***, 1978, John Wiley & Sons, New York.

[8]. N.S.Gopalakrishnan, ***University Algebra***, Second Edition, 1986, Wiley Eastern Limited, India.

[9]. Nathan Jacobson, ***Basic Algebra I***, 1980, Hindustan Publishing Company, India.

[10]. The Internet source " ***Google-Generating Functions***"

[11]. Different abstracts about Generating functions and its combinatorial applications