ON ENTIRE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

By

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Declaration

I, Ataklti Araya, with student number $GSR/2787/05$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

______________________________ Date ____________________

Ataklti Araya

Certificate

I hereby certify that I have read this dissertation prepared by Ataklti Araya under my supervision and recommended that, it should be accepted as fulfilling the dissertation requirement.

______________________________ Date __________________

Prof. Ahmed Mohammed

Abstract

In this thesis, we investigate entire solutions of the quasilinear equation

$$(\dagger) \qquad \Delta_\phi u = h(x, u)$$

where $\Delta_\phi u := \text{div}(\phi(|\nabla u|)\nabla u)$. Under suitable assumptions on the right-hand side we will show the existence of infinitely many positive solutions that are bounded and bounded away from zero in $\mathbb{R}^N$. All these solutions converge to a positive constant at infinity. The analysis that leads to these results is based on a fixed-point theorem attributed to Shcauder-Tychonoff.

Under appropriate assumptions on $h(x,t)$, we will also study ground state solutions of (†) whose asymptotic behavior at infinity is the same as a fundamental solution of the ϕ-Laplacian operator Δ_ϕ. Ground state solutions are positive solutions that decay to zero at infinity.

An investigation of positive solutions of (†) that converge to prescribed positive constants at infinity will be considered when the right-hand side in (†) assumes the form $h(x,t) = a(x)f(t)$. After establishing a general result on the construction of positive solutions that converge to positive constants, we will present simple sufficient conditions that apply to a wide class of continuous functions $f : \mathbb{R} \to \mathbb{R}$ so that the equation $\Delta_\phi u = a(x)f(u)$ admits positive solutions that converge to prescribed positive constants at infinity.

We will also study Cauchy-Liuoville type problems associated with the equation $\Delta_\phi u = f(u)$ in $\mathbb{R}^N$. More specifically, we will study sufficient conditions on $f : \mathbb{R} \to \mathbb{R}$ in order that the equation

$$\Delta_\phi u = f(u)$$

admits only constant positive solution provided that f has at least one real root. Our result in this direction can best be illustrated by taking $\phi(t) = pt^{p-2} + qt^{q-2}$ for some $1 < p < q$ which leads to the so called (p,q)-Laplacian, $\Delta_{(p,q)}u := \Delta_p u + \Delta_q u$.

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Contents

Chapter 1

Introduction

1.1 The Problem

In this dissertation, we wish to study several aspects of solutions of the following quasilinear PDE.

$$\Delta_\phi u = h(x, u), \quad x \in \mathbb{R}^N \tag{1.1.1}$$

where $\Delta_\phi u := \mathrm{div}(\phi(|\nabla u|)\nabla u)$ which is called the ϕ-Laplacian. If $\phi(t) := t^{p-2}$, $t > 0$ for $p > 1$, this reduces to the usual p-Laplacian.
Our specific purpose is to investigate the existence of solutions of (1.1.1), their asymptotic behavior at infinity and to study Cauchy-Liouville type properties of solutions of (1.1.1). This requires various structure conditions on ϕ and the inhomogeneous term $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ which will be explicitly stated below. Our results extend many known results in the literature and in many cases they provide substantial improvements over known results.
To put our results in perspective, let us recall some early works in the special case of $\phi \equiv 1$ that are relevant to this investigation. More specifically, let us consider the PDE

$$\Delta u = a(x)f(u), \ x \in \mathbb{R}^N. \tag{1.1.2}$$

It has long been known that if a is a non-negative and non-trivial function and f is a non-negative function defined on the positive set of real numbers, then Problem (1.1.2) has no positive bounded solution if $N = 2$. In fact, this is due to the fact that there are no bounded sub-harmonic functions in the plane. In contrast, when $N \geq 3$ and $f(t) = t^p$ for $p \neq 1$, N. Kuwano [19], N. Kuwano showed that Equation (1.1.2) admits infinitely many positive bounded solutions in $\mathbb{R}^N$ which are bounded away from zero, provided that $a(x)$, not necessarily non-negative, is a locally Hölder continuous function with $|a(x)| \leq b(|x|)$ in $\mathbb{R}^N$ for some non-negative and non-trivial function b on $[0, \infty)$ such that

$$\int_0^\infty tb(t)dt < \infty \tag{1.1.3}$$

On a closely related topic, the existence of ground state solutions, that is, entire positive solutions that vanish at infinity, has also been the subject of extensive investigations. For instance, we refer to the works [21, 22, 31, 36]. In the paper [21], the authors study ground state solutions of (1.1.2) when $f(t) = t^{-\gamma}$, $0 < \gamma < 1$. Subject to appropriate conditions on a, which may change sign in $\mathbb{R}^N$, it is shown in [21] that a positive entire solution u exists such that $u(x) \approx |x|^{2-N}$ for $|x| \geq 1$. This result was later extended to solutions of other elliptic PDEs with different inhomogeneous terms.

Another interesting result on entire solutions of Equation (1.1.2) was also obtained by M. Naito in [34]. In [34], the author provides sufficient conditions on the possibly sign changing weight a and on f in order that for a given constant $\ell > 0$ of the interval I, depending on a and f, the PDE (1.1.2) admits a positive solution u in $\mathbb{R}^N$ such that $u(x) \to \ell$ as $|x| \to \infty$.

Finally, the well-known result that there are no positive entire harmonic functions has been extended to solutions of the other semilinear equations. While there are many generalizations in the literature, we want to focus here on the work of J. A. McCoy [29, 30] in which it was shown that the only positive solutions to $-\Delta u = -f(u)$ are constants that are roots of f. For instance in [29], McCoy shows, among other results, that the PDE $\Delta u = -u^\gamma$ has no non-trivial solution for $\gamma \leq \frac{N+1}{N-1}$ where $N \geq 2$. Results of this kind hold for many elliptic PDEs and are commonly referred to as Liouville type theorems. See [9] for a similar result when the Laplacian is replaced by the p-Laplacian.

In this dissertation, we wish to extend all the aforementioned results to solutions of (1.1.1) provided suitable conditions hold for the nonlinearity h. In Chapter 2, we will show that Problem (1.1.1) admits infinitely many positive bounded solutions, each of which is bounded away from zero. In some cases, we will show that such entire solutions converge to positive constants at infinity. In Chapter 3, we will establish that under a different set of conditions on h, Problem (1.1.1) admits positive ground state solutions such that $\dfrac{u(x)}{\Gamma(|x|)}$ is bounded and bounded away from zero in appropriate exterior domains. Here Γ is the fundamental solution of the ϕ-Laplacian. In Chapter 4, we will show that Problem (1.1.1) admits positive solutions that are asymptotic at infinity to prescribed positive constants. This will require a suitable structure condition on h. In Chapter 5, we will explore various Cauchy-Liouville type theorems. All our results will generalize known results in the literature, and in some cases they provide improvements to already known results. Finally, we have included an Appendix where we collect some useful facts that are used to support our arguments in the corpus of dissertation. Of course, the results stated in the Appendix are well-known and we cite appropriate references. However, the proofs given in these references are usually either given in very general context, rely on previously proved results or are completely left out altogether. In such cases, we have decided to include shorter proofs for completeness and clarity. As we conclude this introductory section, we list the assumptions needed on the function ϕ that appears in the definition of the ϕ-Laplace operator $\Delta_\phi u := \mathrm{div}(\phi(|\nabla u|)\nabla u)$. Let

$$\Psi(t) := t\phi(t) \text{ and } \Phi(t) := \int_0^t \Psi(s)ds, \quad t \geq 0.$$

The following conditions will be used in parts of this work.

(ϕ-1): Ψ is a strictly increasing C^1 function in $\mathbb{R}^+ := (0, \infty)$.

(ϕ-2) : $\lim_{s\to 0^+} \Psi(s) = 0$, and $\lim_{s\to\infty} \Psi(s) = \infty$.

(ϕ-3) : There are constants $0 < \sigma \leq \rho$ such that

$$\sigma \leq \frac{\Phi''(t)t}{\Phi'(t)} \leq \rho, \quad \forall\, t > 0.$$

As a consequence of (ϕ-3), we notice that

$$\lambda(s)\Psi(t) \leq \Psi(st) \leq \Lambda(s)\Psi(t), \quad \forall\, s, t \in \mathbb{R}_0^+ := [0, \infty), \tag{1.1.4}$$

for some increasing functions $\lambda \leq \Lambda$. In fact,

$$\lambda(s) := \min\{s^\sigma, s^\rho\} \quad \text{and} \quad \Lambda(s) := \max\{s^\sigma, s^\rho\}. \tag{1.1.5}$$

This in turn implies the following.

$$\Lambda^{-1}(\varrho)\Psi^{-1}(\tau) \leq \Psi^{-1}(\varrho\tau) \leq \lambda^{-1}(\varrho)\Psi^{-1}(\tau),\ \forall\, \varrho, \tau \in \mathbb{R}_0^+. \tag{1.1.6}$$

As the inequalities (1.1.4) and (1.1.6) will be important in Chapter 2, 3, and 4, we provide a proof below. Let $s, t > 0$. Consider the case $s > 1$ first so that $t < st$. We integrate both sides of the inequality in (ϕ-3) from t to st to obtain

$$\sigma \int_t^{st} \frac{1}{\tau} d\tau \leq \int_t^{st} \frac{\Phi''(\tau)}{\Phi'(\tau)} d\tau \leq \rho \int_t^{st} \frac{1}{\tau} d\tau$$

That is,

$$\ln s^\sigma \leq \ln\left(\frac{\Phi'(st)}{\Phi'(t)}\right) \leq \ln s^\rho.$$

In other words,

$$\Psi(t)s^\sigma \leq \Psi(st) \leq \Psi(t)s^\rho. \tag{1.1.7}$$

If $0 < s \leq 1$, then integrating from st to t leads to

$$\Psi(t)s^{\rho} \leq \Psi(st) \leq \Psi(t)s^{\sigma}. \tag{1.1.8}$$

Combining (1.1.7) and (1.1.8), we find that

$$\min\{s^{\sigma}, s^{\rho}\}\Psi(t) \leq \Psi(st) \leq \max\{s^{\sigma}, s^{\rho}\} \quad \forall\, s, t \geq 0,$$

which proves (1.1.4). One then obtains (1.1.6) from (1.1.4) as follows. In the left inequality of (1.1.4), replace s and t by $\lambda^{-1}(\varrho)$ and $\Psi^{-1}(\tau)$, respectively, to obtain

$$\varrho\tau \leq \Psi(\lambda^{-1}(\varrho)\Psi^{-1}(\tau)), \text{ that is }, \Psi^{-1}(\varrho\tau) \leq \lambda^{-1}(\varrho)\Psi^{-1}(\tau).$$

Similarly, we obtain

$$\Lambda^{-1}(\varrho)\Psi^{-1}(\tau) \leq \Psi^{-1}(\varrho\tau),$$

and this proves (1.1.6). We remark that

$$\lambda^{-1}(t) = \max\{t^{\frac{1}{\sigma}}, t^{\frac{1}{\rho}}\}, \text{ and } \Lambda^{-1}(t) = \min\{t^{\frac{1}{\sigma}}, t^{\frac{1}{\rho}}\}.$$

On multiplying both sides of (1.1.4) by s and then integrating on $(0, t)$ for $t > 0$, we find that

$$s\lambda(s)\int_0^t \Psi(\tau)d\tau \leq \int_0^t \Psi(s\tau)sd\tau \leq s\Lambda(s)\int_0^t \Psi(\tau)d\tau.$$

That is,

$$\widetilde{\lambda}(s)\Phi(t) \leq \Phi(st) \leq \widetilde{\Lambda}(s)\Phi(t), \tag{1.1.9}$$

where

$$\widetilde{\lambda}(s) = \lambda(s)s, \quad \text{and} \quad \widetilde{\Lambda}(s) = \Lambda(s)s.$$

Then it follows that

$$\widetilde{\Lambda}^{-1}(s)\Phi^{-1}(t) \leq \Phi^{-1}(st) \leq \widetilde{\lambda}^{-1}(s)\Phi^{-1}(t). \tag{1.1.10}$$

In the Appendix, we include a remark on comparing Condition (ϕ-3) with other conditions used in the literature. It will be instructive to keep several examples in mind (see [37, 44]).

Example 1.1. (a) $\phi(t) = pt^{p-2}$ for $p > 1$. In this case $\sigma = \rho = p - 1$.
(b) $\phi(t) = pt^{p-2} + qt^{q-2}$ for $1 < p < q$. Here $\sigma = p - 1$ and $\rho = q - 1$.
(c) $\phi(t) = 2p(1 + t^2)^{p-1}$ for $p > \frac{1}{2}$. Then $\sigma = \min\{1, 2p - 1\}$ and $\rho = \max\{1, 2p - 1\}$.
(d) $\phi(t) = p(\sqrt{t^2 + 1} - 1)^{p-1}(t^2 + 1)^{-\frac{1}{2}}$, $p > 1$. Then $\sigma = p - 1$ and $\rho = 2p - 1$
(e) $\phi(t) = pt^{p-2}\log^q(1 + t) + qt^{p-1}(1 + t)^{-1}\log^{q-1}(1 + t)$, for $p > 1, q > 0$. Here $\sigma = p - 1$ and $\rho = p + q - 1$.

When ϕ is as in Example (b) above, Problem (1.1.1) with ϕ appears in quantum physics ([5]) while Problem (1.1.1) models nonlinear elasticity problems ([17]) for the choice of ϕ as in Example (c). With ϕ as in Example (d), problem (1.1.1) is also used to model nonlinear elasticity (see [10] for the case $p = 1$ and [16] when $p > 1$). Finally, Problem (1.1.1) appears in plasticity when ϕ is as in Example (e), and we refer the reader to [17]. Similar applications appear in the papers [7, 14, 15, 39, 40, 42, 45].

1.2 On a Subsolution-Supersolution Theorem

In this section, we will develop the necessary framework for the study of the PDE (1.1.1). Here we introduce the Orlicz and Orilcz-Sobolev spaces associated with the N-function ϕ, study the energy functional associated to it, and establish existence of solutions to Dirichlet problems related to the ϕ-Laplacian on bounded domains. An important tool in our investigation is the subsolution-supersolution method for entire solutions of (1.1.1). Intuitively, given an entire sub-solution v and an entire super-solution w of (1.1.1) such that $v \leq w$, there is an entire solution u such that $v \leq u \leq w$. To establish such result, we need to develop some background work on appropriate function spaces that will set the stage for the necessary argument leading up to the results.
We refer the reader to the appendix for the basic notions necessary to define these function spaces. For more details we suggest the monograph [1].
The assumptions (ϕ-1) and (ϕ-2) show that Φ is an N-function and that Condition (ϕ-3) allows us to conclude that Φ satisfies the Δ_2-condition, namely there is a constant $c > 0$ such that

$$\Phi(2t) \leq c\Phi(t), \quad \forall\, t > 0.$$

In fact this follows immediately from (1.1.9). Therefore, for a given open set $\Omega \subseteq \mathbb{R}^N$, the Orlicz space

$$L^{\Phi}(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is measuarble and } \int_{\Omega} \Phi\left(|u(x)|\right) dx < \infty\}.$$

is a Banach space under the norm (Luxumberg norm)

$$\|u\|_{\Phi} := \inf\{\tau > 0 : \int_{\Omega} \Phi\left(\frac{|u(x)|}{\tau}\right) dx \leq 1\}.$$

Thus $u \in L^{\Phi}(\Omega)$ if and only if $\Phi(|u|) \in L^1(\Omega)$. The definition of $\|u\|_{\Phi}$ shows for any given $u \in L^{\Phi}(\Omega)$ and for $\epsilon > 0$

$$\int_{\Omega} \Phi\left(\frac{|u|}{\|u\|_{\Phi} + \epsilon}\right) dx \geq 1.$$

Therefore,

$$\begin{aligned}\int_\Omega \Phi(|u|)dx &= \int_\Omega \Phi\bigg(\frac{|u|(\|u\|_\Phi+\epsilon)}{\|u\|_\Phi+\epsilon}\bigg)dx\\ &\geq \widetilde{\lambda}(\|u\|_\Phi+\epsilon)\int_\Omega \Phi\bigg(\frac{|u|}{\|u\|_\Phi+\epsilon}\bigg)dx \quad \text{from(1.1.9)}\\ &\geq \widetilde{\lambda}(\|u\|_\Phi+\epsilon).\end{aligned}$$

Letting $\epsilon \to 0$, we find that

$$\widetilde{\lambda}(\|u\|_\Phi) \leq \int_\Omega \Phi(|u|). \tag{1.2.1}$$

On the other hand, for any $u \in L^\Phi(\Omega)$ with $\|u\|_\Phi \neq 0$, we see that

$$\int_\Omega \Phi(|u|) = \int_\Omega \Phi\bigg(\frac{|u|\|u\|_\Phi}{\|u\|_\Phi}\bigg) \leq \widetilde{\Lambda}(\|u\|_\Phi)\int_\Omega \Phi\bigg(\frac{|u|}{\|u\|_\Phi}\bigg) \leq \widetilde{\Lambda}(\|u\|_\Phi).$$

We have used (7.2.5) in the last inequality. This together with (1.2.1) shows that

$$\widetilde{\lambda}(\|u\|_\Phi) \leq \int_\Omega \Phi(|u|) \leq \widetilde{\Lambda}(\|u\|_\Phi), \quad \forall\, u \in L^\Phi(\Omega). \tag{1.2.2}$$

The convexity of Φ, together with Jensen's inequality, implies that $L^\Phi(\Omega) \subseteq L^1(\Omega)$ for bounded $\Omega \subseteq \mathbb{R}^N$. Moreover, $L^\infty(\Omega) \subseteq L^\Phi(\Omega)$ for bounded $\Omega \subseteq \mathbb{R}^N$.

The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is defined as the set of all $u \in L^\Phi(\Omega)$ such that the weak derivatives of $D^\alpha u$ belongs to $L^\Phi(\Omega)$ for all $|\alpha| \leq 1$. Obviously, $u \in W^{1,\Phi}(\Omega)$ if and only if $u \in W^{1,\Phi}(\mathcal{O})$ for every open subset $\mathcal{O} \subseteq \Omega$. The spaces $L^\Phi_{loc}(\Omega)$ and $W^{1,\Phi}_{loc}(\Omega)$ are defined by

$$L^\Phi_{loc}(\Omega) := \{u : u \in L^\Phi(\mathcal{O}),\ \forall\, \mathcal{O} \subset\subset \Omega\} \text{ and } W^{1,\Phi}_{loc}(\Omega) := \{u : u \in W^{1,\Phi}(\mathcal{O}),\ \forall\, \mathcal{O} \subset\subset \Omega\}.$$

We remark that if $\Omega \subseteq \mathbb{R}^N$ is unbounded subset, then $u \in W^{1,\Phi}_{loc}(\Omega)$ if and only if $u \in W^{1,\Phi}(\mathcal{O})$ for every open bounded subset $\mathcal{O} \subseteq \Omega$.

The space $W^{1,\Phi}(\Omega)$ is a Banach space under the norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|u\|_\Phi + \|\nabla u\|_\Phi.$$

As in the case of the usual Sobolev space, the function space $W^{1,\Phi}_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in the Banach space $W^{1,\Phi}(\Omega)$. The following is the analogous version of the Poincaré inequality in the usual Sobolev spaces

$$\|u\|_\Phi \leq C\|\nabla u\|_\Phi, \quad \forall\, u \in W^{1,\Phi}_0(\Omega). \tag{1.2.3}$$

We also note that the dual $(L^\Phi(\Omega))^*$ is $L^{\widetilde{\Phi}}(\Omega)$, where we recall, $\widetilde{\Phi}$ is the complement of Φ as defined in (7.2.3), that is

$$\widetilde{\Phi}(t) = \int_0^t \Psi^{-1}(s)ds, \quad t > 0.$$

Let us suppose that (ϕ-1) holds. Then

$$\begin{aligned} \widetilde{\Phi}(\Psi(t)) &= \int_0^{\Psi(t)} \Psi^{-1}(s)ds \\ &\leq t\Psi(t), \quad \forall\, t \geq 0. \end{aligned} \tag{1.2.4}$$

Moreover,

$$\begin{aligned} \Phi(2t) &= \int_0^{2t} \Psi(s)ds \geq \int_t^{2t} \Psi(s)ds \\ &\geq t\Psi(t), \quad \forall\, t \geq 0. \end{aligned} \tag{1.2.5}$$

Therefore, (1.2.4) and (1.2.5) imply the following holds for all $t \geq 0$.

$$\widetilde{\Phi}(\Psi(t)) \leq \Phi(2t). \tag{1.2.6}$$

If, in addition to (ϕ-1), we also assume that (ϕ-3) holds, we find from (1.1.9) and (1.2.6) the following.

$$\widetilde{\Phi}(\Psi(t)) \leq \widetilde{\Lambda}(2)\Phi(t), \quad \forall\, t \geq 0. \tag{1.2.7}$$

The assumption (ϕ-3) shows that $\widetilde{\Phi}$ satisfies a global Δ_2-condition. In fact, this can be seen easily by integrating the right-hand side of the inequality in (1.1.6). Therefore, according to Theorem 7.12, $W^{1,\Phi}(\Omega)$ is a reflexive Banach space and a sequence $\{u_j\}$ in $W^{1,\Phi}(\Omega)$ converges weakly to $u \in W^{1,\Phi}(\Omega)$ (we use the notation $u_j \rightharpoonup u$ to denote weak convergence) if and only if $u_j \rightharpoonup u$ in $L^{\Phi}(\Omega)$ and $\nabla u_j \rightharpoonup \nabla u$ in the $(L^{\Phi}(\Omega))^N$.
Given $k \in L^{\infty}(\Omega \times \mathbb{R})$, we consider the equation

$$\operatorname{div}(\phi(|\nabla u|)\nabla u) = k(x,u) \quad \text{in} \quad \Omega. \tag{1.2.8}$$

With Equation (1.2.8), we associate an energy functional $J : W^{1,\Phi}(\Omega) \to \mathbb{R} \cup \{\pm\infty\}$ as follows:

$$J(u) := \int_\Omega \Phi(|\nabla u|) + \int_\Omega K(x,u)dx \tag{1.2.9}$$

where

$$K(x,t) = \int_0^t k(x,s)ds.$$

Let us first show that J is weakly lower semi-continuous. For this, let $\{v_j\}$ be a sequence in $W^{1,\Phi}(\Omega)$ such that $v_j \rightharpoonup v$ for some $v \in W^{1,\Phi}(\Omega)$. We need to show that

$$J(v) \leq \liminf_{j\to\infty} J(v_j).$$

Let

$$\beta := \liminf_{j\to\infty} \int_\Omega \Phi(|\nabla v_j|)dx.$$

There is a subsequence of $\{v_j\}$ which we continue to denote by $\{v_j\}$ such that

$$\lim_{j\to\infty}\int_\Omega \Phi(|\nabla v_j|)dx = \beta. \tag{1.2.10}$$

By Mazur's Theorem (Theorem 7.3) there is a function $N : \mathbb{N} \to \mathbb{N}$ and corresponding to each $n \in \mathbb{N}$ there are non-negative real numbers $\gamma_k(n)$ for $k = n, \cdots, N(n)$ with $\sum_{k=n}^{N(n)} \gamma_k(n) = 1$ and such that the sequence $\{w_n\}$ defined by

$$w_n := \sum_{k=n}^{N(n)} \gamma_k(n) v_k$$

converges (strongly) to v in $W^{1,\Phi}(\Omega)$. In particular $\{w_n\}$ contains a subsequence, which we still denote by $\{w_n\}$ such that $|\nabla w_n - \nabla v| \to 0$ a.e. in Ω and hence $|\nabla w_n| \to |\nabla v|$ a.e. in Ω. Clearly, we have

$$|\nabla w_n| \leq \sum_{k=n}^{N(n)} \gamma_k(n)|\nabla v_k|.$$

Therefore, the convexity of Φ together with Fatou's lemma shows that

$$\begin{aligned} \int_\Omega \Phi(|\nabla v|) &\leq \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla w_n|) \\ &\leq \liminf_{n\to\infty} \sum_{k=n}^{N(n)} \gamma_k(n) \int_\Omega \Phi(|\nabla v_k|)dx \end{aligned} \tag{1.2.11}$$

Let $\epsilon > 0$ be given. Then from (1.2.10) there is a positive integer N_0 such that

$$\beta - \epsilon < \int_\Omega \Phi(|\nabla v_k|)dx < \beta + \epsilon, \quad \forall\, k \geq N_0. \tag{1.2.12}$$

Therefore, for $n \geq N_0$, after multiplying (1.2.12) by $\gamma_k(n)$ and
adding over $k = n, \cdots, N(n)$, we find that

$$\beta - \epsilon \leq \sum_{k=n}^{N(n)} \gamma_k(n) \int_\Omega \Phi(|\nabla v_k|)dx \leq \beta + \epsilon.$$

This shows that

$$\lim_{n\to\infty} \sum_{k=n}^{N(n)} \gamma_k(n) \int_\Omega \Phi(|\nabla v_k|)dx = \beta.$$

Using this in (1.2.11) we find that

$$\int_\Omega \Phi(|\nabla v|) \leq \beta = \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla v_n|)dx. \tag{1.2.13}$$

To proceed as in the above, let us set

$$c := \liminf_{j\to\infty} \int_\Omega K(x, v_j)dx.$$

In fact, let us pick a subsequence of $\{v_j\}$, still denoted by $\{v_j\}$ such that

$$c = \lim_{j\to\infty} \int_\Omega K(x, v_j)dx.$$

Since $v_j \rightharpoonup v$, we see that $\{v_j\}$ is bounded in $W^{1,\Phi}(\Omega)$. By Remark (7.11), we recall that $W^{1,\Phi}(\Omega) \subset\subset L^1(\Omega)$. Therefore,$\{v_j\}$ has a subsequence, which we continue to denote by $\{v_j\}$, such that $v_j \to v$ in $L^1(\Omega)$. Since $k \in L^\infty(\Omega\times\mathbb{R})$, we note that $|K(x,t)-K(x,s)| \leq \beta|t-s|$ for all $s, t \in \mathbb{R}$. Therefore, by the Generalized Hölder Inequality

$$\begin{aligned} \left| \int_\Omega K(x, v_j) - \int_\Omega K(x, v) \right| &\leq \int_\Omega |K(x, v_j) - K(x, v)|dx \\ &\leq \beta \int_\Omega |v_j - v|dx. \end{aligned}$$

Thus, we conclude

$$\int_\Omega K(x, v)dx = \lim_{j\to\infty} \int_\Omega K(x, v_j)dx = c.$$

Consequently,

$$\int_\Omega K(x, v)dx \leq \liminf_{j\to\infty} \int_\Omega K(x, v_j)dx. \tag{1.2.14}$$

Therefore, from (1.2.13) and (1.2.14), we conclude that

$$\begin{aligned} J(v) &= \int_\Omega \Phi(|\nabla v|) + \int_\Omega K(x, v)dx \\ &\leq \liminf_{j\to\infty} \int_\Omega \Phi(|\nabla v_j|) + \liminf_{j\to\infty} \int_\Omega K(x, v_j(x))dx \\ &\leq \liminf_{j\to\infty} J(v_j). \end{aligned}$$

We now show that J is Gâteaux differentiable on $W^{1,\Phi}(\Omega)$. Let $v, w \in W^{1.\Phi}(\Omega)$. Note that for $t \neq 0$ (in fact it is no loss of generality in supposing that $0 < t \leq 1$).

$$\begin{aligned} &\frac{J(v + tw) - J(v)}{t} \\ &= \frac{1}{t} \int_\Omega (\Phi(|\nabla(v + tw)|) - \Phi(|\nabla v|))dx + \frac{1}{t} \int_\Omega (K(x, v + tw) - K(x, v))dx. \end{aligned} \tag{1.2.15}$$

Recalling that $k \in L^\infty(\Omega \times \mathbb{R})$ and that $w \in L^1(\Omega)$, by Remark (7.10), we invoke the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{t\to 0} \frac{1}{t} \int_\Omega (K(x, v + tw) - K(x, v))dx = \int_\Omega k(x, v)wdx \tag{1.2.16}$$

Next, we focus on the limit involving the first integral in (1.2.15). Let us first observe that

$$\frac{1}{t} \int_\Omega (\Phi(|\nabla(v + tw)|) - \Phi(|\nabla v|))dx = \int_\Omega \frac{1}{t} \left(\int_{|\nabla v|}^{|\nabla v + t\nabla w|} \Psi(s)ds \right) dx$$

Recalling $0 < t \le 1$, we easily estimate

$$\left| \frac{1}{t} \int_{|\nabla v|}^{|\nabla v + t\nabla w|} \Psi(s)ds \right| \le \Psi(|\nabla v| + |\nabla w|)|\nabla w|.$$

Moreover, recalling (1.2.7), we find that

$$\widetilde{\Phi}(\Psi(|\nabla v| + |\nabla w|)) \le \widetilde{\Lambda}(2)\Phi(|\nabla v| + |\nabla w|). \tag{1.2.17}$$

From this, we see that $\Psi(|\nabla v|+|\nabla w|) \in L^{\widetilde{\Phi}}(\Omega)$. Since $|\nabla w| \in L^{\Phi}(\Omega)$, by the Generalized Hölder Inequality, Theorem 7.9, we conclude that $\Psi(|\nabla v|+|\nabla w|)|\nabla w|$ belongs to $L^1(\Omega)$. Therefore by the Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t} \int_\Omega (\Phi(|\nabla(v+tw)|) - \Phi(|\nabla v|))dx &= \lim_{t\to 0^+} \int_\Omega \left(\frac{1}{t} \int_{|\nabla v|}^{|\nabla v + t\nabla w|} \Psi(s)ds \right) dx \\ &= \int_\Omega \phi(|\nabla v|)\nabla v \cdot \nabla w dx. \end{aligned} \tag{1.2.18}$$

The last limit is a consequence of

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t} \int_{|\xi|}^{|\xi + t\zeta|} \Psi(s)ds &= \lim_{t\to 0^+} \Psi(|\xi + t\zeta|) \frac{(\xi + t\zeta)\cdot\zeta}{|\xi + t\zeta|} \\ &= \lim_{t\to 0^+} \phi(|\xi + t\zeta|)(\xi + t\zeta)\cdot\zeta \\ &= \phi(|\xi|)\xi\cdot\zeta \quad \forall\, \xi, \zeta \in \mathbb{R}^N \quad \xi \ne 0. \end{aligned}$$

Obviously the statement is true when $t = 0$.

Therefore, from (1.2.15), (1.2.16) and (1.2.18), we conclude that J is Gâteaux differentiable at v and that

$$J'(v; w) = \int_\Omega \phi(|\nabla v|)\nabla v \cdot \nabla w dx + \int_\Omega k(x, v)w dx, \quad \forall\, w \in W^{1,\Phi}(\Omega). \tag{1.2.19}$$

The Hölder Inequality shows that map $w \mapsto J'(v; w)$ is a continuous linear functional on $W^{1,\Phi}(\Omega)$, denoted by $J'(v)$.

Now let $g \in W^{1,\Phi}(\Omega)$, and consider the set

$$\mathcal{A} := \{u \in W^{1,\Phi}(\Omega) : u \in g + W_0^{1,\Phi}(\Omega)\}.$$

We show that there is $u \in \mathcal{A}$ such that

$$J(u) = \min\{J(w) : w \in \mathcal{A}\}.$$

To see this, let

$$m := \min\{J(w) : w \in \mathcal{A}\}.$$

Note that $g \in \mathcal{A}$ and $-\infty < J(g) < \infty$. Therefore, $-\infty \le m < \infty$. First we claim that $m > -\infty$. If not, we can find a sequence $\{u_j\}$ in $\mathcal{A}$ such that $J(u_j) \to -\infty$. If $\{u_j\}$ is a bounded sequence in $W^{1,\Phi}(\Omega)$, then it has a subsequence, which we still denote by $\{u_j\}$, such that $u_j \rightharpoonup v$ for some $v \in W^{1,\Phi}(\Omega)$. Since J is weakly lower semi-continuous we have

$$J(v) \leq \liminf_{j\to\infty} J(u_j) = -\infty.$$

This is a contradiction. So $\{u_j\}$ must be unbounded in $W^{1.\Phi}(\Omega)$. Let us put $w_j := u_j - g$. As a consequence of this, and Poincaŕe inequality, we note that $\{\|\nabla w_j\|_\Phi\}$ is unbounded, also. We observe that, since $|K(x,t)| \leq C_1|t|$ on $\Omega \times \mathbb{R}$

$$\begin{aligned} J(u_j) &= \int_\Omega \Phi(|\nabla u_j|) + \int_\Omega K(x,u_j) \geq \widetilde{\lambda}(\|\nabla u_j\|_\Phi) - C_1 \int_\Omega |u_j| \text{ by (1.2.2)} \\ &\geq \widetilde{\lambda}(\|\nabla u_j\|_\Phi) - C_2\|u_j\|_\Phi \\ &\geq \widetilde{\lambda}(|\|\nabla w_j\|_\Phi - \|\nabla g\|_\Phi|) - C_2\|w_j\|_\Phi - C_2\|g\|_\Phi \\ &\geq \widetilde{\lambda}(|\|\nabla w_j\|_\Phi - \|\nabla g\|_\Phi|) - C_3\|\nabla w_j\|_\Phi - C_3\|\nabla g\|_\Phi, \text{ by Poincaré inequality..} \end{aligned}$$

Therefore, for sufficiently large j, we see that

$$\begin{aligned} J(u_j) &\geq |\|\nabla w_j\|_\Phi - \|\nabla g\|_\Phi|^{\sigma+1} - C_3\|\nabla w_j\|_\Phi - C_3\|\nabla g\|_\Phi \\ &= \|\nabla w_j\|_\Phi^{\sigma+1}\left[\left(\left|1 - \frac{\|\nabla g\|_\Phi}{\|\nabla w_j\|_\Phi}\right|\right)^{\sigma+1} - \frac{C}{\|\nabla w_j\|_\Phi^\sigma} - \frac{C\|\nabla g\|_\Phi}{\|\nabla w_j\|_\Phi^{\sigma+1}}\right]. \end{aligned} \tag{1.2.20}$$

Since $\sigma > 0$, it follows that $J(u_j) \to \infty$ as $j \to \infty$. Contradicting the fact that $J(u_j) \to -\infty$. Therefore, we have $m > -\infty$ as claimed.

Suppose again $\{u_j\}$ is a sequence in $W^{1,\Phi}(\Omega)$ such that $J(u_j) \to m$, so that $\{J(u_j)\}$ is bounded. We claim that $\{u_j\}$ is a bounded sequence in $W^{1,\Phi}(\Omega)$. Otherwise, it contains a subsequence, which for convenience we still denote by $\{u_j\}$ such that $\|u_j\|_{W^{1,\Phi}(\Omega)} \to \infty$. Then $\|w_j\|_{W^{1,\Phi}(\Omega)} \to \infty$. But, the inequality (1.2.20) shows that $J(u_j) \to \infty$, in contradiction with the fact that $J(u_j) \to m < \infty$. Therefore, indeed $\{u_j\}$ is bounded in $W^{1,\Phi}(\Omega)$ and hence has a subsequence, still denoted by $\{u_j\}$ that converges weakly to $u \in W^{1,\Phi}(\Omega)$. We show that $u \in \mathcal{A}$. To see this, note that $w_j \in W_0^{1,\Phi}(\Omega)$. Since $W_0^{1,\Phi}(\Omega)$ is a weakly closed linear subspace of $W^{1,\Phi}(\Omega)$, by Mazur's theorem (Theorem 7.2), $W_0^{1,\Phi}(\Omega)$ is weakly closed subspace. Since $w_j \rightharpoonup u - g$, it follows that $u - g \in W_0^{1,\Phi}(\Omega)$. Therefore $u \in \mathcal{A}$, as claimed. Since J is weakly lower semi-continuous, we have

$$J(u) \leq \liminf_{j\to\infty} J(u_j) = m.$$

Therefore $u \in \mathcal{A}$ and $J(u) = m$.

By (1.2.7) and (1.2.2), respectively, we have

$$\int_\Omega \widetilde{\Phi}(\Psi(|\nabla u|))dx \leq \widetilde{\Lambda}(2)\int_\Omega \Phi(|\nabla u|) \leq \widetilde{\Lambda}(2)\tilde{\Lambda}(\|\nabla u\|_\Phi) := \kappa < \infty \tag{1.2.21}$$

for all $u \in W^{1,\Phi}(\Omega)$. Therefore the convexity of $\widetilde{\Phi}$ together (1.2.21) shows that

$$\int_\Omega \widetilde{\Phi}\left(\frac{\phi(|\nabla u|)|\nabla u|}{\max\{1,\kappa\}}\right) \leq \frac{1}{\max\{1,\kappa\}}\int_\Omega \widetilde{\Phi}(\Psi(|\nabla u|))dx \leq 1.$$

That is,

$$\|\Psi(|\nabla u|)\|_{\widetilde{\Phi}} \leq \max\{1,\kappa\} < \infty.$$

By the Generalized Hölder Inequality, Theorem 7.9, it follows that

$$\begin{aligned} \left|\int_\Omega \phi(|\nabla u|)\nabla u\cdot\nabla\varphi\right| &\leq \int_\Omega \phi(|\nabla u|)|\nabla u||\nabla\varphi| = \int_\Omega \Psi(|\nabla u|)|\nabla\varphi| \quad (1.2.22)\\ &\leq 2\|\Psi(|\nabla u|)\|_{\widetilde{\Phi}}\|\nabla\varphi\|_\Phi \\ &\leq 2\max\{1,k\}\|\varphi\|_\Phi. \quad (1.2.23)\end{aligned}$$

We are now ready to introduce the notion of solution to the PDE

$$\Delta_\phi u = g(x,u), \quad x\in\Omega \tag{1.2.24}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set and $g : \Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function. A weakly differentiable function $v : \Omega \to \mathbb{R}$ is said to be a sub-solution of (1.2.24) in Ω if and only if for any open and bounded subset $\mathcal{O} \subseteq \Omega$, we have $v \in W^{1,\Phi}(\mathcal{O})$ with $g(x, v(x)) \in L^{\widetilde{\Phi}}(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \phi(|\nabla v|)\nabla v\cdot\nabla\varphi \leq -\int_{\mathcal{O}} g(x,v)\varphi, \quad \forall\, 0 \leq \varphi \in W_0^{1,\Phi}(\mathcal{O}). \tag{1.2.25}$$

A weakly differentiable function $w : \Omega \to \mathbb{R}$ is said to be a super-solution of (1.2.24) in Ω if and only if for every open and bounded subset $\mathcal{O} \subseteq \Omega$, we have $w \in W^{1,\Phi}(\mathcal{O})$ with $g(x, w(x)) \in L^{\widetilde{\Phi}}(\mathcal{O})$ such that the reverse inequality holds in (1.2.25) for all non-negative $\varphi \in W_0^{1,\Phi}(\mathcal{O})$. A weakly differentiable function $u : \Omega \to \mathbb{R}$ is said to be a solution of (1.2.24) in Ω if u is both a sub-solution and a super-solution of (1.2.24) in Ω.
We follow common practice and write

$$\Delta_\phi v \geq g(x,v) \quad \text{in}\,\Omega \quad \text{and} \quad \Delta_\phi w \leq g(x,w) \quad \text{in} \quad \Omega$$

to indicate the fact that v is a sub-solution and w is a super-solution of (1.2.24), respectively in Ω.
Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and $k \in W^{1,\Phi}(\Omega)$. We consider the following boundary value problem.

$$\begin{cases} \Delta_\phi u = g(x,u) & \text{in} \quad \Omega \\ \qquad u = k & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2.26}$$

For the boundary value problem (1.2.26), we say $v \in W^{1,\Phi}(\Omega)$ is a sub-solution of (1.2.26) if $(v-k)^+ \in W_0^{1,\Phi}(\Omega)$ and v is a sub-solution of (1.2.24). We say $w \in W^{1,\Phi}(\Omega)$ is a super-solution of (1.2.26) if $(w-k)^- \in W_0^{1,\Phi}(\Omega)$ and w is a super-solution of (1.2.24). We say $u \in W^{1,\Phi}(\Omega)$ is a solution of (1.2.26) if u is a sub-solution and a super-solution of (1.2.26).

Remark 1.2. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, and suppose $\varphi \in W_0^{1,\Phi}(\Omega)$ and $\{\varphi_j\}$ is a sequence in $C_0^\infty(\Omega)$ such that $\varphi_j \to \varphi$ in $W^{1,\Phi}(\Omega)$. Then for any $u \in W^{1,\Phi}(\Omega)$

$$\lim_{j\to\infty} \int_\Omega \langle \phi(|\nabla u|)\nabla u, \nabla\varphi_j\rangle = \int_\Omega \langle \phi(|\nabla u|)\nabla u, \nabla\varphi\rangle .$$

This follows from (1.2.23). If u is a measurable function in Ω such that $h(x,t) \in L^{\widetilde{\Phi}}(\Omega)$, then we also have

$$\lim_{j\to\infty} \int_\Omega h(x,u)\varphi_j = \int_\Omega h(x,u)\varphi.$$

Again, this follows from the Generalized Hölder inequality.

The following lemma will be useful in the sequel. Let us note that $W^{1,\infty}_{loc}(\mathbb{R}^N) \subseteq W^{1,\Phi}_{loc}(\mathbb{R}^N)$.

Lemma 1.3. Let $N > 1$ and $u \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ be a sub-solution (resp., super-solution) of (1.2.24) in $\mathbb{R}^N \setminus \{0\}$. Then u is a sub-solution (resp.,super-solution) of (1.2.24) in $\mathbb{R}^N$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be non-negative and $\vartheta \in C_c^\infty(B(0,2))$ such that $0 \le \vartheta \le 1$ with $\vartheta \equiv 1$ on $B(0,1)$. For each positive integer j, let $\vartheta_j(x) := \vartheta(jx)$. Since u is a sub-solution (resp., super-solution) of (1.2.24) in $\mathbb{R}^N \setminus \{0\}$ and $(1-\vartheta)\varphi \in C_c^\infty(B(0,2)) \setminus \{0\})$ we have

$$\begin{aligned} \int \phi(|\nabla u|)\nabla u \cdot \nabla\varphi \quad &= \quad \int \phi(|\nabla u|)\nabla u \cdot \nabla[(1-\vartheta_j)\varphi] + \int \phi(|\nabla u|)\nabla u \cdot \nabla(\vartheta_j\varphi) \\ &\le (\ge) \quad -\int g(x,u)(1-\vartheta_j)\varphi + \int \phi(|\nabla u|)\nabla u \cdot \nabla(\vartheta_j\varphi) \\ &= \quad E_j + F_j. \end{aligned}$$

Since $g(x,u)\varphi \in L^{\widetilde{\Phi}}_{loc}(\mathbb{R}^N)$ and $1-\vartheta_j \to 1$ a.e. in $\mathbb{R}^N$, it follows by the Dominated Convergence theorem that

$$E_j \to -\int g(x,u)\varphi.$$

On the other hand,

$$\begin{aligned} |F_j| \quad &\le \quad \int \Psi(|\nabla u|)|\nabla\vartheta_j||\varphi| + \int \Psi(|\nabla u|)|\nabla\varphi|\vartheta_j \\ &\le \quad Cj^{1-N}\int |\nabla\vartheta|\varphi + C\int |\nabla\varphi|\vartheta_j. \end{aligned}$$

Again, by the Dominated Convergence theorem, we see that $F_j \to 0$ as $j \to \infty$. □

For the next lemma, we suppose that $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous.

Lemma 1.4. Suppose $z \in C^1((0,\infty)) \cap W^{1,\infty}((0,\infty))$ is a distributional solution of

$$(r^{N-1}\phi(|z'|)z')' = (resp., \leq, \geq) r^{N-1} g(r,z), \quad r > 0. \tag{1.2.27}$$

Then $u(x) = z(|x|)$ satisfies $\triangle_\phi u = (resp., \leq, \geq) g(|x|, u)$ in $\mathbb{R}^N$.

Proof. It is clear that $\nabla u(x) = z'(|x|)\frac{x}{|x|}$ for $x \neq 0$. Therefore

$$\phi(|\nabla u|)\nabla u = \phi(|z'(|x|)|)z'(|x|)\frac{x}{|x|} = -\Psi(|z'(|x|)|)\frac{x}{|x|}, \quad x \neq 0.$$

For an open set $\mathcal{O} \subseteq \mathbb{R}^N$, let us note that if $v, w \in C^1(\mathcal{O})$ with v harmonic on $\mathcal{O}$, then we have the identity

$$\nabla v \cdot \nabla w = \operatorname{div}(w \nabla v) \quad \text{in} \quad \mathcal{O}. \tag{1.2.28}$$

Now, let $\varphi \in C_c^1(\mathbb{R}^N \setminus \{0\})$ be non-negative. Then on $\mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned}
\phi(|\nabla u|)\nabla u \cdot \nabla \varphi &= -\Psi(|z'|)\frac{x}{|x|} \cdot \nabla\varphi \\
&= -|x|^{N-1}\Psi(|z'|)\frac{x}{|x|^N} \cdot \nabla\varphi \\
&= -|x|^{N-1}\Psi(|z'|)\nabla\left(\frac{1}{2-N}|x|^{2-N}\right) \cdot \nabla\varphi \\
&= -|x|^{N-1}\Psi(|z'|)\operatorname{div}\left(\varphi\nabla\left(\frac{1}{2-N}|x|^{2-N}\right)\right) \quad \text{by (1.2.28)}.
\end{aligned}$$

Since $\operatorname{supp}(\varphi)$ is a compact subset of $\mathbb{R}^N \setminus \{0\}$, let $\mathcal{O} \subseteq \mathbb{R}^N \setminus \{0\}$ be an open subset with C^1 boundary such that $\operatorname{supp}(\varphi) \subseteq \mathcal{O}$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \cdot \nabla\varphi dx &= \int_{\mathcal{O}} \phi(|\nabla u|)\nabla u \cdot \nabla\varphi dx \\
&= -\int_{\mathcal{O}} |x|^{N-1}\Psi(|z'|)\operatorname{div}\left(\varphi\nabla\left(\frac{1}{2-N}|x|^{2-N}\right)\right) dx \\
&= \sum_{i=1}^N \int_{\mathcal{O}} (|x|^{N-1}\Psi(|z'|))' \frac{x_i}{|x|} \cdot \frac{x_i}{|x|^N}\varphi dx \\
&= \int_{\mathcal{O}} (|x|^{N-1}\Psi(|z'|))' \frac{1}{|x|^{N-1}}\varphi dx \\
&= (\text{resp.}, \geq, \leq) - \int_{\mathcal{O}} |x|^{N-1} g(|x|, z(|x|)) \frac{1}{|x|^{N-1}}\varphi dx, \text{ from (1.2.27)} \\
&= (\text{resp.}, \geq, \leq) - \int_{\mathbb{R}^N} g(|x|, u)\varphi dx.
\end{aligned}$$

Therefore, $u \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ is a solution (resp., super-solution, sub-solution) of (1.2.27) in $\mathbb{R}^N \setminus \{0\}$. By Lemma 1.3, we conclude that $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ is a solution (resp., super-solution, sub-solution) of (1.2.27) in $\mathbb{R}^N$ as claimed. ∎

Lemma 1.5. Assume that conditions (ϕ-1), (ϕ-2,)and (ϕ-3) hold. Suppose $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $k \in W^{1,\Phi}(\Omega)$ and $g \in C(\bar{\Omega} \times \mathbb{R})$. Let $w, v \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ such that v is a sub-solution and w is a super-solution of Problem (1.2.26). Then Problem (1.2.26) has a solution $u \in W^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ such that $v \le u \le w$ a.e. in Ω. Moreover, g is in $L^\infty(\Omega \times \mathbb{R})$, then $u \in C^{1,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$.

This Lemma can be found in [37] when g is a constant. The proof of the lemma for general k follows along the same lines. For completeness we supply a proof.

Proof. Let us introduce a nonlinearity as follows.

$$z(x,t) = \begin{cases} g(x, v(x)) & \text{if } t \le v(x) \\ g(x,t) & \text{if } v(x) \le t \le w(x) \\ g(x, w(x)) & \text{if } t \ge w(x) \end{cases}$$

We note that $z \in C(\overline{\Omega} \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$. We use the energy functional

$$J(u) := \int_\Omega \Phi(|\nabla u|) + \int_\Omega Z(x,u)dx, \quad u \in \mathcal{A},$$

where, for $(x,t) \in \Omega \times \mathbb{R}$, $Z(x,t) = \int_0^t z(x,s)ds$. Let

$$\mathcal{A} := \{u \in W^{1,\Phi}(\Omega) : u - k \in W_0^{1,\Phi}(\Omega)\}.$$

As we have seen already, J is weakly lower semicontinuous functional on $W^{1,\Phi}(\Omega)$ that attains a minimum at some $u \in \mathcal{A}$. Then u is a critical point of the functional point, and hence u satisfies the Euler-Lagrange equation $\operatorname{div}(\phi(|\nabla u|)\nabla u) = z(x,u)$. It remains to show that $v \le u \le w$ in Ω.

Let us note that

$$0 \le (u-w)^+ = (u - k + k - w)^+ \le (u-k)^+ + (k-w)^+.$$

Since $u - k \in W_0^{1,\Phi}(\Omega)$, it follows from Lemma 7.15 that $(u-k)^+ \in W_0^{1,\Phi}(\Omega)$. Moreover, by definition, we note that $(k-w)^+ \in W_0^{1,\Phi}(\Omega)$. Therefore, we invoke Lemma 7.18 to conclude that $(u-w)^+ \in W_0^{1,\Phi}(\Omega)$. See the Appendix for justification of these assertions. We use $(u-w)^+$ as test function and we get

$$\begin{aligned} \int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla (u-w)^+ &= -\int_\Omega z(x,u)(u-w)^+ \\ &= -\int_\Omega z(x,w)(u-w)^+ \\ &\le \int_\Omega \phi(|\nabla w|)\nabla w \cdot \nabla (u-w)^+ \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\int_{u>w} \langle \phi(|\nabla u|)\nabla u - \phi(|\nabla w|)\nabla w, \nabla(u-w)\rangle \\
(1.2.29) \qquad &\leq \int_{\Omega} \left\langle \phi(|\nabla u|)\nabla u - \phi(|\nabla w|)\nabla w, \nabla(u-w)^{+}\right\rangle \leq 0.
\end{aligned}
$$

But, according to [35, Theorem 2.4.1], we have

$$\langle \phi(|\xi|)\xi - \phi(|\zeta|)\zeta, \xi - \zeta\rangle > 0,\ \forall\ \xi \neq \zeta \text{ in } \mathbb{R}^N. \tag{1.2.30}$$

As a consequence of this and (1.2.29), we conclude that $\nabla(u-w)^{+} = 0$ a.e. in Ω. Since $(u-w)^{+} \in W_0^{1,\Phi}(\Omega)$, it follows, Lemma 7.14, that $(u-w)^{+} = 0$ a.e. in Ω. Consequently, $u \leq w$ a.e. in Ω. Similarly, one can show that $u \geq v$ a.e. in Ω. Therefore $z(x,u) = g(x,u)$. Recalling that u is a solution of $\mathrm{div}(\phi(|\nabla u|)\nabla u) = z(x,u)$, we obtain the desired result. □

Finally we have the following theorem on sub-solution and super-solution method for entire solutions.

Theorem 1.6. Assume that conditions (ϕ-1), (ϕ-2) and (ϕ-3) hold. Let $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $v, w \in W^{1,\Phi}(\mathbb{R}^N) \cap L^{\infty}_{loc}(\mathbb{R}^N)$ are weak solutions of

$$\Delta_\phi v \geq g(x,v) \quad \text{and} \quad \Delta_\phi w \leq g(x,w)$$

in $\mathbb{R}^N$. If $v \leq w$ a.e. in $\mathbb{R}^N$, there is an entire solution $u \in C^1(\mathbb{R}^N)$ of

$$\Delta_\phi u = g(x,u) \tag{1.2.31}$$

in $\mathbb{R}^N$ such that $v \leq u \leq w$ a.e. in $\mathbb{R}^N$.

Proof. For each positive integer j, let $B_j := B(0,j)$ be the ball centered at the origin $0 \in \mathbb{R}^N$ and of radius j. Fix $k \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ such that $v \leq k \leq w$ a.e. in $\mathbb{R}^N$. For each positive integer j, we invoke Lemma 1.5 to find a solution $z_j \in W^{1,\Phi}(B_j)$ of

$$\begin{cases} \Delta_\phi u = g(x,u) & \text{in} \quad B_j \\ \quad u = k & \text{on} \quad \partial B_j. \end{cases} . \tag{1.2.32}$$

such that $v \leq z_j \leq w$ for a.e. on B_j. Note that $g_j(x) := g(x, z_j(x))$ is in $L^{\infty}(B_j)$ and

$$\|g_j\|_{L^{\infty}(B_j)} \leq M_j := \max\{|g(x,t)| : (x,t) \in \overline{B_j} \times [I_j, S_j]\}$$

where

$$I_j := \inf_{B_j} v, \quad \text{and} \quad S_j := \sup_{B_j} w.$$

Therefore, by [16, Lemma 3.3] we note that $z_j \in C^{1,\alpha}(\overline{B_j})$ for some $0 < \alpha < 1$, depending on σ and ρ, and that

$$\|z_j\|_{C^{1,\alpha}(\overline{B}_j)} \leq M_j.$$

Let $u_j \in C_0^{1,\alpha}(\mathbb{R}^N)$ be the extension of $z_j \in C^{1,\alpha}(\overline{B_j})$ such that

$$\|u_j\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq \|z_j\|_{C^{1,\alpha}(\overline{B_j})}.$$

We refer to [18, Lemma 6.37] for the existence of such extensions. Given a positive integer m, we note that for $j \geq m$,

$$\|u_j\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq M_m, \ \forall\, j \geq m.$$

We invoke the Arzelá-Ascoli's theorem to draw a subsequence that converges in $C^1(\mathbb{R}^N)$. The Dominated Convergence theorem shows that the limit is a solution of (1.2.31) in B_m. By a diagonal argument we find a subsequence $\{u_{n_j}\}$ of $\{u_j\}$ that converges to some $u \in C^1(\mathbb{R}^N)$ such that $v \leq u \leq w$ a.e. in $\mathbb{R}^n$, and is a solution of (1.2.31) in $\mathbb{R}^N$. $\square$

Chapter 2

Bounded Entire Solutions

2.1 Infinitely Many Positive Bounded Solutions

In this chapter, we study the existence of infinitely many bounded solutions to (1.1.1) under the assumption that $h : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and satisfies

$$\text{(h-1):} \qquad |h(x,t)| \leq b(|x|)f(t), \quad \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R}^+$$

for some continuous function $b : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and monotone function $f : \mathbb{R} \to \mathbb{R}$ which satisfy some appropriate conditions that will be described below. Condition (h-1) will be assumed throughout Chapter 2 without further mention.

Moreover, we will require b is a radial function that decays at infinity at a rate dictated by the following condition.

$$\text{(b-1):} \quad B := \int_0^\infty \Psi^{-1}\left(t^{1-N} \int_0^t s^{N-1} b(s)ds\right)dt < \infty.$$

Let $\sigma > 0$ be the parameter in Condition (ϕ-3). We remark that $N > \sigma + 1$ is necessary for the condition (b-1) to hold when $b \geq 0$ and $b \neq 0$. To see this, suppose $\sigma \geq N-1$ and that $b(t_0) > 0$ for some $t_0 \geq 0$. We show then that Condition (b-1) can not hold. Indeed, let $\tau > t_0$ such that $b(\tau) > 0$, and set

$$0 < c := \int_0^\tau s^{N-1} b(s)ds.$$

Then for $t > \max\{1, \tau\}$ we have

$$\frac{1}{t^{N-1}} \int_0^t s^{N-1} b(s)ds \geq \frac{1}{t^{N-1}} \int_0^\tau s^{N-1} b(c)ds = \frac{c}{t^{N-1}}.$$

Therefore, for $t > \max\{1, \tau\}$,

$$\begin{aligned} \Psi^{-1}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} b(s)ds\right) &\geq \Psi^{-1}\left(\frac{c}{N-1}\right) \\ &\geq \Psi^{-1}(c)\Lambda^{-1}\left(\frac{1}{t^{N-1}}\right) \\ &= \Psi^{-1}(c)t^{-(N-1)/\sigma}. \end{aligned}$$

Recalling that $(N-1)\sigma^{-1} \leq 1$, we conclude that Condition (b-1) cannot hold. Therefore, in any subsequent discussion where Condition (b-1) is needed, we will always assume that $N > \sigma + 1$. For instance, $N > 2$ when $\phi(t) = 1$.

As an example, let us note that

$$b(s) := \frac{1}{(s^N + 1)^m}, \quad s \geq 0$$

satisfies Condition (b-1) for any $m \geq 1$. To see this it is enough to consider $m = 1$ only. So assuming $m = 1$, and fix $0 < \theta < \frac{N-1-\sigma}{N}$. Such a choice is possible since $\sigma > N - 1$. Then for sufficiently large $t_0 > 1$ and $t > t_0$, we have

$$\begin{aligned} \Psi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t s^{N-1}b(s)ds\right) &= \Psi^{-1}\left(\frac{1}{N}\frac{\log(t^N+1)}{t^{N-1}}\right) \leq \Psi^{-1}\left(\frac{t^{\theta N}}{t^{N-1}}\right) \\ &= \frac{1}{t^{(N-1-\theta N)/\sigma}}. \end{aligned}$$

On the other hand, since $0 \leq b(s) \leq 1$ for $s \geq 0$, we have for $0 \leq t \leq t_0$,

$$\Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right) \leq \Psi^{-1}\left(\int_0^t b(s)ds\right) \leq \Psi^{-1}(t_0).$$

Therefore, since $\frac{N-1-\theta N}{\sigma} > 1$, we have

$$\begin{aligned} &\int_0^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt \\ &= \int_0^{t_0} \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt + \int_{t_0}^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt \\ &\leq \Psi^{-1}(t_0) + \int_{t_0}^\infty t^{-(N-1-\theta N)/\sigma}dt < \infty. \end{aligned}$$

Let us see how Condition (1.1.3) used in N. Kawano's paper, [19], compares with Condition (b-1) when the Principal part of Equation (1.1.1) is the Laplacian, that is when $\phi \equiv 1$ in (1.1.1). In this case, we see that $\Psi(t) = t$ for $t > 0$ and Condition (b-1) reduces to

$$\int_0^r \left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt < \infty \tag{2.1.1}$$

Furthermore $\sigma = 1$ in (ϕ-3), and thus the necessary condition $N > \sigma + 1$ for (**b-1**)to hold becomes $N > 2$, a condition that was required by N. Kawano for his work in [19]. We claim that when $N > 2$, the Conditions 1.1.3 and (2.1.1) are equivalent. To see this we assume $N > 2$ and proceed as follows. Using integration by parts and L'Hospital's Rule, we find that for all $r > 0$

$$\int_0^r \left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt = \frac{r^{2-N}}{2-N}\int_0^r s^{N-1}b(s)ds + \frac{1}{N-2}\int_0^r tb(t)dt.$$

Therefore, for all $r > 0$, we have

$$\frac{1}{r^{N-2}}\int_0^r s^{N-1}b(s)ds = \int_0^r sb(s)ds - (N-2)\int_0^r \left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt \tag{2.1.2}$$

$$= P(r) - Q(r).$$

Since P and Q are increasing, we note that both $P(\infty)$ and $Q(\infty)$ belong to $[0,\infty]$. Obviously as can be seen from the relation (2.1.2), the possibility $P(\infty) < \infty$ and $Q(\infty) = \infty$ can not hold. So, suppose $P(\infty) = \infty$, and $Q(\infty) < \infty$. But then the equation (2.1.2) shows that

$$\lim_{r\to\infty} \frac{1}{r^{N-2}} \int_0^r s^{N-1} b(s) ds = \infty.$$

This implies that for some sufficiently large $\tau > 0$, we have

$$t^{1-N} \int_0^t s^{N-1} b(s) ds \geq \frac{1}{t}, \quad \forall\, t \geq \tau.$$

This shows that $r > \tau$

$$\int_\tau^r t^{1-N} \int_0^t s^{N-1} b(s) ds \geq \int_\tau^r \frac{1}{t} dt = \log\left(\frac{r}{\tau}\right).$$

However this contradicts the assumption that $Q(\infty) < \infty$.

Therefore $P(\infty)$ and $Q(\infty)$ are either both finite or both infinity, establishing our claim.

Remark 2.1. When $N = 1, 2$, the equivalence fails. In fact, let b be defined as

$$b(s) = \begin{cases} \dfrac{1}{(s+1)(s^2+1)}, & s \geq 0 \quad \text{when} \quad N = 1 \\ \dfrac{1}{(s^2+1)^2}, & s \geq 0 \quad \text{when} \quad N = 2. \end{cases}$$

Easy computation shows that (2.1.1) holds while (1.1.3) fails.

Concerning the function $f : \mathbb{R} \to \mathbb{R}$ in (h-1), we will always assume that it is continuous and satisfies at least one of the conditions listed below, where B is the constant in Condition (b-1).

(f-1): f is non-decreasing and

$$\lim_{t\to 0^+} \frac{\lambda^{-1}(f(t))}{t} < \frac{1}{B}.$$

(f-2): f is non-decreasing and

$$\lim_{t\to\infty} \frac{\lambda^{-1}(f(t))}{t} < \frac{1}{B}.$$

(f-3): f is non-increasing.

As a first step to study entire solutions of (1.1.1), we investigate radial solutions of the following equations.

$$\text{div}(\phi(|\nabla v|)\nabla v) = -b(|x|)f(v), \ (x \in \mathbb{R}^N). \tag{2.1.3}$$

(2.1.4) $$\operatorname{div}(\phi(|\nabla v|)\nabla v) = b(|x|)f(v),\ (x \in \mathbb{R}^N).$$

In view of the structure condition (h-1), solutions of (2.1.3) and (2.1.4) will provide us with a super-solution (resp., sub-solution) of Problem (1.1.1). To study solutions of (2.1.3) and (2.1.4), we investigate the following two initial value problems for any $\alpha > 0$.

$$(E_\pm) \qquad \begin{cases} (r^{N-1}\phi(|u'|)u')' = \pm r^{N-1}b(r)f(u(r)), \quad r > 0 \\ u(0) = \alpha, \quad u'(0) = 0. \end{cases}$$

These problems are equivalent to solving the integral equations in $X := C(\mathbb{R}_0^+)$.

$$(I_\pm) \qquad u(r) = \alpha \pm \int_0^r \Psi^{-1}(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds)dt,\ r \geq 0.$$

It is clear that any solution $u \in X$ of $(I_\pm)$ is a solution of $(E_\pm)$. To see that a solution u of (E_-) solves the integral equation (I_-), we integrate both sides of (E_-) on $(0, r)$ to find

$$r^{N-1}\phi(|u'|)u' = -\int_0^r s^{N-1}b(s)f(u(s))ds, \quad \text{that is}$$
$$\phi(|u'|)u' = -r^{1-N}\int_0^r s^{N-1}b(s)f(u(s))ds.$$

This shows that $u' < 0$ and hence

$$\phi(-u')(-u') = r^{1-N}\int_0^r s^{N-1}b(s)f(u(s))ds, \quad \text{or equivalently}$$
$$\Psi(-u') = r^{1-N}\int_0^r s^{N-1}b(s)f(u(s))ds$$

Rewriting this, we have

$$u'(r) = -\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)f(u(s))ds\right).$$

We integrate this last expression on $(0, r)$ to find that

$$u(r) = u(0) - \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)dt.$$

Similarly, the equation (E_+) leads to the integral equation (I_+).

We wish to solve Problem $(I_\pm)$ by using the Schauder-Tychonoff fixed point theorem applied to appropriate integral operators defined on the locally convex space $X := C([0,\infty))$, equipped with its uniform convergence metric on every compact subinterval of $J = [0,\infty)$. We will use X_+ to denote the cone of non-negative functions in X. In subsequent proof, we will use the notation $\mathcal{C}$ to denote a convex subset of X whose definition will depend on the condition on f and on whether we are considering Problems (2.1.3) or (2.1.4). Let us recall the Schauder-Tychonoff fixed point theorem [32].

Theorem 2.2. (Schauder-Tychonoff Fixed Point Theorem) Let X be a locally convex linear topological space and let $\mathcal{C} \subseteq X$ be a convex subset. Let $L : \mathcal{C} \mapsto \mathcal{C}$ be a continuous mapping such that

$$L(\mathcal{C}) \subseteq A \subseteq \mathcal{C}$$

for some compact subset A of $\mathcal{C}$. Then L has at least one fixed point.

We now begin by considering Problem (2.1.3).

Lemma 2.3. Suppose that f satisfies any of the Conditions (f-1),(f-2) or (f-3). Then Problem (2.1.3) admits infinitely many positive bounded solutions each of which is bounded away from zero in $\mathbb{R}^N$.

Proof. We will obtain a solution of (2.1.3) in the form $u(x) = y(|x|)$ where y is a solution of (E_-). To find a solution of (E_-), we look for fixed points of the integral operator $T : X_+ \to X$ defined by

$$Tu(r) := \alpha - \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)dt, \quad r \geq 0. \tag{2.1.5}$$

We observe that $Tu \leq \alpha$ for any $u \in X_+$. Let us first suppose that f satisfies (f-1).
In this case, we choose $\alpha_0 > 0$ sufficiently small that for all $0 < \alpha < \alpha_0$

$$\frac{\lambda^{-1}(f(\alpha))}{\alpha} < \frac{1}{B}. \tag{2.1.6}$$

Moreover, let

$$\theta(\alpha) := \alpha - \lambda^{-1}(f(\alpha))B > 0. \tag{2.1.7}$$

We consider the following closed convex subset of X.

$$\mathcal{C} := \{u \in C([0,\infty)) : \theta(\alpha) \leq u(r) \leq \alpha, \quad \forall\, r \geq 0\}. \tag{2.1.8}$$

On recalling that f is non-decreasing, we see that for $u \in \mathcal{C}$

$$Tu(r) \geq \theta(\alpha), \quad r \geq 0.$$

Therefore, when f satisfies (f-1) we have $T : \mathcal{C} \mapsto \mathcal{C}$.
Suppose f satisfies (f-2). In this case we choose $\alpha_\infty > 0$ large enough such that (2.1.6) holds for all $\alpha > \alpha_\infty$. Then with $\theta(\alpha)$ and $\mathcal{C}$ defined as in (2.1.7) and (2.1.8), respectively it follows that $T(\mathcal{C}) \subseteq \mathcal{C}$ also.
Let us now take up the case when f satisfies Condition (f-3). As a consequence of f being non-increasing on $\mathbb{R}^+$, it is clear that

$$\int_0^1 \frac{ds}{\lambda^{-1}(f(s))} < \infty \quad \text{and} \quad \int_0^\infty \frac{ds}{\lambda^{-1}(f(s))} = \infty. \tag{2.1.9}$$

As a consequence of (2.1.9), we note that the following is well-defined function.

$$\zeta(t) = \int_0^t \frac{ds}{\lambda^{-1}(f(s))}, \quad t \geq 0. \tag{2.1.10}$$

Let $\eta : (0, \infty) \to (0, \infty)$ be the inverse function of ζ, that is

$$\int_0^{\eta(t)} \frac{ds}{\lambda^{-1}(f(s))} = t, \quad t > 0. \tag{2.1.11}$$

Note that $\eta'(t) = \lambda^{-1}(f(\eta(t)))$ for all $t > 0$. We invoke (2.1.9) to choose $\alpha_\infty > 0$ large enough such that

$$\zeta(\alpha) - B > 0, \; \forall\, \alpha > \alpha_\infty. \tag{2.1.12}$$

Define the decreasing function

$$\mathcal{E}(r) := \zeta(\alpha) - \int_0^r \Psi^{-1}\left(t^{1-N} \int_0^t s^{N-1} b(s) ds\right) dt, \quad r \geq 0.$$

Note that, as a consequence of (2.1.12) and the definition of $\mathcal{E}$, we have

$$0 < \zeta(\alpha) - B = \mathcal{E}(\infty) \leq \mathcal{E}(r) \leq \zeta(\alpha), \quad \forall\, r \geq 0.$$

Consequently

$$0 < \eta(\zeta(\alpha) - B) \leq \eta(\mathcal{E}(r)) \leq \alpha, \; \forall\, r \geq 0. \tag{2.1.13}$$

Let us consider the following closed and convex subset of X.

$$\mathcal{C} := \{u \in C([0, \infty)) : \eta(\mathcal{E}(r)) \leq u(r) \leq \alpha, \; \forall\, r \geq 0\}.$$

Because of (2.1.13), we see that

$$\mathcal{C} \subseteq \{u \in C([0, \infty)) : \theta(\alpha) \leq u(r) \leq \alpha, \; \forall\, r \geq 0\},$$

where $\theta(\alpha) = \eta(\zeta(\alpha) - B)$. We should note that $\lim_{\alpha \to \infty} \theta(\alpha) = \infty$.
On recalling that f is non-increasing, we see that for any $v \in \mathcal{C}$ and $r \geq 0$

$$\begin{aligned} Tu(r) &= \alpha - \int_0^r \Psi^{-1}\left(t^{1-N} \int_0^t s^{N-1} b(s) f(u(s)) ds\right) dt \\ &\geq \eta(\mathcal{E}(r)), \text{ for all } r \geq 0. \end{aligned} \tag{2.1.14}$$

To see inequality (2.1.14), let

$$\mathcal{P}(r) := \alpha - \int_0^r \Psi^{-1}\left(t^{1-N} \int_0^t s^{N-1} b(s) f(u(s)) ds\right) dt - \eta(\mathcal{E}(r)), \quad r \geq 0.$$

Obviously, $\mathcal{P}(0) = \alpha - \eta(\mathcal{E}(0)) = \alpha - \eta(\zeta(\alpha)) = 0$. Since $\mathcal{E}$ and f are non-increasing and η is non-decreasing for any $u \in \mathcal{C}$, we have

$$f(u(s)) \leq f(\eta(\mathcal{E}(s))) \leq f(\eta(\mathcal{E}(r))), \; 0 \leq s \leq r.$$

Using these facts, we see that for $r > 0$

$$\begin{aligned}
\mathcal{P}'(r) &= -\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)f(u(s))ds\right) - \eta'(\mathcal{E}(r))\mathcal{E}'(r) \\
&\geq -\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)f(\eta(\mathcal{E}(s)))ds\right) - \mathcal{E}'(r)\lambda^{-1}(f(\eta(\mathcal{E}(r)))) \\
&\geq -\lambda^{-1}(f(\eta(\mathcal{E}(r))))\left[\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)ds\right) + \mathcal{E}'(r)\right] \\
&= 0.
\end{aligned}$$

Then $\mathcal{P}$ is non-decreasing and $\mathcal{P}(0) = 0$ implies that $\mathcal{P}(r) \geq 0$ for all $r \geq 0$. Thus, the asserted inequality (2.1.14) holds.

Thus, in all three cases, we have shown that $T(\mathcal{C}) \subseteq \mathcal{C}$.

Let us show that $T : \mathcal{C} \to \mathcal{C}$ is continuous. Suppose $\{u_j\}$ is a sequence in $\mathcal{C}$ and $u \in \mathcal{C}$ such that $u_j \to u$ uniformly on compact subset of $[0, \infty)$. We will show that $Tu_j \to Tu$ uniformly on $[0, R]$ for any given $R > 0$. Now, for $r \in [0, R]$, we have

$$\begin{aligned}
&|Tu_j(r) - Tu(r)| \\
= &\left|\int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u_j(s))ds\right)dt\right. \\
&\left.-\int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)dt\right| \\
\leq &\int_0^r \left|\Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u_j(s))ds\right) - \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)\right|dt.
\end{aligned}$$

Let $\epsilon > 0$ be given. Since f is uniformly continuous on $[\theta(\alpha), \alpha]$ and $u_j \to u$ uniformly on $[0, R]$, there is a positive integer J such that

$$|f(u_j(t)) - f(u(t))| < \frac{\epsilon}{\max\{1, \int_0^R b(s)ds\}}, \ \forall\, t \in [0, R], \text{ and } \forall\, j \geq J.$$

Consequently, for $j \geq J$ and for all $t \in [0, R]$, we have

$$\left|t^{1-N}\int_0^t s^{N-1}b(s)(f(u_j(s)) - f(v(s)))ds\right| \leq \int_0^t b(s)|f(u_j(s)) - f(u(s))|ds < \epsilon.$$

Therefore,

$$t^{N-1}\int_0^t s^{N-1}(f(u_j(s)) - f(u(s)))ds$$

converges uniformly on $[0, R]$. Since Ψ^{-1} is continuous on $[0, c]$ for any $c > 0$, we see that

$$\Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u_j(s))ds\right) \to \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)$$

uniformly on $[0, R]$ as $j \to \infty$. Therefore, $Tu_j \to Tu$ uniformly on $[0, R]$. Thus, we have shown that $\{Tu_j\}$ converges to Tu uniformly on compact subsets of $[0, \infty)$.

We also note from (2.1.5) that

$$(Tu)'(r) = -\Psi^{-1}\left(r^{1-N}\int_0^r t^{N-1}b(t)f(u(t))dt\right),$$

and hence $(Tu)'$ is continuous and non-positive on $(0,\infty)$. For any given $R>0$, we see that

$$|(Tu)'(r)| \le \Psi^{-1}\left(c(\alpha)\int_0^R b(s)ds\right),\ \forall\,(u,r)\in\mathcal{C}\times[0,R],$$

where $c(\alpha) := \max\{f(\theta(\alpha)), f(\alpha)\}$. Therefore, given $R>0$, by the Mean-Value Theorem

$$\begin{aligned} |Tu(r)-Tu(s)| &\le \sup_{\epsilon\in[0,1]} |(Tu)'(\epsilon r+(1-\epsilon)s)||r-s| \\ &\le \Psi^{-1}\left(c(\alpha)\int_0^R b(\tau)d\tau\right)|r-s|,\ r,s\in[0,R]. \end{aligned}$$

This shows that $\{Tu : u\in\mathcal{C}\}$ is equicontinuous on any compact subset of $[0,\infty)$. Thus $\{Tu : u\in\mathcal{C}\}$ is equicontinuous on $[0,\infty)$.

Note that, again from (2.1.5), we have

$$\begin{aligned} |Tu(r)| &\le \alpha+\lambda^{-1}(c(\alpha))\int_0^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt \\ &= \alpha+\lambda^{-1}(c(\alpha))B,\ \forall\,(u,r)\in\mathcal{C}\times[0,\infty). \end{aligned}$$

Therefore, $\{Tu(r) : u\in\mathcal{C}\}$ is bounded in $[0,\infty)$ for each $r\in[0,\infty)$ and hence has compact closure in $[0,\infty)$ for each $r\in[0,\infty)$. Consequently, by Arzela-Ascoli Theorem $T(\mathcal{C})$ is contained in a compact subspace A of $C([0,\infty))$ and hence a compact subspace of $\mathcal{C}$. Therefore, the Schauder-Tychonoff Theorem applies to show that T has a fixed point in $\mathcal{C}$. If y is such a fixed point, we note that $0<a\le y(|x|)\le b$ for some positive constants $a\le b$. Moreover,

$$\begin{aligned} |y'(r)| &\le \lambda^{-1}(f(c))\int_0^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt \\ &\le \lambda^{-1}(f(c))B, \end{aligned}$$

where $c=b$ if f is non-decreasing, and $c=a$, when f is non-increasing. Therefore, we see that $y(|x|)$ belongs to $W^{1,\Phi}_{loc}(\mathbb{R}^N)$ and $b(|x|)f(y(|x|))\in L^{\widetilde{\Phi}}_{loc}(\mathbb{R}^N)$. Thus, by Lemma 1.4 we conclude that $u(x):=y(|x|)$ is a solution of (2.1.3).

Let us now summarize our findings. There are positive constants α_0 and α_∞, depending on f and B such that for all $0<\alpha<\alpha_0$ in the case when f satisfies (f-1), and for all $\alpha_\infty<\alpha<\infty$ in the case when f satisfies (f-2), the corresponding fixed point y_α of T satisfies

(2.1.15) $$\theta(\alpha)\le y_\alpha(r)\le\alpha,\ r\in[0,\infty).$$

Here $\theta(\alpha) := \alpha - \lambda^{-1}(f(\alpha))B$. On the other hand, when f satisfies (f-3), there is a positive constant α_∞, depending on f and B such that for all $\alpha_\infty < \alpha < \infty$ the corresponding fixed point y_α of T satisfies

$$\theta(\alpha) \leq y_\alpha(r) \leq \alpha,\ r \in [0,\infty). \tag{2.1.16}$$

Here $\theta(\alpha) := \eta(\zeta(\alpha) - B)$ where ζ and η, defined in (2.1.10) and (2.1.11) respectively, are inverses of each other.

We also recall that

$$\begin{cases} \lim_{\alpha\to 0^+}(\alpha - \lambda^{-1}(f(\alpha))B) = 0 \text{ if } f \text{ satisfies Condition (f-1)} \\ \lim_{\alpha\to\infty}(\alpha - \lambda^{-1}(f(\alpha))B) = \infty \text{ if } f \text{ satisfies Condition (f-2)} \\ \lim_{\alpha\to\infty}(\eta(\zeta(\alpha) - B)) = \infty \text{ if } f \text{ satisfies Condition (f-3).} \end{cases}.$$

Therefore, (2.1.15), (2.1.16) and the above limits show that Equation (2.1.3) has infinitely many solutions. □

Next, we consider the existence of a solution to Problem (2.1.4). For this, we study (E_+), where we use $u(0) = \beta$ in place of $u(0) = \alpha$, under any of the assumptions (f-1), (f-2) or (f-3).

Lemma 2.4. Suppose that f satisfies any of the Conditions (f-1), (f-2) or (f-3). Then equation (2.1.4) admits infinitely many bounded positive entire solutions each of which is bounded away from zero in $\mathbb{R}^N$.

Proof. As in Lemma 2.3, we find solutions of (2.1.4) in the form $v(x) = z(|x|)$, where z is a solution of (E_+). To solve Problem (E_+), we consider the integral operator $T : X_+ \to X$ given by

$$Tv(r) := \beta + \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(v(s))ds\right)dt,\ r \geq 0 \tag{2.1.17}$$

We point out that $Tv \geq \beta$ for all $v \in X_+$. Let us start by assuming f satisfies Condition (f-1). Then, we note that

$$\int_0^1 \frac{ds}{\lambda^{-1}(f(s))} = \infty. \tag{2.1.18}$$

As a consequence of (2.1.18), we introduce two well-defined functions: a decreasing function $\xi : (0,1) \to (0,\infty)$ and its inverse function $\nu : (0,\infty) \to (0,1)$ as follows:

$$\xi(t) := \int_t^1 \frac{ds}{\lambda^{-1}(f(s))}, \quad \text{and} \quad \int_{\nu(t)}^1 \frac{ds}{\lambda^{-1}(f(s))} = t,\ t > 0. \tag{2.1.19}$$

As a consequence of (2.1.18), let us fix $0 < \beta_0 < 1$ such that $\xi(\beta) > B$ for $0 < \beta \le \beta_0$, we define

$$\mathcal{F}(r) := \xi(\beta) - \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt, \ r \ge 0.$$

Note that $0 < \xi(\beta) - B \le \mathcal{F}(r) \le \xi(\beta)$ for all $r \ge 0$ and hence

$$\beta \le \nu(\mathcal{F}(r)) \le \nu(\xi(\beta) - B) \text{ for all } r \ge 0. \tag{2.1.20}$$

Let us consider the following closed and convex subset of X.

$$\mathcal{C} := \{v \in X : \beta \le v(r) \le \nu(\mathcal{F}(r)), \ r \ge 0\}. \tag{2.1.21}$$

As a consequence of (2.1.20), the following inclusion holds:

$$\mathcal{C} \subseteq \{v \in X : \beta \le v(r) \le \Theta(\beta), \ r \ge 0\},$$

where

$$\Theta(\beta) := \nu(\xi(\beta) - B). \tag{2.1.22}$$

We remark that

$$\lim_{\beta\to 0^+} \Theta(\beta) = 0.$$

With T as in (2.1.17), we claim that for any $v \in \mathcal{C}$, the following holds:

$$\begin{aligned} Tv(r) &= \beta + \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(v(s))ds\right)dt \\ &\le \nu(\mathcal{F}(r)), \forall\, r \ge 0. \end{aligned} \tag{2.1.23}$$

This can be justified with a similar argument used in Lemma 2.3. More specifically, let

$$\mathcal{P}(r) = -\beta - \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(v(s))ds\right)dt + \nu(\mathcal{F}(r)), \ r \ge 0.$$

Then, for $v \in \mathcal{C}$ and for $r > 0$, we have

$$\begin{aligned} \mathcal{P}'(r) &= -\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)f(v(s))ds\right) - \lambda^{-1}(f(\nu(\mathcal{F}(r))))\mathcal{F}'(r) \\ &\ge -\lambda^{-1}(f(\nu(\mathcal{F}(r))))\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)ds\right) - \lambda^{-1}(f(\nu(\mathcal{F}(r))))\mathcal{F}'(r) \\ &= -\lambda^{-1}(f(\nu(\mathcal{F}(r))))\left[\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)ds\right) + \mathcal{F}'(r)\right] \\ &= 0. \end{aligned}$$

Thus, $\mathcal{P}'(r) \ge 0$ and since $\mathcal{P}(0) = -\beta + \nu(\mathcal{F}(0)) = -\beta + \nu(\xi(\beta)) = 0$, we see that $\mathcal{P}(r) \ge 0$ for all $r \ge 0$ and this justifies the inequality (2.1.23). Therefore, when f

satisfies (f-1), we see that $T : \mathcal{C} \mapsto \mathcal{C}$.

Let us now suppose that condition (f-2) holds. Then, we note that

$$\int_1^\infty \frac{ds}{\lambda^{-1}(f(s))} = \infty.$$

As before introduce two functions κ and μ that are inverses of each other, as follows:

$$\kappa(t) := \int_1^t \frac{ds}{\lambda^{-1}(f(s))}, \quad \text{and} \quad \int_1^{\mu(t)} \frac{ds}{\lambda^{-1}(f(s))} = t, \ \forall\, t > 0. \tag{2.1.24}$$

Given any $\beta > \beta_\infty := 1$, let

$$\mathcal{I}(r) := \kappa(\beta) + \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt, \ r \geq 0.$$

We note that $\kappa(\beta) \leq \mathcal{I}(r) < \kappa(\beta) + B$ for all $r \in \mathbb{R}_0^+$. Consequently, we have

$$\beta \leq \mu(\mathcal{I}(r)) \leq \mu(\kappa(\beta) + B), \ \forall\, r \geq 0. \tag{2.1.25}$$

Let $\mathcal{C}$ be defined as

$$\mathcal{C} := \{v \in X : \beta \leq v(r) \leq \mu(\mathcal{I}(r)), \ \forall\, r \geq 0\}. \tag{2.1.26}$$

On recalling (2.1.25) we see that

$$\mathcal{C} \subseteq \{v \in X : \beta \leq v(r) \leq \Theta(\beta), \ \forall\, r \geq 0\},$$

where

$$\Theta(\beta) := \mu(\kappa(\beta) + B). \tag{2.1.27}$$

Let us also remark that $\lim_{\beta\to\infty} \Theta(\beta) = \infty$.

We let $T : X_+ \to X$ be as in (2.1.17). Clearly, $Tv(r) \geq \beta$ for all $(v, r) \in X_+ \times \mathbb{R}_0^+$. To show that $Tv(r) \leq \mu(\mathcal{I}(r))$ for all $(v, r) \in \mathcal{C} \times [0, \infty)$, we introduce

$$\mathcal{P}(r) := \beta + \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(v(s))ds\right)dt - \mu(\mathcal{I}(r)), \ r \geq 0.$$

Then $\mathcal{P}(0) = \beta - \mu(\mathcal{I}(0)) = \beta - \mu(\kappa(\beta)) = 0$ and for $u \in \mathcal{C}$, we have

$$\begin{aligned}
\mathcal{P}'(r) &= \Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)f(v(s))ds\right) - \lambda^{-1}(f(\mu(\mathcal{I}(r))))\mathcal{I}'(r) \\
&\leq \lambda^{-1}(f(\mu(\mathcal{I}(r))))\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)ds\right) - \lambda^{-1}(f(\mu(\mathcal{I}(r))))\mathcal{I}'(r) \\
&= \lambda^{-1}(f(\mu(\mathcal{I}(r))))\left[\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)ds\right) - \mathcal{I}'(r)\right] \\
&= 0.
\end{aligned}$$

Consequently, $\mathcal{P}(r) \leq \mathcal{P}(0) = 0$ for all $r \geq 0$. Thus, indeed $Tv(r) \leq \mu(\mathcal{I}(r))$ on $\mathbb{R}_0^+$ whenever $v \in \mathcal{C}$ and hence $Tv \in \mathcal{C}$. Therefore, in case f satisfies (f-2), we have $T : \mathcal{C} \mapsto \mathcal{C}$.
Suppose now f is non-increasing on $\mathbb{R}^+$. Given any positive constant β, let

$$\Theta(\beta) := \beta + \lambda^{-1}(f(\beta))B = \beta\left(1 + \frac{\lambda^{-1}(f(\beta))B}{\beta}\right).$$

We note that

$$\lim_{\beta\to\infty} \Theta(\beta) = \lim_{\beta\to\infty} \beta\left(1 + \frac{\lambda^{-1}(f(\beta))B}{\beta}\right) = \infty.$$

Let us set

$$\mathcal{C} := \{v \in X : \beta \leq v(r) \leq \Theta(\beta),\ \forall\, r \geq 0\}. \tag{2.1.28}$$

Then with $T : X_+ \to X$ defined as in (2.1.17) and recalling that f is non-increasing, we have

$$\begin{aligned} Tu(r) &= \beta + \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)dt \\ &\leq \beta + \lambda^{-1}(f(\beta))B = \Theta(\beta). \end{aligned}$$

Therefore, in case f satisfies (f-3), we still have $T(\mathcal{C}) \subseteq \mathcal{C}$, as desired.
Thus, in all cases we have shown that $T : \mathcal{C} \mapsto \mathcal{C}$, where $\mathcal{C}$ is the closed convex subspace of X given by (2.1.21), (2.1.26) or (2.1.28) when f satisfies condition (f-1), (f-2) or (f-3), respectively. As in the proof of Lemma 2.3, we can show that T is continuous and that $T(\mathcal{C})$ is relatively compact in $X = C([0,\infty))$. Therefore, an application of Schauder-Tychonoff Theorem allows us to conclude that T has a fixed point z in $\mathcal{C}$. Therefore, the initial value problem (E_+) has a solution z such that $0 < a \leq z(|x|) \leq b$ for some positive constants $a \leq b$. Moreover, as observed in the proof of Lemma 2.3, we have $|z'(r)| \leq \lambda^{-1}(f(c))B$ for some positive constant c. Therefore, we see that $z(|x|)$ and $b(|x|)f(z(|x|))$ belong to $W^{1,\Phi}_{loc}(\mathbb{R}^N)$ and $L^{\widetilde{\Phi}}_{loc}(\mathbb{R}^N)$, respectively. Again, we invoke Lemma 1.4 to conclude that $u(x) = z(|x|)$ is a solution of (2.1.4).
We recapture the above discussion as follows: When f satisfies (f-1), there is a positive constant β_0 depending on f and B such that for each $0 < \beta < \beta_0$, the fixed point z_β of T satisfies

$$\beta \leq z_\beta(r) \leq \mu(\xi(\beta) - B),\ \text{for } r \in [0,\infty) \tag{2.1.29}$$

for all $0 < \beta < \beta_0$. Similarly, there is a positive constant β_∞ such that $\beta > \beta_\infty$, the fixed point z_β of T that satisfies

$$\beta \leq z_\beta(r) \leq \Theta(\beta),\ \text{for } r \in [0,\infty). \tag{2.1.30}$$

Here $\Theta(\beta) := \mu(\kappa(\beta) + B)$ if f satisfies (f-2) and $\Theta(\beta) := \beta + \lambda^{-1}(f(\beta))B$ when f satisfies (f-3). We also note that

$$(2.1.31) \qquad \begin{cases} \lim_{\beta\to 0^+} \nu(\xi(\beta) - B) = 0 \text{ if } f \text{ satisfies Condition (f-1)} \\ \lim_{\beta\to\infty} \mu(\kappa(\beta) + B) = \infty \text{ if } f \text{ satisfies Condition (f-2)} \\ \lim_{\beta\to\infty}(\beta + \lambda^{-1}(f(\beta))B) = \infty \text{ if } f \text{ satisfies Condition (f-3).} \end{cases}.$$

In conclusion, (2.1.29), (2.1.30) and the limits in (2.1.31) show that Equation (2.1.4) has infinitely many bounded positive solutions, each bounded away from zero in $\mathbb{R}^N$. $\square$

We are now in a position to state existence theorems to Problem (1.1.1). We will consider different cases, depending on the sign of the inhomogeneous term $h(x,t)$.
Lemma 2.3 leads to the following theorem.
Theorem 2.5. Suppose $h(x,t) \leq 0$ on $\mathbb{R}^N \times (0,\infty)$. If f satisfies any of the conditions (f-1), (f-2) or (f-3), then Equation (1.1.1) has infinitely many positive bounded solutions, each of which is bounded away from zero in $\mathbb{R}^N$ and converges to a constant as $|x| \to \infty$.

Proof. Suppose f satisfies (f-1). The proof of Lemma 2.3 shows that there is $\alpha_0 > 0$ such that for each $0 < \alpha < \alpha_0$, Problem (E_-) has a solution y_α on $J := [0,\infty)$ with $\theta(\alpha) \leq y_\alpha(r) \leq \alpha$ where $\theta(\alpha) = \alpha - \lambda^{-1}(f(\alpha))B$. When f satisfies (f-2) then again according to Lemma 2.3, there is a constant $\alpha_\infty > 0$ such that for each $\alpha > \alpha_\infty$, Problem (E_-) has a solution y_α with $\theta(\alpha) \leq y_\alpha(r) \leq \alpha$ on J, where $\theta(\alpha)$ is as in the above. Likewise, if f satisfies (f-3), there is $\alpha_\infty > 0$ such that for each $\alpha > \alpha_\infty$ Problem (E_-) admits a solution y_α on J with $\theta(\alpha) \leq y_\alpha \leq \alpha$ on J. Here $\theta(\alpha) = \eta(\zeta(\alpha) - B)$ where η and ζ are as defined in (2.1.10) and (2.1.11), respectively. We also recall that $\theta(\alpha) \to 0$ as $\alpha \to 0^+$ when f satisfies (f-1), while $\theta(\alpha) \to \infty$ as $\alpha \to \infty$ when f satisfies (f-2) or (f-3). Lemma 1.4 shows that $w_\alpha(x) = y_\alpha(|x|)$ is a super-solution of (1.1.1). Since y_α is decreasing, we have $\lim\limits_{r\to\infty} y_\alpha(r) = y_\alpha(\infty)$ for some constant $y_\alpha(\infty) \geq \theta(\alpha)$. Using the assumption that $h \leq 0$ on $\mathbb{R}^N \times \mathbb{R}^+$, we see that $v_\alpha(x) := y_\alpha(\infty)$ is a sub-solution of (1.1.1). Thus we invoke Theorem 1.6 to conclude that Problem (1.1.1) admits a solution u_α such that $v_\alpha \leq u_\alpha \leq w_\alpha$ in $\mathbb{R}^N$. Clearly $u_\alpha(x) \to y_\alpha(\infty)$ as $|x| \to \infty$. Taking the limit behavior of $\theta(\alpha)$ noted above into account, we conclude that there are infinitely many such solutions. $\square$

A result analogous to theorem 2.5 holds when h is non-negative in $\mathbb{R}^N$. This is the essence of the next theorem, which is a consequence of Lemma 2.4.

Theorem 2.6. Suppose $h(x,t) \geq 0$ on $\mathbb{R}^N \times (0,\infty)$. If f satisfies any of the conditions (f-1), (f-2) or (f-3), then Equation (1.1.1) has infinitely many positive bounded solutions which are bounded away from zero in $\mathbb{R}^N$ and converges to a constant as $|x| \to \infty$.

Proof. Suppose f satisfies condition (f-1). According to the proof of Lemma 2.4, there is $\beta_0 > 0$ such that for every $0 < \beta < \beta_0$, Problem (E_+) has a solution z_β on $J := [0,\infty)$ with $\beta \leq z_\beta(r) \leq \Theta(\beta)$ where $\Theta(\beta) := \nu(\xi(\beta) - B)$. Similarly, if f satisfies (f-2) or (f-3), then the proof of Lemma 2.4 shows that there is $\beta_\infty > 0$ such that for every $\beta > \beta_\infty$, Problem (E_+) has a solution z_β on J with $\beta \leq z_\beta \leq \Theta(\beta)$, where $\Theta(\beta) := \mu(\kappa(\beta) + B)$ when f satisfies (f-2) and $\Theta(\beta) := \beta + \lambda^{-1}(f(\beta))B$ when f satisfies (f-3). Here ν and ξ are as defined in (2.1.19), while κ and μ are as defined in (2.1.24). By Lemma 1.4, we note that $v_\beta(x) = z_\beta(|x|)$ is a sub-solution of (1.1.1) in $\mathbb{R}^N$. Since z_β is increasing, we note that $\lim_{r\to\infty} z_\beta(r) = z_\beta(\infty)$ for some $z_\beta(\infty) \leq \Theta(\beta)$. Recalling that $h \geq 0$ on $\mathbb{R}^N \times \mathbb{R}^+$, we find that $w_\beta(x) = z_\beta(\infty)$ is a super-solution of (1.1.1). Therefore by Theorem 1.6, we conclude that Problem (1.1.1) admits a solution u_β such that $v_\beta \leq u_\beta \leq w_\beta$ in $\mathbb{R}^N$ and $u_\beta(x) \to w_\beta(\infty)$ as $|x| \to \infty$. In view of the limit relations in (2.1.31), we see that Equation (1.1.1) admits infinitely many solutions. □

Let us record the following remark that will prove useful in the proof of the next result when h changes sign.

Remark 2.7. Suppose f is a non-decreasing function such that f satisfies Condition (f-1). According to the proof of Lemma 2.3 we fix $\alpha > 0$ such that Problem (2.1.3) has a solution y with $y \geq \theta(\alpha)$ where $\theta(\alpha)$ is as given in (2.1.7). Similarly, the proof of Lemma 2.4 shows that, for sufficiently small $\beta > 0$, Problem (2.1.4) admits a solution z with $z \leq \Theta(\beta)$ where $\Theta(\beta)$ is given by (2.1.22). Since $\Theta(\beta) \to 0$ as $\beta \to 0^+$, we can choose β sufficiently small that $z \leq \Theta(\beta) \leq \theta(\alpha) \leq y$. Suppose now f satisfies Condition (f-2). For fixed $\beta > 1$, the argument employed to prove Lemma 2.4 allows us to get a solution z of Problem (2.1.4) such that $z \leq \Theta(\beta)$, where this time $\Theta(\beta)$ is given as in (2.1.17). Let us consider $\theta(\alpha)$ given by (2.1.7). In this case we recall that $\theta(\alpha) \to \infty$ as $\alpha \to \infty$. Therefore, we can choose $\alpha > 0$ such that $\Theta(\beta) \leq \theta(\alpha)$. The corresponding solution y of Problem (2.1.3) given by Lemma 2.3 satisfies $z \leq \Theta(\beta) < \theta(\alpha) \leq y$. In conclusion, when f satisfies (f-1) or (f-2), we can find a solution $v(x) := z(|x|)$ of (2.1.4) and a solution $w(x) := y(|x|)$ of (2.1.3) such that $v \leq w$ on $\mathbb{R}^N$. The above procedure shows that one can choose such a pair $v \leq w$ in an infinitely many ways.

We now consider the case when $h(x,t)$ changes sign in $\mathbb{R}^N$. We have the following theorem.

Theorem 2.8. Suppose f satisfies any of the Conditions (f-1), (f-2) or (f-3). Then Equation (1.1.1) admits infinitely many positive bounded solutions that are bounded away from zero in $\mathbb{R}^N$.

Proof. The hypotheses allow us to invoke Theorem 2.5 to find an entire solution w_0 of Equation (2.1.3) such that $0 < a \leq w_0 \leq c$ on $\mathbb{R}^N$ for some positive constants a and c. Similarly, by Theorem 2.6, we take an entire solution v_0 of Equation (2.1.4) such that $0 < d \leq v_0 \leq e$ for some constants d and e. Let us first suppose that f satisfies Condition (f-3), that is f is non-increasing. Set $w := w_0 + e$, and $v := v_0$. Since f is non-increasing, we have

$$\Delta_\phi w = -b(|x|)f(w_0) \leq -b(|x|)f(w) \leq h(x,w),\ x \in \mathbb{R}^N.$$

Clearly,

$$\Delta_\phi v = b(|x|)f(v) \geq h(x,v),\ x \in \mathbb{R}^N.$$

Moreover $v \leq w$ in $\mathbb{R}^N$, and therefore, we invoke Theorem 1.6 to find a solution u of Problem (1.1.1) in $\mathbb{R}^N$ such that $v \leq u \leq w$ in $\mathbb{R}^N$. Let us now suppose that f is non-decreasing and f satisfies either (f-1) or (f-2). By Remark 2.7, we can find two positive and bounded functions v and w, both bounded away from zero such that v is an entire solution of Problem (2.1.3) and w is an entire solution of Problem (2.1.4). Moreover $v \leq w$. Then by Theorem 1.6, we conclude that there is a solution u of Problem (1.1.1) such that $v \leq u \leq w$ in $\mathbb{R}^N$. □

2.2 Infinitely Many Sign-Changing Bounded Solutions

For our final result in this section, we consider nonlinearities f in Problem (1.1.1) which do not satisfy any of the conditions (f-1) through (f-3). We find that Problem (1.1.1) admits infinitely many bounded solutions which are not necessarily positive. Since the proofs follow along similar lines to those of the previous theorems, we will be brief in our discussion.

Let us suppose that $f : \mathbb{R} \to (0,\infty)$ is non-decreasing function such that

$$\text{(f-4)} : \quad \int_t^\infty \frac{ds}{\lambda^{-1}(f(s))} < \infty,\ \forall\, t \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^\infty \frac{ds}{\lambda^{-1}(f(s))} = \infty.$$

For instance given $\kappa \in \mathbb{R}$ the function $f_\kappa(t) = (t^2+\kappa^2)^k \exp t$, with $t \in \mathbb{R}$ is non-decreasing on $\mathbb{R}$ and satisfies (f-4). However none of the conditions (f-1), (f-2), (f-3) holds for f_κ.

Theorem 2.9. Suppose f is a non-decreasing function that satisfies Condition (f-4). Then Problem (2.1.3) has infinitely many bounded solutions.

Proof. Given $\alpha \in \mathbb{R}$, define $T : X \to X$ as

$$Tu(r) := \alpha - \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)dt,\ r \geq 0. \tag{2.2.1}$$

Since f is non-negative, it is clear that $Tu(r) \leq \alpha$ for all $r \geq 0$ and all $u \in X := C([0,\infty))$.

Let $\theta(\alpha) := \alpha - \lambda^{-1}(f(\alpha))B$, and set

$$\mathcal{C} := \{u \in C([0,\infty)) : \theta(\alpha) \leq u(r) \leq \alpha,\ \forall\, r \geq 0\}.$$

Since f is non-decreasing on $\mathbb{R}$, we also see that for $u \in \mathcal{C}$,

$$Tu(r) \geq \theta(\alpha),\ r \geq 0.$$

Therefore $T : \mathcal{C} \to \mathcal{C}$. As before, one can show that T is continuous and $T(\mathcal{C})$ relatively compact in X. Therefore by Schauder-Tychonoff Theorem, T has a fixed point $y \in \mathcal{C}$, and hence by Lemma 1.4, $u(x) := y(|x|)$ gives the desired solutions of Problem (2.1.3). □

Theorem 2.10. Suppose f is a non-decreasing function that satisfies (f-4). Then Problem (2.1.4) has infinitely many bounded solutions.

Proof. Let $\zeta : \mathbb{R} \to (0,\infty)$ and $\vartheta : (0,\infty) \to \mathbb{R}$ be defined as follows:

$$\zeta(t) = \int_t^\infty \frac{ds}{\lambda^{-1}(f(s))}, \quad \text{and} \quad \int_{\vartheta(t)}^\infty \frac{ds}{\lambda^{-1}(f(s))} = t.$$

We note that ζ is a decreasing function such that $\zeta(\infty) = 0$ and $\zeta(-\infty) = \infty$.

Fix $\beta \in \mathbb{R}$ such that $\zeta(\beta) > B$, where B is the constant in (b-1). Let us define

$$Tu(r) := \beta + \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)f(u(s))ds\right)dt, \quad r \geq 0. \tag{2.2.2}$$

Clearly, $Tu(r) \geq \beta$ for all $r \geq 0$. Let

$$\mathcal{E}(r) = \zeta(\beta) - \int_0^r \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)dt.$$

Notice that $0 < \zeta(\beta) - B < \mathcal{E}(r) \leq \zeta(\beta)$ and hence $\beta \leq \vartheta(\mathcal{E}(r)) \leq \vartheta(\zeta(\beta) - B)$ for all $r \geq 0$.

We consider the following convex subset of X.

$$\mathcal{C} := \{u \in C([0,\infty)) : \beta \leq u(r) \leq \vartheta(\mathcal{E}(r)),\ \forall\, r \geq 0\}.$$

We note that

$$\mathcal{C} \subseteq \{u \in C([0,\infty)) : \beta \le u(r) \le \tau(\beta) \; \forall \, r \ge 0\} \tag{2.2.3}$$

where $\tau(\beta) := \vartheta(\zeta(\beta) - B)$. We claim that for $u \in \mathcal{C}$,

$$Tu(r) \le \vartheta(\mathcal{E}(r)). \tag{2.2.4}$$

To see this, let

$$\mathcal{P}(r) = -Tu(r) + \vartheta(\mathcal{E}(r)), \; r \ge 0.$$

Note that $\mathcal{P}(0) = -\beta + \vartheta(\mathcal{E}(0)) = -\beta + \vartheta(\zeta(\beta)) = 0$.

Now, for all $r > 0$, we have

$$\begin{aligned}
\mathcal{P}'(r) &= -\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)f(u(s))ds\right) - \lambda^{-1}(f(\vartheta(\mathcal{E}(r))))\mathcal{E}'(r) \\
&\ge -\lambda^{-1}(f(\vartheta(\mathcal{E}(r))))\left[\Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}b(s)ds\right) + \mathcal{E}'(r)\right] \\
&= 0.
\end{aligned}$$

Therefore, $\mathcal{P}(r) \ge \mathcal{P}(0) = 0$ for all $r \ge 0$ and this proves (2.2.4). Thus $T : \mathcal{C} \mapsto \mathcal{C}$. Therefore, Schauder-Tychonoff Theorem shows that T has a fixed point $z \in \mathcal{C}$ and according to Lemma 1.4, $w(x) = z(|x|)$ is a solution of (2.1.4). □

Theorem 2.11. Suppose $h(x,t)$ does not change sign in $\mathbb{R}^N \times \mathbb{R}$, and ($h$-1) holds. If f is a non-decreasing function that satisfies (f-4), then Problem (1.1.1) has infinitely many bounded solutions that converge to constant.

Proof. Suppose $h(x,t) \le 0$ in $\mathbb{R}^N \times \mathbb{R}$. Corresponding to a given $\alpha \in \mathbb{R}$, let y_α be the solution of (2.1.3) given in Theorem 2.9 such that $\theta(\alpha) \le y_\alpha \le \alpha$. Since $y'_\alpha < 0$, we see that $\theta(\alpha) \le y_\alpha(\infty) \le y_\alpha(r) \le \alpha$ on $\mathbb{R}^+$. Let $w(x) := y_\alpha(|x|)$ and $v(x) := y_\infty(\infty)$.
Then w is a super-solution of (1.1.1) and v is a sub-solution of (1.1.1) such that $v \le w$ in $\mathbb{R}^N$. By Theorem 1.6, Problem (1.1.1) has a solution u such that $v \le u \le w$ in $\mathbb{R}^N$, and clearly $u(x) \to y_\alpha(\infty)$ as $|x| \to \infty$. Obviously, there are infinitely many such solutions.
The case when $h(x,t) \ge 0$ in $\mathbb{R}^N \times \mathbb{R}$ is proved similarly using Theorem 2.10, and is therefore omitted. □

Finally, we have the following theorem whose proof proceeds along the same lines as that of Theorem 2.8.

Theorem 2.12. Suppose f satisfies (f-4). Then equation (1.1.1) admits infinitely many positive bounded solutions.

Chapter 3

Ground State Solutions

3.1 Some Preliminaries

In this chapter, we study positive solutions of (1.1.1) that converge to zero at infinity. Such solutions are called ground state solutions of (1.1.1). As a preliminary step towards this goal, let us establish some results that will be useful for us later.

Let us remark that, for $N > \rho + 1$, we have in light of (1.1.6),

$$\int_1^\infty \Psi^{-1}(t^{1-N})dt \le \Psi^{-1}(1)\int_1^\infty \lambda^{-1}(t^{1-N})dt \le \Psi^{-1}(1)\int_1^\infty \frac{dt}{t^{(N-1)/\rho}} < \infty$$

In this section, we assume that $N > \rho + 1$. Consider the function

$$G(r) := -\int_r^\infty \Psi^{-1}\left(s^{1-N}/\omega_{N-1}\right) ds, \quad r > 0. \tag{3.1.1}$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Remark 3.1. As a side note we remark that $G(|x|)$ is a fundamental solution of Δ_ϕ in $\mathbb{R}^N$. To see this, let us first note that

$$\nabla G(|x|) = \Psi^{-1}(\omega_{N-1}^{-1}|x|^{1-N})\frac{x}{|x|}, \quad x \neq 0.$$

Therefore,

$$\begin{aligned} \phi(|\nabla G|)\nabla G &= \phi(\Psi^{-1}(w_{N-1}^{-1}|x|^{1-N}))\Psi^{-1}(w_{N-1}^{-1}|x|^{1-N})\frac{x}{|x|} \\ &= \frac{1}{\omega_{N-1}}\frac{x}{|x|^N} = \nabla\left(\frac{|x|^{2-N}}{\omega_{N-1}(2-N)}\right) = \nabla G_0, \end{aligned}$$

On observing that

$$G_0 := \frac{|x|^{2-N}}{\omega_{N-1}(2-N)}$$

is the fundamental solution for the Laplacian, we have

$$\int_{\mathbb{R}^N} \phi(|\nabla G|)\nabla G\cdot\nabla\varphi = \int_{\mathbb{R}^N}\nabla G_0\cdot\nabla\varphi = \varphi(0), \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^N).$$

This shows that G is a fundamental solution of Δ_ϕ.

In our subsequent discussion, it will be convenient to use the following variant of G.

$$\Gamma(r) := \int_r^\infty \Psi^{-1}(s^{N-1})ds, \quad \forall\, r > 0.$$

Since we assume $N > \rho + 1$, we remark that $\Gamma(r) \to 0$ as $r \to \infty$. For the purpose of the next lemma, given a non-negative function π on $(0, \infty)$, we assume that

$$\chi := \int_0^\infty s^{N-1}\pi(s)ds < \infty. \tag{3.1.2}$$

We note that if $\pi \geq 0$, then

$$\int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}\pi(s)ds\right)dt \leq \lambda^{-1}(\chi)\Gamma(r), \quad \forall\, r > 0.$$

As a straightforward applications of the inequality in (1.1.6), we have the following.

Lemma 3.2. Let π be a non-negative function for which (3.1.2) holds. Then

$$\int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}\pi(s)ds\right)dt \leq \mathcal{A}\int_r^\infty \Psi^{-1}(t^{1-N})dt, \quad \forall\, r > 0. \tag{3.1.3}$$

$$\mathcal{B}\int_r^\infty \Psi^{-1}(t^{1-N})dt \leq \int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}\pi(s)ds\right)dt, \quad \forall\, r \geq 1. \tag{3.1.4}$$

where

$$\mathcal{A} := \lambda^{-1}\left(\int_0^\infty s^{N-1}\pi(s)ds\right) \quad \text{and} \quad \mathcal{B} := \Lambda^{-1}\left(\int_0^1 s^{N-1}\pi(s)ds\right).$$

Let

$$\kappa(t) := \begin{cases} \Gamma(1) & \text{if} \quad 0 \leq t < 1 \\ \Gamma(t) & \text{if} \quad t \geq 1. \end{cases} \tag{3.1.5}$$

Remark 3.3. We remark that the following estimate holds for all $t \geq 1$.

$$\frac{C_\sigma}{t^{(N-(\sigma+1))/\sigma}} \leq \kappa(t) \leq \frac{C_\rho}{t^{(N-(\rho+1))/\rho}},$$

where

$$C_\sigma := \frac{\Psi^{-1}(1)\sigma}{N-(\sigma+1)} \quad \text{and} \quad C_\rho := \frac{\Psi^{-1}(1)\rho}{N-(\rho+1)}.$$

3.2 Ground State Solutions and Their Asymptotic Behavior

Let us now consider $h : \mathbb{R}^N \times \mathbb{R}^+ \to [0, \infty)$ such that

(h-2): $\quad g(|x|, t) \leq h(x, t) \leq f(|x|, t), \quad \forall\, (x, t) \in \mathbb{R}^N \times \mathbb{R}^+.$

and for some continuous functions $g, f : \mathbb{R}_0^+ \times \mathbb{R}^+ \to [0, \infty)$. In this section, we study entire solutions of

$$\Delta_\phi u = -h(x, u), \quad x \in \mathbb{R}^N, \tag{3.2.1}$$

under the following conditions on g and f.

(f-5): For each $s \in \mathbb{R}_0^+$, the function $t \mapsto g(s,t)$ and $t \mapsto f(s,t)$ are non-decreasing on $\mathbb{R}^+$ and there is $0 < \theta < \sigma$, such that

$$g(s, \varrho t) = \varrho^\theta g(s,t), \quad \text{and} \quad f(s,\varrho t) = \varrho^\theta f(s,t) \quad \forall (s,t,\varrho) \in \mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

(f-6): $\int_1^\infty s^{N-1} f(s,\kappa(s))ds < \infty$ and $g(s,1) > 0$ for some $0 \le s \le 1$.

Let us first note that

$$\begin{aligned}
&\int_0^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1} f(s,\kappa(s))ds\right)dt \\
&= \int_0^1 \Psi^{-1}\left(\int_0^t \left(\frac{s}{t}\right)^{N-1} f(s,\kappa(1))ds\right)dt + \int_1^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1} f(s,\kappa(s))ds\right)dt \\
&\le \Psi^{-1}\left(\int_0^1 f(s,\kappa(1))ds\right) + \lambda^{-1}\left(\int_0^\infty s^{N-1} f(s,\kappa(s))ds\right)\int_1^\infty \Psi^{-1}(t^{1-N})dt. \qquad (3.2.2)
\end{aligned}$$

Theorem 3.4. Suppose h satisfies (h-2) where g and f satisfy (f-5) and (f-6). Then Problem (3.2.1) has a solution $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ such that for some positive constant $C \ge 1$ we have

$$C^{-1}\Gamma(|x|) \le u(x) \le C\Gamma(|x|) \quad \forall\, |x| \ge 1. \qquad (3.2.3)$$

Proof. In the proof, we will use the following easily verifiable inequalities.

$$\lambda^{-1}(ts) \le \lambda^{-1}(t)\lambda^{-1}(s) \quad \text{and} \quad \Lambda^{-1}(st) \ge \Lambda^{-1}(t)\Lambda^{-1}(s), \quad \forall\, s,t \ge 0. \qquad (3.2.4)$$

Let

$$y(r) = \int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1} f(s, c\kappa(s))ds\right)dt, \qquad (3.2.5)$$

where $c > 1$ is a constant to be determined. For $0 \le r \le 1$, we use (3.2.2) to get

$$\begin{aligned}
y(r) &\le \lambda^{-1}(c^\theta)\left[\Psi^{-1}\left(\int_0^t f(s,\kappa(1))ds\right) + \lambda^{-1}\left(\int_0^t s^{N-1} f(s,\kappa(s))ds\right)\kappa(1)\right] \\
&\le c^{\frac{\theta}{\sigma}}\left[\frac{1}{\kappa(1)}\Psi^{-1}\left(\int_0^1 f(s,\kappa(1))ds\right) + \lambda^{-1}\left(\int_0^t s^{N-1} f(s,\kappa(s))ds\right)\right]\kappa(1).
\end{aligned}$$

Now, we assume that $r \ge 1$. Then by (3.1.3) of Lemma 3.2, we see that

$$\begin{aligned}
y(r) &\le \lambda^{-1}\left(\int_0^\infty s^{N-1} f(s,c\kappa(s))ds\right)\kappa(r) = c^{\frac{\theta}{\sigma}}\lambda^{-1}\left(\int_0^\infty s^{N-1} f(s,\kappa(s))ds\right)\kappa(r) \\
&\le c^{\frac{\theta}{\sigma}}\left[\frac{1}{\kappa(1)}\Psi^{-1}\left(\int_0^1 f(s,\kappa(s))ds\right) + \lambda^{-1}\left(\int_0^\infty s^{N-1} f(s,\kappa(s))ds\right)\right]\kappa(r).
\end{aligned}$$

In conclusion, we have shown that for all $r \ge 0$

$$y(r) \le c^{\frac{\theta}{\sigma}}\left[\frac{1}{\kappa(1)}\Psi^{-1}\left(\int_0^1 f(s,\kappa(s))ds\right) + \lambda^{-1}\left(\int_0^\infty s^{N-1} f(s,\kappa(s))ds\right)\right]\kappa(r).$$

Now, we choose $c > 1$ sufficiently big such that

$$c \geq \left[\frac{1}{\kappa(1)}\Psi^{-1}\left(\int_0^1 f(s,\kappa(s))ds\right) + \lambda^{-1}\left(\int_0^\infty s^{N-1}f(s,\kappa(s))ds\right)\right]^{\frac{\sigma}{\sigma-\theta}}.$$

Such a choice of c leads to

$$y(r) \leq c\kappa(r), \quad \forall\, r \geq 0 \tag{3.2.6}$$

and as a consequence, we have

$$(r^{N-1}\Psi(|y'|))' = r^{N-1}f(r, c\kappa(r)) \geq r^{N-1}f(r, y(r)), \quad r > 0.$$

In other words, we have

$$(r^{N-1}\phi(|y'|)y')' \leq -r^{N-1}f(r, y(r)), \quad r > 0. \tag{3.2.7}$$

From (3.2.5), we see that

$$\begin{aligned} 0 \leq |y'(r)| &= \Psi^{-1}\left(r^{1-N}\int_0^r s^{N-1}f(s, c\kappa(s))ds\right) \\ &\leq \begin{cases} \Psi^{-1}\left(\int_0^1 f(s, c\kappa(1))ds\right) & \text{if} \quad 0 < r \leq 1 \\ \Psi^{-1}\left(\int_0^\infty s^{N-1}f(s, c\kappa(s))ds\right) & \text{if} \quad r \geq 1. \end{cases} \end{aligned}$$

Therefore, for all $r \geq 0$,

$$|y'(r)| \leq \max\left\{\Psi^{-1}\left(\int_0^1 f(s, c\kappa(1))ds\right), \Psi^{-1}\left(\int_0^\infty s^{N-1}f(s, c\kappa(s))ds\right)\right\}.$$

Recalling Condition (ϕ-2), we also note that $y'(0+) = 0$. This together with (f-6) shows that y' is bounded on $[0, \infty)$.

Next, let

$$z(r) = \int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}g(s, a\kappa(s))ds\right)dt,$$

where $0 < a < 1$ is a constant to be determined. If $0 < r \leq 1$, then by Lemma 3.2

$$\begin{aligned} z(r) &\geq \int_1^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}g(s, a\kappa(s))ds\right)dt \\ &\geq \Lambda^{-1}\left(\int_0^1 s^{N-1}g(s, a\kappa(s))ds\right)\int_1^\infty \Psi^{-1}(t^{1-N})dt \\ &= \Lambda^{-1}\left(a^\theta \int_0^1 s^{N-1}g(s, \kappa(1))ds\right)\kappa(1) \\ &= a^{\frac{\theta}{\sigma}}\Lambda^{-1}\left(\int_0^1 s^{N-1}g(s, \kappa(1))ds\right)\kappa(r). \end{aligned}$$

Suppose now $r \geq 1$. Then

$$z(r) \geq \int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}g(s, a\kappa(s))ds\right)dt$$

$$\geq \Lambda^{-1}\left(\int_0^1 s^{N-1}g(s,a\kappa(1))ds\right)\int_r^\infty \Psi^{-1}(t^{1-N})dt$$
$$\geq a^{\frac{\theta}{\sigma}}\Lambda^{-1}\left(\int_0^1 s^{N-1}g(s,\kappa(1))ds\right)\kappa(r).$$

In any case, we have shown that

$$z(r) \geq a^{\frac{\theta}{\sigma}}\Lambda^{-1}\left(\int_0^1 s^{N-1}g(s,\kappa(1))ds\right)\kappa(r), \quad \forall\, r > 0.$$

Now, we choose $0 < a < \min\{1, c\}$ small enough such that

$$0 < a \leq \left[\Lambda^{-1}\left(\int_0^1 s^{N-1}g(s,\kappa(1))ds\right)\right]^{\frac{\sigma}{\sigma-\theta}}.$$

Such a choice of the constant a leads to the estimate

$$z(r) \geq a\kappa(r), \quad \forall\, r \geq 0. \tag{3.2.8}$$

Consequently, we have

$$(r^{N-1}\Psi(|z'|))' = r^{N-1}g(r, a\kappa(r)) \leq r^{N-1}g(r, z(r)), \quad r > 0.$$

Thus

$$(r^{N-1}\phi(|z'|)z')' \geq -r^{N-1}g(r, z(r)), \quad r > 0. \tag{3.2.9}$$

Clearly, we have $z'(0+) = 0$. The same argument used to show that y' is bounded shows that z' is bounded on $(0, \infty)$. In fact, on recalling $a \leq 1$, we have

$$|z'(r)| \leq \max\left\{\Psi^{-1}\left(\int_0^1 g(s,\kappa(1))ds\right), \Psi^{-1}\left(\int_0^\infty s^{N-1}g(s,\kappa(s))ds\right)\right\}$$

holds on $[0, \infty)$.

Now, let $v(x) := z(|x|)$ and $w(x) := y(|x|)$ for $x \in \mathbb{R}^N$. Then $v, w \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ and Lemma 1.4 shows that w is a super-solution and v is a sub-solution of (3.2.1) in $\mathbb{R}^N$. Condition (h-2) shows that $v \leq w$ in $\mathbb{R}^N$. By Theorem 1.6, there is $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ such that $v \leq u \leq w$ in $\mathbb{R}^N$. But from (3.2.6) and (3.2.8), we conclude that

$$a\Gamma(|x|) \leq z(|x|) = v(x) \leq u(x) \leq w(x) = y(|x|) \leq c\Gamma(|x|), \quad \forall\, |x| \geq 1.$$

This concludes the proof of Theorem 3.4. □

Our next result shows that the above theorem remains true if $g(s,t)$ and $f(s,t)$ are both non-increasing in $t > 0$ and are homogeneous with negative degree of homogeneity in the t variable. More precisely, let us replace (f-5) by the following conditions on f and g.

(f-5)* : For each $s \in \mathbb{R}^+$, the function $t \mapsto g(s,t)$ and $t \mapsto f(s,t)$ are non-increasing on $\mathbb{R}^+$ and a constant $\vartheta \geq 0$ such that

$$g(s, \varrho t) = \varrho^{-\vartheta} g(s,t), \quad \text{and} \quad f(s, \varrho t) = \varrho^{-\vartheta} f(s,t), \quad \forall\, (s,t,\varrho) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

Theorem 3.5. Suppose h satisfies (h-2) where g and f satisfy (f-5)* and (f-6). Then Problem (3.2.1) has a solution $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ such that for some positive constant C we have

$$C^{-1}\Gamma(|x|) \leq u(x) \leq C\Gamma(|x|), \quad \forall\, |x| \geq 1.$$

Proof. Let

$$y(r) := \int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1} f(s, c\kappa(s))ds\right) dt \tag{3.2.10}$$

where $0 < c < 1$ is a constant to be determined. For $r \geq 1$, we have

$$\begin{aligned} y(r) &\geq \Lambda^{-1}\left(\int_0^1 s^{N-1} f(s, c\kappa(s))ds\right)\kappa(r) \\ &= \Lambda^{-1}\left(c^{-\vartheta}\int_0^1 s^{N-1} f(s, \kappa(s))ds\right)\kappa(r) \\ &\geq \Lambda^{-1}(c^{-\vartheta})\Lambda^{-1}\left(\int_0^1 s^{N-1} f(s, \kappa(s))ds\right)\kappa(r) \\ &\geq c^{-\frac{\vartheta}{\rho}}\Lambda^{-1}\left(\int_0^1 s^{N-1} f(s, \kappa(s))ds\right)\kappa(r). \end{aligned}$$

For $0 < r \leq 1$,

$$\begin{aligned} y(r) &\geq \int_1^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1} f(s, c\kappa(s))ds\right) dt \\ &\geq c^{-\frac{\vartheta}{\rho}}\Lambda^{-1}\left(\int_0^1 s^{N-1} f(s, \kappa(s))ds\right)\Gamma(1). \end{aligned}$$

Therefore,

$$y(r) \geq c^{-\frac{\vartheta}{\rho}}\Lambda^{-1}\left(\int_0^1 s^{N-1} f(s, \kappa(s))ds\right)\kappa(r), \quad r \geq 0.$$

We choose $c > 0$ small enough such that

$$0 < c \leq \min\left\{1, \left[\Lambda^{-1}\left(\int_0^1 s^{N-1} f(s, \kappa(s))ds\right)\right]^{\frac{\rho}{\rho+\vartheta}}\right\}.$$

This choice shows that

$$y(r) \geq c\kappa(r), \quad r \geq 0,$$

and as recalling that $f(s,t)$ is non-increasing in t for each fixed $s \geq 0$, we have

$$(r^{N-1}\Psi(|y'|))' = r^{N-1} f(r, c\kappa(r)) \geq r^{N-1} f(r, y(r)), \quad r > 0.$$

In other words, we have

$$(r^{N-1}\phi(|y'|)y')' \le -r^{N-1}f(r,y(r)), \quad r>0. \tag{3.2.11}$$

Let

$$z(r) := \int_r^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}g(s,d\kappa(s))ds\right)dt, \tag{3.2.12}$$

where $d>1$ will be chosen shortly.

Again, by (3.1.3) of Lemma 3.2, we have

$$\begin{aligned} z(r) &\le \lambda^{-1}\left(\int_0^\infty s^{N-1}g(s,d\kappa(s))ds\right)\Gamma(r) \\ &\le \lambda^{-1}(d^{-\vartheta})\lambda^{-1}\left(\int_0^\infty s^{N-1}g(s,\kappa(s))ds\right)\Gamma(r) \\ &= d^{-\frac{\vartheta}{\sigma}}\lambda^{-1}\left(\int_0^\infty s^{N-1}g(s,\kappa(s))ds\right)\Gamma(r). \end{aligned}$$

Let us now choose d such that

$$d \ge \max\left\{1, \left[\lambda^{-1}\left(\int_0^\infty s^{N-1}g(s,\kappa(s))ds\right)\right]^{\frac{\rho}{\rho+\vartheta}}\right\}.$$

Such a choice of d leads to the inequality

$$z(r) \le d\kappa(r), \quad r\ge 0.$$

Then, we have

$$(r^{N-1}(\Psi^{-1}(|z'|))' = r^{N-1}g(r,d\kappa(r)) \le r^{N-1}g(r,z(r)), \quad r>0.$$

That is,

$$(r^{N-1}\phi(|z'|)z')' \ge -r^{N-1}g(r,z(r)), \quad r>0.$$

Let $v(x) := z(|x|)$ and $w(x) := y(|x|)$ for $x\in\mathbb{R}^N$. The choice of the positive constants c and d shows that $v\le w$ in $\mathbb{R}^N$. Furthermore, v is a sub-solution while w is a super-solution of Problem (3.2.1). Therefore, Problem (3.2.1) admits a solution u for which the stated estimates hold for $|x|\ge 1$. □

As an example, let us consider the problems

$$\Delta_\phi u = -b(|x|)u^\gamma, \quad x\in\mathbb{R}^N \quad \text{and} \tag{3.2.13}$$

$$\Delta_\phi u = -b(|x|)u^{-\gamma}, \quad x\in\mathbb{R}^N, \tag{3.2.14}$$

where $b:[0,\infty)\to(0,\infty)$ is continuous.

Let us first consider Problem (3.2.13). Note that Condition (f-5) holds if $0\le\gamma<\sigma$, using Remark 3.3, we see that

$$s^{N-1}f(\kappa(s)) = s^{N-1}\kappa(s)^{\gamma} \leq C_{\rho}s^{((\rho-\gamma)(N-1)+\gamma\rho)/\rho}, \ s \geq 1.$$

Therefore, (f-6) holds if b satisfies

$$\int_1^{\infty} s^{((\rho-\gamma)(N-1)+\gamma\rho)/\rho}b(s)ds < \infty. \tag{3.2.15}$$

Theorem 3.4 guarantees a solution u of (3.2.13) that satisfies (3.2.3) if $0 < \gamma < \sigma$ and b decays at infinity according (3.2.15).

On the other hand, Theorem 3.5 shows that for any $\gamma > 0$, Problem (3.2.14) has a solution that satisfies the estimate (3.2.3) provided b decays according to (3.2.15) with γ replaced by $-\gamma$ in Condition (3.2.15).

For the special case of the Laplacian, that is, when $\phi(t) = 1$, Theorem 3.4 shows that Problem (3.2.13) admits a solution that satisfies (3.2.3) if $0 \leq \gamma < 1$ and b satisfies

$$\int_1^{\infty} s^{N-1-\gamma(N-2)}b(s)ds < \infty. \tag{3.2.16}$$

On the other hand, for the Laplacian, Problem (3.2.14) admits a solution that satisfies (3.2.3) for any $\gamma \geq 0$ and b satisfying (3.2.16) with γ replaced by $-\gamma$.

Chapter 4

Entire Solutions with Prescribed Limits at Infinity

4.1 A General Result

In this chapter, we consider positive solutions $u(x)$ of

$$\Delta_\phi u = a(x)f(u), \quad x \in \mathbb{R}^N. \tag{4.1.1}$$

that converge to predetemined constants as $|x| \to \infty$. The case $\phi \equiv 1$ was discussed by M. Naito [34]. Here, and throughout this section, f is a continuous and non-decreasing function on $(0, \infty)$ such that $f(t) > 0$ for $t > 0$. We also assume that a is a continuous function such that

$$-a_*(|x|) \leq a(x) \leq a^*(|x|), \quad x \in \mathbb{R}^N, \tag{$\triangle$}$$

for some continuous functions $a_*(t)$ and $a^*(t)$ on $\mathbb{R}_0^+$ that are non-negative for $t \geq 0$ and $a_*(t)\, a^*(t) > 0$ for some $t > 0$. We assume both of the following hold.

$$\textbf{(a-1):}\ A_* := \int_0^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}a_*(s)ds\right)dt < \infty.$$

$$\textbf{(a-2):}\ A^* := \int_0^\infty \Psi^{-1}\left(t^{1-N}\int_0^t s^{N-1}a^*(s)ds\right)dt < \infty.$$

As noted in Chapter 2, $\sigma < N-1$ is necessary for these to hold. It will be convenient to introduce the following functions defined on $[0, \infty)$.

$$E^*(\xi) := \int_0^\infty \Psi^{-1}\left(\xi t^{1-N}\int_0^t s^{N-1}a^*(s)ds\right)dt, \quad \text{and}$$

$$E_*(\xi) := \int_0^\infty \Psi^{-1}\left(\xi t^{1-N}\int_0^t s^{N-1}a_*(s)ds\right)dt.$$

Note that $\Lambda^{-1}(\xi)A_* \leq E_*(\xi) \leq \lambda^{-1}(\xi)A_*$ and $\Lambda^{-1}(\xi)A^* \leq E^*(\xi) \leq \lambda^{-1}(\xi)A^*$ for all $\xi \geq 0$.

Following [34], we consider the following two sets of positive numbers in terms of which we state the main result of this section.

$$\Xi^* := \{\ell > 0 : \ell - E^*(f(\ell)) > 0\} \quad \text{and}$$

$$\Xi_* := \{\ell > 0 : \ell = c - E_*(f(c)) \quad \text{for some} \quad c > 0\}.$$

Our main result of this section is the following theorem.

Theorem 4.1. If $\ell \in \Xi_* \cap \Xi^*$, then there is a positive solution $u \in W^{1,\Phi}(\mathbb{R}^N)$ of (4.1.1) such that

$$\lim_{|x|\to\infty} u(x) = \ell. \tag{4.1.2}$$

Proof. Let $\ell \in \Xi_* \cap \Xi^*$. Then $\ell - E^*(f(\ell)) \geq 0$, $\ell > 0$ and there is $c > 0$ such that $\ell = c - E_*(f(c))$. We consider the following two functions.

$$y(r) := \ell - \int_r^\infty \Psi^{-1}\left(f(\ell)t^{1-N}\int_0^t s^{N-1}a^*(s)ds\right)dt, \quad r \geq 0 \tag{4.1.3}$$

$$z(r) := c - \int_0^r \Psi^{-1}\left(f(c)t^{1-N}\int_0^t s^{N-1}a_*(s)ds\right)dt, \quad r \geq 0. \tag{4.1.4}$$

Since $\ell - E^*(f(\ell)) > 0$ and $c - E_*(f(c)) = \ell > 0$, we see that $y(r) > 0$ and $z(r) > 0$ for all $r \geq 0$. Moreover, we note that

$$y'(r) \geq 0, \quad \forall\, r > 0, \quad \lim_{r\to 0^+} y(r) = \ell - E^*(f(\ell)), \quad \text{and} \quad \lim_{r\to\infty} y(r) = \ell.$$

Therefore y satisfies

$$\begin{aligned} &(r^{N-1}\Psi(|y'(r)|))' = r^{N-1}a^*(r)f(\ell), \quad r > 0 \\ &0 \leq \ell - E^*(f(\ell)) \leq y(r) \leq \ell, \quad \forall\, r \geq 0. \end{aligned}$$

Let us note that

$$(r^{N-1}\Psi(|y'(r)|))' = r^{N-1}a^*(r)f(\ell) \geq r^{N-1}a^*(r)f(y(r)), \quad r > 0. \tag{4.1.5}$$

Similarly,

$$z'(r) \leq 0, \quad \forall\, r > 0, \quad \lim_{r\to 0^+} z(r) = c, \quad \text{and} \quad \lim_{r\to\infty} z(r) = c - E_*(f(c)).$$

Therefore, z satisfies

$$\begin{aligned} &(r^{N-1}\Psi(|z'(r)|))' = r^{N-1}a_*(r)f(c), \quad r > 0 \\ &0 < c - E_*(f(c)) \leq z(r) \leq c, \quad \forall\, r \geq 0. \end{aligned}$$

Here also we notice that

$$(r^{N-1}\Psi(|z'(r)|))' = r^{N-1}a_*(r)f(c) \geq r^{N-1}a_*(r)f(z(r)), \quad r > 0. \tag{4.1.6}$$

Now, let $v(x) := y(|x|)$ for $x \in \mathbb{R}^N$, and $w(x) := z(|x|)$ for all $x \in \mathbb{R}^N$. By the choice of c in the definition (4.1.4) of z, we see that

$$\ell - E^*(f(\ell)) \le v(x) \le \ell, \quad \ell = c - E_*(f(c)) \le w(x) \le c, \quad (x \in \mathbb{R}^N).$$

From (4.1.5) and (4.1.6), we see that, for $r > 0$,

$$(r^{N-1}\phi(|y'|)y')' \ge r^{N-1}a^*(r)f(y(r)) \quad \text{and} \quad (r^{N-1}\phi(|z'|)z')' \le -r^{N-1}a_*(r)f(z(r)).$$

It follows that w and v are both positive functions that belong to $W^{1,\Phi}_{loc}(\mathbb{R}^N)$, and on recalling Lemma 1.4 and Condition($\triangle$), we see the following hold.

$$\begin{aligned} \Delta_\phi v &\ge a(x)f(v), \quad (x \in \mathbb{R}^N), \quad \lim_{|x|\to\infty} v(x) = \ell \\ \Delta_\phi w &\le a(x)f(w), \quad (x \in \mathbb{R}^N), \quad \lim_{|x|\to\infty} w(x) = \ell. \end{aligned}$$

Therefore, we invoke Theorem 1.6 to conclude the existence of a solution $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ of (4.1.1) such that $v \le u \le w$ in $\mathbb{R}^N$, and hence $u(x) \to \ell$ as $|x| \to \infty$. □

4.2 Some Sufficient Conditions and an Example

Let us now describe some sufficient conditions on f that would ensure the set $\Xi_* \cap \Xi^*$ is non-empty.

(f-7) : $\quad t \mapsto \dfrac{\lambda^{-1}(f(t))}{t}$ is non-increasing in $(0,\infty)$, and

$$\lim_{t\to\infty} \frac{\lambda^{-1}(f(t))}{t} < \min\left\{\frac{1}{A_*}, \frac{1}{A^*}\right\} < \lim_{t\to 0^+} \frac{\lambda^{-1}(f(t))}{t}.$$

(f-8) : $\quad t \mapsto \dfrac{\lambda^{-1}(f(t))}{t}$ is non-decreasing in $(0,\infty)$, and

$$\lim_{t\to 0^+} \frac{\lambda^{-1}(f(t))}{t} < \min\left\{\frac{1}{A_*}, \frac{1}{A^*}\right\} \le \max\left\{\frac{1}{A_*}, \frac{1}{A^*}\right\} < \lim_{t\to\infty} \frac{\lambda^{-1}(f(t))}{t}.$$

Theorem 4.2. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing continuous function.
(1) If f satisfies (f-7), then there is τ such that for each $\ell > \tau$, Problem (4.1.1) has a positive solution for which (4.1.2) holds.
(2) f satisfies (f-8), then there is τ such that for each $0 < \ell < \tau$, Problem (4.1.1) has a positive solution satisfying (4.1.2).

Proof. We start by remarking that, as a consequence of the dominated convergence theorem, $t \mapsto E_*(f(t))$ is a continuous function in $\mathbb{R}^+$.

Let us assume first that condition (f-7) holds. Then, we note that the function $g(t) := t - \lambda^{-1}(f(t))\max\{A_*, A^*\}$ is non-decreasing on $\mathbb{R}^+$, is negative near $t = 0$ and positive at infinity. Therefore, there is $\tau \geq 0$ such that $g(\tau) = 0$ and $g(t) > 0$ for $t > \tau$. Moreover, we see that

$$\begin{aligned}
\lim_{t\to\infty}[t - E_*(f(t))] &\geq \lim_{t\to\infty}(t - \lambda^{-1}(f(t))A_*) \\
&\geq \max\{A_*, A^*\}\lim_{t\to\infty} t\left(\frac{1}{\max\{A_*, A^*\}} - \frac{\lambda^{-1}(f(t))}{t}\right) = \infty. \qquad (4.2.1)
\end{aligned}$$

Let $\ell > \tau$ so that $\ell > \tau - E_*(f(\tau))$. From this and (4.2.1), we see that there is $c > \tau$ such that $c - E_*(f(c)) = \ell$. That is, $\ell \in \Xi_*$. Taking $\ell > \tau' > \tau$ and on using Condition (f-7), we have

$$\begin{aligned}
\ell - E^*(f(\ell)) &\geq \ell - \lambda^{-1}(f(\ell))\max\{A_*, A^*\} = \ell\left(1 - \frac{\lambda^{-1}(f(\ell))\max\{A_*, A^*\}}{\ell}\right) \\
&\geq \ell\left(1 - \frac{\lambda^{-1}(f(\tau'))\max\{A_*, A^*\}}{\tau'}\right) \\
&> 0,
\end{aligned}$$

and hence $\ell \in \Xi^*$. An appeal to Theorem 4.1 shows that (4.1.2) holds in this case. Suppose now Condition (f-8) holds. Note that $f(0) = 0$ in this case. Moreover $g(t) := t - \lambda^{-1}(f(t))\max\{A_*, A^*\}$ is negative at infinity, positive in $(0, \delta)$ for sufficiently small $\delta > 0$, and $g(0) = 0$. Therefore, let

$$\tau := \sup_{t>0}(t - \lambda^{-1}(f(t))\max\{A_*, A^*\}) > 0. \qquad (4.2.2)$$

We pick $\gamma > 0$ such that

$$\tau = \gamma - \lambda^{-1}(f(\gamma))\max\{A_*, A^*\}. \qquad (4.2.3)$$

Given $0 < \ell < \tau$ from (4.2.3), we see that

$$0 < \ell < \gamma - E_*(f(\gamma)).$$

Thus, there is $0 < c < \gamma$ such that $c - E_*(f(c)) = \ell$ and hence $\ell \in \Xi_*$. Since $0 < \ell < \tau < \gamma$, it follows from Condition (f-8) that

$$\begin{aligned}
\ell - E^*(f(\ell)) &\geq \ell\left(1 - \frac{\lambda^{-1}(f(\ell))\max\{A_*, A^*\}}{\ell}\right) \geq \ell\left(1 - \frac{\lambda^{-1}(f(\gamma))\max\{A_*, A^*\}}{\gamma}\right) \\
&= \frac{\ell\tau}{\gamma} > 0.
\end{aligned}$$

Therefore $\ell \in \Xi^*$. We invoke Theorem 4.1 to prove existence of positive solution that satisfies (4.1.2). □

By way of illustrating the above theorem, let $f(t) = t^\kappa$ for $\kappa > 0$ and let $a(x)$ be a continuous function that satisfies the assumptions described at the beginning of this section. For simplicity, we assume that $A := A_* = A^*$. Suppose $0 < \kappa < \sigma$. Then f satisfies Condition (f-7). According to the proof of Theorem 4.2, we choose $\tau > 0$ such that

$$0 = \tau - \lambda^{-1}(f(\tau))A = \tau - \max\{\tau^{\kappa/\sigma}, \tau^{\kappa/\rho}\}A.$$

Thus,

$$\tau := \begin{cases} A^{\sigma/(\sigma-\kappa)} & \text{if} \quad A > 1 \\ A^{\rho/(\rho-\kappa)} & \text{if} \quad A \le 1. \end{cases}$$

Consequently, Theorem 4.2 shows that for any $\ell > \tau$, Problem (4.1.1) admits a positive solution u that satisfies (4.1.2).

Suppose $\kappa = \sigma$. If $A < 1$, then Condition (f-7) holds. In this case given any $\ell > 0$, Problem (4.1.1) admits a solution that satisfies (4.1.2).

Suppose now $\kappa > \rho$. In this case Condition (f-8) holds. Following the proof of Theorem (4.2), we find that

$$\tau := \begin{cases} \left(\dfrac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)} \left(1 - \dfrac{\sigma}{\kappa}\right) & \text{if} \quad 0 < A \le \dfrac{\sigma}{\kappa} \\ 1 - A & \text{if} \quad \frac{\sigma}{\kappa} < A < \frac{\rho}{\kappa} \\ \left(\dfrac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)} \left(1 - \dfrac{\rho}{\kappa}\right) & \text{if} \quad A \ge \dfrac{\rho}{\kappa}. \end{cases}$$

According to Theorem 4.1, given $0 < \ell < \tau$, we see that Problem (4.1.1) has a positive solution u that satisfies (4.1.2). To see this, note that

$$\begin{aligned} G(t) &:= t - \lambda^{-1}(f(t))A = t - \lambda^{-1}(t^\kappa)A \\ &= \begin{cases} t - t^{\kappa/\rho}A & \text{if} \quad 0 \le t \le 1 \\ t - t^{\kappa/\sigma}A & \text{if} \quad t > 1 \end{cases} \end{aligned}$$

is continuous on $\mathbb{R}_0^+$, is differentiable on $\mathbb{R}^+ \setminus \{1\}$ and

$$G'(t) = \begin{cases} 1 - \frac{\kappa}{\rho} t^{(\kappa-\rho)/\rho} A & \text{if} \quad 0 < t < 1 \\ 1 - \frac{\kappa}{\sigma} t^{(\kappa-\sigma)/\sigma} A & \text{if} \quad t > 1. \end{cases}$$

Suppose $0 < A < \dfrac{\sigma}{\kappa}$ so that $\dfrac{\sigma}{\kappa A} > 1$. Note that G' vanishes at

$$\left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)}.$$

Note that $\left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)} \geq \left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)}$. Hence G is increasing in $0 < t < \left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)}$, and decreasing in $\left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)} < t < \infty$. Therefore, G attains its maximum at $\left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)}$ and this maximum is

$$\left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)} \left(1 - \frac{\sigma}{\kappa}\right).$$

Now, suppose $\frac{\sigma}{\kappa} < A < \frac{\rho}{\kappa}$. Then G is increasing on $0 < t < 1$ since $g(t) = t - t^{\kappa/\rho}A$ is increasing in $0 < t < \left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)}$ and $\left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)} > 1$.

On the other hand, $h(t) = t - t^{\kappa/\sigma}A$ is decreasing in $t > \left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)}$ and hence on $(1, \infty)$, since $\left(\frac{\sigma}{\kappa A}\right)^{\sigma/(\kappa-\sigma)} < 1$.

Therefore, in this case G takes its maximum value of $1 - A$ at $t = 1$. Finally, suppose $A \geq \frac{\rho}{\kappa}$. In this case, note that

$$\frac{\sigma}{\kappa A} \leq \frac{\rho}{\kappa A} \leq 1.$$

Therefore, both $g(t) = t - t^{\kappa/\rho}A$ and $h(t) = t - t^{\kappa/\sigma}A$ are decreasing on

$$\left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)} \leq t < \infty,$$

and G is increasing in $0 < t < \left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)}$. Therefore G takes its maximum value

$$\left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)} \left(1 - \frac{\rho}{\kappa}\right) \quad \text{at} \quad \left(\frac{\rho}{\kappa A}\right)^{\rho/(\kappa-\rho)}.$$

Finally, suppose $\kappa = \rho$ and $A < 1$. If $\sigma < \rho$, then Condition (f-8) holds. Even if $\rho = \sigma$, note that (4.2.2) and the left-hand side of Condition (f-8) holds. As a result, the argument used in the proof of Theorem 4.2 under the assumption of Condition (f-8) is still valid. Therefore, in this case given any $\ell > 0$, Problem (4.1.1) has a positive solution that satisfies (4.1.2).

Chapter 5

Cauchy-Liouville Type Theorems

In this chapter, we focus on Cauchy-Liouville type results for solutions of the PDE

$$\Delta_\phi u = \pm f(u)$$

where f is a continuous function.

The following comparison principle will be useful in our proof of Theorem 5.3 below.

Lemma 5.1. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and suppose ϕ satisfies (ϕ-1) and (ϕ-2) and $g \in L^\infty_{loc}(\Omega \times \mathbb{R})$ such that $t \mapsto g(x,t)$ is a non-decreasing function in $\mathbb{R}$ for each $x \in \Omega$. Suppose $u, v \in W^{1,\Phi}(\Omega) \cap C(\Omega)$ satisfy

$$\Delta_\phi u \geq g(x,u),\ x \in \Omega \quad \text{and} \quad \Delta_\phi v \leq g(x,v),\ x \in \Omega.$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Recall that by our definition of sub-solution and super-solution, we have $g(x,u) \in L^{\widetilde{\Phi}}(\Omega)$ and $g(x,v) \in L^{\widetilde{\Phi}}(\Omega)$. We note that $(u-v)^+ \in W_0^{1,\Phi}(\Omega)$ and hence using this as a test function, we have

$$\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla(u-v)^+ \leq -\int_\Omega g(x,u)(u-v)^+ \quad \text{and}$$
$$\int_\Omega \phi(|\nabla v|)\nabla v \cdot \nabla(u-v)^+ \geq -\int_\Omega g(x,v)(u-v)^+.$$

Combining these two inequalities we find

$$\begin{aligned}
&\int_{u>v} (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v) \\
&= \int_\Omega (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v) \cdot \nabla(u-v)^+ \\
&\leq -\int_\Omega (g(x,u) - g(x,v))(u-v)^+ \\
&= -\int_{u>v} (g(x,u) - g(x,v))(u-v).
\end{aligned}$$

Since $g(x,\cdot)$ is non-decreasing for each $x \in \Omega$, we see that $g(x,u(x)) \geq g(x,v(x))$ on the set $\{x \in \Omega : u(x) \geq v(x)\}$. Therefore we conclude that

$$\int_{u>v} (\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v) \leq 0.$$

On the other hand, the inequality (1.2.30) shows that on the set $\{x \in \Omega : u(x) > v(x)\}$

$$(\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v) \cdot (\nabla u - \nabla v) = 0.$$

But, in view of (1.2.30), this is impossible. Therefore we must have $u \leq v$ on Ω. □

5.1 Absorption Terms of Keller-Osserman Type

We begin this section by making note of some consequence of (ϕ-3) that will be needed for our investigation.

On multiplying both sides of (1.1.4) by s and then integrating on $(0,t)$ for $t > 0$, we find that

$$s\lambda(s)\int_0^t \Psi(\tau)d\tau \leq \int_0^t \Psi(s\tau)sd\tau \leq s\Lambda(s)\int_0^t \Psi(\tau)d\tau.$$

That is,

$$\widetilde{\lambda}(s)\Phi(t) \leq \Phi(st) \leq \widetilde{\Lambda}(s)\Phi(t), \quad s,t > 0,$$

where

$$\widetilde{\lambda}(s) = \lambda(s)s, \quad \text{and} \quad \widetilde{\Lambda}(s) = \Lambda(s)s.$$

Then, it follows that

$$\widetilde{\Lambda}^{-1}(s)\Phi^{-1}(t) \leq \Phi^{-1}(st) \leq \widetilde{\lambda}^{-1}(s)\Phi^{-1}(t). \tag{5.1.1}$$

Since $\sigma \leq \rho$, where σ and ρ are the parameters in Condition (ϕ-3) for $s > 1$, we have

$$\Lambda^{-1}(s) = \min\{s^{1/\sigma}, s^{1/\rho}\} = s^{1/\rho} \quad \text{and} \quad \lambda^{-1}(s) = \max\{s^{1/\sigma}, s^{1/\rho}\} = s^{1/\sigma}.$$

Therefore, for $s > \varrho > 1$, we have

$$\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})} = \frac{\varrho^{(N-1)(\rho+1)/\rho}}{s^{(N-1)(\sigma+1)/\sigma}}. \tag{5.1.2}$$

Since $(\rho+1)/\rho \leq (\sigma+1)/\sigma$, we see that the right-hand side of (5.1.2) is less than one whenever $s > \varrho > 1$. Now, let us note that

$$\widetilde{\Lambda}(s) = \Lambda(s)s = \max\{s^{\sigma+1}, s^{\rho+1}\}.$$

Therefore,

$$\widetilde{\Lambda}^{-1}(s) = \min\{s^{1/(\sigma+1)}, s^{1/(\rho+1)}\}.$$

Consequently, for $0 < s < 1$, we have

$$\widetilde{\Lambda}^{-1}(s) = s^{1/(\sigma+1)}.$$

Applying this to (5.1.2), we find that

$$\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)=\frac{\varrho^{(N-1)(\rho+1)/\rho(\sigma+1)}}{s^{(N-1)/\sigma}}.$$

If $\sigma\geq N-1$, then it is clear that

$$\lim_{r\to\infty}\int_{\varrho}^{r}\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds=\varrho^{(N-1)(\rho+1)/\rho(\sigma+1)}\lim_{r\to\infty}\int_{\varrho}^{r}s^{\frac{-(N-1)}{\sigma}}ds=\infty$$

On the other hand, suppose $\sigma<N-1$. If, in addition, $\dfrac{\rho-\sigma}{\rho\sigma(\sigma+1)}<\dfrac{1}{N-1}$, then we have

$$\int_{\varrho}^{\infty}\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds=\varrho^{(N-1)(\rho+1)/\rho(\sigma+1)}\int_{\varrho}^{\infty}s^{\frac{-(N-1)}{\sigma}}ds$$
$$=\frac{\sigma}{N-1-\sigma}\varrho^{1-(N-1)(\rho-\sigma)/\rho\sigma(\sigma+1)}ds.$$

Consequently, in this case we find

$$\lim_{\varrho\to\infty}\int_{\varrho}^{\infty}\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds=\infty.$$

The above discussion leads us to consider the following condition on the parameter σ and ρ in Condition (ϕ-3).

$$(\phi\text{-}4):\qquad \frac{\rho-\sigma}{\rho\sigma(\sigma+1)}<\frac{1}{N-1}.$$

For easy reference, let us summarize the above discussion vis-à-vis Condition (ϕ-4) in the following remark.

Remark 5.2. Let us suppose that Condition (ϕ-4) holds. Rewriting this condition as

$$\frac{\rho-\sigma}{\rho(\sigma+1)}<\frac{\sigma}{N-1}.$$

We consider two cases:

$$\text{Case (I)}\quad \frac{\sigma}{N-1}\geq 1 \quad\text{and}\quad \text{Case (II)}\quad \frac{\sigma}{N-1}<1.$$

Direct computation leads to the following conclusions.

(1) If (I) occurs, then for each $\varrho>1$

$$\lim_{r\to\infty}\int_{\varrho}^{r}\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds=\infty.$$

(2) On the other hand, under Case (II)

$$\lim_{\varrho\to\infty}\int_{\varrho}^{\infty}\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds=\infty.$$

Let us now consider a non-decreasing function $f : (0, \infty) \to (0, \infty)$ such that

$$\int_1^\infty \frac{ds}{\Phi^{-1}(F(s))} < \infty \quad \text{where} \quad F(t) := \int_0^t f(s)ds. \tag{5.1.3}$$

Condition (5.1.3) is easily recognized as a generalization of the well-known Keller-Osserman condition when $\phi \equiv 1$. We point out that Condition (5.1.3) is equivalent to

$$\int_t^\infty \frac{ds}{\Phi^{-1}(F(s) - F(t))} < \infty, \quad \forall\, t > 0. \tag{5.1.4}$$

That (5.1.4) implies (5.1.3) is obvious. Therefore, we only need to show that (5.1.4) is implied by (5.1.3). For this, we first observe that

$$F(s) - F(t) \geq f(t)(s-t), \quad \text{and} \quad F(s) - F(t) \geq F(s-t), \quad \forall\, s \geq t.$$

Let $t > 0$ and fix $0 < \theta < 1$. Using (5.1.1), together with the above inequalities, we have

$$\begin{aligned} \int_t^\infty \frac{ds}{\Phi^{-1}(F(s) - F(t))} &= \int_t^{\theta+t} \frac{ds}{\Phi^{-1}(F(s) - F(t))} + \int_{\theta+t}^\infty \frac{ds}{\Phi^{-1}(F(s) - F(t))} \\ &\leq \frac{1}{\Phi^{-1}(f(t))} \int_t^{\theta+t} \frac{ds}{\widetilde{\Lambda}^{-1}(s-t)} + \int_{\theta+t}^\infty \frac{ds}{\Phi^{-1}(F(s-t))} \\ &\leq \frac{1}{\Phi^{-1}(f(t))} \int_t^{\theta+t} \frac{ds}{(s-t)^{1/(\sigma+1)}} + \int_\theta^\infty \frac{ds}{\Phi^{-1}(F(s))} \\ &= \frac{\theta^{\sigma/(\sigma+1)}}{\Phi^{-1}(f(t))} + \int_\theta^\infty \frac{ds}{\Phi^{-1}(F(s))}. \end{aligned} \tag{5.1.5}$$

Thus, Condition (5.1.3) on f implies that the right-hand side of (5.1.5) is finite for any $t > 0$. Therefore (5.1.4) holds.

Let us now consider the following equation

$$\Delta_\phi u = f(u). \tag{5.1.6}$$

Our first Liouville-type result in this section is provided by the following.

Theorem 5.3. Suppose (ϕ-4) holds and that f satisfies conditions (5.1.3). If $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a non-negative sub-solution of (5.1.6) in $\mathbb{R}^N$, then $u \equiv 0$ in $\mathbb{R}^N$.

Proof. Let $z \in \mathbb{R}^N$ and $\epsilon > 0$ be arbitrary. Let v be a solution of

$$\begin{cases} (r^{N-1}\phi(|v'|)v')' = r^{N-1} f(v(r)), \quad r > 0 \\ v(0) = \epsilon,\ v'(0) = 0. \end{cases} \tag{5.1.7}$$

Let $(0, R)$ be the maximal interval of existence of v. We show that $R < \infty$. Note that

$$r^{N-1}\phi(v'(r))v'(r) = \int_0^r s^{N-1} f(v(s))ds, \quad 0 < r < R. \tag{5.1.8}$$

Therefore, $v' > 0$ in $(0, R)$. Moreover, according to (5.1.8), we note that $r^{N-1}\Psi(v'(r))$ is a non-decreasing function in $(0, R)$, and hence $\Psi^{-1}(r^{N-1}\Psi(v'(r)))$ is a non-decreasing on $(0, R)$. We now multiply both sides of (5.1.8) by $\Psi^{-1}(r^{N-1}\Psi(v'(r)))$ and for any $\varrho < r < R$, we have

$$\begin{aligned}
\Psi^{-1}(r^{N-1}\Psi(v'(r)))r^{N-1}\Psi(v'(r)) &= \Psi^{-1}(r^{N-1}\Psi(v'(r)))\int_0^r s^{N-1}f(v(s))ds \\
&\geq \int_0^r s^{N-1}\Psi^{-1}(s^{N-1}\Psi(v'(s)))f(v(s))ds \\
&\geq \int_\varrho^r s^{N-1}\Lambda^{-1}(s^{N-1})f(v(s))v'(s)ds \qquad (5.1.9)\\
&= \varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})\int_\varrho^r f(v(s))v'(s)ds \\
&= \varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})(F(v(r)) - F(v(\varrho))). \qquad (5.1.10)
\end{aligned}$$

We have used (1.1.6) in obtaining (5.1.9). The last inequality (5.1.10), together with the inequality (which is a consequence of (1.1.6)),

$$\Psi^{-1}(r^{N-1}\Psi(v'))r^{N-1}\Psi(v') \leq \lambda^{-1}(r^{N-1})r^{N-1}\Psi(v'(r))v'(r), \quad \varrho < r < R$$

shows that

$$\Psi(v'(r))v'(r) \geq \frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{r^{N-1}\lambda^{-1}(r^{N-1})}(F(v(r)) - F(v(\varrho))), \quad \varrho < r < R.$$

On noting that $\Phi(2v'(r)) \geq \Psi(v'(r))v'(r)$, we find that

$$\begin{aligned}
v'(r) &\geq \frac{1}{2}\Phi^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{r^{N-1}\lambda^{-1}(r^{N-1})}(F(v(r)) - F(v(\varrho)))\right) \\
&\geq \frac{1}{2}\widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{r^{N-1}\lambda^{-1}(r^{N-1})}\right)\Phi^{-1}(F(v(r)) - F(v(\varrho))), \quad 0 < \varrho < r.
\end{aligned}$$

Integrating this last inequality on (ϱ, r) shows that

$$\int_{v(\varrho)}^{v(r)} \frac{ds}{\Phi^{-1}(F(s) - F(v(\varrho)))} \geq \frac{1}{2}\int_\varrho^r \widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds.$$

Consequently, we see that

$$\frac{1}{2}\int_\varrho^r \widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds \leq \int_{v(\varrho)}^\infty \frac{ds}{\Phi^{-1}(F(s) - F(v(\varrho)))}. \qquad (5.1.11)$$

Note that

$$\mathcal{Q}(t) = \int_t^\infty \frac{ds}{\Phi^{-1}(F(s) - F(t))} = \int_0^\infty \frac{d\zeta}{\Phi^{-1}(\zeta)F^{-1}(\zeta + F(t))}$$

is non-increasing function. Therefore, since $\epsilon = v(0) \leq v(\varrho)$, we have $\mathcal{Q}(\epsilon) \geq \mathcal{Q}(v(\varrho))$. Thus, we obtain the following from inequality (5.1.11):

$$\int_\varrho^r \widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right)ds \leq 2\int_\epsilon^\infty \frac{ds}{\Phi^{-1}(F(s) - F(\epsilon))}, \quad \forall\, \varrho < r < R. \qquad (5.1.12)$$

Thus, Condition (5.1.3) on f implies that the left hand side of (5.1.12) is finite. Now assume that $R = \infty$. Then, we can take $\varrho > 1$ in (5.1.12). If $\sigma \geq N-1$, then taking the limit as $r \to \infty$ in (5.1.12) leads to a contradiction, by Remark 5.2.

Now, let us suppose that $\sigma < N-1$. Taking the limit in (5.1.12) as $r \to \infty$, and we find that

$$\int_{\varrho}^{\infty} \widetilde{\Lambda}^{-1}\left(\frac{\varrho^{N-1}\Lambda^{-1}(\varrho^{N-1})}{s^{N-1}\lambda^{-1}(s^{N-1})}\right) ds \leq 2\int_{\epsilon}^{\infty} \frac{ds}{\Phi^{-1}(F(s)-F(\epsilon))}. \tag{5.1.13}$$

Then, we use Condition (ϕ-4), and Remark 5.2 to see that Inequality (5.1.13) leads to a contradiction upon taking the limit as $\varrho \to \infty$.

Therefore, we conclude that indeed $R < \infty$, and as a consequence $v(r) \to \infty$ as $r \to R^{-}$. Fix an arbitrary $z \in \mathbb{R}^N$, and let $w(x) := v(|x-z|)$ for $x \in \mathbb{R}^N$. Given a non-negative sub-solution u of (5.1.6) in $\mathbb{R}^N$, let $0 < \delta < 1$ with $1-\delta$ is sufficiently small such that $u < w$ on $\partial B(z, \delta R)$. We also note that w is a solution of (5.1.6) in the ball $B(z, \delta R)$. We invoke the comparison principle, Lemma 5.1, to conclude that

$$u(x) \leq w(x), \quad x \in B(z, \delta R).$$

In particular, we have $0 \leq u(z) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we find that $u(z) = 0$, and since $z \in \mathbb{R}^N$ is arbitrary, we conclude $u \equiv 0$ in $\mathbb{R}^N$. ☐

Theorem 5.4. Suppose Condition (ϕ-4) holds. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) > 0$ for $t > 0$ and $f(0) = 0$, and that f satisfies (5.1.3).

(a) If u is a sub-solution of (5.1.6), then $u \leq 0$ in $\mathbb{R}^N$.

(b) If f is an odd function and u satisfies (5.1.6), then $u \equiv 0$ in $\mathbb{R}^N$.

Proof. (a) Let u be any sub-solution of (5.1.6) in $\mathbb{R}^N$. To show that $u \leq 0$ in $\mathbb{R}^N$, it suffices to demonstrate that u^+ is a sub-solution of (5.1.6) in $\mathbb{R}^N$. Once, this is proven, then we can invoke Theorem 5.3 to conclude that $u^+ \equiv 0$, and hence $u = -u^- \leq 0$ in $\mathbb{R}^N$. First, since $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$, let us notice that $u^+ \in W^{1,\Phi}_{loc}(\mathbb{R}^N)$ (see Lemma 7.15 of the Appendix). To see that u^+ is a sub-solution of (5.1.6), let

$$\Omega := \{x \in \mathbb{R}^N : u(x) > 0\}.$$

Of course, we suppose that Ω is non-empty for otherwise there is nothing to prove. Let $\vartheta \in C^1(\mathbb{R})$ such that

$$\vartheta'(t) > 0 \quad \text{on} \quad (0,1), \quad \vartheta \equiv 1 \quad \text{on} \quad [1,\infty) \quad \text{and} \quad \vartheta \equiv 0 \quad \text{on} \quad (-\infty, 0].$$

Let $\vartheta_j(t) := \vartheta(jt)$ for any positive integer j, and any $t \in \mathbb{R}$. Given $\mathcal{O} \subset\subset \mathbb{R}^N$, let $\varphi \in W_0^{1,\Phi}(\mathcal{O})$ with $\varphi \geq 0$ in $\mathcal{O}$, we have

$$\int_{\mathcal{O}} \left\langle \phi(|\nabla u^+|)\nabla u^+, \nabla\varphi \right\rangle = \int_{\Omega\cap\mathcal{O}} \left\langle \phi(|\nabla u|)\nabla u, \nabla\varphi \right\rangle, \text{ by Lemma 7.15}$$
$$= \lim_{j\to\infty} \int_{\Omega\cap\mathcal{O}} \vartheta_j(u) \left\langle \phi(|\nabla u^+|)\nabla u^+, \nabla\varphi \right\rangle$$
$$= \lim_{j\to\infty} \int_{\Omega\cap\mathcal{O}} \left\langle \phi(|\nabla u|)\nabla u, \vartheta_j(u)\nabla\varphi \right\rangle$$
$$= \lim_{j\to\infty} \int_{\mathcal{O}} \left\langle \phi(|\nabla u|)\nabla u, \vartheta_j(u)\nabla\varphi \right\rangle$$
$$= \lim_{j\to\infty} \int_{\mathcal{O}} \left\langle \phi(|\nabla u|)\nabla u, \nabla(\vartheta_j(u)\varphi) \right\rangle - \lim_{j\to\infty} \int_{\mathcal{O}} \varphi \left\langle \phi(|\nabla u|)\nabla u, \nabla(\vartheta_j(u)) \right\rangle$$
$$\leq -\lim_{j\to\infty} \int_{\mathcal{O}} \vartheta_j(u)f(u)\varphi - \lim_{j\to\infty} \int_{\mathcal{O}} \varphi \left\langle \phi(|\nabla u|)\nabla u, \nabla(\vartheta_j(u)) \right\rangle.$$

To obtain the last inequality, we have used the assumption that u is a sub-solution of (5.1.6) in $\mathbb{R}^N$.

Since $\phi(t)t^2 \geq 0$ for all $t \geq 0$ and $\vartheta' \geq 0$ on $\mathbb{R}$, we have

$$\int_{\mathcal{O}} \varphi \left\langle \phi(|\nabla u|)\nabla u, \nabla(\vartheta_j(u)) \right\rangle = \int_{\mathcal{O}} \varphi\vartheta_j'(u) \left\langle \phi(|\nabla u|)\nabla u, \nabla u \right\rangle \geq 0.$$

Therefore, on recalling that $f(0) = 0$, the last two inequalities lead to the following.

$$\int_{\mathcal{O}} \left\langle \phi(|\nabla u^+|)\nabla u^+, \nabla\varphi \right\rangle \leq -\lim_{j\to\infty} \int_{\mathcal{O}} \vartheta_j(u)f(u)\varphi$$
$$= -\lim_{j\to\infty} \int_{\Omega\cap\mathcal{O}} \vartheta_j(u)f(u)\varphi = -\int_{\Omega\cap\mathcal{O}} f(u)\varphi = \int_{\mathcal{O}} f(u^+)\varphi.$$

Therefore, u^+ is a sub-solution of (5.1.6), as desired.

(b) Suppose f is an odd function, and that u is a solution of (5.1.6) in $\mathbb{R}^N$. Therefore, $u \leq 0$ in $\mathbb{R}^N$, by what was proved in (a). On the other hand, it is easily seen that $-u$ is a solution of (5.1.6) in $\mathbb{R}^N$. Therefore, $-u \leq 0$ in $\mathbb{R}^N$ again. Thus, we have shown that $u \equiv 0$ in $\mathbb{R}^N$.

□

Let us illustrate the above results for the ϕ-Laplacian with some of the functions ϕ from Example 1.1.

5.2 Some Examples

(1) For $\phi(t) = pt^{p-2}$ with $p > 1$. The ϕ-Laplacian is essentially the standard p-Laplacian. In this case, we see that Condition (ϕ-4) holds. Condition (5.1.3)

reduces to the requirement that

$$\int_t^\infty \frac{ds}{(F(s))^{\frac{1}{p}}} < \infty, \quad \forall\, t > 0.$$

Therefore, in this case both Theorem 5.3 and Theorem 5.4 hold for any $p > 1$.

(2) Let us know consider $\phi(t) = pt^{p-2} + qt^{q-2}$ for $1 < p < q$. Note that, in this case

$$\Phi(t) = \frac{t^p}{p} + \frac{t^q}{q} \leq 2t^q, \quad t > 1.$$

As a consequence of this, we see that

$$\int_t^\infty \frac{ds}{\Phi^{-1}(F(s))} \leq 2^{\frac{1}{q}} \int_t^\infty \frac{ds}{(F(t))^{\frac{1}{q}}}, \quad \forall\, t > 0.$$

Therefore, if the right-hand side integral in the above inequality is finite for some $t > 0$, then Condition (5.1.3) holds, and therefore Theorem 5.3 and Theorem 5.4 both hold provided that (ϕ-4) is satisfied, that is,

$$\frac{\rho - \sigma}{\rho\sigma(\sigma + 1)} = \frac{q - p}{p(p-1)(q-1)} < \frac{1}{N-1}$$

(3) Let $\phi(t) = pt^{p-1}\log^q(1+t) + qt^{p-1}(1+t)^{-1}\log^{q-1}(1+t)$ for $p > 1$ and $q > 0$. In this case, $\Phi(t) = t^p \log^q(1+t)$. Note that given $\epsilon > 0$, there is t_ϵ, sufficiently large, such that

$$\Phi(t) \leq t^{p+\epsilon}, \quad \forall\, t > t_\epsilon.$$

Therefore, if there is $r > p$ such that

$$\int_t^\infty \frac{ds}{(F(s))^{\frac{1}{r}}} < \infty$$

for some $t > 0$, then Condition (5.1.3) holds. Thus,if

$$\frac{\rho - \sigma}{\rho\sigma(\sigma + 1)} = \frac{q}{p(p-1)(p+q-1)} < \frac{1}{N-1}, \tag{5.2.1}$$

then Theorem 5.3 and Theorem 5.4 both apply. For instance, if q is sufficiently large that

$$\frac{N-1}{(q-1)(2q-1)} < 1,$$

then, for $p > q$, we have

$$\frac{N-1}{(p-1)(p+q-1)} \leq 1 \leq \frac{p}{q},$$

and therefore, (5.2.1) holds for such pair (p, q).

5.3 Sign-changing Absorption Terms

This section is devoted to the study of some Liouville-type property of positive solutions of the following equation for a given $f : \mathbb{R} \to \mathbb{R}$.

$$\Delta_\phi u = -f(u) \quad \text{in } \mathbb{R}^N. \tag{5.3.1}$$

Solutions of (5.3.1) are invariant under rotations. To see this, let A be any orthogonal transformation of $\mathbb{R}^N$, and define $v(x) := u(Ax)$ on $\mathbb{R}^N$. Computation shows that

$$\nabla v(x) = A^T \nabla u(Ax).$$

Given $\Omega \subset\subset \mathbb{R}^N$, and any $\varphi \in W_0^{1,\Phi}(\Omega)$, let $\psi(y) := \varphi(A^T y)$ for $y \in \mathcal{O} := A(\Omega)$. Then $A^T(\mathcal{O}) = \Omega$ and it is easily noted that $\psi \in W_0^{1,\Phi}(\mathcal{O})$. On using change of variables, and using $\langle\ ,\ \rangle$ for the Euclidean inner product, we have

$$\begin{aligned}
\int_\Omega \phi(|\nabla v|)\, \langle \nabla v, \nabla\varphi\rangle\, dx &= \int_{A^T(\mathcal{O})} \phi(|\nabla v|)\, \langle \nabla v, \nabla\varphi\rangle\, dx \\
&= \int_{A^T(\mathcal{O})} \phi(|\nabla u|)\, \left\langle A^T\nabla u, \nabla\varphi\right\rangle dx \\
&= \int_{A^T(\mathcal{O})} \phi(|\nabla u|)\, \langle \nabla u, A\nabla\varphi\rangle\, dx \\
&= \int_{\mathcal{O}} \phi(|\nabla u|)\, \langle \nabla u, \nabla\psi\rangle\, dy = \int_{\mathcal{O}} f(u)\psi dy = \int_{A(\Omega)} f(u)\psi dy \\
&= \int_\Omega f(v)\varphi dx.
\end{aligned}$$

To study Liouville-type properties of solutions to (5.3.1), we start by making some suitable assumption on ϕ. Specifically, we require that $\phi \in C^2(0,\infty)$ and satisfies following.

$$(\phi\text{-}5): \quad -\infty < \inf_{t>0} \varpi(t) \le \sup_{t>0} \varpi(t) < \left(\frac{N}{N-1}\right)\sigma^2 - 3\rho + 2,$$

where

$$\varpi := \frac{\phi''(t)t^2}{\phi(t)}, t > 0.$$

In main result of this section is the following Liouville-type theorem for solution of (5.3.1). This theorem extends the result in the [6] where the special case $\phi(t) = t^{p-2}, p > 1$ was considered.

In the proof of the theorem, we will observe the Einstein summation convention over repeated indices. That is, in any expression containing subscripted variables appearing twice (and only twice) in any term, the subscripted variables are assumed to be summed over. If circumstances demand clarity, we will resort to usage of an explicit summation notation. For instance $x_{ij}y_j$ for $j = 1, \cdots, N$ stands for $\displaystyle\sum_{j=1}^{N} x_{ij}y_j$.

Theorem 5.5. Suppose Conditions (ϕ-1), (ϕ-3), and (ϕ-5) hold, and that $f \in C^{1,\gamma}_{loc}(\mathbb{R})$ for some $0 < \gamma < 1$. If f satisfies

$$f'(t) \le \sigma \left(\frac{N+1}{N-1}\right) \frac{f(t)}{t}, \quad \forall\, t > 0, \tag{5.3.2}$$

then any positive solution $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ of (5.3.1) in $\mathbb{R}^N$ is a constant in $\mathbb{R}^N$.

Remark 5.6. Suppose ϕ satisfies the assumptions of the theorem. If $f(t) = t^\theta - t^\vartheta$ for some $0 \le \theta \le \sigma(N+1)/(N-1) \le \vartheta$, then f satisfies (5.3.2) and therefore, the only positive solution of (5.3.1) in $\mathbb{R}^N$ is $u \equiv 1$. We should also note that if f is a non-negative and non-increasing function on $\mathbb{R}$, then f satisfies (5.3.2) and therefore, in this case only constants are the possible solutions of (5.3.1).

Proof. Let us first note that, as a consequence of the continuity of u and [16, Lemma 3.3], we see that $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$. Therefore, it follows that the equation (5.3.1) is uniformly elliptic on open sets that are compactly contained in $\mathcal{O}$ where $\mathcal{O} := \{x \in \mathbb{R}^N : |\nabla u(x)| > 0\}$. Consequently, it follows that $u \in C^3$ in any open set that is compactly contained in $\mathcal{O}$ (see [28, Corollary 2.2] where we take $p = q = 2$). We refer the reader to the Appendix for a detailed justification of the assertion.

Let $x_0 \in \mathbb{R}^N$ be an arbitrary but fixed point. We wish to show $\nabla u(x_0) = 0$. Given $a > 0$ let us set

$$J(x) := (a^2 - r^2)^2 \Theta, \text{ where } \Theta = \frac{|\nabla u|^2}{u^2}, \text{ and } \quad r = |x - x_0|.$$

We note that $J \ge 0$ in $B := B(x_0, a)$ and $J|_{|x-x_o|=a} = 0$. Therefore, J attains its maximum value on $\overline{B}$ at some interior point $x^* \in B$. Suppose $\nabla u(x^*) = 0$. Then Θ (and hence J) will be zero at x^*. But then J (and therefore Θ) will be zero on $\overline{B}$. This would imply $|\nabla u| = 0$ in $\overline{B}$, and in particular $\nabla u(x_0) = 0$. Therefore we assume that $|\nabla u| > 0$ at x^*. We recall that u is $C^3_{loc}(\mathcal{O})$, where $\mathcal{O} := \{x \in \Omega : |\nabla u(x)| > 0\}$. Now, at x^* we have

$$0 = J_j = -2(a^2 - r^2)(r^2)_j \Theta + (a^2 - r^2)^2 \Theta_j, \quad j = 1, \cdots, N \quad \text{and} \tag{5.3.3}$$

$$D^2 J \le 0. \tag{5.3.4}$$

Let us first show that the $N \times N$ matrix

$$H := I_N + \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)} |\nabla u|^{-2} \nabla u \nabla u^T$$

is positive definite. Here I_N is the $N \times N$ identity matrix, and A^T stands for the transpose of matrix A. To see that H is positive definite, we first note that for any $\xi \in \mathbb{R}^N \setminus \{0\}$

$$\xi^T H \xi = |\xi|^2 + \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)} \frac{(\xi^T \nabla u)^2}{|\nabla u|^2}.$$

If $\frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)} \geq 0$, then $\xi^T H \xi \geq |\xi|^2 > 0$. If, on the other hand, $-1 < \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)} < 0$, then on noting that $|\xi^T \nabla u| \leq |\xi||\nabla u|$ and therefore,

$$\frac{(\xi^T \nabla u)^2}{|\nabla u|^2} \leq |\xi|^2,$$

we have (recalling Condition (ϕ-3))

$$\xi^T H \xi \geq |\xi|^2 + \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)}|\xi|^2 = |\xi|^2 \left(1 + \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)}\right) > 0.$$

Thus, in any case we have shown that $H > 0$.

Now recalling that $D^2 J \leq 0$ at x^*, the product matrix $HD^2 J \leq 0$ at x^*. Therefore, at x^* we have

$$0 \geq \sum_{j=1}^{N} e_j^T (HD^2 J) e_j = \Delta J + \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)} J_{ij} u_i u_j |\nabla u|^{-2},$$

where e_j is the unit vector in $\mathbb{R}^N$ with 1 in the j^{th} position. In the last equation, we have used the simple fact that $e_k^T B e_k = b_{kk}$ where $B = [b_{ij}]$ is an $N \times N$ matrix. Thus, at x^* we have

$$\Delta J + \frac{\phi'(|\nabla u|)|\nabla u|}{\phi(|\nabla u|)} J_{ij} u_i u_j |\nabla u|^{-2} \leq 0. \tag{5.3.5}$$

Recalling that (5.3.1) is invariant under rotation, we choose coordinates so that at the point x^*, we have

$$|\nabla u| = u_1, \quad u_j = 0, \quad j = 2, \cdots, N. \tag{5.3.6}$$

Through direct computation, we note that

$$\begin{aligned}
\Delta J + \frac{\phi'(u_1)u_1}{\phi(u_1)} J_{11} &= 2|\nabla(r^2)|^2 \Theta - 2(a^2 - r^2)\Delta(r^2)\Theta - 4(a^2 - r^2)\nabla(r^2) \cdot \nabla\Theta \\
&\quad + (a^2 - r^2)^2 \Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}[2((r^2)_1)^2 \Theta \\
&\quad - 4(a^2 - r^2)\Theta - 4(a^2 - r^2)(r^2)_1 \Theta_1 + (a^2 - r^2)^2 \Theta_{11}].
\end{aligned}$$

This, together with Inequality (5.3.5), implies

$$\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)} \Theta_{11} \leq -\frac{8r^2}{(a^2 - r^2)^2}\Theta + \frac{4N}{(a^2 - r^2)}\Theta + \frac{4\nabla(r^2) \cdot \nabla\Theta}{a^2 - r^2}$$

$$+\frac{\phi'(u_1)u_1}{\phi(u_1)}\left[-\frac{2((r^2)_1)^2}{(a^2-r^2)^2}\Theta+\frac{4}{a^2-r^2}\Theta+\frac{4(r^2)_1}{a^2-r^2}\Theta_1\right]. \tag{5.3.7}$$

To obtain (5.3.7), we have used the easily verifiable identities

$$|\nabla(r^2)|^2=4r^2,\quad \Delta(r^2)=2N. \tag{5.3.8}$$

Moreover, from (5.3.3) we obtain

$$\Theta_1=\frac{2(r^2)_1}{a^2-r^2}\Theta \quad\text{and}\quad \nabla\Theta=\frac{2\Theta}{a^2-r^2}\nabla(r^2).$$

Using these and (5.3.8) in (5.3.7) we find

$$\Delta\Theta+\frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11}\le\Theta\left[\frac{24r^2}{(a^2-r^2)^2}+\frac{4N}{a^2-r^2}+\frac{\phi'(u_1)u_1}{\phi(u_1)}\left(\frac{6((r^2)_1)^2}{(a^2-r^2)^2}+\frac{4}{a^2-r^2}\right)\right].$$

Therefore, at x^*, we have

$$\frac{\Delta\Theta+\frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11}}{\Theta}\le\frac{24r^2}{(a^2-r^2)^2}+\frac{4N}{a^2-r^2}+\left|\frac{\phi'(u_1)u_1}{\phi(u_1)}\right|\left(\frac{24r^2}{(a^2-r^2)^2}+\frac{4}{a^2-r^2}\right).$$

Since $r^2\le a^2$ in $B(x_0,a)$, we have

$$\begin{aligned}&\frac{\Delta\Theta+\frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11}}{\Theta}\\ &\le\frac{24a^2}{(a^2-r^2)^2}+\frac{4Na^2}{(a^2-r^2)^2}+\left|\frac{\phi'(u_1)u_1}{\phi(u_1)}\right|\left(\frac{24a^2}{(a^2-r^2)^2}+\frac{4a^2}{(a^2-r^2)^2}\right).\end{aligned} \tag{5.3.9}$$

Let us now notice that for $i,j=1,2,\cdots,N$, we have

$$\Theta_i=2u_{ij}u_ju^{-2}-2|\nabla u|^2u_iu^{-3} \tag{5.3.10}$$

$$\Theta_{ii}=2u_{iij}u_ju^{-2}+2u_{ij}u_{ij}u^{-2}-8u_{ij}u_iu_ju^{-3}-2|\nabla u|^2u^{-3}u_{ii}+6|\nabla u|^2u^{-4}u_i^2. \tag{5.3.11}$$

Upon using (5.3.6), we obtain the following from (5.3.11).

$$\begin{aligned}\Theta_{11}&=2u_{111}u_1u^{-2}+2u_{1j}u_{1j}u^{-2}-10u_{11}u_1^2u^{-3}+6u_1^4u^{-4}\\ \Delta\Theta&=2(\Delta u)_1u_1u^{-2}+2u_{ij}u_{ij}u^{-2}-8u_{11}u_1^2u^{-3}-2u_1^2u^{-3}\Delta u+6u_1^4u^{-4}.\end{aligned}$$

From the above computation, we see that, at x^*,

$$\begin{aligned}\Delta\Theta+\frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11}&=2u_1u^{-2}\left[(\Delta u)_1+\frac{\phi'(u_1)u_1}{\phi(u_1)}u_{111}\right]+2u^{-2}\left(u_{ij}u_{ij}+\frac{\phi'(u_1)u_1}{\phi(u_1)}u_{1j}u_{1j}\right)\\ &-2u_1^2u^{-3}\left[\Delta u+\left(4+5\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_{11}\right]+6\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4u^{-4}.\end{aligned} \tag{5.3.12}$$

Let us now proceed to find alternative forms for the expressions in the parentheses in (5.3.12). We start by rewriting the equation (5.3.1) in any open set $\mathcal{O}'$ that contains x^* where u is C^3- smooth.

$$\phi(|\nabla u|)\Delta u + \frac{\phi'(|\nabla u|)}{|\nabla u|}u_{ij}u_iu_j = -f(u). \tag{5.3.13}$$

On recalling (5.3.6), we obtain the following from (5.3.13) at x^*

$$\phi(u_1)\Delta u + \phi'(u_1)u_{11}u_1 = -f(u).$$

That is,

$$\Delta u + \frac{\phi'(u_1)u_1}{\phi(u_1)}u_{11} = -\frac{f(u)}{\phi(u_1)}. \tag{5.3.14}$$

We now differentiate (5.3.13) with respect to the variable x_1. On evaluating the resulting expression at x^*, we find

$$\begin{aligned}&\phi(u_1)(\Delta u)_1 + \phi'(u_1)u_{11}\Delta u + \phi''(u_1)u_1u_{11}^2 - \phi'(u_1)u_{11}^2 + \phi'(u_1)u_{111}u_1\\ &+2\phi'(u_1)\sum_{j=1}^{N}u_{1j}^2 = -u_1f'(u).\end{aligned}$$

That is

$$\begin{aligned}\phi(u_1)(\Delta u)_1 + \phi'(u_1)u_1u_{111} &= -2\phi'(u_1)\sum_{j=2}^{N}u_{1j}^2 - \phi'(u_1)u_{11}\Delta u - \phi''(u_1)u_1u_{11}^2\\ &\quad -\phi'(u_1)u_{11}^2 - u_1f'(u).\end{aligned} \tag{5.3.15}$$

On dividing both sides of (5.3.15) by $\phi(u_1)$, using (5.3.14) to replace Δu in the resulting expression, and rearranging we find

$$\begin{aligned}&(\Delta u)_1 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\\ &= -2\frac{\phi'(u_1)}{\phi(u_1)}\sum_{j=2}^{N}u_{1j}^2 + \left[\frac{f(u)}{\phi(u_1)} - \left(1 - \frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_{11}\right]\frac{\phi'(u_1)u_1}{\phi(u_1)}u_{11}\\ &-\frac{\phi''(u_1)u_1}{\phi(u_1)}u_{11}^2 - \frac{u_1f'(u)}{\phi(u_1)}.\end{aligned} \tag{5.3.16}$$

Let us observe the following.

$$\begin{aligned}&u_{ij}u_{ij} + \frac{\phi'(u_1)u_1}{\phi(u_1)}u_{1j}u_{1j}\\ &= 2\sum_{j=2}^{N}u_{1j}^2 + u_{11}^2 + \sum_{j=2}^{N}u_{jj}^2 + \sum_{i,j\geq 2, i\neq j}^{N}u_{ij}^2 + \frac{\phi'(u_1)u_1}{\phi(u_1)}u_{11}^2 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\sum_{j=2}^{N}u_{1j}^2\end{aligned}$$

$$(5.3.17) \quad \geq \left(2 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) \sum_{j=2}^N u_{1j}^2 + \left(1 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}^2 + \sum_{j=2}^N u_{jj}^2.$$

To proceed further, we need the following inequality.

Lemma 5.7.

$$\left(1 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}^2 + \sum_{j=2}^N u_{jj}^2 \geq \left(\frac{N}{N-1} + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}^2 - \frac{2}{N-1} u_{11}\Delta u + \frac{1}{N-1}(\Delta u)^2.$$

Proof. Let us note that

$$\begin{aligned} \left(\sum_{j=2}^N u_{jj}\right)^2 &= \sum_{j=2}^N u_{jj}^2 + 2 \sum_{2\leq i<j}^N u_{ii}u_{jj} \\ &\leq \sum_{j=2}^N u_{jj}^2 + \sum_{2\leq i<j}^N (u_{ii}^2 + u_{jj}^2) = \sum_{j=2}^N u_{jj}^2 + (N-2) \sum_{j=2}^N u_{jj}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j=1}^N u_{jj}^2 \geq \frac{1}{N-1} \left(\sum_{j=2}^N u_{jj}\right)^2 &= \frac{1}{N-1}(\Delta u - u_{11})^2 \\ &= \frac{1}{N-1}((\Delta u)^2 - 2u_{11}\Delta u + u_{11}^2). \end{aligned}$$

The proof is complete on adding

$$\left(1 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}^2$$

to both sides of the last inequality above. □

Using Lemma 5.7 in equality (5.3.17), we obtain the following

$$(5.3.18) \qquad u_{ij}u_{ij} + \frac{\phi'(u_1)u_1}{\phi(u_1)} u_{1j}u_{1j} \geq \left(2 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) \sum_{j=2}^N u_{1j}^2$$
$$+ \left(\frac{N}{N-1} + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}^2 - \frac{2}{N-1} u_{11}\Delta u + \frac{1}{N-1}(\Delta u)^2.$$

We use (5.3.16) and (5.3.18) in (5.3.12) to get

$$\begin{aligned} \Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11} \geq 2u_1u^{-2} \Bigg\{ &-2\frac{\phi'(u_1)u_1}{\phi(u_1)} \sum_{j=2}^N u_{1j}^2 \\ &+ \left[\frac{f(u)}{\phi(u_1)} - \left(1 - \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}\right] \frac{\phi'(u_1)u_1}{\phi(u_1)} u_{11} - \frac{\phi''(u_1)u_1}{\phi(u_1)} u_{11}^2 - \frac{u_1 f'(u)}{\phi(u_1)} \Bigg\} \\ &+2u^{-2} \left[\left(2 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) \sum_{j=2}^N u_{1j}^2 + \left(\frac{N}{N-1} + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right) u_{11}^2 - \frac{2}{N-1} u_{11}\Delta u \right. \end{aligned}$$

$$+\frac{1}{N-1}(\Delta u)^2\Big] - 2u_1^2u^{-3}\left[\Delta u + \left(4+5\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_{11}\right] + 6\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4u^{-4}.$$

Rearranging the terms in the last inequality, we find

$$\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11} \geq 2u^{-2}\Bigg\{\left(2-\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)\sum_{j=2}^{N}u_{1j}^2 + \frac{\phi'(u_1)u_1}{\phi(u_1)}\cdot\frac{f(u)}{\phi(u_1)}u_{11} - \frac{f'u_1^2}{\phi(u_1)}$$

$$-\frac{2}{N-1}(\Delta u)u_{11} + \frac{1}{N-1}(\Delta u)^2 + \left[\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2 - \frac{\phi''(u_1)u_1^2}{\phi(u_1)} + \frac{N}{N-1}\right]u_{11}^2$$

$$-\left[\Delta u + \left(4+5\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_{11}\right]u_1^2u^{-1} + 3\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4u^{-2}\Bigg\}. \tag{5.3.19}$$

We recall from (5.3.10) that for $j = 2,\cdots,N$, we have $\Theta_j = 2u_{1j}u_1u^{-2}$ and therefore the following holds.

$$2u^{-2}\sum_{j=2}^{N}u_{1j}^2 = \frac{\sum_{j=2}^{N}\Theta_j^2}{2\left(\frac{u_1}{u}\right)^2} = \frac{\sum_{j=2}^{N}\Theta_j^2}{2\Theta} \leq \frac{|\nabla\Theta|^2}{2\Theta}.$$

We use the above inequality and (5.3.15) to estimate Inequality (5.3.19) as follows.

$$\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11} \geq -\left|2-\frac{\phi'(u_1)u_1}{\phi(u_1)}\right|\frac{|\nabla\Theta|^2}{2\Theta}$$

$$+2u^{-2}\Bigg\{\frac{\phi'(u_1)u_1}{\phi(u_1)}\cdot\frac{f(u)}{\phi(u_1)}u_{11} - \frac{f'u_1^2}{\phi(u_1)} + \left[\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2 - \frac{\phi''(u_1)u_1^2}{\phi(u_1)} + \frac{N}{N-1}\right]u_{11}^2$$

$$+\frac{2}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}u_{11} + \frac{f(u)}{\phi(u_1)}\right)u_{11} + \frac{1}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}u_{11} + \frac{f(u)}{\phi(u_1)}\right)^2$$

$$-\left[-\frac{\phi'(u_1)u_1}{\phi(u_1)}u_{11} - \frac{f(u)}{\phi(u_1)} + \left(4+5\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_{11}\right]u_1^2u^{-1}$$

$$+3\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4u^{-2}\Bigg\}.$$

Rearranging the terms in the above inequality, we find

$$\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11} \geq -\left|2-\frac{\phi'(u_1)u_1}{\phi(u_1)}\right|\frac{|\nabla\Theta|^2}{2\Theta}$$

$$+2u^{-2}\Bigg\{-\frac{f'u_1^2}{\phi(u_1)} + \Bigg[\frac{N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2 + \frac{2}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)} - \frac{\phi''(u_1)u_1^2}{\phi(u_1)}$$

$$+\frac{N}{N-1}\Bigg]u_{11}^2 + \frac{1}{N-1}\left(\frac{f(u)}{\phi(u_1)}\right)^2 + \left[\frac{2}{N-1}\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right) + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right]\frac{f(u)}{\phi(u_1)}u_{11}$$

$$-\left[4\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_{11} - \frac{f(u)}{\phi(u_1)}\right]u_1^2u^{-1} + 3\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4u^{-2}\Bigg\}.$$

We recall from (5.3.10) that $\Theta_1 = 2u_1u_{11}u^{-2} - 2u_1^3u^{-3}$. That is,

$$u_{11} = \frac{\Theta_1u^2}{2u_1} + \frac{u_1^2}{u}.$$

Inserting this in the last equality, we have

$$\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11} \geq -\left|2 - \frac{\phi'(u_1)u_1}{\phi(u_1)}\right| \frac{|\nabla\Theta|^2}{2\Theta} + 2u^{-2}\left\{-\frac{f'u_1^2}{\phi(u_1)}\right.$$
$$+\left[\frac{N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2 + \frac{2}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)} - \frac{\phi''(u_1)u_1^2}{\phi(u_1)} + \frac{N}{N-1}\right]\left(\frac{\Theta_1 u^2}{2u_1} + \frac{u_1^2}{u}\right)^2$$
$$+\frac{1}{N-1}\left(\frac{f(u)}{\phi(u_1)}\right)^2 + \left[\frac{2}{N-1}\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right) + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right]\left(\frac{\Theta_1 u^2}{2u_1} + \frac{u_1^2}{u}\right)\frac{f(u)}{\phi(u_1)}$$
$$\left.-\left[4\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)\left(\frac{\Theta_1 u^2}{2u_1} + \frac{u_1^2}{u}\right) - \frac{f(u)}{\phi(u_1)}\right]u_1^2 u^{-1} + 3\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4 u^{-2}\right\}.$$

Further rearrangement leads to

$$\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11} \geq -\left|2 - \frac{\phi'(u_1)u_1}{\phi(u_1)}\right| \frac{|\nabla\Theta|^2}{2\Theta}$$
$$+\left[\frac{N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2 + \frac{2}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)} - \frac{\phi''(u_1)u_1^2}{\phi(u_1)} + \frac{N}{N-1}\right]\left(\frac{2\Theta_1 u_1}{u} + \frac{2u_1^4}{u^4}\right)$$
$$+\frac{2}{N-1}\left(\frac{f(u)}{u\phi(u_1)}\right)^2 + \left[\frac{2}{N-1}\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right) + \frac{\phi'(u_1)u_1}{\phi(u_1)}\right]\left(\frac{\Theta_1 f(u)}{u_1\phi(u_1)}\right)$$
$$-4\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)\frac{\Theta_1 u_1}{u} - 2\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)u_1^4 u^{-4}$$
$$+2\left[\frac{N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2 + \frac{2}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)} - \frac{\phi''(u_1)u_1^2}{\phi(u_1)} + \frac{N}{N-1}\right]\left(\frac{\Theta_1 u^2}{2uu_1}\right)^2$$
$$+2u^{-2}\left[-f'(u) + \frac{N+1}{N-1}\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)\frac{f(u)}{u}\right]\frac{u_1^2}{\phi(u_1)}. \tag{5.3.20}$$

Conditions (5.3.2), together with Condition (ϕ-3), implies that

$$-f'(u) + \frac{N+1}{N-1}\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)\frac{f(u)}{u} \geq -f'(u) + \sigma\left(\frac{N+1}{N-1}\right)\frac{f(u)}{u} \geq 0. \tag{5.3.21}$$

To continue, it will be convenient to introduce the following notations.

$$A(t) := -\left|2 - \frac{\phi'(t)t}{\phi(t)}\right|$$
$$B(t) := \frac{2N}{N-1}\left(\frac{\phi'(t)t}{\phi(t)}\right)^2 - \frac{2(N-3)}{N-1}\frac{\phi'(t)t}{\phi(t)} - \frac{2\phi''(t)t^2}{\phi(t)} + \frac{2}{N-1}$$
$$C := \frac{2}{N-1},\ D(t) := \frac{2}{N-1}\left(1+\frac{\phi'(t)t}{\phi(t)}\right) + \frac{\phi'(t)t}{\phi(t)}$$
$$E(t) := \frac{2N}{N-1}\left(\frac{\phi'(t)t}{\phi(t)}\right)^2 - \frac{4(N-2)}{N-1}\frac{\phi'(t)t}{\phi(t)} - \frac{2\phi''(t)t^2}{\phi(t)} - \frac{2(N-2)}{N-1} \quad \text{and}$$
$$F(t) := \frac{N}{N-1}\left(\frac{\phi'(t)t}{\phi(t)}\right)^2 + \frac{2}{N-1}\frac{\phi'(t)t}{\phi(t)} - \frac{\phi''(t)t^2}{\phi(t)} + \frac{N}{N-1}.$$

Using the Inequality (5.3.21) last inequality, and then dividing both sides of (5.3.22) by Θ, and rearranging, we find

$$\frac{\Delta\Theta + \frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11}}{\Theta} \geq -\left|2 - \frac{\phi'(u_1)u_1}{\phi(u_1)}\right| \frac{|\nabla\Theta|^2}{2\Theta^2}$$

$$+\left[\frac{2N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2-\frac{2(N-3)}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)}-\frac{2\phi''(u_1)u_1^2}{\phi(u_1)}+\frac{2}{N-1}\right]\frac{u_1^2}{u^2}$$
$$+\frac{2}{N-1}\left(\frac{f(u)}{u_1\phi(u_1)}\right)^2+\left[\frac{2}{N-1}\left(1+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)+\frac{\phi'(u_1)u_1}{\phi(u_1)}\right]\left(\frac{\Theta_1 f(u)}{\Theta u_1\phi(u_1)}\right)$$
$$+\left[\frac{2N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2-\frac{4(N-2)}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)}-\frac{2\phi''(u_1)u_1^2}{\phi(u_1)}-\frac{2(N-2)}{N-1}\right]\frac{\Theta_1 u_1}{\Theta u}$$
$$+2\left[\frac{N}{N-1}\left(\frac{\phi'(u_1)u_1}{\phi(u_1)}\right)^2+\frac{2}{N-1}\frac{\phi'(u_1)u_1}{\phi(u_1)}-\frac{\phi''(u_1)u_1^2}{\phi(u_1)}+\frac{N}{N-1}\right]\frac{\Theta_1^2}{4\Theta^2}$$
$$=A(u_1)\frac{|\nabla\Theta|^2}{2\Theta^2}+B(u_1)\frac{u_1^2}{u^2}+C\left(\frac{f(u)}{u_1\phi(u_1)}\right)^2+D(u_1)\left(\frac{\Theta_1 f(u)}{\Theta u_1\phi(u_1)}\right)$$
$$+E(u_1)\frac{\Theta_1 u_1}{\Theta u}+F(u_1)\frac{\Theta_1^2}{4\Theta^2}$$
$$\geq A(u_1)\frac{|\nabla\Theta|^2}{2\Theta^2}+B(u_1)\frac{u_1^2}{u^2}+C\left(\frac{f(u)}{u_1\phi(u_1)}\right)^2 \tag{5.3.22}$$
$$+D(u_1)\left(\frac{\Theta_1 f(u)}{\Theta u_1\phi(u_1)}\right)+E(u_1)\frac{\Theta_1 u_1}{\Theta u}.$$

Note that we have used Condition (ϕ-5) and $F \geq B > 0$ in the penultimate inequality[1] Using Cauchy-Schwartz inequality, we estimate (5.3.21) as follows. We will suppress the dependence of $A, B, D,$ and E on u_1.

$$\frac{\Delta\Theta+\frac{\phi'(u_1)u_1}{\phi(u_1)}\Theta_{11}}{\Theta}\geq A\frac{|\nabla\Theta|^2}{2\Theta^2}+B\frac{u_1^2}{u^2}+C\left(\frac{f(u)}{u_1\phi(u_1)}\right)^2-\frac{D^2}{2C}\left(\frac{\Theta_1^2}{2\Theta^2}\right)$$
$$-C\left(\frac{f(u)}{u_1\phi(u_1)}\right)^2-\frac{E^2}{2B}\left(\frac{\Theta_1^2}{2\Theta^2}\right)-\frac{B}{2}\frac{u_1^2}{u^2}$$
$$=A\frac{|\nabla\Theta|^2}{2\Theta^2}+\frac{B}{2}\frac{u_1^2}{u^2}-\frac{1}{2}\left(\frac{D^2}{C}+\frac{E^2}{B}\right)\frac{\Theta_1^2}{2\Theta^2}. \tag{5.3.23}$$

We now combine inequalities (5.3.9) and (5.3.23) to get

$$A\frac{|\nabla\Theta|^2}{2\Theta^2}+\frac{B}{2}\frac{u_1^2}{u^2}-\frac{1}{2}\left(\frac{D^2}{C}+\frac{E^2}{B}\right)\frac{\Theta_1^2}{2\Theta^2}$$
$$\leq\frac{24a^2}{(a^2-r^2)^2}+\frac{4Na^2}{(a^2-r^2)^2}+\left|\frac{\phi'(u_1)u_1}{\phi(u_1)}\right|\left(\frac{24a^2}{(a^2-r^2)^2}+\frac{4a^2}{(a^2-r^2)^2}\right).$$

In other words, we have

$$0\leq\frac{u_1^2}{u^2}\leq\frac{2}{B}\left[\frac{4a^2(6+N)}{(a^2-r^2)^2}+\left|\frac{\phi'(u_1)u_1}{\phi(u_1)}\right|\frac{28a^2}{(a^2-r^2)^2}+\left|\frac{1}{2}\left(\frac{D^2}{C}+\frac{E^2}{B}\right)-A\right|\frac{|\nabla\Theta|^2}{2\Theta^2}\right].$$

[1] Let $\omega := \frac{\phi'(t)t}{\phi(t)}$ and $\varpi := \frac{\phi''(t)t^2}{\phi(t)}$. Then we see that

$$B=\frac{2N}{N-1}\omega^2-\frac{2(N-3))}{N-1}\omega-2\varpi+\frac{2}{N-1}=2\left[\frac{N}{N-1}(1+\omega)^2-3(1+\omega)+2-\varpi\right],$$

and

$$F=\frac{2N}{N-1}\omega^2+\frac{4}{N-1}\omega-2\varpi+\frac{2N}{N-1}=2\left[\frac{N}{N-1}(1+\omega)^2-2(1+\omega)+2-\varpi\right].$$

From (5.3.3), we recall that at x^*, we have

$$\frac{\nabla\Theta}{\Theta} = \frac{\nabla r^2}{(a^2-r^2)^2}, \quad \text{so that} \quad \frac{|\nabla\Theta|^2}{2\Theta^2} = \frac{8r^2}{(a^2-r^2)^2}.$$

Therefore, at x^* we have the estimate

$$0 \le \frac{|\nabla u|^2}{u^2} \le \frac{C_0(u_1)a^2}{(a^2-r^2)^2},$$

where

$$C_0(u_1) := \frac{8}{B}\left[6 + N + 7(\rho+1) + 2\left|\frac{1}{2}\left(\frac{D^2(u_1)}{C} + \frac{E^2(u_1)}{B(u_1)}\right) - A(u_1)\right|\right]$$

Note that by Conditions (ϕ-3) and (ϕ-5), $C_0(u_1)$ is bounded by a positive constant $\mathcal{M}$. Moreover, we observe that

$$J(x^*) = \frac{|\nabla u|^2}{u^2}(a^2-r^2)^2 \le \mathcal{M}a^2.$$

Since $J(x_0) \le J(x^*)$, we conclude that at x_0 we have

$$\frac{|\nabla u|^2}{u^2}a^4 = J(x_0) \le J(x^*) \le \mathcal{M}a^2.$$

Thus, at x_0 we have the estimate

$$\frac{|\nabla u|^2}{u^2} \le \frac{\mathcal{M}}{a^2}.$$

Letting $a \to \infty$, we find that $|\nabla u| = 0$ at x_0. Since x_0 was arbitrary, we conclude that $\nabla u \equiv 0$ on $\mathbb{R}^N$, as desired. □

5.4 Some Examples

Let us illustrate the above theorem with some examples. The simplest case occurs when $\phi(t) = pt^{p-2}$ for some $p > 1$. In this case, $\sigma = \rho = p - 1$, and we note

$$\frac{N}{N-1}\sigma^2 - 3\rho + 2 = \frac{(p-1)^2}{N-1} + (p-2)(p-3).$$

Therefore, we immediately see that Condition (ϕ-5) holds. This has been investigated in [9].

Now, let us consider $\phi(t) = pt^{p-2} + qt^{q-2}$ for $1 < p < q$. Recall that in this case $\sigma = p-1$ and $\rho = q - 1$. Let us note that[2]

$$\varpi(t) := \frac{\phi''(t)t^2}{\phi(t)} = \frac{p(p-2)(p-3))t^{p-2} + q(q-2)(q-3)t^{q-2}}{pt^{p-2} + qt^{q-2}}$$

[2] If $a, b \in \mathbb{R}$. Since $\min\{a,b\} \le a, b \le \max\{a,b\}$, then we have $\min\{a,b\} \le \theta a + (1-\theta)b \le \max\{a,b\}$ for all $0 \le \theta \le 1$.

$$\leq \max\{(p-2)(p-3),(q-2)(q-3)\}.$$

$$\begin{aligned}
&\frac{N}{N-1}\sigma^2 - 3\rho + 2 - \varpi \\
(5.4.1) \qquad &\geq \frac{N}{N-1}(p-1)^2 - 3(q-1) + 2 - \max\{(p-2)(p-3),(q-2)(q-3)\}.
\end{aligned}$$

Let us observe that

$$\begin{aligned}
(q-2)(q-3) &= (q-p+p-2)(q-p+p-3) \\
&= (q-p)^2 + (q-p)(2p-5) + (p-2)(p-3) \\
&= (q-p)(q+p-5) + (p-2)(p-3) \\
&\geq (q-p)(2p-5) + (p-2)(p-3).
\end{aligned}$$

Therefore, we see that

$$\max\{(p-2)(p-3),(q-2)(q-3)\} = \begin{cases} (p-2)(p-3) & \text{if } p < 5/2 \\ (q-2)(q-3) & \text{if } p \geq 5/2. \end{cases}$$

So let us first suppose that $1 < p < 5/2$. Then Inequality (5.4.1) reduces to

$$\frac{N}{N-1}\sigma^2 - 3\rho + 2 - \bar{\omega} \geq \frac{N}{N-1}(p-1)^2 - 3(q-1) + 2 - (p-2)(p-3).$$

Thus, if

$$\begin{aligned}
p < q &< \frac{1}{3}\left[\frac{N}{N-1}(p-1)^2 - (p-2)(p-3) + 5\right] \\
&= \frac{1}{3}\left[\frac{N}{N-1}(p-1)^2 - (p-1)^2 + 3p\right] \\
&= \frac{1}{3}\left[\frac{(p-1)^2}{N-1} + 3p\right],
\end{aligned}$$

then Condition (ϕ-5) holds.

Now suppose $p \geq \frac{5}{2}$. Then Inequality (5.4.1) becomes

$$\begin{aligned}
&\frac{N}{N-1}\sigma^2 - 3\rho + 2 - \varpi \\
&\geq \frac{N}{N-1}(p-1)^2 - 3(q-1) + 2 - (q-2)(q-3) \\
&= \frac{N}{N-1}(p-1)^2 - q^2 + 2q - 1 \\
&= \frac{N}{N-1}(p-1)^2 - (q-1)^2.
\end{aligned}$$

Therefore, if

$$p < q < (p-1)\sqrt{\frac{N}{N-1}} + 1,$$

the Condition (ϕ-5) holds.

Now, given $p > 1$, let us set p_* as follows.

$$p_* := \begin{cases} \dfrac{1}{3}\left[\dfrac{(p-1)^2}{N-1} + 3p\right] & \text{if } p < \frac{5}{2} \\ (p-1)\sqrt{\dfrac{N}{N-1}} + 1 & \text{if } p \geq \frac{5}{2}. \end{cases}$$

We remark that $p < p_*$ for $p > 1$. We now summarize the above discussion in the following corollary.

Corollary 5.8. Given $p > 1$, suppose f satisfies Condition (5.3.2) with $\sigma = p - 1$. If $1 < p < q < p_*$, then any non-negative entire solution

$$\Delta_p u + \Delta_q u = -f(u)$$

is a constant on $\mathbb{R}^N$.

As another example, let us look at $\phi(t) = 2p(1+t^2)^{p-1}$ for $1/2 < p \leq 2$.

We recall that

$$\sigma = \min\{1, 2p-1\} \quad \text{and} \quad \rho = \max\{1, 2p-1\}$$

Moreover, direct computation shows that

$$\begin{aligned} \varpi(t) &= \frac{\phi''(t)t^2}{\phi(t)} \\ &= 4(p-1)(p-2)\left(\frac{t^2}{t^2+1}\right)^2 + 2(p-1)\frac{t^2}{t^2+1}. \end{aligned}$$

Suppose first $1/2 < p \leq 1$. Then

$$\begin{aligned} \varpi(t) &\leq [4(p-1)(p-2) + 2(p-1)]\frac{t^2}{t^2+1} \\ &= 2(p-1)(2p-3)\frac{t^2}{t^2+1}. \end{aligned}$$

Then in this case, we have $2p - 1 = \sigma \leq \rho = 1$ and therefore,

$$\begin{aligned} \frac{N}{N-1}\sigma^2 - 3\rho + 2 - \varpi &\geq \frac{N}{N-1}(2p-1)^2 - 1 - 2(p-1)(2p-3) \\ &= \frac{N}{N-1}(2p-1)^2 - (2p-1)^2 + 3(2p-1) - 3 \\ &= \frac{1}{N-1}(2p-1)^2 + 3(2p-1) - 3 \end{aligned}$$

which is positive if and only if

$$1 \geq p > \frac{-3(N-1) + \sqrt{(N-1)(9N+3)}}{2}.$$

On the other hand, suppose $1 < p \leq 2$. Then

$$\varpi(t) \leq 2(p-1)$$

and since

$$1 \leq \sigma \leq \rho = 2p-1,$$

we have

$$\begin{aligned} \frac{N}{N-1}\sigma^2 - 3\rho + 2 - \varpi &\geq \frac{N}{N-1} - 3(2p-1) + 2 - 2(p-1) \\ &= \frac{N}{N-1} - 8p + 7. \end{aligned}$$

Therefore, for

$$1 < p < 1 + \frac{1}{N+1}$$

$\frac{N}{N-1}\sigma^2 - 3\rho + 2 - \varpi$ is positive. Therefore, as a consequence of Theorem 5.5, we have the following

$$\frac{-3(N-1) + \sqrt{(N-1)(9N+3)}}{2} < p < 1 + \frac{1}{N-1}. \tag{5.4.2}$$

Corollary 5.9. Suppose f satisfies Condition (5.3.2) with $\sigma = \min\{1, 2p-1\}$. If p satisfies (5.4.2), then any non-negative entire solution of

$$\operatorname{div}((1 + |\nabla u|^2)^{p-1}\nabla u) = -\frac{1}{2p} f(u)$$

is a constant on $\mathbb{R}^N$.

Chapter 6

Future Work

There are some interesting problems related to this dissertation that one can investigate. Here, we cite a couple of problems.

(I) One can ask questions similar to those addressed in chapter 2, 3 and 4 of this dissertation when the ϕ-Laplacian is replaced by a non-divergence structure and uniformly elliptic operator:

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_j(x)u_{x_j} + c(x)u.$$

We suppose that the coefficients of L are all continuous in $\mathbb{R}^N$ and that L is uniformly elliptic in $\mathbb{R}^N$, in the sense that there are constants $0 < \theta \leq \Theta$ such that

$$\theta|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \Theta|\xi|^2, \quad \forall\,(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

(II) Another natural problem to consider would be to investigate if some or most of the results extend to equations of the form

$$\Delta_\phi u = h(x, u, \nabla u)$$

where $h : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfy the structure condition

$$|h(x,t,p)| \leq b(|x|)f(t,|p|), \quad (x,t,p) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N.$$

It is reasonable to expect that many of the results in Chapter 2 continue to hold under the following assumptions.

$b : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is non-trivial continuous function that satisfies Condition (b-1) and $f : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies any of the following conditions. The function $f(t,r)$ is

(f-1)* : non-decreasing in t, r and $\lim\limits_{t\to 0^+} \dfrac{\lambda^{-1}(f(t,r))}{t} < \dfrac{1}{B}$ for each $r \geq 0$.

(f-2)* : non-decreasing in t, r and $\lim\limits_{t\to \infty} \dfrac{\lambda^{-1}(f(t,r))}{t} < \dfrac{1}{B}$ for each $r \geq 0$.

(f-3)* : non-increasing in t, non-decreasing in r.

Moreover $f(r,r)$ is non-increasing in r.

For instance, $f(t,r) = (1+r)^p(1+t)^{-q}$ satisfies conditions

$$\begin{cases} \text{(f-1)}^* & \text{if} \quad (p,q) \in \mathbb{R} \times (-\sigma, \infty) \\ \text{(f-2)}^* & \text{if} \quad (p,q) \in \mathbb{R} \times (-\infty, -\sigma) \\ \text{(f-3)}^* & \text{if} \quad 0 < p \leq q \end{cases}$$

Chapter 7

Appendix

This appendix has two parts. In the first part, we recall some basic background material used to develop the work in this dissertation. In the second part, we prove some results that were used in proofs of some results in the thesis.

Let X be a real Banach space. As is standard, we denote by X^* the Banach space of all bounded linear functionals on X. A sequence $\{x_n\}$ in X is said to converge weakly to some $x \in X$, denoted by $x_n \rightharpoonup x$ if and only if

$$f(x_n) \to f(x) \quad \forall\, f \in X^*.$$

Note that if $\{x_n\}$ is a sequence in X such that $x_n \to x$ for $x \in X$, then $x_n \rightharpoonup x$. Furthermore, it is well-known that any weakly convergent sequence is bounded.

Any bounded sequence in a reflexive Banach space contains a weakly convergent subsequence. We state this fact in the following theorem.

Theorem 7.1. (Weak Compactness) Let X be a reflexive Banach space and suppose that $\{x_k\}$ is a bounded sequence. Then there exists a subsequence $\{x_{k_j}\}$ and $x \in X$ such that $x_{k_j} \rightharpoonup x$.

To state another useful theorem, we start by developing the basic notions needed in its statement. Let X be a Banach space, and $\ell \in X^*$. It is easily seen that $C := \{u \in X : \ell(u) \leq \alpha\}$ is a convex subset of X, and clearly is a weakly closed subset of X. We refer to such a set as a closed half-space. Since ℓ is continuous, it is clear that C is a closed subset of X as well. Since arbitrary intersections of convex sets are convex, and arbitrary intersections of closed sets are closed. we see that for any $\mathcal{I} \subseteq X^*$ the set

$$\bigcap_{\ell \in \mathcal{I}} \{u \in X : \ell(u) \leq \alpha_\ell\},$$

where α_ℓ are constants, is a weakly closed as well as a weakly convex subset of X. Mazur's Theorem states that every closed and convex subset of X arises in this way, see [12, Corollary 1.4]. In other words

Theorem 7.2. (Mazur's Theorem) Let X be a Banach space. Then every closed and convex subset of X is the intersection of the closed half-spaces that contain it. In other words, any closed and convex subset of X is weakly closed.

Another useful reformation of the above theorem is the following.

Theorem 7.3. (Mazur's Theorem) Let X be a Banach space, and suppose $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u \in X$. Then there is $N : \mathbb{N} \to \mathbb{N}$ and for each $n \in \mathbb{N}$ and there are real numbers $\gamma_k(n),\ k = n, \cdots, N(n)$ such that the sequence

$$w_n = \sum_{k=n}^{N(n)} \gamma_k(n) u_k \tag{7.0.1}$$

converges (strongly) to u in X.

To see the equivalence, let us first assume Theorem 7.2 and prove Theorem 7.3. So suppose $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u$ for some $u \in X$. As a consequence, it is clear that for each $n \in \mathbb{N}$, u belongs to the weak closure of H_n, where H_n is the convex hull of $\{u_k : k \geq n\}$. By Theorem 7.2, the weak closure of H_n is the same as the closure $\overline{H}_n$ of H_n. Thus, $u \in \overline{H}_n$ for all $n \in H_n$. Now, for all n, we choose $w_n \in H_n$ such that $\|u - w_n\| < \frac{1}{n}$. Thus $w_n \to u$ strongly and each w_n is of the desired form.

Now let us assume that Theorem 7.3 holds, and let $A \subseteq X$ be a closed and convex set. Let S be the weak closure of A. Clearly, $A \subseteq S$. Let $u \in S$. Then there is a sequence $\{u_n\}$ in A that converges weakly to u. By Theorem 7.3, there is a sequence $\{w_n\}$ of elements of X of the form (7.0.1) such that $w_n \to u$. Since A is convex, we note that $w_n \in A$ for all n. Since A is closed, we see that $u \in A$. Thus, we have shown that $A = S$, as claimed.

Let X and Y be Banach spaces with $X \subseteq Y$. We say

(a) X is continuously imbedded in Y, indicated, $X \hookrightarrow Y$ if there is a constant $C > 0$ such that $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$, and

(b) X is compactly imbedded in Y, indicated by $X \subset\subset Y$ if

  (i) X is continuously imbedded in Y and

  (ii) Each bounded sequence in X has a subsequence that converges in Y.

This concept can best be illustrated by recalling the Rellich-Kondrachov Theorem (see [1, Theorem 6.3, Part I]). Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set that satisfies the cone condition and suppose $1 \leq p < n$. Then, for any q, $1 < q < p^* = n/(n-p)$, we have

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega).$$

In particular,

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega), \quad \forall\, 1 \leq p < n.$$

A functional $J : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be weakly lower semicontinuous at $x \in X$ if and only if for every sequence $\{x_n\}$ that converges weakly to x, we have

$$J(x) \leq \lim_{n\to\infty} \inf J(x_n).$$

A functional J on a separable Banach space is called coercive if and only if

$$\lim_{\|x\|\to\infty} J(x) = \infty.$$

7.1 The Schauder-Tychonoff Fixed-Point Theorem

Let X be a topological space and (Y, ρ) be metric space. By $\mathcal{C}(X, Y)$, we denote the space of continuous functions from X into Y. Let $\mathcal{F} \subseteq \mathcal{C}(X, Y)$. We say $\mathcal{F}$ is equicontinuous at $x_0 \in X$ if and only if for each $\epsilon > 0$, there is a neighborhood U of x_0 in X such that

$$\rho(f(x), f(x_0)) < \epsilon, \quad \forall\, (x, f) \in U \times \mathcal{F}.$$

We say $\mathcal{F}$ is equicontinuous on X if $\mathcal{F}$ is equicontinuous at each point $x \in X$.
One can introduce a topology on Y^X, the set of all functions from X into Y as follows. Given $f \in Y^X$, a compact set $K \subseteq X$ and $r > 0$, let

$$B_K(f, r) = \left\{ g \in Y^X : \sup_{x\in K} \rho(f(x), g(x)) < r \right\}.$$

One can show that the sets $B_K(f, r)$ form a basis for a topology on Y^X. This is called the topology of compact convergence. The use of "compact convergence" in this definition is justified by the following theorem.
Theorem 7.4. A sequence $f_n : X \to Y$ of functions converges to $f : X \to Y$ in the topology of compact convergence of Y^X if and only if $f_n \to f$ uniformly on K for each compact subset $K \subseteq X$.

We are now ready to state a more general version of the Arzela-Ascoli theorem. See [33, Theorem 47.1].
Theorem 7.5. Let X be a topological space and (Y, d) be a metric space, and suppose $\mathcal{C}(X, Y)$ is given the topology of compact convergence. Let $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ such that

(i) $\mathcal{F}$ is equicontinuous at each $x \in X$ and

(ii) $\mathcal{F}_z = \{f(z) : f \in \mathcal{F}\}$ has compact closure for each $z \in X$.

Then $\mathcal{F}$ is contained in a compact subspace of $\mathcal{C}(X, Y)$.

The Schauder-Tychonoff Theorem 2.2 is stated for locally convex linear topological spaces X. We recall these notations below.

Definition 7.6. By a linear topological space X, we mean a linear space X over $\mathbb{R}$ with a topology τ with respect to which every singleton is closed and the linear space operations $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$

$$(x, y) \mapsto x + y \quad \text{and} \quad (\alpha, x) \mapsto \alpha \cdot x$$

are continuous. A linear topological space X is said to be locally convex if it has a local base whose elements are convex.

7.2 Orlicz and Orlicz-Sobolev Spaces

In this part of the appendix, we will recall some basic notions and results on Orlicz and Orlicz-Sobolev spaces.

Definition 7.7. Let $a : [0, \infty) \to \mathbb{R}$ be a function such that

(i) $a(0) = 0$ and $a(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} a(t) = \infty$;

(ii) a is non-decreasing;

(iii) a is right continuous on $[0, \infty)$.

Then, the real-valued function

$$A(t) = \int_0^t a(s)ds$$

is called an N-function.

One can easily verify that if A is such an N-function, then the following properties hold.

(a) A is continuous on $[0, \infty)$;

(b) A is strictly increasing;

(c) A is convex;

(d) $\lim_{t\to 0^+} \frac{A(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{A(t)}{t} = \infty$;

(e) $t \to \frac{A(t)}{t}$ is strictly increasing on $(0, \infty)$.

It is noteworthy to see that if $A : [0, \infty) \to \mathbb{R}$ is a function that satisfies properties (a)-(d), then A is an N-function. To see this, suppose $x < y < z$ and set

$$\alpha := \frac{y - x}{z - x} \quad \text{and} \quad \beta := \frac{z - y}{z - x}.$$

Note that $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$. Moreover, by convexity we have

$$A(y) = A(\alpha z + \beta x) \leq \alpha A(z) + \beta A(x).$$

Thus, we obtain

$$A(y)\frac{z-y}{z-x} + A(y)\frac{y-x}{z-x} \leq A(z)\frac{y-x}{z-x} + A(x)\frac{z-y}{z-x}.$$

Rearranging, we find that

$$(A(y) - A(x))\frac{z-y}{z-x} \leq (A(z) - A(y))\frac{y-x}{z-x}.$$

In other words, we find that

$$\frac{A(y) - A(x)}{y - x} \leq \frac{A(z) - A(y)}{z - y}. \tag{7.2.1}$$

Given $a < b$, we see that by a repeated applications of (7.2.1), for $c < a < x < y < b < d$, we have

$$\frac{A(a) - A(c)}{a - c} \leq \frac{A(y) - A(x)}{y - x} \leq \frac{A(d) - A(b)}{d - b}.$$

In other words, for given $a < b$, there is a constant M depending on a and b only such that $x, y \in [a, b]$

$$|A(x) - A(y)| \leq M|x - y|.$$

That is, A is locally Lipschits and hence for any $c > 0$, A is absolutely continuous on $[0, c]$ and hence is differentiable a.e. on $[0, \infty)$ and A' is integrable on $[0, c]$. Moreover,

$$A(x) = \int_0^x A'(t)dt.$$

Condition (d) shows that $A'(0) = 0$ and that $A'(t) \to \infty$ as $t \to \infty$. To see the latter, note that (7.2.1) implies

$$\frac{A(a)}{a} \leq \frac{A(y) - A(x)}{y - x}, \quad \forall\, 0 < a < x < y.$$

Given $M > 0$, we choose $a > 0$ large enough such that

$$M \leq \frac{A(a)}{a} \leq \frac{A(y) - A(x)}{y - x}, \quad \forall\, 0 < a < x < y.$$

This shows that $A'(x) \geq M$ for $x > a$. The inequality (7.2.1) also shows that A' is increasing. The other conditions in Definition (7.7) for $a(t) := A'(t)$ can also be shown to hold, thus showing that A is indeed an N-function.

Let A be an N-function with the corresponding function a that satisfies conditions (i)-(iii). Given $s > 0$, note that $\{t \geq 0 : a(t) \leq s\}$ is bounded. Let

$$\widetilde{a}(s) := \sup\{t : a(t) \leq s\}.$$

One can easily check that the function $\widetilde{a}$ satisfies all the properties (i)-(iii) in Definition (7.7). Furthermore, we note that

$$a(t) = \sup\{s : \widetilde{a}(s) \le t\}. \tag{7.2.2}$$

If a is strictly increasing, then

$$\widetilde{a}(a(t)) = \sup\{s : a(s) \le a(t)\} = t.$$

Therefore $\widetilde{a} = a^{-1}$. Let

$$\widetilde{A}(s) := \int_0^s \widetilde{a}(\sigma) d\sigma. \tag{7.2.3}$$

The N-functions A and $\widetilde{A}$ are called complementary. On the $\tau - \sigma$ axis, $\widetilde{A}(s)$ represents the area of the region to the left of the graph $\sigma = a(\tau)$ (or more precisely $\tau = \widetilde{a}(\sigma)$) from $\sigma = 0$ to $\sigma = s$ while $A(t)$ represents the area of the region below the graph of $\sigma = a(\tau)$ from $\tau = 0$ to $\tau = t$. From these graphical considerations, it is easily noted that the following inequality (usually called Young's Inequality) holds.

$$st \le A(t) + \widetilde{A}(s). \tag{7.2.4}$$

An analytic proof runs as follows.
On the other hand using (7.2.2) and $a(t) \ge s$, we have

$$\begin{aligned} A(t) &= \int_0^t a(\tau) d\tau \\ &= \int_0^{a(t)} \sigma d\, \widetilde{a}(\sigma), \quad \text{here we used the change of variable} \quad \sigma = a(\tau) \\ &\ge \int_0^s \sigma d\, \widetilde{a}(\sigma) = \sigma\, \widetilde{a}(\sigma)|_0^s - \int_0^s \widetilde{a}(\sigma) d\sigma = s\widetilde{a}(s) - \widetilde{A}(s) \ge a(t)\widetilde{a}(s) - \widetilde{A}(s). \end{aligned}$$

Note that equality holds if and only if

$$\int_0^{a(t)} \sigma d\, \widetilde{a}(\sigma) = \int_0^s \sigma d\, \widetilde{a}(\sigma),$$

which occurs if and only if $s = a(t)$ (or equivalently $t = \tilde{a}(s)$). Writing Inequality (7.2.4) as

$$\widetilde{A}(s) \ge st - A(t)$$

and noting that equality holds when $t = \widetilde{a}(s)$, we have

$$\widetilde{A}(s) = \max_{t \ge 0}(st - A(t)).$$

This last relationship could have been used as the definition of the N-function complementary to A.

Definition 7.8. An N-function A is said to satisfy a global Δ_2-condition if there exists a positive constant k such that

$$A(2t) \leq kA(t), \quad \forall\, t \geq 0.$$

It can be shown that an N-function A associated to a satisfies the global Δ_2-condition if and only if there is a positive c such that

$$\frac{1}{c} ta(t) \leq A(t) \leq ta(t), \quad \forall\, t \geq 0.$$

We are now ready to introduce the Orlicz and Orlicz-Sobolev spaces. Let $\Omega \subseteq \mathbb{R}^N$ be a non-empty open set in $\mathbb{R}^N$, and let A be an N-function that satisfies the global Δ_2-condition. The Orlicz class $L^A(\Omega)$ is the set of all (equivalence classes modulo equality a.e. in Ω.) of measurable functions u defined on Ω that satisfy

$$\int_\Omega A(|u(x)|)dx < \infty.$$

Note that if $u \in L^A(\Omega)$, then for k sufficiently large, the convexity of A shows that

$$\int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \leq \frac{1}{k} \int_\Omega A(|u(x)|)dx \leq 1.$$

One can use the Δ_2-condition to show that $L^A(\Omega)$ is a vector space, see [1, Chapter 8]. In fact, it is a Banach space under the following norm, called the Luxemburg norm:

$$\|u\|_A = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \leq 1\right\}.$$

The infimum is attained. In fact, if $\{k_j\}$ is a sequence of positive real numbers that decreases to $\|u\|_A$, then

$$\frac{|u(x)|}{k_j} \to \frac{|u(x)|}{\|u\|_A}, \quad x \in \Omega.$$

Therefore, by the monotone convergence theorem, we have

$$\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right) dx = \lim_{j\to\infty} \int_\Omega A\left(\frac{|u(x)|}{k_j}\right) dx \leq 1.$$

Consequently, we have

$$\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right) dx \leq 1. \tag{7.2.5}$$

For the rest of this section, we will assume that A is an N-function that satisfies a global Δ_2-condition. We start with the following useful inequality which is an immediate consequence of Young's Inequality, (7.2.4).

Theorem 7.9. (Generalized Hölder Inequality) Let A and $\widetilde{A}$ be complementary N-functions. Then

$$\left|\int_\Omega u(x)v(x)dx\right| \leq 2\|u\|_A\|v\|_{\widetilde{A}}.$$

Proof. Clearly, the inequality holds if either u or v is zero. So, let us suppose that both $\|u\|_A$ and $\|v\|_{\widetilde{A}}$ are non-zero. Let

$$s = \frac{|u(x)|}{\|u\|_A}, \quad \text{and} \quad t = \frac{|v(x)|}{\|v\|_{\widetilde{A}}}.$$

Then

$$\frac{|u(x)|}{\|u\|_A}\frac{|v(x)|}{\|v\|_{\widetilde{A}}} \leq A\left(\frac{|u(x)|}{\|u\|_A}\right) + \widetilde{A}\left(\frac{|v(x)|}{\|v\|_{\widetilde{A}}}\right).$$

Integrating both sides on Ω and utilizing (7.2.5), we find that

$$\begin{aligned}\int_\Omega \frac{|u(x)|}{\|u\|_A}\frac{|v(x)|}{\|v\|_{\widetilde{A}}}dx &\leq \int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right)dx + \int_\Omega \widetilde{A}\left(\frac{|v(x)|}{\|v\|_{\widetilde{A}}}\right)dx \\ &\leq 2.\end{aligned}$$

This leads to the claimed inequality. □

Remark 7.10. If $u \in L^A(\Omega)$ and $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, then by the Generalized Hölder Inequality, we have

$$\int_\Omega |u(x)|dx \leq 2\|1\|_{\widetilde{A}}\|u\|_A < \infty.$$

This shows that $L^A(\Omega) \hookrightarrow L^1(\Omega)$, with $\|u\|_{L^1} \leq 2\|1\|_{\widetilde{A}}\|u\|_A$ for all $u \in L^A(\Omega)$.
A sequence $\{u_j\}$ in $L^A(\Omega)$ is said to converge in mean to $u \in L^A(\Omega)$ if and only if

$$\lim_{j\to\infty}\int_\Omega A(|u_j(x) - u(x)|)dx = 0.$$

Since A is convex and $A(0) = 0$, we note that, for each $0 < \epsilon < 1$, we have

$$\begin{aligned}\int_\Omega A(|u_j(x) - u(x)|)dx &= \int_\Omega A\left(\frac{\epsilon|u_j(x) - u(x)|}{\epsilon}\right)dx \\ &\leq \epsilon\int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right)dx.\end{aligned} \tag{7.2.6}$$

Suppose $\|u_j - u\|_A \to 0$ as $j \to \infty$, and let $0 < \epsilon \leq 1$ be given. Then there is $J \in \mathbb{N}$ such that $\|u_j - u\|_A < \epsilon$ for all $j \geq J$. Note that if

$$\int_\Omega A\left(\frac{|u_\ell(x) - u(x)|}{\epsilon}\right)dx > 1,$$

for some $\ell \geq J$, then for all $0 < \delta \leq \epsilon$, we have

$$\int_\Omega A\left(\frac{|u_\ell(x) - u(x)|}{\delta}\right) dx \geq \int_\Omega A\left(\frac{|u_\ell(x) - u(x)}{\epsilon}\right) dx > 1,$$

and hence

$$\|u_\ell - u\|_A = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u_\ell(x) - u(x)|}{\epsilon}\right) dx \leq 1\right\} \geq \epsilon,$$

contrary to assumption. Therefore, we must have

$$\int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right) dx \leq 1, \quad \forall\, j \geq J.$$

Using this in (7.2.6) we find that

$$\int_\Omega A(|u_j(x) - u(x)|)dx \leq \epsilon, \quad \forall j \geq J.$$

For the converse, suppose $\epsilon > 0$ is given. We fix a positive integer k such that $2^k \geq \frac{1}{\epsilon}$. Then,

$$A(\epsilon^{-1}t) \leq A(2^k t) \leq c^k A(t), \quad \forall\, t \geq 0. \tag{7.2.7}$$

By assumption, there is J, large enough such that

$$\int_\Omega A(|u_j(x) - u(x)|)dx < c^{-k}, \quad \forall\, j \geq J.$$

Now on using (7.2.7), for $j \geq J$,

$$\begin{aligned}\int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right) dx &< c^k \int_\Omega A(|u_j(x) - u(x)|)dx \\ &< 1.\end{aligned}$$

Hence, by definition, we have

$$\|u_j - u\|_A < \epsilon, \quad \forall\, j \geq J.$$

For a positive integer k, we define the Orlicz-Sobolev space $W^{k,A}(\Omega)$ to be the set of all $u \in L^A(\Omega)$ such that the weak derivatives $D^\alpha u$ belong to $L^A(\Omega)$ for all multi-induces $|\alpha| \leq k$. This is a Banach space under the norm

$$\|u\|_{k,A} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_A.$$

We denote by $W_0^{k,\Phi}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,\Phi}(\Omega)$.
The space $W^{1,A}(\Omega)$ share many properties with the standard Sobolev spaces $W^{1,p}(\Omega)$. We first make the following useful remark.

Remark 7.11. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open subset. By Remark 7.10, $W^{1,\Phi}(\Omega) \subseteq W^{1,1}(\Omega)$. In fact, $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,1}(\Omega)$. By Rellich-Kondachov Compactness Theorem, we have $W^{1,1}(\Omega) \subset\subset L^1(\Omega)$. Therefore for any bounded set $\Omega \subseteq \mathbb{R}^N$ we have

$$W^{1,\Phi}(\Omega) \subset\subset L^1(\Omega). \tag{7.2.8}$$

Below, we highlight these properties as they are useful in this thesis.
We start with the following theorem which gives some basic properties of Orlicz-Sobolev spaces. The proofs are straightforward generalizations of the proofs of the analogous properties for the standard Sobolev spaces. See [1].

Theorem 7.12. Let $\Omega \subseteq \mathbb{R}^N$ be a non-empty bounded open set, and let A be an N-function that satisfies a global Δ_2-condition.

(a) If $\widetilde{A}$ satisfies a global Δ_2-condition, then $W^{1,A}(\Omega)$ is reflexive.

(b) Each element F of the dual space $(W^{1,A}(\Omega))^*$ is given by

$$F(u) = \int_\Omega (\nabla u \cdot \mathbf{v} + uv)$$

for some $\mathbf{v} \in \left(L^{\widetilde{A}}(\Omega)\right)^N$ and $v \in L^{\widetilde{A}}(\Omega)$.

(c) $C^\infty(\Omega) \cap W^{1,A}(\Omega)$ is dense in $W^{1,A}(\Omega)$.

The following Poincaré inequality is useful (see [15, Lemma 2.4]).

Theorem 7.13. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open subset with diameter d, and suppose A is an N-function that satisfies the global Δ_2-condition. Then

$$\|u\|_A \leq d\|\nabla u\|_A, \quad \forall\, u \in W_0^{1,A}(\Omega). \tag{7.2.9}$$

Proof. Suppose $\Omega \subseteq [a, a+d]^N$ where $d = \operatorname{diam}(\Omega)$. Then, for $\varphi \in C_c^\infty(\Omega)$ we have

$$\varphi(x', x_N) = \int_a^{x_N} \varphi_{x_N}(x', t)dt.$$

Therefore, for arbitrary constant $\varrho > 0$, we have

$$\frac{|\varphi(x', x_N)|}{\varrho d} \leq \frac{1}{\varrho d}\int_a^{x_N} |\varphi_{x_N}(x', t)|dt \leq \frac{1}{d}\int_a^{a+d} \frac{|\nabla\varphi(x', x_N)|}{\varrho} dx_N.$$

By Jensen's Inequality, we have

$$A\left(\frac{|\varphi(x)|}{\varrho d}\right) \leq \frac{1}{d}\int_a^{a+d} A\left(\frac{|\nabla\varphi(x', x_N)|}{\varrho}\right) dx_N.$$

We now integrate both sides with respect to x' over $[a, a+d]^{N-1}$ to find

$$\int_{[a,a+d]^{N-1}} A\left(\frac{|\varphi(x)|}{\varrho d}\right) dx' \leq \frac{1}{d}\int_{[a,a+d]^N} A\left(\frac{|\nabla\varphi(x', x_N)|}{\varrho}\right) dx.$$

Finally, we integrate both sides on $[a, a+d]$ with respect x_N. We get

$$\int_{[a,a+d]^N} A\left(\frac{|\varphi(x)|}{\varrho d}\right) dx \leq \int_{[a,a+d]^N} A\left(\frac{|\nabla\varphi(x)|}{\varrho}\right) dx.$$

We now take $\varrho = \|\nabla\varphi\|_A$ to get

$$\int_\Omega A\left(\frac{|\varphi(x)|}{d\|\nabla\varphi\|_A}\right) dx \leq \int_\Omega A\left(\frac{|\nabla\varphi(x)|}{\|\varphi\|_A}\right) dx \leq 1.$$

By definition of the norm $\|\varphi\|_A$, we see that

$$\|\varphi\|_A \leq d\|\nabla\varphi\|_A. \tag{7.2.10}$$

If $u \in W_0^{1,A}(\Omega)$, then we pick a sequence $\{\varphi_n\}$ in $C_c^\infty(\Omega)$ such that $\varphi_n \to u$ in $W^{1,A}(\Omega)$ and we apply (7.2.10) to the $\varphi_n's$ and then take the limits as $n \to \infty$ to get Inequality (7.2.9). □

The next lemma is a consequence of the above Poincaré inequality.

Lemma 7.14. Suppose A is an N-function that satisfies the global Δ_2-condition, and let $\Omega \subseteq \mathbb{R}^N$ be a domain (i.e. connected open set) and $u \in W_0^{1,A}(\Omega)$. If $\nabla u = 0$ in Ω, then $u = 0$.

Proof. Let $\{\varphi_j\}$ be a sequence in $C_c^\infty(\Omega)$ such that $\varphi_j \to u$ in $W^{1,A}(\Omega)$. In particular, $\|\varphi_j - u\|_A \to 0$ and $\|\nabla\varphi_j\|_A = \|\nabla\varphi - \nabla u\|_A \to 0$ as $j \to \infty$. Consequently, using Poincaré Inequality, we have

$$\|u\|_A \leq \|\varphi_j - u\|_A + \|\varphi_j\|_A \leq \|\varphi_j - u\|_A + d\|\nabla\varphi_j\|_A, \quad \forall\, j. \tag{7.2.11}$$

Taking the limit in (7.2.11) as $j \to \infty$, we conclude that $\|u\|_A = 0$ which gives the desired result. □

Lemma 7.15. Let $u \in W^{1,A}(\Omega)$ (resp., $W_0^{1,A}(\Omega)$). Then $u^+ \in W^{1,A}(\Omega)$ (resp., $W_0^{1,A}(\Omega)$). Moreover, we have

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases} \tag{7.2.12}$$

Proof. The validity of (7.2.12) follows from [18, Lemma 7.6]. Since $0 \leq u^+ \leq |u|$ and $|\nabla u^+| \leq |\nabla u|$, the assertion that $u^+ \in W^{1,\Phi}(\Omega)$ is evident. □

Corollary 7.16. Let $u, v \in W^{1,A}(\Omega)$. Then $\max\{u, v\} \in W^{1.A}(\Omega)$ and

$$\nabla \max\{u, v\}(x) = \begin{cases} \nabla u(x) & \text{if } u(x) \geq v(x) \\ \nabla v(x) & \text{if } u(x) \leq v(x). \end{cases} \tag{7.2.13}$$

Proof. The assertion follows from Lemma 7.15 and the relation

$$\max\{u, v\} = (u - v)^+ + v.$$

□

Lemma 7.17. Suppose $\{u_j\}$ is a sequence in $W^{1,A}(\Omega)$ such that $u_j \to u$ in $W^{1,A}(\Omega)$. Then $u_{k_j}^+ \to u^+$ in $W^{1,A}(\Omega)$ for some subsequence $\{u_{k_j}\}$.

Proof. Using the fact that $|f^+ - g^+| \leq |f - g|$ for any functions[1] f and g, we see that $A(|u_j^+ - u^+|) \leq A(|u_j - u|)$ for all j, from which it follows that $u_j^+ \to u^+$ in $L^A(\Omega)$. Next, we proceed to show that $\nabla u_{k_j}^+ \to \nabla u^+$ for some subsequence $\{u_{k_j}\}$ such that $u_{k_j} \to u$ a.e. in Ω. For convenience, we continue to denote $\{u_{k_j}\}$ simply as $\{u_j\}$. Let χ be the characteristic function of $\mathbb{R}^+ := (0, \infty)$. Then

$$\nabla u_j^+ = \chi(u_j)\nabla u_j, \quad \text{and} \quad \nabla u^+ = \chi(u)\nabla u.$$

Therefore,

$$\begin{aligned} \|\nabla u_j^+ - \nabla u^+\|_A &= \|\chi(u_j)\nabla u_j - \chi(u)\nabla u\|_A \\ &\leq \|\chi(u_j)\nabla u_j - \chi(u_j)\nabla u + \chi(u_j)\nabla u - \chi(u)\nabla u\|_A \\ &\leq \|\chi(u_j)(\nabla u_j - \nabla u)\|_A + \|(\chi(u_j) - \chi(u))\nabla u\|_A = I_j + II_j. \end{aligned}$$

It is clear that $I_j \to 0$ as $u_j \to u$ in $W^{1,A}(\Omega)$. Moreover, $II_j \to 0$ by dominated convergence theorem. □

[1] To see this let $x \in \Omega$. Suppose first $f(x) < 0$ and $g(x) < 0$. Then $|f^+(x) - g^+(x)| = 0 \leq |f(x) - g(x)|$. Suppose $f(x) \geq 0$ or $g(x) \geq 0$. Then

$$\begin{aligned} |f(x) - g(x)| &= |f^+(x) - g^+(x) - (f^-(x) - g^-(x))| \geq \begin{cases} |f^+(x) - g^+(x)| - f^-(x) + g^-(x) \\ |f^+(x) - g^+(x)| - g^-(x) + f^-(x) \end{cases} \\ &= \begin{cases} |f^+(x) - g^+(x)| + g^-(x) & \text{if } f(x) \geq 0 \\ |f^+(x) - g^+(x)| + f^-(x) & \text{if } g(x) \geq 0 \end{cases} \\ &\geq |f^+(x) - g^+(x)|. \end{aligned}$$

Lemma 7.18. *Let* $v \in W^{1,A}(\Omega)$*.*

(i) If v *has compact support, then* $v \in W_0^{1,A}(\Omega)$*.*

(ii) If $0 \leq v \leq u$ *for some* $u \in W_0^{1,A}(\Omega)$*, then* $v \in W_0^{1,A}(\Omega)$*.*

Proof. (i) Let $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \equiv 1$ on the support of v. According to Theorem 7.12, we can pick a sequence $\{\varphi_j\}$ in $C^\infty(\Omega) \cap W^{1,A}(\Omega)$ such that $\varphi_j \to v$ in $W^{1,A}(\Omega)$. Then $\varphi\varphi_j \in C_c^\infty(\Omega)$ and $\varphi\varphi_j \to \varphi v = v$ in $W^{1,A}(\Omega)$ and $\varphi \in W_0^{1,A}(\Omega)$.

(ii) Let $\{\varphi_j\}$ be a sequence in $C_c^\infty(\Omega)$ such that $\varphi_j \to u$ in $W^{1,A}(\Omega)$. Then according to Corollary 7.16, we have $\min\{\varphi_j, v\} \in W^{1,A}(\Omega)$. Since $\min\{\varphi_j, v\}$ has compact support in Ω, it follows from part (i) that $\min\{\varphi_j, v\} \in W_0^{1,A}(\Omega)$ for all j. Moreover, by the above lemma there is a subsequence, still denoted by $\{\min\{\varphi_j, v\}\}$ such that $\min\{\varphi_j, v\} \to \min\{u, v\} = v$ in $W^{1,A}(\Omega)$. Therefore $v = \min\{u, v\} \in W_0^{1,A}(\Omega)$.

□

7.3 On Regularity of Solutions to $\Delta_\phi u = f(u)$

In the paper [25], Gary M. Lieberman discusses the regularity of solutions to the quasilinear PDE

$$\operatorname{div}(\mathbf{A}(x, u, \nabla u)) + B(x, u, \nabla u) = 0 \tag{7.3.1}$$

in an open subset of $\Omega \subseteq \mathbb{R}^N$, under the following structure conditions.

(A-1): $a^{ij}(x, z, p)\xi_i\xi_j \geq \phi(|p|)|\xi|^2$

(A-2): $|a^{ij}(x, z, p)| \leq K\phi(|p|)$

(A-3): $|\mathbf{A}(x, z, p) - \mathbf{A}(y, w, p)| \leq K_1(1 + \Psi(|p|))[|x - y|^\alpha + |z - w|^\alpha]$

(A-4): $|B(x, z, p)) \leq K_1(1 + \phi(|p|))$

Here

$$a^{ij}(x, z, p) := \frac{\partial A^i(x, z, p)}{\partial p_j}(x, z, p).$$

The following regularity result, in the context of Orlicz-Sobolev spaces, is due to Gary M. Lieberman.

Theorem 7.19. Let $\Omega \subseteq \mathbb{R}^N$ be open, and assume that ϕ satisfies Conditions (ϕ-1) and (ϕ-3). Moreover, assume that conditions (A-1), (A-2), (A-3) and (A-4) all hold for some positive constants α, K, K_1 for all $(x, z, p), (y, w, p) \in \Omega \times [-M, M] \times \mathbb{R}^N$ for some positive constant $M > 0$, then any solution $u \in W^{1,\Phi}(\Omega)$ of (7.3.1) with $|u| \leq M$ in Ω belongs to $C^{1,\beta}(\Omega)$.

As a corollary we have the following.

Corollary 7.20. Assume that ϕ satisfies Conditions (ϕ-1) and (ϕ-3). If $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a solution of (5.3.1), then given $\Omega \subset\subset \mathbb{R}^N$, we have $u \in C^{1,\beta}(\Omega)$ for some $0 < \beta < 1$.

Proof. Since $u \in C(\overline{\Omega})$, let $M = \sup_\Omega |u|$. To prove the corollary we need to show that $\mathrm{A}(x, z, p) := \phi(|p|)p$ satisfies Conditions (A-1), (A-2), (A-3) and (A-4) for some positive constants α, K, K_1 for all $(x, z, p), (y, w, p) \in \Omega \times [-M, M] \times \mathbb{R}^N$.

Now, then

$$
\begin{aligned}
a^{ij}(x,z,p) &= \frac{\partial A^i(x,z,p)}{\partial p_j} = \frac{\partial(\phi(|p|)p_i)}{\partial p_j} \\
&= \phi'(|p|)\frac{p_i p_j}{|p|} + \phi(|p|)\delta_{ij} \qquad (7.3.2)
\end{aligned}
$$

Thus, for $p \neq 0$ we have

$$
\begin{aligned}
a^{ij}(x,z,p)\xi_i\xi_j &= \phi'(|p|)\frac{p_i\xi_i p_j\xi_j}{|p|} + \phi(|p|)\delta_{ij}\xi_i\xi_j \\
&= \phi'(|p|)\frac{(p\cdot\xi)^2}{|p|} + \phi(|p|)|\xi|^2 \\
&\geq \phi(|p|)|\xi|^2\left[\left(\frac{\phi'(|p|)|p|}{\phi(|p|)} + 1\right)\cdot\frac{(p\cdot\xi)^2}{(|p||\xi|)^2} + \left(1 - \frac{(p\cdot\xi)^2}{(|p||\xi|)^2}\right)\right] \\
&\geq \phi(|p|)|\xi|^2\left[(\sigma-1)\frac{(p\cdot\xi)^2}{(|p||\xi|)^2} + 1\right] \\
&\geq \min\{\sigma, 1\}\phi(|p|)|\xi|^2.
\end{aligned}
$$

Rewriting (7.3.2), we have

$$
\begin{aligned}
|a^{ij}(x,z,p)| &= \left|\frac{\phi'(|p|)|p|}{\phi(|p|)}\frac{p_i p_j}{|p|^2} + \delta_{ij}\right|\phi(|p|) \\
&= \left(\frac{|\phi'(|p|)|p|}{2\phi(|p|)} + 1\right)\phi(|p|) \\
&\leq \frac{1}{2}(3+\rho)\phi(|p|).
\end{aligned}
$$

Since $\mathrm{A}(x, z, p) = \phi(|p|)p$ does not depend on x or z, we note that (A-3) is obviously true. Obviously, (A-4) is also true if we define $B(x, z, p) := f(x)\chi_E(z)$ where $E = [-M, M]$. To establish the desired regularity result that is appropriate for our work, it would be easier to rely on the following result due to Paolo Marcellini [28, Corollary 2.2] for solutions of

$$
\mathrm{div}\mathrm{A}(x, \nabla u) = g(x). \qquad (7.3.3)
$$

□

The following conditions of [28] (with $p = q = 2$) are needed (see also [27]). Suppose $\Omega \subseteq \mathbb{R}^N$ is bounded. We assume that there are positive constants m, M and K such that the following hold for all $x \in \Omega$ and all $p, \xi \in \mathbb{R}^N$.

(a-1): $a^{ij}(x,p)\xi_i\xi_j \geq m|\xi|^2$

(a-2): $|a^{ij}(x,p)| \leq M$ for all $1 \leq i,j \leq N$.

(a-3): $|a^{ij}(x,p) - a^{ji}(x,p)| \leq M$ for all $1 \leq i,j \leq N$.

(a-4): $|a^i_{x_k}(x,p)| \leq K(1+|p|^2)^{\frac{1}{2}}$ for all $1 \leq i,j \leq N$.

We restate [28, Corollary 2.2] in the following theorem.

Theorem 7.21. Suppose Conditions (a-1) to (a-4) all hold. Let us also suppose that $a^i \in C^{k,\alpha}_{loc}(\Omega \times \mathbb{R}^N)$ and $g \in C^{k-1,\alpha}_{loc}(\Omega) \cap L^\infty(\Omega)$ for all $i = 1, \cdots, N$, some $k \geq 1$ and some $0 < \alpha < 1$. If $u \in W^{1,2}_{loc}(\Omega)$ is a solution to (7.3.3), then $u \in C^{k+1,\alpha}_{loc}(\Omega)$.

Suppose now $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a solution of (5.3.1) and let $\Omega := \{x \in \mathbb{R}^N : |\nabla u(x)| > 0\}$, and suppose $\mathcal{O} \subset\subset \Omega$.
Fix $0 < a < \alpha \leq \beta < b$ where

$$\alpha := \inf_{\mathcal{O}} |\nabla u|, \quad \text{and} \quad \beta := \sup_{\mathcal{O}} |\nabla u|.$$

Let $\eta \in C^{1,1}(\mathbb{R})$ such that

(i) $\eta(t) = t$ for $\alpha \leq t \leq \beta$

(ii) $\eta \equiv$ constant on $(-\infty, 0)$ and on (b, ∞)

(iii) $0 \leq \eta'(t) \leq \frac{\eta(t)}{t}$ for all $t \geq 0$.

Such function can be easily constructed. First, we define η on $[a,b]$ as follows

$$\eta(t) := \begin{cases} t + \dfrac{1}{2(\alpha - a)}(t-\alpha)^2 & \text{if } a \leq t \leq \alpha \\ t & \text{if } \alpha \leq t \leq \beta \\ t - \dfrac{1}{2(b-\beta)}(t-\beta)^2 & \text{if } \beta \leq t \leq b. \end{cases}$$

We extend η to $\mathbb{R}$ by defining η on $(-\infty, a]$ and $[b, \infty)$ to be the constants $\eta(a)$ and $\eta(b)$, respectively. One can easily check that $\eta \in C^{1,1}(\mathbb{R})$ and that η has the properties (i)-(iii).

Now, let $a(x,p) := \phi(\eta(|p|))p$. Then

$$\begin{aligned} a^{ij}(x,p) &= \frac{\partial(\phi(\eta(|p|))p_i)}{\partial p_j} \\ &= \phi'(\eta(|p|))\eta'(|p|)\frac{p_i p_j}{|p|} + \phi(\eta(|p|))\delta_{ij}. \end{aligned}$$

Therefore, for any $p, \xi \in \mathbb{R}^N$, we have

$$\begin{aligned} a^{ij}(x,p)\xi_i\xi_j &= \phi'(\eta(|p|))\eta(|p|)\frac{\eta'(|p|)|p|}{\eta(|p|)}\frac{(p\cdot\xi)^2}{|p|^2} + \phi(\eta(|p|))|\xi|^2 \\ &= \phi(\eta(|p|))\left[\left(\frac{\phi'(\eta(|p|))\eta(|p|)}{\phi(\eta(|p|))}\right)\frac{\eta'(|p|)|p|}{\eta(|p|)}\frac{(p\cdot\xi)^2}{|p|} + |\xi|^2\right] \\ &\geq \phi(\eta(|p|))\left[(\sigma-1)\frac{\eta'(|p|)|p|}{\eta(|p|)}\frac{(p\cdot\xi)^2}{|p|^2} + |\xi|^2\right] \\ &\geq \min\{1,\sigma\}\phi(\eta(|p|))|\xi|^2. \end{aligned}$$

In the case, $0 < \sigma \leq 1$, we have used the inequality

$$\frac{\eta'(|p|)|p|}{\eta(|p|)}\frac{(p\cdot\xi)^2}{|p|^2} \leq |\xi|^2.$$

Therefore, Conditions (a-1) holds. It is easy to show that Conditions (a-2)-(a-4) hold as well. Now suppose $u \in W^{1,\Phi}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a solution of (5.3.1). Then, as noted in Corollary 7.19, we see that $u \in C^{1,\gamma}(\Omega)$ for some $0 < \gamma < 1$. Since $|\nabla u|$ is bounded in $\mathcal{O}$, it is obvious that $u \in W^{1,2}_{loc}(\Omega)$. Recalling the assumption made just before the statement of Condition (ϕ-5), we have ϕ is C^2 in $(0,\infty)$ and therefore, we conclude that $a^i \in C^{1,1}(\mathcal{O} \times \mathbb{R}^N)$ for all $i = 1, 2, \cdots, N$. Moreover, since $f \in C^{1,\gamma}_{loc}(0,\infty)$, we observe that $g(x) := f(u(x))$, $x \in \mathcal{O}$ belongs to $C^{1,\gamma}(\mathcal{O})$. We invoke Theorem 7.21 to conclude that $u \in C^3_{loc}(\mathcal{O})$.

7.4 A Remark on the Definition of Sub(Super)-solution

Given open sets $\mathcal{O} \subseteq \Omega$, suppose v is a measurable function on $\mathcal{O}$. We define the zero extension $\widetilde{v} : \Omega \to \Omega$ by

(7.4.1) $$\widetilde{v}(x) := \begin{cases} v(x) & \text{if} \quad x \in \mathcal{O} \\ 0 & \text{if} \quad x \in \Omega \setminus \mathcal{O}. \end{cases}$$

Now, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. $u \in W^{1,\Phi}(\Omega)$ is a sub-solution (resp.,super-solution) of (1.2.24) if and only if $g(x,u) \in L^{\widetilde{\Phi}}(\Omega)$ and

(7.4.2) $$\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla\varphi \leq (\text{resp.}, \geq) - \int_\Omega g(x,u(x))\varphi, \quad \forall\, \varphi \in W^{1,\Phi}_0(\Omega).$$

To see this, assume that u is a sub-solution. Then (1.2.25) holds for $\mathcal{O} = \Omega$ and hence (7.4.2) holds. So, suppose (7.4.2) holds, and let $\mathcal{O}$ be any subset of Ω and let $\varphi \in W_0^{1,\Phi}(\mathcal{O})$. Let us first show that $\widetilde{\varphi} \in W_0^{1,\Phi}(\Omega)$. In fact, it is enough to show that $\widetilde{\varphi}$ is weakly differentiable on Ω, and $D^\alpha \widetilde{\varphi} = \widetilde{D\varphi}$. For this, let $\eta \in C_c^\infty(\Omega)$ and let α be any multi-index with $|\alpha| \leq 1$. Suppose $\{\varphi_j\}$ is a sequence in $C_c^\infty(\mathcal{O})$ such that $\varphi_j \to \varphi$ in $W^{1,\Phi}(\mathcal{O})$. Then

$$\begin{aligned}
\int_\Omega \widetilde{\varphi} D^\alpha \eta &= \int_{\mathcal{O}} \varphi D^\alpha \eta = \lim_{j\to\infty} \int_{\mathcal{O}} \varphi_j D^\alpha \eta \quad \text{by the the Generalized Hölder Inequality} \\
&= (-1)^{|\alpha|} \lim_{j\to\infty} \int_{\mathcal{O}} (D^\alpha \varphi_j)\eta \\
&= (-1)^{|\alpha|} \int_{\mathcal{O}} (D^\alpha \varphi)\eta \quad \text{by the Generalized Hölder Inequality} \\
&= (-1)^{|\alpha|} \int_\Omega (\widetilde{D^\alpha \varphi})\eta.
\end{aligned}$$

Therefore, $\widetilde{\varphi}$ is weakly differentiable in Ω and $D^\alpha \widetilde{\varphi} = \widetilde{D^\alpha \varphi}$ a.e. in Ω. Clearly, for each j, the extension $\widetilde{\varphi}_j$ belongs to $C_c^\infty(\Omega)$. Let us now show that $\widetilde{\varphi}_j \to \widetilde{\varphi}$ in $W^{1,\Phi}(\Omega)$. For this, we observe that

$$\lim_{j\to\infty} \int_\Omega \Phi(|\widetilde{\varphi}_j - \widetilde{\varphi}|)dx = \lim_{j\to\infty} \int_{\mathcal{O}} \Phi(|\varphi_j - \varphi|)dx \to 0.$$

Similarly $D^\alpha \widetilde{\varphi}_j = \widetilde{D^\alpha \varphi_j} \to \widetilde{D\varphi} = D^\alpha \widetilde{\varphi}$ in $L^\Phi(\Omega)$. Therefore, $\widetilde{\varphi}$ belongs to $W_0^{1,\Phi}(\Omega)$ as claimed.

Now, suppose $u \in W^{1,\Phi}(\Omega)$ such that (7.4.2) holds. We show that u is a sub-solution of (1.2.24). So let $\mathcal{O}$ be a subset of Ω, and let $\varphi \in W_0^{1,\Phi}(\Omega)$. Then, as shown above $\widetilde{\varphi} \in W_0^{1,\Phi}(\Omega)$ and hence by (7.4.2) we have

$$\begin{aligned}
\int_{\mathcal{O}} \phi(|\nabla u|)\nabla u \cdot \nabla\varphi &= \int_\Omega \phi(|\nabla u|)\nabla u \cdot \widetilde{\nabla\varphi} = \int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla\widetilde{\varphi} \\
&\leq -\int_\Omega g(x,u)\widetilde{\varphi} = -\int_{\mathcal{O}} g(x,u)\varphi.
\end{aligned}$$

Therefore, u is a sub-solution. A similar argument shows that analogous claim holds when u is a super-solution.

7.5 Remark on Condition (ϕ-3)

It is common in the literature to see condition (ϕ-1), (ϕ-2) and (ϕ-3) used together with the following condition.

$(\phi\text{-}3)^*$: There are constants $0 < \sigma^* \le \rho^*$ such that

$$\sigma^* \le \frac{\Phi'(t)t}{\Phi(t)} \le \rho^*, \quad \forall\, t > 0.$$

We refer to the papers [16, 17, 36, 44] for such conditions. However, it is easy to see that Condition $(\phi\text{-}3)^*$ is a consequence of $(\phi\text{-}3)$ as we show below. Let us assume that Condition $(\phi\text{-}3)$ holds. Suppose $t > 0$. Then subtracting $\Phi(t)$ from all sides of (1.1.9) and then dividing by $s-1$ for $s > 1$, we find that

$$\frac{\widetilde{\lambda}(s)-1}{s-1}\Phi(t) \le \frac{\Phi(st)-\Phi(t)}{s-1} \le \frac{\widetilde{\Lambda}(s)-1}{s-1}\Phi(t). \tag{7.5.1}$$

Since Φ is differentiable we note that

$$\begin{aligned}\lim_{s\to 1^+}\frac{\Phi(st)-\Phi(t)}{s-1} &= \lim_{r\to t^+}\frac{\Phi(r)-\Phi(t)}{r/t-1}\\ &= t\lim_{r\to t^+}\frac{\Phi(r)-\Phi(t)}{r-t}\\ &= t\Phi'(t).\end{aligned}$$

Therefore, taking the limits as $s \to 1^+$ in (7.5.1) we have

$$\widetilde{\lambda}'(1)\Phi(t) \le \Phi'(t)t \le \widetilde{\Lambda}'(1)\Phi(t),$$

thus verifying Condition $(\phi\text{-}3)^*$ holds with $\sigma^* = \widetilde{\lambda}'(1)$ and $\rho^* = \widetilde{\Lambda}'(1)$. In fact, when λ and Λ are given as in (1.1.5), then $\sigma^* = \sigma + 1$ and $\rho^* = \rho + 1$.

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