THE STUDY OF COEXISTANCE OF SUPERCONDUCTIVITY AND FERROMAGNETISM IN URHGE AND UCOGE

By

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Abstract

Superconductivity and ferromagnetism are usually antagonstic because of the computative nature between the superconducting screening (Meissner effect) and the internal fields generated by magnetic ordering.However,the discovery of magnetic superconductor has allowed for better understanding of how magnetic order and superconducting can coexist.So, in this work we have demonstrated the coexistance of ferromagnetism and superconductivity in the intermetallic compounds URhGe and UCoGe.

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Introduction

Superconductivity is a phenomena of zero resistance. It has many unusual property not observed in normal metals. Since the discovery of superconductivity, ferromagnetism and superconductivity had been thought to be mutually competitive phenomena, the large internal field easily destroys the cooper-pairs for conventional superconductivity. An important step forward in superconductor physics is the observation of the coexistence of superconductivity and ferromagnetism in the spin-triplet pairing superconductors such as UGe_2, $ZrZn_2$ and URhGe.So, after the discovery of superconductivity in UGe_2 under pressure, the coexistence of superconductivity and ferromagnetism becomes one of the major topics in condensed matter Physics. In this work we study theorticaly the coexistence of ferromagnetism and superconductivity in URhGe and UCoGe.

Out line of the thesis: The first Chapter (1) presents a general review on superconductivity, ferromagnetism, and their interaction. It also presents a review on the intermetallics compounds URhGe and UCoGe. In Chapter (2), a comprehensive presentation of the mathematical formalism used in the work will be given . This addresses the topics such as retarded double time Green's function and equations of motion for Green's function. In Chapter (3) a comprehensive presentation of the theoretical formulation and calculations of the problem is presented. Chapter 4 is a short presentation on the results of the calculation and graphs drawn. The 5^{th} Chapter is give conculusion of the work.

Chapter 1

Review Littrature

1.1 Superconductivity

1.1.1 History of superconductivity

Superconductivity was discovered by Kamerlingh Onnes in Holland in 1911 as a result of his investigations leading to the liquefaction of helium gas. The transition of a normal metal into the superconducting state is revealed by the total disappearance of the electrical resistivity at a critical temperature T_c [1,4,5]. Another fundamental property of the superconducting state was discovered in 1933 when Walther Meissner and his Ph. D.student Robert Ochsenfeld demonstrated that superconductors expel any residual magnetic field. Similarly superconductivity can be destroyed by applying magnetic field that exceeds the critical value B_c.[3] The microscopic theory of superconductivity was created by John Bardeen, Leon Cooper and Robert Schrieffer in 1957 [2,6]. According to this the so called BCS theory, the electrons form pairs, known as cooper-pairs, due to interaction with the crystal lattice at low temperature .Electrons in this cooper-pairs have opposite values of momentum, meaning that the pairs themselves generally have zero orbital angular momenta add up to zero. The formation of cooper pairs leads to a superconducting energy gap, which means that single electron can not occupy states near the Fermi surface. Such energy gaps which are essentially equal the energy needed to

break up the cooper-pairs. Advancement came in 1962 when Brain Josephson, a graduate student at Cambridge University predicted that electrical current would flow between two superconducting materials, even when they are separated by an insulator. In 1987, a dream of many scientists was realized with the discovery of superconducting compounds containing copper- oxygen layers that are superconducting above a critical temperature of 30K. The revolutionary discovery of superconducting in this class of compounds
(the cuprates) Georg Bednorz and Alex Mueller the Nobel prize. In February 2008, a group under Prof. Hideo Hosono Tokyo Institute of Technology discovered that an Fe-As compound, LaFeAsO, displays a superconducting transition at an absolute temperature of 26K.Inspit of the fact that the compound contains iron, which had been considered disadvantageous for superconducting because it is magnetic, this material attracted worldwide attention due its relatively high superconducting transition temperature (T_c) and suprred boom in research on Fe-based superconductor. Immediately after this discovery, it was found that SmFeAsO, in which La is replaced with Sm, displays an even higher T_c of 55K. As a result, high expectation have been placed on the Fe-based materials as new high temperature superconductors succeeding the copper-oxide -based materials [4,7].

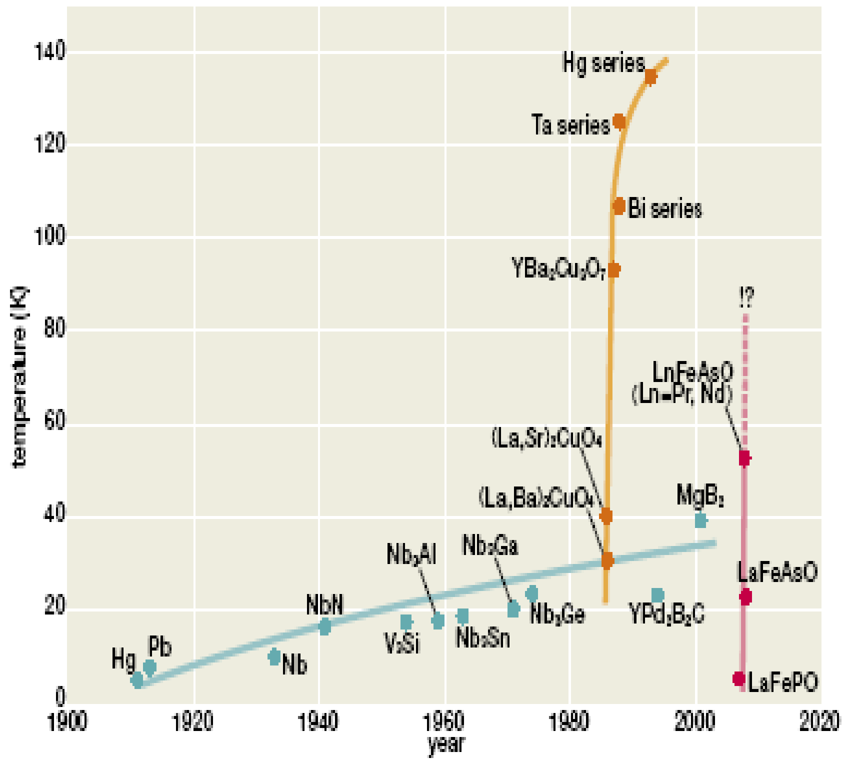

Figure 1.1: As researchers pursue room-temperature superconductivity, superconductors with high critical temperatures are discovered every year [4].

1.1.2 Meissner-Ochsenfeld effect

In addition to zero resistivity (i.e. infinite conductivity), the superconductor exhibits another striking property: it expels the magnetic field from its interior. This is not a consequence of infinite conductivity, it is another intrinsic characteristic property of the susuperconducting state which shall now be discussed in some detail[8]. As already illustrated in Fig. 1.2, in the normal state at temperatures above T_c the field lines pass through the metallic specimen. Upon cooling below T_c, a phase transition into the superconducting state takes place and the magnetic flux gets expelled out of the interior of the metallic sample. The Meissner-Ochsenfeld effect [3] cannot be deduced from the infinite conconductivity of a superconductor. The exclusion of the magnetic field from the interior of a superconducting specimen is a direct evidence that the superconducting state is not simply one of zero resistance. If it were so, then a superconductor cooled in the magnetic field through T_c would have trapped the field in its interior. When the external field is re- moved, the induced persistent eddy currents would nevertheless preserve the trapped field in the interior of the specimen.

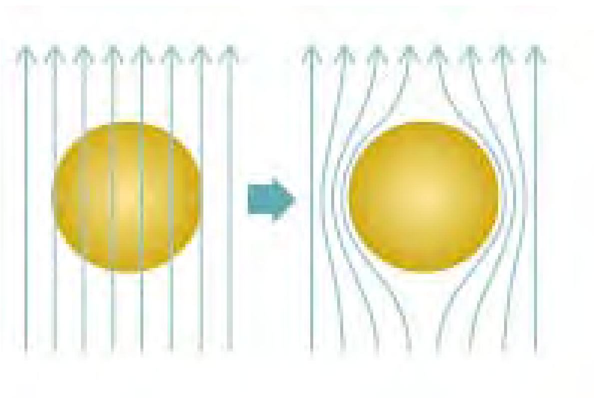

Figure 1.2: The Meissner effect [4].

1.1.3 Type-I and Type-II superconducting materials

Superconducting materials that completely expel magnetic flux until they become completely normal are called type-I superconductors. With the exception of V and Nb, all superconducting elements and most of their alloys in the dilute limit, are type-I superconductors. The strength of the applied magnetic field required to completely destroy the state of perfect dia magnetism in the interior of the superconducting specimen is called the thermodynamic critical field B_c. For a type-II superconductor there are two critical fields. The lower B_{c1} and the upper B_{c2}. The flux is completely expelled only up to the field B_{c1}.So, in applied fields smaller than B_{c1}, the type-Il superconductor behaves just like a type-I superconductor below B_c.Above B_{c1} the flux partially penetrates into the material until the upper critical field B_{c2}, is reached. Above B_{c2} the material returns to the normal state. Between B_{c1} and $B_{c2}$the superconductor is said to be in the mixed state. The Meissner effect is only partial. For all applied fields $B_{c1} < B < B_{c2}$, magnetic flux partially penetrates the superconducting specimen in the form of tiny microscopic filaments called vortices [8].

1.1.4 Wide spread applications for superconductors

Applications for superconductors have emerged in diverse areas of science and Technology including energy, transportation, medical care, and the environment,owing to their ability to conduct an electric current at a density of ten thousands to one million times that of copper with no loss using this property, superconducting can handle high currents in power applications [4]. A typical application is a system employing a powerful magnetic field generated by a superconducting material having a superconducting wire as the winding for example, magnetic resonance imaging (MRI) equipment used in hospital, a superconducting magnetic levitation (maglev) trains, a nuclear magnetic resonance (NMR) spectrometer used for analysis, such as the analysis of protein structure. Superconductors are used findings in important applications in electronics, including filters, antennas, resonators, magnetometers (SQUID), voltage storages, arithmetic circuits, and memory circuits. SQUID in particular is capable of detecting externally weak magnetic fields and, thus, is used for studying magnetic properties of materials and for observing weak magnetic fields emitted by hair or brain.

1.2 Magnetic ordering and ferromagnetism

1.2.1 Magnetic ordering

In magnetic material there exists a spontaneous long-range ordering of the microscopic moment. This is due to so called exchange interactions between the moment carriers [15]. There are two major classes of magnetic materials exhibiting spontaneous order ferromagnet and anti ferromagnetic. In ferromagnetic materials, the exchange interactions tend to align the moments in one direction, giving the material a non-zero magnetization In contrast to ferromagnetism, the exchange interactions in anti ferromagnetism material tend to periodically order the moments in such away that there is no over all magnetization of the system [11,12,13].

1.2.2 Ferromagnetism

Ferromagnetism is a phenomena by which a material can exhibit a spontaneous magnetization and is one of the strongest form of magnetism. The distinct characters of ferromagnetic material are those spontaneous magnetization provided by the exchange interactions and existence of magnetic ordering temperature. Ferromagnetism differs from the weaker diamagnetism and para magnetism is that the electrons of neighboring atoms interact with one another in a process called exchange coupling [16]. A Ferromagnet may be divided into microscopic volumes called domains,each possesing one oriented magnetic moment. The application of an external magnetic field results in an expansion of the domain with moments align with the field at the expense of those with anti-aligned moment.

1.3 Antagonism of superconductivity and magnetism

For conventional superconductors the superconducting state corresponds to a condensate of cooper-pairs made up from opposite sign electrons. An applied magnetic field tends to destroy conductivity. Two causes of this distraction can be distinguished: The orbital effect corresponds to the action of the magnetic field on the electron charge. As the field is increased the electrons charge discribs circular motion with decreasing radius. Roughly, when this radius is smaller than the coherence length of the order parameter the superconductivity is destroyed. The critical field can be written as [14]

$$\hat{H_{c2}}^{orbital} = \frac{\phi_o}{2\pi\xi^2(T)} \tag{1.3.1}$$

where $\phi_o = \frac{h}{2e}$ is the magnetic flux quantum and ξ is the coherence length. The coherence length is proportional to the inverse of the electron effective mass $(m^*)^{-1}$, and thus the orbital critical field is proportional to the square of the effective mass. $H_{c2}^{\hat{orbital}} \propto (m^*)^2$. The paramagnetic effect corresponds to the action of magnetic field on the electron spin. The field tends to align the spins of the electrons that make up the cooper-pairs. At some point, as the field is increased, it is essentially favorable to lose the condensation energy by breaking the cooper-pairs to align the spins with the field. At zero temperature, the critical field can be expressed as

$$\hat{H_{c2}}^{paramagnetic}(T=0) = \sqrt{2}\frac{\Delta(T=0)}{g\mu_B} \tag{1.3.2}$$

where $\Delta(T=0)$ is the superconducting gap at zero temperature,μ_B is the Bohr magneton and g is the Landau-g factor for the electrons .With in the BCS theory (and for g =2) the paramagnetic field is expressed in Tesla as $\hat{H}_{c2}^{para}(T=0) = 1.85T_c$, where T_c is the critical temperature for superconductivity expressed in kelvin.

1.4 Coexistence of superconductivity and magnetism

The problem of coexistence of superconductivity and ferromagnetism, a priori, two antagonistic properties, was raised in 1957 by Ginzburg [14]. From the critical field value and the spontaneous magnetization measured for ferromagnets at that time it was an expected to observe this coexistence. In 1958 ,Mathian's and Collaborators demonstrated that even a small concentration of magnetic impurities was enough to destroy Lanthanum's superconductivity [2,30] . Then,in $E_rRh_4B_4$ and $HoMo_6S_6$ ferromagnetism and superconductivity were observed simultaneously,but the two properties turned out to be antagonistic [3,18].Both compounds become superconductors ferromagnetic order. Around 1980, it was recognized that under special conditions superconductivity may coexist with anti ferromagnetic order, where neighboring electron spins arrange in an anti parellel conflagration. The discovery of the first superconducting ferromagnet ($T_{sc} < T_c$) UGe_2 in the year 2000 came as big surprise .In this material, superconductivity is realized well bellow the Curie temperature, with out expelling the ferromagnetic order [a].Since then, three other superconducting ferromagnet have been discovered: UIr_2 ,URhGe, and UCoGe. This material have in common that ferromagnetic order is due to the uranium 5f magnetic moments and has a strong itinerant character [17].

1.5 Mechanism of the coexistance of superconductivity and ferromagnetism

In contrast to the standard s-wave pairing in usual (conventional) superconductors, where the electron pairs are formed by an attractive electron-electron interaction due to a virtual phonon exchange, the widely accepted mechanism of the cooper-pairing in super-fluid ^{3}He is based on an attractive interaction between the fermions (^{3}He) atoms due to a virtual exchange of spin fluctuations [18,19]. In ternary compounds the ferro magnetism comes from the localized 4f electrons where as the s-wave cooper-pairs are formed by conduction electrons. However, in U-based compounds UGe_2, UIr, URhGe and UCoGe, the 5f electrons of U atoms form both superconductivity and ferromagnetic order and have a strong itinerant character more over,superconducting occures close to a magnetic insensibility. But, in $ZrZn_2$ the same twofold role is played by the 4d electrons of Zr. The coexistence of superconductivity and ferro magnetism in U-based compounds can be understood in terms of spin fluctuation model: in the vicinity of a ferromagnetic quantum critical point, critical magnetic fluctuations can mediate superconductivity by pairing the electrons in spin-triplet cooper-pairs that is the equal spin pairing (ESP) states $|\uparrow\uparrow>$ $(L = 1, S_z = 1)$ and $|\downarrow\downarrow>$ $(L = 1, S_z = -1)$, and the state $\frac{(|\uparrow\downarrow>+|\downarrow\uparrow>)}{\sqrt{2}}$ with orbital moment L =1 and projection of spin momentum $S_z = 0$ [21,27].

1.6 Superconductivity and ferromagnetism in URhGe and UCoGe

The inter metallic compounds URhGe [22] and UCoGe[27] belong to the family of ferromagnetic superconductors, which attracts much attention. The crystal structures are a low symmetry, orthorhombic, which results in a strong uniaxial anisotropy of the electronic and magnetic properties.

1.6.1 URhGe

This material belongs to the large family of uranium 1 :1 :1 inter-metallics. The crystal structures is Orthorhombic (a=6.87A^o ,b=4.33A^o, c=7.51A^o)[23,24,25]. Ferromagnetic order is observed bellow $T_c = 9.5K$ and the uniaxial spontaneous moment of $0.42\mu_B$ per U atom is directed along the c-axis [29] . Spin -triplet superconductivity is observed at atmospheric pressure deep in the ferromagnetic phase bellow T$_{sc}$ = 0.25K .Under hydrostatic pressure, ferromagnetic order is not suppressed as shown in Fig. 1.4 but T_c increases at a rate of 0.65 kGPa^{-1} up to the highest pressure measured (13 GPa)[17].The phase diagram is distinctly different when compared to the P-T diagrams of UGe$_2$,UIr, and UCoGe, which obey the more commonly obseerved Doniach-like behavior for magnetic order in correlated metals. The magnetic transition temperature is reduced when the product JB(ε_F) increases under the influence of mechanical pressure (here J is the exchange interaction and $N(\varepsilon_F)$ is the density of state at the Fermi level) .While T$_c$ steadily increases under hydrostatic pressure ,superconductivity is depressed and vanishes near 3.0 Gpa .Solid evidence for triplet superconductivity has extracted from measurements of the upper critical field B$_{c2}$ .At 0K , B$_{c2}$ exceeds the paramagnetic Pauli limit and the temperature variation B$_{c2}$(T) is well described by the model function for a superconducting gap with a line node (polar gap) and the maximum gap parallel to the a - axis. Surprisingly, a highly interesting phenomena occurs for strong magnetic fields directed along the orthorhombic b-axis. Superconductivity is first suppressed at $B_{c2} \sim 2T$, but reappears when the applied field exceeds 12T. The field induced superconductivity phase is connected to a spin re-orientation process. When the component of the field along the b-axis reaches 12T, the ordered moment rotates from the c- axis to wards the b- axis [31].

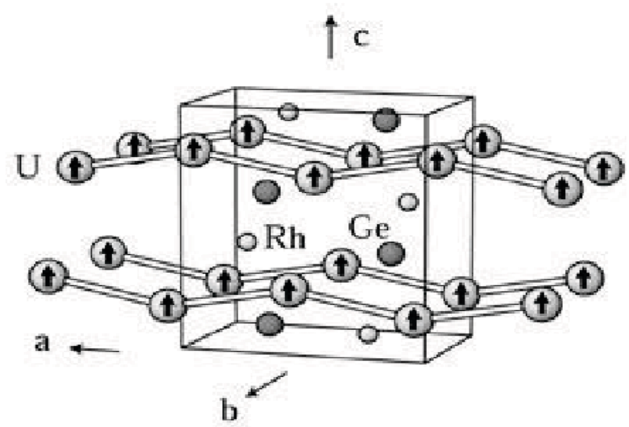

Figure 1.3: Crystal stracture of URhGe [32].

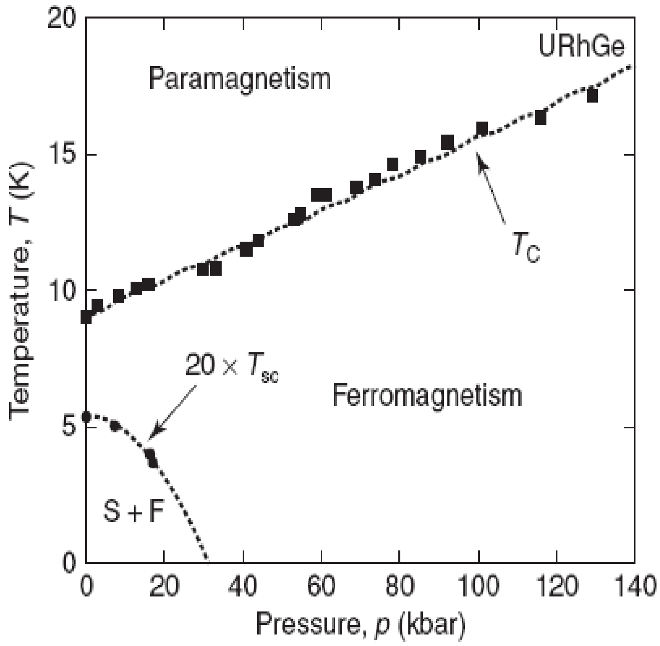

Figure 1.4: Temperature verssus presure phase diagram of URhGe [29].

1.6.2 UCoGe

This uranium 1:1:1 compound forms in the same orthorhombic crystal structure with (a=6.845A^o, b=4.206A^o, c=7.222A^o) as URhGe. Itinerant ferromagnetic order is weak, with a Curie temperature of 3K and a small ordered moment of 0.07μ_B per U-atom. The ferromagnetic structure is uniaxial with m$_o$ $||c$. Superconductivity is observed at atmospheric pressure in the ferromagnetic phase with T$_{sc}$ = 0.6K. The ratio $\frac{T_{sc}}{T_C} \approx 0.2$ is the large among the superconducting ferromagnet. For UIr, $\frac{T_{sc}}{T_C}$ is 0.1 at 2.7GPa where T$_{sc}$ is maximum, while for UGe_2 and URhGe the ratio is almost one order of magnitude smaller ($\sim 0.02 - 0.03$). Measurements of the upper critical field B$_{c2}$ support triplet superconductivity and point to an axial superconducting gap function with node along the c-axis, that is, the direction of the ordered moment m_o [[28,27].

The B_{c2} curves show an unusual upward curvature ($B\|b$) or tink ($B\|a$) ,which is possibly due to a computation between the equal-spin pairing states $|\uparrow\uparrow>$ and $|\downarrow\downarrow>$, expected for a two band ferromagnetic superconductor . Under hydrostatic pressure, as shown in Fig. 1.5. However, for $P > 1.0GPa$, ferromagnetic order is no longer observed. This P-T phase diagram differs from the diagrams measured for the other superconducting ferromagnet, notable because superconducting survives in the paramagnetic regime up to the highest pressure (2.2GPa). In the ferromagnetic phase time-reversal symmetry is broken, and spin-orbit coupling restricts the cooper states to the EQS states $|\uparrow\uparrow>$ $or|\downarrow\downarrow>$. The high pressure ($P > 1.0GPa$) superconducting phase dose not break time reversal symmetry and is possibly a planner spin - triplet or a conventional spin-triplet.

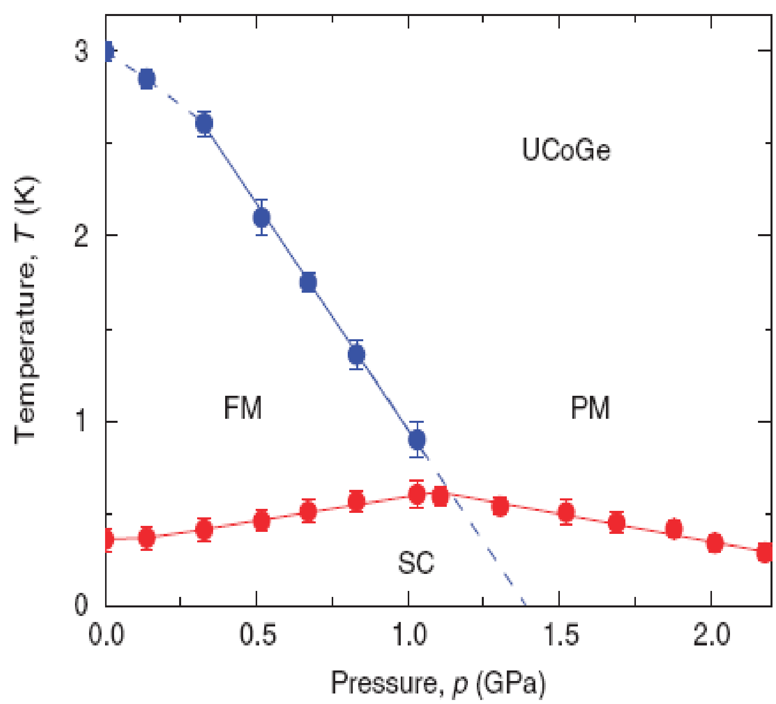

Figure 1.5: Temperature verssus presure phase diagram of UCoGe [29].

1.7 Bardeen - Cooper -Schrieffer theory

In 1957, Bardeen, Cooper,and Schieffer (BCS) proposed a general theory of superconductivity that qualitatively a satisfactory explanation of the phenomenon [8,9]. There are various levels of approximation in which the BCS theory has been applied. The mechanical underpinning of the BCS-theory is so complex that it will not be of much benefit to summarize general formulation, so this section will enphesize predictions that are often compared with experiments. This predictions arise mainly from th homogeneous, isotopic,phonon-mediated,square well, s- wave coupling simplification of the BCS-theory, and many superconductors, to a greater or lesser extent, have been found to satisfy these predictions. Some of them are follows. The isotopic effect involves the claim that for a particular element the transition ends on the mass M of the isotope as follows;
$M^{\alpha}T_c = constant$
The weak coupling BCS limit gives the value $\alpha = \frac{1}{2}$, which have been observed in some superconducting elements, but not in all of them. A superconductor has an energy gap $E_g = 2\Delta(k)$. Consider the square-wall electron-electron potential V_o and an energy gap $\Delta(k)$ that is equal to in the neighborhood of the Fermi-surface, $\Delta(k) = \Delta_o, -\hbar\omega_b \leq \epsilon(k) \leq \hbar\omega_b$ and is 0 elsewhere.

The Debby frequency ω_D determines the range of ϵ because it is assumed that cooper-pair formation is mediated by phonon s. The energy gap Δ_o in this approximation is given by

$$\Delta_o = \frac{\hbar\omega_D}{sinh(\frac{1}{V_o D_n(o)})}. \qquad (1.7.1)$$

In the weak coupling (small V_o) limit $V_o D_n(o) << 1, k_B T_c << \hbar\omega_D$, we obtain the dimensionless ratios;

$$\frac{E_g}{k_B T_c} = \frac{2\Delta_o}{k_B T_c} = 3.53. \qquad (1.7.2)$$

This ratio approximate experimental measurements that have been made on many superconductors. Superconductors are characterized as having weak ($\lambda << 1$), intermediate

($\lambda \approx 1$), and strong ($\lambda >> 1$) coupling. The electron-electron interaction potenitial V_o for cooper-pair bonding has an attractive electron-phonon part measured by λ and a repulsive screened coulomb part μ_o^* to give $V_0 D_n(0) = \lambda - \mu_o^*$ and this provides the well known formula for the critical temperature T_c.

$$T_c = 1.13\theta_D exp(\frac{-1}{\lambda - \mu_o^*})[10]. \tag{1.7.3}$$

Where θ_D is the Debby temperature related to ω_D by $\hbar\omega_D = k_B\theta_D, \theta_D$ rages from 100K to 500K .This range of θ_D (and $\lambda - \mu_o^* \approx 0.3$) implies a maximum BCS value of $T_c \sim 25K$. A number of related formulate for the dependence of T_c on λ and μ_o^* have applied in in the literature , eg .the $M_cMillaequation$.

$$T_c = \frac{\theta_D}{1.45} exp(\frac{-1.04(1+\lambda)}{\lambda - \mu_o^*(1+0.62\lambda)})[26]. \tag{1.7.4}$$

Chapter 2

Mathematical Methods

The method of modern quantum field theory have recently more and more penetrated into statistical physics. This is connected with the fact that the basic problems in both fields are very much the same [33]. The problems of the particle interacting with a quantized field, or that of a system of interacting fields, is formulated in terms of second-quantized Hamiltonian's (Lagrangian's) just as the basic problems of Statistical mechanics, that of a system of interacting particles. One of the basic concepts of quantum field theory is that of the Green functions, which are convenient for the study of the particles of interacting quantized fields. The use of these concepts turns to out to be useful also in statistical mechanics. The application of Green functions turns out to be useful in these case where one can sum some type of perturbation theory diagrams. Tasks of this kind are performed more simply with Green functions. So in this study we have used a Green's function to obtain the expression for superconducting transition temperature -T_C and superconducting and magnetic order parameters (Δ, η).

2.1 Retarded double - time Green's function

The Green functions in statistical mechanics are the appropriate generalization of the concept of correlation functions. They are just as intimately connected with the evaluation of observed quantities and they have well-known advantages when equations are formulated and solved [34,35]. We can consider in statistical mechanics as in the quantum theory of fields , different kinds of Green functions, for instance, the retarded double-time and advanced Green functions $G_r(t,t')$ and $G_a(t,t')$. They are defined as

$$G_r(t,t') = \ll \hat{A}(t); \hat{B}(t') \gg_r = -i\theta(t-t') < [\hat{A}(t); \hat{B}(t')] >_r . \quad (2.1.1)$$

$$G(t,t')_a = \ll \hat{A}(t); \hat{B}(t') \gg_a = -i\theta(t-t') < [\hat{A}(t); \hat{B}(t')] >_a . \quad (2.1.2)$$

Where $\ll ... \gg$ abbreviated notation for the Green's function where as $< .... >$ indicates that one should average over grand canonical ensemble. $\theta(t,t')$ is a heviside step functions. It also called a unit step function whose values is zero for negative argument and one for positive argument i .e

$$\theta(t-t') = \begin{cases} 1, & t > t' \\ 0, & t < t' \end{cases} \quad (2.1.3)$$

$\hat{A(t)}$ and $\hat{B(t')}$ are operators in the Heisenberg representations of the operators. They are expressed in terms of a product of quantized functions (or of particle creation and annihilation operators) and can be expressed as

$$\hat{A(t)} = exp(i\hat{H}t)\hat{A(0)}exp(-i\hat{H}t). \quad (2.1.4)$$

For $\hbar = 1$, $[\hat{A(t)}, \hat{B(t')}]$ indicates the commutation or anti commutation i . e

$$[\hat{A(t)}, \hat{B(t')}] = \hat{A(t)}\hat{B(t')} - \eta\hat{B(t')}\hat{A(t)}, \quad (2.1.5)$$

where $\eta = \pm 1$ positive for Bose operators and negative for Fermi operators .

2.2 Equations for Green's function

We shall obtain a set of equations for the Green's function. $\hat{A(t)}$ and $\hat{B(t')}$ satisfy equation of the form

$$i\frac{d\hat{A(t)}}{dt} = \hat{A(t)}\hat{H} - \hat{H}\hat{A(t)} = [\hat{A(t)}, \hat{H}]. \quad (2.2.1)$$

The right hand side of Eq. 2 .1.1 can be written in more detail using the explicit form of the Hamiltonian and the commutation relations for the operators. In order to obtain the equation of motion, we differentiate Eq. 2.1.1 with respect to t as

$$
\begin{aligned}
& i\frac{dG(t-t')_r}{dt} = i\frac{d}{dt} \ll \hat{A}(t); \hat{B}(t') \gg \\
& = i\frac{d}{dt}(-i\theta(t,t') < [\hat{A}(t); \hat{B}(t')] >) \\
& = \frac{d}{dt}\theta(t,t') < [\hat{B}(t); \hat{B}(t')] > -i\theta(t,t') < [i\frac{d}{dt}\hat{A}(t); \hat{B}(t')] > \\
& = \frac{d}{dt}\theta(t,t') < [\hat{A}(t); \hat{B}(t')] > -i\theta(t,t') < [\hat{A}(t), H]; \hat{B}(t') > \\
& = \frac{d}{dt}\theta(t,t') < [\hat{A}(t); \hat{B}(t')] > + \ll [\hat{A}(t), \hat{H}]; \hat{B}(t') \gg . \qquad (2.2.2)
\end{aligned}
$$

Using the relation

$$\theta(t-t') = \int_{-\infty}^{t} \delta(t-t')dt, \qquad (2.2.3)$$

and

$$\frac{d}{dt}\theta(t-t') = \delta(t-t'). \qquad (2.2.4)$$

Then

$$
\begin{aligned}
& i\frac{d}{dt}G_t(t,t') = \delta(t-t') < [\hat{A}(t); \hat{B}(t')] > + \ll [\hat{A}(t), \hat{H}]; \hat{B}(t') \gg \\
& =< [\hat{A}(t); \hat{B}(t')] > + \ll [\hat{A}(t), \hat{H}]; \hat{B}(t') \gg, \qquad (2.2.5)
\end{aligned}
$$

where $t > t'$.

To solve this equation, it is convenient to use Fourier transformation of the Green's function. Let $G_r(\omega)$ be the Fourier transform of $G_r(t-t')$, then

$$G_r(t-t') = \int_{-\infty}^{\infty} G_r(\omega)exp(-i\omega(t-t'))d\omega, \qquad (2.2.6)$$

and

$$G_r(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_r(t-t')exp(i\omega(t-t'))d(t-t'). \qquad (2.2.7)$$

$\delta(t-t')$ can be defined as

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} exp(-i\omega(t-t'))d\omega. \tag{2.2.8}$$

Thus after differentaiting we can have

$$\frac{dG}{dt} = -i\omega\int G(\omega)exp(-i\omega(t-t'))d\omega$$
$$= -iw \times (fouriertransformationofG(t,t')). \tag{2.2.9}$$

Finally Eq. 2.2.1 can be written as

$$\omega \ll \hat{A}(t);\hat{B}(t') \gg_{\omega} =< [\hat{A}(t);\hat{B}(t')] > + \ll [\hat{A}(t),\hat{H}];\hat{B}(t') \gg . \tag{2.2.10}$$

Since $\ll \hat{A}(t);\hat{B}(t') \gg_{\omega}$ denotes the Fourier transform of the Green's function involving the operator $\hat{A}(t)$ and $\hat{B}(t')$. It satisfies the equations of motion, where the double brackets $\ll ..... \gg$ indicates the Fourier transform of the corresponding Green's function.The single brackets $< ...... >$ indicates the thermal average over the canonical ensemble. To obtain the superconducting properties, we have defind superconducting order parameters, Δ indicates the thermal average over the canonical ensemble or spin and momentum by

$$\Delta = \sum_k V < a_{k\uparrow}, a_{-k\downarrow} >, \tag{2.2.11}$$

$$\Delta^* = \sum_k V < a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} >, \tag{2.2.12}$$

where $\Delta^* = \Delta$(real quantity).

The value of transition temperature T_c is calculated by using the condition $T \rightarrow T_c$ as $\Delta \rightarrow 0$.

From the BCS Hamiltonian which is given by

$$\hat{H_{BCS}} = \sum_{k,\sigma} \epsilon_k a^{\dagger}_{k\sigma} a_{k\sigma} - \sum_{k,k'} V(k,k') a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} a_{k'\downarrow} a_{-k'\uparrow}, \tag{2.2.13}$$

one can write the equation of motion as

$$\omega \ll a_{k\uparrow}, a^{\dagger}_{k\uparrow} \gg = \delta_{kk'} + \ll [a_{k\uparrow}, \hat{H_{BCS}}]; a^{\dagger}_{k\uparrow} \gg, \tag{2.2.14}$$

where $\delta_{kk'} = 1$ for $k = k'$ and 0 otherwise.

To solve the above equation, let we first evaluate the following anti commutator relation for fermions with the use of operators A, B and C as

$$[A, BC] = [A, B]C - B[A, C], \tag{2.2.15}$$

$$[AB, C] = A[B, C] - [A, C]B, \tag{2.2.16}$$

and

$$[a_{k\downarrow}, a^{\dagger}_{k'\uparrow}] = \delta_{kk'}, \tag{2.2.17}$$

$$[a_{k\uparrow}, a_{k\uparrow}] = [a^{\dagger}_{k\uparrow}, a^{\dagger}_{k\uparrow}] = o, \tag{2.2.18}$$

where $\delta_{kk'} = 1$ if $k = k'$ otherwise 0.

Now let we calculate the commutation as follows

$$\begin{aligned}&[a_{k\uparrow}, \sum_{p,\sigma} \epsilon_p a^{\dagger}_{k\sigma} a_{k\sigma}] = \sum_{p,\sigma} \epsilon_p (\{a_{k\downarrow}, a^{\dagger}_{k\sigma}\} a_{k\sigma} - a^{\dagger}_{k\sigma} \{a_{k\uparrow}, a_{k\sigma}\}) \\ &= \epsilon_k a_{k\uparrow}.\end{aligned} \tag{2.2.19}$$

Similarly

$$\begin{aligned}[a_{k\uparrow}, -\sum_{p,p'} V(p,p') a^{\dagger}_{p\uparrow} a^{\dagger}_{-\downarrow} a_{p'\downarrow} a_{-p'\uparrow}] = \quad & -\sum_{p,p'} V(p,p')([a_{k\uparrow}, a^{\dagger}_{p\uparrow} a^{\dagger}_{-\downarrow}] a_{p'\downarrow} a_{-p'\uparrow} \\ & + \sum_{p,p'} V(p,p') a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} [a_{k\uparrow}, a_{p'\downarrow} a_{-p'\uparrow}]) \\ & = -\sum_{p'} V a^{\dagger}_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}.\end{aligned} \tag{2.2.20}$$

Substituting Eqs. 2.2.19 and 2.2.20 into Eq. 2.2.14, we obtain

$$\ll a_{k\uparrow}, a^{\dagger}_{k\uparrow} \gg = \frac{1}{\omega - \epsilon_k} - \frac{\Delta}{\omega - \epsilon_k} \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg, \tag{2.2.21}$$

where $\Delta = \sum_{p'} V < a_{p'\downarrow} a_{-p'\uparrow} >$.

One can also obtain the equation of motion for $\ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg$, using the equation

$$\omega \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg = 0 + \ll [a^{\dagger}_{-k\downarrow}, H\hat{_{BCS}}]; a^{\dagger}_{k\uparrow} \gg . \tag{2.2.22}$$

The commutation

$$[a^\dagger_{-k\downarrow}, \sum_{p,\sigma} \epsilon_p a^\dagger_{k\sigma} a_{k\sigma}] = -\epsilon_k a^\dagger_{-k\downarrow}. \quad (2.2.23)$$

$$[a^\dagger_{-k\downarrow}, -\sum_{p,p'} V(p,p') a^\dagger_{p\uparrow} a^\dagger_{-\downarrow} a_{p'\downarrow} a_{-p'\uparrow}] = -\sum_{p} V a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} a_{k\uparrow}. \quad (2.2.24)$$

Using Eqs. 2.2.23 and 2.2.24 into Eq. 2.2.22, we obtain

$$\ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = -\frac{\Delta}{\omega + \epsilon_k} \ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg . \quad (2.2.25)$$

Finally ,using Eqs. 2.2.21 and 2.2.25, we obtain

$$\ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = -\frac{\Delta}{\omega^2 - \epsilon_k^2 - \Delta^2}. \quad (2.2.26)$$

$$\ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg = \frac{\omega + \epsilon_k}{\omega^2 - \epsilon_k^2 - \Delta^2}. \quad (2.2.27)$$

The superconducting order parameter, Δ can be given as

$$\Delta = \frac{V}{\beta} \sum_{k,n} \ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = -\frac{V}{\beta} \sum_{k,n} \frac{\Delta}{\omega^2 - \epsilon_k^2 - \Delta^2}. \quad (2.2.28)$$

Changing the sum into an integral by introducing the density of state $N(\epsilon)$ as

$$\sum_k \to \int d^3k = \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon). \quad (2.2.29)$$

Which implies

$$\Delta = -\frac{V}{\beta} \sum_n \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon) \frac{\Delta}{\omega^2 - \epsilon_k^2 - \Delta^2}. \quad (2.2.30)$$

Since attractive intraction is effective for the region $-\hbar\omega_D < \epsilon < \hbar\omega_D$, and assuming the density of states does not vary over this region, the above expression becomes

$$1 = -2\frac{V}{\beta} N(o) \sum_n \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\omega^2 - \epsilon_k^2 - \Delta^2}. \quad (2.2.31)$$

Changing $\omega \to i\omega_n$, with the Metsubara frequency $\omega_n = (2n+1)\frac{\pi}{\beta}$, and using the relation $\frac{1}{2x} tanh(\frac{x}{2}) = \sum_n \frac{1}{(2n+1)^2\pi^2 + x^2}$, one can write

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_D} d\epsilon \frac{1}{\sqrt{\epsilon_k^2 + \Delta^2}} tanh(\frac{\beta\sqrt{\epsilon_k^2 + \Delta^2}}{2}), \quad (2.2.32)$$

where $\lambda = N(o)V$.

When $T \rightarrow T_c$, $\Delta \rightarrow 0$.

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_D} d\epsilon \frac{1}{\epsilon} tanh(\frac{\beta\epsilon}{2}). \tag{2.2.33}$$

Integrating the above integral gives

$$k_B T_C = 1.14\hbar\omega_D exp(-\frac{1}{\lambda}). \tag{2.2.34}$$

Which is the BCS expression .

Chapter 3

Theoretical Formulation of the Problem

In this chapter we formulate the model Hamiltonian and study theoretically the coexistence of ferro magnetism and superconductivity in the compounds URhGe and UCoGe in general, and obtain expressions for transition temperature and order parameters in particular.

3.1 The model Hamiltonian

The model Hamiltonian of the system is given by

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4, \tag{3.1.1}$$

where

$$\hat{H}_1 = \sum_{k\sigma} \epsilon_k a^\dagger_{k\sigma} a_{k\sigma} - \sum_{k,k'} V(k,k') a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} a_{k'\downarrow} a_{-k'\uparrow}, \tag{3.1.2}$$

which is the kinetik energy and pairing intraction Hamiltonian of the conduction electrons.

$$\hat{H}_2 = \sum_{l} \epsilon_l b^\dagger_{l\sigma} b_{l\sigma}, \tag{3.1.3}$$

which is the localized electron energy.

$$\hat{H}_3 = U \sum_{l} n_{l\uparrow} n_{l\downarrow}, \tag{3.1.4}$$

which is the intra coulomb repulsion energy of localized electrons.

$$\hat{H}_4 = \sum_{k,l,l'} \gamma_{l,l'}(k) a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} b_{l\uparrow} b_{l'\uparrow} + hc, \tag{3.1.5}$$

which is the interaction term between conduction electrons and localized electrons due to some unspecified mechanism as may be due to spin fluctuation, with coupling constant γ, where V(k,k') defines the matrix element of the interaction potential, $a^{\dagger}_{k\sigma}(a_{k\sigma})$ is the creation (annihilation) operators of an electron specified by the wave vector k and the spin σ, ϵ_k is the one electron energy measured relative to the chemical potential. $b^{\dagger}_{l\sigma}(b_{l\sigma})$ are creation and (annihilation) operators of localized electrons.

3.2 Equation of motion

In order to obtain the equation of motion, we differentiate the Green's function defined as $G_r(t,t') = \ll \hat{A}(t), \hat{B}(t') \gg = -i\theta(t,t) < \hat{A}(t), \hat{B}(t') >$ with respect to time t and taking the Fourier transform of this equation. After differentiating and taking the Fourier transform, we get

$$\omega \ll \hat{A}(t); \hat{B}(t') \gg_{\omega} = < [\hat{A}(t), \hat{B}(t')] > + \ll [\hat{A}(t), \hat{H}], \hat{B}(t') \gg_{\omega} . \tag{3.2.1}$$

3.2.1 Equation of motion for conduction electrons

The equation of motion for conduction electron is

$$\omega \ll a_{k\uparrow}, a^{\dagger}_{k\uparrow} \gg = 1 + \ll [a_{k\uparrow}, \hat{H}], a^{\dagger}_{k\uparrow} \gg_{\omega} . \tag{3.2.2}$$

Since electrons are fermions, they satisfy the anti commutaion rule as

$$\{a_{k\uparrow}, a_{k\uparrow}\} = \{b_{l\uparrow}, b_{l\uparrow}\} = \{a_{k\uparrow}, b_{l\uparrow}\} = 0. \tag{3.2.3}$$

$$\{a^\dagger_{k\uparrow}, a^\dagger_{k\uparrow}\} = \{b^\dagger_{l\uparrow}, b^\dagger_{l\uparrow}\} = 0. \tag{3.2.4}$$

Now to obtain the equation of motion, let us first calculate the commutator $[a_{k\uparrow}, \hat{H}]$ using the value of the model Hamiltonian written at the beginning of this chapter.

$$\begin{aligned}
&[a_{k\uparrow}, \sum_{p,\sigma} \epsilon_p a^\dagger_{p\sigma} a_{p\sigma}] = \sum_{p,\sigma} \epsilon_p (\{a_{k\uparrow}, a^\dagger_{p\sigma}\} a_{p\sigma} - a^\dagger_{p\sigma}\{a_{k\uparrow}, a_{p\sigma}\}) \\
&= \sum_{p,\sigma} \epsilon_p \delta_{kp} \delta_{\uparrow\sigma} a_{p\sigma} \\
&= \epsilon_k a_{k\uparrow},
\end{aligned} \tag{3.2.5}$$

and

$$\begin{aligned}
&[a_{k\uparrow}, -\sum_{p,p'} V(p,p') a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}] = -\sum_{p,p'} V(p,p')[a_{k\uparrow}, a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}] \\
&= -\sum_{p,p'} V(p,p')([a_{k\uparrow}, a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow}] a_{p'\downarrow} a_{-p'\uparrow} \\
&+ a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow}[a_{k\uparrow}, a_{p'\downarrow} a_{-p'\uparrow}]) = -\sum_{p,p'} V(p,p')[a_{k\uparrow}, a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow}] a_{p'\downarrow} a_{-p'\uparrow} \\
&= -\sum_{p,p'} V(p,p')(\{a_{k\uparrow}, a^\dagger_{p\uparrow}\} a^\dagger_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow} - a^\dagger_{p\uparrow}\{a_{k\uparrow}, a^\dagger_{-p\downarrow}\} a_{p'\downarrow} a_{-p'\uparrow}) \\
&= -\sum_{p,p'} V(p,p')(\delta_{kp}\delta_{\uparrow\uparrow} a^\dagger_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow} - a^\dagger_{p\uparrow}\delta_{k-p}\delta_{\uparrow\downarrow} a_{p'\downarrow} a_{-p'\uparrow}) \\
&= -\sum_{p'} V a^\dagger_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}.
\end{aligned} \tag{3.2.6}$$

The commutation with the interaction Hamiltonian of localized electron is calculated as follows;

$$\begin{aligned}
&[a_{k\uparrow}, \hat{H}_2] = [a_{k\uparrow}, \sum_{l\sigma} \epsilon_l b^\dagger_{l\sigma} b_{l\sigma}] \\
&= \sum_{l\sigma} \epsilon_l [a_{k\uparrow}, b^\dagger_{l\sigma} b_{l\sigma}] \\
&= \sum_{l\sigma} \epsilon_l (\{a_{k\uparrow}, b^\dagger_{l\sigma}\} b_{l\sigma} - b^\dagger_{l\sigma}\{a_{k\uparrow}, b_{l\sigma}\}) \\
&= 0,
\end{aligned} \tag{3.2.7}$$

and

$$
\begin{aligned}
[a_{k\uparrow}, \hat{H}_3] &= [a_{k\uparrow}, U \sum_l n_{l\uparrow} n_{l\downarrow} \\
&= U \sum_l ([a_{k\uparrow}, n_{l\uparrow}] n_{l\downarrow} + n_{l\uparrow} [a_{k\uparrow}, n_{l\downarrow}]) \\
&= U \sum_l ([a_{k\uparrow}, b^{\dagger}_{l\uparrow} b_{l\uparrow}] b^{\dagger}_{l\downarrow} b_{l\downarrow} + b^{\dagger}_{l\uparrow} b_{l\uparrow} [a_{k\uparrow}, b^{\dagger}_{l\downarrow} b_{l\downarrow}]) \\
&= 0. \qquad (3.2.8)
\end{aligned}
$$

Finally

$$
\begin{aligned}
[a_{k\uparrow}, \hat{H}_4] = \quad & [a_{k\uparrow}, \sum_{p,l,l'} \gamma_{ll'} a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow}] + [a_{k\uparrow} hc] \\
&= \sum_{p,l,l'} \gamma_{ll'} [a_{k\uparrow}, a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow}] + 0 \\
&= \sum_{p,l,l'} \gamma_{ll'} ([a_{k\uparrow}, a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow}] b_{l\uparrow} b_{l'\uparrow} + a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} [a_{k\uparrow}, b_{l\uparrow} b_{l'\uparrow}]) \\
&= \sum_{p,l,l'} \gamma_{ll'} (\{a_{k\uparrow}, a^{\dagger}_{p\uparrow}\} a^{\dagger}_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow} - a^{\dagger}_{p\uparrow} \{a_{k\uparrow}, a^{\dagger}_{-p\downarrow}\} b_{l\uparrow} b_{l'\uparrow}) \\
&= \sum_{p,l,l'} \gamma_{ll'} (\delta_{kp} \delta_{\uparrow\uparrow} a^{\dagger}_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow} - a^{\dagger}_{-p\downarrow} \delta_{k-p} \delta_{\uparrow\downarrow} b_{l\uparrow} b_{l'\uparrow}) \\
&= \sum_{l,l'} \gamma_{ll'} a^{\dagger}_{-k\downarrow} b_{l\uparrow} b_{l'\uparrow}. \qquad (3.2.9)
\end{aligned}
$$

Substituting Eqs. 3.2.5, 3.2.6 and 3.2.9 into Eq. 3.2.2, yeilds

$$
\begin{aligned}
(\omega - \epsilon_k) \ll a_{k\uparrow}, a^{\dagger}_{k\uparrow} \gg = \quad & 1 - \sum_{p'} V < a_{p'\downarrow}, a_{-p'\uparrow} > \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg \\
+ \sum_{l,l'} \gamma_{ll'} < b_{l\uparrow}, b_{l'\uparrow} > \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg & \\
&= 1 - (\Delta - \eta) \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg, \qquad (3.2.10)
\end{aligned}
$$

where $\Delta = V < a_{p'\downarrow}, a_{-p'\uparrow} >$ and $\eta = \sum_{l,l'} \gamma_{ll'} < b_{l\uparrow}, b_{l'\uparrow} >$.

Solving for $\ll a_{k\uparrow}, a^{\dagger}_{k\uparrow} \gg$, we obtain

$$
\ll a_{k\uparrow}, a^{\dagger}_{k\uparrow} \gg = \frac{1 - (\Delta - \eta) \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg}{\omega - \epsilon_k}. \qquad (3.2.11)
$$

In order to obtain the equation of motion for higher order Green's function $\ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg$ using the equation

$$\omega \ll a^{\dagger}_{-k\downarrow}, a^{\dagger}_{k\uparrow} \gg = 0 + \ll [a^{\dagger}_{-k\downarrow}, \hat{H}], a^{\dagger}_{k\uparrow} \gg_{\omega}, \tag{3.2.12}$$

we have to first calculate the commutator $[a^{\dagger}_{-k\downarrow}, \hat{H}]$ as follows

$$[a^{\dagger}_{-k\downarrow}, \hat{H}_1] = [a^{\dagger}_{-k\downarrow}, \sum_{p,\sigma} \epsilon_p a^{\dagger}_{p\sigma} a_{p\sigma}] + [a^{\dagger}_{-k\downarrow}, \sum_{p,p'} V(p,p') a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}], \tag{3.2.13}$$

where

$$\begin{aligned}
&[a^{\dagger}_{-k\downarrow}, \sum_{p,\sigma} \epsilon_p a^{\dagger}_{p\sigma} a_{p\sigma}] = \sum_{p,\sigma} \epsilon_p [a^{\dagger}_{-k\downarrow}, a^{\dagger}_{p\sigma} a_{p\sigma}] \\
&= \sum_{p,\sigma} \epsilon_p (\{a^{\dagger}_{-k\downarrow}, a^{\dagger}_{p\sigma}\} a_{p\sigma} - a^{\dagger}_{p\sigma} \{a^{\dagger}_{-k\downarrow}, a_{p\sigma}\}) \\
&= -\sum_{p,\sigma} \epsilon_p a^{\dagger}_{p\sigma} \delta_{-kp} \delta_{\downarrow\sigma} \\
&= -\epsilon_{-k} a^{\dagger}_{-k\downarrow},
\end{aligned} \tag{3.2.14}$$

and

$$\begin{aligned}
&[a^{\dagger}_{-\downarrow}, -\sum_{p,p'} V(p,p') a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}] = -\sum_{p,p'} V(p,p') [a^{\dagger}_{-k\downarrow}, a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} a_{p'\downarrow} a_{-p'\uparrow}] \\
&= -\sum_{p,p'} V(p,p') ([a^{\dagger}_{-k\downarrow}, a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow}] a_{p'\downarrow} a_{-p'\uparrow} + a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} [a^{\dagger}_{-k\downarrow}, a_{p'\downarrow} a_{-p'\uparrow}]) \\
&= 0 - \sum_{p,p'} V(p,p') (a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} \{a^{\dagger}_{-k\downarrow}, a_{p'\downarrow}\} a_{-p'\uparrow} - a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} a_{p'\downarrow} \{a^{\dagger}_{-k\downarrow}, a_{-p'\uparrow}\}) \\
&= -\sum_{p,p'} V(p,p') (a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} \delta_{-kp'} \delta_{\downarrow\downarrow} a_{-p'\uparrow} - a^{\dagger}_{p\uparrow} a^{\dagger}_{-p\downarrow} \delta_{-k-p'} \delta_{\downarrow\uparrow}) \\
&= -\sum_{p} V a^{\dagger}_{p\uparrow} a^{\dagger}_{-p'\downarrow} a_{k\uparrow}.
\end{aligned} \tag{3.2.15}$$

Similarly

$$\begin{aligned}
&[a^{\dagger}_{-k\downarrow}, \hat{H}_3] = [a^{\dagger}_{-k}, U \sum_{l} n_{l\uparrow} n_{l\downarrow} \\
&= U \sum_{l} ([a^{\dagger}_{-k\downarrow}, n_{l\uparrow}] n_{l\downarrow} + n_{l\uparrow} [a^{\dagger}_{-k\downarrow}, n_{l\downarrow}]) \\
&= U \sum_{l} ([a^{\dagger}_{-k\downarrow}, b^{\dagger}_{l\uparrow} b_{l\uparrow}] b^{\dagger}_{l\downarrow} b_{l\downarrow} + b^{\dagger}_{l\uparrow} b_{l\uparrow} [a^{\dagger}_{-k\downarrow}, b^{\dagger}_{l\downarrow} b_{l\downarrow}]) \\
&= 0.
\end{aligned} \tag{3.2.16}$$

Finally

$$[a^\dagger_{-k\downarrow}, \hat{H}_4] = [a^\dagger_{-k\downarrow}, \sum_{p,l,l'} \gamma_{ll'} a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow}] + [a^\dagger_{-k\downarrow}, hc], \tag{3.2.17}$$

where

$$\begin{aligned}
[a^\dagger_{-k\downarrow}, \sum_{p,l,l'} \gamma_{ll'} a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow}] = \quad & \sum_{p,l,l'} \gamma_{ll'} [a^\dagger_{-k\downarrow}, a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow}] \\
& = \sum_{p,l,l'} \gamma_{ll'} ([a^\dagger_{-k\downarrow}, a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow}] b_{l\uparrow} b_{l'\uparrow} + a^\dagger_{p\uparrow} a^\dagger_{-p\downarrow} [a^\dagger_{-k\downarrow}, b_{l\uparrow} b_{l'\uparrow}]) \\
& = \sum_{p,l,l'} \gamma_{ll'} (\{a^\dagger_{-k\downarrow}, a^\dagger_{p\uparrow}\} a^\dagger_{-p\downarrow} b_{l\uparrow} b_{l'\uparrow} - a^\dagger_{p\uparrow} \{a^\dagger_{-k\downarrow}, a^\dagger_{-p\downarrow}\} b_{l\uparrow} b_{l'\uparrow}) \\
& = 0,
\end{aligned} \tag{3.2.18}$$

and

$$\begin{aligned}
[a^\dagger_{-k\downarrow}, hc] = \quad & \sum_{p,l,l'} \gamma^*_{ll'} [a^\dagger_{-k\downarrow}, a_{-p\downarrow} a_{p\uparrow} b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow}] \\
& = \sum_{p,l,l'} \gamma^*_{ll'} ([a^\dagger_{-k\downarrow}, a_{-p\downarrow} a_{p\uparrow}] b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow} + a_{-p\downarrow} a_{p\uparrow} [a^\dagger_{-k\downarrow}, b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow}]) \\
& = \sum_{p,l,l'} \gamma^*_{ll'} (\{a^\dagger_{-k\downarrow}, a_{-p\downarrow}\} a_{p\uparrow} b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow} - a_{-p\downarrow} \{a^\dagger_{-k\downarrow}, a_{p\uparrow}\} b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow}) \\
& = \sum_{p,l,l'} \gamma^*_{ll'} (\delta_{-k-p} \delta_{\downarrow\downarrow} a_{p\uparrow} b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow}) - a_{-p\downarrow} \delta_{-kp} \delta_{\downarrow\uparrow} b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow}) \\
& = \sum_{l,l'} \gamma^*_{ll'} a_{k\uparrow} b^\dagger_{l\uparrow} b^\dagger_{l'\uparrow}.
\end{aligned} \tag{3.2.19}$$

Taking Eqs. 3.2.14, 3.2.15 and 3.2.19 into the equation

$$\omega \ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = 0 + \ll [a^\dagger_{-k\downarrow}, \hat{H}], a^\dagger_{k\uparrow} \gg, \tag{3.2.20}$$

we obtain

$$\begin{aligned}
\omega \ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = \quad & -\epsilon_{-k} \ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg - \sum_p V < a^\dagger_{p\uparrow}, a^\dagger_{-p\downarrow} > \ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg \\
& + \sum_{ll'} \gamma^* < b^\dagger_{l\uparrow}, b^\dagger_{l\uparrow} > \ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg \\
& = -\epsilon_{-k} \ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg - (\Delta - \eta) \ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg,
\end{aligned} \tag{3.2.21}$$

which implies

$$\ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = \frac{-(\Delta - \eta) \ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg}{\omega + \epsilon_k}, \tag{3.2.22}$$

where $\Delta^* = \sum_p V < a^\dagger_{p\uparrow}, a^\dagger_{-p\downarrow} >$.

$\eta^* = \sum_{ll'} \gamma^* < b^\dagger_{l\uparrow}, b^\dagger_{l\uparrow} >$

Since energy and order parameters are real, $\epsilon_k = \epsilon_{-k}$, $\Delta^* = \Delta$ and $\gamma = \gamma^*$.

From Eqs. 3.2.11 and 3.2.22, we obtain

$$\ll a_{k\uparrow}, a^\dagger_{k\uparrow} \gg = \frac{\omega + \epsilon_k}{\omega^2 - \epsilon_k^2 - (\Delta - \eta)^2}. \tag{3.2.23}$$

$$\ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg = \frac{-(\Delta - \eta)}{\omega^2 - \epsilon_k^2 - (\Delta - \eta)^2}. \tag{3.2.24}$$

The above two equations are called the equations of motion for conduction electrons.

The order parameters of a superconductor is defined as

$$\Delta = \frac{V}{\beta} \sum_{k,n} \ll a^\dagger_{-k\downarrow}, a^\dagger_{k\uparrow} \gg, \tag{3.2.25}$$

where the summation over k and n includes all order pairs.

Substituting Eq. 3.2.24 into Eq. 3.2.25, we obtain

$$\Delta = -\frac{V}{\beta} \sum_{k,n} \frac{(\Delta - \eta)}{\omega^2 - \epsilon_k^2 - (\Delta - \eta)^2}. \tag{3.2.26}$$

Now one can change the summation over k into an integral, by introducing the density of state $N(\epsilon)$,

$$\sum_k \rightarrow \int d^3k = \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon). \tag{3.2.27}$$

Taking Eq. 3.2.27 into Eq. 3.2.26, it becomes

$$\Delta = -\frac{V}{\beta} \sum_n \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon) \frac{(\Delta - \eta)}{\omega^2 - \epsilon_k^2 - (\Delta - \eta)^2}. \tag{3.2.28}$$

Using $\omega \rightarrow i\omega_n$ and the Metsubara frequency $\omega_n = (2n+1)\frac{\pi}{\beta}$,the above equation becomes

$$\begin{aligned} \Delta &= V\beta \sum_n \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon) \frac{(\Delta - \eta)}{(2n+1)^2\pi^2 + (\epsilon_k^2 + (\Delta - \eta)^2)\beta^2} \\ &= V\beta \sum_n \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon) \frac{(\Delta - \eta)}{(2n+1)^2\pi^2 + E^2\beta^2}, \end{aligned} \tag{3.2.29}$$

where E $= \epsilon_k^2 + (\Delta - \eta)^2$.

Since attractive intraction is effective in the region $-\hbar\omega_b < \epsilon < \hbar\omega_b$,and taking the density of state is constant in this region Eq. (3.2.29) becomes

$$\Delta = 2V\beta N(o)\sum_n \int_0^{\hbar\omega_b} d\epsilon \frac{(\Delta - \eta)}{(2n+1)^2\pi^2 + E^2\beta^2}. \tag{3.2.30}$$

Using the relation $\frac{1}{2x}tanh(\frac{x}{2}) = \sum_n \frac{1}{(2n+1)^2\pi^2 + x^2}$,
one can write $\frac{1}{2\beta E}tanh(\frac{\beta E}{2}) = \sum_n \frac{1}{(2n+1)^2\pi^2 + E^2\beta^2}$.
Then Eq. 3.2.30 becomes

$$\Delta = N(o)V\int_0^{\hbar\omega_b} d\epsilon \frac{(\Delta - \eta)}{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}} tanh(\frac{\beta\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}}{2}). \tag{3.2.31}$$

Now let us study the properties of Eq. (3.2.31), when $T \to 0$ and $T \to T_c$.

Case I, $T \to 0$

when $T_c \to 0$,$\beta \to \infty$,which implies that $tanh(\frac{\beta E}{2}) \to 1$,then

$$\Delta = \lambda\int_0^{\hbar\omega_b} d\epsilon \frac{(\Delta - \eta)}{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}}. \tag{3.2.32}$$

using the relation $\int \frac{a}{\sqrt{a^2 = x^2}}dx = asinh^{-1}(\frac{x}{a})|_0^{\hbar\omega_b}$, then

$$\begin{aligned} \frac{1}{\lambda} = \quad & (1 - \frac{\eta}{\Delta})sinh^{-1}(\frac{\epsilon}{\Delta - \eta})|_0^{\hbar\omega_b} \\ & = (1 - \frac{\eta}{\Delta})sinh^{-1}(\frac{\hbar\omega_b}{\Delta - \eta}) \\ & \approx (1 - \frac{\eta}{\Delta})ln\{\frac{\hbar\omega_b}{\Delta - \eta} + \sqrt{(\frac{\hbar\omega_b}{\Delta - \eta})^2 + 1}\} \\ & \approx (1 - \frac{\eta}{\Delta})ln\{2\frac{\hbar\omega_b}{\Delta - \eta}\}. \end{aligned} \tag{3.2.33}$$

Solving for $\Delta - \eta$, we obtain

$$\Delta - \eta = 2\hbar\omega_b exp(-\frac{1}{\lambda(1 - \frac{\eta}{\Delta})}). \tag{3.2.34}$$

From BCS at T = 0, $2\Delta(T = 0) = 3.53k_BT_c$. Which implies

$$\eta \simeq 1.75k_BT_c - 2\hbar\omega_b exp(-\frac{1}{\lambda(1 - \frac{\eta}{1.75k_BT_c})}). \tag{3.2.35}$$

Case II, as $T \to T_c$, $\Delta \to 0$

$$\frac{\Delta}{\lambda} = \int_0^{\hbar\omega_b} d\epsilon \frac{\Delta - \eta}{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}} tanh(\beta \frac{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}}{2}), \tag{3.2.36}$$

which implies

$$\begin{aligned} \frac{1}{\lambda} = & \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}} tanh(\beta \frac{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}}{2}) \\ & - \int_0^{\hbar\omega_b} d\epsilon \frac{\eta}{\Delta\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}} tanh(\beta \frac{\sqrt{\epsilon_k^2 + (\Delta - \eta)^2}}{2}) \\ & = I_1 - I_2. \end{aligned} \tag{3.2.37}$$

Let us calculate the above integral separately at $T = T_c, \Delta = 0$.

$$\begin{aligned} I_1 = & \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\sqrt{\epsilon_k^2 + \eta^2}} tanh(\beta \frac{\sqrt{\epsilon_k^2 + \eta^2}}{2}) \\ & = \int_0^{\hbar\omega_b} d\epsilon \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \epsilon_k^2 + \eta^2}. \end{aligned} \tag{3.2.38}$$

Taking Lapplace's transformation, and Metsubara frequency, $\omega_n = (2n+1)\frac{\pi}{\beta}$, the above integral becomes

$$I_1 = \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\epsilon} tanh(\beta \frac{\epsilon}{2}) - \int_0^{\hbar\omega_b} d\epsilon \eta^2 \frac{2}{\beta} \sum_{-\infty}^{\infty} \frac{1}{(((2n+1)\frac{\pi}{\beta})^2 + \epsilon^2)^2} + ... \tag{3.2.39}$$

Since $\sum_{-\infty}^{\infty} \frac{1}{(((2n+1)\frac{\pi}{\beta})^2 + \epsilon^2)^2} = 2\sum_0^{\infty} \frac{1}{(((2n+1)\frac{\pi}{\beta})^2 + \epsilon^2)^2}$,

and $2\sum_0^{\infty} \frac{1}{(a^2 + \epsilon^2)^2} = 2\sum_0^{\infty} \frac{1}{a^4(1 + \frac{\epsilon}{a^2}^2)^2}$.

Then

$$I_1 = \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\epsilon} tanh(\beta \frac{\epsilon}{2}) - \int_0^{\hbar\omega_b} d\epsilon \eta^2 \frac{2}{\beta} 2 \sum_0^{\infty} \frac{1}{a^4(1 + \epsilon^2)^2} + ... \tag{3.2.40}$$

For $y = \frac{\beta}{2}\epsilon$,the integral becomes

$$\begin{aligned} \int_0^{\hbar\omega_b} d\epsilon \frac{1}{y} tanh(\beta \frac{\epsilon}{2}) = & \int_0^y dy \frac{1}{y} tanh(y) \\ & = lnytanhy|_0^y - \int_0^y \frac{lny}{codh^2 y}. \end{aligned} \tag{3.2.41}$$

For low temperature, $tanh(\frac{\hbar\omega_b}{2k_BT}) \to 1$, so

$$\int_0^{\hbar\omega_b} d\epsilon\frac{1}{\epsilon}tanh(\beta\frac{\epsilon}{2}) = ln(\frac{\beta}{2}\hbar\omega_b) - ln(\frac{\pi}{4})$$
$$= ln1.14\frac{\hbar\omega_b}{k_BT_c}, \tag{3.2.42}$$

and

$$\int_0^{\hbar\omega_b} \eta^2 d\epsilon\frac{2}{\beta}2\sum_{n=0}^{\infty}\frac{dx}{a^4(1+x^2)^2} + ... = 4\eta^2\sum_{n=0}^{\infty}(\frac{\beta}{\pi(2n+1)})^3\int_0^{\infty}\frac{dx}{(1+x^2)^2} + ...$$
$$= \frac{4\beta^2}{\pi^3}\eta^2\frac{7}{8}\zeta(3)\frac{\pi}{4}$$
$$= (\frac{\eta}{\pi k_BT_c})^2\frac{8.414}{8}, \tag{3.2.43}$$

where $\sum_0^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ and $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^p} = (1-2^{-p})\zeta(3)$, here p =3, $\zeta(3) = 1.202$,where ζ is a zeta function. Substituting Eqs. 3.2.42 and 3.2.43 into Eq. 3.2.40 we obtain,

$$I_1 = ln1.14\frac{\hbar\omega_b}{k_BT_c} - \eta^2(\frac{1}{\pi k_BT_c})^2\frac{8.414}{8}. \tag{3.2.44}$$

Inorder to integrate the second integral, we apply L'Hoppital's rule. There fore

$$I_2 = \int_0^{\hbar\omega_b} lim_{\Delta\to 0}[d\epsilon\frac{\eta}{\Delta\sqrt{\epsilon_k^2+(\Delta-\eta)^2}}tanh(\frac{\beta\sqrt{\epsilon_k^2+(\Delta-\eta)^2}}{2})]'$$
$$= \int_0^{\hbar\omega_b} lim_{\Delta\to}d\epsilon[\eta\frac{sech^2(\frac{\beta\sqrt{\epsilon^2+(\Delta-\eta)^2}}{2})\frac{\beta(\Delta-\eta)}{2\sqrt{\varepsilon^2+(\Delta-\eta)^2}}}{\sqrt{\varepsilon^2+(\Delta-\eta)^2}+\Delta\frac{(\Delta-\eta)}{\sqrt{\epsilon^2+(\Delta-\eta)^2}}}]$$
$$= -\int_0^{\hbar\omega_b}\eta^2\beta\frac{sech^2(\frac{\beta\sqrt{\epsilon^2+\eta^2}}{2})}{2(\epsilon^2+\eta^2)}. \tag{3.2.45}$$

Combining the first and second integral

$$\frac{1}{\lambda} = ln(1.14\frac{\hbar\omega_b}{k_BT_c}) - \eta^2(\frac{1}{\pi k_BT_c})^2\frac{8.414}{8} + \int_0^{\hbar\omega_b} d\epsilon\eta^2\beta\frac{sech^2(\frac{\beta\sqrt{\epsilon^2+\eta^2}}{2})}{2(\epsilon^2+\eta^2)}. \tag{3.2.46}$$

Using the relation $sech^2x = 1 - tanh^2x$, one can write

$$\frac{1}{\lambda} = ln(1.14\frac{\hbar\omega_b}{k_B T_C}) - \eta^2(\frac{1}{\pi k_B T_c})^2\frac{8.414}{8} + \int_0^{\hbar\omega_b}\frac{\eta^2}{2k_B T_c(\epsilon^2+\eta^2)}d\epsilon - \int_0^{\hbar\omega_b}\frac{\eta^2 tanh^2(\frac{\beta\sqrt{\epsilon^2+\eta^2}}{2})}{2k_B T_C(\epsilon^2+\eta^2)}d\epsilon. \tag{3.2.47}$$

But

$$\int_0^{\hbar\omega_b} d\epsilon \frac{\eta^2}{2k_B T_c(\epsilon^2+\eta^2)} = \frac{\eta^2 arctan(\frac{\hbar\omega_b}{\eta})}{2k_B T_c} = \frac{\eta}{4k_B T_c} ln(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b}). \tag{3.2.48}$$

The fourth term rhs of Eq. 3.2.47, can be integrated using the help of FORTRAN LANGUGE and using the approximation $\hbar\omega_b \approx \hbar\omega_D = 10mev$, (for BCS) and. using superconducting critical temperature for URhGe. $T_c = 0.25K$, so $\frac{\hbar\omega_b}{2k_B T_c} \approx 0.00071$, using $\eta = a = 0.05$ and $k_B = 1$. With the afformantioned method and the above approximation we obtain

$$\int_{10^{-10}}^{0.00071} tanh^2(\frac{0.714\sqrt{x^2+a^2}}{2(x^2+a^2)})dx = 7.5\times 10^{-18} \tag{3.2.49}$$

Substituting Eqs. 3.2.48 and 3.2.49 into Eq. 3.2.47 and simplifying gives that

$$\frac{1}{\lambda} = ln(1.14\frac{\hbar\omega_b}{k_B T_c}) + \frac{\eta}{4k_B T_c}ln(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b}) - \eta^2(\frac{1}{\pi k_B T_c})^2\frac{8.414}{8} - \frac{7.5\times 10^{-18}\eta^2}{(2k_B T_c)^2}. \tag{3.2.50}$$

For small η we can ignore the η^2 term, then Eq. 3.2.50 becomes

$$\frac{1}{\lambda} = ln(1.14\frac{\hbar\omega_b}{k_B T_c}) + \frac{\eta}{4k_B T_c}ln(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b}), \tag{3.2.51}$$

which implies

$$k_B T_c = 1.14\hbar\omega_b e^{-(\frac{1}{\lambda}-a\eta)}, \tag{3.2.52}$$

where $a = \frac{1}{4k_B T_c}ln(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b})$.

3.2.2 Equation of motion for localized electrons

The equation of motion for localized electrons using retarded double time Green's function can be written as follows,

$$\omega \ll b_{l\uparrow}, b_{l\uparrow}^{\dagger} \gg = 1 + \ll [b_{l\uparrow}, \hat{H}]; b_{l\uparrow}^{\dagger} \gg_{\omega} . \tag{3.2.53}$$

Let we first evaluate the commutation using the model Hamiltonian as follows

$$\begin{aligned} [b_{l\uparrow}, \hat{H}_1] = \quad & [b_{l\uparrow}, \sum_{k\sigma} \epsilon_k a_{k\sigma}^{\dagger} a_{k\sigma} - \sum_{k,k'} V(k,k') a_{k\uparrow}^{\dagger} a_{-k\downarrow}^{\dagger} a_{k'\downarrow} a_{-k'\uparrow}] \\ & = \sum_{k\sigma} \epsilon_k [b_{l\uparrow}, a_{k\sigma}^{\dagger} a_{k\sigma}] - \sum_{k,k'} V(k,k') [b_{l\uparrow}, a_{k\uparrow}^{\dagger} a_{-k\downarrow}^{\dagger} a_{k'\downarrow} a_{-k'\uparrow}] \\ & = 0, \end{aligned} \tag{3.2.54}$$

and

$$\begin{aligned} [b_{l\uparrow}, \hat{H}_2] = \quad & [b_{l\uparrow}, \sum_{t} \epsilon_t b_{t\sigma}^{\dagger} b_{t\sigma}] \\ & = \sum_{t} \epsilon_t [b_{l\uparrow}, b_{t\sigma}^{\dagger} b_{t\sigma}] \\ & = \sum_{t} \epsilon_t (\{b_{l\uparrow}, b_{t\sigma}^{\dagger}\} b_{t\sigma} - b_{t\sigma}^{\dagger} \{b_{l\uparrow}, b_{t\sigma}\}) \\ & = \sum_{t} \epsilon_t \delta_{lt} \delta_{\uparrow\sigma} b_{lt} \\ & = \epsilon_l b_{l\uparrow}. \end{aligned} \tag{3.2.55}$$

The commutation with the intra coulomb repulsion interaction Hamiltonian is also calculated as

$$\begin{aligned}
[b_{l\uparrow}, \hat{H}_3] = \quad & [b_{l\uparrow}, U\sum_t n_{l\uparrow}n_{\downarrow}] \\
&= U\sum_t [b_{l\uparrow}, n_{t\uparrow}n_{t\downarrow}] \\
&= U\sum_t ([b_{l\uparrow}, n_{t\uparrow}]n_{t\downarrow} + n_{t\uparrow}[b_{l\uparrow}, n_{l\downarrow}]) \\
&= U\sum_t ([b_{l\uparrow}, b^\dagger_{t\uparrow}b_{t\uparrow}]b^\dagger_{t\downarrow}b_{t\downarrow} + b^\dagger_{t\uparrow}b_{t\uparrow}[b_{l\uparrow}, b^\dagger_{t\downarrow}b_{t\downarrow}]) \\
&= U\sum_t (\{b_{l\uparrow}, b^\dagger_{t\uparrow}\}b_{t\uparrow}b^\dagger_{t\downarrow}b_{t\downarrow} - b^\dagger_{t\uparrow}\{b_{l\uparrow}, b_{t\uparrow}\}b^\dagger_{t\downarrow}b_{t\downarrow} + b^\dagger_{t\uparrow}b_{t\uparrow}\{b_{l\uparrow}, b^\dagger_{t\downarrow}\}b_{t\downarrow} - b^\dagger_{t\uparrow}b_{t\uparrow}b^\dagger_{t\downarrow}\{b_{l\uparrow}, b_{t\downarrow}\}) \\
&= U\sum_t \delta_{lt}\delta_{\uparrow\uparrow}b_{t\uparrow}b^\dagger_{t\downarrow}b_{t\downarrow} \\
&= Ub_{l\uparrow}b^\dagger_{l\downarrow}b_{l\downarrow}, \qquad (3.2.56)
\end{aligned}$$

and

$$[b^\dagger_{l'\uparrow}, \hat{H}_4] = [b_{l\uparrow}, \sum_{k,t,l'} \gamma_{tl}a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}b_{t\uparrow}b_{l'\uparrow} + hc]. \qquad (3.2.57)$$

Commuting separatelly we have

$$\begin{aligned}
[b_{l\uparrow}, \sum_{k,t,l'} \gamma_{tl}a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}b_{t\uparrow}b_{l'\uparrow}] = \quad & \sum_{k,t,l'} \gamma_{tl'}([b_{l\uparrow}, a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}]b_{t\uparrow}b_{l'\uparrow} + a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}[b_{l\uparrow}, b_{t\uparrow}b_{l'\uparrow}]) \\
&= \sum_{k,t,l'} \gamma_{tl}(a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}\{b_{l\uparrow}, b_{t\uparrow}\}b_{l'\uparrow} - a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}b_{t\uparrow}\{b_{l\uparrow}, b_{l'\uparrow}\}) \\
&= 0. \qquad (3.2.58)
\end{aligned}$$

Finally

$$\begin{aligned}
[b^\dagger_{l'\uparrow}, \sum_{k,t,l'} \gamma^*_{tl'}a_{-k\downarrow}a_{k\uparrow}b^\dagger_{t\uparrow}b^\dagger_{l'\uparrow}] = \quad & \sum_{kt,l'} \gamma^*_{tl'}([b_{l\uparrow}, a_{-k\downarrow}a_{k\uparrow}b^\dagger_{t\uparrow}b^\dagger_{l'\uparrow}] \\
&= \sum_{k,t,l'} \gamma^*_{tl'}([b_{l\uparrow}, a_{-k\downarrow}a_{k\uparrow}]b^\dagger_{t\uparrow}b^\dagger_{l'\uparrow} + a_{-k\downarrow}a_{k\uparrow}[b_{l\uparrow}, b^\dagger_{t\uparrow}b^\dagger_{l'\uparrow}]) \\
&= \sum_{k,t,l} \gamma^*_{tl'}a_{-k\downarrow}a_{k\uparrow}[b_{l\uparrow}, b^\dagger_{t\uparrow}b^\dagger_{l'\uparrow}] \\
&= \sum_{k,t,l} \gamma^*_{tl'}a_{-k\downarrow}a_{k\uparrow}(\{b_{l\uparrow}, b^\dagger_{t\uparrow}\}b^\dagger_{l'\uparrow} + b^\dagger_{t\uparrow}\{b_{l\uparrow}, b^\dagger_{l'\uparrow}\}) \\
&= \sum_{k,t,l'} \gamma^*_{tl'}a_{-k\downarrow}a_{k\uparrow}\delta_{lt}\delta_{\uparrow\uparrow}b^\dagger_{l'\uparrow} - \sum_{k,t,l'} \gamma^*_{tl'}a_{-k\downarrow}a_{k\uparrow}\delta_{ll'}\delta_{\uparrow\uparrow}b^\dagger_{t\uparrow} \\
&= -\sum_{k,l'} \gamma^*_{tl'}a_{-k\downarrow}a_{k\uparrow}b^\dagger_{l\uparrow}, \qquad (3.2.59)
\end{aligned}$$

for $l = l'$.

Substituting Eqs. 3.2.55 ,(3.2.56 and 3.2.59 into Eq. 3.2.53 which revealed that

$$\begin{aligned}\omega \ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg = \quad & 1 + \epsilon_l \ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg +U < b^{\dagger}_{l\downarrow} b_{l\downarrow} > - \sum_{k,l'} \gamma^{*}_{l,l'} < a_{-k\downarrow} a_{k\uparrow} >\ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg \\ & = 1 + \epsilon_l \ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg +U < n_{l\downarrow} >\ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg \\ & -\Delta^{*}_{l} \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg, \end{aligned} \tag{3.2.60}$$

which implies

$$(\omega - \epsilon_l - U < n_{l\downarrow} >) \ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg = 1 - \Delta^{*}_{l} \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg, \tag{3.2.61}$$

where $\Delta^{*}_{l} = \sum_k \gamma^{*}_{ll'} < a_{-k\downarrow} a_{k\uparrow} >$. and we obtain

$$\begin{aligned}\ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg = \quad & \frac{1}{\omega - \epsilon_l - U < n_{l\downarrow} >} - \frac{\Delta^{*}_{l} \ll b^{+}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg}{\omega - \epsilon_l - U < n_{l\downarrow} >} \\ & = \frac{1}{\omega - (\epsilon_l + U < n_{l\downarrow} >)} - \frac{\Delta^{*}_{l} \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg}{\omega - (\epsilon_l + U < n_{l\downarrow} >)}. \end{aligned} \tag{3.2.62}$$

We can also write the equation of motion for localized electrons using higher order Green's function as

$$\omega \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg_{\omega} = < [b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow}] > + \ll [b^{\dagger}_{l'\uparrow}, \hat{H}], b^{\dagger}_{l\uparrow} \gg \tag{3.2.63}$$

The commutator can be calculated as follows;

$$\begin{aligned}[b^{\dagger}_{l'\uparrow}, \hat{H}_1] = \quad & [b^{\dagger}_{l'\uparrow}, \sum_{k\sigma} \epsilon_k a^{\dagger}_{k\sigma} a_{k\sigma} - \sum_{k,k'} V(k,k') a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} a_{k'\downarrow} a_{-k'\uparrow}] \\ & = \sum_{k\sigma} \epsilon_k [b^{\dagger}_{l'\uparrow}, a^{\dagger}_{k\sigma} a_{k\sigma}] - \sum_{k,k'} V(k,k') [b^{\dagger}_{l'\uparrow}, a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} a_{k'\downarrow} a_{-k'\uparrow}] \\ & = 0, \end{aligned} \tag{3.2.64}$$

and

$$\begin{aligned}
[b^{\dagger}_{l'\uparrow}, \hat{H}_2] = \quad & [b^{\dagger}_{l'\uparrow}, \sum_t \epsilon_t b^{\dagger}_{t\sigma} b_{t\sigma}] \\
&= \sum_t \epsilon_t [b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\sigma} b_{t\sigma}] \\
&= \sum_t \epsilon_t (\{b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\sigma}\} b_{t\sigma} - b^{\dagger}_{t\sigma} \{b^{\dagger}_{l'\uparrow}, b_{t\sigma}\}) \\
&= -\sum_t \epsilon_t \delta_{l't} \delta_{\uparrow\uparrow} b^{\dagger}_{t\sigma} \\
&= -\epsilon_{l'} b^{\dagger}_{l'\uparrow}. \qquad (3.2.65)
\end{aligned}$$

The commutation with the intra coulomb repulsion interaction Hamiltonian is also calculated as;

$$\begin{aligned}
[b^{\dagger}_{l'\uparrow}, \hat{H}_3] = \quad & [b^{\dagger}_{l'\uparrow}, U \sum_t n_{t\uparrow} n_{t\downarrow}] \\
&= U \sum_t [b^{\dagger}_{l'\uparrow}, n_{t\uparrow} n_{t\downarrow}] \\
&= U \sum_t ([b^{\dagger}_{l'\uparrow}, n_{t\uparrow}] n_{t\downarrow} + n_{t\uparrow} [b^{\dagger}_{l'\uparrow}, n_{t\downarrow}]) \\
&= U \sum_t ([b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\uparrow} b_{t\uparrow}] b^{\dagger}_{t\downarrow} b_{t\downarrow} + b^{\dagger}_{t\uparrow} b_{t\uparrow} [b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\downarrow} b_{t\downarrow}]) \\
&= U \sum_t (\{b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\uparrow}\} b_{t\uparrow} b^{\dagger}_{t\downarrow} b_{t\downarrow} - b^{\dagger}_{t\uparrow} \{b^{\dagger}_{l'\uparrow}, b_{t\uparrow}\} b^{\dagger}_{t\downarrow} b_{t\downarrow} + b^{\dagger}_{t\uparrow} b_{t\uparrow} \{b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\downarrow}\} b_{t\downarrow} - b^{\dagger}_{t\uparrow} b_{t\uparrow} b^{\dagger}_{t\downarrow} \{b^{\dagger}_{l'\uparrow}, b_{t\downarrow}\}) \\
&= -U \sum_t b^{\dagger}_{t\uparrow} \delta_{l't} \delta_{\uparrow\uparrow} b^{\dagger}_{t\downarrow} b_{t\downarrow} \\
&= -U b^{\dagger}_{l'\uparrow} b^{\dagger}_{l'\downarrow} b_{l'\downarrow}. \qquad (3.2.66)
\end{aligned}$$

Finally

$$[b^{\dagger}_{l\uparrow}, \hat{H}_4] = [b^{\dagger}_{l\uparrow}, \sum_{k,t,t'} \gamma_{tl} a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} b_{t\uparrow} b_{t'\uparrow} + hc], \qquad (3.2.67)$$

commuting separatelly we have

$$[b^{\dagger}_{l'\uparrow}, \sum_{k,t,t'} \gamma_{tl} a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} b_{t\uparrow} b_{t'\uparrow}] = \sum_{k,t,l'} \gamma_{tl'}([b^{\dagger}_{l'\uparrow}, a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow}] b_{t\uparrow} b_{t'\uparrow} + a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow}[b^{\dagger}_{l'\uparrow}, b_{t\uparrow} b_{t'\uparrow}])$$
$$= \sum_{k,t,l'} \gamma_{tl}(a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow}\{b^{\dagger}_{l'\uparrow}, b_{t\uparrow}\} b_{t'\uparrow} - a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} b_{t\uparrow}\{b^{\dagger}_{l'\uparrow}, b_{t'\uparrow}\})$$
$$= -\sum_{k,l} \gamma_{ll'} a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} \delta_{l't'} \delta_{\uparrow\uparrow} b_{l\uparrow}$$
$$= -\sum_{k,l} \gamma_{ll'} a^{\dagger}_{k\uparrow} a^{\dagger}_{-k\downarrow} b_{l\uparrow}, \tag{3.2.68}$$

and

$$[b^{\dagger}_{l'\uparrow}, \sum_{k,t,l'} \gamma^{*}_{tl'} a_{-k\downarrow} a_{k\uparrow} b^{\dagger}_{t\uparrow} b^{\dagger}_{t'\uparrow}] = \sum_{kt,l'} \gamma^{*}_{tl'}[b^{\dagger}_{l'\uparrow}, a_{-k\downarrow} a_{k\uparrow} b^{\dagger}_{t\uparrow} b^{\dagger}_{t'\uparrow}]$$
$$= \sum_{k,t,l'} \gamma^{*}_{tl'}([b^{\dagger}_{l'\uparrow}, a_{-k\downarrow} a_{k\uparrow}] b^{\dagger}_{t\uparrow} b^{\dagger}_{t'\uparrow} + a_{-k\downarrow} a_{k\uparrow}[b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\uparrow} b^{\dagger}_{t'\uparrow}])$$
$$= \sum_{k,t,l} \gamma^{*}_{tl'} a_{-k\downarrow} a_{k\uparrow}[b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\uparrow} b^{\dagger}_{t'\uparrow}]$$
$$= \sum_{k,t,l} \gamma^{*}_{tl'} a_{-k\downarrow} a_{k\uparrow}(\{b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t\uparrow}\} b^{\dagger}_{t'\uparrow} - b^{\dagger}_{t\uparrow}\{b^{\dagger}_{l'\uparrow}, b^{\dagger}_{t'\uparrow}\})$$
$$= 0. \tag{3.2.69}$$

Substituting Eqs. 3.2.65, 3.2.66 and 3.2.68 into Eq. 3.2.63, yeilds

$$\omega \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg = -\epsilon_{l'} \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg -U < b^{\dagger}_{l'\uparrow} b^{\dagger}_{l'\downarrow} > b_{l'\downarrow} - \sum_{k,l} \gamma_{ll'} < a^{\dagger}_{k\uparrow}, a^{\dagger}_{-k\downarrow} >\ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg$$
$$= -\epsilon_{l'} \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg -U < n_{l'\downarrow} >\ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg$$
$$-\Delta_l \ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg . \tag{3.2.70}$$

Solving for $\ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg$, we obtain

$$\ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg = \frac{-\Delta_l}{(\omega + \epsilon_{l'} + U < n_{l'\downarrow} >)} \ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg . \tag{3.2.71}$$

From Eqs. 3.2.62 and 3.2.71, we obtain

$$\ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg = \frac{-\Delta_l}{\omega^2 - (\epsilon_l + U < n_{l\downarrow} >)^2 - \Delta_l^2}. \tag{3.2.72}$$

$$\ll b_{l\uparrow}, b^{\dagger}_{l\uparrow} \gg = \frac{(\omega + \epsilon_l + U < n_{l'\downarrow} >)}{\omega^2 - (\epsilon_l + U < n_{l\downarrow} >)^2 - \Delta_l^2} \tag{3.2.73}$$

Equations that show the correlation between conduction and mobile electrons

The magnetic order parameter is defined as

$$\eta = -\frac{\alpha}{\beta}\sum_{k,n} \ll b^{\dagger}_{l'\uparrow}, b^{\dagger}_{l\uparrow} \gg . \tag{3.2.74}$$

Taking Eq. 3.2.72 into Eq. 3.2.74

$$\eta = -\frac{\alpha}{\beta}\sum_{k,n} \frac{\Delta_l}{\omega^2 - (\epsilon_l + U < n_{l\downarrow} >)^2 - \Delta_l^2}. \tag{3.2.75}$$

Changing the sum into an integral by introducing the density of state $N(\epsilon)$, $\sum_k \to \int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon)$. then

$$\eta = -\frac{\alpha}{\beta}\sum_{n}\int_{-\varepsilon_F}^{\infty} d\epsilon N(\epsilon) \frac{\Delta_l}{\omega^2 - (\epsilon_l + U < n_{l\downarrow} >)^2 - \Delta_l^2}. \tag{3.2.76}$$

Assuming the density of state is constant in the effective attraction region

$$\begin{aligned}\eta = \quad & -2\frac{\alpha}{\beta}N(o)\sum_{n}\int_{0}^{\hbar\omega_b} d\epsilon \frac{\Delta_l}{\omega^2 - (\epsilon_l + U < n_{l\downarrow} >)^2 - \Delta_l^2} \\ & = -2\frac{\alpha}{\beta}N(o)\sum_{n}\int_{0}^{\hbar\omega_b} d\tilde{\epsilon} \frac{\Delta_l}{\omega^2 - \tilde{\epsilon}^2 - \Delta_l^2},\end{aligned} \tag{3.2.77}$$

where $\tilde{\epsilon} = \epsilon + U < n_{l\downarrow} >$ and $d\tilde{\epsilon} = d\epsilon$. Using $\omega \to i\omega_n$, the Metesubara frequency $\omega_n = (2n+1)\frac{\pi}{\beta}$, the above equation becomes

$$\eta = -2N(o)\alpha\beta\sum_{n}\int_{0}^{\hbar\omega_b} d\tilde{\epsilon} \frac{\Delta_l}{(2n+1)^2\pi^2 + E^2\beta^2}, \tag{3.2.78}$$

where E $= \tilde{\epsilon}^2 - \Delta_l^2$. Using the relation $\frac{1}{2x}tanh(\frac{x}{2}) = \sum_n \frac{1}{(2n+1)^2\pi^2 + x^2}$, the above equation becomes

$$\eta = -2N(o)\alpha\beta\int_{0}^{\hbar\omega_b} d\tilde{\epsilon}\frac{\Delta_l}{2\beta E}tanh(\frac{\beta E}{2}). \tag{3.2.79}$$

This implies

$$\eta = -\lambda_l\int_{0}^{\hbar\omega_b} d\tilde{\epsilon}\Delta_l \frac{1}{\sqrt{\tilde{\epsilon}^2 + \Delta_l^2}} tanh(\frac{\beta\sqrt{\tilde{\epsilon}^2 + \Delta_l^2}}{2}). \tag{3.2.80}$$

One can also write the above equation as

$$\eta = -\lambda_l \Delta_l \int_0^{\hbar\omega_b} \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \frac{d\tilde{\epsilon}}{\omega_n^2 + \tilde{\epsilon}^2 + \Delta_l^2}. \tag{3.2.81}$$

Using the Lapplace transformation, and Metusbara frequency $\omega_n = (2n+1)\frac{\pi}{\beta}$, the above equation becomes

$$\begin{aligned} -\frac{\eta}{\lambda\Delta_l} = & \int_0^{\hbar\omega_b} d\tilde{\epsilon}\frac{2}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \tilde{\epsilon}^2} \\ & - \int_0^{\hbar\omega_b} \Delta_l^2 d\tilde{\epsilon}\frac{2}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{(\omega_n^2 + \tilde{\epsilon}^2)^2} + ... \\ & = \int_0^{\hbar\omega_b} \frac{1}{\tilde{\epsilon}} d\tilde{\epsilon} tanh(\frac{\beta\tilde{\epsilon}}{2}) \\ & - \int_0^{\hbar\omega_b} d\tilde{\epsilon}\Delta_l^2 \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{(((2n+1)\frac{\pi}{\beta})^2 + \tilde{\epsilon}^2)^2}. \end{aligned} \tag{3.2.82}$$

Let $a^2 = ((2n+1)\frac{\pi}{\beta})^2$

$$-\frac{\eta}{\lambda\Delta_l} = \int_0^{\hbar\omega_b} \frac{1}{\tilde{\epsilon}} d\tilde{\epsilon} tanh(\frac{\beta\tilde{\epsilon}}{2}) - \int_0^{\hbar\omega_b} d\tilde{\epsilon}\Delta_l^2 \frac{2}{\beta} 2 \sum_0^{\infty} \frac{dx}{a^4(1+x^2)^2} + ..., \tag{3.2.83}$$

where $x^2 = \frac{a^2}{\tilde{\epsilon}^2}$.

$$-\frac{\eta}{\lambda\Delta_l} = I_1 + I_2. \tag{3.2.84}$$

$$I_1 = \int_0^{\hbar\omega_b} d\tilde{\epsilon}\frac{1}{\tilde{\epsilon}} tanh(\frac{\beta\tilde{\epsilon}}{2}). \tag{3.2.85}$$

Let$y = \frac{\beta\tilde{\epsilon}}{2}$, then $dy = \frac{\beta}{2} d\tilde{\epsilon}$

$$\begin{aligned} I_1 = & \int_0^y dy \frac{tanhy}{y} \\ & = lnytanhy - \int_0^y dy \frac{lny}{cosh^2 y}. \end{aligned} \tag{3.2.86}$$

For low temperature, $tanh(\frac{\tilde{\epsilon}}{2k_B T}) \to 1$, which implies

$$\begin{aligned} I_1 = & ln(\frac{\tilde{\epsilon}}{2k_B T_m}) - ln(\frac{\pi}{4}) \\ & = ln 1.14 \frac{\hbar\omega_b}{k_b T_m}. \end{aligned} \tag{3.2.87}$$

$$\begin{aligned}
I_2 = \quad & -\int_0^{\hbar\omega_b} \Delta_l^2 d\check{\epsilon} \frac{2}{\beta} 2 \sum_{n=0}^{\infty} \frac{1}{a^4(1+x^2)} dx + .. \\
& = -4\Delta_l^2 \sum_{n=0}^{\infty} (\frac{\beta}{\pi(2n+1)^2})^2 \int_0^{\hbar\omega_b} \frac{dx}{a^4(1+x^2)} + .. \\
& = -\frac{4}{\pi^3}\beta^2 \frac{7}{8}\zeta(3)\frac{\pi}{4} \\
& = -(\frac{\Delta_l^2}{\pi k_B T_c})^2 \frac{8.414}{8}, \qquad (3.2.88)
\end{aligned}$$

where $\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ and $\sum_0^{\infty} \frac{1}{(2n+1)^p} = (1-2^{-p})\zeta(p)$, here p =3 ,$\zeta(3) = 1.202$,ζ is the zeta function.

$$\begin{aligned}
\eta \approx \quad & -\lambda_l \Delta_l (ln 1.14 \frac{\hbar\omega_b}{k_B T_m} - \Delta_l^2 (\frac{1}{\pi k_B T_m})^2 \frac{8.414}{8}) \\
& \approx -\lambda_l \Delta_l ln 1.14 \frac{\hbar\omega_b}{k_B T_m} + \lambda_l \Delta_l^3 (\frac{1}{\pi k_B T_m})^2 \frac{8.414}{8}. \qquad (3.2.89)
\end{aligned}$$

Since Δ_l is very small,we can ignore the second term

$$\eta \approx -\lambda_l \Delta_l ln 1.14 \frac{\hbar\omega_b}{k_B T_m}, \qquad (3.2.90)$$

which implies

$$k_B T_m = 1.14 \hbar\omega_b exp(\frac{\eta}{\lambda_l \Delta_l}). \qquad (3.2.91)$$

3.2.3 Pure superconducting system

For pure superconducting (when magnetic effect is zero) we can ignore the magnetic parameter (η) term in Eq. (3.2.36) and one can calculate the following result.
Using Eq. (3.2.36),as $T \to 0, \eta \to 0$ and $tanh(\frac{\beta E}{2}) \to 1$

$$\frac{\Delta}{\lambda} = \int_0^{\hbar\omega_b} d\epsilon \frac{\Delta}{\sqrt{(\epsilon_k^2 + \Delta^2)}}. \qquad (3.2.92)$$

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\sqrt{(\epsilon_k^2 + \Delta^2)}} = sinh^{-1}(\frac{\hbar\omega_b}{\Delta}). \qquad (3.2.93)$$

As $T \rightarrow T_c$, for $\eta = 0$,

$$\begin{aligned} \frac{1}{\lambda} = & \int_0^{\hbar\omega_b} d\epsilon \frac{tanh(\frac{\beta\tilde{\epsilon}}{2})}{\epsilon} \\ &= \int_0^{\hbar\omega_b} dy \frac{tanhy}{y} \\ &= lnytanhy|_0^y - \int_0^y \frac{lny}{cosh^2y} dy, \end{aligned} \tag{3.2.94}$$

where $y = \frac{\beta\epsilon}{2}$.

For low temperature, $tanh(\frac{\hbar\omega_b}{2k_BT}) \rightarrow 1$,

then

$$\begin{aligned} \frac{1}{\lambda} = & ln\frac{\beta\hbar\omega_b}{2} - ln(\frac{\pi}{4}) \\ &= ln1.14\frac{\hbar\omega_b}{k_BT_c}, \end{aligned} \tag{3.2.95}$$

which implies

$$k_BT_c = 1.14\hbar\omega_b e^{-\frac{1}{\lambda}}. \tag{3.2.96}$$

In order to obtain temperature dependence of energy gap, we follow the same technique to solve the integral.

$$\begin{aligned} \frac{1}{\lambda} = & \int_0^{\hbar\omega_b} d\epsilon \frac{1}{\sqrt{\epsilon^2+\Delta^2}} tanh(\frac{\beta\sqrt{\epsilon^2+\Delta^2}}{2}) \\ &= ln1.14\frac{\hbar\omega_b}{k_BT} - \Delta^2(\frac{1}{\pi k_BT})^2\frac{8.414}{8} + ... \end{aligned} \tag{3.2.97}$$

But from (BCS)at $T = T_c$,$\frac{1}{\lambda} = ln1.14\frac{\hbar\omega_b}{k_BT_c}$ and $\omega_b = \omega_D$, from this we obtain

$$ln(\frac{T}{T_c}) = -\Delta^2(\frac{1}{\pi k_BT})^2\frac{8.414}{8}. \tag{3.2.98}$$

Using ,$ln(1-x) = -x - \frac{(1-x)^2}{2} + ...$

$$\begin{aligned} ln(\frac{T}{T_c}) = & ln(1-(1-\frac{T}{T_c})) \\ &= -(1-\frac{T}{T_c}) - \frac{(1-\frac{T}{T_c})^2}{2} + ..., \end{aligned} \tag{3.2.99}$$

which implies that

$$-(1-\frac{T}{T_c}) \approx -\Delta^2(\frac{1}{\pi k_B T})^2\frac{8.414}{8}, \tag{3.2.100}$$

then

$$\Delta(T) = 3.06 k_B T_c (1-\frac{T}{T_c})^{\frac{1}{2}}. \tag{3.2.101}$$

Chapter 4

Result and Discussion

This part of our study deals with the investigation of the effect of magnetic ordering on the superconductor parameter (Δ), on the transition temperature (T_c) and on the ferromagnetic temperature (T_c). For this purpose, we use the model Hamiltonian of the system and the retarded double time Green's function technique to obtain an expression for superconducting order parameter (Δ), ferromagnetic order parameter (η), superconducting temperature (T_c) and ferromagnetic ordering temperature (T_c)
For pure superconductor i.e when the magnetic order parameter ($\eta = 0$), the expression we obtain Eq. 3.2.96 is in agreement with the BCS expression. As we have illustrated in Fig. 4.1 the superconducting order parameter (Δ) for URhGe decreases with increasing temperature and even vanishes at a critical temperature $T_c = 0.25K$.It is also shown from Fig. 4.3, the superconducting temperature T_c decreases with increasing magnetic order η for URhGe. However, the magnetic order temperature T_m increases with increasing η for this compound. By merging Figs. 4.2 and 4.3, we have demonstrated the the region where both superconductivity and ferromagnetism can coexists in Fig. 4.4.

In similar to URhGe, the superconducting order parameter Δ decreases with increasing temperature and even vanishes at a critical temperature $T_c = 0.6K$ for the compound UCoGe. However, the superconducting temperature decreases with increasing magnetic order parameter η for this compound. But,the magnetic order temperature T_c increases

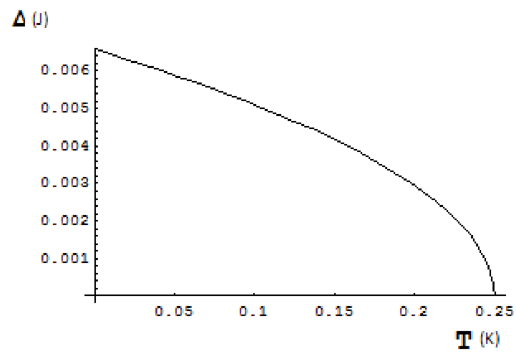

Figure 4.1: Superconducting order parameter vs temperature for URhGe.

with increasing magnetic order parameter η.By merging FigS. (4.6) and (4.7) to gather,we have demonstrated the region where both phenomenas can coexist. As we compared Figs. (4.4) and (4.8) for URhGe and UCoGe,respectively,both graphs have similar character although the interval where coexistence uppers are different.The unit of temperature in the graphs bellow is kelvin (K) and mili electron volt (mev) for magneting order.

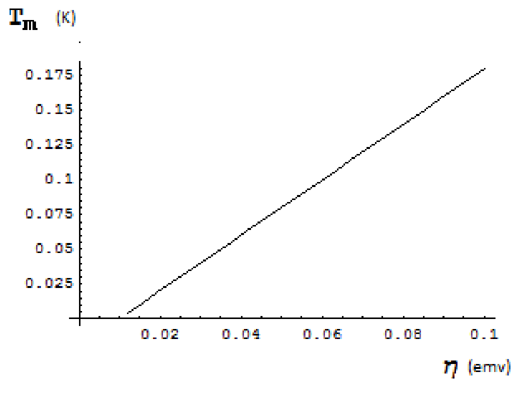

Figure 4.2: Magnetic ordering temperature vs magnetic order parameter for URhGe.

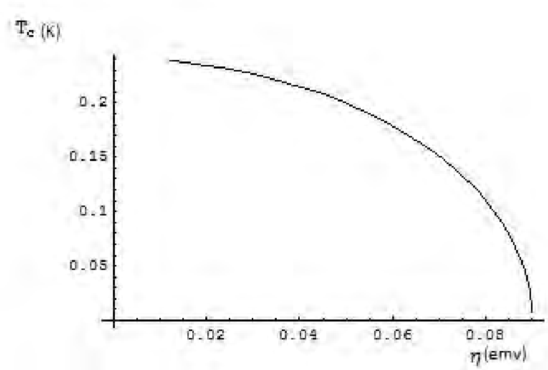

Figure 4.3: Superconducting temperature vs magnetic order parameter for URhGe .

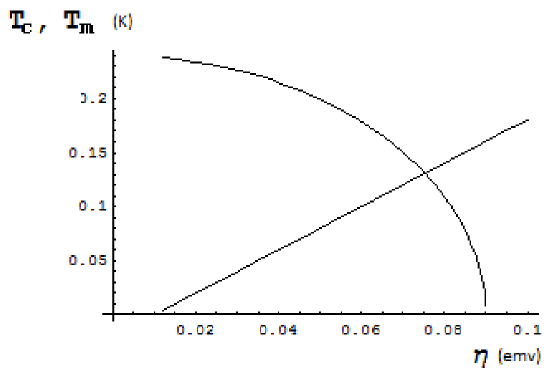

Figure 4.4: Superconducting temperature,magnetic ordering temperature vs magnetic order parameter for URhGe.

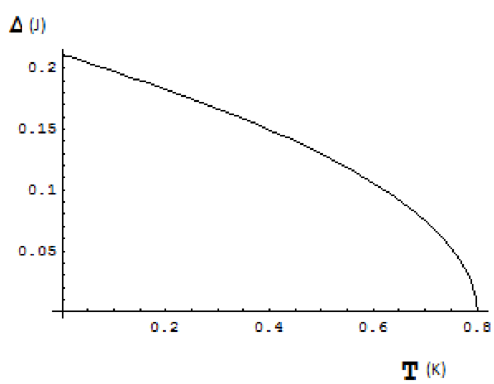

Figure 4.5: Superconducting order parameter vs temperature for UCoGe.

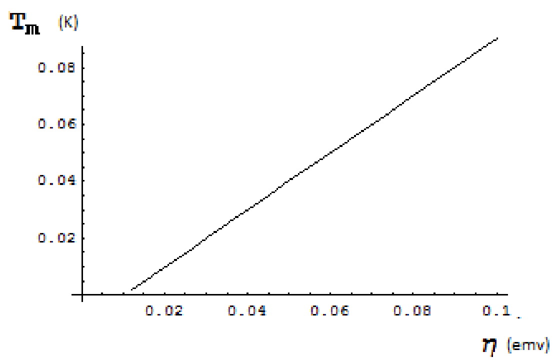

Figure 4.6: Magnetic ordering temperature vs magnetic order parameter for UCoGe.

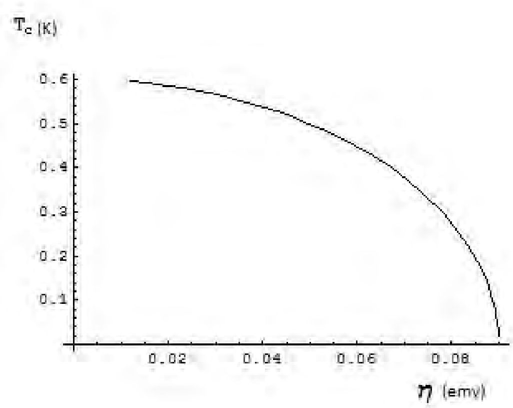

Figure 4.7: Superconducting temperature vs magnetic order parameter for UCoGe.

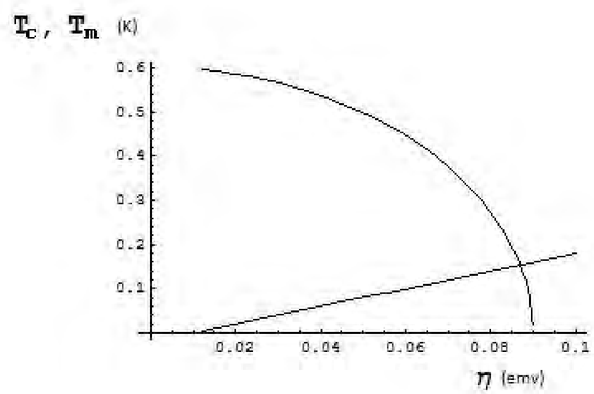

Figure 4.8: Superconducting temperature,magnetic ordering temperature vs magnetic order parameter for UCoGe.

Chapter 5

Conclusion

In the present work we have introduced the basic concept of superconductivity and ferromagnetism like the Meissner effect,types of superconductivity and ferromagnetic ordering parameter. Though superconductivity and magnetism are generally antagonistic to each other because of the effect of magnetic field on the electron's charge and spin for unconventional superconductors, however,we have demonstrated the coexistence of superconductivity and ferro magnetism in the inter metallic compounds URhGe and UCoGe.In U-based compounds like URhGe and UCoGe,the 5f electrons of U-atom form both superconductivity and ferro magnetism order. The coexistence of superconductivity and ferro magnetism in these compounds can be understood in terms of spin fluctuation model. As we have shown in figures,the magnetic ordering temperature increases with increasing of magnetic order parameter. But the superconducting temperature decreases with increasing the magnetic temperature T

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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Place and time of submission: Addis Ababa University, June 2010

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