Statistical Mechanics in an Expanding Maximally Symmetric Universe

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By

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Abstract

The universe was radiation dominated at red shifts higher than $Z_eq \cong 3.9x10^4\Omega h^2$. In the radiation dominated phase, the temperature will be greater $T_eq \cong 9.2\Omega h^2 eV$ and will be increasing as $(1+Z)$. As we go to earlier than phases of the universe the radiation will produce particle-antiparticle pairs of different kinds. These elementary particles will be interacting with each other via different processes which will try to maintain statistical equilibrium between the particles. In order to study the early phases of the universe, one would like to understand how the material content of the universe changes as it expands.

Introduction

Most of the universe is seen by us in the electromagnetic spectrum. The fact arises from a redshift in the spectra of distant galaxies mainly interprets the universe expansion as the scale factor today is larger than when the photons were emitted by the observed galaxies which was discovered by **Edwin P. Hubble** (1929) [6]. Our present understanding of the laws of physics allows us to talk about the earliest moment at the so-called Planck time $t_{pl} \sim 10^{-43}$ when the temperature of the universe at the Planck scale $T \sim m_{pl} \equiv 1.22 \times 10^{19} GeV$ [1].

The temperature at very early times had been so high that stable matter as we know today would not have been existed. The best way of viewing the early universe is therefore as a hot 'primordial soup'of radiation (photons) and elementary particles ilke **baryons** (neutrons and protons) and **leptons** (electrons, muons, neutrinos, and their antiparticles). Each particle has antiparticle associate with it. During collision, they annihilate each other and decay into photons. If too much radiation is present, some of it will decay back into particles. In this way, the distribution of particles in the early universe reaches thermal equilibrium.

The cosmic microwave background (**CMB**) photons, first observed by A. A. Penzias and R. W. Wilson (1965), found to be isotropic, at about one part in 10^4 [7]. It can be detected in a receiver tuned to microwaves. Of course, we know the universe is in fact not smooth (the same at all points); because, it contains galaxies, stars, planets, etc. This assumption is self-consistent in that early times the universe can be very smooth indeed, with gravitational growth gradually forming that we see today.

Statistical mechanics allows us trying to understand the laws of thermodynamics and the properties of matter arise from a statistical analysis of the collective behaviors of many particles. In addition, particles state of equilibrium is characterized completely by the temperature and thermodynamic chemical potential. This fact that the early universe was in thermodynamic equilibrium leads to a tremendous simplification compared to the present epoch.

It is the statistical and thermal properties that such plasma should have and the role that causal horizons play in the final outcomes of the early universe expansion. For instance, of crucial importance is the time at which certain particles left **out of equilibrium** with the plasma. Hence, one can trace the evolution of the universe from its origin till today.

The importance of this thesis is to analyze the influence of the statistical description in the standard picture we have for the evolution of the early universe and its layout is as follows. The first chapter starts by discussing about the nature of an expanding maximally symmetric universe and review about the universe evolution. Photons statistics and basic evolution equations presented in chapter two. And then, both chapters (3 and 4) include the derivation of thermodynamic quantities in and out of equilibrium respectively on the basis of statistical mechanics. Finally, the last chapter is accounted for discussion and conclusion.

Note that in this thesis, we will always use natural units where $\hbar = \kappa_B = c = 1$ unless the contrary is implied by the specified units.

Chapter 1

Expanding Universe

The most important feature of our universe is that it expands. This expansion is according to the Hubble law, and this scientific argument help to realizing that it would have been smaller in the past.

Obviously, to describe the entire universe, with all its parts that are not accessible to us, we need a hypothesis which will allow us to extrapolate from what we know about our **"space-time"** to the entire universe.

1.1 Maximally Symmetric Universe

In order to define and characterize maximally symmetric spaces, we need to obtain first the information of **killing vectors**.

A metric $g_{\mu\nu}(x)$ is said to form invariant under a given coordinate transformation $x \to x'$ when $g'_{\mu\nu}(x')$ is the same function of its argument x'^{μ} as the original metric $g_{\mu\nu}$ was of its argument x^{μ}, i.e $g'_{\mu\nu}(y) = g_{\mu\nu}(y)$. If the components of the metric are all independent of a particular coordinate, say y, then $V = \partial y$ is a killing vector, i.e., ∂_y $g_{\mu\nu}$ $\forall g_{\mu\nu} \Longrightarrow V = \partial y$. Killing vectors $K_\mu(x)$ are determined by the values $K_\mu(x_0)$ and $\nabla_\mu K_\nu(x_0)$ at a single point x_0 [11].

Maximally symmetric space is a space with a metric with the maximal number $n(n+1)/2$ of killing vectors.

There are only three species of maximally symmetric spaces (for any n), namely flat space $\Re^n$, the sphere S^n, and its negatively curved counterpart i.e the n-dimensional pseudosphere or hyperboloid we will call H^n [4].

Homogeneous space has infinitesimal isometries that carry any given point x_0 into any other point in its immediate neighborhood for arbitrary killing vectors $K_\mu(x_0)$. As the result, n-dimensional space admits n translational killing vectors. A space to be **isotropic** at a point x_0 if it has isometries that leave the given point x_0 fixed and such that they can rotate any vector at x_0 into any other vector at x_0. Thus, the metric must admit killing vectors such that $K_\mu(x_0) = 0$ but $\nabla_\mu K_\nu(x_0)$ is an arbitrary anti-symmetric matrix. This means that the n-dimensional space admits $n(n-1)/2$ translational killing vectors [11].

Generally, in an n-dimensional space-time there can be at most n linearly independent vectors $K_\mu(x_0)$ at a point, and at most $n(n-1)/2$ independent anti- symmetric matrices $\nabla_\mu K_\nu(x_0)$, we reach the conclusion that there are at most $n(n+1)/2$ independent killing vectors in an n-dimensional space [11]. This result allows that homogeneous and isotropic space is maximally symmetric. In addition, a space that is isotropic is also homogeneous and maximally symmetric [4]. In practice, this characterization of a maximally symmetric space which is easiest to use because it requires consideration of only one type of symmetries, namely rotational symmetries.

Einstein (1917) and others, in the early days of relativistic cosmology, justify a procedure in modeling the universe on the basis of observational data. According to this model, the universe appears to be spatially homogeneous and isotropic on the large-scale $\gtrsim 10^8$ parsecs, i.e., $\gtrsim 3 \times 10^8$ light years [3].

The fundamental **cosmological principle** says that "no point in the universe is preferred". Thus, the universe needs to be **Isotropic** (i.e. look the same in all directions, rotationally symmetric) and **Homogeneous** (the same at all points, symmetric under translations) [2]. This indicates that the universe is expanding uniformly and isotropically-all galaxies see the same things in all directions.

As the universe expands, the wavelengths of light rays are stretched out in proportion to the distance between the galaxies. Even though the sphere (universe) expands, the $X's$ (galaxies) remain at the same spatial coordinates. These trajectories are geodesics and hence the $X's$ (galaxies) can be considered to be in free fall as shown in the fig.(1.1) below. It also shows that it is the number density per unit coordinate volume that is conserved, not the density per unit proper volume.

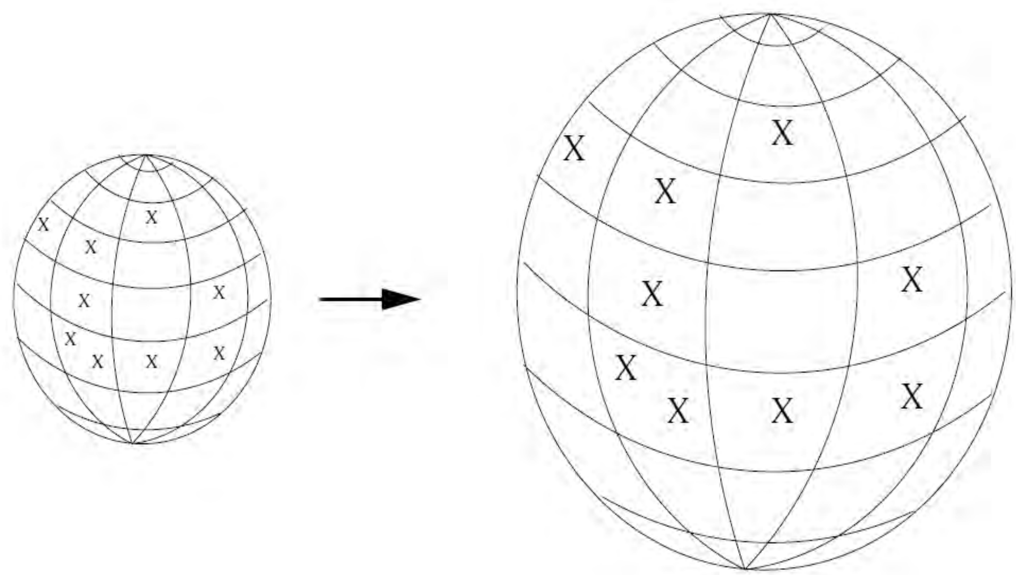

Figure 1.1: Illustration of Homogeneous and Isotropic Expansion in a Comoving Coordinate System

Here we have used the spatial coordinates that move with the particles. In other words, a given particle, which is at rest with respect to the expansion will have constant coordinates; and the expansion of the universe is reflected in the change of the scale of the universe as a whole. This coordinate system is called **comoving system**. Hence, the coordinate system is falling with comoving, and the proper time along such geodesics coincides with the coordinate time, $d\tau = dt$ [11].

Our aim to make maximal use of the symmetries that cosmological model should have is to find a simple picture for the metric.

Consider the metric of a spatially flat universe $ds^2 = dt^2 - a(t)^2\mathbf{x}^2$, where the scale factor $a(t)$ gives physical size to the spatial coordinates $\mathbf{x}^2$, and the expansion is nothing but a change of scale with time. In the context of a Friedmann-Robertson-Walker (**FRW**) metric, the observed universe expansion is characterized by Hubble rate of expansion, whose value today is denoted by vt, and it can be written here in terms of the invariant geodesic distance $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in four dimensions, in which case the matrix $g_{\mu\nu}$ is called the metric tensor [6].

A **Friedmann universe** is one kind of maximally symmetric universe. One of the distinct features of the Friedmann universe is that they give three possible geometries for the universe, which would be represented by the following line element in three dimensions: $ds^2 = dt^2 - a(t)^2\left(\dfrac{dr^2}{1-Kr^2} + r^2 d\Omega\right)$ has the required symmetries [5]. It is useful to define a scale factor $a(t)$ determining the overall scale of the universe. Here $d\Omega = d\theta^2 + sin^2\theta d\phi^2$ is the two dimensional angular volume element, and $K = 0, \pm 1$. The angles ϕ and θ are the usual azimuthal and polar angles of spherical cooridinates, with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$.

Aparticularly useful quantity to define from the scale factor is the Hubble parameter (sometimes called the Hubble constant), given by $H \equiv \dfrac{\dot{a}}{a}$. The Hubble parameter relates how fast the most distant galaxies are receding from us their distance from us via Hubble's law, $v \simeq Hd$. This is the relationship that was discovered by Edwin Hubble, and has been verified to high accuracy by modern observational methods [13].

Geometrically, the parmeter K describes the curvature of the spatial sections (slices at constant cosmic time). The case K=1 corresponds to a **closed** space-time with a Spherical spatial geometry, and the case K=0 corresponds to an infinite (**flat**) space-time with Eucledian spatial geometry. Finally, the case K=-1 corresponds to an **open** space-time with Hyperbolic spatial geometry [2]. These three qualitatively distinict values for the three curvature can be summarized as: open($K < 0$), flat ($K = 0$) and closed ($K > 0$). Note that if $K = 0$ and $a(t)$=1 we have the **Minkowski metric**.

The three possibile types of expanding universes are called open, flat, closed universes. If the universe were **open**, it would expand forever. If the universe were **flat**, it would also expand forever, but the expansion rate would slow to zero after an infinite amount of time. If the universe were **closed**, it would eventually stop expanding and recollapse on itself, possibly leading to another big bang. In all three cases, the expansion slows, and the force that causes the slowing is gravity [13].

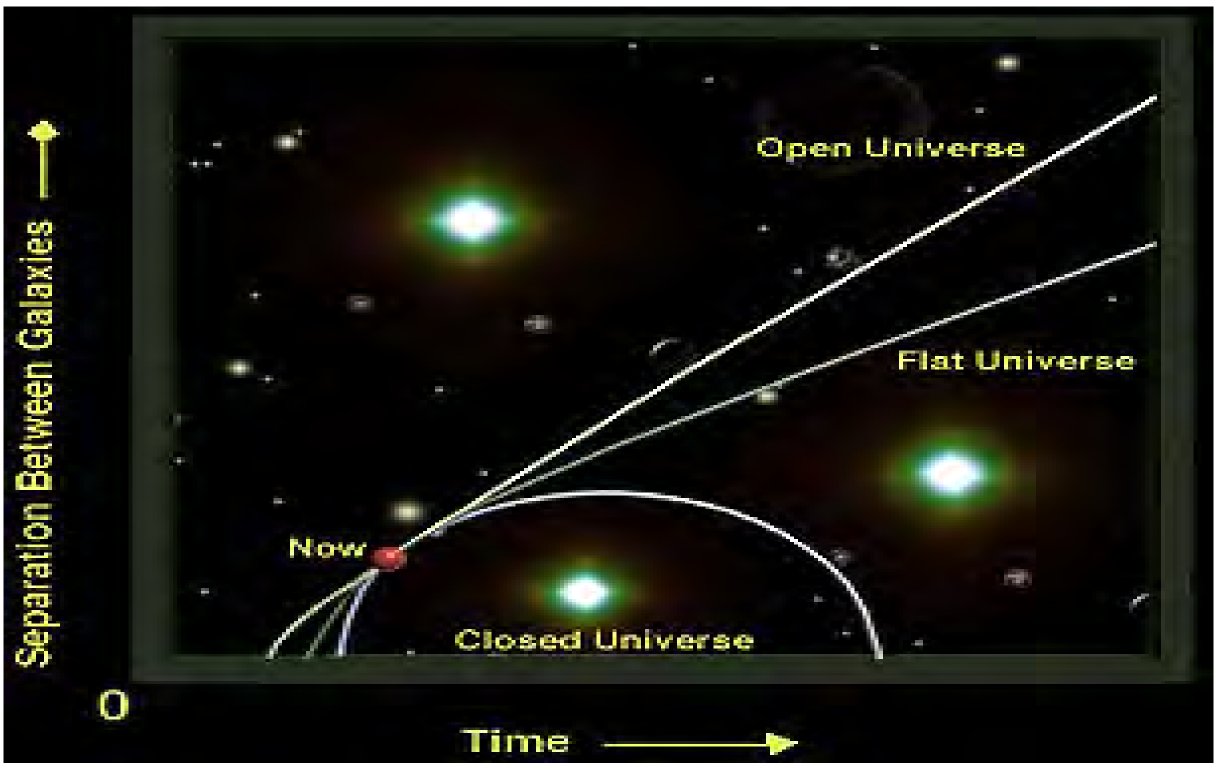

Figure 1.2: Properties of the Expanding Friedmann Maximally Symmetric Universe

1.2 Universe Evolution

The evolution of our zero-order cosmological model is highly dependent on the equation of state $P(\rho)$; and that equation of state, in turn, depends on the types of matter and fields that fill the universe, i.e., the universe's constituents: Cold matter, Radiation, and Dark energy.

Cold matter has an idealized equation of state $P_M = 0$. It includes the baryonic matter of which people, planets, stars, galaxies, and intergalactic gas are made, as well as so-called cold, dark matter which is known to exist in profusion and might be predominantly fundamental particles (axions or neutralinos). **Radiation**, i.e. material with equation of state $P_R = \rho_R/3$. This contains the CMB (primordial photons), primordial gravitons, primordial neutrinos when their temperatures exceed their rest masses, and other finite-rest mass particles when the temperature is sufficiently high (very early in the universe). The observational evidence shows that **Dark energy** is present today in profusion. We do not yet know for sure its nature or its equation of state, but the most likely candidate is a nonzero stress-energy tensor associated with the vacuum, for which the equation of state is $P_\wedge = -\rho_\wedge$ [3].
There are several strong evidences, which show the possibility of an evolution of universe. According to more accepted views, I try to mention its evolution as follows.

The universe originated at the **Planck era** $(10^{19}GeV, 10^{-43}s)$ from a quantum gravity fluctuation. Then after, the universe reached the Grand Unified Theories (**GUT**) era $(10^{16}GeV, 10^{-35}s)$. The huge energy density of the inflation field was converted into particles, which soon thermalized and became the origin of the hot **Big Bang**. Since then, the universe became radiation dominated. It is a matter of speculation whether **Baryogenesis** could have occurred at energies as low as the electroweak scale $(100GeV, 10^{-10}s)$. As the universe cooled down, it may had gone through the **Quark-Gluon** phase transition $(100MeV, 10^{-5}s)$ when baryons (mainly protons and neutrons) formed from their constituent quarks [6].

The furthest window we have on the ealy universe at the moment is that of primordial **Nucleosynthesis** $(1-0.1MeV, 1s-3min)$, when protons and neutrons were cold enough that bound systems could form, giving rise to the lightest elements, soon after **Neutrino decoupling** process. Afterwards $(0.5MeV, 1min)$, electron-positron annihilation occurs and all their energy goes into photons.

Matter and radiation have equal energy densities at about $1eV, \sim 10^5 yr$. Immediately afterwards ($0.3eV, \sim 3x10^5 yr$), electrons become bound to nuclei to form atoms; and photons decoupled from the plasma travelling freely since then [6]. They have cooled further as the universe has expanded and now have a temperature of about $2.7K$ and a wavelength in the microwave region [7]. The small inhomogeneity generated during inflation have grown, via gravitational collapse, to become galaxies, clusters of galaxies, and super clusters characterized by structure formation ($\sim 1 - 10Gyr$). Finally ($3K, 13Gyr$), the Sun, the Earth, and biological life originated from previous generations of stars, and from a primordial soup of organic compounds, respectively [6]. Correspondingly, the extrapolation backwards suggest age of modern universe on the order of $(10 - 20) \times 10^9$ years [8].

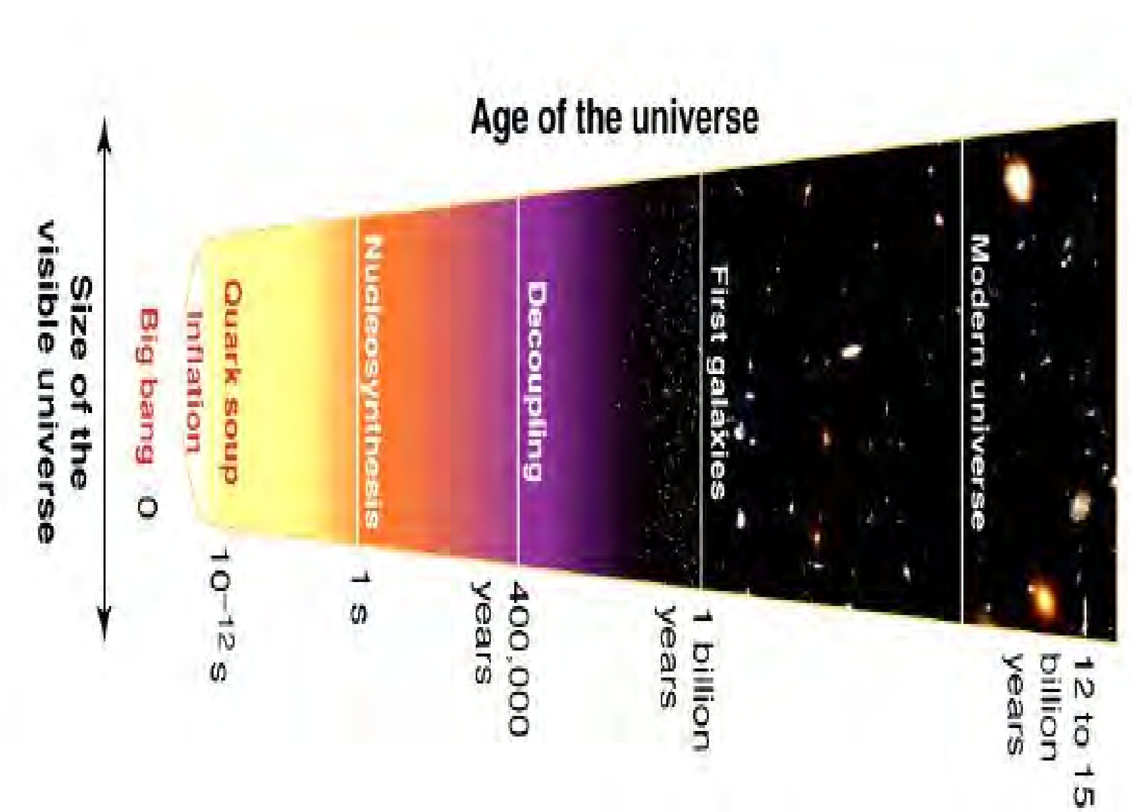

Figure 1.3: Thermal Evolution of the Universe

Chapter 2

Photon Statistics and Evolution Equations

In this chapter, we shall determine the photons energy density with black body distribution and shall derive the evolution equations in which the expansion changes with time.

2.1 Photon Statistics

The photons were in thermal equilibrium with other forms of matter in the early universe; and thus had a planckian spectrum with a very accurate black body distribution.

A photon of frequency ω has energy $\hbar\omega$ and the total energy of the system can be given by:

$$E_{\{n_{\mathbf{p},\vec{\varepsilon}}\}} = \sum_{\mathbf{p},\vec{\varepsilon}} \hbar\omega n_{\mathbf{p},\vec{\varepsilon}}, \tag{2.1.1}$$

where $\{n_{\mathbf{p},\vec{\varepsilon}}\} \equiv$ set of occupation numbers $(n_1, n_2, ...)$ of momemtum $\mathbf{p}$ and polarization $\vec{\varepsilon}$, which tell us how many particles are in each single-particle eigen state and $\hbar \equiv$ planck's constant.

When evaluating the partition function, we must be sure to do the sum over the allowed occupation numbers for the type of particles being considered.

According to the definition of partition function, the total number of states of the system [11]

$$Q = \sum_{\{n_{\mathbf{p},\vec{\varepsilon}}\}} exp[-\beta E_{\{n_{\mathbf{p},\vec{\varepsilon}}\}}], \tag{2.1.2}$$

where we have defined β (inverse temperature, T) as $\beta = 1/\kappa_B T$ and κ_B is Boltzmann constant. Here we have

$$Q = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}...\sum exp[-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + ...)], \tag{2.1.3}$$

where the sum is over all $n_1, n_2, ...$ If the particles are fermions each $n_{\mathbf{p},\vec{\varepsilon}}$ can either be zero or one, where as if the particles are bosons, each $n_{\mathbf{p},\vec{\varepsilon}}$ can take on any value.

Since **photons** have spin one, they are bosons and their occupation numbers can take on any value. As the result, the partition function is found by carrying out an unrestricted sum over all values of $n_1, n_2, ...$ etc. Thus, for photons, we see that

$$\begin{aligned} Q &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}...\sum exp[-\beta(n_1\hbar\omega + n_2\hbar\omega + ...)] \\ &= \left(\sum_{n_1=0}^{\infty} exp[-\beta n_1\hbar\omega]\right)\left(\sum_{n_2=0}^{\infty} exp[-\beta n_2\hbar\omega]\right)... \\ &= \prod_{\mathbf{p},\vec{\varepsilon}}\left(\sum_{n=0}^{\infty} exp[-\beta n\hbar\omega]\right) \end{aligned} \tag{2.1.4}$$

Each infinite sum is a simple geometric series which may be summed explicitly, giving the result

$$Q = \prod_{\mathbf{p},\vec{\varepsilon}} \frac{1}{1 - e^{-\beta\hbar\omega}} \tag{2.1.5}$$

Further,

$$lnQ = -\sum_{\mathbf{p},\vec{\varepsilon}} ln(1 - e^{-\beta\hbar\omega}) = -2\sum_{\mathbf{p}} ln(1 - e^{-\beta\hbar\omega}), \tag{2.1.6}$$

where factor 2 due to we have two values for polarization for a given value of $\mathbf{p}$.

From the expression (2.1.6) for partition function we shall proceed to find the thermodynamic quantities.

Let us compute the expectation value of the particle number of momentum **p** in a single-particle energy eigenstate when the system is in equilibrium. Although $n_{\mathbf{p}}$ can take on any value for bosons, the probability that $n_{\mathbf{p}}$ takes on different values is given by the Boltzmann distribution, with

$$P(n_{\mathbf{p}}) = \frac{exp(-\beta\hbar\omega n_{\mathbf{p}})}{Q} \tag{2.1.7}$$

The **mean occupation number** of single-particle energy eigenstate is thus

$$\langle n_{\mathbf{p}}\rangle = \sum_{\mathbf{p}} n_{\mathbf{p}} P(n_{\mathbf{p}}) = -\frac{1}{\beta}\frac{\partial \ln Q}{\partial(\hbar\omega)} \tag{2.1.8}$$

and from the result of (2.1.6), we find

$$\begin{aligned}\frac{\partial \ln Q}{\partial(\hbar\omega)} &= \frac{\partial}{\partial(\hbar\omega)}\Big[-2\sum_{\mathbf{p}} ln(1-e^{-\beta\hbar\omega})\Big]\\ &= -\frac{2\beta e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\\ &= -\frac{2\beta}{e^{\beta\hbar\omega}-1}\end{aligned} \tag{2.1.9}$$

Combining these, we see that

$$\langle n_{\mathbf{p}}\rangle = \frac{2}{e^{\beta\hbar\omega}-1} \tag{2.1.10}$$

Correspondingly, we can see the mean energy and pressue.

The **mean energy** is given by

$$U = -\frac{\partial \ln Q}{\partial\beta} \tag{2.1.11}$$

Performing the differentiation, which gives

$$\begin{aligned}\frac{\partial \ln Q}{\partial\beta} &= -\frac{\partial}{\partial\beta}\Big[-2\sum_{\mathbf{p}} ln(1-e^{-\beta\hbar\omega})\Big]\\ &= -\sum_{\mathbf{p}}\hbar\omega\Big\{\frac{2}{e^{\beta\hbar\omega}-1}\Big\}\\ &= -\sum_{\mathbf{p}}\hbar\omega\langle n_{\mathbf{p}}\rangle,\end{aligned} \tag{2.1.12}$$

and which confirms that

$$U = \sum_{\mathbf{p}}\hbar\omega\langle n_{\mathbf{p}}\rangle \tag{2.1.13}$$

The **pressure** is

$$P = \frac{1}{\beta}\frac{\partial \ln Q}{\partial V} \tag{2.1.14}$$

For a photon, the energy depends on the frequency via Planck's relationship $\varepsilon = \hbar\omega$. Inorder to determine the available frequencies, it is only necessary to ask what standing waves can propagate with in the box subject to the boundary conditions.

Using a cube with a side of length L, we see that there must be an integer number of half wave lengths in L for each of the 3-directions. Hence the propagation vector is $\mathbf{k}$, with cartesian components (k_x, k_y, k_z), we must have that

$$k_x = \frac{n_x\pi}{L} \quad , \quad k_y = \frac{n_y\pi}{L} \quad and \quad k_z = \frac{n_z\pi}{L}, \tag{2.1.15}$$

where (n_x, n_y, n_z) are from the set $\{1, 2, 3, ...\}$. Associated with each $\mathbf{k}$, the energy of the photon is $\hbar\omega = \hbar \mid \mathbf{k} \mid c$ where c is the speed of light.

The frequency of a photon, with $V = L^3$ is the volume of the box, is thus

$$\omega = \frac{n\pi c}{V^{1/3}} \tag{2.1.16}$$

Carrying out the diffrentiation, we find that

$$\begin{aligned}
\frac{\partial \ln Q}{\partial V} &= \frac{\partial}{\partial V}\Big[-2\sum_{\mathbf{p}} ln(1 - exp[-\beta\hbar n\pi c V^{-1/3}])\Big] \\
&= \frac{\beta}{3V}\sum_{\mathbf{p}} \hbar\omega\Big\{\frac{2}{e^{\beta\hbar\omega} - 1}\Big\} \\
&= \frac{\beta}{3V}\sum_{\mathbf{p}} \hbar\omega\langle n_p\rangle,
\end{aligned} \tag{2.1.17}$$

and using (2.1.14) in, we obtain

$$P = \frac{1}{3V}\sum_{\mathbf{p}} \hbar\omega\langle n_p\rangle \tag{2.1.18}$$

Comparing (2.1.13) with (2.1.18), we get the equation of state interms of energy density (ρ) as:

$$P = \frac{1}{3}\left(\frac{U}{V}\right) = \frac{1}{3}\rho \tag{2.1.19}$$

Now let us derive the expression of **energy density** in an infinite volume.

In this limit we can replace the summation by integration over a **p** space, hence,

$$\sum_{\mathbf{p}} \longrightarrow \frac{V}{(2\pi\hbar)^3}\int dp^3, \tag{2.1.20}$$

where the allowed number of states in an infinitesimal momentum is

$$\frac{V}{(2\pi\hbar)^3}dp^3$$

In the interval between p and $p + dp$, the momenta element is

$$dp^3 = 4\pi p^2 dp \tag{2.1.21}$$

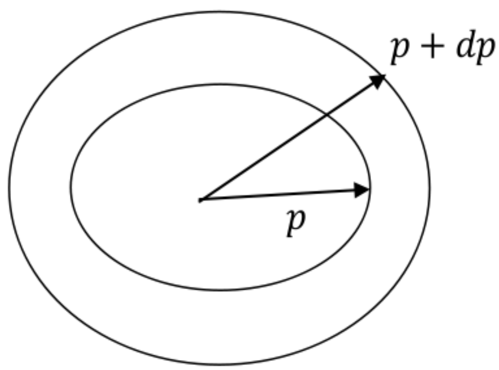

Figure 2.1: Summation Range over Momenta in the Phase Space

By using equations (2.1.13), (2.1.20) and (2.1.21) it is clear to write the energy density with the natural units as:

$$\begin{aligned}\rho &= \frac{2}{(2\pi)^3}\int_0^\infty \frac{\omega}{exp(\omega/T)-1}4\pi p^2 dp \\ &= \frac{1}{\pi^2}\int_0^\infty \frac{\omega^3}{exp(\omega/T)-1}d\omega \\ &= \frac{1}{\pi^2}T^4\int_0^\infty \frac{x^3}{e^x-1}dx \\ &= \frac{1}{\pi^2}T^4\int_0^\infty x^3 e^{-x}(1-e^{-x})^{-1}dx,\end{aligned} \tag{2.1.22}$$

Here we have set $\omega = p$ and $x = \omega/T$ and to solve this integral we have to expand the expression $(1 - e^{-x})^{-1}$ as an infinite geometric series provided that $e^{x} \ll 1$. We can therefore proceed as follows:

$$\begin{aligned} \int_0^\infty x^3 e^{-x}(1-e^{-x})^{-1}dx &= \int_0^\infty x^3 \left(e^{-x} + e^{-2x} + e^{-3x} + ...\right)dx \\ &= \int_0^\infty x^3 \left(\sum_{n=1}^{\infty} e^{-nx}\right)dx \\ &= \sum_{n=1}^{\infty} \int_0^\infty x^3 e^{-nx} dx \end{aligned} \tag{2.1.23}$$

But we have

$$\int_0^\infty x^a e^{-nx} dx = \frac{a!}{n^{a+1}}, \tag{2.1.24}$$

inserting $a = 3$ in, we obtain

$$\int_0^\infty x^3 e^{-x}(1-e^{-x})^{-1}dx = 6\sum_{n=1}^{\infty} \frac{1}{n^4}, \tag{2.1.25}$$

and the sum is by definition the **Riemann Zeta** function of 4,

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{2.1.26}$$

Therefore,

$$\int_0^\infty x^3 e^{-x}(1-e^{-x})^{-1}dx = \frac{\pi^4}{15} \tag{2.1.27}$$

Finally the energy density of photons can be written as

$$\rho = \left(\frac{\pi^2}{15}\right)T^4 \tag{2.1.28}$$

2.2 Evolution Equations

The observational data indicates that the **dark energy** can not have had a very significant influence on the universe expansion at several key epochs in the past, and we shall idealize the universe as containing only cold matter and radaition [3].

The evolution expresses interms of cosmological **redshift** as follows.

The photon's wavelength is redshifted in direct proportional to the expansion factor of the inverse.

$$\frac{\lambda_o}{\lambda} = \frac{a_o}{a} \tag{2.2.1}$$

It is conventional to speak of the redshift z as the ratio of the functional change in wavelength to that when emitted, so

$$z \equiv \frac{\lambda_o - \lambda}{\lambda} = \frac{a_o}{a} - 1 \tag{2.2.2}$$

Note that $\frac{a_o}{a}$ is the ratio of the linear size of the universe today to the size of the univese at some time in the past. Each volume element, comoving with the homogeneous observers, expands in length by $\frac{a_o}{a}$ from then until now, expands in volume by $\left(\frac{a_o}{a}\right)^3$.

To compute the evolution equation in an **energy density**, we would bring the fundamental relation of thermodynamics for a system in equilibrium with negligible chemical potential (no change in particle number) as

$$TdS = dE + PdV, \tag{2.2.3}$$

where S represent the entropy of a given system.

Applying this law to the expansion of universe, having $E = \rho a^3$ in a comoving volume a^3, we get

$$\begin{aligned} dE + PdV &= d(\rho a^3) + Pd(a^3) \\ &= 3a^2\dot{a}\rho + a^3\dot{\rho} + 3Pa^2\dot{a} \\ &= a^3\left[\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P)\right], \end{aligned} \tag{2.2.4}$$

where the dot represents a derivative with respect to the homogeneous observers' proper time t.

From the homogeneity and isotopy nature of early universe, we can deduce that the universe is expanding adiabatically, so that the entropy ($S = sa^3$) is conserved. That is,

$$dS = d(sa^3) = 0 \tag{2.2.5}$$

The use of this result in (2.2.4) yields,

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + P) \tag{2.2.6}$$

Equipped with the general equation of state $P = \omega\rho$, where ω is constant, we can proceed to solve ρ as a function of scale factor a, $\rho(a)$, as:

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(1 + \omega)\rho \tag{2.2.7}$$

Now, recall that

$$\dot{\rho} = \frac{d\rho}{dt} \qquad \textit{and} \qquad \dot{a} = \frac{da}{dt} \tag{2.2.8}$$

Then we can rewrite (2.2.7) as the differential equation

$$\left(\frac{d\rho}{dt}\right) = -3(1+\omega)\left(\frac{\rho}{a}\right)\left(\frac{d\rho}{dt}\right), \tag{2.2.9}$$

which we rearrange to,

$$\left(\frac{d\rho}{\rho}\right) = -3(1+\omega)\left(\frac{da}{a}\right) \tag{2.2.10}$$

Hence, we have to specify the boundary conditions at the present time t_o, and chosen $\rho(t_o) = \rho_o$ and $a(t_o) = a_o$, we find

$$\int_{\rho_o}^{\rho} \frac{d\rho'}{\rho'} = -3(1+\omega)\int_{a_o}^{a} \frac{da'}{a'}, \tag{2.2.11}$$

This result in

$$\ln(\rho/\rho_o) = -3(1+\omega)\ln(a/a_0), \tag{2.2.12}$$

Therefore, the energy densities evolved as

$$\rho = \rho_o\left(\frac{a_o}{a}\right)^{3(1+\omega)} \tag{2.2.13}$$

For radiation, using the equation of state in (2.1.19), we find

$$\rho = \rho_o\left(\frac{a_o}{a}\right)^{4} \tag{2.2.14}$$

It is also instuctive to examine the **energy density** evolution interms of proper time t. According to the **first Freidmann equation** with natural units [2]

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho, \tag{2.2.15}$$

where

$K \equiv$ the constant spatial curvature of the universe

$G \equiv$ Newton's gravitational constant

$H \equiv \frac{\dot{a}}{a}$, the Hubble constant

Further, (2.2.15) reduced to:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \tag{2.2.16}$$

where in the radiation dominated era the evolution was independent of K to high precession and substituting for ρ in (2.2.14), we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_o\left(\frac{a_0}{a}\right)^4 \tag{2.2.17}$$

This can be solved as

$$ada = \left[\left(\frac{8\pi G}{3}\right)\rho_o a_o^4\right]^{1/2} dt, \tag{2.2.18}$$

which corresponds to,

$$a^2 = \left\{\left(\frac{32\pi G}{3}\right)\rho_o a_o^4\right\}^{1/2} t = Ct, \tag{2.2.19}$$

where the constant C is given by

$$C = \left\{\left(\frac{32\pi G}{3}\right)\rho_o a_o^4\right\}^{1/2}$$

Finally we have

$$a \propto t^{1/2}, \tag{2.2.20}$$

and again it can be rephrase as,

$$\left(\frac{1}{a}\right)\left(\frac{da}{dt}\right) = \frac{1}{2}t^{-1} \tag{2.2.21}$$

Equivalently,

$$H = \frac{1}{2t} \tag{2.2.22}$$

Here the origin of time, $t = 0$ is taken to be at the moment when the expansion of the univese began: the "**big-bang**".

Chapter 3

Statistical Mechanics in Equilibrium

In this chapter we are going to derive basic quantities in an early universe when particle species in equilibrium, by using statistical mechanics and applying high and low temperature approximations.

In the early universe, it is usually assumed that the distribution functions are the standard ones; and considering the evolution in a large theoretical framework will allow to test this assumption and to place limits to the range of its validity.

Consider some of early epoch at which the universe was populated by different species of particles such as electrons, positrons, muons, neutrinos etc. For a particle species **A** (with mass m) in statistical equlibrium, the distribution function is $f_{\mathbf{A}}(\mathbf{x,p},t)$. This is defined so that in a 3-momentum element $d\mathbf{p}^3$ and the spatial volume element $d\mathbf{x}^3$ there are $f_{\mathbf{A}}(\mathbf{x,p},t)d\mathbf{p}^3d\mathbf{x}^3$ particles.

$$dN = f_{\mathbf{A}}(\mathbf{x,p},t)d\mathbf{p}^3d\mathbf{x}^3 \tag{3.0.1}$$

We only consider the **homogeneous** case here, so

$$f_{\mathbf{A}}(\mathbf{x,p},t) = f_{\mathbf{A}}(\mathbf{p},t), \tag{3.0.2}$$

and statistical **isotropy** implies

$$f_{\mathbf{A}}(\mathbf{p},t) = f_{\mathbf{A}}(\mid \mathbf{p} \mid,t) \equiv f_{\mathbf{A}}(p,t) \tag{3.0.3}$$

To determine the form of $f_{\mathbf{A}}(\mathbf{x},\mathbf{p},t)$ we may reason as follows. The different species of particles will be interacting constantly through various forces, scattering off each other and exchanging energy and momentum. If the rate of these reactions, $\Gamma(t)$, is much higher than the rate of expansion of the universe, $H(t) = (\dot{a}/a)$, then these interaction can produce (and maintain) thermodynamic equilibrium among the interaction particles with some temperature $T(t)$. All these interactions which occur between the particles have a short range. (The Coulomb force between charged particles has a long range; but, in a plasma, the process of Debye shielding reduces this range, making it effectively a short-range force.) Therefore, we may assume that the role of these interactions is limited to providing a mechanism for thermalization, and ignore their effects in deciding the form of the distribution function. In that case, the particles may be treated as an ideal **(Bose and Fermi)** gas, with the equilibrium distribution function:

$$f_{\mathbf{A}}(p,t)dp^3 = \frac{g_A}{(2\pi)^3}\left\{exp\left(\frac{E_p - \mu_p}{T_A(t)}\right) \pm 1\right\}^{-1} dp^3, \tag{3.0.4}$$

where $\mu_{\mathbf{A}}(T)$ is a chemical potential, $g_{\mathbf{A}}$ is the spin degeneracy factor, $E(p) = (p^2+m^2)^{1/2}$ and $T_{\mathbf{A}}$ is the temperature charcterizing this species at time t. The upper sign (+1) corresponds to fermions and the lower sign (-1) is for bosons. Moreover, the term $(2\pi)^3$ comes from the density of states in momentum space in natural units.

The distribution function tell us what fraction of the particle is in a state with momentum p at a given temprature T, and it is given by

$$f(p) = \frac{g}{(2\pi)^3}\frac{1}{exp\left\{(E-\mu)/T\right\} \pm 1} \tag{3.0.5}$$

At any instant in time, the universe will also contain a black body distribution of photons with some characteristic temperature $T_\gamma(t)$. If a particular species couples to the photon directly or indirectly, and if the rate of these $\mathbf{A}-\gamma$ interactions is high enough ($i.e$ $\Gamma_{A\gamma} \ll H$), then these particles will have the same temperature as photons: $T_{\mathbf{A}} = T_\gamma$. Since this is usually the case, one often refers to the photon temperature by the term, 'temperature of the universe '.

Of course, any set of particle species **A,B,C**,... which are interacting themselves at a high enough rate will also have the same temperature $T_{\mathbf{A}} = T_{\mathbf{B}} = T_{\mathbf{C}}$.... But, ingeneral, this may not be the case and the universe could be populated by different species of particles, each with its own temperature.

As the universe evolves, the temperature $T(t)$ changes due to expansion on a time scale of the order $H^{-1} \equiv (\dot{a}/a)^{-1}$; the rate at which the temperature is given by $H(t)$. The rate of interaction (per particle) can be expressed as $\Gamma \equiv n\langle\sigma v\rangle$ where n is the number density of target particles, v is the relative velocity and σ is the interaction cross-section. As long as $\Gamma \gg H$, the interaction can maintain equilibrium. In that case, f will evolve adiabatically, maintaining the form of the equilibrium distribution function given at the very beginning, with the temperature corresponding to the instantaneous value.

Once the distribution function is given, we can then calculate the **number density** n, **energy density** ρ and **pressure** p as integrals over the distribution function $f(k)$. Suppressing the time dependence and the subscript **A** for simplicity, we have

$$n = \int f(k)dk^3 \tag{3.0.6}$$

$$\rho = \int E(k)f(k)dk^3 \tag{3.0.7}$$

$$p = \int \left\{\frac{k^2}{3E(k)}\right\} f(k)dk^3 \tag{3.0.8}$$

We have used the symbol k to denote the momentum and the letter p to denote the pressure. For the collection of relativistic particles, it is known that the pressure p corresponding to velocity v is $p = m\left[v^2/3\sqrt{(1-v^2)}\right]$, which is the same as $p = (k^2/3E)$. In the other way, pressure is the change in momentum per unit time per unit area; momentum in the x-direction is k_x per particles, hence a change in momentum of $2|k_x|$ if it hits the perpendicular area dA. The volume swept out in unit time is $|v_x|dA = (|k_x|/E)dA$, so the average pressure over the distribution for particles is $2k_x^2/E$, which correponds for isotropic distribution ($k_x = k_y = k_z$) if the averge over all momenta as $k^2/3E$.

Using (2.1.21) and (3.0.5) in, we can further simplified as

$$n = \frac{g}{2\pi^2} \int_0^\infty \frac{k^2}{exp\Big\{(E-\mu)/T\Big\} \pm 1} dk \tag{3.0.9}$$

$$\rho = \frac{g}{2\pi^2} \int_0^\infty \frac{E(k)k^2}{exp\Big\{(E-\mu)/T\Big\} \pm 1} dk \tag{3.0.10}$$

$$p = \frac{g}{2\pi^2} \frac{1}{3} \int_0^\infty \frac{E^{-1}(k)k^4}{exp\Big\{(E-\mu)/T\Big\} \pm 1} dk \tag{3.0.11}$$

It is also useful to write integrals above as the integrals over the particle energy E instead of the momentum k.

Now we can define the equation

$$k^2 = E^2 - m^2, \tag{3.0.12}$$

implying,

$$dk = \sqrt{E^2 - m^2} dE \tag{3.0.13}$$

Then, because of (3.0.12) and (3.0.13) we get

$$n = \frac{g}{2\pi^2} \int_m^\infty \frac{\sqrt{E^2 - m^2}}{exp\Big\{(E-\mu)/T\Big\} \pm 1} EdE \tag{3.0.14}$$

$$\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{\sqrt{E^2 - m^2}}{exp\Big\{(E-\mu)/T\Big\} \pm 1} E^2 dE \tag{3.0.15}$$

$$p = \frac{g}{2\pi^2} \frac{1}{3} \int_m^\infty \frac{\sqrt{(E^2 - m^2)^3}}{exp\Big\{(E-\mu)/T\Big\} \pm 1} dE \tag{3.0.16}$$

Differentiating the above expression in (3.0.8) for pressure with respect to T, and treating μ as some specified function of T, we obtain

$$\frac{dp}{dT} = \frac{4\pi}{3} \int_0^\infty \frac{k^4 dk}{E} f^2 \left[exp\left\{ \frac{(E-\mu)}{T} \right\} \right] \left[\frac{E}{T^2} + \frac{d}{dT}\left(\frac{\mu}{T}\right) \right] \tag{3.0.17}$$

Using the relation

$$\frac{df}{dk} = -\frac{k}{ET} f^2 \left[exp\left\{ \frac{(E-\mu)}{T} \right\} \right], \tag{3.0.18}$$

we can rewrite equation (3.0.17) as:

$$\frac{dp}{dT} = -\frac{4\pi}{3}\int_0^\infty dk(k^3T)\left(\frac{df}{dk}\right)\left[\frac{E}{T^2} + \frac{d}{dT}\left(\frac{\mu}{T}\right)\right] \tag{3.0.19}$$

Integrating by parts and using the definitions of ρ and p we find

$$\frac{dp}{dT} = \frac{1}{T}(\rho + p) + nT\frac{d}{dT}\left(\frac{\mu}{T}\right) \tag{3.0.20}$$

The **Friedmann** metric is given by

$$ds^2 \cong \left(1 - \frac{\ddot{a}}{a}\frac{R^2}{c^2}\right)c^2dT^2 - \left(1 + \frac{k}{a^2}R^2 + \frac{\dot{a}^2}{a^2}R^2\right)dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.0.21}$$

Einstein's field equations of general relativity are [2]

$$G^i_k = R^i_k - \frac{1}{2}\sigma^i_k R = 8\pi G T^i_k, \tag{3.0.22}$$

where G^i_k is the Einstein tensor, given by a combination of Ricci tensor $R^i_k \equiv R^\alpha_{i\alpha k}$ and the Ricci scalar $R \equiv g^{ik}R_{ik}$ which are related to the curvature of space-time, and are determined by metric:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} + g_{\mu\nu,\beta}),$$

$$R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\nu\beta,\mu} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\delta_{\nu\beta}\Gamma^\alpha_{\mu\delta} - \Gamma^\delta_{\mu\beta}\Gamma^\alpha_{\nu\delta},$$

$$R^0_0 = -3\ddot{a}/a,$$

$$R^k_i = -[\ddot{a}/a + 2(\dot{a}/a)^2 + 2K/a^2]g^i_k,$$

$$R = -6[\ddot{a}/a + (\dot{a}/a)^2 + K/a^2].$$

The assumption of homogeneity and isotropy of space implies that the T^μ_0 s must be zero and that the spatial components T^β_α must have with diagonal form with T^1_1=T^2_2=T^3_3. As a result the non-trivial components of G^i_k for the Friedmann metric will be:

$$G^0_0 = \frac{3}{a^2}(\dot{a}^2 + k), \tag{3.0.23}$$

$$G^\mu_\nu = \frac{1}{a^2}(2a\ddot{a} + \dot{a}^2 + k)\sigma^\nu_\mu \tag{3.0.24}$$

Thus Einstein's equations give two independent equations:

$$\frac{\dot{a}^2+k}{a^2}=\frac{8\pi G}{3}\rho, \tag{3.0.25}$$

$$\frac{2\ddot{a}}{a}+\frac{\dot{a}^2+k}{a^2}=-8\pi Gp \tag{3.0.26}$$

Substituting (3.0.25) into (3.0.26) and making some rearrangements we obtain

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(\rho+3p) \tag{3.0.27}$$

From (3.0.25) we see that,

$$\rho a^3=\left(\frac{3}{8\pi G}\right)a\left[(\dot{a})^2+k\right] \tag{3.0.28}$$

Differentiating this expression and using (3.0.26), we get

$$\frac{d}{dt}\left(\rho a^3\right)=-3a^2\dot{a}p=-p\left(\frac{da^3}{dt}\right), \tag{3.0.29}$$

which is identical to

$$d(\rho a^3)=-pd(a^3)=-[d(pa^3)-a^3dp], \tag{3.0.30}$$

and this gives,

$$d[(p+\rho)a^3]=a^3dp \tag{3.0.31}$$

Hence we derive

$$\frac{d}{dT}\left[(p+\rho)a^3\right]=a^3\left(\frac{dp}{dT}\right) \tag{3.0.32}$$

Now upon substituting for (dp/dT) from (3.0.20) and rearranging terms, we finally obtain

$$d(sa^3)\equiv d\left\{\frac{a^3}{T}(\rho+p-n\mu)\right\}, \tag{3.0.33}$$

where s is the specific entropy in general, and μ is the increase in internal energy due to the addition of a particle when no heat flow or perform of work accompanies the change. This provides the thermodynamic definition of the chemical potential.

So, using (3.0.29), we may easily show that

$$\frac{d}{dt}\left\{a^3(\rho+p)\right\}=0 \tag{3.0.34}$$

This equation is equivalent to$\bigtriangledown_\nu T_{\mu\nu} = 0$ and is usually said to express the conservation of energy.

Applying this result in (3.0.33), we get

$$d(sa^3) \equiv d\left\{\frac{a^3}{T}(\rho + p - n\mu)\right\} = \left(-\frac{\mu}{T}\right)d(na^3) \tag{3.0.35}$$

In most cases of interset in cosmology, either (na^3) will be approximately constant or we will have $\mu \ll T$ (non-degenerate system). The above relation shows that in either case, the quantity (sa^3) will be conserved and the expression for s reduces to $T^{-1}(\rho+p)$. Expansion under such circumstances will be **iso-entropic or adiabatic**.

3.1 Relativistics Species

In the early universe, for relativistic particles $(T \gg m)$ which are created and annihilated continuously, the chemical potential can be neglected. We will therefore only consider the case non-degenerate $(\mu \ll T)$. The total energy E of the particles for high temprature approximation gives $E \approx k$.

Now using these approximations, let us derive the expressions of thermodynamic quantities for different species.

Energy Density

From (3.0.10), we can write the energy density of relativistic **Bosons** as

$$\rho_B = \frac{g_B}{2\pi^2}\int_0^\infty \frac{k^3}{exp(k/T)-1}dk, \tag{3.1.1}$$

introducing the substitution $x = k/T$, it gives

$$\begin{aligned} \rho_B &= \frac{g_B}{2\pi^2}T^4\int_0^\infty \frac{x^3}{e^x-1}dx \\ &= \frac{g_B}{2\pi^2}T^4\int_0^\infty x^3e^{-x}(1-e^{-x})^{-1}dx, \end{aligned} \tag{3.1.2}$$

and by using the integral evaluated in (2.1.27), we see that

$$\rho_B = g_B\left(\frac{\pi^2}{30}\right)T^4 \tag{3.1.3}$$

For relativistic **Fermions**,

$$\rho_F = \frac{g_F}{2\pi^2} T^4 \int_0^\infty \frac{x^3}{e^x + 1} dx \tag{3.1.4}$$

To relate the integrals for **Bosons** and **Fermions**, one can use the following trick.

Let

$$I_n^\pm \equiv \int_0^\infty \frac{x^n}{e^x \pm 1} dx \tag{3.1.5}$$

Then,

$$I_n^- - I_n^+ = \int_0^\infty x^n \left(\frac{2}{e^{2x} - 1} \right) dx \tag{3.1.6}$$

If we write the variable $y = 2x$, we find that

$$I_n^- - I_n^+ = \frac{1}{2^n} \int_0^\infty \left(\frac{y^n}{e^y - 1} \right) dy = \left(\frac{1}{2^n} \right) I_n^- , \tag{3.1.7}$$

where we have used the fact that the integration variable is just 'dummy variable' to rename the integration variable from y to x and giving

$$I_n^+ = I_n^- \left(1 - \frac{1}{2^n} \right) \tag{3.1.8}$$

This accounts for (7/8) factor when n=3. Thus the energy density of relativistic **Fermions** can be expressed as

$$\begin{aligned} \rho_F &= \frac{g_F}{2\pi^2} T^4 \left(\frac{7}{8} \int_0^\infty \frac{x^3}{e^x - 1} dx \right) \\ &= \frac{7}{8} g_F \left(\frac{\pi^2}{30} \right) T^4 \end{aligned} \tag{3.1.9}$$

Number density

We now compute the expressions of the number density for different particle species in the relativistic limit.

From the general equation (3.0.9), and consider the case of **Bosons** first, we have that

$$\begin{aligned} n_B &= \frac{g_B}{2\pi^2} T^3 \int_0^\infty \frac{x^2}{e^x - 1} dx \\ &= \frac{g_B}{2\pi^2} T^3 \int_0^\infty x^2 e^{-x} (1 - e^{-x})^{-1} dx, \end{aligned} \tag{3.1.10}$$

where we have set the variable $x = k/T$ and then, using (2.1.23) we get

$$\begin{aligned} n_B &= \frac{g_B}{2\pi^2}T^3 \sum_{n=1}^{\infty} \int_0^{\infty} x^2 e^{-nx} dx \\ &= \frac{g_B}{2\pi^2}T^3 \sum_{n=1}^{\infty} \frac{1}{n^3} dx \\ &= g_B \left(\frac{\zeta(3)}{\pi^2}\right) T^3, \end{aligned} \tag{3.1.11}$$

where we used equation (2.1.24) at $a = 2$ and defined the summation as **Riemann Zeta** function of order 3, $\zeta(3) \cong 1.202$.

Let us repeat the calculation for **Fermions**,

$$n_F = \frac{g_F}{2\pi^2}T^3 \int_0^{\infty} \frac{x^2}{e^x + 1} dx \tag{3.1.12}$$

To solve the integral, we have to use the trick in (3.1.8) at n=2. In this case,

$$I_2^+ = I_2^- \left(1 - \frac{1}{2^2}\right) = \frac{3}{4} I_2^- \tag{3.1.13}$$

Then, (3.1.12) yields

$$n_F = \frac{g_F}{2\pi^2}T^3 \left(\frac{3}{4} \int_0^{\infty} \frac{x^2}{e^x - 1} dx\right), \tag{3.1.14}$$

and using the integral result in (3.1.10), we find that the number density of relativistic **Fermions** as

$$n_F = g_F \frac{3}{4} \left(\frac{\zeta(3)}{\pi^2}\right) T^3 \tag{3.1.15}$$

Now combination of (3.1.3), (3.1.9), (3.1.11), and (3.1.15) leads to find the **mean energy** of the particles $\langle E \rangle \equiv (\rho/n)$.

For**Bosons**,

$$\langle E_B \rangle = \frac{g_B \left(\frac{\pi^2}{30}\right) T^4}{g_B \left(\frac{\zeta(3)}{\pi^2}\right) T^3} \simeq 2.7T, \tag{3.1.16}$$

and for **Fermions**,

$$\langle E_F \rangle = \frac{\frac{7}{8} g_F \left(\frac{\pi^2}{30}\right) T^4}{g_F \frac{3}{4} \left(\frac{\zeta(3)}{\pi^2}\right) T^3} \simeq 3.15T \tag{3.1.17}$$

Pressure

In the ultra-relativistic limit, taking (3.0.11) with $x = k/T$, we can compute the equation of pressure for **Bosons**.

Then,

$$p_B = \frac{1}{3}\left(\frac{g_B}{2\pi^2}T^4\int_0^\infty \frac{x^3}{e^x-1}dx\right) \tag{3.1.18}$$

By comparing (3.1.18) with (3.1.2), we obtain

$$p_B = \frac{1}{3}\rho_B = g_B\left(\frac{\pi^2}{90}\right)T^4, \tag{3.1.19}$$

and in case of **Fermions**,

$$\begin{aligned} p_F &= \frac{1}{3}\left(\frac{g_F}{2\pi^2}T^4\int_0^\infty \frac{x^3}{e^x+1}dx\right) \\ &= \frac{7}{8}g_F\left(\frac{\pi^2}{90}\right)T^4, \end{aligned} \tag{3.1.20}$$

where the expression in the bracket compared with (3.1.4).

Entropy Density

Entropy counts the number of ways we can rearrange microstates to get the same macrostate. The universe has far more photons than baryons, so the entropy of a uniform universe is dominated by that of the relativistic particles.

Using (2.2.3) in a cosmological volume V, we have $E = \rho V$ and it is also useful to consider the entropy density $S = sV$, so

$$d(\rho V) = Td(sV) - pdV, \tag{3.1.21}$$

where ρ and p are the equilibrium energy density and pressure. Moreover, the differentiation gives

$$Vd\rho + \rho dV = TVds + TsdV + pdV, \tag{3.1.22}$$

implying,

$$d\rho - Tds = \frac{dV}{v}\left(Ts - \rho - p\right) \tag{3.1.23}$$

For a system at equilibrium the entropy density, energy density and pressure can be written as functions only of the temperature, $\rho = \rho(T), s = s(T), p = p(T)$. Since ds and $d\rho$ are intensive quantities proportional to dT , the coefficients of the dT and dV terms must separately be zero (for example under a volume change at constant temperature the dV term must be zero). For the dV coefficient we get an expression for the entropy density

$$s = \frac{1}{T}\Big(\rho + p\Big), \tag{3.1.24}$$

Since the pressure due to relativistic species is $p = \rho/3$; and then the entropy density in the relativistic **Bosons** case becomes,

$$s_B = \left(\frac{4}{3T}\right)\rho_B = g_B\left(\frac{2\pi^2}{45}\right)T^3, \tag{3.1.25}$$

and for relativistic **Fermions**,

$$s_F = \left(\frac{4}{3T}\right)\rho_F = \frac{7}{8}g_F\left(\frac{2\pi^2}{45}\right)T^3 \tag{3.1.26}$$

3.2 Non-relativistic Species

At a given temperature T, the main contribution to the (kinetic) energy comes from momenta $k \sim T$. In the non-relativistic (massive) limit ($T \ll m$), we can expand the energy of the particle in powers of the momentum k.

$$\begin{aligned} E &= m[1 + (k/m)^2]^{1/2} \\ &= m[1 + \frac{1}{2}(k/m)^2 - \frac{1}{8}(k/m)^4 + \frac{1}{32}(k/m)^6 - ...], \end{aligned} \tag{3.2.1}$$

which to second order gives,

$$E \approx m + k^2/2m \tag{3.2.2}$$

Furthermore, the exponential in the denominator is very large, $exp(m/T) \gg \pm 1$, so that we can drop ± 1. Thus, the number density, energy density and pressure are the same for **Bose** and **Fermi** species.

Number Density

From (3.0.9) and (3.2.2), the number density for all species can be given by

$$\begin{aligned} n &= \frac{g}{2\pi^2}\int_0^\infty exp\Big[-\frac{(m-\mu)}{T}\Big]exp\Big(-\frac{k^2}{2mT}\Big)k^2dk \\ &= \frac{g}{2\pi^2}exp\Big\{-\frac{1}{T}(m-\mu)\Big\}\int_0^\infty exp\Big(-\frac{k^2}{2mT}\Big)k^2dk \\ &= \frac{g}{2\pi^2}exp\Big\{-\frac{1}{T}(m-\mu)\Big\}\Big(2mT\Big)^{3/2}\int_0^\infty x^2e^{-x^2}dx \end{aligned} \tag{3.2.3}$$

Here we have made substitution $x^2 = k^2/2mT$ and to solve the integral we have to introduce the Gamma function.

$$\int_0^\infty x^n e^{-ax^2}dx = a^{-(n+1)/2}\frac{1}{2}\Gamma\Big(\frac{n+1}{2}\Big), \tag{3.2.4}$$

and its properties

$$\Gamma(n+1) = n\Gamma(n) \qquad and \qquad \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}$$

Imposing $n = 2$ at $a = 1$, the integral becomes

$$\int_0^\infty x^2e^{-x^2}dx = \frac{1}{2}\Gamma\Big(\frac{3}{2}\Big) \tag{3.2.5}$$

But

$$\Gamma\Big(\frac{3}{2}\Big) = \Gamma\Big(1+\frac{1}{2}\Big) = \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) = \frac{\sqrt{\pi}}{2} \tag{3.2.6}$$

Therefore,

$$n = g\Big(\frac{mT}{2\pi}\Big)^{3/2}exp\Big\{-\frac{1}{T}(m-\mu)\Big\} \tag{3.2.7}$$

Energy Density

Using (3.0.10) in the low temprature approximation , then energy density is given by

$$\begin{aligned} \rho &= \frac{g}{2\pi^2}exp\Big\{-\frac{1}{T}(m-\mu)\Big\}\int_0^\infty \frac{k^2(m+k^2/2m)}{exp(k^2/2mT)}dk \\ &= \frac{g}{2\pi^2}exp\Big\{-\frac{1}{T}(m-\mu)\Big\}\Big[\int_0^\infty \frac{mk^2}{exp(k^2/2mT)}dk \\ &\quad + \int_0^\infty \frac{k^4/2m}{exp(k^2/2mT)}dk\Big] \end{aligned} \tag{3.2.8}$$

And ignoring the higher order term (k^4) , it follows

$$\rho \simeq mg\left(\frac{mT}{2\pi}\right)^{3/2} exp\left\{-\frac{1}{T}(m-\mu)\right\} \tag{3.2.9}$$

Comparing this with (3.2.7), this may be written as

$$\rho \simeq nm \tag{3.2.10}$$

Pressure

In the same way, using (3.0.11) and (3.2.2), we can write the equation of pressure in the relativistic limit as

$$\begin{aligned} p &= \frac{g}{6m\pi^2} exp\left\{-\frac{1}{T}(m-\mu)\right\} \int_0^\infty \frac{k^4(1+k^2/2m^2)^{-1}}{exp(k^2/2mT)} dk \\ &\simeq \frac{g}{6m\pi^2} exp\left\{-\frac{1}{T}(m-\mu)\right\} \int_0^\infty \frac{k^4}{exp(k^2/2mT)} dk \\ &= \frac{g}{6m\pi^2} exp\left\{-\frac{1}{T}(m-\mu)\right\} (2mT)^{5/2} \int_0^\infty x^4 e^{-x^2} dx, \end{aligned} \tag{3.2.11}$$

where approximation is in the lowest k^4 term and set $x^2 = k^2/2mT$.

But using (3.2.4), we find that

$$\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8} \tag{3.2.12}$$

Substituting (3.2.12) for the integral value in (3.2.11), it becomes

$$p \simeq T\left\{g\left(\frac{mT}{2\pi}\right)^{3/2} exp\left[-\frac{1}{T}(m-\mu)\right]\right\}, \tag{3.2.13}$$

and relating with (3.2.7) to obtain

$$p \simeq nT, \tag{3.2.14}$$

which justifies our use of $p = 0$ as the equation of state for non-relativistic matter.

3.3 Quantities in Early Universe

In general, the energy density and pressure of the universe gets contributions from many different species of particles, which can be both ultra-relativistic and non-relativistic.

We note from our earlier results that the quantity of non-relativistic particles is exponentially suppressed compared to ultra-relativistic particles. In the early universe (up to matter-radiation equality) it is therefore a good approximation to include only the contributions from ultra-relativistic particles in the sums.

3.3.1 Thermodynamic Quantities

In equilibrium, it is straight forward to compute quantities like Energy Density, Number density, Pressure and Entropy Density.

Energy Density

The contribution to the energy density of a given particle species **A** of mass m_A and temperature T_A can be written as:

$$\rho_A = \frac{g_A}{2\pi^2} T_A^4 \int_{x_A}^{\infty} \frac{\sqrt{u^2 - x_A^2}}{e^u \pm 1} u^2 du, \tag{3.3.1}$$

where $u \equiv E/T_A$, $x_A \equiv m_A/T_A$, and neglecting the chemical potentails ($\mu_A \ll T_A$).

From (2.1.28), (3.1.3) and (3.1.9), we can see that the energy density of photons and all relativistic particles behave like the fourth power of temperature. It is convenient to express the total energy density in terms of the photon temperature, T, since the photons are still with us, whereas other particles, like muons and positrons, have long since annihilated.

If we have a collection of relativistic species, each of them in equilibrium at different temperatures T_A, we can write the total energy density ρ_{total}, summing over all the contributions, as:

$$\begin{aligned} \rho_{total} &= T^4 \sum_{A=B,F} \frac{g_A}{2\pi^2} \left(\frac{T_A}{T}\right)^4 \left\{ \int_{x_A}^{\infty} \frac{\sqrt{u^2 - x_A^2}}{e^u \pm 1} u^2 du \right\} \\ &= T^4 \sum_{B} \frac{g_B}{2\pi^2} \left(\frac{T_B}{T}\right)^4 \left\{ \int_{x_B}^{\infty} \frac{\sqrt{u^2 - x_B^2}}{e^u - 1} u^2 du \right\} \\ &\quad + T^4 \sum_{F} \frac{g_F}{2\pi^2} \left(\frac{T_F}{T}\right)^4 \left\{ \int_{x_F}^{\infty} \frac{\sqrt{u^2 - x_F^2}}{e^u + 1} u^2 du \right\} \end{aligned} \tag{3.3.2}$$

But in the relativistic limit, $m_A \ll T_A$, the term x_A ignored and using equation (3.1.8) it can be written as

$$\rho_{total} = T^4 \sum_B \frac{g_B}{2\pi^2}\left(\frac{T_B}{T}\right)^4 \left\{\int_0^\infty \frac{u^3}{e^u - 1}du\right\} + T^4 \sum_F \frac{g_F}{2\pi^2}\left(\frac{T_F}{T}\right)^4 \left\{\int_0^\infty \frac{u^3}{e^u + 1}du\right\}$$
$$= \left\{\sum_B g_B\left(\frac{T_B}{T}\right)^4 + \frac{7}{8}\sum_F g_F\left(\frac{T_F}{T}\right)^4\right\}\left\{\int_0^\infty \frac{u^3}{e^u - 1}du\right\}\left(\frac{1}{2\pi^2}\right)T^4, \tag{3.3.3}$$

and then from the result of (2.1.27) it is clear to write

$$\rho_{total} = \left\{\sum_B g_B\left(\frac{T_B}{T}\right)^4 + \frac{7}{8}\sum_F g_F\left(\frac{T_F}{T}\right)^4\right\}\left(\frac{\pi^2}{30}\right)T^4, \tag{3.3.4}$$

which we can write more compactly as

$$\rho_{total} = g_{total}\left(\frac{\pi^2}{30}\right)T^4, \tag{3.3.5}$$

where g_{total} is the effective number of relativistic degrees of freedom, given by:

$$g_{total} = \sum_B g_B\left(\frac{T_B}{T}\right)^4 + \frac{7}{8}\sum_F g_F\left(\frac{T_F}{T}\right)^4$$

Here T_B,T_F to the temperature characterizing the distributions functions of species of **Boson** or **Fermion** and note that g_{total} is a function of the temperature, but usually the dependence on T is weak.

In writing g_{total}, we have explicitly taken into account the possiblity that even though all the species may have a thermal distribution they may have not the same temperature. If all species have the same temperature, then $g = g_B + (7/8)g_F$.

Pressure

Similarly, the total pressure can be written as

$$p = T^4 \sum_{A=B,F} \frac{g_A}{6\pi^2}\left(\frac{T_A}{T}\right)^4 \left\{\int_{x_A}^\infty \frac{(u^2 - x_A^2)^{3/2}}{e^u \pm 1}du\right\} \tag{3.3.6}$$

For all ultra-relativistic particles, it reduced to

$$p = T^4 \sum_B \frac{g_B}{6\pi^2}\left(\frac{T_B}{T}\right)^4 \left\{\int_0^\infty \frac{u^3}{e^u - 1}du\right\} + T^4 \sum_F \frac{g_F}{6\pi^2}\left(\frac{T_F}{T}\right)^4 \left\{\int_0^\infty \frac{u^3}{e^u + 1}du\right\}$$
$$= \left(\frac{1}{3}\right)\left\{\sum_B g_B\left(\frac{T_B}{T}\right)^4 + \frac{7}{8}\sum_F g_F\left(\frac{T_F}{T}\right)^4\right\}\left\{\int_0^\infty \frac{u^3}{e^u - 1}du\right\}\left(\frac{1}{2\pi^2}\right)T^4 \tag{3.3.7}$$

Thus comparing with (3.3.3), we also have

$$p = \left(\frac{1}{3}\right)\rho_{total}$$
$$= g_{total}\left(\frac{\pi^2}{90}\right)T^4 \tag{3.3.8}$$

Number Density

Hence, the total number of density in equilibrium is given by

$$n = T^3 \sum_{A=B,F} \frac{g_A}{2\pi^2}\left(\frac{T_A}{T}\right)^3 \left\{ \int_{x_A}^{\infty} \frac{(u^2 - x_A^2)^{1/2}}{e^u \pm 1} u du \right\} \tag{3.3.9}$$

This can be further simplified as:

$$n = T^3 \sum_B \frac{g_B}{2\pi^2}\left(\frac{T_B}{T}\right)^3 \left\{ \int_0^{\infty} \frac{u^2}{e^u - 1} du \right\} + T^4 \sum_F \frac{g_F}{2\pi^2}\left(\frac{T_F}{T}\right)^3 \left\{ \int_0^{\infty} \frac{u^2}{e^u + 1} du \right\}$$
$$= \left\{ \sum_B g_B \left(\frac{T_B}{T}\right)^3 + \frac{3}{4}\sum_F g_F \left(\frac{T_F}{T}\right)^3 \right\} \left[\int_0^{\infty} \frac{u^2}{e^u - 1} du \right] \left(\frac{1}{2\pi^2}\right) T^3, \tag{3.3.10}$$

where we have used the trick in equation (3.1.13). Then by substituting the result of integral using (3.1.10), we obtain

$$n = \left\{ \sum_B g_B \left(\frac{T_B}{T}\right)^3 + \frac{3}{4}\sum_F g_F \left(\frac{T_F}{T}\right)^4 \right\} \left(\frac{\zeta(3)}{\pi^2}\right) T^3, \tag{3.3.11}$$

Let

$$\lambda = \sum_B g_B \left(\frac{T_B}{T}\right)^3 + \frac{3}{4}\sum_F g_F \left(\frac{T_F}{T}\right)^3, \tag{3.3.12}$$

and therefore

$$n = \lambda \left(\frac{\zeta(3)}{\pi^2}\right) T^3 \tag{3.3.13}$$

For relativistic species, either **Fermions or Bosons**, the number density goes as the third power of temperature (T^3).

Entropy Density

The total entropy density s, summing over all possible contributions is given by:

$$s = \sum_{A=B,F} s_A = \sum_B s_B + \sum_F s_F \tag{3.3.14}$$

On the account of (3.1.25) and (3.1.26), one can write

$$s = \left\{ \sum_B g_B \left(\frac{T_B}{T}\right)^3 + \frac{7}{8} \sum_F g_F \left(\frac{T_F}{T}\right)^3 \right\} \left(\frac{2\pi^2}{45}\right) T^3 = q \left(\frac{2\pi^2}{45}\right) T^3, \tag{3.3.15}$$

where

$$q \equiv q_{tot} = \sum_B g_B \left(\frac{T_B}{T}\right)^3 + \frac{7}{8} \sum_F g_F \left(\frac{T_F}{T}\right)^3$$

Clearly, q_{tot} is equal to g_{tot} only when all the relativistic species are in equilibrium at the same temperature.

Using the entropy conservation law equation (2.2.5), we have then:

$$d(sa^3) = d\left[q\left(\frac{2\pi^2}{45}\right) T^3 a^3\right] = 0 \tag{3.3.16}$$

Note that s is proportional to the number density of relativistic particles, if all species have the same temperature; infact, $s \cong 1.8 q n_\gamma$, where n_γ is the photon number density.

3.3.2 Radiation Temperature

As the universe evolves the matter density will catch up with the radiation density. Let us say, the equality of matter and radiation energies occured at some time $t = t_{eq}$. Actually it can be shown that $t_{eq} \simeq 1.57x10^{10}(\Omega h^2)^{-2}s$, where $\Omega = \rho/\rho_{critical}$ determines the spatial geometry of the universe. For $t < t_{eq}$, in the radiated dominated phase, we may ignore the contribution of non-relativistic particles to ρ.

During radiation dominated era to see the expression of radiation temperature we can use (3.3.5) in the first **Friedmann** equation, which gives

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3} g \left(\frac{\pi^2}{30}\right) T^4 \\ &= Gg\left(\frac{4\pi^3}{45}\right) T^4 \end{aligned} \tag{3.3.17}$$

Applying the definition of the Planck mass in natural units [7]

$$m_{pl} = \frac{1}{\sqrt{G}} \tag{3.3.18}$$

Then,

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{4\pi^3}{45}\right)\frac{gT^4}{m_{pl}^2}, \qquad (3.3.19)$$

it can be approximated as:

$$H(T) \cong 1.66g^{1/2}\left(\frac{T^2}{m_{pl}}\right) \qquad (3.3.20)$$

With the aid of (2.2.22), equation (3.3.19) reduces to

$$4t^2 = \left(\frac{45}{4\pi^3}\right)\left(\frac{m_{pl}^2}{gT^4}\right), \qquad (3.3.21)$$

which implies

$$t = \left(\frac{45}{16\pi^3}\right)^{1/2}\left(\frac{m_{pl}}{g^{1/2}T^2}\right), \qquad (3.3.22)$$

This can be done numerically, giving

$$t \cong 0.3g^{-1/2}\left(\frac{m_{pl}}{T^2}\right), \qquad (3.3.23)$$

and using the value $m_{pl} = 1.22x10^{22}MeV \cong 1.85x10^{43}/s$, we can write this result in the useful form

$$t \cong 1s(\frac{T}{1MeV})^{-2}g^{-1/2}, \qquad (3.3.24)$$

where t is measured in seconds and T_{MeV} in units of MeV.

Therefore, the temperature of the universe at a time t is given by the expansion

$$t \cong 0.3g^{-1/2}(\frac{m_{pl}}{T^2}) \cong 1s(\frac{T}{1MeV})^{-2}g^{-1/2} \qquad (3.3.25)$$

Thus, we see that the universe was a few seconds old when the photon temperature had dropped to $T = 1MeV$, if $g^{1/2}$ was not much larger than one at that time.

Chapter 4

Statistical Mechanics out of Equilibrium

The distribution function f maintains the form of equlibrium distribution if a particle of species **A** must be in statistical equilibrium and decuple from the thermal bath when the interaction rate drops below the rate of expansion ($\Gamma_A \ll H$). But once the species, **A**, is completely decoupled, each of the-**A** particles will be travelling along **geodesics** in the space-time.

In this cahpter we will apply statistical mechanics to derive thermodynamic quantities of the universe for decoupled species on the basis of equilibrium distribution and the redshift correction.

4.1 Derivation of Redshifted Momentum

Here we can see that the redshifting effect is generally from the geodesic equation for a freely propagating particle (massive particle that feels gravity only).

In 3D sapce the path of a force-free particle, the straight line, has the property of being the shortest curve between any two points lying on it or 2D space;i.e., for that curve whose arc length S is a minimum for given intial-point (P_I) and (P_E):

$$S = \int_{P_I}^{P_E} dS = extremum \tag{4.1.1}$$

Let us represent the locally inertial coordinate system by ε^α which is expected to be a function of the general coordinate system x^μ, i.e $\varepsilon^\alpha = \varepsilon^\alpha(x^\mu)$. So, for a freely falling particle under gravity we can write its equation of motion as,

$$\frac{d^2\varepsilon^\alpha}{d\tau^2} = 0, \tag{4.1.2}$$

where τ is the proper time interval (time interval between two ticks).

Since $\varepsilon^\alpha = \varepsilon^\alpha(x^\mu)$, which may be written as

$$\frac{d}{d\tau}\left(\frac{\partial\varepsilon^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = 0 \tag{4.1.3}$$

This expands to,

$$\left(\frac{\partial\varepsilon^\alpha}{\partial x^\mu}\right)\left(\frac{d^2x^\mu}{d\tau^2}\right) + \left(\frac{\partial^2\varepsilon^\alpha}{\partial x^\mu\partial x^\nu}\right)\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right) = 0, \tag{4.1.4}$$

where

$$\frac{d}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\nu}$$

Multiplying (4.1.4) by $\left(\frac{\partial x^\lambda}{\partial\varepsilon^\alpha}\right)$, we get

$$\left(\frac{\partial x^\lambda}{\partial\varepsilon^\alpha}\right)\left(\frac{\partial\varepsilon^\alpha}{\partial x^\mu}\right)\left(\frac{d^2x^\mu}{d\tau^2}\right) + \left(\frac{\partial x^\lambda}{\partial\varepsilon^\alpha}\right)\left(\frac{\partial^2\varepsilon^\alpha}{\partial x^\mu\partial x^\nu}\right)\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right) = 0 \tag{4.1.5}$$

and using the identity,

$$\left(\frac{\partial x^\lambda}{\partial\varepsilon^\alpha}\right)\left(\frac{\partial\varepsilon^\alpha}{\partial x^\mu}\right) = \delta^\lambda_\mu,$$

provided that δ^λ_μ=1 if λ=μ and δ^λ_μ=0 if $\lambda \neq \mu$ and which leads to,

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{4.1.6}$$

where $\Gamma^\lambda_{\mu\nu}$ is the affine connection, defined by

$$\Gamma^\lambda_{\mu\nu} = \left(\frac{\partial x^\lambda}{\partial\varepsilon^\alpha}\right)\left(\frac{\partial^2\varepsilon^\alpha}{\partial x^\mu\partial x^\nu}\right)$$

Hence, the **geodesic equation** in arbitrary gravitational field with a suitable variable parameterizing positions s along the space-time curve, proportional to τ for massive particles is given by

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{4.1.7}$$

Introducing the **four-momentum** of particles with mass m as:

$$p^{\mu} = mu^{\mu} = m\left(\frac{dx^{\mu}}{ds}\right), \tag{4.1.8}$$

where u^{μ} is the four-velocity of particles.

If we choose to normalize the affine parameter λ, so that p^{μ} becomes

$$p^{\mu} = \frac{dx^{\mu}}{d\lambda}, \tag{4.1.9}$$

where $d\lambda = ds/m$ and using (4.1.7), we get the geodesic equation in the form

$$\frac{dp^{\mu}}{d\lambda} + \Gamma^{\mu}_{\alpha\beta}p^{\alpha}p^{\beta} = 0 \tag{4.1.10}$$

The **four-velocity** particles interms of proper time is given by

$$u^{\alpha} = \left(\frac{dx^{\alpha}}{d\tau}\right), \tag{4.1.11}$$

and inserting $\mu = 0$ gives,

$$u^{0} = \left(\frac{dx^{0}}{d\tau}\right) = \left(\frac{dt}{d\tau}\right), \tag{4.1.12}$$

where $x^0 \equiv t$, spatial component along space-time curve.

But, in special relativity we have that the relation equation between spatial component (t) and temporal components ($\mathbf{x}$) as

$$d\tau^2 = dt^2 - d\mathbf{x}^2 \qquad and \qquad d\mathbf{x} = (\mathbf{v}dt) \tag{4.1.13}$$

From this follows,

$$d\tau^2 = dt^2[1 - v^2], \tag{4.1.14}$$

which we can further simplified as,

$$\left(\frac{dt}{d\tau}\right) = [1 - v^2]^{-1/2} \tag{4.1.15}$$

By substituting (4.1.12) and (4.1.15) into four-momentum particles equation at $\mu = 0$, we find

$$p^0 = mu^0 = m[1 - v^2]^{-1/2} \tag{4.1.16}$$

This component may be identified by making a binomial expansion in powers of v/c ($\ll 1$) as follows:

$$p^0 = m + \frac{1}{2}mv^2 - \frac{3}{8}mv^4 + ... \qquad (4.1.17)$$

Here we note that p^0 refers the **total energy** (E) comprises infinte terms which the first two are m is the rest mass energy and $\frac{1}{2}mv^2$ is the kinetic energy.
By ignoring the higher order terms ($v^4, ...$) and making square on both sides, this gives

$$(p^0)^2 \simeq m^2 + m^2v^2, \qquad (4.1.18)$$

and the use of total energy (E) in yields,

$$E^2 = m^2 + \mathbf{p}^2 \qquad (4.1.19)$$

Taking $\mu = 0$ component for the evolution of $p^0 = E$ in (4.1.10) then gives

$$\frac{dE}{d\lambda} + \Gamma^0_{rr}p^r p^r = 0, \qquad (4.1.20)$$

where we can align our FRW metric coordinates so that the partciles is moving along a purely radial direction, as a result $p^\theta = p^\phi = 0$.
In the full space-time the **FRW** is given by [6]

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1 - Kr^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2\right] \qquad (4.1.21)$$

This put in a matrix form as

$$ds^2 = g_{\mu v}dx^\mu dx^v, \qquad (4.1.22)$$

where $g_{\mu v}$ is the metric, defined by

$$g_{\mu v} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & -ar^2 & 0 \\ 0 & 0 & 0 & -a^2r^2sin^2\theta \end{bmatrix}$$

In which case the metric becomes diagonal, with the components of $g_{\mu v}$

$$g_{00} = g^{00} = 1,$$

$$g_{rr} = -\frac{a^2}{1-Kr^2},$$

$$g_{r0} = g^{r0} = 0, \ \ldots$$

According to the relationship between metric $g_{\mu v}$ and affine connection $\Gamma^{\mu}_{\alpha\beta}$ in general relativity is given by [4]

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}\left(\frac{\partial g_{\beta\lambda}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\lambda}}\right) \tag{4.1.23}$$

From this we have

$$\Gamma^{0}_{rr} = \frac{1}{2}g^{\lambda o}\left(\frac{\partial g_{r\lambda}}{\partial x^{r}} + \frac{\partial g_{r\lambda}}{\partial x^{r}} + \frac{\partial g_{rr}}{\partial x^{\lambda}}\right) \tag{4.1.24}$$

Now Γ^{0}_{rr}, using the value of index $\lambda = 0$ and $\lambda = r$, may be expanded to:

$$\Gamma^{0}_{rr} = \frac{1}{2}g^{00}\left(\frac{\partial g_{r0}}{\partial x^{r}} + \frac{\partial g_{r0}}{\partial x^{r}} + \frac{\partial g_{rr}}{\partial x^{0}}\right) + \frac{1}{2}g^{r0}\left(\frac{\partial g_{rr}}{\partial x^{r}} + \frac{\partial g_{rr}}{\partial x^{r}} + \frac{\partial g_{rr}}{\partial x^{r}}\right), \tag{4.1.25}$$

and this gives

$$\Gamma^{0}_{rr} = \frac{1}{2}\left(\frac{\partial g_{rr}}{\partial x^{0}}\right) = \frac{1}{2}\left(\frac{\partial g_{rr}}{\partial t}\right) \tag{4.1.26}$$

Again, using the expression of g_{rr} we have

$$\begin{aligned}\Gamma^{0}_{rr} &= \frac{1}{2}\left(\frac{d}{dt}\left[-\frac{a^2}{1-Kr^2}\right]\right)\\ &= \left(\frac{a}{1-Kr^2}\right)\left(\frac{da}{dt}\right)\\ &= \left(-\frac{1}{a}g_{rr}\right)\left(\frac{da}{dt}\right)\end{aligned} \tag{4.1.27}$$

Substituting (4.1.27) into (4.1.20) gives

$$\left(\frac{dE}{dt}\right)\left(\frac{dt}{d\lambda}\right) - \left(\frac{1}{a}\frac{da}{dt}\right)\left(g_{rr}p^{r}p^{r}\right) = 0, \tag{4.1.28}$$

which implies,

$$\left(\frac{dE}{dt}\right)E - \left(\frac{1}{a}\frac{da}{dt}\right)\left(p_{r}p^{r}\right) = 0 \tag{4.1.29}$$

Note that in the above equation (4.1.29) we have used $\mu = 0$ in (4.1.9), which gives $p^0 = E = \dfrac{dt}{d\lambda}$ and tensor contraction,

$$g_{rr}p^r p^r = p_r p^r \tag{4.1.30}$$

According to the general expression for E can also be written in covariant form as [4]

$$p_\mu p^\mu = m^2 = E^2 - p^2, \tag{4.1.31}$$

Since the particle is moving along a purely radial direction, by taking only $\mu = 0$ and $\mu = r$ components we can expand $p_\mu p^\mu$ as:

$$p_\mu p^\mu = p_0 p^0 + p_r p^r = E^2 - p^2 \tag{4.1.32}$$

The use of this result yields,

$$p_r p^r = -p^2, \tag{4.1.33}$$

and differentiating equation (4.1.31) implies,

$$E\left(\frac{dE}{dt}\right) = p\left(\frac{dp}{dt}\right) \tag{4.1.34}$$

Inserting (4.1.33) and (4.1.34) into (4.1.29) inorder to obtain,

$$\left(\frac{dp}{dt}\right)p + \left(\frac{1}{a}\right)\left(\frac{da}{dt}\right)p^2 = 0 \tag{4.1.35}$$

We then get the geodesic equation as

$$\left(\frac{1}{p}\frac{dp}{dt}\right) = -\left(\frac{1}{a}\frac{da}{dt}\right), \tag{4.1.36}$$

whose solution is

$$p \propto \frac{1}{a} \tag{4.1.37}$$

Hence, the momentum of freely propagating particles decays with the scale factor.

4.2 Relativistic Decoupling

Consider dN (freely propagating) particles in a proper volume $dX^3 \propto a^3$ with momentum element $dp^3 \propto a^{-3}$. Between a time t_D and a latter time t, assuming the particles are completely decoupled and don't decay, the momentum redshifts so

$$\begin{aligned} dN &\propto f(X, p, t) d^3X d^3p \\ &= f(X_D, p_D, t_D) d^3X_D d^3p_D \\ &= f(X_D, \frac{pa}{a_D}, t_D) d^3X d^3p, \end{aligned} \tag{4.2.1}$$

where the factors in phase space volume elements cancel and D indicates the time of decoupling.

Since the number of particles dN is conserved, it follows that f is also conserved. This allows us to obtain the function f_{dec} after the species has decoupled, from the known form of f_{eq} before decoupling.

Approximating decoupling occured instantaneously at $t = t_D$, when the temperature was T_D and the expansion factor a_D. For $t < t_D$, the distribution function is given by

$$f_{\mathbf{A}}(p,t)dp^3 = \frac{g_A}{(2\pi)^3}\left\{exp\left(\frac{E_p - \mu_A}{T_A(t)}\right) \pm 1\right\}^{-1} dp^3 \tag{4.2.2}$$

At some latter time, $t > t_D$, let the distribution function be $f_{dec}(p,t)$. Because of the red-shift in momentum in (4.1.37), all particles with momentum p at time t must have had momentum $p[a(t)/a(t_D)]$ at $t = t_D$. Therefore,

$$f_{dec}(p,t) = f_{eq}\left\{p\left(\frac{a}{a_D}\right), t_D\right\} \qquad (for \quad t > t_D), \tag{4.2.3}$$

where f_{eq} is the equilibrium distribution function of f_A given in (4.2.1). Thus, as long as the species A was in equilibrium at some time, we can determine its distribution function at all later times.

The above expression in $f_{dec}(p,t)$ simplifies considerably if the decoupling occurs either when the species is ultra-relativistic ($T_D \gg m$) or when it is non-relativistic ($T_D \ll m$).

If decoupling occurs when the species is ultra-relativistic ($E \simeq p$), then:

$$f_{dec}(p) = f_{eq}\left(p\frac{a(t)}{a(t_D)}\right), T_D\Big\} \cong \frac{g}{(2\pi)^3}\left[exp\left\{\frac{1}{T_D}\left(p\frac{a(t)}{a(t_D)}\right)\right\} \pm 1\right]^{-1} \tag{4.2.4}$$

This has the same form as the distribution function f_{eq} for a relativistic species with temperature

$$T(t) = T_D\left[\frac{a(t_D)}{a(t)}\right], \tag{4.2.5}$$

eventhough this species is not in thermodynamic equilibrium any longer. The 'temperature' in this distribution function falls strictly as a^{-1}; the entropy of these particles, $S_{\mathbf{A}} = (s_{\mathbf{A}}a^3)$ is conserved separately. Note that for the species which are still in thermal equilibrium, $T \propto q^{-\frac{1}{3}}a^{-1}$ falls more slowly.

Now let us derive the expressions of number density and energy density for redshifted correction based on equilibrium distribution.

Number Density

From (3.3.13), the number density of decoupled partciles is defined by

$$n(t) = g_{eff}\left(\frac{\zeta(3)}{\pi^2}\right)T^3(t) \tag{4.2.6}$$

Then by substituting (4.2.5) into (4.2.6), we can get

$$n(t) = g_{eff}\left(\frac{\zeta(3)}{\pi^2}T_D^3\right)\left(\frac{a_D}{a}\right)^3 = n(t_D)\left(\frac{a_D}{a}\right)^3, \tag{4.2.7}$$

where we have defined $n(t_D)$, the number density at decoupling, as

$$n(t_D) \equiv g_{eff}\left(\frac{\zeta(3)}{\pi^2}T_D^3\right),$$

where g_{eff}=$(3g/4)$ for fermions and g_{eff}=(g) for bosons. (Here g refers to the spin degeneracy factor of the particular species which has decoupled.) This number density will be comparable to the number density of photons at any given time. In particular, any such decoupled species will continue to exist in our universe today as a relic background, with number densities comparable to the number density of photons.

Energy Density

In estimating the energy density contributed by the decoupled species, the following point should be noted.

Suppose that a species with mass m decouples at the temperature T_D with $T_D \gg m$. At the time of decoupling, most of these particles will be ultra-relativistic and their (mean) momentum $p(t_D)$ and energy $E(t_D) = [p^2(t_D) + m^2]^{1/2} \cong p(t_D)$ will be of order T_D. Their distribution at $t = t_D$ is well approximated by the f_{eq} of zero-mass particles. Decoupling 'freezes' the distribution function in this form.

At a later time ($t > t_D$), the mean momentum of the particles will be red-shifted to a value $p(t) = p(t_D)(a_D/a) \cong T_D(a_D/a)$. Hence, using (3.3.5), we have to find the energy density of **decoupled** species which is given by

$$\rho(t) = g_D \left(\frac{\pi^2}{30}\right) T^4(t), \tag{4.2.8}$$

and applying the result in (4.2.5), we obtain

$$\rho(t) = g_D \left(\frac{\pi^2}{30} T_D^4\right) \left(\frac{a_D}{a}\right)^4 = \rho(t_D) \left(\frac{a_D}{a}\right)^4, \tag{4.2.9}$$

where $\rho(t_D)$ is the energy density of relativistic particle at decoupling, given by

$$\rho(t_D) \equiv g_D \left(\frac{\pi^2}{30} T_D^4\right)$$

For $t \gg t_D$ most of the particles will have momentum $p(t)$, which is much smaller than m. Thus, the individual particles would have become non-relativistic when the universe expanded sufficiently which happens when the temperature of the universe drops below $T_{nr} \cong m$; that is, when $(a/a_D) \geq (T_D/m)$. The energy of each of these particles will now be $E(t) = [p^2(t) + m^2]^{1/2} \cong m$. But the distribution function (and the number density) of particles will still be given by the (frozen-in) form which corresponds to relativistic particles but the energy density will be that of non-relativistic particles: $\rho_{dec} \cong nm$.

4.3 Non-relativistic Decoupling

Consider that of a species which decouples when most of the particles are already non-relativistic: ($T_D \ll m$). In this case,

$$f_{dec}(p) = f_{eq}\left(\frac{pa}{a_D}, T_D\right) \tag{4.3.1}$$

In the view of (3.0.5) and (3.2.2), the equilibrium distribution function for non-relativistic species can be given by

$$f_{eq}(p) = \frac{g}{(2\pi)^3} exp\left\{-\frac{(m-\mu)}{T}\right\} exp\left\{-\frac{p^2}{2mT}\right\} \tag{4.3.2}$$

Now introducing (4.3.2) into (4.3.1), we obtain

$$\begin{aligned} f_{dec}(p) &\simeq \frac{g}{(2\pi)^3} exp\left[-\frac{(m-\mu)}{T_D}\right] exp\left[-\frac{p^2}{2m}\frac{1}{T_D}\left(\frac{a}{a_D}\right)^2\right] \\ &= \frac{g}{(2\pi)^3} e^{-m/T_D} exp\left[-\frac{p^2}{2mT_D}\left(\frac{a}{a_D}\right)^2\right], \end{aligned} \tag{4.3.3}$$

where we have further assumed $\mu \ll T_D$. This distribution function has the same form as that of a non-relativistic Maxwell-Boltzmann gas with a 'temperature' $T = T_D(a_D/a)^2$, which decreases as the square of the expansion factor.

Then, the distribution is similar to an equilibrium but with a new temperature

$$T(t) = T_D\left(\frac{a_D}{a}\right)^2 \tag{4.3.4}$$

Let us now derive the energy density from the calculated number density equation.

Number Density

By using (3.2.7), the number density of a non-relativistic particle at decoupling can be written as:

$$n(t) = g\left(\frac{mT(t)}{2\pi}\right)^{3/2} exp\left[-\frac{\{m-\mu\}}{T(t)}\right], \tag{4.3.5}$$

on the basis of equation (4.3.3) we have

$$\begin{aligned} n(t) &= g\left\{\frac{mT_D}{2\pi}\left(\frac{a_D}{a}\right)^2\right\}^{3/2} exp\left\{-\frac{(m-\mu)}{T_D}\right\} \\ &\cong g\left(\frac{mT_D}{2\pi}\right)^{3/2} e^{-m/T_D}\left(\frac{a_D}{a}\right)^3 \qquad (for \quad \mu \ll T_D) \end{aligned} \tag{4.3.6}$$

Energy Density

Correspondingly, this limit ($m \gg T_D$) gives the energy density as

$$\rho(t) \cong n(t)m \tag{4.3.7}$$

Then using (4.3.6) in, which yields

$$\rho(t) \cong mg\left(\frac{mT_D}{2\pi}\right)^{3/2} e^{-m/T_D}\left(\frac{a_D}{a}\right)^3 \tag{4.3.8}$$

Note that as is to be expected, $n \propto a^{-3}$. To any species of particle which is not being created or destroyed $n \propto a^{-3}$, we can assign a conserved number $N \propto na^3$; since $a^3 \propto s^{-1}$, we can conveniently define this number to be $N \equiv (n/s)$. From equation (3.3.15) we have that,

$$s \cong \frac{1}{T}(\rho + p) = \frac{2\pi^2}{45} qT^3 \tag{4.3.9}$$

On the basis of (3.3.13), in the **relativistic limit**, one can write the number density as:

$$n = g_{eff}\left(\frac{\zeta(3)}{\pi^2}\right)T^3, \tag{4.3.10}$$

and by taking the ratio of (4.3.10) to (4.3.9) we easily find,

$$N = \frac{g_{eff}\left(\frac{\zeta(3)}{\pi^2}\right)T^3}{q\left(\frac{2\pi^2}{45}\right)T^3} \cong 0.28(g_{eff}/q), \tag{4.3.11}$$

where $g_{eff} = g$ for Bosons and $g_{eff} = (3g/4)$ for Fermions.

On the other hand, in the **non-relativistic** case , we see from (3.2.7) that

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}, \tag{4.3.12}$$

which is valid for $\mu \ll T$ and then it follows

$$N = \frac{g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}}{q\left(\frac{2\pi^2}{45}\right)T^3} \cong 0.15(g/q)(mT)^{3/2}e^{-m/T} \tag{4.3.13}$$

Chapter 5

Discussion and Conclusion

5.1 Discussion

The universe gets contributions from many different species of particles, relativistic or non-relativistic. However, as the temperature rises as $T = (1 + Z)T_o$, the average energy of particles increases, allowing more and more kinds of interactions to take place, and which result in many particle species kept in thermal equilibrium. From (3.3.16) as far as q is constant, which gives $a^{-3} \propto T^3$ and as a consequence, the scale factor grows inversely proportional to the temperature of the universe, $a \propto \frac{1}{T}$. The energy density , as shown in (3.2.9), is exponentially suppressed as compared to relativistic particles, which decay as $\rho \propto \frac{1}{T^4}$. It is therefore a good approximation to include only the contributions from relativistic particles, which in turn indicates that the universe was **radiation dominated**. Particles species typically do not stay in equilibrium forever; eventually if their interactions become much less as compared with universe expansion, they possibly fall out of equilibrium (decoupling).

To see the difference between relativistic and non-relativistic species after decoupling, consider the evolution of the phase space distribution of a massive, non-relativistic particle species that was in thermal equilibrium, and decuples when $t = t_D$, $T = T_D$, and $a = a_D$. The momentum of each particle red shifts as the universe expands, from which it follows that the temperature red shifts as a^{-2}: $T \propto \frac{1}{a^2}$.

Further, the energy density which is easy to understand: since the energy density is proportional to the matter density and no matter disappears, the density decreases inversely proportional to the volume, which in turn is proportional to a^3.

In contrast, the energy of each massless particle is red shifted by the expansion of the universe: $E(t) = E(t_D)\{a(t_D)/a(t)\}$, and with the temperature red shifted as a^{-1}. As the result, the number density of particles decreases. In addition, the energy density decay as $\rho \propto \frac{1}{a^4}$; this shows that the relativistic species of the decreases with in additional expansion factor a as compared to the non- relativistic species. This leads us to see that non-relativistic **matter dominated** the universe. Even if matter dominates today, back in time it would have been radiation dominated, our result also shows this.

5.2 Conclusion

Generally, this thesis covered how the statistical description that we suppose is valid in the early moments of the universe along the thermal history of our basic cosmological model. In particular, starting from finding universe quantities in an equilibrium distribution, we explored the consequence introduced in the processes of decoupling.

It can be concluded as equations (4.2.9) and (4.3.8) predict that, the universe is expanding in shifting its phase from radiation dominated to matter dominated. These could be resulted from obtained expansion factor in an energy density on the basis of the well-known equilibrium distribution function. In statistical mechanics, as we have seen in detail the behavior and interactions of realistic particles at high temperature, that governing the expansion of the universe, this then allows us to gain a qualitative understanding of the early universe.

Finally, the observational fact shows that the universe is expanding today implies that in the past the universe must have been much hotter and denser; and that in the future it will become much colder and dilute.

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Declaration

I, the undersigned, declare that this thesis is my original work and it has not been presented before for a degree in any other University. Moreover, I declare that all the sources of material used for the thesis have been dully acknowledged.

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This thesis has been submitted for examination with my approval as
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