TEMPERATURE DEPENDENCE VISCOSITY OF THE NEUTRON STAR MATTER

By

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Abstract

In this work, we review the basics of both ideal and non ideal fluid dynamics. Fundamental principles of fluid dynamics,solution of nonlinear Boltzmann equation, transport coefficients, and moment transfer, are discussed. To solve the collision integral we have used the concept of time reversal symmetry. The basic equations governing fluid motion , energy and momentum are discussed. The influence of viscosity and its time and temperature dependency in fluid motion is also discussed. These concepts are then applied systematically in Neutron star matter to obtain the general balance equation and to investigate the motion of fluid particles between two parallel planes moving with a constant relative velocity . Difference between ideal and non ideal fluids are discussed.

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Boltzmann equation

Chapter 1

Introduction

In kinetic theory we work with distribution functions for points representative of the system in the phase space at a given time. A distribution function obeys its evolution equation built on the support of classical or quantum equations of motion for particles com praising the system. In this sense, the kinetic theory of matter is mesoscopic theory ranked, in the heirarky of theories between microscopic theory of matter for a few particles, and the macroscopic theory that has no molecular picture but assume some hypothesis of a statistical nature of that are alien to Newtonian or quantum mechanics of a few isolated particles and combines them with few-body systems.

As a result, one arrives at an evolution equation for the distribution function, but this equation no longer can be deemed completely mechanical in the sense that Newton's or Schroedinger's equation is for a few-body systems such as a hydrogen or Helium atom, for it has as one of its basic elements in the statical concepts that are not mechanical and, moreover, as a correctly be haved evolution equation for a macroscopic system, must have its time reversal invariance broken. This later feature, is in stark contrast to the equations of motion, classical or quanta l, for a few (molecular or atomic) system which are time reversal invariant. It then is important to find such an equation in the kinetic theory.

One conspicuous, universal behavior exhibited by non equilibrium system is their uni-directional tendency, when their evolution is left unhindered, to approach spontaneously

to ward the state of equilibrium. There fore, evolution of such systems is irreversible and the evolution equations for the probability distribution functions must be broken of their time reversal symmetry. we will call the evolution equations for the probability distribution functions the kinetic equations, considered in this work will be irreversible, namely, of broken time reversal invariance. Such a kinetic equations entails contraction of information as a non equilibrium state tends to ward that of equilibrium. Such a kinetic equations was discovered for dilute mono-atomic fluids (gases) by J.C.Maxwell and L.Boltzmann in the later part of nineteenth century. Boltzmann equation, by considering the collision dynamics of two particles with in the frame work of Newtonian mechanics and combining it with a statistical assumption which appears in the guise of molecular chaos assumption. This assumption is not of a mechanical origin and there fore can not be derived from mechanical principle alone. So the Boltzmann equation is not mechanical equation as Newtons equation of motion.

Boltzmann equation, rationalize it from Liouville's equations, and its solution methods, and the irreversible thermodynamics implied by its solutions. Finally we shall expect that the collision dynamics of particles phenomena results the transport coefficients namely thermal conductivity, shear viscosity and bulk viscosity of a fluid during energy and momentum transfer through a given time and temperature in a closed system.

In our discussion we shall start with the simple system such as ideal fluids, where, the intermolecular potential can be neglected, and progressed to more complicated system where the intermolecular potential plays a key role. And also we shall derive the molecular transport coefficient such as thermal conductivity(K), shear viscosity (η) and bulk viscosity (ζ) which are the result of particle scattering in the equilibrium situation and non equilibrium situation. Our aim is to show that the tansport coefficient is inversely proportional to temperature.

Calculation of viscosity is important study the behavior of the neutron star matter. The dissipative processes play an important role in the evolution of neutron stars. This

process are governed by transport coefficients such as heat conductivities, shear and bulk viscosity.

1.1 Objective:

At the end of this essay some one should be able to calculate :

- Viscosity from collision integral in the hydrodynamic regime.
- Fluid viscosity vanishes in the non-hydrodynamic regimes because of the absence of collision.

Chapter 2

PARTICLE DISTRIBUTION FUNCTION

2.1 One particle distribution function

The microscopic level a fluid, can be thought of as a collection of molecules. Ignoring the internal structure of molecules, are can be specify the state of any molecule of mass $'m'$ by giving of its position $\vec{x}$ and a molecules' momentum $\vec{P} = m\vec{v}$.
Let $dN = f(\vec{x}, \vec{p}, t)d^3\vec{x}d^3\vec{p}$ denotes the number of molecules in a phase volume $d^3\vec{p}$ at time, 't'. We are interested in the form and evolution of this distribution function (d.f). The distribution function can be changes due to two kinds of forces. These are :

1. Due to the macroscopic external force.
2. Due to the collision of the particles.

Macroscopic external force field, for example the gravitational field can be exert forces on the molecules and influence their motion. Such a force $\vec{F}_{ext}(\vec{x}, t) = \vec{\nabla}U_{ext}(\vec{x}, t)$ will vary soothly over the macroscopic level and can be derived from a suitable potential $U(\vec{x}, t)$. particular molecules also will experience the force , $\vec{F}_{coll}$ due to collisions. Molecule whenever it is close to neighbor molecule, the collisions are assumed to be uncorrelated in space and time, and conserved the total momentum and energy of the colliding particles.

The dynamics of such collision can be completely characterized by a differential cross section of $\sigma(\Omega)$ where Ω denotes the two angular coordinates θ and φ. If a system under consideration is in the classical kinetic theory of a fluid (diluted gas) of 'N' molecules enclosed in a box of volume 'V' with temperature is sufficiently high and density is sufficiently low for a molecule to be localized wave packets whose extension are small compared to the average intermolecular distance. For this to be realized the average de Broglie wave length of a molecules must be much smaller than the average intermolecular separation. Under such conditions each molecule may be considered as a classical particle with a rather well defined position and momentum. The molecules interact with each other through collisions whose nature is specified through a differential scattering cross section $\sigma(\Omega)$.

Let consider the system of one kind of molecule, in which the theory of none-qulibrium phenomena (nonlinear Boltzmann equation) tells us that how the observed irreversibly of macroscopic system emerges from the microscopic dynamics. Boltzmann's theory lies at a level some what intermediate between the purely stochastic analysis and the fully microscopic treatment. It faces the dynamical problem in its simplest aspect (i.e., the two body problem) but avoids its most delicate feature (i.e the apparent randomness of molecular motion) by introducing the extra mechanical, probabilistic assumptions. These assumption conserved the distribution functions of the fluids, not the individual motion of each molecule. Now we are not interested in the motion of each molecule in detail, rather we are interested the distribution function, f$(\vec{x}, \vec{p}, t)$.

$$dN = f(\vec{x}, \vec{p}, t) d^3\vec{x} d^3\vec{p} \tag{2.1.1}$$

is the number of molecules which at time 't', have position lying with in a volume element $d^3\vec{x}$ about $\vec{x}$ and momenta lying with in a momentum space element $d^3\vec{p}$ about $\vec{p}$. The volume elements $d^3\vec{x}$ and $d^3\vec{p}$ are not taken to be mathematical infinitesimal quantities. They are macroscopically small and microscopically large enough. To make the definition

of distribution function (d.f) more precise. Let us consider the six dimensional space called μ space, spanned by the coordinates $(\vec{x}, \vec{p})$ is referred to as a molecule's coordinate.

A point in μ space represents a states of a molecule. At any instant of time, the state of entire system of 'N' molecules is represented by 'N' points in μ space. dN is the number of molecules in the volume of $d^3\vec{x}d^3\vec{p}$ is $f(\vec{x}, \vec{p}, t)d^3\vec{x}d^3\vec{p})$ stated in equation (2.1.1) above. If the density of the points does not vary rapidly from one element to a neighboring element, then $f(\vec{x}, \vec{p}, t)$ may be regarded as a continuous function. we can approximate equation (2.1.1) over the entire μ space as follow,

$$\sum f(\vec{x}, \vec{p}, t)d^3\vec{x}d^3\vec{p} \approx \int f(\vec{x}, \vec{p}, t)d^3\vec{x}d^3\vec{p} \tag{2.1.2}$$

through the normalization condition. The total number of particles in the entire system is can be obtained from the above equation by;

$$N = \int f(\vec{x}, \vec{p}, t)d^3\vec{x}d^3\vec{p} \tag{2.1.3}$$

If molecules are uniformly distributed in space, the distribution function 'f' is independent of position $\vec{x}$ then the equation (2.1.3) is written as;

$$\int f(\vec{x}, \vec{p}, t)d^3\vec{x}d^3\vec{p} = \frac{N}{V} \tag{2.1.4}$$

where, $\int d^3\vec{x} = V$. Now the aim of kinetic theory is to find the distribution function (d.f) for a given form of molecular interaction. The limiting form of $f(\vec{x}, \vec{p}, t)$ as time goes to infinite $(t \to \infty)$ would then contain all the equilibrium properties of the the system. In our aim of kinetic theory we include the derivation of thermodynamics of a fluid.
First, we should have to drive the equation of motion for the distribution function which changes with time, because molecules constantly enter and leave a given volume element in μ-space. The total distribution function satisfies the Liouvile's equation.

2.2 Phase space and Liouvile's theorem

Consider a system of N-particles which move in accordance of the Hamiltonians equation of motion.

$$\dot{\vec{X}} = \frac{\partial H}{\partial \vec{P}_i} \tag{2.2.1}$$

$$\dot{\vec{p}} = \frac{\partial H}{\partial \vec{x}_i} \tag{2.2.2}$$

where, i = 1,2,....,N . And H denotes Hamiltonian, $\vec{x}_i$ and $\vec{p}_i$ are the coordinates and conjugate of momenta in the i^{th} particle. The solution of this equation may represent by so cold phase point of coordinates $\vec{x}$ and $\vec{p}$ in the 6N-dimensional phase space. As the time elapse, the phase point moves around in phase space according to equations of motion, but the trajectory does not intersect it self at any point, because of the uniqueness of the solution of the differential equations. In general the number of particles in a macroscopic system is very large, and the equation of motion are too complicated for rigorous solution. Moreover in most case, the precise initial conditions needed to solve the differential equations are not known. Furtherer more, if we are concerned the macroscopic properties of the system, only the average behavior of the particles are important. The distribution function defines an average value of a given function in the μ-space. For instance let $G(\vec{x}, \vec{p})$ is a given function and its average value is given by:

$$\langle G \rangle = \frac{1}{N} \int G(\vec{x}, \vec{p}) f(\vec{x}, \vec{p}, t) d^3\vec{x} d^3\vec{p} \tag{2.2.3}$$

where, N is the normalization constant of $f(\vec{x}, \vec{p}, t)$ and $\langle G \rangle$ time independent.

If there is no sink or source the total number of systems in ensemble is conserved and the number of phase points leaving any volume must equal to the rate of decreasing number of phase points (representative points) in the same volume. Let 'V' is an arbitrary volume and $\vec{S}$ be its surface and if we denote by $\vec{v}$ the six-dimensional vector whose components are:

$$\vec{v} \equiv (\dot{\vec{p}}_1, \dot{\vec{p}}_2, ..., \dot{\vec{p}}_N, \dot{\vec{x}}_1, \dot{\vec{x}}_2, ..., \dot{\vec{x}}_N) \tag{2.2.4}$$

and $\hat{n}$ is the vector locally normal to the surface $\vec{S}$ is:

$$-\frac{d}{dt}\int_V dV\rho = \int_s ds\hat{n}\cdot\vec{v}\rho \tag{2.2.5}$$

where, $\vec{v}\rho$ is the current vector.

dV is the volume element.

ρ is the mass density.

$d\vec{s}$ is the surface element..

The surface integral can be converted to volume integral with the help of divergence theorem in 6N-dimensional space. These equation become;

$$-\frac{d}{dt}\int_v dV\rho = \int_v dV(\vec{\nabla}.(\vec{v}\rho)) \tag{2.2.6}$$

$$-\frac{d}{dt}\int_v dV\rho - \int_v dV(\vec{\nabla}.(\vec{v}\rho)) = 0 \tag{2.2.7}$$

by ignoring the volume integral over the volume:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot v\vec{\rho} = 0 \tag{2.2.8}$$

This equation denotes the conservation law (mass continuity equation). The operator $\vec{\nabla}$ in the 6N-dimensional gradient operator is:

$$\vec{\nabla} \equiv \left(\frac{d}{d\vec{p}_1}, \frac{d}{d\vec{p}_2}, ..., \frac{d}{d\vec{p}_n}, \frac{d}{d\vec{x}_1}, \frac{d}{d\vec{x}_2}, ..., \frac{d}{d\vec{x}_n}\right) \tag{2.2.9}$$

by substitution equation (2.2.9) in to equation (2.2.8) we obtain:

$$\frac{\partial\rho}{\partial t} = \sum_{i=1}^{N}\left(\frac{\partial}{\partial\vec{p}_i}(\dot{\vec{p}}_i\rho) + \frac{\partial}{\partial\vec{x}_i}(\dot{\vec{x}}_i\rho)\right) \tag{2.2.10}$$

$$\frac{\partial\rho}{\partial t} = \sum_{i=1}^{N}\left(\frac{\partial\rho}{\partial\vec{p}_i}\cdot\dot{\vec{p}}_i + \frac{\partial\dot{\vec{p}}}{\partial\vec{p}_i}\cdot\rho + \frac{\partial\rho}{\partial\vec{x}_i}\cdot\dot{\vec{x}}_i + \frac{\partial\dot{\vec{x}}}{\partial\vec{x}_i}\cdot\rho\right) \tag{2.2.11}$$

$$\frac{\partial\rho}{\partial t} = \sum_{i=1}^{N}\left(\frac{\partial\rho}{\partial\vec{p}_i}\cdot\dot{\vec{p}}_i + \frac{\partial\rho}{\partial\vec{x}_i}\cdot\dot{\vec{x}}_i\right) + \sum_{i=1}^{N}\rho\left(\frac{\partial\dot{\vec{p}}}{\partial\vec{p}}_i + \frac{\partial\dot{\vec{x}}}{\partial\vec{x}}_i\right) \tag{2.2.12}$$

we consider $\rho(\vec{x},\vec{p},t)$ is the distribution function of density of particles, and we can denote it by $f = f(\vec{x},\vec{p},t)$ which is the distribution function of an arbitrary dynamical quantity.

Thus,

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial \vec{p}_i} \cdot \dot{\vec{p}}_i + \frac{\partial f}{\partial \vec{x}_i} \cdot \dot{\vec{x}}_i \right) + \sum_{i=1}^{N} f \left(\frac{\partial \dot{\vec{p}}}{\partial \vec{p}_i} + \frac{\partial \dot{\vec{x}}}{\partial \vec{x}_i} \right) \tag{2.2.13}$$

The second term on the right hand side vanishes, from the equation of motion. By substitution of equation (2.2.1) and (2.2.2) in to (2.2.13) we obtain that:

$$\frac{\partial}{\partial \vec{p}} \cdot \dot{\vec{p}} + \frac{\partial}{\partial \vec{x}} \cdot \dot{\vec{x}} = -\frac{\partial^2 H}{\partial \vec{x}_i \partial \vec{p}_i} + \frac{\partial^2 H}{\partial \vec{p}_i \partial p \vec{x}_i} = 0 \tag{2.2.14}$$

so equation (2.2.13) be comes :

$$\frac{\partial f}{\partial t} + [f, H] = 0 \tag{2.2.15}$$

Hence,

$$[f, H] = \left(\frac{\partial f}{\partial \vec{x}_i} \cdot \frac{\partial H}{\partial \vec{p}_i} - \frac{\partial f}{\partial \vec{x}_i} \cdot \frac{\partial H}{\partial \vec{p}_i} \right) \tag{2.2.16}$$

This is called the Liouville's theorem.

2.3 Interaction of fluid particles under molecular collisions

The distribution function can be changes due to macroscopic external force fields and collisoins. Now we suppose first, as there were no molecular collisions (i.e., no cross section scattering, $\sigma(\Omega) = 0$). The molecules at coordinate $((\vec{x} + \vec{v}\delta t + \delta t), (\vec{p} + \vec{F}\delta t), t + \delta t)$ at the instant of $t + \delta t$. where $\vec{F}$ is the external force acting on a molecule. We take δt is an infinitesimal quantity. The probability of finding of particle in the μ-space is conserved and all the molecules contained in a μ-space element $d^3\vec{x}d^3\vec{p}$ at $(\vec{x}, \vec{p})$ at the instant of time t will be found $d^3\vec{x}d^3\vec{p}$ at $((\vec{x} + \vec{v}\delta t + \delta t), (\vec{p} + \vec{F}\delta t), t + \delta t)$ at the time $t + \delta t$. In the absence of collision the probability of i^{th} particle in the range of $\vec{v}_i \sim \vec{v}_i + d\vec{v}_i$ and $\vec{x} \sim \vec{x} + d\vec{x}$ at time t is conserved. This indicates that the number of particles in the volume element is conserved.

let, N = number of particles before collision

N' = number of particles after collisions

If there is no collision the initial number of molecules are equal with the final number of molecules. so,

$$dN = dN' \tag{2.3.1}$$

In phase volume element the number of molecules, dN, are described by :

$$dN = f(\vec{x}, \vec{p}, t) d^3\vec{x} d^3\vec{p} \tag{2.3.2}$$

The equation can be written as;

$$f(\vec{x}, \vec{p}, t) d^3\vec{x} d^3\vec{p} = f(\vec{x} + \vec{v}\delta t, \vec{p} + \vec{F}\delta t, t + \delta t) d^3\vec{x} d^3\vec{p} \tag{2.3.3}$$

Since we consider as no collision the volume elements are equal in the final states and in the initial states .i.e.,

$$d^3\vec{x'} d^3\vec{p'} = d^3\vec{x} d^3\vec{p} \tag{2.3.4}$$

The primed quantity indicates that, for the final states. We assume that the external force depends only on the position. The volume element is invariant under the dynamical evolution of time 't'. Now, we can think as, $d^3\vec{x}^3\vec{p}$ to be a six-dimensional cube, be cause we have three components of the coordinate (X_1, X_2, X_3) and the three components of momenta (P_x, P_y, P_z) in the μ-space. The area of any projection of the cube $(dxdp_x, dydp_y, dxdp_y, ...)$ does not change. For instance the projection of $dxdp_x$, originally a square be come a parallelogram of the same area (i.e.,a rhombus) in the time interval δt. This invariance is valid as long as $(\vec{x}, \vec{p})$ are canonically conjugate generalized coordinates.

Since the distribution function changes with time and the number of particles are conserved, we have an equation of;

$$f(\vec{x} + \vec{v}\delta t, \vec{p} + \vec{F}\delta t, t + \delta t) - f(\vec{x}, \vec{p}, t) = \delta f \tag{2.3.5}$$

Expanding the left hand side by using Taylor's series to the first order of δt we get,

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}_x+\vec{F}\cdot\vec{\nabla}_p\right)f+f=f+\delta f \tag{2.3.6}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}_x+\vec{F}\cdot\vec{\nabla}_p\right)f=\delta f \tag{2.3.7}$$

where $\delta f = C[f]$ is the collision term and $\vec{\nabla}_x$ and $\vec{\nabla}_p$ are gradient operators of position and momenta respectively. the external force field depends on the external potential.

$$\left(\frac{\partial f}{\partial t}+\vec{v}\cdot\vec{\nabla}f-\vec{\nabla}\phi\cdot\frac{\partial f}{\partial\vec{p}}\right)\triangle\,\vec{x}\,\triangle\,\vec{p}\,\triangle\,t=C[f]\,\triangle\,\vec{x}\,\triangle\,\vec{p}\,\triangle\,t \tag{2.3.8}$$

Where , $\phi = u_{ext}$ is the external potential. The left hand side equation (1.3.8) is the change due to collisions, in the time interval Δt of the number of molecules lying in the element $\Delta\vec{x}\Delta\vec{p}$ around the point $(\vec{x},\vec{p})$ in μ-space.

In the law density limit, $nx_o^3 << 1$,where, x_o is a parameter characterizing the range of the intermolecular force (eg. $x_o = a$, for hard sphere), and it is very natural to assume that only binary collisions have to be retained; in deed the probability of having more than two molecules in their mutual range should be much small, by a factor proportional to nx_o^3. Moreover such a binary collision takes place over a distance of order $\vec{x}_o$, which is very small compared to define the distribution function f; similarly the duration of this collision of order $\tau_c \approx \frac{x_o}{\langle v\rangle}$ is also very small compared to other thing time Δt used in defining f.

To define the collision term $C[f]$, we consider two points in a μ-space; these may be A and B, at $(\vec{x},\vec{p},t)$ and $(\vec{x}+\vec{v}\delta t,\vec{p}+\vec{F}\delta t,t+\delta t)$ respectively, where δt tends to zero. During the time interval δt some molecules in A will be removed from A by collision. We regarded A as so small that any collision that a molecule in A suffers will knock it out of A. Such molecules will not reach B. On the other hand, there are molecules out side A which, through collisions, will get in to A during the time interval δt. There will be

in B. Therefore the number of molecules in B at $t + \delta t$, as $(\delta t \to 0)$, equals in A due to collisions during the time interval δt. This can be expressed as:

$$\left(\frac{\partial f}{\partial t}\right)_{coll} \delta t = C[f]\delta t = (\bar{R} - R)\delta t \tag{2.3.9}$$

$R\delta t d^3\vec{x} d^3\vec{p}$ = the number of collisions during the time between t and $t + \delta t$ in the initial states of volume $d^3\vec{x} d^3\vec{p}$.

$\bar{R}\delta t d^3\vec{x} d^3\vec{p}$ = the number of collisions during the time between t and $t + \delta t$ in which one of the final states of volume $d^3\vec{x} d^3\vec{p}$ about $(\vec{x}, \vec{p})$. The error in this will be negligible because of smallness of $d^3\vec{p}$ and if we assume a molecule qualifies under this description, none of it parameters in collision qualifies.

2.4 Binary collisions

Let consider an elastic collison in free space between two spineless molecules of respective mass m_1, m_2. The momenta of molecule in the initial states are denoted by $\vec{p_1}$ and $\vec{p_2}$, respectively, and energies $\in_1$ and $\in_2$ with,

$$\in_i = \frac{p_i^2}{2m_i} \tag{2.4.1}$$

The corresponding quantities in the final states are indicated by a prime. Momentum conservation requires that:

$$\vec{p_1} + \vec{p_2} = \vec{p_1'} + \vec{p_2'} \tag{2.4.2}$$

$$E = \in_1 + \in_2 = \in_1' + \in_2' \tag{2.4.3}$$

where, E is the total energy. We define the total mass by M and the reduced mass by μ

$$M = m_1 + m_2 \tag{2.4.4}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.4.5}$$

And the total momentum, $\vec{P}_{tot}$ and relative momentum $\vec{p}$ are given by:

$$\vec{P}_{tot} = \vec{p_1} + \vec{p_2} \tag{2.4.6}$$

$$\vec{p} \equiv \vec{P}_{tot}\frac{m_1}{M} - \vec{p_1} \tag{2.4.7}$$

$$\vec{p} \equiv \frac{m_2\vec{p_1} - m_1\vec{p_2}}{m_1 + m_2} \tag{2.4.8}$$

$$\vec{p} = \mu(\vec{v_1} - \vec{v_2}) \tag{2.4.9}$$

where , $\vec{v_i} = \frac{\vec{p_i}}{m_i}$ is the velocity of i^{th} particles.

$$\vec{p_1} = \frac{m_1}{M}\vec{P}_{tot} - \vec{p} \tag{2.4.10}$$

$$\vec{p_2} = \frac{m_2}{M}\vec{P}_{tot} + \vec{p} \tag{2.4.11}$$

It can be verified that the total energy is given by:

$$E = \frac{\vec{P}^2_{tot}}{2M} + \frac{\vec{p^2}}{2\mu} \tag{2.4.12}$$

The conditions for momentum energy conservation be comes, simply,

$$\vec{P}_{tot} = \vec{P'}_{tot} \tag{2.4.13}$$

$$|p| = |p'| \tag{2.4.14}$$

The collision marely rotates the relative momentum with out changing its magnitude. Let the angle between $\vec{p'}$ and $\vec{p}$ be θ and the azimuthal angle of $\vec{p'}$ about $\vec{p}$ be Φ. These angles are completely specifies the kinematics of the collision. They are collectively denoted by Ω, and are called scattering angles. If the potential responsible for the scattering is a central potential (i.e., dependent only on the magnitude of the distance between the molecules), then the scattering is independent of Φ. The dynamical aspects of the collision are contained in the differential cross section $\frac{d\sigma}{d\Omega}$ which is defined as follow. Consider a beam of particle 2 incident on a particle 1, regarded as the target. The incident flux I is defined as the number of incident particles crossing a unit area per second , from the view point of the target.

$$I \equiv n \mid \vec{v_1} - \vec{v_2} \mid \tag{2.4.15}$$

where , n is the density of the molecules in the incident beam. The differential cross section is defined by the statement that: N_{inci} is the number of incident molecules scattered per second in the solid angle element $d\Omega$ about the direction Ω

$$I\left(\frac{d\sigma}{d\Omega}\right) \equiv N_{inci} \tag{2.4.16}$$

The differential cross section has the dimension of the area. The number of molecules scattered in to $d\Omega$ per second is equal to the number of molecules in the incident beam crossing an area $\frac{d\sigma}{d\Omega}$ per second. The total cross section is the number of molecules scattered per second, regardless of the scattering angles:

$$\sigma_{tot} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right) \tag{2.4.17}$$

where, σ_{tot} is the total cross section area.

In classical mechanics the differential cross section can be calculated from the intermolecular potential as follow. First we transform the coordinate system to the center of mass system, in which the total momentum is zero. Since we are considering only the non relativistic domain, this involves a trivial translation of all velocities by a constant amount. We only need to follow the trajectory of one of the particles, which will move along an orbit, as if it were scattered by a fixed center of force 'O'.

If a particle approaches point 'O' with momentum $\vec{p}$ the relative momentum and will recede from 'O' with a momentum $\vec{p'}$, the rotated relative momentum. The normal distance between the line of approach and 'O' is called the impact parameter denoted by b. By conservation of angular momentum, it is also the normal distance between the line of recession and 'O'. This is indicated as :

$$I\left(\frac{d\sigma(\Omega)}{d\Omega}\right) d\Omega = I b db d\Phi \tag{2.4.18}$$

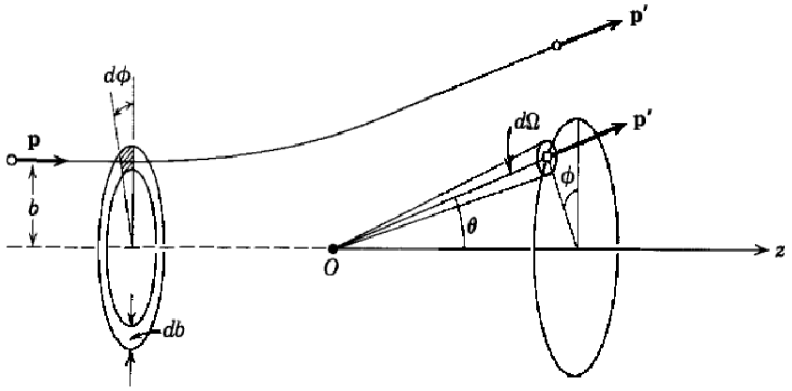

Figure 2.1: classical scattering of a molecule by a fixed center of force O

We can find the relation ship between b and the scattering angles from the classical orbit equation, there may obtaining $\frac{d\sigma}{d\Omega}$ as a function of scattering angles. The use of classical mechanics to calculate the differential cross section in this problem predating quantum mechanics. Now, it is better to, when we use quantum mechanics because when molecules collide their wave functions necessarily over lap and they see each other as a plane waves of definite momenta rather than wave packets of well defined positions. Further more, formulating the scattering problem quantum mechanically, makes the kinematics and symmetries of the problem more obvious.

In quantum mechanics the basic quantity in scattering problem is the transition matrix (T- matrix), whose elements are the matrix element of a certain operator T(E) between initial state (i) and final state (f). so, the equation is:

$$T_{fi} = \langle 1', 2'|T|1, 2\rangle \tag{2.4.19}$$

A collision is a transition from the initial state to the set of final state. We know that the square of transition matrix is determined as:

$$\mid T_{fi} \mid^2 = \delta_{if} \tag{2.4.20}$$

where, the value of δ_{if} is 1 for $i = f$ and zero for $i \neq f$. For final states in the infinitesimal momentum-space element $d^3\vec{x}d^3\vec{p}$, the rate is:

$$dP_{12\to 1'2'} = Id\sigma \tag{2.4.21}$$

$$d\vec{P}_{12\to 1'2'} = d^3\vec{p'}_1 d^3\vec{p'}_2 \delta^4(\vec{P}_f - \vec{P}_i) \mid T_{fi} \mid^2 \tag{2.4.22}$$

where, $\delta^4(\vec{P}_f - \vec{P}_i) = \delta^3(\vec{P'} - \vec{P})\delta(E' - E)$

$\vec{P}_f$ and $\vec{P}_i$ are the final and initial total momentum respectively.

The transition rate in to any region of momentum space can be obtained by integrating the above over the appropriate region. To obtain the differential cross section, integrate over the recoil momentum $\vec{p}_1$ which is fixed determined by energy conservation and magnitude $\vec{p'}_2$ to obtain

$$I(\frac{d\sigma}{d\Omega}) = \int d^3\vec{p'}_1 \int d\vec{p'}_2 \vec{p'}_2^{\,2} \delta^4(\vec{P}_f - \vec{P}_i) \mid T_{fi} \mid^2 \tag{2.4.23}$$

The integration are trivial, and yield a factor representing the density of final states, because the δ-function that in force momentum energy conservation. The transition matrix (T-matrix) is invariant under the spatial rotation, reflection, and under time reversal, because all molecular interactions originate in the electromagnetic interaction, which have these invariance properties, explicitly we have

$$\langle \vec{p'}_2, \vec{p'}_1 | T | \vec{p}_1, \vec{p}_2 \rangle = \langle R\vec{p'}_2, R\vec{p'}_1 | T | R\vec{p}_1, R\vec{p}_2 \rangle = \langle -\vec{p'}_2, -\vec{p'}_1 | T | -\vec{p}_1, -\vec{p}_2 \rangle \tag{2.4.24}$$

where $R_{\vec{p}}$ is the vector obtained from $\vec{p}$ after performing a spatial rotation about an axis, and a reflection with respect to a plane. For elastic scattering the density of states are the same in the initial and final state. Thus, the invariance of the T-matrix directly implies corresponding invariance of differential cross section. If the molecules have spin, the above equation (1.4.24) remains valid , provided we inter prate $\vec{p}$ to include the spin , a rotation

rotates both spin and momentum but reflection does not affect the spin. Under time reversal both the momentum and the spin change sign. Now from the equation (1.4.24) we can deduce that the inverse collision, defines as the collision with initial and final states interchanged, has the same T-matrix and hence the same cross section.

$$\langle \vec{p'}_2, \vec{p'}_1 | T | \vec{p}_1, \vec{p}_2 \rangle = \langle \vec{p}_2, \vec{p}_1 | T | \vec{p'}_1, \vec{p'}_2 \rangle \tag{2.4.25}$$

This transition phase leads us to find out the nonlinearity equation of states (the nonlinear Boltzmann equation of state.) which have the exact and none exact solutions and to work out its solutions is some what complicated.

Chapter 3

TRANSPORT PHENOMENA OF SYSTEM

3.1 Boltzmann transport equation

To drive the Boltzmann transport equation we assume that only binary collisions needed to taken in to account. The effect of external forces on a collision are ignored on the assumption that these forces if present , would vary a little over the range of inter molecular potential. The number of transitions 1 2 $\rightarrow$ 1' 2' in a volume element $d^3\vec{x}$ at $\vec{x}$ owing to collisions during the time interval δt is:

$dN_{12}d\vec{P}_{12\rightarrow 1'2'}\delta t$

where, dN_{12} is the initial number of colliding pairs $(\vec{p}_1, \vec{p}_2)$. Now we introduce the two particle correlation function F by;

$$dN_{12} \equiv F(\vec{x}, \vec{p}_1, \vec{p}_2, t)d^3\vec{x}d^3\vec{p}_1d^3\vec{p}_2 \tag{3.1.1}$$

Thus, in the notation of equation(2.3.9), we have,

$$R\delta t d^3\vec{x}d^3\vec{p}_1 = \delta t d^3\vec{x}d^3\vec{p}_1 \int d^3\vec{p}_2 dP_{12\rightarrow 1'2'}F(\vec{x}, \vec{p}_1, \vec{p}_2, t) \tag{3.1.2}$$

Using equation (2.4.21) and (2.4.22) we obtain that:

$$R = \int d^3\vec{p}_2 d^3\vec{p'}_1 d^3\vec{p'}_2 \delta^4(\vec{P}_f - \vec{P}_i) \mid T_{fi} \mid^2 F(\vec{x}, \vec{p}_1, \vec{P}_2, t) \tag{3.1.3}$$

Similarly for $\bar{R}$ we find that :

$$\bar{R} = \int d^3\vec{p}_2 d^3\vec{p'}_1 d^3\vec{p'}_2 \delta^4(\vec{P}_f - \vec{P}_i) \mid T_{fi} \mid^2 F(\vec{x}, \vec{p'}_1, \vec{P'}_2, t) \tag{3.1.4}$$

The δ-function in equation (3.1.3) and (3.1.4) is the same be cause $T_{if} = T_{fi}$. so,

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \bar{R} - R = \int d^3\vec{p}_2 d^3\vec{p'}_1 d^3\vec{p'}_2 \delta^4(\vec{P}_f - \vec{P}_i) \mid T_{fi} \mid^2 (F_{1'2'} - F_{12}) \tag{3.1.5}$$

where ; $F_{12} = F(\vec{x}, \vec{p}_1, \vec{p}_2, t)$ and $F_{1'2'} = F(\vec{x}, \vec{p'}_1, \vec{p'}_2, t)$

Note that we can integrate over the vector $\vec{p'}_1$ and the magnitude of $\vec{p}_2$, so the integral cross section appears in the integrand equation of (3.1.5) be comes

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = C[f] = \int d^3\vec{p}_2 d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (\frac{d\sigma}{d\Omega})(F_{1'2'} - F_{12}) \tag{3.1.6}$$

$$C[f] = \int d^3\vec{p}_2 \sigma(\Omega) d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (F_{1'2'} - F_{12}) \tag{3.1.7}$$

Now , let introduce the crucial assumption that :

$F(\vec{x}, \vec{p}_1, \vec{p}_2, t) \approx f(\vec{x}, \vec{p}_1, t) f(\vec{x}, \vec{p}_2, t) = f_1 f_2$

$F(\vec{x}, \vec{p'}_1, \vec{p'}_2, t) \approx f(\vec{x}, \vec{p'}_1, t) f(\vec{x}, \vec{p'}_2, t) = f'_1 f'_2$

This says that the momenta of two particles in the volume element of $d^3\vec{x}$ are uncorrelated , so that the probability of finding them simultaneously is the product of probability of finding each alone. This is known as the assumption of molecular 'chaos'. It is necessary to obtain a closed equation for the distribution function.

$$C[f] = \int d^3\vec{p}_2 \sigma(\Omega) d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (f'_1 f'_2 - f_1 f_2) \tag{3.1.8}$$

By substitution, the Boltzmann transport equation will be obtained;

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p_1}}{m} \cdot \vec{\nabla}_x - \vec{\nabla}\phi \cdot \vec{\nabla}_{p1}\right) f = \int d^3\vec{p}_2 \sigma(\Omega) d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (f'_1 f'_2 - f_1 f_2) \tag{3.1.9}$$

Which is a nonlinear integro-differential equation for the distribution. In this case we have considered only the case of a single species of spin less molecules. If we consider different types of molecules, then we have to introduce a separate distribution function

for each type, and the collision term will couple them if different types of molecules can scatter each other. Now if we eliminate the subscript 1 and replacing 2 by 1.

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}_x-\vec{\nabla}\phi\cdot\vec{\nabla}_p\right)f=\int d^3\vec{p}_1\sigma(\Omega)d\Omega\mid\vec{v}-\vec{v}_1\mid(f'f'_1-ff_1) \tag{3.1.10}$$

3.2 Mean free path

If we consider a fluid (gas) is not in equilibrium when the distribution function is different from Maxwell-Boltzmann distribution function. The most common case of a non equilibrium situation is that in which the temperature, density,and average velocity are not constant through out the system. To approach the equilibrium , non-uniformalities have to be ironed out through the transport of energy, mass and momentum from one part of the system to an other. The mechanism of transport is molecular collision, and the average distance over which molecular properties can be transported in one collision is the mean free path. It is the average distance traveled by a molecule between successive collisions. The number of collisions happening per second per unit volume is at the point in the path given by:

$$Z=\int d^3\vec{p}_1d^3\vec{p}_2d^3\vec{p'}_1d^3\vec{p'}_2\delta^4(\vec{P}_f-\vec{P}_i)\mid T_{fi}\mid^2(ff_1) \tag{3.2.1}$$

where, $f(\vec{x},\vec{p},t)$ is the distribution function. The integration is over $\vec{p'}_1$ and $\vec{p'}_2$ can be immediately to yield

$$Z=\int d^3\vec{p}_1d^3\vec{p}_2\sigma(\Omega)d\Omega\mid\vec{v}_1-\vec{v}_2\mid(ff_1) \tag{3.2.2}$$

A free mean path is defined as the distance traveled by a molecules between two successive collisions. Since it takes two molecules to make a collision, every collision terminates two free paths. The total number of free paths occurring per second per unit volume is 2Z. Since there is n molecules per unit volume, the average number of free paths traveled by a molecules per second is $\frac{2Z}{n}$. The mean free path, which is the average length of a free path is given by;

$$\lambda = \frac{n}{2Z}\langle\vec{v}\rangle \tag{3.2.3}$$

Where, k_b is Boltzmann constant, and $\langle\vec{v}\rangle = \sqrt{2K_bT}$ is the most probability speed of a molecule of gas. The average duration of free mean path is called collision time and is given by;

$$\tau = \frac{\lambda}{\langle\vec{v}\rangle} \tag{3.2.4}$$

finaly we obtain that

$$Z = \frac{\sigma_{tot}}{m}\int d^3\vec{p}_1 \int d^3\vec{p}_2 \mid \vec{v_1} - \vec{v} \mid f(\vec{p}_1)f(\vec{p}_2) \tag{3.2.5}$$

Frequency of the collision also can be calculated by $\frac{1}{\tau}$ that is the inverse of the collision time.

$$\nu = \frac{1}{\tau} \tag{3.2.6}$$

Where ν is the frequency of the collision.

3.3 Collision invariants

Since Boltzmann equation is nonlinear, its solutions not known, in general. Nevertheless some general and important properties of the collision terms can be proved , allowing to establish some remarkable features of this solution. To stress the nonlinearity of the collision term, it is continent to write :

$$(\partial_t f_1) = C[f] \equiv J(f_1, f_1) \tag{3.3.1}$$

Where, the quantity $J(f_1, f_1) = J$ is defined in general by:

$$J(f_1, f_1) = \frac{1}{2}\int d^3\vec{p}_1 \int \sigma(\Omega)d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (f'h'_1 + f'_1h - fh_1 - f_1h) \tag{3.3.2}$$

Here, f and h are arbitrary function of $\vec{x}$ and $\vec{p}$. Example $h_1 = h(\vec{x}, \vec{p}_1, t)$. we have the obvious symmetry property.

$$J(f, h) = J(h, f) \tag{3.3.3}$$

Let us consider the following integrals

$$I_u = \int d^3\vec{p} J(f,h)\psi(\vec{p}) \tag{3.3.4}$$

$$I_u = \frac{1}{2}\int d^3\vec{p}d^3\vec{p}_1 \int \sigma(\Omega)d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (f'h'_1 + f'_1h - fh_1 - f_1h)\psi(\vec{p}) \tag{3.3.5}$$

Where , $\psi(\vec{p})$ is an arbitrary function of $\vec{p}$ or $\vec{v}$ in this integral we 1^{st} make the change of variables, $\vec{p} \leftrightarrows \vec{p}_1$ and $\vec{p'} \leftrightarrows \vec{p'}_1$, because of symmetry. So the right hand side of the a bave equation is unchanged except that $\psi(\vec{p}) \rightarrow \psi(\vec{p}_1)$. Next we use $\vec{p'}$ and $\vec{p'}_1$ as a new integration variable; again recover a similar expressions except for the change $\psi(\vec{p}) \rightarrow \psi(\vec{p'})$ and change of sign. Finaly we exchange $\vec{p} \rightarrow \vec{p'_1}$ and $\vec{p_1} \rightarrow \vec{p'}$ with the result $\psi(\vec{p}) \rightarrow -\psi(\vec{p'})$, taking $\frac{1}{4}$ of the four equivalent expression.

$$I_u = \frac{1}{8}\int d^3\vec{p}d^3\vec{p}_1 \int \sigma(\Omega)d\Omega \mid \vec{v}_1 - \vec{v}_2 \mid (f'h'_1 + f'_1h - fh_1 - f_1h)(\psi(\vec{p}) + \psi(\vec{p}_1) - \psi(\vec{p'}) - \psi(\vec{p'}_1)) \tag{3.3.6}$$

The important consequence of this equation is that I_u identically vanishes for any function $\psi(\vec{p})$ such that

$$\psi(\vec{p}) + \psi(\vec{p}_1) \equiv \psi(\vec{p'}) + \psi(\vec{p'}_1) \tag{3.3.7}$$

Functions that have this property are called collision invariants. It can be shown rigorously that the most general collision invariant is a linear combination.

$$\psi(\vec{p}) = \sum_{\alpha=1}^{5} \psi_\alpha(\vec{p}) \tag{3.3.8}$$

of the five linearly independent collision invariants

$$\psi_1 = 1 \tag{3.3.9}$$

$$\psi_2 = \vec{v}_i \tag{3.3.10}$$

Where ,i = 2,3,4,=x,y,z

$$\psi_5 = \frac{v_i^2}{2} \tag{3.3.11}$$

$\vec{p}$ can be replaced by $\vec{v}$. These ψ_α are indeed a collisions invariants of little momentum, and of kinetic energy in bindery collision. What is less obvious is that they are the only linearly independent ones.

3.4 Conservation theorems (balanced equations)

The conservation properties of the collision invariants suggest that their average values

$$\langle \psi(\vec{x},t) \rangle = \frac{\int d^3\vec{p}\psi_\alpha(\vec{x},\vec{p})f(\vec{x},\vec{p},t)}{\int d^3\vec{p}f(\vec{x},\vec{p},t)} \tag{3.4.1}$$

obey simple equation of evolution. Moreover the average values are directly related to the macroscopic variable n, $\vec{u}$, T used to describe the hydrodynamical behavior of a simple fluid, and can be expect the study of their evolution to lead us to a microscopic understanding of hydrodynamics when potential energy is vanishes. Now to investigate the non equilibrium phenomena, we solve the Boltzmann transport equation, with given initial conditions, to obtain the distribution function as a function of time. Let consider , $\psi(\vec{x},\vec{p})$ be any quantity associated with a molecule of velocity $\vec{p}$ located at $\vec{x}$, such that in any collision $\vec{p}_1,\vec{p}_2 \rightarrow \vec{p'}_1,\vec{p'}_2$ taking place at $\vec{x}$, stated in equation (3.3.7)

$$\psi_1 + \psi_2 = \psi_1' + \psi_2' \tag{3.4.2}$$

Where, $\psi_1 = \psi(\vec{x},\vec{p}_1,t)$,.....etc. We call ψ is a conserved quantity. The conservation equation is derived from this:

$$\int d^3\vec{p}\psi(\vec{x},\vec{p})J(f,h) = 0 \tag{3.4.3}$$

$$\int d^3\vec{p}\psi(\vec{x},\vec{p})\left(\frac{\partial f}{\partial t}\right)_{coll} = 0 \tag{3.4.4}$$

$$\int d^3\vec{p}\psi(\vec{x},\vec{p})C[f] = 0 \tag{3.4.5}$$

$$\int d^3\vec{p}\psi(\vec{x},\vec{p})C[f] = \int d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p'}_1 d^3\vec{p'}_2 \delta^4(\vec{P}_f - \vec{P}_f) \mid T_{fi} \mid^2 (f_2'f_1' - f_2f_1)\psi(\vec{x},\vec{p}) \tag{3.4.6}$$

$$\int d^3\vec{p}\psi(\vec{x},\vec{p})C[f] = \int d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p'}_1 d^3\vec{p'}_2 \delta^4(\vec{P}_f - \vec{P}_f) \mid T_{fi} \mid^2 (f'_2 f'_1 - f_2 f_1) \times$$
$$(\psi_1(\vec{p}) + \psi_2(\vec{p}) - \psi_1(\vec{p}) - \psi_2(\vec{p})) \tag{3.4.7}$$

Now conservation theorem relevant to the Boltzmann transport equation is obtained by multiplying the Boltzmann transport equation on both side by $\psi(\vec{p})$ and then integrating over $\vec{p}$. The collision term vanishes by virtue of the collision invariants.

$$\int d^3\vec{p}\psi(\vec{x},\vec{p})\left(\frac{\partial}{\partial t} + \frac{\vec{P}_i}{m}\frac{\partial}{\partial \vec{x}_i} + \vec{F}\frac{\partial}{\partial \vec{v}_i}\right) f(\vec{x},\vec{p},t) = 0 \tag{3.4.8}$$

This can be written as follow

$$\frac{\partial}{\partial t}\int d^3\vec{p}\psi(\vec{x},\vec{p})f + \frac{\partial}{\partial \vec{x}_i}\int d^3\psi(\vec{x},\vec{p})\frac{\vec{p}_i}{m}f - \int d^3\vec{p}\frac{\partial\psi(\vec{x},\vec{p})}{\partial \vec{x}_i}\frac{\vec{p}_i}{m}f$$
$$+ \int d^3\vec{p}\frac{\partial\psi((\vec{x},\vec{p})\vec{F}_i f)}{\partial \vec{p}_i} - \int d^3\vec{p}\frac{\partial\psi(\vec{x},\vec{p})}{\partial \vec{p}_i}\vec{F}_i f - \int d^3\vec{p}\psi(\vec{x},\vec{p})\frac{\partial \vec{F}_i}{\partial \vec{p}_i}f = 0 \tag{3.4.9}$$

The fourth term in this equation vanishes if $F(\vec{x},\vec{p},t)$ is assumed to vanishes when $\mid \vec{p} \mid \rightarrow \infty$. This conservation theory is most useful in hydrodynamics, Where the velocity is $\vec{v} = \frac{p}{m}$. It is possible to express the expression interns of $\vec{v}$ rather than $\vec{p}$. We define the average value of any quantity G by

$$\langle G\rangle = \frac{\int d^3\vec{p}Gf}{d^3\vec{p}} = \frac{1}{n}\int d^3\vec{p}Gf \tag{3.4.10}$$

where, $n = n(\vec{x},t)$, with the help of this equation we can express the above equation of conservation as

$$\langle n\psi\rangle + \frac{\partial\langle nv_i\rangle}{\partial x_i} - n\langle v_i\frac{\partial\psi}{\partial x_i}\rangle - \frac{n}{m}\langle\vec{F}_i\frac{\partial\psi}{\partial v_i}\rangle - \frac{n}{m}\langle\frac{\partial F_i}{\partial v_i}\psi\rangle = 0 \tag{3.4.11}$$

Where, ψ is conserved quantity and $\langle nG\rangle = n\langle G\rangle$ be cause n is independent of velocity function. From now on let restrict our selves (attention) to velocity independent external force, so that the last equation on the above equation is dropped out. For simple molecules

the independent conserved quantities are mass, momentum and energy. For charged molecules we also include the charge , but this extension is trivial.

$$\psi(\vec{x},\vec{p}) = m \qquad (mass) \tag{3.4.12}$$

$$\psi(\vec{x},\vec{p}) = mv_i \qquad (momentum) \tag{3.4.13}$$

$$\psi(\vec{x},\vec{p}) = \frac{1}{2}m \mid \vec{v} - \vec{u} \mid^2 \qquad (thermal \quad energy) \tag{3.4.14}$$

Where, $\vec{u} = \langle v \rangle$ is the local or relative velocity of fluid particles.

3.5 Conservation of mass

For $\psi_i = m$, by using the conservation theorem immediately we get,

$$\frac{\partial}{\partial t}(nm) + \frac{\partial}{x_i}\langle nmv_i \rangle = 0 \tag{3.5.1}$$

We introduce a new ingredient that is mass density.

$$\rho(\vec{x},t) = mn(\vec{x},t) \tag{3.5.2}$$

so the continuity mass equation can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\cdot(\rho\vec{u}) = 0 \tag{3.5.3}$$

3.6 Conservation of momentum

For $\psi_i = mv_i$, by using the conservation theorem immediately we get, momentum conservation;

$$\frac{\partial\langle \rho v_i \rangle}{\partial t} + \frac{\partial\langle \rho v_i v_j \rangle}{\partial x_j} - \frac{1}{m}\rho\vec{F_i} \tag{3.6.1}$$

to reduce this further let us write ;

$$\begin{aligned} \langle v_i v_j \rangle &= \langle (v_i - u_i)(v_j - u_j) \rangle + \langle v_i \rangle u_i + u_i \langle v_j \rangle - u_i u_j \\ &= \langle (v_i - u_i)(v_j - u_j) \rangle + u_i u_j \end{aligned} \tag{3.6.2}$$

Now by substitution of this quantities to the equation of conservation of momentum, we obtain;

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{1}{m}\rho\vec{F_i}-\frac{\partial}{\partial x_j}\langle\rho(v_i-u_i)(v_j-u_j)\rangle \tag{3.6.3}$$

$$\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)u_i=\frac{F_i}{m}-\frac{1}{\rho}\frac{\partial}{x_j}P_{ij} \tag{3.6.4}$$

where we have introduced the pressure tensor with components P_{ij}, the thermal velocity or the peculiar velocity. $\vec{\xi}=\vec{v}-\vec{u}=v_i-u_i$ and $\langle\vec{v}\rangle=\vec{u}$

$$\langle\vec{\xi}\rangle=\langle\vec{v}-\vec{u}\rangle=0 \tag{3.6.5}$$

$$P_{ij}=\langle\rho(v_i-u_i)(v_j-u_j)\rangle=\rho\langle(v_i-u_i)(v_j-u_j)\rangle \tag{3.6.6}$$

$$P_{ij}=\rho\langle\vec{\xi}\vec{\xi}\rangle=m\int d^3\vec{p}\vec{\xi}\vec{\xi}f \tag{3.6.7}$$

In the absence of external force we obtain an equation of ;

$$\partial_t(\rho\vec{u})+\frac{\partial}{\partial\vec{x}}(\rho\vec{u}\vec{u}+P)\qquad\qquad P=pressure\quad tensor \tag{3.6.8}$$

3.7 Conservation of energy

For $\psi_i=E$ where $E=\frac{1}{2}m\mid\vec{v}-\vec{u}\mid^2$ then we obtain the equation of ;

$$\frac{1}{2}\frac{\partial}{\partial t}\langle\rho\mid\vec{v}-\vec{u}\mid^2\rangle+\frac{1}{2}\frac{\partial}{\partial x_i}\langle\rho v_i\mid\vec{v}-\vec{u}\mid^2\rangle-\frac{1}{2}\rho\langle v_i\frac{\partial}{\partial x_i}\mid\vec{v}-\vec{u}\mid^2\rangle=0 \tag{3.7.1}$$

We define the temperature by:

$$k_bT\equiv\frac{1}{3}m\langle\mid\vec{v}-\vec{u}\mid^2\rangle \tag{3.7.2}$$

and heat flux by:

$$\vec{q}=\frac{1}{2}\rho\langle(\vec{v}-\vec{u})\mid\vec{v}-\vec{u}\mid^2\rangle \tag{3.7.3}$$

$$\rho\langle v_i\mid\vec{v}-\vec{u}\mid\rangle=\frac{1}{2}\rho\langle(v_i-ui)(v_j-u_j)\rangle+\rho u_i\langle v_j-u_j\rangle=P_{ij} \tag{3.7.4}$$

$$\begin{aligned}\frac{1}{2}\rho\langle v_i \mid \vec{v}-\vec{u}\mid^2\rangle &= \frac{1}{2}\rho\langle(v_i-u_i)(v_j-u_j)\rangle + \frac{1}{2}\rho u_i\langle\mid v_j-u_j\mid^2\rangle \\ &= \vec{q_i} + \frac{1}{2\rho}u_i(\frac{3k_bT}{m}) \\ &= \vec{q_i} + \frac{3}{2}\rho u_i\frac{k_bT}{m} \end{aligned} \tag{3.7.5}$$

Now by substitution of these on the energy conservation equation we obtain that;

$$\frac{3}{2}\frac{\partial}{\partial t}(\rho k_{bT}) + \frac{\partial\vec{q_i}}{\partial x_i} + \frac{3}{2}\frac{\partial}{\partial x_i}(\rho k_bTu_i) + mP_{ij}\frac{\partial u_i}{\partial x_i} = 0 \tag{3.7.6}$$

since, $P_{ij} = P_{ji}$, by symmetry property we obtain that

$$\begin{aligned} mP_{ij}\frac{\partial u_j}{\partial x_i} &= P_{ji}\frac{m}{2}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}) \\ &= P_{ij}\Lambda_{ij} \end{aligned} \tag{3.7.7}$$

The momentum tensor can be expressed in terms of velocity tensor, τ^{ik}

$$\begin{aligned} P_{ij} &= \rho\langle\vec{\xi}\vec{\xi}\rangle \\ &= \rho\tau^{ik} \end{aligned} \tag{3.7.8}$$

where , τ^{ik} is velocity tensor

$F_i = -\nabla\phi$=external force

The momentum conservation equation be comes , identical to the former is :

$$\frac{\partial}{\partial t}\langle\rho v^i\rangle + \frac{\partial}{x^k}\tau^{ik} + \frac{\rho}{m}\frac{\partial\phi}{\partial x^i} = 0 \tag{3.7.9}$$

And the energy conservation equation becomes as :

$$\frac{(\partial n\bar{\in})}{\partial t} + \vec{\nabla}\cdot\vec{q} + \rho\frac{\partial\phi}{\partial x^k}\bar{v}^k = 0 \tag{3.7.10}$$

where, $\bar{v}^k = \langle v^k\rangle$ and $\bar{\in} = \langle E\rangle$

$$n = \frac{\rho}{m} = \int d^3\vec{p}f(\vec{x},\vec{p},t) \tag{3.7.11}$$

$$\langle\vec{v}\rangle = \frac{1}{n}\int\vec{v}d^3\vec{p} \tag{3.7.12}$$

$$\langle E\rangle = \frac{1}{n}\int E d^3\vec{p} \tag{3.7.13}$$

$$\begin{aligned} P_{ij} &= \int m\vec{\xi^i}\vec{\xi^k} d^3\vec{p} \\ &= m\tau^{ik} \end{aligned} \tag{3.7.14}$$

$$\vec{q} = n\langle E\vec{\xi}\rangle \tag{3.7.15}$$

where, $\vec{\xi}$ is the peculiar or thermal velocity of a fluid particles. From the equation (3.7.6) results;

$$\frac{3}{2}\frac{\partial}{\partial t}(\rho k_b T) + \frac{\partial \vec{q}}{\partial x_i} + \frac{3}{2}\frac{\partial}{\partial x_i}(\rho k_b T u_i) = -mP_{ij}\frac{\partial u_j}{\partial x_i} \tag{3.7.16}$$

If we consider n is a constant that means particles are uniformly distributed on the μ-space then ρ, the density of the particle distribution is constant. So that this equation can be written as :

$$\rho(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i})k_b T + \frac{2}{3}\frac{\partial}{\partial x_i}\vec{q} - \frac{2}{3}\Lambda_{ij}P_{ij} \tag{3.7.17}$$

the three conservation equations are :

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \qquad for \quad mass \tag{3.7.18}$$

$$\rho(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})\vec{u} = \frac{\rho}{m}\vec{F} - \vec{\nabla}P_{ij} \qquad for \quad momentum \tag{3.7.19}$$

$$\rho(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})k_b T = -\frac{2}{3}\vec{\nabla}\cdot\vec{q} - \frac{2}{3}P_{ij}\Lambda_{ij} \qquad for \quad energy \tag{3.7.20}$$

The auxiliary quantities are defined as follow.

$$\rho(\vec{x},t) = nm = m\int d^3\vec{p} f(\vec{x},\vec{p},t) \qquad mass \quad density \tag{3.7.21}$$

$$\vec{u}(\vec{x},t) = \langle\vec{v}\rangle = \int d^3\vec{p}\vec{v} f(\vec{x},\vec{p},t) \qquad average \quad velocity \tag{3.7.22}$$

$$\vartheta(\vec{x},t) = \frac{m}{3}\langle\mid \vec{v} - \vec{u} \mid^2\rangle \qquad Temperature \tag{3.7.23}$$

Where, $\vartheta = Tk_b$ and k_b is the Boltzmann constant and T is temperature.

$$\begin{aligned} \vec{q} \quad &= \int d^3\vec{p} E \vec{\xi} f(\vec{x}, \vec{p}, t) \\ &= \frac{\rho}{2n} \langle (\vec{v} - \vec{u}) \mid \vec{v} - \vec{u} \mid^2 \rangle \qquad\qquad heat \qquad flux \end{aligned} \tag{3.7.24}$$

The expression of equation of (3.7.3), (3.7.15) and (3.7.24) are equivalent that expresses heat flux.

Chapter 4

THE NONLINEAR BOLTZMANN EQUATION

4.1 Microscopic conservation equation

Let start from the Boltzmann equation in the absence of an external force; we multiply it by $\psi(\vec{p})$ instead of $\psi(\vec{x},\vec{p})$ and integrate over the velocity $\vec{p}$, with its invariant properties and the equation of

$$\int d^3\vec{p}\psi_\alpha(\vec{p})J(f_1,f_1)=0 \tag{4.1.1}$$

By substitution this equation to the Boltzmann immediately we get:

$$\begin{aligned}\frac{\partial}{\partial t}\int d^3\vec{p}\psi_\alpha f_1+\frac{\partial}{\partial\vec{x}}\cdot\int d^3\vec{p}\vec{v}\psi_\alpha f_1 \quad &= \partial_t n\langle\psi_\alpha(\vec{x},t)\rangle+\frac{\partial}{\partial\vec{x}}\cdot\langle\vec{v}\psi_\alpha(\vec{x},t)\rangle \\ &= 0\end{aligned} \tag{4.1.2}$$

In many experimental situation, we have no way to measure directly the one particle distribution function; we have control only over macroscopic quantities , defined as average values taken over the distribution function. For example the local density $n(\vec{x},t)\Delta\vec{x}$ is the number of particles in the element $\Delta\vec{x}$ is given by:

$$n(\vec{x},t)=\int d^3\vec{p}f_1(\vec{x},\vec{p},t) \tag{4.1.3}$$

The total number of particles in the system expressed in equation (2.1.3) that is

$$N = \int d^3\vec{x} n(\vec{x},t) \qquad = \int d^3\vec{x} d^3\vec{p} f_1(\vec{x},\vec{p},t) \tag{4.1.4}$$

Similarly the local velocity can be defined as

$$\vec{u}(\vec{x},t) = \frac{1}{n(\vec{x},t)} \int d^3\vec{p}\vec{v} f_1(\vec{x},\vec{p},t) \tag{4.1.5}$$

More generally, any local macroscopic quantity $\langle G\rangle_{x;t}$ appears as

$$\langle G\rangle_{x;t} = \frac{1}{n(\vec{x},t)} \int d^3\vec{p} G(\vec{p}) f_1(\vec{x},\vec{p},t) \tag{4.1.6}$$

of the suitable chosen function $G(\vec{p})$. Even though, most observations only involve such averages, the one particle distribution function remains with $\psi_1 = 1$ results the mass continuity equation of (1.5.3) and for ($\psi_i = v_i$ where , i =2,3,4 = x,x,z) the momentum conservation equation i.e,

$$\partial_{x;t}\langle\vec{v}\rangle_{x;t} + \frac{\partial}{\partial\vec{x}} \cdot n\langle\vec{v}\vec{v}\rangle_{x;t} = 0 \tag{4.1.7}$$

Since, $\langle\vec{v}\rangle_{x;t} = \vec{u}(\vec{x},t)$ and $\vec{v}$ can be decompose in to two parts.

$$\vec{v} = \vec{u} + \vec{\xi} \tag{4.1.8}$$

$$\begin{aligned} \langle\vec{v}\vec{v}\rangle_{x;t} \qquad &= \langle(\vec{v}+\vec{u})(\vec{v}+\vec{u})\rangle_{x;t} \\ &= \vec{u}\vec{u} + \langle\vec{\xi}\vec{\xi}\rangle_{x;t} \end{aligned} \tag{4.1.9}$$

Now working out with the mass density, we obtain,

$$\partial_t(\rho\vec{u}) + \frac{\partial}{\partial\vec{x}}(\vec{u}\vec{u} + P) = 0 \tag{4.1.10}$$

Where is the pressure tensor with its competent P_{ij} and

P_{ij}=$\rho\langle\vec{\xi_i}\vec{\xi_j}\rangle$ where ,$(i, j\epsilon x, y, z)$

similarly, for $\psi_5 = \frac{v^2}{2}$, we find that

$$\partial_t n\langle\frac{v^2}{2}\rangle_{x;t} + \frac{\partial}{\partial x} n\langle\vec{v}\frac{v^2}{2}\rangle_{x;t} = 0 \tag{4.1.11}$$

$$\langle \vec{v}\frac{v^2}{2}\rangle = \vec{u}[\frac{\vec{u}^2}{2} + \langle\ \frac{\vec{\xi}^2}{2}\rangle_{x;t}] + \vec{u}\cdot\langle\vec{\xi}\vec{\xi}\rangle_{x;t} + \langle\vec{\xi}\frac{\vec{\xi}^2}{2}\rangle_{x;t} \tag{4.1.12}$$

By substituting equation (4.1.12) in to (4.1.11) we obtain the following equation

$$\partial_t(\frac{\vec{u}^2}{2} + e) + \frac{\partial}{\partial\vec{x}}\cdot[\rho\vec{u}(\frac{\vec{u}^2}{2} + e) + P\cdot\vec{u} + \vec{q}_i] = 0 \tag{4.1.13}$$

where the heat current is defined as;

$$\vec{q}_i = \int d^3\vec{p}\vec{\xi}(\frac{m}{2})\vec{\xi}^2 f_1 \tag{4.1.14}$$

And the thermal energy per unit mass is given by;

$$e = \frac{\langle E\rangle_{x;t}}{m} \tag{4.1.15}$$

We can recognize in equation of (3.5.3, 4.1.11, 4.1.13) the fundamental conservation equations of classical fluid dynamics, which express the time variation of the local density of the conserved variables (mass momentum and energy) as the divergence of the corresponding flow. In the case of mass the flow is $m\vec{u}$. For the momentum the flow tensor is the sum of the two terms. These are :

(i) The macroscopic convection flow of momentum density $\vec{u}(\rho\vec{u})$.

(ii) The pressure tensor P_{ij} which is seem to represent the microscopic momentum flow of the molecules, calculated in a reference frame moving at a local velocity $\vec{u}$. The term pressure tensor is justified because at equilibrium, the momentum flow through a unit surface, which is then perpendicular to this surface defines the scalar hydrostatic pressure.

$$P_{ij}^{eq} = P\delta_{ij} \tag{4.1.16}$$

Where, $P_{ij}^{eq} = P_{ij}^{(0)}$ is the pressure at equilibrium. More over ,for diluted gas , which have f_1^{eq}, distribution function at equilibrium, and $\langle E\rangle$ the local thermal kinetic energy as the average kinetic energy per particle in a reference frame moving at local velocity $\vec{u}$

$f_1^{(0)} = f_1^{eq}$, which is the Boltzmann distribution function.

$$f_1^{(0)} = n\left(\frac{m}{2\pi k_b T}\right)^{\frac{3}{2}} exp\left[\frac{-m\mid\vec{v}-\vec{u}\mid^2}{2k_bT}\right] \tag{4.1.17}$$

$$\begin{aligned}\bar{e}(\vec{x},t) &= \frac{1}{n(\vec{x},t)}\int d^3\vec{p}\frac{m}{2} \mid \vec{v}-\vec{u} \mid^2 f_1(\vec{x},\vec{p},t) \\ &= \frac{m}{2}\langle\vec{v}^2\rangle - \frac{m}{2}\langle\vec{u}^2\rangle \end{aligned} \tag{4.1.18}$$

Where, $\bar{e} = e(m)$ m , is mass of the fluid particle.

We more over assume that in a dilute gas this quantity defines the local temperature through the relation of

$$\bar{e}(\vec{x},t) = \frac{3}{2}k_bT(\vec{x},t) = \langle E\rangle \tag{4.1.19}$$

which generalizes locally the trivial theorem for equilibrium pressure, $P = nk_bT = \frac{3}{2}n\bar{e}$. Where n, $\vec{u}$, and ,T generally depend on $\vec{x}$ and t (position and time). Thus

$$\begin{aligned}P_{ij}^{eq} &= m\int d^3\vec{p}v_iv_jf_1^{eq} \\ &= \delta_{ij}\frac{2n\bar{e}}{3} \\ &= \delta_{ij}nk_bT\end{aligned} \tag{4.1.20}$$

Which is consistent with the low of perfect gases. For energy density, the sum of the microscopic energy and of the internal energy (which is already mentioned , defines temperature through $e = \frac{3k_bT}{2m}$) the flow is made of three terms:

(i) The macroscopic convection flow.

(ii) The work done by pressure force.

(iii) The heat flow (i.e., the average microscopic energy flow measured again in a reference frame moving with fluid). Although the Boltzmann equation leads to thes conservation laws, this circumstance hardly represents a successful application of the theory; the law of mechanics and statistical definitions of macroscopic quantities already suffice to establish them; intact, provided the definitions of e, P and $\vec{q}_i$ are properly generalized to include the interactions between the molecules, they remain valid in dense liquids. With out further approximations, however, these macroscopic conservation equations are of little use

because they are not closed; as long as P and $\vec{q_i}$ are not depend as integral over f_1, which can be calculated from the Boltzmann equation. The problem closure does not arise at least in principle.

4.2 Classical hydrodynamics

To make the conservation equations are useful, we must supplement them by constitutive equations relating the pressure tensor and heat flux (heat flow) to the hydrodynamic variables. Since this equation take essentially the same form for any isotropic fluid, weather a diluted gas or a dense liquid, we present the theory in its general form, which allows us to apply it in dense liquids.

The simplest approximation corresponds to ideal fluid, where it is postulated that heat flow vanishes, while the pressure tensor the form of equation (4.1.16)

$$P_{ij}^{eq} = P\delta_{ij} \tag{4.2.1}$$

Where ,

$$P = P(e, \rho) \tag{4.2.2}$$

is the same function of internal energy and the mass density as it is at equilibrium. This is the so called local-equilibrium assumption, Which appears, as a fundamental ingredient of hydrodynamics. This ideal fluid approximation (the momentum equation is then called Euler's equation) describe motions with out friction and is limited applicability. The next step takes in to account is a phenomenological way the irreversible process due to internal motion. It is an experimental fact that a temperature gradient creates a heat flow , and this permits us to write Fourier's law.

$$\vec{q} = -K\frac{\partial T}{\partial x} \tag{4.2.3}$$

Where the coefficients 'K' is the thermal conductivity, as a positive definite scalar (be cause the fluid is isotropic) that depends in principle on the local variables ρ and

e. In general equations (4.2.3) seems to introduce a new variable T (temperature) in the problem because except in the dilute gases there no reason for the simple relation equation (4.1.12) between temperature and eternal energy, to be valid. We then restore again to the local equilibrium assumption, supposing that the entropy per unit mass can be defined and is related to e and ρ:

$$s = s(e, \rho) \tag{4.2.4}$$

is the same way as at equilibrium, and the temperature is then given by the thermodynamic relation

$$\frac{1}{T} = \left(\frac{\partial s}{\partial e}\right)_{\rho} \tag{4.2.5}$$

and depends on $\vec{x}$ and T only through the local variables E and ρ.

for the pressure tensor we write.

$$P_{ij} = P\delta_{ij} + P'_{ij} \tag{4.2.6}$$

Where, P'_{ij} describes internal frictions and has the following properties.

(i) It vanishes if $\vec{u}$ is constant, hence in a first approximation, it has to proportional to the first derivatives $\frac{\partial u_i}{\partial x_j}$.

(ii) It may depend only on the condition

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{4.2.7}$$

be cause no friction can appear in a rotation of the fluid with uniform angular velocity $\vec{\omega}$, such that $\vec{u} = \vec{\omega} \times \vec{x}$. The most general tensor that satisfies this condition is ,

$$P'_{ij} = a\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + b\left(\sum_{l=x,y,z} \frac{\partial u_l}{\partial x_l}\right)\delta_{ij} \tag{4.2.8}$$

Where, we have taken in to account that the intrinsic properties of an isotropic fluid may appear through scalars only (a and b). It is convenient to write this equation as ;

$$P'_{ij} = -\eta\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\sum_{l=x,y,z} \frac{\partial u_l}{\partial x_l})\delta_{ij}\right] - \zeta(\sum_{l=x,y,z} \frac{\partial u_l}{\partial x_l})\delta_{ij} \tag{4.2.9}$$

in such away that the trace of the first term is zero.

$$\sum_{i,i=x,y,z}\left[(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})\delta_{ij}-\frac{2}{3}(\sum_{l=x,y,z}\frac{\partial u_l}{\partial x_l})\delta_{ij}\right]=0 \tag{4.2.10}$$

The so called Newtonian law equation (4.2.9) introduce two phenomenological coefficients; η is the shear viscosity ans ζ the bulk viscosity; they are both positive and depends on the ρ and e. With the help of equations of (4.2.3, 4.2.6 and ,4.2.9) the hydrodynamic of a real fluid are obtained at once from the conservation laws of equation (3.5.3, 3.6.8 and 3.7.16) we obtain that;

$$\partial_t\rho+\frac{\partial}{\partial\vec{x}}\cdot(\rho\vec{u})=0 \qquad Continuity \qquad equation \tag{4.2.11}$$

$$\partial_t(\rho\vec{u})+\frac{\partial}{\partial\vec{x}}\cdot(\rho\vec{u}\vec{u})+\frac{\partial}{\partial\vec{x}}P=\frac{\partial}{\partial\vec{x}}\cdot\{\eta[(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})+(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})^{+}-\frac{2}{3}U(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})]\}+\frac{\partial}{\partial\vec{x}}[\zeta(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})] \tag{4.2.12}$$

This is Navier-Stokes equation.

$$\begin{aligned}\partial_t[\rho(\frac{u^2}{2}+e)]+\frac{\partial}{\partial\vec{x}}\cdot[\rho\vec{u}(\frac{u^2}{2}+e+\frac{P}{\rho})] \quad &=\frac{\partial}{\partial\vec{x}}\cdot K\frac{\partial T}{\partial\vec{x}}+\frac{\partial}{\partial\vec{x}}\cdot\{\eta[(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})+(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})^{+}\\ &-\frac{2}{3}U(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})]\cdot\vec{u}\}+\frac{\partial}{\partial\vec{x}}[\zeta\vec{u}(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})]\end{aligned} \tag{4.2.13}$$

Where ; $\delta_{ij}=U$, in matrix form, and equation (4.2.12) and (4.2.13) and the transposed tensor $\frac{\partial\vec{u}}{\partial\vec{x}}$ is defined as;

$$(\frac{\partial}{\partial\vec{x}}\cdot\vec{u})^{+}_{ij}=\frac{\partial u_i}{\partial x_j} \tag{4.2.14}$$

When supplemented by equation (4.2.2, 4.4.4 ,and 4.2.5) this equation forms a closed system. The system of this equation is very complicated and have extremely rich varieties of solutions; because of its nonlinear character; it describes phenomena as different as the turbulence of liquids and on set of instabilities in stars. However, in one limiting case it be come fairly small: namely when a phenomena is nearly close to equilibrium; that is

$$\rho(\vec{x},t)=\rho+\delta\rho(\vec{x},t) \tag{4.2.15}$$

$$\vec{u}(\vec{x},t) = \delta\vec{u}(\vec{x},t) \tag{4.2.16}$$

$$e(\vec{x},t) = e + e(\vec{x},t) \tag{4.2.17}$$

We use the conservation that when the $(\vec{x},t)$ depends on the thermodynamic variable (ρ,e ,T,...) is not explicitly displayed, we mean that its absolute equilibrium value. For a small deviations we can replace the system of nonlinear equations (4.2.11 ,4.2.12, and 4.2.13) by its linearized approximation (see chapter 5); in doing this ,the transport coefficients K, η, ζ be come absolute constants, depending only on the equilibrium reference system, and the equilibrium relations (4.2.2 ,4.2.4 and 4.2.4) can be replaced by the differential formula;

$$\delta P = \left(\partial\frac{P}{\partial T}\right)_{\rho}\delta T + \left(\partial\frac{P}{\partial\rho}\right)_{T}\delta\rho \tag{4.2.18}$$

$$\delta e = c_v\delta T + \left[\frac{P}{\rho^2} - \frac{T}{\rho^2}(\frac{\partial P}{\partial T})_{\rho}\right]\delta\rho \tag{4.2.19}$$

where, c_v is the specific heat per unit mass at constant volume. By definition we have ;

$$\delta e = c_v\delta T + \left(\frac{\delta e}{\delta\rho}\right)_{T}\delta\rho \tag{4.2.20}$$

and from the second principle of thermodynamics , we can write ;

$$\begin{aligned}\delta e \quad &= T\delta s + \left(\frac{P}{\rho^2}\right) \\ &= T\left(\frac{\partial s}{\partial T}\right)_{\rho}\delta T + \left[T(\frac{\partial s}{\partial\rho})_T + \frac{P}{\rho^2}\right]\delta\rho\end{aligned} \tag{4.2.21}$$

using in addition the thermodynamic quantities :

$$\left(\frac{\partial s}{\partial\rho}\right)_{T} = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_{\rho} \tag{4.2.22}$$

by elementary manipulation this result we arrive at the following system of equations, which describes the linearized hydrodynamics.

$$\partial_t\delta\rho(\vec{x},t) + \rho\frac{\partial}{\partial\vec{x}}\cdot\delta\vec{u}(\vec{x},t) = 0 \tag{4.2.23}$$

$$\partial_t \delta\vec{u}(\vec{x},t) + \frac{1}{\rho}(\frac{\partial P}{\partial T})_\rho \frac{\partial}{\partial \vec{x}}\delta T(\vec{x},t) + \frac{1}{\rho}(\frac{\partial P}{\partial \rho})_T \frac{\partial}{\partial \vec{x}}\rho(\vec{x},t) = \frac{\eta}{\rho}[\frac{\partial}{\partial \vec{x}}] + \frac{1}{\rho}(\zeta + \frac{\eta}{3})\frac{\partial}{\partial \vec{x}}(\frac{\partial}{\partial \vec{x}}\cdot \delta\vec{u}) \tag{4.2.24}$$

$$\partial_t \delta T(\vec{x},t) + \frac{T}{\rho c_v}(\frac{\partial P}{\partial T})_\rho \frac{\partial}{\partial \vec{x}}\cdot \delta\vec{u}(\vec{x},t) = \frac{K}{\rho c_v}\frac{\partial}{\partial \vec{x}}\cdot\frac{\partial}{\partial \vec{x}}\delta T \tag{4.2.25}$$

This equation is fairly easy to solve the integral; but despite this simplicity, they contain irreversible transport phenomena that appeared in the nonlinear theory. This means that the determination of the transport coefficients is a well posed problem even for small deviations from equilibrium. Finally let us point out that the equations of (4.2.23 ,4.2.24, and 4.2.24) applying also to dilute gases. The thermodynamic properties are particularly simple in such cases:

$$\left(\frac{\partial P}{\partial \rho}\right)_T = \frac{k_b T}{m} \tag{4.2.26}$$

$$\left(\frac{\partial P}{\partial T}\right)_\rho = \frac{k_b \rho}{m} \tag{4.2.27}$$

$$c_v = \frac{3k_b}{2m} \tag{4.2.28}$$

where as it turns out that the bulk viscosity vanishes.

$$\zeta = 0 \tag{4.2.29}$$

Similarly in the nonlinear equations, simplifications occur from the equation of the states:

$$P = \frac{\rho k_b T}{m} = \frac{2\rho e}{3} \tag{4.2.30}$$

We shall carried out more calculations and derivations of the transport coefficients next.

4.3 Equilibrium situation(solution of $J(f_0,f_0)$)

Since the existence of collision invariants uniquely determines the nature of the nonlinear equation $J(f_0, f_0) = 0$, these solutions are of central importance in determining the approach to equilibrium in fluid (dilute) gases, we have the following inequality.

$$\int d^3\vec{p} J(f_0, f_0) \ln f_1 \leq 0 \tag{4.3.1}$$

For any distribution function f_1 and the equality sign holds if and only if :

$$f_0 = A\exp[-\alpha(\vec{v}-\vec{u})^2] \tag{4.3.2}$$

where; A, α ,v_o are parameters that may depend on $\vec{x}$ and t, but not on $\vec{v}$. Let us start from equation (2.3.6) with $\psi(\vec{v}) = \ln f_1(\vec{v})$ and f=h=f_0 .We get;

$$\int d^3\vec{p}J(f_1,f_1)\ln f_1 = \frac{1}{4}\int d^3\vec{p}\int d^3\vec{p}_1\int d\Omega\sigma(\Omega)\mid \vec{v}-\vec{v}_1\mid [f_0'f_{01}'-f_0f_{01}]\ln(\frac{f_0f_{01}}{f_0'f_{01}'}) \tag{4.3.3}$$

But $\sigma(\Omega)\mid \vec{v}-\vec{v}_1\mid \geq 0$ and f_0 also semi-positive definite moreover it is easy to check that;

$$(x-y)\ln(\frac{y}{x}) \leq 0 \tag{4.3.4}$$

and that the equality sign only holds if x=y. Hence the inequality equation (4.3.2) is valid and becomes as an equality only if

$$f_0f_{01} = f_0'f_{01}' \tag{4.3.5}$$

or

$$\ln f_0 + \ln f_{01} = \ln f_0' + \ln f_{01}' \tag{4.3.6}$$

This means that $\ln f_0$ is a linear combination of collision invariants which can convinently be written as ;

$$\ln f_0 = \ln A - \alpha v_o^2 + 2\alpha\vec{v}_o\cdot\vec{v} - \alpha v^2 \tag{4.3.7}$$

This is identical to equation of (4.3.2). An important implication of this result is that $J(f_0,f_0)$ vanishes if and only if f_0 is Gaussian that is equation (4.3.2). The meaning of A, α,$\vec{v}_o$ can be related to the macroscopic observable quantities by considering equations (4.1.18) and (4.1.19) which generalize the viral theorem for the equilibrium pressure $P = nk_bT = \frac{2}{3}n\bar{e}$, with equation (4.3.2) the integral leading to n, $\vec{u}$, and $\bar{e}$ are elementary.

$$n = A(\frac{\pi}{\alpha})^{\frac{3}{2}}$$

$$\vec{v} \quad = \vec{v}_o$$

$$k_b \quad = \frac{m}{2\alpha}$$

This allows us to write equations (4.3.2) as the local equilibrium Maxwell-Boltzmann distribution function $f_0(\vec{u}, T)$ stated in equation of (4.1.17) where parameters n, $\vec{u}$,T generally depends on $\vec{x}$ and t. The fundamental in equality equation (4.3.2) leads to the famous Boltzmann H-theorem, which establishes the irreversible of the evolution of the particle distribution functions to ward an equilibrium state of the Maxwell-Boltzmann type, where n, $\vec{u}$ and T are absolute constant. For a system that does not exchange entropy with external world , the Boltzmann equation describes a monotonous irreversible approach to ward the absolute equilibrium distribution.

4.4 Equation of fluid dynamics

At this stage we shall see the hydrodynamic equations of fluid dynamics before we manipulate the zero order approximation, first order approximation and transport of coefficients η, ζ and K in detail. Let we consider a closed system with out any external force field, that is ϕ=U_{ext} vanishes. To firest approximation, we may assume that the fluid in different region of space is in local thermodynamic equilibrium, even through, the full system has not reached. Then the distribution function in each volume element can be set equal to equilibrium to corresponding to the density, temperature and macroscopic velocity $\langle\vec{v}\rangle$, which exists at that region. At this local approximation we shall ignore the gradients in temperature and velocity and thermal conductivity .... etc. We can evaluate τ^{ik} which the velocity tensor form of pressure tensor and $\vec{q_i}$ which is the heat flux to this order of accuracy i.e.,

$\tau^{ik} = \rho\bar{v}^i\bar{v}^k + P\delta_{ik}$

$\vec{q_i} = \rho(\bar{v}^2 + w)v^i$

where ; P is presure at equilibrium and $w = (e + \frac{P}{\rho})$ is enthalpy density of fluid.

Let assume a volume 'V' surrounded by a surface $\vec{S}$ with a flux of particles flowing in and out of the volume through the surface, $d\vec{s}$ with normal out from the volume. The mass flowing out from the volume per unit time is $\rho\vec{v}d\vec{S}$. This flow integrated over the whole surface is of course the same as the mass decreasing in the surface.

$$-\int \frac{\partial\rho}{\partial t}dV = \int \rho\vec{v}d\vec{S} \tag{4.4.1}$$

with Gausses theorem the surface integral can be converted to volume integral.

$$-\int \frac{\partial\rho}{\partial t}dV = \int \vec{\nabla}\cdot\rho\vec{v}dV \tag{4.4.2}$$

Where, dV is the volume element.
From this we obtain the continuity equation of mass described in equation (3.5.3). If we consider the force acting on the surface through the pressure -$Pd\vec{s}$ (note sign). Again using Gauss theorem we have

$$-\int Pd\vec{S} = -\int \nabla PdV \tag{4.4.3}$$

The force on dV as it moves around is the force -∇PdV so that

$$\rho\frac{d\vec{u}}{dt} = -\nabla P + \rho\nabla\phi \tag{4.4.4}$$

where, ϕ potential (may be gravitational potential) .This is the Lagrangian (co moving), derivative, so in Euler coordinates we get,

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = -\nabla P + \rho\nabla\phi \tag{4.4.5}$$

This is an Euler equation identical to equation of (3.6.4). This equation can also written as the conservation equation law as we have seen in section (3.6) equation (3.6.4) i.e., the conservation law of momentum per unit volume, ρv^i, similar to that of mass. Using the mass and momentum conservation law , we can get the following;

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho v^i) \quad &= \rho\frac{\partial v^i}{\partial t} + v^i\frac{\partial \rho}{\partial t} \\
&= -\rho\sum_j \frac{\partial v^i}{\partial x_i} - \frac{\partial P}{\partial x_i} + \rho\frac{\partial \phi}{\partial x_i} - v_i\sum_j \frac{\partial}{\partial x_j}(\rho v_i) \\
&= -\sum_j \frac{\partial}{\partial x_i}(\rho v_i v_j) - \frac{\partial P}{\partial x_i} + \rho\frac{\partial \phi}{\partial x_i} \\
&= \sum_i \frac{\partial}{\partial x_i}(\rho v_i v_j + P) + \rho\frac{\partial \phi}{\partial x_i} \\
&= -\frac{\partial}{\partial x_i}(\tau^{ij}) + \rho\frac{\partial \phi}{\partial x_i}
\end{aligned} \tag{4.4.6}$$

Where , $v^i = \vec{\xi^i}$ and $v^j = \vec{\xi^j}$

$$\tau^{ij} = \rho v^i v^j + P\delta_{ij} \tag{4.4.7}$$

If ϕ=0,

$$\frac{\partial}{\partial t}(\rho v_i) = -\sum_i \frac{\partial \tau^{ik}}{\partial x_j} \tag{4.4.8}$$

if $\tau^{ij} = \tau^{ik}$ and it is identical to equation (4.2.6).

The tensor τ^{ik} gives the momentum flux of the j momentum component in the direction i including both the mass flux and pressure. It is usually called energy momentum tensor. Now again we consider the energy of the volume. This energy is the sum of the kinetic internal energy, $\rho\frac{v^2}{2} + \rho\epsilon$, where ϵ is the internal energy per unit mass that is e. Now let us consider the evolution of this quantity with time,

$$\frac{\partial}{\partial t}\left[(\frac{1}{2}\rho v^2) + \rho\epsilon\right] = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \sum_i \rho v_i\frac{\partial v_i}{\partial t} + \epsilon\frac{\partial \rho}{\partial t} + \rho\frac{\partial \epsilon}{\partial t} \tag{4.4.9}$$

Which is clearly has the form of conservation law, and is analogous the mass conservation law. The connection between internal energy , pressure, volume and heat loss Tds is:

$$d\epsilon = -PdV + Tds \tag{4.4.10}$$

Where 'V' is the specific volume (volume per unit mass). so the volume, V = $\frac{1}{\rho}$ and 's' is the entropy also per unit mass. Using this together with the mass conservation and momentum conservation equations, can be written the heat loss term as :

$$\begin{aligned} \rho T\left(\frac{1}{2}\frac{ds}{dt} + \sum_i \frac{\partial s}{\partial x_i}\right) &= \rho T \frac{ds}{dt} \\ &= \Lambda \end{aligned} \tag{4.4.11}$$

One can transform equation of (4.4.9) to:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{v}^2 + \rho\epsilon\right) = -\sum_i \frac{\partial}{\partial x_i}\left[\rho v_j(\frac{1}{2}\bar{v}^2 + w)\right] + v_j\frac{\partial \phi}{\partial x_j} + \Lambda \tag{4.4.12}$$

Where,

$$\begin{aligned} w &= \epsilon + \frac{P}{\rho} \\ &= \frac{\gamma}{(\gamma-1)}(\frac{P}{\rho}) \\ &= heat \quad function \quad or \quad ethalphy \quad density \end{aligned} \tag{4.4.13}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{v}^2 + \rho\epsilon\right) = \sum_i \frac{\partial}{\partial x_i}\vec{q_i} + v_j\frac{\partial \phi}{\partial x_j} + \Lambda \tag{4.4.14}$$

where ;

$$\vec{q^i} = \rho v^i\left(\frac{1}{2}\bar{v}^2 + w\right) \tag{4.4.15}$$

Generally this equation is equivalent to:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\bar{v}^2 + \rho\epsilon\right] = -\vec{\nabla}\cdot\left[\vec{v}(\frac{1}{2}\rho v^2 + \rho\epsilon + P)\right] + \rho\vec{v}\cdot\vec{\nabla}\phi + \Lambda \tag{4.4.16}$$

This equation says that the change in total energy density in a volume is equal to flux of the kinetic energy and internal energy through its surface the $(\rho\vec{v}(\frac{1}{2}v^2+\epsilon \quad term))$ plus the work done on this volume by the pressure (the $\vec{v}P$ term) the heat lost by other process, like conduction or radiation equations (3.5.3, 3.6.4 ,4.2.6 and 4.4.16) constitute the complete set of hydrodynamic equations, a case which often is important in that of a stationary, symmetric flow. In this case all $\frac{\partial}{\partial t}$ terms as well as angular derivatives are zero. Furthermore;

$$\vec{\nabla} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) \tag{4.4.17}$$

since

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\varphi} = 0 \tag{4.4.18}$$

where θ and φ are scattering angles. Thus;

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}[\vec{v}r^2(\frac{1}{2}\rho v^2 + \rho\epsilon P)] + \rho\frac{\partial\phi}{\partial r} + \Lambda \tag{4.4.19}$$

For terrestrial application it is often a good approximation to consider the fluid as incomprensible, that is density does not changes as a fluid element moves. This means the a bove $\vec{\nabla}\cdot\vec{v} = 0$, $\frac{d}{dt} = 0$ finaly we obtain

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} \quad &= \frac{\partial\rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{v} \\ &= \frac{\partial\rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{v} \\ &= 0\end{aligned}$$

4.5 Zero order approximation

We shall work with the hydrodynamic regime, the mean free path is small compared to other characteristic length. This means that the fluid molecules make a large number of collisions with in small space, consequently they come to local equilibrium rapidly. In

the lowest order approximation it is natural to assume that the gas has a local Maxwell-Boltzmann distribution with slow varying temperature, density, and average velocity:

$$f(\vec{x},\vec{p},t) \approx f^{(0)}(\vec{x},\vec{p},t) \tag{4.5.1}$$

Where, $f^{(0)}$ is the Boltzmann distribution, stated in equation (4.1.17) that is

$$f^{(0)} \quad = n\left(\frac{m}{2\pi k_b T}\right)^{\frac{3}{2}} \exp\left[\frac{-m}{2k_b T}(\vec{v}-\vec{u})^2\right]$$

Where, n,T, and $\vec{u}$ do not depend on $\vec{v}$ but depend on $\vec{x}$ and t, are slowly varying. The equation of (4.1.17) clearly can not exact solution of the Boltzmann transport equation. So we have ;

$$\left(\frac{\partial f^{(0)}}{\partial t}\right)_{coll} = C[f] = 0 \tag{4.5.2}$$

because n, T , and $\vec{u}$ do not depend on $\vec{v}$. But it is clear that integral;

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_x + \frac{\vec{F}}{m}\cdot\vec{\nabla}_v\right) f^{(0)}(\vec{x},\vec{p},t) \neq 0 \tag{4.5.3}$$

Now, let us calculate $\vec{q_i}$, P_{ij} to the lowest order, and denote $q_i^{\vec{(0)}}$ and $P_{ij}^{(0)}$.

Let $C(\vec{x},t) = n(\frac{m}{2k_b T\pi})^{\frac{3}{2}}$

$A(\vec{x},t) = \frac{m}{2k_b T}$

$$\begin{aligned} q_i^{\vec{(0)}} \quad &= \frac{1}{2}\frac{\rho}{n}\int d^3\vec{p}(\vec{v}-\vec{u})(\vec{v}-\vec{u})^2 C(\vec{x},t)\exp[-A(\vec{x},t)\mid \vec{v}-\vec{u}\mid^2] \\ &= \frac{1}{2}\frac{\rho}{n}\int d^3\vec{p}\vec{\xi}\vec{\xi^2}\exp[-A\vec{\xi^2}] = 0 \end{aligned}$$

$$\begin{aligned} P_{ij}^{(0)} \quad &= \frac{\rho}{n}C(\vec{x},t)\int d^3\vec{p}(\vec{v}-\vec{u})(\vec{v}-\vec{u})^2\exp[-A(\vec{x},t)\mid \vec{v}-\vec{u}\mid^2] \\ &= mc(\vec{x},t)\int d^3\vec{p}\vec{\xi}\vec{\xi^2}\exp[-A\vec{\xi^2}] \\ &= \delta_{ij}P \end{aligned}$$

Where ,

$$\begin{aligned} P &= \frac{1}{3}\rho\left(\frac{m}{2k_b\pi T}\right)^{\frac{3}{2}} \int d^3\vec{\xi}\vec{\xi^2} \exp\left[-A\vec{\xi^2}\right] \\ &= nk_bT \end{aligned} \tag{4.5.4}$$

Which P is local hydrostatic pressure. By substituting this equation in to (4.5.3) we obtain that;

$$\vec{\nabla}\cdot P^{(0)} = \nabla P \qquad\qquad and \tag{4.5.5}$$

$$P^{(0)}\cdot \Lambda_{ij} = P\sum_{i=1}^{3}\Lambda_{ii} = mP\vec{\nabla}\cdot\vec{u} \tag{4.5.6}$$

finally we obtain the three conservation equations are ;

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{u}\right) = 0 \qquad\qquad for \qquad mass \tag{4.5.7}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} + \frac{1}{\rho}\nabla P = \frac{\vec{F}}{m} \qquad\qquad Euler's \qquad Equation \tag{4.5.8}$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)k_bT = \frac{1}{c_v}(\vec{\nabla}\cdot\vec{u})k_bT \qquad\qquad for \qquad energy \tag{4.5.9}$$

Where , $c_v = \frac{3}{2}$. These are hydrodynamic equations for none viscous flow of fluids. They posses solutions describing flow pattern that persists indefinitely. Thus, in this approximation, the local Maxwell-Boltzmann distribution, never decay to true Maxwell-Boltzmann distribution. The equations (4.5.7, 4.5.8, 4.5.9) are also use full for fluids. The quantity $(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})X$ is known as the substantial derivative of X because it is the time rate of change of X to an observer moving with the local average velocity $\vec{u}$. Such an observer is said to be moving along a streamline. So we shall show the zero order approximation of a fluid under goes only adiabatic transformations to an observer moving along a streamline. so, equations (4.5.7 and 4.5.9) be comes as ;

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\rho = -\rho\vec{\nabla}\cdot\vec{u} \tag{4.5.10}$$

$$-\frac{3}{2}\frac{\rho}{T}\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)T=\rho\vec{\nabla}\cdot\vec{u} \tag{4.5.11}$$

Now, adding the two equations we obtain;

$$\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)\rho+-\frac{3}{2}\frac{\rho}{T}\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)T=0 \tag{4.5.12}$$

$$\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)\rho T^{\frac{-3}{2}}=0 \tag{4.5.13}$$

Using equation $P=\frac{\rho k_b T}{m}$ then equation (4.5.13) be can be convert to;

$$P\rho^{\frac{-5}{2}}=constant \qquad along \qquad (a\ \ streamline) \tag{4.5.14}$$

This is a condition for adiabatic transformation for the ideal gas, since $\frac{c_p}{c_v}=\frac{5}{3}$. Then we consider the case of steady flow under the influence of conservative external force field, i.e., under conditions of;

$$\vec{F}=-\nabla U_{ext}=-\nabla\phi \tag{4.5.15}$$

$$\frac{\partial\vec{u}}{\partial t}=0 \tag{4.5.16}$$

using vector identity;

$$\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}=\frac{1}{2}\vec{\nabla}(\vec{u^2})-\vec{u}\times(\vec{\nabla}\times\vec{u}) \tag{4.5.17}$$

Now the Euler equation can be written as;

$$\nabla\left(\frac{1}{2}\vec{u^2}+\frac{1}{\rho}P+\frac{1}{m}\phi\right)=\vec{u}\times(\vec{\nabla}\times\vec{u})-\frac{k_bT}{m}\frac{\nabla\rho}{\rho} \tag{4.5.18}$$

We must consider two cases; These are;

(i) In the case of uniform density and irrotational flow , namely $\nabla\rho=0$ and $\vec{\nabla}\times\vec{u}=0$ we get,

$$\nabla\left(\frac{1}{2}\vec{u^2}+\frac{1}{\rho}P+\frac{1}{m}\phi\right)=0 \tag{4.5.19}$$

Which is Bernoulli's equation.

(ii) In the case of uniform temperature and irrotational flow namely; $\nabla T=0$ and $\vec{\nabla}\times\vec{u}=0$, we get;

$$\nabla\left(\frac{1}{2}\vec{u^2}+\frac{1}{m}\phi\right)=\frac{-T}{m}\nabla(\ln\rho) \tag{4.5.20}$$

Which may be immediately integrated to yield;

$$\rho = \rho_0 \exp(\frac{-1}{2k_bT} m\vec{u^2} + \phi) \tag{4.5.21}$$

where ρ_0 is an arbitrary constant.

4.6 First order approximation

For simplicity we shall start with the dilute gases and finally we shall generalize to dense fluid. If we consider the thermal energy is given by $E = \frac{1}{2}\vec{\xi^2}$, then, let $f(\vec{x}, \vec{p}, t)$ be the exact distribution function, and

$$\delta f = f(\vec{x}, \vec{p}, t) - f^{(0)}(\vec{x}, \vec{p}, t) \tag{4.6.1}$$

Now, we are interested in the magnitude of δf as compared to $f^{(0)}$. First let us estimate the order of magnitude of

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = C[f] \tag{4.6.2}$$

By definition, we can write as,

$$\left(\frac{\partial(\delta f)}{\partial t}\right)_{coll} = \left(\frac{\partial f}{\partial t}\right)_{coll} - \frac{\partial f^{(0)}}{\partial t} \tag{4.6.3}$$

Now we know that from the definition of $J(f^{(0)}, f^{(0)}) = 0$ the term $C[f^{(0)}] = \frac{\partial f^{(0)}}{\partial t} = 0$, there fore the equation can be comes as;

$$\begin{aligned} C[\delta f] \quad &= C[f] \\ &= \int d^3\vec{p_2} d^3\vec{p_1'} d^3\vec{p_2'} \delta^4\Big(P_f - P_i\Big) |T_{fi}|^2 \Big(f_2' f_1' - f_2 f_1\Big) \\ &\approx \int d^3\vec{p_2} d^3\vec{p_1'} d^3\vec{p_2'} \delta^4\Big(P_f - P_i\Big) |T_{fi}|^2 \Big(f_2'^{(0)} \delta f_1'^{(0)} - f_2^{(0)} \delta f_1 + \delta f_2' f_2^{(0)} - \delta f_1 f_1^{(0)}\Big) \end{aligned} \tag{4.6.4}$$

Since $C[f^{(0)}] = 0$, and the assumption is that δf is small quantity whose squared can be neglected. An order of magnitude of the above equation (4.6.4) is only the 2^{nd} term is

contribute, and using the idea of

$$f = f_2^{(0)}\left(1 + \delta f + \frac{(\delta f f_2^{(0)})^2}{2!} + ... + ..\right) \tag{4.6.5}$$

the equation (4.6.4) can be approximated as ;

$$\begin{aligned} C[\delta f] \quad &= \int d^3\vec{p}d^3\vec{p_1'}d^3\vec{p_2'}\delta^4\Big(P_f - P_i\Big)|T_{fi}|^2\Big(-(\delta f)f^{(0)}f_2^{(0)}\Big) \\ &= -\delta f \int d^3\vec{p}_2\sigma(\Omega)d\Omega \mid \vec{v} - \vec{v}_1 \mid f_2^{(0)} \end{aligned} \tag{4.6.6}$$

By using equation of (3.2.1) and (3.2.2) we obtain that

$$\begin{aligned} C[\delta f] \quad &= C[f] \\ &= -\frac{\delta f}{\tau} \\ &= -\frac{(f - f^{(0)})}{\tau} \end{aligned} \tag{4.6.7}$$

where is the number of order of magnitude of collision time. The Boltzmann transport equation be comes;

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}_x + \frac{1}{m}\vec{F}\cdot\vec{\nabla}_v\right)(f^{(0)} + \delta f) = -\frac{(f - f^{(0)})}{\tau} \tag{4.6.8}$$

We assume that $\delta f << f^{(0)}$, we can neglect δf on the left side of equation (4.6.8). We assume that $f^{(0)}$ varies by significant amount (i.e., of the order of it self) only when $\mid \vec{X} \mid$ varies by a distance 'L'. Then equation (3.6.8) furnishes the estimate.

$$\frac{\langle\vec{v}\rangle f^{(0)}}{L} = -\frac{\delta f}{\tau} \tag{4.6.9}$$

or

$$\frac{\delta f}{f^{(0)}} = -\frac{\lambda}{\tau} \tag{4.6.10}$$

Where λ is a length of order of mean free path. From this considerations we conclude that $f^{(0)}$ is approximation if the local density, temperature and velocity have characteristic

wave length 'L' much larger than the mean free path λ. The corrections to $f^{(0)}$ would be of the order of $\frac{\lambda}{L}$. A systematic expansion of f in power of $\frac{\lambda}{L}$ is furnished by Chapman-Enskog expansion, which is some what complicated. We put now

$$f = f^{(0)} + \delta f \tag{4.6.11}$$

So, equation of (4.6.8) be approximate as;

$$\delta f \equiv -\tau\left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}_x + \frac{1}{m}\vec{F}\cdot\vec{\nabla}_v\right)f^{(0)} \tag{4.6.12}$$

Now, to calculate δf, we note that $f^{(0)}$ depends on $\vec{x}$ and t, only through the functions, $\rho(\vec{x},t)$, $T(\vec{x},t)$ and $u(\vec{x},t)$. Thus we need the derivatives,

$$\frac{\partial f^{(0)}}{\partial \rho} = \frac{f^{(0)}}{\rho} \tag{4.6.13}$$

$$\frac{\partial f^{(0)}}{\partial \vartheta} = \frac{1}{\vartheta}\left[\frac{m\vec{\xi^2}}{2\vartheta} - \frac{3}{2}\right]f^{(0)} \tag{4.6.14}$$

$$\frac{\partial f^{(0)}}{\partial \vec{u}} = \frac{m}{\vartheta}\vec{\xi}f^{(0)} \tag{4.6.15}$$

$$\frac{\partial f^{(0)}}{\partial v_i} = -\frac{m}{\vartheta}\vec{\xi}f^{(0)} \tag{4.6.16}$$

where, $\vartheta = k_b T$, $\vec{\xi} = \vec{v} - \vec{u}$ which is the peculiar velocity. Thus,

$$\delta f = -\tau\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} + \frac{F_i}{m}\frac{\partial}{\partial v_i}\right)f^{(0)} \tag{4.6.17}$$

Let,

$$D(\Psi) = (\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i})\Psi \tag{4.6.18}$$

When we use zero order of hydrodynamic equations of (4.5.7, 4.5.8, 4.5.9) we obtain that,

$$D(\rho) = -\rho(\vec{\nabla}\cdot\vec{u}) + \vec{\xi}\cdot\vec{\nabla}\rho \tag{4.6.19}$$

$$D(\vartheta) = -\frac{2}{3}\vartheta(\vec{\nabla}\cdot\vec{u}) + \vec{\xi}\cdot\vec{\nabla}\vartheta \tag{4.6.20}$$

$$D(u_j) = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \frac{\vec{F_i}}{m} + \vec{\xi_i}\partial\frac{u_j}{\partial x_i} \tag{4.6.21}$$

where, $P = \frac{\rho\vartheta}{m}$, then,

$$\delta f = -\tau f^{(0)}\left[\frac{1}{\rho}D(\rho) + \frac{1}{\vartheta}\left(\frac{m}{2\vartheta}\vec{\xi}^2 - \frac{3}{2}\right)D(\vartheta) + \frac{m}{\vartheta}\vec{\xi}_j D(u_j) - \frac{1}{\vartheta}(\vec{F}\cdot\vec{\xi})\right] \tag{4.6.22}$$

By substitution of equation (4.6.19, 4.6.20, and 4.6.21) in to equation of (4.6.22) we obtain that

$$\begin{aligned}\delta f \quad &= -\tau f^{(0)}\bigg[-(\vec{\nabla}\cdot\vec{u}) + \vec{\xi}\cdot\vec{\nabla}\rho\frac{1}{\rho} + \frac{1}{\vartheta}(\frac{m}{2\vartheta}\vec{\xi}^2 - \frac{3}{2})\left(\frac{-2}{3}\vartheta\vec{\nabla}\cdot\vec{u} + \vec{\xi}\cdot\vec{\nabla}\vartheta\right)\\ &+\frac{m}{\vartheta}\left(-\vec{\xi}\cdot\vec{\nabla}P\frac{1}{\rho} + \vec{\xi}\cdot\vec{F}\frac{1}{m} + \vec{\xi}_i\vec{\xi}_j\frac{\partial u_j}{\partial x_i}\right) - \frac{1}{\vartheta}\vec{F}\cdot\vec{\xi}\bigg]\end{aligned} \tag{4.6.23}$$

After some manipulation and canceling terms we get the following results, for dilute gas, $E = \frac{m}{2}\vec{\xi}^2$ and the average speed $\langle\vec{v}\rangle = \sqrt{\frac{2k_bT}{m}}$, $c_v = \frac{3}{2}$ and $c_p = \frac{5}{2}$.

$$\delta f = -\tau\left[\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial x_i}\vec{\xi}\left(\frac{m}{2\vartheta}\vec{\xi}^2 - \frac{5}{2}\right) + \frac{1}{\vartheta}\Lambda_{ij}\left(\vec{\xi}_i\vec{\xi}_j - \frac{1}{3}\delta_{ij}\vec{\xi}^2\right)\right]f^{(0)} \tag{4.6.24}$$

note that do not forget to use the symmetry property.

Where, $\Lambda_{ij} \equiv \frac{1}{2}m(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.

by using equation of (3.2.6) the above equation can be written as,

$$\delta f = -\frac{1}{\nu}\left[\frac{1}{T}\frac{\partial T}{\partial x_i}\vec{\xi}\left(\frac{m}{2k_bT}\vec{\xi}^2 - \frac{5}{2}\right) + \frac{1}{k_bT}\Lambda_{ij}\left(\vec{\xi}_i\vec{\xi}_j - \frac{1}{3}\delta_{ij}\vec{\xi}^2\right)\right]f^{(0)} \tag{4.6.25}$$

In this first order approximation we can calculate the heat flux $q^{(\vec{1})}$ and the pressure P_{ij} tensor with the help of, $\delta f = f - f^{(0)}$, to find the equation of hydrodynamics to the firest order.

$$\begin{aligned}\vec{q} \quad &= \frac{1}{2}m\rho\langle(\vec{v} - \vec{u} \mid \vec{v} - \vec{u} \mid)^2\rangle\\ &= \frac{1}{2}\frac{m\rho}{n}\int d^3\vec{p}\vec{\xi}\vec{\xi}^2 f(\vec{x},\vec{p},t)\\ &= \frac{1}{2}m^2\int d^3\vec{p}\vec{\xi}\vec{\xi}^2(f^{(0)} + \delta f)\\ &= q^{(\vec{0})} + \frac{1}{2}m^2\int d^3\vec{p}\vec{\xi}\vec{\xi}^2\delta f\end{aligned} \tag{4.6.26}$$

The heat flux at local equilibrium vanishes, so the equation can be written as,

$$\vec{q} = \frac{1}{2}m^2 \int d^3\vec{p}\vec{\xi}\vec{\xi}^2 \delta f \tag{4.6.27}$$

By substituting of equation of (4.6.25) in to equation of (4.6.27) we get that,

$$\begin{aligned}\vec{q} \quad &= \frac{1}{2}m^2 \int d^3\vec{p}\vec{\xi}\vec{\xi}^2 \Bigg[-\frac{1}{\nu}\Bigg[\frac{1}{T}\frac{\partial T}{\partial x_i}\vec{\xi}\Bigg(\frac{m}{2k_bT}\vec{\xi}^2 - \frac{5}{2}\Bigg) + \frac{1}{k_bT}\Lambda_{ij}\Bigg(\vec{\xi}_i\vec{\xi}_j - \frac{1}{3}\delta_{ij}\vec{\xi}^2\Bigg)\Bigg] f^{(0)}\Bigg] \\ &= -\tau\frac{1}{2}m^2 \int d^3\vec{p}\vec{\xi}\vec{\xi}^2 \Bigg[\Bigg[\frac{1}{T}\frac{\partial T}{\partial x_i}\vec{\xi}\Bigg(\frac{m}{2k_bT}\vec{\xi}^2 - \frac{5}{2}\Bigg) + \frac{1}{k_bT}\Lambda_{ij}\Bigg(\vec{\xi}_i\vec{\xi}_j - \frac{1}{3}\delta_{ij}\vec{\xi}^2\Bigg)\Bigg] f^{(0)}\Bigg]\end{aligned} \tag{4.6.28}$$

The second them in equation (4.6.28) does not contribute to this integral of heat flux; so, we find that, for

$$\begin{aligned}\vec{q} \quad &= -\frac{m^2\tau}{2}\int d^3\vec{p}\vec{\xi}\vec{\xi}^2[\frac{1}{T}\frac{\partial T}{\partial x_i}\vec{\xi}(\frac{m}{2k_bT}\vec{\xi}^2 - \frac{5}{2})]f^{(0)} \\ &= -\frac{\tau m^5}{2}\int d^3\vec{p}\vec{\xi}\vec{\xi}^2(\frac{m}{2k_bT}\vec{\xi}^2 - \frac{5}{2})\frac{1}{T}\vec{\xi}\frac{\partial T}{\partial x_i}f^{(0)}\end{aligned} \tag{4.6.29}$$

Let introduce the quantity 'K' which is the coefficient of heat conductivity.

$$\vec{q} = -K\frac{\partial T}{\partial x} = -K\nabla T \tag{4.6.30}$$

This equation is identical to equation (4.2.3) whrere,

$$\begin{aligned}K \quad &= \frac{\tau m^5}{2T}\int d^3\vec{p}\vec{\xi}\vec{\xi}^2(\frac{m}{2k_bT}\vec{\xi}^2 - \frac{5}{2})\vec{\xi}f^{(0)} \\ &= \frac{5}{2}\tau k_bTn\end{aligned} \tag{4.6.31}$$

The magnitude of thermal conductivity $\vec{q}$ is small quantity of the first order, being of order of $\frac{\lambda}{L}$. For the pressure tensor P_{ij} only the second term of equation (4.6.25) is contributes;

$$\begin{aligned}P_{ij} \quad &= \frac{\rho}{n}\int d^3\vec{p}\vec{\xi}\vec{\xi}(f^{(0)} + \delta f) \\ &= \delta_{ij}P + P'\end{aligned} \tag{4.6.32}$$

This equation is identical to equation of (4.2.6) so,

$$P'_{ij} = -\frac{\tau\rho m^4}{k_b T n}\Lambda_{ij}\int d^3\vec{p}\vec{\xi_i}\vec{\xi_j}(\vec{\xi_i}\vec{\xi_j} - \frac{1}{3}\delta_{ij}\vec{\xi^2})f^{(0)} \tag{4.6.33}$$

Where, $P = \frac{\rho k_n T}{m}$. To evaluate this note that P_{ij} is a symmetric tensor zero trace (i.e., $\sum_{i=1}^{3}\Lambda_{ii} = 0$), and is depends linearly on the symetry tensor Λ_{ij}. There fore P'_{ij} must have the form of

$$P'_{ij} = \frac{2\eta}{m}\left(\Lambda_{ij} - \frac{m}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right) \tag{4.6.34}$$

Where, m$\vec{\nabla}\cdot\vec{u}$ is none other than the trace of Λ_{ij}

$$\sum_{i=1}^{3}\Lambda_{ii} = m\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_j} = m\vec{\nabla}\cdot\vec{u} \tag{4.6.35}$$

And η is a constant. It remains to calculate η and ζ vanishes for dilute gases. For instace for $i = 1$ and $j = 2$ we obtain that;

$$\begin{aligned} P'_{12} \quad &= -\frac{\tau m^4}{k_b T}\Lambda_{12}\int d^3\vec{p}\vec{\xi_1}\vec{\xi_2}(\vec{\xi_1}\vec{\xi_2} - \frac{1}{3}\delta_{12}\vec{\xi^2})f^{(0)} \\ &= -\frac{\tau m^4}{k_b T}(2)\Lambda_{12}\int d^3\vec{p}\vec{\xi_1}\vec{\xi_2}\vec{\xi_1}\vec{\xi_2}f^{(0)} \end{aligned} \tag{4.6.36}$$

$$\eta = -\frac{\tau m^5}{k_b T}\int d^3\vec{p}\vec{\xi_1}^2\vec{\xi_2}^2 f^{(0)} \approx \tau n k_b T \tag{4.6.37}$$

$$P_{ij} = \delta_{ij}P - \frac{2\eta}{m}(\Lambda_{ij} - \frac{m}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}) \tag{4.6.38}$$

The second term in the order of $\frac{\lambda}{L}$ and η is the coefficient of the viscosity. we have to note that equation of (4.6.6) also can can be written as ;

$$C[\delta f] = -\delta f(\vec{x},\vec{p},t)\sigma_{tot}n\langle\mid \vec{v}_{rel} \mid\rangle \tag{4.6.39}$$

where, $\langle\mid \vec{v}_{rel} \mid\rangle$ is the average velocity between colliding particles. And $\sigma_{tot}n\langle\mid \vec{v}_{rel} \mid \rangle$ represents the average scattering rate of particles of speed $\mid \vec{u} \mid$. If the system is approximately steady state and there are no external force then,

$$\mid \vec{u} \mid \frac{f^{(0)}}{L} \approx \frac{\mid \delta f \mid}{\tau} \tag{4.6.40}$$

$$\mid \delta f \mid \approx \frac{\mid \vec{u} \mid \tau}{L} f^{(0)} \approx \frac{\lambda}{L} f^{(0)} \tag{4.6.41}$$

These equations are identical to equations of (4.6.9) and (4.6.10) and the expression of equations of (4.6.40) (4.6.41) is for the sake of showing the order $\frac{\lambda}{L}$ which λ and L are the mean wave length and characteristic distance of fluid particles respectively. Let L is the gradient scale of height of the system, for $\lambda << L$. By using $\alpha = \frac{\lambda}{L}$ as small expression parameter, we can write,

$$f = f^{(0)} + \alpha f^{(1)} + \alpha^2 f^{(1)} + ... \tag{4.6.42}$$

Chapter 5

BOLTZMANN LINEARIZED EQUATION

5.1 Boltznamm linearized equation

To steady the dissipative process, it is necessary to consider the next order of approximation. Assume that $f = f^{(0)} + \delta f$ where δf is a small perturbation. Hence, $f^{(0)} \propto \exp\left(\frac{-E}{T}\right)$ (where E denotes energy) corresponding to the distribution in local thermodynamic equilibrium, we shall write δf, in the form of

$$\delta f \equiv -\frac{\partial f}{\partial E} h = \frac{f^{(0)}}{T} h \tag{5.1.1}$$

separating out the factor $\frac{f^{(0)}}{T}$ from δf the moment of h vanishes. Since the distribution function

$$f = f^{(0)}(1 + h) \tag{5.1.2}$$

must satisfy the condition of ;

$$\int m f d^3\vec{p} = m \tag{5.1.3}$$

$$\begin{aligned} \int m\vec{v} f d^3\vec{p} \quad &= \langle m\vec{v} \rangle \\ &= \langle \vec{p} \rangle \end{aligned} \tag{5.1.4}$$

$$\int Ef d^3\vec{p} = \langle E \rangle \tag{5.1.5}$$

From equation (3.2.6) and (4.6.7) we have that

$$\begin{aligned} C[\delta f] \quad &= \nu(f^{(0)} - f) \\ &= \nu(f^{(0)} - f^{(0)} - f^{(0)}h) \end{aligned} \tag{5.1.6}$$

where, h approximated from a combination of linear terms of the expanded series

$$h \approx h^{(1)} + h^{(2)} + h^{(3)} + ... + h^{(n)} \tag{5.1.7}$$

By substituting of equation (3.2.6) in to equation of (4.6.8) we obtain the following formula,

$$\frac{\partial f}{\partial t} + \vec{v_j}\frac{\partial f}{\partial t} + \frac{1}{m}\frac{\partial}{\partial v_j}(\vec{F_j}f) = \nu(f^{(0)} - f) \tag{5.1.8}$$

When we take the moment transfer of right hand side equation (5.1.8) that corresponds to the collisional invariant, m, $\vec{p}$ and E then the equations of transfer that obtained are zero. And using equation of (4.6.11), we have,

$$\int \nu(f^{(0)} - f) d^3\vec{p} = 0 \tag{5.1.9}$$

$$\int \vec{p}\nu(f^{(0)} - f) d^3\vec{p} = \nu[\langle \vec{p} \rangle - \langle \vec{p} \rangle] = 0 \tag{5.1.10}$$

$$\int E\nu(f^{(0)} - f) d^3\vec{p} = \nu[\langle E \rangle - \langle E \rangle] = 0 \tag{5.1.11}$$

satisfies the equation rests on the relation of $\langle \vec{v}^2 \rangle = \langle \vec{v} \rangle^2 + \langle \vec{\xi}^2 \rangle$ and the moment equation of (5.1.3) (5.1.4) and (5.1.5). Since equation of (5.1.3) (5.1.4) and (5.1.5) satiesfies by $f^{(0)}$ a lone, we must have

$$\int fh d^3\vec{p} = \langle h \rangle = 0 \tag{5.1.12}$$

$$\int \vec{p}fh d^3\vec{p} = \langle \vec{p}h \rangle = 0 \tag{5.1.13}$$

$$\int Efh d^3\vec{p} = \langle Eh \rangle = 0 \tag{5.1.14}$$

By comparing equation of (4.6.11) and (5.1.2) we get that,

$$\delta f \approx h f^{(0)} \tag{5.1.15}$$

By using equation (3.2.6) and (4.6.8)

$$\frac{\partial f^{(0)}}{\partial t} + \vec{v_j}\frac{\partial f^{(0)}}{\partial x_j} + \frac{\vec{F_j}}{m}\frac{\partial f^{(0)}}{\partial \vec{v}_j} = -\nu(\delta f) \tag{5.1.16}$$

Now, using equations (4.6.7) (4.6.12) and (5.1.2) we obtain that

$$\frac{\partial f^{(0)}}{\partial t} + \vec{v_j}\frac{\partial f^{(0)}}{\partial x_j} + \frac{\vec{F_j}}{m}\frac{\partial f^{(0)}}{\partial \vec{v}_j} = \frac{f^{(0)}}{T} I[h] \tag{5.1.17}$$

where, C[h]=I[h] is the scattering integral. Using equations (3.1.10) (3.3.2) and (4.6.7) we obtain that,

$$I[h] = \int d^3\vec{p}\sigma(\Omega)d\Omega \mid \vec{v_1} - \vec{v_2} \mid f_{12}^{(0)} h \tag{5.1.18}$$

and using (4.6.25) and (5.1.2) we can write that

$$\begin{aligned} h \quad &= -\frac{1}{\nu f^{(0)}}\left(\frac{\partial f^{(0)}}{\partial t} + \vec{v}\frac{\partial f^{(0)}}{\partial x_j} + \frac{\vec{F_j}}{m}\frac{\partial f^{(0)}}{\partial \vec{v_j}}\right) \\ &= -\frac{1}{\nu}\left(\vec{\xi}\left(\frac{m\vec{\xi}^2}{2k_bT} - \frac{5}{2}\right)\frac{\partial \ln T}{\partial x_j} + \frac{m}{k_bT}\left(\vec{v_j}\vec{v_k} - \frac{1}{3}\vec{\xi}^2\delta_{jk}\right)\frac{\partial u_j}{\partial x_k}\right) \end{aligned} \tag{5.1.19}$$

By considering the distribution function is

$$f^{(0)} \propto \exp\left(\frac{-E}{T}\right) \tag{5.1.20}$$

We obtain that,

$$h = -\frac{1}{\nu}\left(\vec{\xi}\left(\frac{E(\vec{p})}{T} - \frac{5}{2}\right)\frac{\partial \ln T}{\partial x_j} + \frac{m}{T}\left(\vec{v_j}\vec{v_k} - \frac{1}{3}\vec{\xi}^2\delta_{jk}\right)\frac{\partial u_j}{\partial x_k}\right) \tag{5.1.21}$$

Using equations of (3.1.10) (3.3.2) and (4.6.7) and, where $\vec{p_1} = \vec{p_2} = \vec{p}$ and $f_1^{(0)} f_2^{(0)} = f_1^{(0)} f_2^{(0)} = f_{12}^{(0)}$ we obtain that,

$$\begin{aligned} \frac{\partial f}{\partial t} + \vec{v_j}\frac{\partial f}{\partial x_j} + \frac{F_j}{m}\frac{\partial f}{\partial v_j} \quad &= \int d^3\vec{p}\sigma(\Omega)d\Omega \mid \vec{v_1} - \vec{v_2} \mid f_{12}^{(0)} h \\ &= I[h] \end{aligned} \tag{5.1.22}$$

From equations (5.1.17) and (5.1.22) we obtained that

$$\begin{aligned}\frac{\partial f}{\partial t}+\vec{v_j}\frac{\partial f}{\partial x_j}+\frac{F_j}{m}\frac{\partial f}{\partial v_j} &= \left(\vec{\xi}\left(\frac{m\vec{\xi^2}}{2k_bT}-\frac{5}{2}\right)\frac{\partial \ln T}{\partial x_j}+\frac{m}{k_bT}\left(\vec{v_j}\vec{v_k}-\frac{1}{3}\vec{\xi^2}\delta_{jk}\right)\frac{\partial u_j}{\partial x_k}\right)\\ &= \frac{f^{(0)}}{T}I[h]\end{aligned} \tag{5.1.23}$$

Using the fact that $E=\frac{1}{2}m\vec{\xi^2}$, $c_p=\frac{5}{2}$ and $c_v=\frac{3}{2}$ where c_p and c_v are specific heat at constant pressure and volume respectively, equation (5.1.23) results

$$\frac{f^{(0)}}{T}I[h]=\frac{1}{T}\left(\vec{\xi}\left(\frac{E(\vec{p})}{T}-c_p\right)\nabla T+\left(m\vec{v_j}\vec{v_k}-\frac{2}{3}E(\vec{p})\delta_{jk}\right)\frac{\partial u_j}{\partial x_k}\right)f^{(0)} \tag{5.1.24}$$

$$I[h]=\left(\vec{\xi}\left(\frac{E(\vec{p})-c_pT}{T}\right)\nabla T+\left(m\vec{v_j}\vec{v_k}-\frac{E(\vec{p})}{c_v}\delta_{jk}\right)\frac{\partial u_j}{\partial x_k}\right) \tag{5.1.25}$$

The first correction to $f^{(0)}$ in the form of δf will change the expression for P_{ij} and $\vec{q}$, the result obtained in the zero order approximation in which the gradient of T and $\vec{u}$ were ignored. In the first order P_{ij} and $\vec{q}$ will contain terms which are proportional to the gradients $\frac{\partial T}{\partial x_j}$ and $\frac{\partial u_j}{\partial x_j}$. We will write these expression, correct to linear order as,

$$\tau_{ij}=\rho\vec{v}_i\vec{v}_j+P\delta_{ij}-P'_{ij} \tag{5.1.26}$$

The third term in equation of (5.1.26) expressed in equation (4.2.9)

$$\vec{q_i}=\left(\rho\frac{1}{2}\vec{v_i^2}+w\right)\vec{v}_i+Q_i \tag{5.1.27}$$

The first order approximation to the Boltzmann transport equation (BTE) derived in equation (4.1.25) can be used to obtain the correction to P_{ij} and $\vec{q}$ due to spatial gradients. The linearized BTE we will use,

$$h=g_1^i\frac{\partial T}{\partial x_j}+g^{ij}\vec{v}_{ik}+g_3\frac{\partial \bar{u}_i}{\partial x_j} \tag{5.1.28}$$

Where, g_1^i, g^{ik} and g_3 depend only on the v^i and are to be determined by solving the linearized Boltzmann equation. Note that by definition, g_{ik} is symmetric; it may also be taken to be trace less since any part of g_{ik} proportional to δ_{ik} absorbed in to the third

term by redefining $g_3(v)$. We use symmetry conditions to show that g_1^i and g_{ik} must have the forms,

$$g_1^i = \frac{\vec{v}^i}{\vec{\xi}} g_1(\vec{\xi}) \tag{5.1.29}$$

$$g^{ik} = (\vec{v}^i \vec{v}^k - \frac{1}{3}\delta_{ik}) g_2(\vec{\xi}) \tag{5.1.30}$$

From equation (3.7.13) in chapter 2 pressure tensor that is $P_{ij} = \tau_{ik}$ we have

$$\begin{aligned} \tau_{ik} &= \int m\vec{v}_i\vec{v}_k f(1 - \frac{h}{T}) d^3\vec{p} \\ &= \rho\vec{v}_i\vec{v}_k + P\delta_{ik} - \sigma_{ik} \end{aligned} \tag{5.1.31}$$

Where,

$$\sigma_{ik} = -\frac{m}{T}\int f^{(0)}\vec{v}_i\vec{v}_k \left(\frac{\vec{v}_a}{\vec{\xi}} g_1(\vec{\xi}) \frac{\partial T}{\partial x_a} + (\vec{v}^a\vec{v}^b - \frac{1}{3}\vec{\xi}^2\delta_{ik}) g_2(\vec{\xi})\vec{v}_{ab} + g_3\vec{\nabla}\cdot\vec{u} \right) d^3\vec{p} \tag{5.1.32}$$

Note that $\tau_{ik} = P_{ik}$ is the pressure tensor. The form of this integrals can be determined by general symmetry consideration. The first integral has the form of

$$\begin{aligned} A_{ik} &= -\frac{m}{T}\left(\frac{\partial T}{\partial x_a}\right) \int d^3\vec{P} f^{(0)} \vec{v}_i\vec{v}_k\vec{v}_a \\ &= -\frac{m}{T}\left(\frac{\partial T}{\partial x_a}\right) \gamma_{aik} \end{aligned} \tag{5.1.33}$$

In evaluating this integral we may assume that we are working in a frame with expectation value of the thermal velocity vanishes i.e., $\langle\vec{\xi}\rangle$=0, since the contribution for $\vec{\xi}$ already taken in to account. This implies that the completely symmetric tensor γ_{aik} has to be constructed entirely from δ_{mnl} the only tensor available in the problem. No such γ_{aik} can be constructed from δ_{mn}. Hence,

$$A_{ik} = 0 \tag{5.1.34}$$

Consider now the second term; it has the form

$$\begin{aligned} B_{ik} &= (\frac{m}{T})\vec{v}^{ab}\int d^3\vec{p}\vec{v}_i\vec{v}_k\left(\vec{v}_a\vec{v}_b - \frac{1}{3}\vec{\xi}^2\delta_{ab}\right)g_2(\vec{\xi}) \\ &= \vec{v}^{ab}\gamma_{ikab} \end{aligned} \tag{5.1.35}$$

The tensor γ_{abik} is symmetric in (a,b) and (i,k) and is trace less in (a,b). Again it has to be constructed from δ_{mn}. The only tensor with this properties is

$$\gamma_{ikab} = \eta\left[\delta_{ai}\delta_{bk} + \delta_{ak}\delta_{bi} - \frac{2}{3}\delta_{ab}\delta_{ik}\right] \tag{5.1.36}$$

where, η is some constant. On seting, i=a and K=b, in this expression and using the equation just before we obtain

$$\begin{aligned} \gamma_{ikab} &= \eta\left[9 + 3 - \frac{2}{3}\times 3\right] \\ &= 10\eta \\ &= -\frac{2}{3}\left(\frac{m}{T}\right)\int d^3\vec{p}\vec{\xi}^4 g_2(\vec{\xi}) \end{aligned} \tag{5.1.37}$$

Substituting for γ_{ikab} in the expression for B_{ik} we find that

$$\begin{aligned} B_{ik} &= \eta\left[\vec{v}_{ik} + \vec{v}_{kj} - \frac{2}{3}\delta_{ik}\vec{\nabla}\cdot\vec{u}\right] \\ &= 2\eta\left[\vec{v}_{ik} - \frac{1}{3}\delta_{ik}\vec{\nabla}\cdot\vec{u}\right] \end{aligned} \tag{5.1.38}$$

Finally the third term has the form

$$C_{ik} = -\frac{m}{T}\vec{\nabla}\cdot\vec{u}\int f^{(0)}\vec{v}_i\vec{v}_k d^3\vec{p} \tag{5.1.39}$$

By similar arguments we conclude that of C_{ik} should be proportional to δ_{ik}. Hence,

$$C_{ik} = \zeta(\vec{\nabla}\cdot\vec{u})\delta_{ik} \tag{5.1.40}$$

Where,

$$\zeta = -\frac{m}{T}\langle \vec{\xi}^2 g_3 \rangle \tag{5.1.41}$$

This indicates the bulk viscosity of the fluid expressed in equation (4.2.9) to satisfies the Navier-Stokes equation. Adding the contribution of B_{ik} and γ_{ik} we obtain the final form of the stress-tensor correct to linear order in the gradient of the velocity expressed in equation (5.0.31) ,where,

$$\sigma_{ik} = 2\eta\left(\vec{v}_{ik} - \frac{1}{3}\delta_{ik}\vec{\nabla}\cdot\vec{u}\right) + \zeta\delta_{ik}\vec{\nabla}\cdot\vec{u} \tag{5.1.42}$$

The analysis for determining the energy flux is similar. From the basic definition,(i.e.,3.7.24) we have,

$$\begin{aligned} \vec{q_i} \quad &= \int E\vec{v}_i f^{(0)}(1 + \frac{h}{T})d^3\vec{p} \\ &= \rho(\frac{1}{2}\vec{\xi}^2 + w)\vec{v}_i + Q_i \end{aligned} \tag{5.1.43}$$

Where, Q_i defines the correction to the flux in the case of the ideal gas.

$$Q_i = \frac{1}{T}\int d^3\vec{p}E\vec{v}_i\left[\frac{\vec{v}_a}{\vec{\xi}}g_1(\vec{\xi})\frac{\partial T}{\partial x_a} + g_{ab}\vec{v}^{ab} + g_3\vec{\nabla}\cdot\vec{u}\right] \tag{5.1.44}$$

By arguments similar to the one given above, we can determine the form the first term. It reduces to $-K\frac{\partial T}{\partial x_i}$ with

$$K = -\frac{1}{3T}\langle E\vec{\xi}g_1(\vec{\xi})\rangle \tag{5.1.45}$$

identical to equation of (4.6.30) and (4.6.31) approximated in the first order approximation for dilute gas. The shear viscosity is already calculated in equation (5.0.37) that of

$$\eta = -\frac{m}{15T}\langle \vec{\xi}^4 g_3(\vec{\xi})\rangle \tag{5.1.46}$$

In the expression for Q_i, the second and third terms represents the contributions to the energy flux due to gradients in velocity. The value of this terms are most easily obtained

by the following conditions.

Since the energy density contributed by velocity gradients is $-\sigma_{ik}$ as given in equation (5.1.32), the corresponding energy flux along the direction -i must be have the form,

$$Q_i = -K\frac{\partial T}{\partial x_i} - \vec{v}_k\delta_{ik} \tag{5.1.47}$$

Note that the second term vanishes when computed in a frame of reference with $\vec{v}_i = 0$. Hence it needs to be obtained in a frame of reference with $\vec{v}_i \neq 0$ in which we can not the general symmetry arguments used before. In the ideal fluid, the flux energy is described in equation (4.4.15). In the absence of temperature and velocity gradients we expect an additional flow of energy since the range energy of of molecules at different location will now be different. To the lowest at which we are working we can expand $\vec{q}_i$ as a Taylor series in the gradients and retain only the first non-trivial terms. This will contribute to $\vec{q}$ a term in the form $-K\vec{\nabla}T$ in addition to terms using from velocity gradient. This is what we found in the lowest part. The contribution to τ_{ik} uisng from spatial variation of $\vec{\xi}$ are a bit complicated. We again expand τ_{ik} in a Taylor series and retaining terms liear in $\frac{\partial \vec{v}_i}{\partial x_k}$. We can write this gradient as a sum of symmetric and antisymmetry tensors.

$$\frac{\partial \vec{v}_i}{\partial x_k} = \vec{v}_{ik} + \frac{1}{2}\left(\frac{\partial \vec{v}_i}{\partial x_k} - \frac{\partial \vec{v}_k}{\partial x_i}\right) \tag{5.1.48}$$

However, we expect any viscous dissipation when the entire fluid is in a state of uniform rigid rotation, for which

$$\vec{u} = \left(\vec{\omega} \times \vec{X}\right). \tag{5.1.49}$$

where, $\vec{\omega}$ is angular velocity. In this case $\vec{v}_{ik}$=0, but the antisymmetric part does not vanish. It follows that τ_{ik} should not have any contribution from the antisymmetric part if this should be no viscous dissipation in a state of uniform rotation. Hence τ_{ik} must be a linear combination of $\vec{v}_k$ and $\delta_{ik}(\vec{\nabla}\cdot\vec{u})$, which are the only two symmetric independent, second rank tensors linear in the gradient of $\vec{u}$. Such a linear combination can be written

as $\alpha\vec{v}_{ik} + \beta\delta_{ik}(\vec{\nabla}\cdot\vec{u})$. It is however conventional to separate it trace out from $\vec{v}_{ik}$ and rewrite the expression in the form of

$$\sigma_{ik} = 2\eta\left(\vec{v}_{ik} - \frac{1}{3}\delta_{ik}\vec{\nabla}\cdot\vec{u}\right) + \zeta\delta_{ik}\vec{\nabla}\cdot\vec{u} \tag{5.1.50}$$

This is the general form of the viscous energy tensor which we have obtained. Note that it is parameterized by two coefficients of viscosity, η and ζ. The linear order in the gradients of $\vec{u}$ and T no other contributions to τ_{ik} can arise. Our system is completly determined by three constants K, η and ζ.

Chapter 6

VISCOSITY OF FLUIDS

6.1 Viscosity

Viscosity is a property of a fluid by virtue of which it offers a resistance to shear or deformation. When any fluid flows over a solid surface the velocity is not uniform at any cross section; it is zero (no slip) at the solid surface and progressively approaches the free steam velocity in the fluid layer far away from the solid surface. This aspect of velocity profile (a curve connecting the type of velocity vector) indicates the existence of some resistance to flow due to friction between fluids layers and solid surface and also between adjacent layers of the fluid it self. Again the velocity gradient (the spatal rate change of $\frac{d\vec{u}}{dy}$) is large at the solid surface and gradualy diminishes to zero with distance from the wall. The resistance between the fluid and the surface is greater when compared to that of between fuids layer them selves. So the resistance to flow between the internal energy is called viscous resistance, and the properties which enables the fuild to offer resistace to relative motion between adjacent layers is called viscosity of fluid. We can say that viscosity is a measure of resistance to relative transitional motion of adjacent layers of fluid.

6.2 Newton's law of viscosity

Let we consider two adjacent layers at an infinitesimal distance dy apart and moving with velocity $\vec{u}$ and $\vec{u} = d\vec{u}$. Let the upper layer moves with velocity $\vec{u} = d\vec{u}$ drags the lower layer by external force $\vec{F}$. The maximum shear stress disappears where the velocity gradient is zero. The velocity gradient at the solid boundary has a finite value. The velocity profile can not be asymptotic to the boundary because that would imply an infinite velocity gradient and in turn an infinite shear stress. The velocity gradient be come less steep ($\frac{d\vec{u}}{dy}$ small) with distance from boundary. Consequently, maximum value of shear stress occurs at the boundary and progressively decrease with distance from the boundary.

6.3 Effect of temperature on viscosity

There exist a distinct difference between fluids of liquids and gaseous nature in the effect of temperature on the value of their dynamical viscosity. Increasing of temperature causes a decrease in the viscosity of a fluid (see equation of 5.1.41 and 5.1.46) where as viscosity of gas increase with temperature growth (see equation 4.6.37). The difference in the behavior can be explained by considering the basic mechanics that give raise to viscosity. The viscous forces in a fluid are the out come of the intermolecular cohesion and molecular momentum transfer. In a liquid the molecules are cooperatively more closely packed; molecular activity is a rather small and so the viscosity permanently due to molecular cohesion. The molecular cohesion is decrease with growth of temperature and consequently, the viscosity of the fluids drop at elevated temperature.

In gases the molecular cohesive forces are negligibly small and viscosity results primarily from molecular momentum transfer. This molecular activity increases with arise with temperature and so does the gas viscosity. High pressure also affect viscosity of fluids, for liquid viscosity increases with increasing pressure. This may attribute to the fact

that with pressure growth, there occurs an increasing with energy required for relative movement of molecules.

Chapter 7

CONCLUSION

In this essay we have derived basic equations of both ideal and non ideal fluids. Through the use of these equations, the evolution of many dynamic phenomena can be studied in great detail from their initial configurations through to their final configurations. Examples of phenomena that can be usefully examined using these equations include the motion of super fluid in the core of Neutron star [7], the motions of molten rock inside the earth, the motions in the outer layers of the sun, the explosion of supernovae, aerodynamics, and the swirling of gases in galaxies. In this essay, we have seen that the state of any moving ideal fluid can be completely determined by five quantities: the three components of the velocity u, the pressure p and the density. For ideal fluid the equations obtained, Eulers equations and the equation of continuity, together with the adiabatic equation give a complete system of equations for describing ideal fluids. The absence of heat exchange between different parts of the ideal fluid implies that the motion is adiabatic throughout the whole fluid, and that the rate of change of entropy is, therefore, constant. In the case of non ideal fluids, the entropy of the fluid is not constant, as we have seen from their general adiabatic equation. This implies that the entropy of non ideal fluids increases as a result of the irreversible processes of thermal conduction and internal friction [3]. From the Euler's equation and Navier-Stokes equations we observe that the

equations of motion for both types of fluids are not the same. What makes them different is the presence of internal friction,heat conductivity in non ideal fluids. The equation of continuity, as we would expect, is the same for both types of fluids. The energy equation of ideal fluids differs from that of non ideal fluids only because of the addition of viscous and heat conduction terms, resulting from the processes of internal friction and thermal conduction respectively. So the equations of real fluid governed by the Na vier-Stokes's equation. For the dissipative process, we have proved that the transport coefficients are inversely proportional to temperature. Shear viscosity reflects the transitional molecular motion but bulk viscosity reflects the relaxation of the rotational degree of molecular freedom and reaction to compression.

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