COHERENTLY DRIVEN TWO-LEVEL LASER

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By

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Abstract

In this thesis, we study the quantum properties of the light generated by a coherently driven two-level laser in which two level atoms available in a cavity coupled to a vacuum reservoir are pumped to the upper level at a rate r_{ca}. Applying the master equation for our system,we obtain the quantum Langevin equations for the cavity mode and atomic operators. With the aid of the solution of these equations,we calculate the mean and variance of the photon number. Moreover, we determine the power spectrum and quadrature variance for the cavity light.

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Chapter 1

Introduction

A two -level laser is a source of coherent or chaotic light emitted by two-level atoms inside a cavity coupled to a vacuum reservoir via a single-port mirror [1]. In one model of such a laser, two-level atoms initially in the upper level are injected at a constant rate into a cavity and removed after they decayed due to spontaneous emission [1,2]. In another model, the two-level atoms available in a cavity are pumped to the upper level by some convenient means such as electron bombardment [2, 3]. We seek here to analyze the quantum properties of the light emitted by two-level atoms available in a cavity driven by a coherent beam and coupled to vacuum reservoir via a single port-mirror . We study the case in which the atoms as well as the cavity mode interact with vacuum reservoir. We denote the top, bottom, and ground levels of the two-level atom by $|a_j\rangle$, $|b_j\rangle$ and $|c_j\rangle$ respectively. We also consider the case in which a two -level atom is pumped to the upper level $|a_j\rangle$ from the ground level $|c_j\rangle$ at the rate of r_{ca} . A two-level atom may make a transition from level $|a_j\rangle$ to level $|b_j\rangle$ by emitting a photon of frequency ω . Alternatively, the atom may decay from level $|a_j\rangle$ or $|b_j\rangle$ spontaneously to the ground level $|c_j\rangle$ at the rate γ [4]. Employing the master equation for the system under consideration, we derive equations of evolution for a cavity mode and atomic operators. Applying the solutions of these equations along with the correlation properties of the noise operators, we calculate the mean and variance of the photon number for the cavity light. Moreover, we determine the quadrature variance and power spectrum for the light generated by the two-level laser.

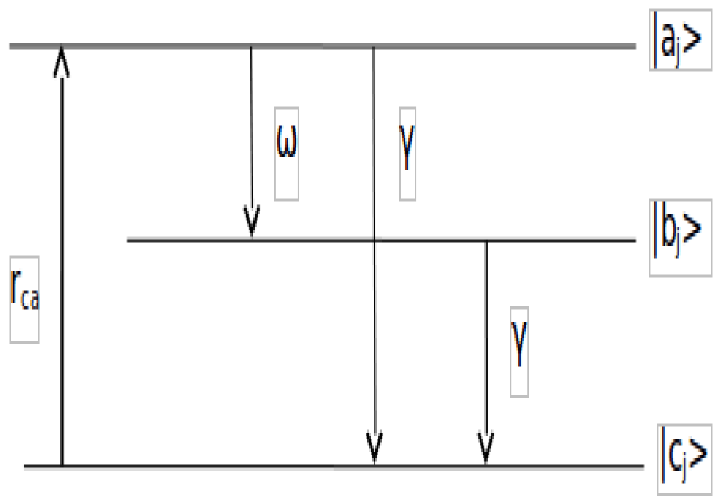

Figure 1.1: *A two-level atom.*

Chapter 2

Operator Dynamics

In this chapter we seek to obtain the quantum Langevin equations for cavity mode and atomic operators, employing the pertinent master equation. Moreover, we determine correlation properties of cavity mode and atomic noise operators .

2.1 Equations of evolution for the expectation values of cavity mode and atomic operators

The Hamiltonian describing the interaction of a cavity mode with deriving coherent light represented by a real c-number constant ε is [4]

$$\hat{H}_1 = i\varepsilon(\hat{a}^\dagger - \hat{a}), \tag{2.1.1}$$

where $\hat{a}$ is the annihilation operator for the cavity mode. The interaction of a two -level atom with a cavity mode is describable at resonance by the Hamiltonian[6]

$$\hat{H}_2 = ig(\hat{a}^\dagger|b_j\rangle\langle a_j| - |a_j\rangle\langle b_j|\hat{a}), \tag{2.1.2}$$

where g is the atom -cavity mode coupling constant. The total Hamiltonian describing the interaction of a cavity mode with deriving coherent light as well as with a two-level atom can thus be written as $\hat{H} = \hat{H}_1 + \hat{H}_2$

$$= i(\varepsilon(\hat{a}^\dagger - \hat{a}) + g(\hat{a}^\dagger|b_j\rangle\langle a_j| - |a_j\rangle\langle b_j|\hat{a})). \tag{2.1.3}$$

The master equation describing the interaction of a two -level atom and a cavity mode with each other and with vacuum reservoir is expressible as [4]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j| - |a_j\rangle\langle a_j\hat{\rho} - \hat{\rho}|a_j\rangle\langle a_j| \\ + 2|c_j\rangle\langle b_j\hat{\rho}|b_j\rangle\langle c_j| - |b_j\rangle\langle b_j|\hat{\rho} - \hat{\rho}|b_j\rangle\langle b_j|). \tag{2.1.4}$$

Upon substituting Eq.(2.1.3) into Eq.(2.1.4) the master equation becomes

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \varepsilon(\hat{a}^\dagger\hat{\rho} - \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}) \\ &+ g(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}) \\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j| - |a_j\rangle\langle a_j|\hat{\rho} - \hat{\rho}|a_j\rangle\langle a_j| \\ &+ 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j| - |b_j\rangle\langle b_j\hat{\rho} - \hat{\rho}|b_j\rangle\langle b_j|).\end{aligned} \tag{2.1.5}$$

The equation of evolution for cavity mode operator is obtained employing the master equation (2.1.5) along with the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right). \tag{2.1.6}$$

It then follows that $\frac{d}{dt}\langle\hat{a}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a})$

$$\begin{aligned}&= \varepsilon Tr(\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^2) \\ &+ gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}\hat{a} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a} + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}^2) \\ &+ \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &+ \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|\hat{a} - |a_j\rangle\langle a_j|\hat{\rho}\hat{a} - \hat{\rho}|a_j\rangle\langle a_j|\hat{a} \\ &+ 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|\hat{a} - |b_j\rangle\langle b_j|\hat{\rho}\hat{a} - \hat{\rho}|b_j\rangle\langle b_j|\hat{a}).\end{aligned} \tag{2.1.7}$$

Using the cyclic property of trace operation,we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= \varepsilon Tr(\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^2)\\ &+ gTr(\hat{\rho}\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}|a_j\rangle\langle b_j|\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a} + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}^2)\\ &+ \frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2)\\ &+ \frac{\gamma}{2}Tr(2\hat{\rho}|a_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle a_j| - \hat{\rho}\hat{a}|a_j\rangle\langle a_j| - \hat{\rho}|a_j\rangle\langle a_j|\hat{a}\\ &+ 2\hat{\rho}|b_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle b_j| - \hat{\rho}\hat{a}|b_j\rangle\langle b_j| - \hat{\rho}|b_j\rangle\langle b_j|\hat{a}),\end{aligned} \tag{2.1.8}$$

it then reduces to

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{\kappa}{2}\langle\hat{a}\rangle + g\langle|b_j\rangle\langle a_j|\rangle + \varepsilon. \tag{2.1.9}$$

The equation of evolution for the expectation value of the atomic operator $|b_j\rangle\langle a_j|$ is obtainable using the master equation described by Eq.(2.1.5) in the relation

$$\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle = Tr(\frac{d}{dt}\rho|b_j\rangle\langle a_j|). \tag{2.1.10}$$

We thus see that

$$\begin{aligned}\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle &= \varepsilon(\hat{a}^\dagger\hat{\rho}|b_j\rangle\langle a_j| - \hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| + \hat{\rho}\hat{a}|b_j\rangle\langle a_j|)\\ &+ g(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|b_j\rangle\langle a_j| - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|b_j\rangle\langle a_j| + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|b_j\rangle\langle a_j|)\\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| - \hat{a}^\dagger\hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|)\\ &+ \frac{\gamma}{2}(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|b_j\rangle\langle a_j| - |a_j\rangle\langle a_j|\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}|a_j\rangle\langle a_j|b_j\rangle\langle a_j|\\ &+ 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|b_j\rangle\langle a_j| - |b_j\rangle\langle b_j\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}|b_j\rangle\langle b_j|b_j\rangle\langle a_j|).\end{aligned} \tag{2.1.11}$$

Applying the cyclic property of trace operation we have

$$\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle = \varepsilon(\hat{\rho}|b_j\rangle\langle a_j|\hat{a}^\dagger - \hat{\rho}|b_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| + \hat{\rho}\hat{a}|b_j\rangle\langle a_j|)$$
$$+ g(\hat{\rho}|b_j\rangle\langle a_j|\hat{a}^\dagger|b_j\rangle\langle a_j| - \hat{\rho}|b_j\rangle\langle a_j|a_j\rangle\langle b_j|\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|b_j\rangle\langle a_j| + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|b_j\rangle\langle a_j|)$$
$$+ \frac{\kappa}{2}(2\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a} - \hat{\rho}|b_j\rangle\langle a_j|\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|)$$
$$+ \frac{\gamma}{2}(2\hat{\rho}|a_j\rangle\langle c_j|b_j\rangle\langle a_j|c_j\rangle\langle a_j| - \hat{\rho}|b_j\rangle\langle a_j|a_j\rangle\langle a_j| - \hat{\rho}|a_j\rangle\langle a_j|b_j\rangle\langle a_j|$$
$$+ 2\hat{\rho}|b_j\rangle\langle c_j|b_j\rangle\langle a_j|c_j\rangle\langle b_j| - \hat{\rho}|b_j\rangle\langle a_j|b_j\rangle\langle b_j - \hat{\rho}|b_j\rangle\langle b_j|b_j\rangle\langle a_j|), \qquad (2.1.12)$$

it ,thus reduces to

$$\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle = -\gamma\langle|b_j\rangle\langle a_j|\rangle + g(\langle|a_j\rangle\langle a_j|\hat{a}\rangle - \langle|b_j\rangle\langle b_j|\hat{a}\rangle). \qquad (2.1.13)$$

Following the same procedure, one readily obtains

$$\frac{d}{dt}\langle|c_j\rangle\langle b_j|\rangle = -\frac{\gamma}{2}\langle|c_j\rangle\langle b_j|\rangle + g\langle\hat{a}^\dagger|c_j\rangle\langle a_j|\rangle, \qquad (2.1.14)$$

$$\frac{d}{dt}\langle|c_j\rangle\langle a_j|\rangle = -\frac{\gamma}{2}\langle|c_j\rangle\langle a_j|\rangle - g\langle|c_j\rangle\langle b_j|\hat{a}\rangle, \qquad (2.1.15)$$

$$\frac{d}{dt}\langle|a_j\rangle\langle a_j|\rangle = -\gamma\langle|a_j\rangle\langle a_j|\rangle - g(\langle|a_j\rangle\langle b_j|\hat{a}\rangle + \langle\hat{a}^\dagger|b_j\rangle\langle a_j|\rangle), \qquad (2.1.16)$$

$$\frac{d}{dt}\langle|b_j\rangle\langle b_j|\rangle = -\gamma\langle|b_j\rangle\langle b_j|\rangle + g(\langle|a_j\rangle\langle b_j|\hat{a}\rangle + \langle|b_j\rangle\langle a_j|\hat{a}^\dagger\rangle), \qquad (2.1.17)$$

$$\frac{d}{dt}\langle|c_j\rangle\langle c_j|\rangle = \gamma(\langle|a_j\rangle\langle a_j|\rangle + \langle|b_j\rangle\langle b_j|\rangle). \qquad (2.1.18)$$

The equations of evolution for the expectation values of cavity mode and atomic operators can be rewritten as

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}\rangle + g\langle\sigma_a^j\rangle + \varepsilon, \qquad (2.1.19)$$

$$\frac{d\langle\hat{\sigma}_a^j\rangle}{dt} = -\gamma\langle\hat{\sigma}_a^j\rangle + g(\langle\hat{a}\hat{\eta}_a^j\rangle - \langle\hat{\eta}_b^j\hat{a}\rangle), \qquad (2.1.20)$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^j\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^j\rangle - g(\langle\hat{a}^\dagger\hat{\sigma}_c^j\rangle, \qquad (2.1.21)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c^j\rangle - g\langle\hat{\sigma}_b^j\hat{a}\rangle, \tag{2.1.22}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -\gamma\langle\hat{\eta}_a^j\rangle - g(\langle\hat{\sigma}_a^{\dagger j}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle), \tag{2.1.23}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^j\rangle = -\gamma\langle\hat{\eta}_b^j\rangle + g(\langle\hat{\sigma}_a^{\dagger j}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle), \tag{2.1.24}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = \gamma(\langle\hat{\eta}_a^j\rangle + \langle\hat{\eta}_b^j\rangle), \tag{2.1.25}$$

where

$$\eta_a^j = |a_j\rangle\langle a_j|, \tag{2.1.26}$$

$$\eta_b^j = |b_j\rangle\langle b_j|, \tag{2.1.27}$$

$$\eta_c^j = |c_j\rangle\langle c_j|, \tag{2.1.28}$$

$$\sigma_a^j = |b_j\rangle\langle a_j|, \tag{2.1.29}$$

$$\sigma_b^j = |c_j\rangle\langle b_j|, \tag{2.1.30}$$

$$\sigma_c^j = |c_j\rangle\langle a_j|, \tag{2.1.31}$$

are atomic operators. On the basis of Eq.(2.1.9), we can write

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} + \varepsilon + g|b_j\rangle\langle a_j| + \hat{g}_a(t), \tag{2.1.32}$$

where $\hat{g}_a$ is noise operator associated with the cavity mode. The conjugate of this equation is

$$\frac{d}{dt}\hat{a}^\dagger = -\frac{\kappa}{2}\hat{a}^\dagger + \varepsilon + g|a_j\rangle\langle b_j| + \hat{g}_a^\dagger(t). \tag{2.1.33}$$

Now taking the expectation values to both sides of Eq.(2.1.32), we find

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{\kappa}{2}\langle\hat{a}\rangle + \varepsilon + g\langle|b_j\rangle\langle a_j|\rangle + \langle\hat{g}_a(t)\rangle. \tag{2.1.34}$$

Comparing this equation with (2.1.9), we see that

$$\langle\hat{g}_a(t)\rangle = 0. \tag{2.1.35}$$

We see that Eqs.(2.1.20) - (2.1.24) are coupled non-linear differential equations.It is thus not possible to obtain their exact time-dependent solutions [7]. We thus apply the large-time approximation scheme to Eqs.(2.1.33) and (2.1.34) and obtain

$$\hat{a}(t) = \frac{2\varepsilon}{\kappa} + \frac{2g}{\kappa}|b_j\rangle\langle a_j| + \frac{2}{\kappa}\hat{g}_a(t) \tag{2.1.36}$$

and

$$\hat{a}^\dagger(t) = \frac{2\varepsilon}{\kappa} + \frac{2g}{\kappa}|a_j\rangle\langle b_j| + \frac{2}{\kappa}\hat{g}_a^\dagger(t). \tag{2.1.37}$$

Upon substituting Eq.(2.1.36) into (2.1.13) ,we have

$$\begin{aligned}\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle &= -(\gamma + \frac{\gamma_c}{2})\langle|b_j\rangle\langle a_j|\rangle + \frac{2g\varepsilon}{\kappa}(\langle|a_j\rangle\langle a_j|\rangle - \langle|b_j\rangle\langle b_j|\rangle) \\ &+ \frac{2g}{\kappa}(\langle\hat{g}_a|a_j(t)\rangle\langle a_j(t)|\rangle - \langle|b_j(t)\rangle\langle b_j(t)|\hat{g}_a(t)\rangle).\end{aligned} \tag{2.1.38}$$

Furthermore substituting Eqs .(2.1.36) and (2.1.37) into Eqs.(2.1.14) -(2.1.17),we arrive at

$$\frac{d}{dt}\langle|c_j\rangle\langle b_j|\rangle = -\frac{\gamma}{2}\langle|c_j\rangle\langle b_j|\rangle + \frac{2g\varepsilon}{\kappa}\langle|c_j\rangle\langle a_j|\rangle + \frac{2g}{\kappa}\langle\hat{g}_{aj}(t)|c_j(t)\rangle\langle a_j(t)|\rangle, \tag{2.1.39}$$

$$\begin{aligned}\frac{d}{dt}\langle|c_j\rangle\langle a_j|\rangle &= -(\frac{\gamma}{2} + \frac{\gamma_c}{2})\langle|c_j\rangle\langle a_j|\rangle \\ &- \frac{2g\varepsilon}{\kappa}\langle|c_j\rangle\langle b_j|\rangle - \frac{2g}{\kappa}\langle|c_j(t)\rangle\langle b_j(t)|\hat{g}_a(t)\rangle,\end{aligned} \tag{2.1.40}$$

$$\begin{aligned}\frac{d}{dt}\langle|a_j\rangle\langle a_j|\rangle &= -(\gamma + \gamma_c)\langle|a_j\rangle\langle a_j|\rangle - \frac{2\varepsilon g}{\kappa}(\langle|a_j\langle b_j|\rangle + \langle|b_j\rangle\langle a_j|\rangle) \\ &- \frac{2g}{\kappa}(\langle|a_j(t)\rangle\langle b_j(t)|\hat{g}_a(t)\rangle + \langle\hat{g}_{aj}^\dagger(t)|b_j(t)\rangle\langle a_j(t)|\rangle),\end{aligned} \tag{2.1.41}$$

$$\begin{aligned}\frac{d}{dt}\langle|b_j\rangle\langle b_j|\rangle &= -\gamma\langle|b_j\rangle\langle b_j|\rangle + \gamma_c\langle|a_j\rangle\langle a_j|\rangle + \frac{2\varepsilon g}{\kappa}(\langle|a_j\rangle\langle b_j|\rangle + \langle|b_j\rangle\langle a_j|\rangle) \\ &+ \frac{2g}{\kappa}(\langle|a_j(t)\rangle\langle b_j(t)|\hat{g}_a(t)\rangle + \langle\hat{g}_a^\dagger(t)|b_j(t)\rangle\langle a_j(t)|\rangle),\end{aligned} \tag{2.1.42}$$

$$\frac{d}{dt}\langle|c_j\rangle\langle c_j|\rangle = \gamma(\langle|a_j\rangle\langle a_j|\rangle + \langle|b_j\rangle\langle b_j|\rangle), \tag{2.1.43}$$

where ,

$$\gamma_c = \frac{4g^2}{\kappa}. \tag{2.1.44}$$

In view of Eqs. (2.1.26) - (2.1.31), we can rewrite Eqs.(2.1.38) - (2.1.43) in the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^j\rangle = -(\gamma + \frac{\gamma_c}{2})\langle\hat{\sigma}_a^j\rangle + \frac{2\varepsilon g}{\kappa}(\langle\hat{\eta}_a^j\rangle - \langle\hat{\eta}_b^j\rangle) + \frac{2g}{\kappa}(\langle\hat{g}_a(t)\hat{\eta}_a^j(t)\rangle - \langle\eta_b^j(t)\hat{g}_a(t)\rangle), \tag{2.1.45}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^j\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^j\rangle + \frac{2\varepsilon g}{\kappa}\langle\hat{\sigma}_c^j\rangle + \frac{2g}{\kappa}\langle\hat{g}_{aj}(t)\hat{\sigma}_c^j(t)\rangle, \tag{2.1.46}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = -(\frac{\gamma}{2} + \frac{\gamma_c}{2})\langle\hat{\sigma}_c^j\rangle - \frac{2\varepsilon g}{\kappa}\langle\hat{\sigma}_b^j\rangle - \frac{2g}{\kappa}\langle\hat{\sigma}_b^j(t)\hat{g}_{aj}(t)\rangle, \tag{2.1.47}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -(\gamma + \gamma_c)\langle\hat{\eta}_a^j\rangle - \frac{2\varepsilon g}{\kappa}(\langle\hat{\sigma}_a^{\dagger j}\rangle + \langle\hat{\sigma}_a^j\rangle) - \frac{2g}{\kappa}(\langle\hat{\sigma}_a^{\dagger j}(t)\hat{g}_{aj}(t)\rangle \ + \langle\hat{g}_{aj}^\dagger(t)\sigma_a^j(t)\rangle), \tag{2.1.48}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^j\rangle = -\gamma\langle\hat{\eta}_b^j\rangle + \gamma_c\langle\hat{\eta}_a^j\rangle + \frac{2g\varepsilon}{\kappa}(\langle\hat{\sigma}_a^{\dagger j}\rangle + \langle\hat{\sigma}_a^j\rangle) + \frac{2g}{\kappa}(\langle\hat{\sigma}_a^{j\dagger}(t)\hat{g}_{aj}(t)\rangle + \langle\hat{g}_{aj}^\dagger(t)\hat{\sigma}_a^j(t)\rangle), \tag{2.1.49}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = \gamma(\langle\hat{\eta}_a^j\rangle + \langle\hat{\eta}_b^j\rangle). \tag{2.1.50}$$

In order to include the contribution of all the atoms to the dynamics of the two-level laser,we sum Eqs.(2.1.45) - (2.1.50) over N-two level atoms.We thus obtain

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -(\gamma + \frac{\gamma_c}{2})\langle\hat{m}_a\rangle + \frac{2\varepsilon g}{\kappa}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle) + \frac{2g}{\kappa}(\langle\hat{g}_a(t)\hat{N}_a(t)\rangle - \langle\hat{N}_b(t)\hat{g}_a(t)\rangle), \tag{2.1.51}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{\gamma}{2}\langle\hat{m}_b\rangle + \frac{2\varepsilon g}{\kappa}\langle\hat{m}_c\rangle + \frac{2g}{\kappa}\langle\hat{g}_a(t)\hat{m}_c(t)\rangle, \tag{2.1.52}$$

$$\frac{d}{dt}\langle \hat{m}_c\rangle = -(\gamma+\gamma_c)\langle \hat{m}_c\rangle - \frac{2\varepsilon g}{\kappa}\langle \hat{m}_b\rangle - \frac{2g}{\kappa}\langle \hat{m}_b(t)\hat{g}_a(t)\rangle, \tag{2.1.53}$$

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -(\gamma+\gamma_c)\langle \hat{N}_a\rangle - \frac{2\varepsilon g}{\kappa}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle) - \frac{2g}{\kappa}(\langle \hat{m}_a^\dagger(t)\hat{g}_a(t)\rangle + \langle \hat{g}_a^\dagger(t)\hat{m}_a(t)\rangle), \tag{2.1.54}$$

$$\frac{d}{dt}\langle \hat{N}_b\rangle = -\gamma\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle + \frac{2\varepsilon g}{\kappa}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle) + \frac{2g}{\kappa}(\langle \hat{m}_a^\dagger(t)\hat{g}_a(t)\rangle + \langle, g_a^\dagger(t)m_a(t)\rangle), \tag{2.1.55}$$

$$\frac{d}{dt}\langle \hat{N}_c\rangle = \gamma(\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle), \tag{2.1.56}$$

where

$$\hat{m}_a = \sum_{j=1}^{N} |b_j\rangle\langle a_j| = \sum_{j}^{N} \hat{\sigma}_a^j, \tag{2.1.57}$$

$$\hat{m}_b = \sum_{j=1}^{N} |c_j\rangle\langle b_j| = \sum_{j}^{N} \hat{\sigma}_b^j, \tag{2.1.58}$$

$$\hat{m}_c = \sum_{j=1}^{N} |c_j\rangle\langle a_j| = \sum_{j}^{N} \hat{\sigma} c^j, \tag{2.1.59}$$

are atomic operators and

$$\hat{N}_a = \sum_{j=1}^{N} |a_j\rangle\langle a_j| = \sum_{j}^{N} \hat{\eta}_a^j, \tag{2.1.60}$$

$$\hat{N}_b = \sum_{j=1}^{N} |b_j\rangle\langle b_j| = \sum_{j}^{N} \hat{\eta}, \tag{2.1.61}$$

$$\hat{N}_c = \sum_{j=1}^{N} |c_j\rangle\langle c_j| = \sum_{j}^{N} \hat{\eta}_c^j, \tag{2.1.62}$$

represent number of two -level atoms in the upper, lower and ground states . In the presence of N two-level atoms, we can rewrite Eq.(2.1.32) as

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} + \varepsilon + \lambda\hat{m}_a + \beta\hat{g}_a(t), \tag{2.1.63}$$

$\hat{g}_a(t)$ is noise operator associated with Langevin equation for cavity mode operator. We now proceed to determine the value of the constant λ and β . On the basis of Eqs.(2.1.36) and (2.1.37), we see that

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}, \tag{2.1.64}$$

$$\begin{aligned}
&= \frac{4\varepsilon^2}{\kappa^2} + \frac{4\varepsilon g}{\kappa^2}|a_j\rangle\langle b_j| + \frac{4\varepsilon}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{4\varepsilon g}{\kappa^2}|b_j\rangle\langle a_j| \\
&+ \frac{4g^2}{\kappa^2}|b_j\rangle\langle a_j|a_j\rangle\langle b_j| + \frac{4g}{\kappa^2}|b_j\rangle\langle a_j|\hat{g}_a^\dagger(t) \\
&+ \frac{4\varepsilon}{\kappa^2}\hat{g}_a(t) + \frac{4g}{\kappa^2}\hat{g}_a(t)|a_j\rangle\langle b_j| + \frac{4}{\kappa^2}\hat{g}_a(t)\hat{g}_a^\dagger(t) \\
&- \frac{4\varepsilon^2}{\kappa^2} - \frac{4\varepsilon g}{\kappa^2}|b_j\rangle\langle a_j| - \frac{4\varepsilon}{\kappa^2}\hat{g}_a(t) - \frac{4\varepsilon g}{\kappa^2}|a_j\rangle\langle b_j| \\
&- \frac{4g^2}{\kappa^2}|a_j\rangle\langle b_j|b_j\rangle\langle a_j| - \frac{4g}{\kappa^2}|a_j\rangle\langle b_j|\hat{g}_a(t) - \frac{4\varepsilon}{\kappa^2}\hat{g}_a^\dagger(t) \\
&- \frac{4g}{\kappa^2}\hat{g}_a^\dagger(t)|b_j\rangle\langle a_j| - \frac{4}{\kappa^2}\hat{g}_a^\dagger(t)\hat{g}_a(t),
\end{aligned} \tag{2.1.65}$$

it then reduces to

$$[\hat{a}, \hat{a}^\dagger] = \frac{4g^2}{\kappa^2}(\hat{\eta}_b^j - \hat{\eta}_a^j) + \frac{4}{\kappa^2}(\hat{g}_a(t)\hat{g}_a^\dagger(t) - \hat{g}_a^\dagger(t)\hat{g}_a(t)). \tag{2.1.66}$$

Now summing over N two-level atoms,we obtain

$$[\hat{a}, \hat{a}^\dagger] = \frac{4g^2}{\kappa^2}(\hat{N}_b - \hat{N}_a) + \frac{4}{\kappa^2}N(\hat{g}_a(t)\hat{g}_a^\dagger(t) - \hat{g}_a^\dagger(t)\hat{g}_a(t)). \tag{2.1.67}$$

On the other hand, applying the large -time approximation scheme to Eq.(2.1.63), we obtain

$$\hat{a} = \frac{2\varepsilon}{\kappa} + \frac{2\lambda}{\kappa}\hat{m}_a + \frac{2\beta}{\kappa}\hat{g}_a(t). \tag{2.1.68}$$

Employing Eq.(2.1.68) and its conjugate, we see that

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}, \tag{2.1.69}$$

$$
\begin{aligned}
&= \frac{4\varepsilon^2}{\kappa^2} + \frac{4\varepsilon\lambda^*}{\kappa^2}\hat{m}_a^\dagger + \frac{4\varepsilon\beta^*}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{4\varepsilon\lambda}{\kappa^2}\hat{m}_a + \frac{4\lambda^2}{\kappa^2}\hat{m}_a\hat{m}_a^\dagger \\
&+ \frac{4\lambda\beta^*}{\kappa^2}\hat{m}_a(t)\hat{g}_a^\dagger(t) + \frac{4\varepsilon\beta^*}{\kappa^2}\hat{g}_a(t) + \frac{4\lambda\varepsilon}{\kappa^2}\hat{m}_a + \frac{4\beta^2}{\kappa^2}\hat{g}_a(t)\hat{g}_a^\dagger(t) \\
&- \frac{4\varepsilon^2}{\kappa^2} - \frac{4\varepsilon\lambda}{\kappa^2}\hat{m}_a - \frac{4\varepsilon\beta}{\kappa^2}\hat{g}_a(t) - \frac{4\varepsilon\beta^*}{\kappa^2}\hat{g}_a^\dagger(t) - \frac{4\lambda^*\beta}{\kappa^2}\hat{m}_a^\dagger(t)\hat{g}_a(t) \\
&- \frac{4\lambda^2}{\kappa^2}\hat{m}_a^\dagger\hat{m}_a - \frac{4\varepsilon\beta^*}{\kappa^2}\hat{g}_a^\dagger(t) - \frac{4\lambda\beta^*}{\kappa^2}\hat{g}_a^\dagger(t)\hat{m}_a - \frac{4\beta^2}{\kappa^2}\hat{g}_a^\dagger(t)\hat{g}_a^\dagger(t),
\end{aligned} \tag{2.1.70}
$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{2.1.71}$$

We therefore observe that

$$\hat{m}_a\hat{m}_a^\dagger = N\hat{N}_b, \tag{2.1.72}$$

in which

$$\hat{N}_b = N|b\rangle\langle b|. \tag{2.1.73}$$

Following the same procedure, one can also establish that

$$\hat{m}_a^\dagger\hat{m}_a = N\hat{N}_a, \tag{2.1.74}$$

with

$$\hat{N}_a = N|a\rangle\langle a|. \tag{2.1.75}$$

Then Eq.(2.1.70) leads to

$$[\hat{a}, \hat{a}^\dagger] = \frac{4\lambda^2}{\kappa^2}N(\hat{N}_b - \hat{N}_a) + \frac{4\beta^2}{\kappa^2}(\hat{g}_a(t)\hat{g}_a^\dagger(t) - \hat{g}_a^\dagger(t)\hat{g}_a^\dagger(t)). \tag{2.1.76}$$

Now comparing Eqs .(2.1.67) and (2.1.76) ,we obtain

$$\lambda = \pm\frac{g}{\sqrt{N}}, \tag{2.1.77}$$

$$\beta = \sqrt{N}. \tag{2.1.78}$$

On account of these results Eq.(2.1.63) can be written as

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} + \varepsilon + \frac{g}{\sqrt{N}}\hat{m}_a + \sqrt{N}\hat{g}_a(t), \tag{2.1.79}$$

or

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} + \varepsilon + \frac{g}{\sqrt{N}}\hat{m}_a + \hat{G}(t), \tag{2.1.80}$$

where $\hat{G}(t) = \sqrt{N}\hat{g}_a(t)$ is cavity mode noise operator when the cavity is interacting with N two-level atoms.

2.2 Correlation properties of noise operators

Here we determine the correlation properties of cavity mode and atomic noise operators. Employing the master equation along with the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{2.2.1}$$

it can be verified that

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle + 2\varepsilon\langle\hat{a}\rangle + 2g\langle\hat{a}\hat{\sigma}_a^j\rangle, \tag{2.2.2}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle + \varepsilon(\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle) + g(\langle\sigma_a^{\dagger j}\hat{a}\rangle + \langle\hat{a}^\dagger\sigma_a^j\rangle), \tag{2.2.3}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle + \varepsilon(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle) + g(\langle\hat{a}\hat{\sigma}_a^{\dagger j}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle + \kappa. \tag{2.2.4}$$

Employing the relation

$$\frac{d}{dt}\langle\hat{a}\hat{a}\rangle = \langle\frac{d\hat{a}}{dt}\hat{a}\rangle + \langle\frac{\hat{a}d\hat{a}}{dt}\rangle, \tag{2.2.5}$$

along with Eq.(2.1.19), we obtain

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle + 2\varepsilon\langle\hat{a}\rangle + 2g\langle\hat{a}\hat{\sigma}_a^j\rangle + \langle\hat{a}(t)\hat{g}_a(t)\rangle + \langle\hat{g}_a(t)\hat{a}(t)\rangle. \tag{2.2.6}$$

Upon comparing Eqs.(2.2.6) and (2.2.2), we note that

$$\langle \hat{a}(t)\hat{g}_a(t)\rangle + \langle \hat{g}_a(t)\hat{a}(t)\rangle = 0. \tag{2.2.7}$$

The solution Eq.(2.1.32) can be written as

$$\begin{aligned}\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \varepsilon e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}dt' + ge^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{\sigma}_a^j(t')dt' \\ + e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{g}_a(t')dt'.\end{aligned} \tag{2.2.8}$$

Multiplying Eq.(2.2.8) by $\hat{g}_a(t)$ on the left and taking the expectation value of the resulting expression we obtain

$$\begin{aligned}\langle \hat{g}_a(t)\hat{a}(t)\rangle = \langle \hat{g}_a(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} + \varepsilon e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t)\rangle dt' + ge^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t)\hat{\sigma}_a^j(t')\rangle dt' \\ + e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t)\hat{g}_a(t')\rangle dt'\end{aligned} \tag{2.2.9}$$

Since noise operator at time t should not affect cavity mode operators at earlier times we can write

$$\langle \hat{g}_a(t)\hat{a}(0)\rangle = \langle \hat{g}_a(t)\rangle\langle \hat{a}(0)\rangle = 0 \tag{2.2.10}$$

$$\langle \hat{g}_a(t)\hat{\sigma}_a^j(t')\rangle = 0 \tag{2.2.11}$$

In view of Eqs.(2.1.35), (2.2.10) and (2.2.11) one can write Eq. (2.2.9) as

$$\langle \hat{g}_a(t)\hat{a}(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t)\hat{g}_a(t')\rangle dt' \tag{2.2.12}$$

Multiplying Eq.(2.2.8) by $\hat{g}_a(t)$ from right and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t has no effect on cavity mode operators at earlier time, we obtain

$$\langle \hat{a}(t)g_a(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t')\hat{g}_a(t)\rangle dt' \tag{2.2.13}$$

Introduction of Eqs.(2.2.12) and (2.2.13) into Eq.(2.2.7) leads to

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle g_a(t)g_a(t')\rangle dt' + e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t')\hat{g}_a(t)\rangle dt' = 0. \tag{2.2.14}$$

Now assuming

$$\langle \hat{g}_a(t)\hat{g}_a(t')\rangle = \langle \hat{g}_a(t')\hat{g}_a(t)\rangle \tag{2.2.15}$$

we can put Eq.(2.2.14) in the form

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\langle \hat{g}_a(t)\hat{g}_a(t')\rangle dt' = 0 \tag{2.2.16}$$

or

$$\langle g_a(t)g_a(t')\rangle = 0. \tag{2.2.17}$$

Moreover, employing Eq.(2.1.32) along with its conjugate in the relation

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{a}\rangle = \langle \frac{d\hat{a}^\dagger}{dt}\hat{a}\rangle + \langle \hat{a}^\dagger\frac{d\hat{a}}{dt}\rangle \tag{2.2.18}$$

we see that

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{a}\rangle = -\kappa\langle \hat{a}^\dagger\hat{a}\rangle + \varepsilon(\langle \hat{a}^\dagger\rangle + \langle \hat{a}\rangle) + g(\langle \hat{\sigma}^\dagger\hat{a}\rangle + \langle \hat{a}^\dagger\hat{\sigma}_a\rangle)$$
$$+ \langle \hat{g}_a^\dagger(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{g}_a(t)\rangle \tag{2.2.19}$$

Comparing Eq .(2.2.19) with Eq.(2.2.3), we obtain

$$\langle \hat{g}_a^\dagger(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{g}_a(t)\rangle = 0. \tag{2.2.20}$$

Now multiplying Eq.(2.2.8) by $\hat{g}_a^\dagger(t)$ from left and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t should not affect cavity mode operators at earlier times, we obtain

$$\langle g_a^\dagger(t)\hat{a}(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt'. \tag{2.2.21}$$

Furthermore multiplying the conjugate of Eq.(2.2.8) by $\hat{g}_a(t)$ from right and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t has no effect on cavity mode operators at earlier times, we obtain

$$\langle \hat{a}^\dagger(t)\hat{g}_a(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t}\langle \hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt'. \tag{2.2.22}$$

Employing Eqs.(2.2.21) and (2.2.22) into (2.2.20), we see that

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt' + e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt' = 0 \tag{2.2.23}$$

Now on assuming

$$\langle \hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle = \langle \hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle \tag{2.2.24}$$

We can put Eq(2.2.23) as

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\langle \hat{g}_{aj}^\dagger(t)\hat{g}_a(t')\rangle dt' = 0. \tag{2.2.25}$$

It then follows that

$$\langle \hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle = 0. \tag{2.2.26}$$

Moreover, again employing Eq.(2.1.32) along with its conjugate in the relation

$$\frac{d}{dt}\langle \hat{a}\hat{a}^\dagger\rangle = \left\langle \frac{d\hat{a}}{dt}\hat{a}^\dagger\right\rangle + \left\langle \frac{\hat{a}d\hat{a}^\dagger}{dt}\right\rangle, \tag{2.2.27}$$

we see that

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}\hat{a}^\dagger\rangle &= -\kappa\langle \hat{a}\hat{a}^\dagger\rangle + \varepsilon(\langle \hat{a}\rangle + \langle \hat{a}^\dagger\rangle) + g(\langle \hat{a}|b_j\rangle\langle a_j|\rangle \\ &+ \langle |a_j\rangle\langle b_j|\hat{a}^\dagger\rangle + \langle \hat{a}(t)\hat{g}_a^\dagger(t)\rangle + \langle \hat{g}_a\hat{a}^\dagger(t)\rangle.\end{aligned} \tag{2.2.28}$$

Now, comparing Eqs.(2.2.4) and (2.2.28), we obtain

$$\langle \hat{a}(t)\hat{g}_a^\dagger(t)\rangle + \langle \hat{g}_a(t)\hat{a}^\dagger(t)\rangle = \kappa. \tag{2.2.29}$$

Furthermore multiplying Eq.(2.2.8) by $\hat{g}_a^{\dagger}(t)$ from right and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t has no effect on cavity mode operators at earlier times, we obtain

$$\langle\hat{a}(t)\hat{g}_a^{\dagger}(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\langle\hat{g}_a(t')\hat{g}_a^{\dagger}(t)\rangle dt'. \tag{2.2.30}$$

Similarly, multiplying the conjugate of Eq.(2.2.8) by $\hat{g}_a^{\dagger}(t)$ from left and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t has no effect on cavity mode operators at earlier times, we obtain

$$\langle\hat{g}_a(t)\hat{a}^{\dagger}(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\hat{\langle}g_a(t)g_a^{\dagger}(t')\rangle dt'. \tag{2.2.31}$$

Now, substituting Eqs. (2.2.30) and (2.2.31) into Eq.(2.2.29), we find

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\langle\hat{g}_a(t')\hat{g}_a^{\dagger}(t)\rangle dt' + e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\hat{\langle}g_a(t)g_a^{\dagger}(t')\rangle dt' = \kappa, \tag{2.2.32}$$

assuming that

$$\langle\hat{g}_a(t')\hat{g}_a^{\dagger}(t)\rangle = \hat{\langle}g_a(t)g_a^{\dagger}(t')\rangle \tag{2.2.33}$$

Then, Eq.(2.2.32) becomes

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{-\frac{\kappa}{2}t'}\hat{\langle}g_a(t)g_a^{\dagger}(t')\rangle dt' = \frac{1}{2}\kappa \tag{2.2.34}$$

On the basis of the relation

$$\int_0^t e^{a(t-t')}\langle\hat{f}(t)\hat{g}(t')\rangle dt' = D, \tag{2.2.35}$$

$$\langle\hat{f}(t)\hat{g}(t')\rangle = 2D\delta(t-t'), \tag{2.2.36}$$

where a is a constant or a function of time , we see that

$$\langle\hat{g}_a(t)\hat{g}_a^{\dagger}(t')\rangle = \kappa\delta(t-t'). \tag{2.2.37}$$

Following a similar procedure one can verify that for N two- level atoms, Eq.(2.2.37) becomes

$$\langle \hat{G}(t)\hat{G}^{\dagger}(t')\rangle = \kappa N\delta(t-t'). \tag{2.2.38}$$

We now proceed to obtain the expectation values of the product of cavity mode noise operator and atomic operator appearing in Eqs.(2.1.45) - (2.1.49). On the basis of Eq.(2.1.20),we can write

$$\frac{d}{dt}\hat{\sigma}_a^j = -\gamma\hat{\sigma}_a^j + g(\hat{\eta}_a^j\hat{a} - \hat{\eta}_b^j\hat{a}) + \hat{f}_{aj}(t), \tag{2.2.39}$$

where, $\hat{f}_{aj}(t)$ is the atomic noise operator. Now employing relation

$$\frac{d}{dt}\langle\hat{a}\hat{\sigma}_a^j\rangle = \langle\hat{a}\frac{d\hat{\sigma}_a^j}{dt}\rangle + \langle\frac{d\hat{a}}{dt}\hat{\sigma}_a^j\rangle \tag{2.2.40}$$

along with Eqs.(2.1.32) and (2.2.39), we find that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\hat{\sigma}_a^j\rangle &= g(\langle\hat{a}^2\hat{\eta}_a^j\rangle - \langle\hat{\eta}_b^j\hat{a}^2\rangle) - \gamma\langle\hat{a}\hat{\sigma}_a^j\rangle - \frac{\kappa}{2}\langle\hat{a}\hat{\sigma}_a^j\rangle + \varepsilon\langle\hat{\sigma}_a^j\rangle \\ &+ \langle\hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{a}(t)\hat{f}_{aj}(t)\rangle\end{aligned} \tag{2.2.41}$$

Further more using the master equation described by Eq.(2.1.5)along with the relation

$$\frac{d}{dt}\langle\hat{a}\hat{\sigma}_a^j\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a}\hat{\sigma}_a^j), \tag{2.2.42}$$

one finds

$$\frac{d}{dt}\langle\hat{a}\hat{\sigma}_a^j\rangle = \varepsilon\langle\hat{\sigma}_a^j\rangle + g(\langle\hat{\eta}_a^j\hat{a}^2\rangle - \langle\hat{a}^2\hat{\eta}_b^j\rangle) - \frac{\kappa}{2}\langle\hat{a}\hat{\sigma}_a^j\rangle - \gamma\langle\hat{a}\hat{\sigma}_a^j\rangle \tag{2.2.43}$$

Upon comparing Eq.(2.2.41) and (2.2.43), we see that

$$\langle\hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{a}(t)\hat{f}_{aj}(t)\rangle = 0 \tag{2.2.44}$$

The solution of Eq.(2.2.39) can be written as

$$\hat{\sigma}_a^j(t) = \hat{\sigma}_a^j(0)e^{-\gamma t} + ge^{-\gamma t}\int_0^t e^{\gamma t'}(\hat{a}(t')(\hat{\eta}_a^j(t') - \hat{\eta}_b^j(t')\hat{a}(t'))dt'$$
$$+ e^{-\gamma t}\int_0^t e^{\gamma t'}\hat{f}_{aj}(t')dt' \tag{2.2.45}$$

Multiplying Eq.(2.2.8) by $\hat{f}_{aj}(t)$ from right and taking the expectation value of the resulting expression and applying the fact that the noise operator at some time has no effect on the cavity mode and atomic operators at earlier time,we obtain

$$\langle\hat{a}(t)\hat{f}_{aj}(t)\rangle = e^{\frac{-\kappa t}{2}}\int_0^t e^{\frac{\kappa t'}{2}}\langle\hat{g}_a(t')\hat{f}_{aj}(t)\rangle dt' \tag{2.2.46}$$

Similarly multiplying Eq.(2.2.45) by $\hat{g}_a(t)$ from left and applying the fact that the noise operator at some time has no effect on the cavity mode and atomic operators at earlier time,we arrive at

$$\langle\hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle = e^{-\gamma t}\int_0^t e^{\gamma t'}\langle\hat{g}_a(t)\hat{f}_{aj}(t')\rangle dt' \tag{2.2.47}$$

Now substituting Eqs.(2.2.46) and (2.2.47) into (2.2.44), we obtain

$$e^{\frac{-\kappa t}{2}}\int_0^t e^{\frac{\kappa t'}{2}}\langle\hat{g}_a(t')\hat{f}_{aj}(t)\rangle dt' + e^{-\gamma t}\int_0^t e^{\gamma t'}\langle\hat{g}_a(t)\hat{f}_{aj}(t')\rangle dt' = 0 \tag{2.2.48}$$

Assuming that

$$\langle\hat{g}_a(t')\hat{f}_{aj}(t)\rangle = \langle\hat{g}_a(t)\hat{f}_{aj}(t')\rangle, \tag{2.2.49}$$

we can write Eq(2.2.48) as

$$\int_0^t (e^{\frac{-\kappa}{2}(t-t')} + e^{\gamma(t-t')})\langle\hat{g}_a(t)\hat{f}_{aj}(t')\rangle dt' = 0 \tag{2.2.50}$$

It then follows that

$$\langle\hat{g}_a(t)\hat{f}_{aj}(t')\rangle = 0 \tag{2.2.51}$$

Now in view of this relation Eq.(2.2.47) becomes

$$\langle \hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle = 0, \tag{2.2.52}$$

so that summing over the N two-level atoms results in

$$\langle \hat{G}(t)\hat{m}_a(t)\rangle = 0. \tag{2.2.53}$$

In a similar procedure, it can be established that

$$\langle \hat{m}_b(t)\hat{G}(t)\rangle = 0, \tag{2.2.54}$$

and

$$\langle \hat{m}_c(t)\hat{G}(t)\rangle = 0. \tag{2.2.55}$$

We now seek to determine the expectation value $\langle g_a(t)\hat{N}_a(t)\rangle$. On the basis of Eq.(2.1.16), one can write

$$\frac{d}{dt}|a_j\rangle\langle a_j| = -\gamma|a_j\rangle\langle a_j| - g(|a_j\rangle\langle b_j|\hat{a} + \hat{a}^{\dagger}|b_j\rangle\langle a_j|) + \hat{F}_{aj}(t), \tag{2.2.56}$$

where $\hat{F}_{aj}(t)$ is atomic noise operator. Employing the relation

$$\frac{d}{dt}\langle \hat{a}|a_j\rangle\langle a_j|\rangle = \langle \hat{a}\frac{d|a_j\rangle\langle a_j|}{dt}\rangle + \langle \frac{d\hat{a}}{dt}|a_j\rangle\langle a_j|\rangle \tag{2.2.57}$$

along with Eqs .(2.1.32) and (2.2.56), we arrive at

$$\frac{d}{dt}\langle \hat{a}\hat{\eta}_a^j\rangle = -g(\langle \hat{a}^2\hat{\sigma}_a^{\dagger j}\rangle + \langle \hat{a}^{\dagger}\hat{a}\hat{\sigma}_a^j\rangle) - \gamma\langle \hat{a}\hat{\eta}_{aj}\rangle - \frac{\kappa}{2}\langle \hat{a}\hat{\eta}_a^j\rangle$$
$$+ \varepsilon\langle \hat{\eta}_a^j\rangle + +\langle \hat{a}(t)\hat{F}_{aj}(t)\rangle + \langle \hat{g}_{aj}(t)\hat{\eta}_a^j(t)\rangle \tag{2.2.58}$$

Applying the master equation along with the relation

$$\frac{d}{dt}\langle \hat{a}|a_j\rangle\langle a_j|\rangle = Tr(\frac{d}{dt}\hat{\rho}\hat{a}|a_j\rangle\langle a_j|), \tag{2.2.59}$$

it can readily verified that

$$\frac{d}{dt}\langle\hat{a}\hat{\eta}_a^j\rangle = \varepsilon\langle\hat{\eta}_a^j\rangle - g(\langle\hat{a}^2\hat{\sigma}_a^{\dagger j}\rangle + \langle\hat{a}^\dagger\hat{a}\hat{\sigma}_a^\dagger\rangle) - \frac{\kappa}{2}\langle\hat{a}\hat{\eta}_a^j\rangle - \gamma\langle\hat{a}\hat{\eta}_a^j\rangle \tag{2.2.60}$$

Now comparison of Eqs .(2.2.58) and (2.2.60) yields

$$\langle\hat{a}(t)\hat{F}_{aj}(t)\rangle + \langle\hat{g}_a(t)\hat{\eta}_a^j(t)\rangle = 0. \tag{2.2.61}$$

The solution of Eq.(2.2.56) can be written as

$$\begin{aligned}\hat{\eta}_a^j(t) = \hat{\eta}_a^j(0)e^{-\gamma t} - ge^{-\gamma t}\int_0^t e^{\gamma t'}(\hat{\sigma}_a^{j\dagger}(t')\hat{a}(t') + \hat{a}^\dagger(t')\hat{\sigma}_a^j(t'))dt' \\ + e^{-\gamma t}\int_0^t e^{\gamma t'}\hat{F}_{aj}(t')dt'.\end{aligned} \tag{2.2.62}$$

Upon multiplying Eq.(2.2.8) by $\hat{F}_{aj}(t)$ from the left and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t has no effect on cavity mode operators at earlier time,we arrive at

$$\langle\hat{a}(t)\hat{F}_{aj}(t)\rangle = e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle\hat{g}_{aj}(t')\hat{F}_{aj}(t)\rangle dt'. \tag{2.2.63}$$

Similarly multiplying Eq.(2.2.62) by $\hat{g}_a(t)$ on the left and taking the expectation value of the resulting expression and applying the fact that a noise operator at some time t has no effect on cavity mode and atomic operators at earlier time, we obtain

$$\langle\hat{g}_a(t)\hat{\eta}_a^j(t)\rangle = e^{-\gamma t}\int_0^t e^{\gamma t'}\langle\hat{g}_a(t)\hat{F}_{aj}(t')\rangle dt'. \tag{2.2.64}$$

Upon substituting Eqs.(2.2.63) and (2.2.64) into Eq.(2.2.61) we find

$$e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle\hat{g}_a(t')\hat{F}_{aj}(t)\rangle dt' + e^{-\gamma t}\int_0^t e^{\gamma t'}\langle\hat{g}_a(t)\hat{F}_{aj}(t')\rangle dt' = 0. \tag{2.2.65}$$

Now on assuming

$$\langle\hat{g}_a(t')\hat{F}_{aj}(t)\rangle = \langle\hat{g}_a(t)\hat{F}_{aj}(t')\rangle, \tag{2.2.66}$$

we can write Eq.(2.2.65) as

$$\int_0^t (e^{\frac{\kappa}{2}(t'-t)} + e^{\gamma(t'-t)})\langle \hat{g}_a(t)\hat{F}_{aj}(t')\rangle = 0. \tag{2.2.67}$$

It then follows that

$$\langle \hat{g}_a(t)\hat{F}_{aj}(t')\rangle = 0. \tag{2.2.68}$$

In view of this, Eq.(2.2.64) becomes

$$\langle \hat{g}_a(t)\hat{\eta}_a^j(t)\rangle = 0, \tag{2.2.69}$$

which for N two-level atoms becomes

$$\langle \hat{G}(t)\hat{N}_a(t)\rangle = 0. \tag{2.2.70}$$

Following a similar procedure, it can be established that

$$\langle \hat{G}(t)\hat{N}_b(t)\rangle = 0, \tag{2.2.71}$$

and

$$\langle \hat{G}(t)\hat{N}_c(t)\rangle = 0. \tag{2.2.72}$$

Taking into account Eqs.(2.2.53)-(2.2.55)and (2.2.70) -(2.2.72),we can write Eqs.(2.1.51) -(2.1.55) as

$$\frac{d}{dt}\langle \hat{m}_a\rangle = -(\gamma + \frac{\gamma_c}{2})\langle \hat{m}_a\rangle + \frac{2g\varepsilon}{\kappa}(\langle \hat{N}_a\rangle - \langle \hat{N}_b\rangle), \tag{2.2.73}$$

$$\frac{d}{dt}\langle \hat{m}_b\rangle = -\frac{\gamma}{2}\langle \hat{m}_b\rangle + \frac{2g\varepsilon}{\kappa}\langle \hat{m}_c\rangle, \tag{2.2.74}$$

$$\frac{d}{dt}\langle \hat{m}_c\rangle = -\gamma\langle \hat{m}_c\rangle - \frac{2g\varepsilon}{\kappa}\langle \hat{m}_b\rangle, \tag{2.2.75}$$

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -(\gamma + \gamma_c)\langle \hat{N}_a\rangle - \frac{2g\varepsilon}{\kappa}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle), \tag{2.2.76}$$

$$\frac{d}{dt}\langle \hat{N}_b\rangle = -\gamma\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle + \frac{2g\varepsilon}{\kappa}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle), \tag{2.2.77}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = \gamma(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle. \tag{2.2.78}$$

The pumping process must certainly affect the time evolution of the expectation value of $\langle\hat{N}_a\rangle$ and $\langle\hat{N}_c\rangle$.Hence, we take into account the effect of the pumping process on the time evolution of the operators $\langle\hat{N}_a\rangle$ and $\langle\hat{N}_c\rangle$ by rewriting Eqs.(2.2.76) and (2.2.78)in the form

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma+\gamma_c)\langle\hat{N}_a\rangle - \frac{2g\varepsilon}{\kappa}(\langle\hat{m}_a^\dagger\rangle + \langle\hat{m}_a\rangle) + r_{ca}\langle\hat{N}_c\rangle \tag{2.2.79}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = \gamma(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle) - r_{ca}\langle\hat{N}_c\rangle \tag{2.2.80}$$

where r_{ca} is the rate at which a single atom is pumped to the upper level. Using the relation

$$\hat{N} = \langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle, \tag{2.2.81}$$

one can put Eq.(2.2.80) in the form

$$\frac{d}{dt}\langle\hat{N}_c\rangle = \gamma N - (\gamma + r_{ca})\langle\hat{N}_c\rangle \tag{2.2.82}$$

The steady-state solution of Eq.(2.2.82) is

$$\langle\hat{N}_c\rangle = \frac{\gamma N}{\gamma + r_{ca}} \tag{2.2.83}$$

Moreover, Eq.(2.2.77)reduces,at steady state,to

$$\gamma\langle\hat{N}_b\rangle - \gamma_c\langle\hat{N}_a\rangle = \frac{2g\varepsilon}{\kappa}(\langle\hat{m}_a^\dagger\rangle + \langle\hat{m}_a\rangle) \tag{2.2.84}$$

Upon introducing Eq.(2.2.84) into Eq.(2.2.79) we find

$$\frac{d\langle\hat{N}_a\rangle}{dt} = -\gamma(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle) + r_{ca}\langle\hat{N}_c\rangle \tag{2.2.85}$$

Taking into account Eq.(2.2.81), the steady-state solution of Eq.(2.2.85) is expressible as

$$\langle \hat{N}_a \rangle = \frac{r_{ca}N}{\gamma + r_{ca}} - \langle \hat{N}_b \rangle \tag{2.2.86}$$

Dropping the expectation values in Eqs.(2.2.73) and (2.2.74), we notice that

$$\frac{d}{dt}\hat{m}_a = -(\gamma + \frac{\gamma_c}{2})\hat{m}_a + \frac{2g\varepsilon}{\kappa}(\hat{N}_a - \hat{N}_b) + \hat{f}_a(t), \tag{2.2.87}$$

$$\frac{d}{dt}\hat{m}_b = -\frac{\gamma_c}{2}\hat{m}_b + \frac{2g\varepsilon}{\kappa}\hat{m}_c + \hat{f}_b(t), \tag{2.2.88}$$

in which $\hat{f}_a(t)$ and $\hat{f}_b(t)$ are atomic noise operators. We next wish to determine the correlation properties of the noise operators $\hat{f}_a(t)$ and $\hat{f}_b(t)$. Employing Eq.(2.2.87)along with its conjugate in the relation

$$\frac{d}{dt}\langle \hat{m}_a^\dagger \hat{m}_a \rangle = \langle \frac{d\hat{m}_a^\dagger}{dt}\hat{m}_a \rangle + \langle \hat{m}_a^\dagger \frac{d\hat{m}_a}{dt} \rangle \tag{2.2.89}$$

and taking into account Eq.(2.1.58), we obtain

$$\begin{aligned}\frac{d}{dt}\langle \hat{N}_a \rangle = &-(2\gamma + \gamma_c)\langle \hat{N}_a \rangle - \frac{2g\varepsilon}{\kappa}(\langle \hat{m}_a^\dagger \rangle + \langle \hat{m}_a \rangle)\\ &+ \frac{1}{N}(\langle \hat{f}_a^\dagger(t)\hat{m}_a(t) \rangle + \langle \hat{m}_a^\dagger(t)\hat{f}_a(t) \rangle)\end{aligned} \tag{2.2.90}$$

Upon comparing Eq.(2.2.79) with (2.2.90) we arrive at

$$\langle \hat{f}_a^\dagger(t)\hat{m}_a(t) \rangle + \langle \hat{m}_a^\dagger(t)\hat{f}_a(t) \rangle = \gamma N \langle \hat{N}_a \rangle + r_{ac}N\langle \hat{N}_c \rangle \tag{2.2.91}$$

The solution of Eq.(2.2.87) can be written as

$$\begin{aligned}\hat{m}_a(t) = \hat{m}_a(0)e^{-(\gamma+\frac{\gamma_c}{2})t} &+ \frac{2g\varepsilon}{\kappa}e^{-(\gamma+\frac{\gamma_c}{2})t}\int_0^t e^{(\gamma+\frac{\gamma_c}{2})t'}(\hat{N}_a(t') - \hat{N}_b(t'))dt'\\ &+ e^{-(\gamma+\frac{\gamma_c}{2})t}\int_0^t e^{(\gamma+\frac{\gamma_c}{2})t'}\hat{f}_a(t')dt'\end{aligned} \tag{2.2.92}$$

Multiplying Eq.(2.2.92) by $f_a^\dagger(t)$ from left and taking the expectation value of the resulting expressions and applying the fact that a noise operator at some time t has no effect on atomic operators at earlier time, we arrive at

$$\langle \hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle = e^{(-\gamma+\frac{\gamma_c}{2})t}\int_0^t e^{(\gamma+\frac{\gamma_c}{2})t'}\langle \hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle dt' \tag{2.2.93}$$

Furthermore, multiplying the conjugate of Eq.(2.2.92) by $\hat{f}_a(t)$ from right and taking the expectation value of the resulting expressions and applying the fact that a noise operator at some time t has no effect on atomic operators at earlier time,we obtain

$$\langle \hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle = e^{-(\gamma+\frac{\gamma_c}{2})t}\int_0^t e^{\gamma t'}\langle \hat{f}_a^\dagger(t')\hat{f}_a(t)\rangle dt' \tag{2.2.94}$$

substitution Eqs.(2.2.93) and (2.2.94) into (2.2.91) ,we leads to

$$\begin{aligned} &e^{-(\gamma+\frac{\gamma_c}{2})t}\int_0^t e^{(\gamma+\frac{\gamma_c}{2})t'}\langle \hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle dt' + e^{-(\gamma+\frac{\gamma_c}{2})t}\int_0^t e^{(\gamma+\frac{\gamma_c}{2})t'}\langle \hat{f}_a^\dagger(t')\hat{f}_a(t)\rangle dt' \\ &= \gamma N\langle \hat{N}_a\rangle + r_{ac}N\langle \hat{N}_c\rangle \end{aligned} \tag{2.2.95}$$

Assuming that

$$\langle \hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle = \langle \hat{f}_a^\dagger(t')\hat{f}_a(t)\rangle \tag{2.2.96}$$

we can express Eq.(2.2.95) in the form

$$e^{-\gamma t}\int_0^t e^{\gamma t'}\langle \hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle dt' = \frac{1}{2}(\gamma N\langle \hat{N}_a\rangle + r_{ac}N\langle \hat{N}_c\rangle). \tag{2.2.97}$$

So that with the aid of the relation described in Eq.(2.2.36), (2.2.95)becomes

$$\langle \hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle = (\gamma N\langle \hat{N}_a\rangle + r_{ac}N\langle \hat{N}_c\rangle)\delta(t-t'). \tag{2.2.98}$$

Following the same procedure, one can also establish that

$$\langle \hat{f}_a(t)\hat{f}_a(t')\rangle = 0, \tag{2.2.99}$$

$$\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle = (\frac{\gamma_c r_{ca} N^2}{\gamma + r_{ca}} + \gamma\langle \hat{N}_b\rangle)N\delta(t-t'), \tag{2.2.100}$$

$$\langle \hat{f}_b(t)\hat{f}_b(t')\rangle = 0, \tag{2.2.101}$$

$$\langle \hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle = \gamma_c\langle \hat{N}_a\rangle N\delta(t-t'), \tag{2.2.102}$$

$$\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle = \frac{\gamma_c + r_{ca}}{\gamma + r_{ca}}\gamma N\delta(t-t'), \tag{2.2.103}$$

$$\langle \hat{f}_a(t)\hat{f}_b(t')\rangle = 0, \tag{2.2.104}$$

$$\langle \hat{f}_a^\dagger(t)\hat{f}_b^\dagger(t')\rangle = 0, \tag{2.2.105}$$

$$\langle \hat{f}_a(t)\hat{f}_b^\dagger(t')\rangle = 0. \tag{2.2.106}$$

We now proceed to obtain the solution of Eq.(2.1.79). Employing Eq.(2.1.79) and its conjugate, we see that

$$\frac{d}{dt}\hat{A}_+(t) = \frac{-\kappa}{2}\hat{A}_+(t) + \frac{g}{\sqrt{N}}\hat{m}_+(t) + 2\varepsilon + \hat{G}_+(t), \tag{2.2.107}$$

$$\frac{d}{dt}\hat{A}_-(t) = \frac{-\kappa}{2}\hat{A}_-(t) + \frac{g}{\sqrt{N}}\hat{m}_-(t) + \hat{G}_-(t), \tag{2.2.108}$$

where,

$$\hat{A}_\pm(t) = \hat{a}^\dagger(t) \pm \hat{a}(t), \tag{2.2.109}$$

$$\hat{m}_\pm = \hat{m}_a^\dagger(t) \pm \hat{m}_a(t), \tag{2.2.110}$$

$$\hat{G}_\pm(t) = \sqrt{N}(\hat{g}_a^\dagger(t) \pm \hat{g}_a(t)). \tag{2.2.111}$$

The solution of Eqs.(2.2.107) and (2.2.108) are expressible as

$$\hat{A}_+(t) = \hat{A}_+(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_+(t')dt' + 2\varepsilon e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}dt'$$
$$+ e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_+(t')dt', \tag{2.2.112}$$

$$\hat{A}_{-}(t) = \hat{A}_{-}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_{-}(t')dt' + e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_{-}(t')dt'. \tag{2.2.113}$$

From Eqs.(2.2.112) and (2.2.113), one finds

$$\hat{a}(t) = \hat{a}(0)B(t) + \hat{C}_{+}(t) - \hat{C}_{-}(t) + \hat{F}_{+}(t) + \hat{F}_{-}(t) + \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t}), \tag{2.2.114}$$

or

$$\hat{a}(t) = \hat{C}_{+}(t) - \hat{C}_{-}(t) + \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t}) + \hat{D}(t), \tag{2.2.115}$$

where ,

$$B(t) = e^{-\frac{\kappa}{2}t}, \tag{2.2.116}$$

$$\hat{C}_{\pm} = \frac{g}{2\sqrt{N}}\int_0^t B(t-t')\hat{m}_{\pm}(t')dt', \tag{2.2.117}$$

$$\hat{F}_{\pm}(t) = \frac{1}{2}\int_0^t B(t-t')\hat{G}_{\pm}(t')dt', \tag{2.2.118}$$

$$\hat{D}(t) = \hat{a}(0)B(t) + \hat{F}_{+}(t) - \hat{F}_{-}(t). \tag{2.2.119}$$

It can be seen from Eq.(2.2.119) that for a cavity mode assumed to be initially in a vacuum state,

$$\langle\hat{D}(t)\rangle = 0. \tag{2.2.120}$$

Chapter 3

Photon Statistics

In this Chapter we study the statistical properties of a light produced by coherently driven two-level laser. To this end, we seek to determine the mean of the photon number with the aid of the solution of the quantum Langevin equations for the cavity mode operator. Moreover, we calculate the variance of the photon number employing the Q function .

3.1 The Q function

The Q function for a single -mode light is expressible in terms of antinormally ordered characteristic function [4]

$$\phi_a(z^*, z, t) = Tr\left(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}\right), \tag{3.1.1}$$

as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2}\int d^2z\phi_a(z^*, z, t)e^{z\alpha^* - z^*\alpha}. \tag{3.1.2}$$

Now applying the Baker-Hausdorff identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}, \tag{3.1.3}$$

which holds for

$$[\hat{A},[\hat{A},\hat{B}]]=[\hat{B},[\hat{A},\hat{B}]]=0, \tag{3.1.4}$$

we can write

$$e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}=e^{-z^*\hat{a}(t)+z\hat{a}^\dagger(t)-\frac{1}{2}z^*z\tau}, \tag{3.1.5}$$

where

$$[\hat{a},\hat{a}]=\tau. \tag{3.1.6}$$

Employing Eq.(3.1.6), it can be established that [4]

$$\hat{a}|n\rangle=\sqrt{n}|n-\tau\rangle, \tag{3.1.7}$$

$$\hat{a}^\dagger|n\rangle=\sqrt{n+\tau}|n+\tau\rangle, \tag{3.1.8}$$

$$\hat{a}^{\dagger n}|0\rangle=\sqrt{n!}\tau^{\frac{n}{2}}|n\tau\rangle. \tag{3.1.9}$$

Moreover, using the Baker-Hausdorff identity and Eq.(3.1.6), one can verify that [4]

$$|\alpha\rangle=e^{-\frac{1}{2}\alpha^*\alpha\tau}\sum_n\frac{(\alpha\tau^{\frac{1}{2}})^n}{\sqrt{n!}}|n\tau\rangle, \tag{3.1.10}$$

$$\hat{I}=\frac{\tau}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|, \tag{3.1.11}$$

$$\hat{a}|\alpha\rangle=\tau\alpha|\alpha\rangle. \tag{3.1.12}$$

Applying the completeness relation described by Eq.(3.1.11), the antinormally ordered characteristic function is expressible as

$$\phi_a(z^*,z,t)=Tr\left(\hat{\rho}(0)e^{-z^*\hat{a}(t)}\hat{I}e^{z\hat{a}^\dagger(t)}\right) \tag{3.1.13}$$

$$=\frac{\tau}{\pi}\int d^2\alpha Tr\left(\hat{\rho}(0)e^{-z^*\hat{a}(t)}|\alpha\rangle\langle\alpha|e^{z\hat{a}^\dagger(t)}\right) \tag{3.1.14}$$

$$=\frac{\tau}{\pi}\int d^2\alpha e^{-z^*\alpha+z\alpha^*}\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{3.1.15}$$

Now expanding the density operator in the normal order, as

$$\hat{\rho} = \sum_{lm} C_{lm} \hat{a}^{\dagger l} \hat{a}^m, \tag{3.1.16}$$

we see that

$$\langle\alpha|\hat{\rho}|\alpha\rangle = \sum_{lm} C_{lm} (\alpha^*\tau)^l (\alpha\tau)^m \langle\alpha|\alpha\rangle, \tag{3.1.17}$$

$$= \sum_{lm} C_{lm} \alpha^{*l} \alpha^m \tau^{l+m} \tag{3.1.18}$$

$$= \pi Q(\alpha^*\tau, \alpha\tau). \tag{3.1.19}$$

Upon combining Eq.(3.1.19) and (3.1.15), we find

$$\phi_a(z^*, z, t) = \tau \int d^2\alpha e^{-z^*\alpha\tau + z\alpha^*\tau} Q(\alpha^*\tau, \alpha\tau). \tag{3.1.20}$$

Introducing $\lambda = \alpha\tau$, the antinormally ordered characteristic function can be put in the form

$$\phi_a(z^*, z, t) = \frac{1}{\tau} \int d^2\lambda e^{-z^*\lambda + z\lambda^*} Q(\lambda^*, \lambda). \tag{3.1.21}$$

We note that the Q function is the inverse Fourier transform of the antinormally ordered characteristic function. It then follows from Eq.(3.1.21) that

$$Q(\lambda^*, \lambda) = \frac{\tau}{\pi^2} \int d^2z \phi_a(z^*, z, t) e^{z^*\lambda - z\lambda^*}. \tag{3.1.22}$$

We now seek to obtain the antinormally ordered characteristic function. To this end, substitution of Eq.(3.1.5) into (3.1.1) leads to

$$\phi_a(z^*, z, t) = e^{-\frac{1}{2}z^*z\tau} Tr\left(\hat{\rho}(0) e^{-z^*\hat{a}(t) + z\hat{a}^\dagger(t)}\right), \tag{3.1.23}$$

or

$$\phi_a(z^*, z, t) = e^{-\frac{1}{2}z^*z\tau} \left\langle e^{-z^*\hat{a}(t) + z\hat{a}^\dagger(t)} \right\rangle. \tag{3.1.24}$$

With the aid of Eq.(2.2.115), we note that

$$\left\langle e^{-z^*\hat{a}(t)+z\hat{a}^\dagger(t)}\right\rangle = \left\langle e^{-z^*(\hat{C}_+(t)-\hat{C}_-(t)+\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t}))+z(\hat{C}_+^\dagger(t)-\hat{C}_-^\dagger(t)+\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t}))}e^{z^*\hat{D}(t)+z\hat{D}^\dagger(t)}\right\rangle \tag{3.1.25}$$

$$= e^{-z^*(\langle\hat{C}_+(t)\rangle-\langle\hat{C}_-(t)\rangle+\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t}))+z(\langle\hat{C}_+^\dagger(t)\rangle-\langle\hat{C}_-^\dagger(t)\rangle+\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t}))}\left\langle e^{z^*\hat{D}(t)+z\hat{D}^\dagger(t)}\right\rangle. \tag{3.1.26}$$

Now,combining Eqs.(3.1.24) and (3.1.26), we see that

$$\begin{aligned}\phi_a(z^*,z,t) = exp(-\frac{1}{2}z^*z\tau - z^*(\hat{C}_+(t)-\hat{C}_-(t)+\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t}))\\ + z(\hat{C}_+^\dagger(t)-\hat{C}_-^\dagger(t)+\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})))\left\langle e^{z^*\hat{D}(t)+z\hat{D}^\dagger(t)}\right\rangle.\end{aligned} \tag{3.1.27}$$

We now proceed to verify that $\hat{D}(t)$ is a Gaussian variable. With the aid of Eqs.(2.2.116) and (2.2.118) we can write Eq.(2.2.119), in the form

$$\hat{D}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{1}{2}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_+(t')dt' - \frac{1}{2}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_-(t')dt'. \tag{3.1.28}$$

Taking the time derivative of Eq. (3.1.28), we have

$$\begin{aligned}\frac{d\hat{D}(t)}{dt} = -\frac{\kappa}{2}\hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{1}{2}(-\frac{\kappa}{2})e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_+(t')dt' - \frac{1}{2}(-\frac{\kappa}{2})e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_-(t')dt'\\ + \frac{1}{2}e^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_+(t')dt' - \frac{1}{2}Ne^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_-(t')dt'.\end{aligned} \tag{3.1.29}$$

Employing the relation

$$\frac{d}{dx}\int_a^x f(x,x')dx' = f(x,x) - f(x,a) + \int_a^x \frac{d}{dx}f(x,x')dx', \tag{3.1.30}$$

we see that

$$e^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_+(t')dt' = e^{-\frac{\kappa}{2}t}(e^{\frac{\kappa}{2}t}\hat{G}_+(t)-\hat{G}_+(0)) = \hat{G}_+(t) - e^{-\frac{\kappa}{2}t}\hat{G}_+(0)), \tag{3.1.31}$$

$$e^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_-(t')dt' = e^{-\frac{\kappa}{2}t}(e^{\frac{\kappa}{2}t}\hat{G}_+(t)-\hat{G}_+(0)) = \hat{G}_-(t) - e^{-\frac{\kappa}{2}t}\hat{G}_-(0)), \tag{3.1.32}$$

Now using Eqs.(3.1.31) and (3.1.32)along with (2.2.111), we obtain

$$\frac{1}{2}e^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_+(t')dt' - \frac{1}{2}e^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{G}_-(t')dt' = \sqrt{N}(\hat{g}_a(t) - e^{-\frac{\kappa}{2}t}\hat{g}_a(0)) \tag{3.1.33}$$

Combining Eqs.(3.1.33) and (3.1.29), we readily find that

$$\frac{d\hat{D}}{dt} = -\frac{k}{2}\hat{D}(t) + \sqrt{N}\hat{g}_a(t) - \sqrt{N}\hat{g}_a(0)e^{-\frac{\kappa}{2}t}. \tag{3.1.34}$$

On the basis of Eqs.(2.2.120) and (3.1.34), we observe that $\hat{D}(t)$ is a Gaussian variable with a vanishing mean. Now $\hat{D}(t)$ is a Gaussian variable with zero mean, we can write

$$\left\langle e^{z^*\hat{D}(t)+z\hat{D}^\dagger(t)}\right\rangle = e^{[\frac{1}{2}\left\langle(z^*\hat{D}(t)+z\hat{D}^\dagger(t))^2\right\rangle]} \tag{3.1.35}$$

Then,

$$\begin{aligned}\left\langle(z^*\hat{D}(t) + z\hat{D}^\dagger(t))(z^*\hat{D}(t) + z\hat{D}^\dagger(t))\right\rangle &= z^{*2}\langle\hat{D}^2(t)\rangle - z^*z\langle\hat{D}(t)\hat{D}^\dagger(t)\rangle \\ &- z^*z\langle\hat{D}^\dagger(t)\hat{D}(t)\rangle + z^2\langle\hat{D}^{\dagger 2}(t)\rangle. \end{aligned} \tag{3.1.36}$$

Employing Eq.(2.2.119)along with Eq.(2.2.116), we have

$$\begin{aligned}\langle\hat{D}^2(t)\rangle &= \langle\hat{a}^2(0)\rangle e^{-\frac{\kappa}{2}t} + \langle\hat{a}(0)\hat{F}_+(t)\rangle e^{-\frac{\kappa}{2}t} - \langle\hat{a}(0)\hat{F}_-(t)\rangle e^{-\frac{\kappa}{2}t} \\ &+ \langle\hat{F}_+(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} + \langle\hat{F}_+(t)\hat{F}_+(t)\rangle - \langle\hat{F}_+(t)\hat{F}_-(t)\rangle \\ &- \langle\hat{F}_-(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} - \langle\hat{F}_-(t)\hat{F}_+(t)\rangle + \langle\hat{F}_-(t)\hat{F}_-(t)\rangle \end{aligned} \tag{3.1.37}$$

On account of Eqs.(2.2.116) and (2.2.118), we have

$$\langle\hat{F}_+(t)\hat{F}_+(t)\rangle = \frac{1}{4}\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_+(t')\hat{G}_+(t'')\rangle, \tag{3.1.38}$$

$$\langle\hat{F}_+(t)\hat{F}_-(t)\rangle = \frac{1}{4}\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_+(t')\hat{G}_-(t'')\rangle, \tag{3.1.39}$$

$$\langle\hat{F}_-(t)\hat{F}_+(t)\rangle = \frac{1}{4}\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_-(t')\hat{G}_+(t'')\rangle \tag{3.1.40}$$

$$\langle \hat{F}_-(t)\hat{F}_-(t)\rangle = \frac{1}{4}\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{G}_-(t')\hat{G}_-(t'')\rangle. \tag{3.1.41}$$

In view of Eq.(2.2.111), we note that

$$\langle \hat{G}_+(t')\hat{G}_+(t'')\rangle = N(\langle(\hat{g}_a^\dagger(t') + \hat{g}_a(t'))(\hat{g}_a^\dagger(t'') + \hat{g}_a(t''))\rangle) \tag{3.1.42}$$

$$= N(\langle \hat{g}_a^\dagger(t')\hat{g}_a^\dagger(t'')\rangle + \langle \hat{g}_a^\dagger(t')\hat{g}_a(t'')\rangle + \langle \hat{g}_a(t')\hat{g}_a^\dagger(t'')\rangle + \langle \hat{g}_a(t')\hat{g}_a(t'')\rangle), \tag{3.1.43}$$

so that using Eqs.(2.2.17), (2.2.26) and (2.2.27), we obtain

$$\langle \hat{G}_+(t')\hat{G}_+(t'')\rangle = \kappa N\delta(t' - t''). \tag{3.1.44}$$

Following the same procedure, one can readily establish that

$$\langle \hat{G}_+(t')\hat{G}_-(t'')\rangle = \kappa N\delta(t' - t''), \tag{3.1.45}$$

$$\langle \hat{G}_-(t')\hat{G}_-(t'')\rangle = -\kappa N\delta(t' - t''), \tag{3.1.46}$$

$$\langle \hat{G}_-(t')\hat{G}_-(t'')\rangle = -\kappa N\delta(t' - t''). \tag{3.1.47}$$

In view of Eqs.(3.1.44)- (3.1.47), we can express (3.1.38)-(3.1.41) in the form

$$\langle \hat{F}_+(t)\hat{F}_+(t)\rangle = \frac{1}{4}N(1 - e^{-\kappa t}), \tag{3.1.48}$$

$$\langle \hat{F}_+(t)\hat{F}_-(t)\rangle = -\frac{1}{4}N(1 - e^{-\kappa t}), \tag{3.1.49}$$

$$\langle \hat{F}_-(t)\hat{F}_+(t)\rangle = \frac{1}{4}N(1 - e^{-\kappa t}), \tag{3.1.50}$$

$$\langle \hat{F}_-(t)\hat{F}_-(t)\rangle = -\frac{1}{4}N(1 - e^{-\kappa t}). \tag{3.1.51}$$

Employing Eqs.(3.1.48) - (3.1.51) along with $\langle \hat{a}(0)\rangle = 0$, $\langle \hat{a}(0)\hat{F}_+(t)\rangle = 0$, $\langle \hat{F}_+(t)\hat{a}(0)\rangle$, $\langle \hat{F}_-(t)\hat{a}(0)\rangle = 0$, Eq.(3.1.37) becomes

$$\langle \hat{D}^2(t)\rangle = \frac{1}{4}N(1-e^{-\kappa t}) - \frac{1}{4}N(1-e^{-\kappa t}) + \frac{1}{4}N(1-e^{-\kappa t}) - \frac{1}{4}N(1-e^{-\kappa t}) = 0 \tag{3.1.52}$$

On introducing Eq.(2.2.116) in Eq.(2.2.119), we find

$$\begin{aligned}\langle\hat{D}(t)\hat{D}^\dagger(t)\rangle &= \langle\hat{a}(0)\hat{a}^\dagger(0)\rangle e^{-\kappa t} + \langle\hat{a}(0)\hat{F}_+(t)\rangle e^{-\frac{\kappa}{2}t} - \langle\hat{a}(0)\hat{F}_-^\dagger(t)\rangle e^{-\frac{\kappa}{2}t} \\ &+ \langle\hat{F}_+(t)\hat{a}^\dagger(0)\rangle e^{-\frac{\kappa}{2}t} + \langle\hat{F}_+(t)\hat{F}_+^\dagger(t)\rangle - \langle\hat{F}_+(t)\hat{F}_-^\dagger(t)\rangle \\ &- \langle\hat{F}_-(t)\hat{a}^\dagger(0)\rangle e^{-\frac{\kappa}{2}t} - \langle\hat{F}_-(t)\hat{F}_+^\dagger(t)\rangle + \langle\hat{F}_-(t)\hat{F}_-^\dagger(t)\rangle. \end{aligned} \tag{3.1.53}$$

For a cavity mode assumed initially to be in a vacuum state, we note that

$$\langle\hat{a}(0)\hat{a}^\dagger(0)\rangle e^{-\kappa t} = Tr(\hat{a}\hat{a}^\dagger|0\rangle\langle 0|)e^{-\kappa t} = \langle 0|\hat{a}\hat{a}^\dagger|0\rangle e^{-\kappa t} = e^{-\kappa t}. \tag{3.1.54}$$

Noting that $\langle\hat{a}(0)\hat{F}_+(t)\rangle = \langle\hat{a}(0)\hat{F}_-^\dagger(t)\rangle = \langle\hat{F}_+(t)\hat{a}^\dagger(0)\rangle = \langle\hat{F}_+(t)\hat{a}^\dagger(0)\rangle = 0$ and taking Eq.(3.1.54) into account, Eq.(3.1.53) reduces to

$$\begin{aligned}\langle\hat{D}(t)\hat{D}^\dagger(t)\rangle &= e^{-\kappa t} + \langle\hat{F}_+(t)\hat{F}_+^\dagger(t)\rangle - \langle\hat{F}_+(t)\hat{F}_-^\dagger(t)\rangle \\ &- \langle\hat{F}_-(t)\hat{F}_+^\dagger(t)\rangle + \langle\hat{F}_-(t)\hat{F}_-^\dagger(t)\rangle. \end{aligned} \tag{3.1.55}$$

With the aid of Eq.(2.2.117), we see that

$$\langle\hat{F}_+(t)\hat{F}_+^\dagger(t)\rangle = \frac{1}{4}N\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_+(t')\hat{G}_+^\dagger(t'')\rangle dt'dt'', \tag{3.1.56}$$

$$\langle\hat{F}_+(t)\hat{F}_-^\dagger(t)\rangle = \frac{1}{4}N\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_+(t')\hat{G}_-^\dagger(t'')\rangle dt'dt'', \tag{3.1.57}$$

$$\langle\hat{F}_-(t)\hat{F}_+^\dagger(t)\rangle = \frac{1}{4}N\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_-(t')\hat{G}_+^\dagger(t'')\rangle dt'dt'', \tag{3.1.58}$$

$$\langle\hat{F}_-(t)\hat{F}_-^\dagger(t)\rangle = \frac{1}{4}N\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_-(t')\hat{G}_-^\dagger(t'')\rangle dt'dt''. \tag{3.1.59}$$

With aid of Eqs. (2.2.17), (2.2.26), (2.2.27) and (2.2.111), one readily establish that

$$\langle\hat{G}_+(t')\hat{G}_+^\dagger(t'')\rangle = \kappa N\delta(t'-t''), \tag{3.1.60}$$

$$\langle\hat{G}_+(t')\hat{G}_-^{\dagger(t'')}\rangle = -\kappa N\delta(t'-t''), \tag{3.1.61}$$

$$\langle\hat{G}_-(t')\hat{G}_-^{\dagger(t'')}\rangle = -\kappa N\delta(t'-t''), \tag{3.1.62}$$

$$\langle\hat{G}_-(t')\hat{G}_-^{\dagger(t'')}\rangle = \kappa N\delta(t'-t''). \tag{3.1.63}$$

Using Eqs.(3.1.60)- (3.1.63), we can reduce Eqs.(3.1.56)-(3.1.59) to

$$\langle\hat{F}_+(t)\hat{F}_+^\dagger(t)\rangle = \frac{1}{4}N(1-e^{-\kappa t}), \tag{3.1.64}$$

$$\langle\hat{F}_+(t)\hat{F}_-^\dagger(t)\rangle = -\frac{1}{4}N(1-e^{-\kappa t}), \tag{3.1.65}$$

$$\langle\hat{F}_-(t)\hat{F}_+^\dagger(t)\rangle = -\frac{1}{4}N(1-e^{-\kappa t}), \tag{3.1.66}$$

$$\langle\hat{F}_-(t)\hat{F}_-^\dagger(t)\rangle = \frac{1}{4}N(1-e^{-\kappa t}). \tag{3.1.67}$$

Upon combining Eqs.(3.1.64)- (3.1.67)and Eq.(3.1.55) and also using steady-state solution, we find

$$\langle\hat{D}(t)\hat{D}^\dagger(t)\rangle = e^{-\kappa t} + \frac{1}{4}N(1-e^{-\kappa t}) + \frac{1}{4}N(1-e^{-\kappa t}) + \frac{1}{4}N(1-e^{-\kappa t}) + \frac{1}{4}N(1-e^{-\kappa t}) = N. \tag{3.1.68}$$

Following a similar procedure, one can also readily obtain

$$\langle\hat{D}^\dagger(t)\hat{D}(t)\rangle = 0 \tag{3.1.69}$$

$$\langle\hat{D}^{\dagger 2}(t)\rangle = 0. \tag{3.1.70}$$

Finally, substituting Eqs.(3.1.52), (3.1.68), (3.1.69) and (3.1.70) into Eq.(3.1.35), we see that

$$\left\langle e^{z^*\hat{D}(t)+z\hat{D}^\dagger(t)}\right\rangle = e^{\frac{1}{2}\left\langle(z^*\hat{D}(t)+z\hat{D}^\dagger(t))^2\right\rangle} = e^{-\frac{1}{2}z^*zN} \tag{3.1.71}$$

Substituting Eq.(3.1.71) into (3.1.27), we obtain

$$\phi_a(z^*,z,t) = e^{-\frac{1}{2}z^*z(N+\tau)}e^{-z^*(\langle\hat{C}_+\rangle-\langle\hat{C}_-\rangle)+z(\langle\hat{C}_+^\dagger\rangle-\langle\hat{C}_-^\dagger\rangle)+(z-z^*)\frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})}. \tag{3.1.72}$$

We note that Eq.(2.2.116) can be written as

$$\hat{C}_{+}(t) = \frac{1}{2}(\hat{a}'^{\dagger}(t) + \hat{a}'(t)), \tag{3.1.73}$$

$$\hat{C}_{-}(t) = \frac{1}{2}(\hat{a}'^{\dagger}(t) - \hat{a}'(t)), \tag{3.1.74}$$

where

$$\hat{a}'(t) = \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} (\hat{m}_a(t')dt'. \tag{3.1.75}$$

Employing the relation described by Eq.(3.1.30), we can write

$$\frac{d}{dt}\hat{a}' = -\frac{\kappa}{2}\frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \hat{m}_a(t')dt' + \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} (e^{\frac{\kappa}{2}t}\hat{m}_a(t) - \hat{m}_a(0))$$

$$+ \int_0^t \frac{d}{dt} e^{\frac{\kappa}{2}t'} \hat{m}_a(t')dt' \tag{3.1.76}$$

$$= -\frac{\kappa}{2}\hat{a}'(t) + \frac{g}{\sqrt{N}}\hat{m}_a(t) - \hat{m}_a(0)e^{-\frac{\kappa}{2}t}. \tag{3.1.77}$$

Applying the large -time approximation scheme to Eq.(3.1.77), we obtain

$$\hat{a}'(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a(t) \tag{3.1.78}$$

In view of Eq.(3.1.78), we can express Eqs.(3.1.73) and (3.1.74) in the form

$$\hat{C}_{+}(t) - \hat{C}_{-}(t) = \hat{a}'(t) \tag{3.1.79}$$

$$\hat{C}_{+}^{\dagger}(t) - \hat{C}_{-}^{\dagger}(t) = \hat{C}_{+}(t) + \hat{C}_{-}(t) = \hat{a}'^{\dagger}(t). \tag{3.1.80}$$

Substituting Eqs.(3.1.79) and (3.1.80) into (3.1.72), we find

$$\phi_a(z^*, z, t) = e^{-\frac{1}{2}z^* z(1+\tau) + \frac{2\varepsilon}{\kappa}(z^*-z)(1-e^{-\frac{\kappa}{2}t}) - z^*\hat{a}'(t) + z\hat{a}'^{\dagger}(t)}. \tag{3.1.81}$$

Upon combining Eqs.(3.1.81) and (3.1.22), the Q function for the cavity mode is expressible as

$$Q(\lambda^*, \lambda) = \frac{\tau}{\pi^2} \int d^2 z e^{-\frac{1}{2}(N+\tau)z^* z + z(\hat{a}'^{\dagger}(t) + \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t} - \lambda^*) - Z^*(\hat{a}'(t) + \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t} - \lambda)} \tag{3.1.82}$$

Carrying out the integration employing the relation

$$\int d^2\alpha e^{-a\alpha^*\alpha+b\alpha+c\alpha^*} = \frac{\pi}{a}e^{\frac{bc}{a}}, \tag{3.1.83}$$

we readily obtain

$$\begin{aligned} Q(\lambda^*,\lambda,t) = \frac{2\tau}{\pi(1+\tau)} exp(\frac{2}{N+\tau}[-\lambda^*\lambda + \lambda^*(\langle\hat{a}'(t)\rangle + \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})) \\ + \lambda(\langle\hat{a}'^{\dagger}(t)\rangle + \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})) - \frac{4\varepsilon^2}{\kappa^2}(1-e^{-\frac{\kappa}{2}t})^2 \\ - \langle\hat{a}'^{\dagger}(t)\hat{a}'(t)\rangle - \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})(\langle\hat{a}'(t)\rangle + \langle\hat{a}'^{\dagger}(t)\rangle)]). \end{aligned} \tag{3.1.84}$$

3.2 The mean of the photon number

Here we want to determine the mean of the photon number of a light produced by two-level laser.The solutions of Eq.(2.1.79) and its conjugate can be written as

$$\begin{aligned} \hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \varepsilon e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}dt' \\ + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_a(t')dt' + \sqrt{N}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{g}_a(t')dt'. \end{aligned} \tag{3.2.1}$$

and

$$\begin{aligned} \hat{a}^{\dagger}(t) = \hat{a}^{\dagger}(0)e^{-\frac{\kappa}{2}t} + \varepsilon e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{t}''}dt'' + \\ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_o^t e^{\frac{\kappa}{2}t''}\hat{m}_a^{\dagger}(t'')dt'' + \sqrt{N}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t''}\hat{g}_a^{\dagger}(t)dt''. \end{aligned} \tag{3.2.2}$$

Now with the aid of Eqs.(3.2.1) and (3.2.2), the mean photon number of the cavity mode is expressible as $\bar{n} = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle$

$$
\begin{aligned}
&= \langle \hat{a}^\dagger(0)\hat{a}(0)\rangle e^{-\kappa t} + \frac{2\varepsilon}{\kappa}\langle \hat{a}^\dagger(0)\rangle e^{-\frac{\kappa}{2}t} - \frac{2\varepsilon}{\kappa}\langle \hat{a}^\dagger(0)\rangle e^{-\kappa t}\\
&+ \frac{g}{\sqrt{N}}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{a}^\dagger(0)\hat{m}_a(t')\rangle dt' + \sqrt{N}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{a}^\dagger(0)\hat{g}_a(t')\rangle dt' +\\
&\frac{2\varepsilon}{\kappa}\langle \hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} + \frac{4\varepsilon^2}{\kappa^2} - \frac{4\varepsilon^2}{\kappa^2}e^{-\frac{\kappa}{2}t}\\
&+ \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{m}_a(t')\rangle dt' + \frac{2\varepsilon}{\kappa}\sqrt{N}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t')\rangle dt' - \frac{2\varepsilon}{\kappa}\langle \hat{a}(0)\rangle e^{-\kappa t} - \frac{4\varepsilon^2}{\kappa^2}e^{-\frac{\kappa}{2}t}\\
&+ \frac{4\varepsilon^2}{\kappa^2}e^{-\kappa t} - \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{m}_a(t')\rangle dt' - \frac{2\varepsilon}{\kappa}\sqrt{N}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{g}_a(t')\rangle dt'\\
&+ \frac{g}{\sqrt{N}}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{m}_a^\dagger(t'')\hat{a}(0)\rangle dt'' + \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{m}_a^\dagger(t'')\rangle dt''\\
&- \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{m}_a^\dagger(t'')\rangle dt'' + \frac{g^2}{N}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\langle \hat{m}_a^\dagger(t'')\hat{m}_a(t')\rangle dt''dt'\\
&+ \frac{g}{\sqrt{N}}\sqrt{N}e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t''+t')}\langle \hat{m}_a^\dagger(t'')\hat{g}_a(t')\rangle dt''dt' + \sqrt{N}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{g}_a^\dagger(t'')\hat{a}(0)\rangle dt''\\
&+ \frac{2\varepsilon}{\kappa}\sqrt{N}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{g}_a^\dagger(t'')\rangle dt'' - \frac{2\varepsilon}{\kappa}\sqrt{N}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{g}_a^\dagger(t'')\rangle dt''\\
&+ \frac{g}{\sqrt{N}}\sqrt{N}e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t''+t')}\langle \hat{g}_a^\dagger(t'')\hat{m}_a(t')\rangle dt''dt'\\
&+ Ne^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\langle \hat{g}_a^\dagger(t'')\hat{g}_a(t')\rangle dt''dt'. \qquad (3.2.3)
\end{aligned}
$$

For a cavity mode initially assumed to be in a vacuum state, we see that

$$\langle \hat{a}(0)\rangle = 0. \qquad (3.2.4)$$

Assuming that the cavity mode and atomic operators are not initially correlated ,we can write

$$\langle \hat{a}^\dagger(0)\hat{m}_a(t)\rangle = \langle \hat{m}_a^\dagger(t)\hat{a}(0)\rangle = 0 \qquad (3.2.5)$$

Moreover, in view of the fact that a noise operator at some time has no effect on atomic and cavity mode operators at earlier times, one can write

$$\langle \hat{a}^{\dagger}(0)\hat{g}_a(t)\rangle = \langle \hat{g}_a^{\dagger}(t)\hat{a}(0)\rangle = 0, \tag{3.2.6}$$

Taking into account the relation

$$\langle \hat{g}_a(t)\rangle = \langle \hat{g}_a^{\dagger}(t)\rangle = 0, \tag{3.2.7}$$

$$\langle \hat{g}_a^{\dagger}(t)\hat{g}_a(t')\rangle = 0, \tag{3.2.8}$$

$$\langle \hat{g}_a(t)\hat{g}_a(t')\rangle = 0 \tag{3.2.9}$$

with the aid of Eqs.(2.2.52) and (3.2.4) - (3.2.9), we can put Eq.(3.2.3) in the form

$$\begin{aligned}\bar{n} = {} & \frac{4\varepsilon^2}{\kappa^2} - \frac{4\varepsilon^2}{\kappa^2}e^{-\frac{\kappa}{2}t} + \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{m}_a(t')\rangle dt' - \frac{4\varepsilon^2}{\kappa^2}e^{-\frac{\kappa}{2}t} \\ & + \frac{4\varepsilon^2}{\kappa^2}e^{-\kappa t} - \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t'}\langle \hat{m}_a(t')\rangle dt' \\ & + \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{m}_a^{\dagger}(t'')\rangle dt'' - \frac{2\varepsilon g}{\kappa\sqrt{N}}e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}t''}\langle \hat{m}_a^{\dagger}(t'')\rangle dt'' \\ & + \frac{g^2}{N}e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\langle \hat{m}_a^{\dagger}(t'')\hat{m}_a(t')\rangle dt''dt' \end{aligned} \tag{3.2.10}$$

And also with aids of Eq.(3.1.75), we write Eq.(3.2.10) in the form

$$\begin{aligned} = {} & \frac{4\varepsilon^2}{\kappa^2} - \frac{8\varepsilon^2}{\kappa^2}e^{-\frac{\kappa}{2}t} + \frac{4\varepsilon^2}{\kappa^2}e^{-\kappa t} + \frac{2\varepsilon}{\kappa}\langle \hat{a}'(t)\rangle - \frac{2\varepsilon}{\kappa}e^{-\frac{\kappa}{2}t}\langle \hat{a}'(t)\rangle \\ & + \frac{2\varepsilon}{\kappa}\langle \hat{a}'^{\dagger}(t)\rangle - \frac{2\varepsilon}{\kappa}e^{-\frac{\kappa}{2}t}\langle \hat{a}'^{\dagger}(t)\rangle + \langle \hat{a}'^{\dagger}(t)\hat{a}'(t)\rangle. \end{aligned} \tag{3.2.11}$$

In view of Eq.(3.1.78) and its conjugate, we find

$$\langle \hat{a}'^{\dagger}(t)\hat{a}'(t)\rangle = \frac{4g^2}{\kappa^2 N}\langle \hat{m}_a^{\dagger}\hat{m}_a\rangle = \frac{4g^2}{\kappa^2}\langle \hat{N}_a\rangle. \tag{3.2.12}$$

The steady -state solution of Eq.(2.2.87) is expressible as

$$\hat{m}_a(t) = \frac{4\varepsilon g}{\kappa(2\gamma+\gamma_c)}(\hat{N}_a - \hat{N}_b) + \frac{2}{2\gamma+\gamma_c}\hat{f}_a(t) \tag{3.2.13}$$

Upon substituting Eq.(3.2.13) into Eq.(3.1.78) and taking the expectation value of the resulting expression, we obtain

$$\langle\hat{a}'(t)\rangle = \frac{2\gamma_c\varepsilon}{\kappa^2(2\gamma+\gamma_c)\sqrt{N}}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle). \tag{3.2.14}$$

Finally, employing Eqs.(3.2.12) and (3.2.14)along with its conjugate into Eq.(3.2.11), we find

$$\begin{aligned}\bar{n} = {} & \frac{4\varepsilon^2}{\kappa^2} - \frac{8\varepsilon^2}{\kappa^2}e^{-\frac{\kappa}{2}t} + \frac{4\varepsilon^2}{\kappa^2}e^{-\kappa t} + \frac{4\gamma_c\varepsilon^2}{\kappa^2\sqrt{N}(2\gamma+\gamma_c)}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle) \\ & - \frac{4\gamma_c\varepsilon^2}{\kappa^2\sqrt{N}(2\gamma+\gamma_c)}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle)e^{-\frac{\kappa}{2}t} + \frac{4\gamma_c\varepsilon^2}{\kappa^2\sqrt{N}(2\gamma+\gamma_c)}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle) \\ & - \frac{4\gamma_c\varepsilon^2}{\kappa^2\sqrt{N}(2\gamma+\gamma_c)}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle)e^{-\frac{\kappa}{2}t} + \frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle\end{aligned} \tag{3.2.15}$$

At the steady-state the mean photon number reduces to

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\gamma_c\varepsilon^2}{\kappa^2\sqrt{N}(2\gamma+\gamma_c)}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle) + \frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle \tag{3.2.16}$$

Taking account Eq.(2.2.86)into Eq.(3.2.16) we obtain

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2}(1 + \frac{2\gamma_c}{\sqrt{N}(2\gamma+\gamma_c)}(2\langle\hat{N}_a\rangle - \frac{r_{ca}N}{\gamma+r_{ca}})) + \frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle \tag{3.2.17}$$

We note from Eq.(3.2.17) that in the absence of deriving coherent light ($\varepsilon = 0$) the mean photon number reduces to $\bar{n} = \frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle$, which is identical to the result obtained in Ref.[6]. Moreover, making use of Eqs.(3.2.1) and (3.2.2)and following a similar procedure, one finds

$$\langle\hat{a}\hat{a}^\dagger\rangle = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\gamma_c\varepsilon^2}{\kappa^2\sqrt{N}(2\gamma+\gamma_c)}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle) + \frac{\gamma_c}{\kappa}\langle\hat{N}_b\rangle + N, \tag{3.2.18}$$

$$\langle\hat{a}\rangle = \frac{2\varepsilon}{\kappa} + \frac{2g}{\kappa\sqrt{N}}\langle\hat{m}_a\rangle, \tag{3.2.19}$$

$$\langle\hat{a}^\dagger\rangle = \frac{2\varepsilon}{\kappa} + \frac{2g}{\kappa\sqrt{N}}\langle\hat{m}_a^\dagger\rangle, \tag{3.2.20}$$

$$\langle \hat{a}^2 \rangle = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\varepsilon g}{\kappa^2\sqrt{N}}\langle \hat{m}_a \rangle, \tag{3.2.21}$$

$$\langle \hat{a}^{\dagger 2} \rangle = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\varepsilon g}{\kappa^2\sqrt{N}}\langle \hat{m}_a^\dagger \rangle. \tag{3.2.22}$$

Based on Eq.(3.2.19) and (3.2.20), one can verify that

$$\langle \hat{a} \rangle^2 = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\varepsilon g}{\kappa^2\sqrt{N}}\langle \hat{m}_a \rangle + \frac{\gamma_c}{\kappa N}\langle \hat{m}_a \rangle^2, \tag{3.2.23}$$

$$\langle \hat{a}^\dagger \rangle^2 = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\varepsilon g}{\kappa^2\sqrt{N}}\langle \hat{m}_a^\dagger \rangle + \frac{\gamma_c}{\kappa N}\langle \hat{m}_a^\dagger \rangle^2, \tag{3.2.24}$$

$$\langle \hat{a}^\dagger \rangle\langle \hat{a} \rangle = \frac{4\varepsilon^2}{\kappa^2} + \frac{8\varepsilon g}{\kappa^2\sqrt{N}}\langle \hat{m}_a \rangle + \frac{\gamma_c}{\kappa N}\langle \hat{m}_a \rangle^2. \tag{3.2.25}$$

3.3 Variance of the photon number

The variance of the photon number for a cavity light can be described as

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \tag{3.3.1}$$

or

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 \tag{3.3.2}$$

The expectation value of an operator $\hat{A}$ is expressible as [4]

$$\langle \hat{A} \rangle = \frac{1}{\tau}\int d^2\lambda Q(\lambda^*, \lambda) A_a(\lambda^*, \lambda), \tag{3.3.3}$$

where $A_a(\lambda^*, \lambda)$ is c-number function corresponding to $\hat{A}$ in the antinormal order. It then follows that

$$\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle = \frac{1}{\tau}\int d^2\lambda Q(\lambda^*, \lambda)\lambda^{*2}\lambda^2 - 3\int d^2\lambda Q(\lambda^*, \lambda)\lambda^*\lambda + \tau\int d^2\lambda Q(\lambda^*, \lambda) \tag{3.3.4}$$

so that on introducing Eq.(3.1.84) into (3.3.4), we find

$$\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle = \frac{2\tau}{\pi(1+\tau)}P[\frac{1}{\tau}\int d^2\lambda e^{\frac{2}{1+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)}\lambda^{*2}\lambda^2 - 3\int d^2\lambda e^{\frac{2}{1+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)}\lambda^*\lambda$$
$$+ \tau\int d^2\lambda e^{\frac{2}{1+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)}], \tag{3.3.5}$$

where,

$$P = e^{\frac{2}{N+\tau}[-\frac{4\varepsilon^2}{\kappa^2}(1-e^{-\frac{\kappa}{2}t})^2 - \hat{a}'^{\dagger}\hat{a} - \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})(\hat{a}'^{\dagger}+\hat{a})]}, \tag{3.3.6}$$

$$m = \hat{a}'^{\dagger} + \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t}), \tag{3.3.7}$$

$$n = \hat{a}' + \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t}). \tag{3.3.8}$$

We can put Eq.(3.3.5) in the form

$$\begin{aligned} \langle \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}\rangle = \frac{2\tau}{\pi(N+\tau)} P[\frac{1}{\tau}\frac{d^2}{dn^2}\frac{d^2}{dm^2}\int d^2\lambda e^{\frac{2}{N+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)} \\ - 3\frac{d}{dn}\frac{d}{dm}\int d^2\lambda e^{\frac{2}{N+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)} + \tau\int d^2\lambda e^{\frac{2}{N+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)}]. \end{aligned} \tag{3.3.9}$$

On carrying out the integration separately, employing the relation

$$\int d^2\lambda e^{-a\lambda^*\lambda + b\lambda + c\lambda^*} = \frac{\pi}{a} e^{\frac{bc}{a}}, \tag{3.3.10}$$

we readily obtain

$$\frac{1}{\tau}\frac{d^2}{dn^2}\frac{d^2}{dm^2}\int d^2\lambda e^{\frac{2}{N+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)} = \frac{(N+\tau)\pi}{2\tau} e^{(\frac{N+\tau}{2})mn}(2 + 2nm + n^2m^2), \tag{3.3.11}$$

$$3\frac{d}{dn}\frac{d}{dm}\int d^2\lambda e^{\frac{2}{N+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)} = \frac{3(N+\tau)\pi}{2} e^{(\frac{N+\tau}{2})mn}(1 + nm), \tag{3.3.12}$$

$$\tau\int d^2\lambda e^{\frac{2}{N+\tau}(-\lambda^*\lambda + m\lambda + n\lambda^*)} = \frac{(N+\tau)\tau\pi}{2} e^{(\frac{N+\tau}{2})mn}. \tag{3.3.13}$$

Substituting Eqs.(3.3.11) - (3.3.13) into Eq. (3.3.9), we arrive at

$$\langle \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}\rangle = \frac{8}{(N+\tau)^2}\left[1 + \frac{4mn}{N+\tau} + \frac{2m^2n^2}{(N+\tau)^2}\right] - \frac{6\tau}{N+\tau}\left[1 + \frac{2mn}{N+\tau}\right] + \tau^2 \tag{3.3.14}$$

Employing Eqs.(3.3.14) along with Eqs.(3.3.7) and (3.3.8), we can express the variance of photon number for $\varepsilon = 0$ and $\tau = 1$ in the form

$$(\Delta n)^2 = \bar{n} - \bar{n}^2. \tag{3.3.15}$$

Chapter 4

Quadrature Variance

In this section we wish to determine quadrature variance for the plus and minus operators. Moreover, the power spectrum of a light generated by two-level laser in any frequency interval will be determined.

4.1 The plus and minus quadrature

The plus and minus quadrature operators are defined by [4]

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a} \tag{4.1.1}$$

and

$$\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a}). \tag{4.1.2}$$

The operators $\hat{a}_{+}$ and $\hat{a}_{-}$ represent physical quantities called the plus and minus quadratures. Taking into account Eqs. (4.1.1) and (4.1.2), it can be readily established that

$$[\hat{a}_{+}, \hat{a}_{-}] = 2i[\hat{a}, \hat{a}^{\dagger}]. \tag{4.1.3}$$

On the basis of Eq.(4.1.3), the uncertainty relation for $\hat{a}_{+}$ and $\hat{a}_{-}$ as

$$\Delta a_{+}\Delta a_{-} \geq \frac{1}{2} \mid \langle[\hat{a}, \hat{a}^{\dagger}]\rangle \mid, \tag{4.1.4}$$

and for minimum uncertainty, we have

$$\Delta a_+ \Delta a_- = \frac{1}{2} \mid \langle [\hat{a}, \hat{a}^\dagger] \rangle \mid$$
$$= \mid \langle [\hat{a}, \hat{a}^\dagger] \rangle \mid$$
$$= \mid \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^\dagger \hat{a} \rangle \mid \tag{4.1.5}$$

With aid of Eqs.(3.2.16) and (3.2.18), we see that

$$\langle [\hat{a}_+, \hat{a}_-] \rangle = 2i \left(\frac{\gamma_c}{\kappa} (\langle \hat{N}_b \rangle - \langle \hat{N}_a \rangle) + N \right). \tag{4.1.6}$$

Now taking into account Eqs.(3.2.16) and (3.2.18), we can rewrite Eq.(4.1.5) in the form

$$\Delta a_+ \Delta a_- = \mid \frac{\gamma_c}{\kappa} (\langle \hat{N}_b \rangle - \langle \hat{N}_a \rangle) + N \mid \tag{4.1.7}$$

The quadrature variance for a light beam is expressible as

$$(\Delta \hat{a}_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2 \tag{4.1.8}$$

On account of Eq.(4.1.1), the variance of the plus quadrature can be written as

$$(\Delta \hat{a}_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2 \tag{4.1.9}$$

$$(\Delta \hat{a}_+)^2 = \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^\dagger + \hat{a} \rangle^2$$

$$= \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle - (\langle \hat{a} \rangle^2 + 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a}^\dagger \rangle^2). \tag{4.1.10}$$

In view of Eqs. (3.2.21) - (3.2.22) and (3.2.23)- (3.2.5) Eq.(4.1.11) reduces to

$$= 2\langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle - 4\langle \hat{a} \rangle^2. \tag{4.1.11}$$

Upon employing Eqs. (3.2.16), (3.2.18), (3.2.21) and (3.2.23) into(4.1.10) into Eq.(4.1.16), we arrive at

$$(\Delta \hat{a}_+)^2 = \frac{\gamma_c}{\kappa} (\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle) + N - \frac{4\gamma_c}{\kappa N} \langle \hat{m}_a \rangle^2, \tag{4.1.12}$$

Following the same procedure,one can also readily find that

$$(\Delta\hat{a}_-)^2 = \frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle) + N \tag{4.1.13}$$

Hence, we observe from Eqs.(4.1.17) and (4.1.18) that the plus and minus quadrature variances of the cavity light produced by the two-level laser are not equal due to the presence of deriving coherent light.

4.2 The power spectrum

In this section we seek to calculate the power spectrum of the light produced by the two-level laser. The power spectrum of a single-mode light with central frequency ω_0 is expressible as [6]

$$P(\omega) = \frac{1}{\pi}Re\int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}, \tag{4.2.1}$$

where the subscript "ss" stands for the steady-state.
Upon integrating both sides of Eq.(4.2.1) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega)d\omega = \bar{n}, \tag{4.2.2}$$

in which $\bar{n}$ is the steady-state mean photon number. On the basis of this result, we assert that $P(\omega)d\omega$ is the steady-state mean photon number in the frequency interval between ω and $\omega + d\omega$ [6]. We now proceed to calculate the two-time correlation function that appears in Eq.(4.2.1) for the cavity light.To this end, we realize that the solution of Eq.(2.1.78) can be written as

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \varepsilon e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}d\tau' \\ &+ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}\hat{m}_a(t+\tau')d\tau' + e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}\hat{g}_a(t+\tau)d\tau'\end{aligned} \tag{4.2.3}$$

On the other hand, the solution of Eq.(2.2.87) is expressible as

$$\hat{m}_a(t+\tau') = \hat{m}(t)e^{-\frac{(2\gamma+\gamma_c)}{2}\tau'} + \frac{2\varepsilon g}{\kappa}e^{-\frac{(2\gamma+\gamma_c)}{2}\tau'}\int_0^{\tau'} e^{\frac{(2\gamma+\gamma_c)}{2}\tau''}(\hat{N}_a(t+\tau'') - \hat{N}_b(t+\tau''))d\tau''$$
$$+ e^{-\frac{(2\gamma+\gamma_c)}{2}\tau'}\int_0^{\tau'} e^{\frac{(2\gamma+\gamma_c)}{2}\tau''}\hat{f}_a(t+\tau'')d\tau'' \tag{4.2.4}$$

Now using Eq.(4.2.4), we can write the third term in Eq.(4.2.3) as

$$\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}\hat{m}_a(t+\tau')d\tau' = \frac{g}{\sqrt{N}}\hat{m}_a(t)(\frac{2}{\kappa-2\mu})e^{-\frac{\kappa}{2}\tau}(e^{(\frac{\kappa}{2}-\mu)\tau}-1)$$
$$+ \frac{\gamma_c\varepsilon}{2\sqrt{N}}e^{-\frac{\kappa}{2}\tau'}\int_0^{\tau} e^{(\frac{\kappa}{2}-\mu)\tau'}[\int_0^{\tau'} e^{-\mu\tau''}d\tau''(\hat{N}_a(t+\tau'') - \hat{N}_b(t+\tau''))]d\tau'$$
$$+ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{-(\frac{\kappa}{2}-\mu)\tau'}[\int_0^{\tau} e^{-\mu\tau''}\hat{f}_a(t+\tau'')d\tau'']d\tau' \tag{4.2.5}$$

in which

$$\mu = \gamma - \frac{\gamma_c}{2}. \tag{4.2.6}$$

Now multiplying Eq.(4.2.3) on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting equation, we have

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger(t)\rangle - \frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger(t)\rangle e^{-\frac{\kappa}{2}\tau'}$$
$$+ \frac{g}{\kappa\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}(\langle\hat{a}^\dagger(t)\hat{m}_a(t+\tau')\rangle)d\tau' \tag{4.2.7}$$

Again multiplying Eq.(4.2.5) on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting expression, we can write the last term in Eq.(4.2.7)as

$$\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}\langle\hat{a}^\dagger(t)\hat{m}_a(t+\tau')\rangle d\tau'$$
$$= \frac{g}{\sqrt{N}}\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle(\frac{2}{\kappa-2\mu})(e^{-\mu\tau} - e^{-\frac{\kappa}{2}\tau}) + \frac{\gamma_c\varepsilon}{2\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{(\frac{\kappa}{2}-\mu)\tau'}$$
$$\times[\int_0^{\tau'} d\tau'' e^{-\mu\tau''}(\langle\hat{a}^\dagger(t)\hat{N}_a(t+\tau'')\rangle - \langle\hat{a}^\dagger(t)\hat{N}_b(t+\tau'')\rangle)]d\tau'$$
$$+ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{-(\frac{\kappa}{2}-\mu)\tau'}[\int_0^{\tau} e^{-\mu\tau''}\langle\hat{a}^\dagger(t)\hat{f}_a(t+\tau'')\rangle d\tau'']d\tau', \tag{4.2.8}$$

Employing Eq.(2.2.78) along with Eq. (2.2.82), we obtain

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -(\gamma+\gamma_c)\langle \hat{N}_a\rangle - \frac{2\varepsilon g}{\kappa}(\hat{m}_a^\dagger + \hat{m}_a) + \frac{r_{ca}\gamma N}{\gamma + r_{ca}}. \tag{4.2.9}$$

The steady-state solution of Eq.(4.2.9) is

$$\langle \hat{N}_a\rangle = -\frac{2g\varepsilon}{\kappa(\gamma+\gamma_c)}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle) + \frac{r_{ca}\gamma N}{(\gamma + r_{ca})(\gamma+\gamma_c)}, \tag{4.2.10}$$

so that using Eq.(4.2.10) in (2.2.83), we obtain

$$\langle \hat{N}_b\rangle = \frac{2\varepsilon g}{\kappa(\gamma+\gamma_c)}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle) + \frac{r_{ca}\gamma_c N}{(\gamma + r_{ca})(\gamma+\gamma_c)}. \tag{4.2.11}$$

Employing Eqs. (4.2.10) and (4.2.11), we find

$$\langle \hat{N}_a\rangle - \langle \hat{N}_b\rangle = -\frac{4\varepsilon g}{\kappa(\gamma+\gamma_c)}(\langle \hat{m}_a^\dagger\rangle + \langle \hat{m}_a\rangle) + \frac{r_{ca}(\gamma-\gamma_c) N}{(\gamma + r_{ca})(\gamma+\gamma_c)}. \tag{4.2.12}$$

In view of Eq.(3.2.13), we note that

$$\langle {\hat{m}_a}^\dagger\rangle = \langle \hat{m}_a\rangle, \tag{4.2.13}$$

and hence Eq.(4.2.12) becomes

$$\langle \hat{N}_a\rangle - \langle \hat{N}_b\rangle = -\frac{8\varepsilon g}{\kappa(\gamma+\gamma_c)}\langle \hat{m}_a\rangle + \frac{r_{ca}(\gamma-\gamma_c) N}{(\gamma + r_{ca})(\gamma+\gamma_c)}. \tag{4.2.14}$$

Substituting Eq.(3.2.13) into (4.2.14), we find

$$\langle \hat{N}_a\rangle - \langle \hat{N}_b\rangle = \frac{\kappa(2\gamma+\gamma_c)(\gamma-\gamma_c)r_{ca}N}{(\kappa(2\gamma+\gamma_c)(\gamma-\gamma_c) + 8\varepsilon^2\gamma_c)(\gamma+r_{ca}))}. \tag{4.2.15}$$

Applying the quantum regression theorem to Eq.(4.2.15), we can write

$$\langle \hat{a}^\dagger(t)\hat{N}_a(t+\tau'')\rangle - \langle \hat{a}^\dagger(t)\hat{N}_b(t+\tau'')\rangle = \frac{\kappa(2\gamma+\gamma_c)(\gamma-\gamma_c)r_{ca}N\langle \hat{a}^\dagger(t)\rangle}{(\kappa(2\gamma+\gamma_c)(\gamma-\gamma_c) + 8\varepsilon^2\gamma_c)(\gamma+r_{ca}))}.$$

Upon substituting Eq.(4.2.16) into (4.2.8) and carrying out the integration, we arrive at

$$\begin{aligned}
&\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}\langle\hat{a}^{\dagger}(t)\hat{m}_a(t+\tau')\rangle d\tau' \\
&= \frac{2g}{\sqrt{N}(\kappa-2\mu)}\langle\hat{a}^{\dagger}(t)\hat{m}_a(t)\rangle(e^{-\mu\tau}-e^{-\frac{\kappa}{2}\tau}) \\
&+ \frac{\varepsilon\gamma_c B}{\mu}[(\frac{2}{\kappa-2\mu})(e^{-\mu\tau}-e^{\frac{\kappa}{2}\tau})-(\frac{2}{\kappa-4\mu})(e^{-2\mu\tau}-e^{-\frac{\kappa}{2}\tau})],
\end{aligned} \tag{4.2.16}$$

in which

$$B = \frac{\kappa(2\gamma+\gamma_c)(\gamma-\gamma_c)r_{ca}N\langle\hat{a}^{\dagger}(t)\rangle}{(\kappa(2\gamma+\gamma_c)(\gamma-\gamma_c)+8\varepsilon^2\gamma_c)(\gamma+r_{ca}))}. \tag{4.2.17}$$

In view of Eq.(2.1.79), we can put Eq.(4.2.17) in the form

$$\begin{aligned}
&\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{\frac{\kappa}{2}\tau'}\langle\hat{a}^{\dagger}(t)\hat{m}_a(t+\tau')\rangle d\tau' \\
&= (\frac{4\varepsilon g}{\kappa(\kappa-2\mu)\sqrt{N}}\langle\hat{m}_a(t)\rangle+\frac{\kappa\bar{n}}{\kappa-2\mu})(e^{-\mu\tau}-e^{-\frac{\kappa}{2}\tau}) \\
&+ \frac{B\gamma_c\varepsilon}{\mu}\left[(\frac{2}{\kappa-2\mu})(e^{-\mu\tau}-e^{-\frac{\kappa}{2}\tau})-(\frac{2}{\kappa-4\mu})(e^{-2\mu\tau}-e^{-\frac{\kappa}{2}\tau})\right],
\end{aligned} \tag{4.2.18}$$

Substituting Eq.(4.2.19) into (4.2.7), we find that

$$\begin{aligned}
\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle &= \bar{n}(e^{-\frac{\kappa}{2}\tau}+\frac{\kappa}{\kappa-2\mu}(e^{\mu\tau}-e^{-\frac{\kappa}{2}\tau}))+\frac{2\varepsilon}{\kappa}\langle\hat{a}^{\dagger}(t)\rangle(1-e^{-\frac{\kappa}{2}\tau}) \\
&+ \frac{B\gamma_c\varepsilon}{\mu}\left[(\frac{2}{\kappa-2\mu})(e^{-\mu\tau}-e^{-\frac{\kappa}{2}\tau})-(\frac{2}{\kappa-4\mu})(e^{-2\mu\tau}-e^{-\frac{\kappa}{2}\tau})\right] \\
&+ \frac{4\varepsilon g}{\kappa\sqrt{N}(\kappa-2\mu)}\langle\hat{m}_a(t)\rangle(e^{-\mu\tau}-e^{-\frac{\kappa}{2}\tau}).
\end{aligned} \tag{4.2.19}$$

Finally, on combining Eq.(4.2.20) with Eq.(4.2.1) and carrying out the integration,we readily obtain

$$\begin{aligned} P(\omega) &= \bar{n}[(\frac{\kappa}{\kappa-2\mu})\frac{\mu/\pi}{\mu^2+(\omega-\omega_0)^2} - (\frac{2\mu}{\kappa-2\mu})\frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}] \\ &+ \frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger\rangle[\delta(\omega-\omega_0) - \frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}] \\ &+ \frac{4\varepsilon g}{\kappa\sqrt{N}(\kappa-2\mu)}\langle\hat{m}_a(t)\rangle[\frac{\mu/\pi}{\mu^2+(\omega-\omega_0)^2} - \frac{\kappa/2\pi}{(\kappa/2)^2+(\omega-\omega_0)^2}] \\ &+ \frac{\gamma_c\varepsilon B}{2\mu\sqrt{N}}(\frac{2}{\kappa-2\mu})[\frac{\mu/\pi}{\mu^2+(\omega-\omega_0)^2} - \frac{\kappa/2\pi}{\kappa/2)^2+(\omega-\omega_0)^2}] \\ &- \frac{\gamma_c\varepsilon B}{2\mu\sqrt{N}}(\frac{2}{\kappa-4\mu})[\frac{2\mu/\pi}{\mu^2+(\omega-\omega_0)^2} - \frac{\kappa/2\pi}{\kappa/2)^2+(\omega-\omega_0)^2}] \end{aligned} \quad (4.2.20)$$

Now for $\varepsilon = 0$, the power spectrum reduces to

$$P(\omega) = \frac{\bar{n}}{\pi}[\frac{\kappa}{\kappa-2\mu}(\frac{\mu}{\mu^2+(\omega-\omega_0)^2}) - \frac{2\mu}{\kappa-2\mu}(\frac{\kappa/2}{(\kappa/2)^2+(\omega-\omega_0)^2})]. \quad (4.2.21)$$

We realize from Eq.(4.2.2) that the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \quad (4.2.22)$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq.(4.2.22) into (4.2.23) and carrying out the integration, applying the relation

$$\int_{-\lambda}^{\lambda} \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan(\frac{x}{a}), \quad (4.2.23)$$

we arrive at

$$\bar{n}_{\pm\lambda} = \bar{n}z(\lambda), \quad (4.2.24)$$

where $z(\lambda)$ is given by

$$z(\lambda) = \frac{2\kappa}{\pi(\kappa-2\mu)}\arctan(\frac{\lambda}{\mu}) - \frac{8\mu}{\pi(\kappa-\mu)}\arctan(\frac{2\lambda}{\kappa}) \quad (4.2.25)$$

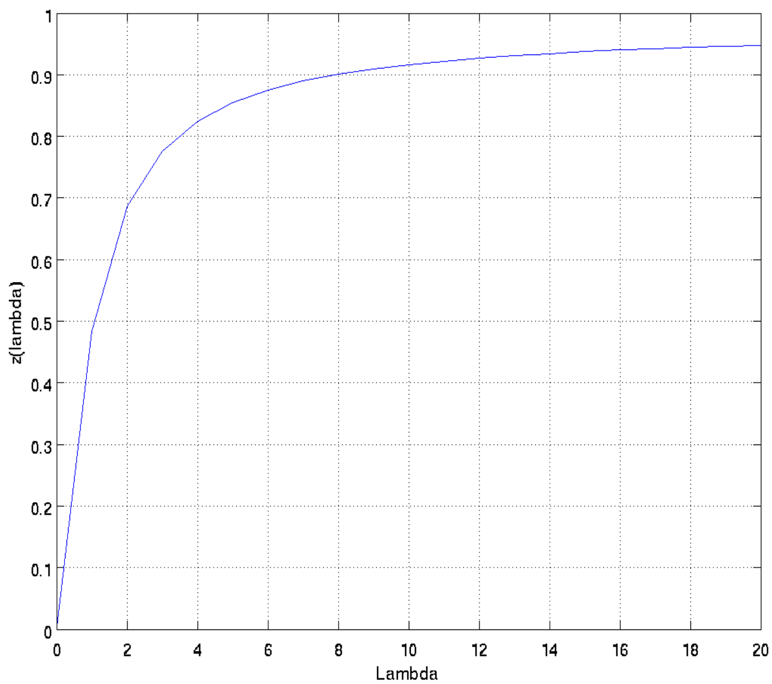

Figure 4.1: $z(\lambda)$ versus λ.

For the values of $\kappa = 14, \mu = 1.2, \pi = 3.14,$. From Figure 2, the value of $z(\lambda)$ described by Eq.(4.2.25) approaches 1 for a relatively small values of λ. This indicates that the total mean photon number is confined in a relatively small frequency interval near the central frequency.

Chapter 5

Conclusion

In this thesis we have studied the statistical properties of the light produced two-level laser driving by coherent light and coupled to vacuum reservoir via a single port-mirror. We considered a two-level laser in which two-level atoms available in the cavity are pumped from the bottom to the top level at a rate r_{ca}. Moreover we consider the case in which the two-level atom interact with the vacuum reservoir. Employing the master equation for the system under consideration, we obtained the Langevin equations for the cavity mode and atomic operators.Applying the solutions of these equations, we have calculated the mean photon number and the quadrature variance as well as the power spectrum for the cavity mode. Employing the Q function, we calculated the variance of the photon number and the photon statistics of the cavity light is found to be sub-Poissonian. We have found that the deriving coherent light increases the mean photon number.

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DECLARATION

I hereby declare that this Thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the Thesis have been duly acknowledged.

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