ON THE PENALTY METHOD
IN OPTIMIZATION;
THEORY AND NUMERICAL REALIZATION.

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Abstract

In this thesis, we deal with a theoretical basis of optimization in general and on numerical methods in particular. Here the penalty methods are in the foreground. Some generalizations for theorems into arbitrary metric space with mild assumptions (Theorem 2.1.3 and 3.1.3) are done. In addition, a numerical computation is presented and tested, with the code written in Mathematica 4.1. Comparisons are made for different choice of the penalty function with respect to convergence speed and objective value.

1. Introduction

The *penalty methods* (*penalty function methods*) have the aim to transform a constrained optimization problem, nonlinear optimization problems with inequality and equality constraints, into (in some sense) a single unconstrained problem or into a sequence of unconstrained problems or problems with simple constraints. The constraints are placed into the objective function via a penalty parameter in a way that penalizes any violation of the constraints. The penalty methods are strong theoretical tools to treat nonlinear optimization problems. These methods can be also applied for those problems which require numerical solutions. We consider two classes of penalty methods:

a) The *interior penalty methods* (*barrier methods*),

b) The *exterior penalty function methods* (*penalty method*).

In this paper both, the interior and the exterior penalty function methods will be considered. Throughout this paper we consider vector spaces X and Y. If a topology is needed we use normed spaces. The space $X = \mathbf{R}^n$ plays an important role in particular for numerical consideration. Unless specified, we use the space $X = \mathbf{R}^n$ and we denote by $S \subseteq X$ the feasible set of the given optimization problem. Consequently, for unconstrained problems we use $S = X$. Furthermore, we denote by C(U) the vector space of all continuous functions

$f: U \to \mathbf{R}$, where $U \subseteq X$, and by $C^{(1)}(U)$ the space of all Frechet-differentiable functions where U is an open set.

We will consider the constrained optimization problem (nonlinear optimization problems with equality and inequality constraints):

$$\text{(P)} \qquad f(x) \to \min, \quad x \in S,$$

$$S := \{ x \in U \subseteq X \mid g(x) \le 0, \; k(x) = 0 \}. \tag{1.1}$$

Here we use $g_i,\ k_j : U \to \mathbf{R}$, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, l\}$, where

$$g(x) := (g_1(x),\ g_2(x),\ \ldots,\ g_m(x))^T, \quad k(x) := (k_1(x),\ k_2(x),\ \ldots,\ k_l(x))^T, \quad x \in U.$$

We used the partial order in $\mathbf{R}^m$ which is defined coordinatewise, i.e. defined by

$g(x) \le 0$ if and only if $g_i(x) \le 0$ for all $i \in \{1, 2, \ldots, m\}$.

$g_i, k_j, f \in C(U)$. The set of all minimum points of f on S is denoted by $M(f, S)$. For describing the penalty method we consider $d > 0$ such that $d \to 0$. The number d is called *penalty parameter.*

Now we consider a function $p : \widetilde{S} \times (\mathbf{R}_+ \setminus \{0\}) \to \mathbf{R}$ defined by

$$p(x,\ d) := f(x) + l(d)h(x), \qquad (x,\ d) \in \widetilde{S} \times (\mathbf{R}_+ \setminus \{0\}) \tag{1.2}$$

where the set $\widetilde{S} \subseteq X$ and the functions l and h will be chosen in a convenient way. The different choice of the triple $(\widetilde{S}, l, h)$ divides the penalty methods into two classes, the *interior* and the *exterior penalty function methods*. The function h is called *penalty function* in the case of exterior method and barrier function in the case of interior method. Sometimes the term $l(d)h(x)$ is called *penalty term.*

By means of the function p we consider the minimization problem

$$p(x, d) \to \min,\ x \in \widetilde{S}, \tag{1.3}$$

and investigate the connection between a sequence (x_d^*), $x_d^* \in M(p,\ \widetilde{S})$ and solutions of the original problem (P). Indeed, we will show that each accumulation point of (x_d^*) is a solution of (P).

2. The Interior Penalty Methods

The interior penalty methods (Barrier methods) are used to transform a constrained problem into an unconstrained problem or into a sequence of unconstrained problems. The barrier functions set a barrier against leaving the feasible region. If the optimal solution occurs at the boundary of the feasible region, the procedure moves from the interior to the boundary.

2.1. Problems with inequality constraints

Consider the following optimization problem

$$(\mathrm{P}_1) \qquad f(x) \to \min, \quad x \in S,$$

$$S := \{x \in U \mid g(x) \leq 0\} \tag{2.1}$$

and the corresponding unconstrained problem

$$(\mathrm{P}^1) \qquad p(x,d) \to \min, \quad x \in \widetilde{S}.$$

Remark:

The problem (P^1) is called unconstrained because the barrier function set a barrier so that the subsequent points lie in the interior of the feasible set so that it is defined at each point.

The interior penalty method is possible to apply, if $\operatorname{int} S \neq \varnothing$. In this case we choose $\widetilde{S} := \{x \in U \mid g_i(x) < 0, \quad i \in \{1, 2, ..., m\}\} = \operatorname{int} S$, $l(d) := d$.

For the choice of the function h there are many possibilities. Some of them are of the form

$$h(x) := \sum_{i=1}^{m} \log|g_i(x)| \quad , \ x \in T, \qquad\qquad h(x) := \sum_{i=1}^{m} \frac{1}{|g_i(x)|} \quad , \ x \in T,$$

$$h(x) := \sum_{i=1}^{m} \frac{1}{(g_i(x))^2} \quad , \ x \in T, \qquad\qquad h(x) := -\sum_{i=1}^{m} \ln\left[\min\{1, \ |g_i(x)|\ \}\right] \ , x \in T,$$

where $T := \{x \in U \mid g_i(x) \neq 0, \ \ for\, all \quad i \in \{1, 2, \ldots, m\}\}$.

In all the cases the value of the barrier function h will tend to infinity as the boundary of S is approached from inside, i.e. the function p will tend to infinity as the boundary of S is approached from inside. Thus, the unconstrained (in certain sense) minimization of p can be started from any point $x_0 \in \operatorname{int} S$. Then the solution of ($P^1$) will always lie in $\operatorname{int} S$, since the constraint boundary of $\operatorname{int} S$ act as barriers during the minimization process. This is the reason why the interior penalty methods are also called barrier methods.

Remark: In the following we use $h(x) := \sum_{i=1}^{m} -\frac{1}{g_i(x)} \quad , \ x \in \operatorname{int} S$.

Now we consider a sequence of minimization problems

$$(P_k) \qquad p_k(x, d_k) = f(x) + d_k \sum_{i=1}^{m} -\frac{1}{g_i(x)} \rightarrow \min, \qquad x \in \operatorname{int} S, \quad k \in \mathbf{N} \qquad (2.2)$$

Theorem 2.1.1: (Convergence Theorem)

Let f and g_i be continuous functions. Furthermore, let $x^* \in S$ and

(i) $\mathrm{M}(f, S) \neq \emptyset$, $\operatorname{int} S \neq \emptyset$,

(ii) $\mathrm{M}(p_k, \operatorname{int} S) \neq \emptyset$ for all $k \in \mathbf{N}$,

(iii) (d_k) be a decreasing sequence and $d_k \xrightarrow{k \to \infty} 0$, $d_k > 0$,

(iv) (x_k^*) be a sequence such that $x_k^* \in \mathrm{M}(p_k, \operatorname{int} S)$.

Then

a) If $x^* \in \mathrm{M}(f, S)$, then

$$\lim_{k \to \infty} f(x_k^*) = f(x^*).$$

b) Each accumulation point of a sequence (x_k^*) is a solution of (P_1) .

Proof: (cf. [9])

a): Let $x^* \in M(f, S)$ be fixed. Then we have

$$f(x^*) \le f(x) \qquad \text{for all } x \in S.$$

In particular $f(x^*) \le f(x)$ for all $x \in \operatorname{int} S$.

Since f is continuous on U, and hence continuous on $\operatorname{int} S$, by (i) there is an $x' \in \operatorname{int} S$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|x' - x^*\| < \delta,\ \delta > 0, \text{ implies } \left| f(x') - f(x^*) \right| < \frac{\varepsilon}{2},$$

i.e.

$$x' \in \operatorname{int} S \cap S(x^*, \delta) \text{ implies } f(x') < f(x^*) + \frac{\varepsilon}{2}. \tag{2.3}$$

Let (x_k^*) be a sequence, where $x_k^* \in M(p_k, \operatorname{int} S)$. We select $k' \in \mathbf{N}$, $k' = k'(\varepsilon)$, such that

$$d_{k'} \le \frac{\dfrac{\varepsilon}{2m}}{\max\{-\dfrac{1}{g_1(x')}, \ldots, -\dfrac{1}{g_m(x')}\}}. \tag{2.4}$$

This is possible since $d_k \xrightarrow{k \to \infty} 0$.

Now, from $f(x) \le f(x) + d_k \sum_{i=1}^{m} -\dfrac{1}{g_i(x)} = p_k(x, d_k)$, for all $x \in \operatorname{int} S$, for all $k \in \mathbf{N}$, by taking minimum on both sides we get, for $x \in \operatorname{int} S$,

$$f(x^*) \le \min_{x \in \operatorname{int} S} f(x) \le \min_{x \in \operatorname{int} S} p_k(x, d_k) = p_k(x_k^*, d_k), \text{ for all } k \in \mathbf{N}. \tag{2.5}$$

Furthermore, for $x_k^* \in M(p_k, \operatorname{int} S)$ we have

$$p_k(x_k^*, d_k) \le p_k(x_{k'}^*, d_k), \tag{2.6}$$

since $x_{k'}^{\bullet} \in \operatorname{int} S$ and $p_k(x_k^{\bullet}, d_k)$ is the minimal value of p_k on $\operatorname{int} S$.

Now, we consider all $k \in \mathbf{N}$ such that $d_k < d_{k'}$ (these are almost all *k*, since $d_k \xrightarrow{k\to\infty} 0$).

Then we have

$$p_k\ (x_k^{\bullet}, d_k) = f(x_k^{\bullet}) - d_k \sum_{i=1}^{m} \frac{1}{g_i(x_k^{\bullet})} \leq p_k\ (x_{k'}^{\bullet}, d_k) = f(x_{k'}^{\bullet}) - d_k \sum_{i=1}^{m} \frac{1}{g_i(x_{k'}^{\bullet})}$$

$$< f(x_{k'}^{\bullet}) - d_{k'} \sum_{i=1}^{m} \frac{1}{g_i(x_{k'}^{\bullet})} = p_{k'}\ (x_{k'}^{\bullet}, d_{k'})\ . \tag{2.7}$$

So we get

$$f(x^{\bullet}) \leq\ p_k\ (x_k^{\bullet}, d_k) \leq\ p_k\ (x_{k'}^{\bullet}, d_k) <\ p_{k'}\ (x_{k'}^{\bullet}, d_{k'}) \tag{2.8}$$

for all k for which $d_k < d_{k'}$.

On the other hand we have (since $x_{k'}^{\bullet}$ is a minimum point of $p_{k'}$ on $\operatorname{int} S$)

$$p_{k'}\ (x_{k'}^{\bullet}, d_{k'}) \leq p_{k'}\ (x', d_{k'}) = f(x') - d_{k'} \sum_{i=1}^{m} \frac{1}{g_i(x')}\ . \tag{2.9}$$

From (2.8) and (2.9) we get

$$f(x^{\bullet}) \leq p_k\ (x_k^{\bullet}, d_k) < p_{k'}\ (x_{k'}^{\bullet},\ d_{k'}) \leq p_{k'}\ (x',\ d_{k'})$$

$$= f(x') - d_{k'} \sum_{i=1}^{m} \frac{1}{g_i(x')}\ . \tag{2.10}$$

From (2.4), using

$$\sum_{i=1}^{m} -\frac{1}{g_i(x')} \leq\ m \cdot \max\left\{-\frac{1}{g_i(x')},\ \ i \in \{1,2,\ldots,\ m\}\right\},$$

we get

$$d_{k'} \sum_{i=1}^{m} -\frac{1}{g_i(x')} \leq\ d_{k'} \cdot m \cdot \max\ \left\{-\frac{1}{g_i(x')},\ \ i \in \{1,2,\ldots,\ m\}\right\}$$

$$\leq m \cdot \frac{\varepsilon}{2m} = \frac{\varepsilon}{2}\ . \tag{2.11}$$

Using (2.3) and (2.11) we get from (2.10)

$$f(x^*) \le p_k(x_k^*, d_k) < f(x') - d_k \sum_{i=1}^{m} \frac{1}{g_i(x')} < (f(x^*) + \frac{\varepsilon}{2}) + \frac{\varepsilon}{2}.$$

This implies

$$0 \le p_k(x_k^*, d_k) - f(x^*) < \varepsilon,$$

i.e.

$$\left| p_k(x_k^*, d_k) - f(x^*) \right| < \varepsilon. \tag{2.12}$$

So we have proved that for each $\varepsilon > 0$ the relationship (2.12) is true for almost all $k \in \mathbf{N}$.

Therefore,

$$\lim_{k \to \infty} p_k(x_k^*, d_k) = f(x^*).$$

Since $x_k^* \in \operatorname{int} S$ and $x^* \in \mathrm{M}(f, S)$, we have $f(x^*) \le f(x_k^*)$. Hence we get from (2.12)

$$0 \le p_k(x_k^*, d_k) - f(x_k^*) \le p_k(x_k^*, d_k) - f(x^*) < \varepsilon, \tag{2.13}$$

i.e.

$$\lim_{k \to \infty} [p_k(x_k^*, d_k) - f(x_k^*)] = 0. \tag{2.14}$$

So we get

$$0 \le f(x_k^*) - f(x^*) = \underbrace{[f(x_k^*) - p_k(x_k^*, d_k)]}_{\to 0} - \underbrace{[f(x^*) - p_k(x_k^*, d_k)]}_{\to 0} \xrightarrow{k \to \infty} 0.$$

This implies

$$\lim_{k \to \infty} f(x_k^*) = f(x^*). \tag{2.15}$$

So we have a).

b): Let $\widetilde{x}$ be an accumulation point of the sequence (x_k^*), where $x_k^* \in \mathrm{M}(p_k, \operatorname{int} S)$, then there is a subsequence $(x_{k_j}^*)$ of the sequence (x_k^*) such that $\lim_{j \to \infty} x_{k_j}^* = \widetilde{x}$.

First we show that $\widetilde{x}$ is a feasible point. Since $x_{k_j}^* \in \operatorname{int} S$ we have

$$g_i(x_{k_j}^*) < 0 \text{ for all } i \in \{1, 2, \ldots, m\}, \text{ for all } j \in \mathbf{N}.$$

Since the functions g_i are continuous, we have

$$0 \geq \lim_{j\to\infty} g_i(x_{k_j}^*) = g_i(\lim_{j\to\infty} x_{k_j}^*) = g_i(\widetilde{x}), \text{ for all } i \in \{1, 2, \ldots, m\}$$

Hence $\widetilde{x} \in S$.

From (2.15) we get, since f is continuous,

$$f(x^*) = \lim_{j\to\infty} f(x_{k_j}^*) = f(\lim_{j\to\infty} x_{k_j}^*) = f(\widetilde{x}).$$

So $\widetilde{x} \in S$ is a minimum point of f on S, i.e. it is a solution of (P_1). //

Remark:

From Theorem 2.1.1, we have

$$\lim_{k\to\infty} p_k(x_k^*, d_k) = f(x^*) \text{ and } \lim_{k\to\infty} f(x_k^*) = f(x^*),$$

i.e. $\lim_{k\to\infty} p(x_k^*, d_k) = \lim_{k\to\infty} f(x_k^*)$.

Using this and $p_k(x_k^*, d_k) - f(x^*) = d_k \sum_{i=1}^{m} -\frac{1}{g_i(x_i)}$ by taking limit for both sides we get

$$\lim_{k\to\infty}[d_k \sum_{i=1}^{m} \frac{-1}{g_i(x_i)}] = \lim_{k\to\infty}\left[p_k(x_k^*, d_k) - f(x_k^*)\right]$$

$$= \lim_{k\to\infty} p_k(x_k^*, d_k) - \lim_{k\to\infty} f(x_k^*) = f(x^*) - f(x^*) = 0.$$

Therefore,

$$\lim_{k\to\infty}\left[-d_k \sum_{i=1}^{m} \frac{1}{g_i(x_k^*)}\right] = 0. \ //$$

Def.2.1.1: Let Y be a metric space and $f : \mathrm{Y} \to \overline{\mathbf{R}}$, where $\overline{\mathbf{R}} := \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$. Then

a) The function f is said to be lower semi-continuous (l.s.c) if and only if the level sets

$$N_\alpha(f) := \{x \in Y \mid f(x) \le \alpha\}$$

are closed for all $\alpha \in \mathbf{R}$.

b) The function f is said to be upper semi-continuous (u.s.c) if and only if $-f$ is lower semi-continuous.

Theorem 2.1.2: Let Y be a metric space and let $f : \mathrm{Y} \to \mathbf{R}$. Then the function f is lower semi-continuous on Y if and only if for each sequence (x_n) in Y, where $x_n \xrightarrow{n\to\infty} x_0$, it follows $f(x_0) \le \underline{\lim}_{n\to\infty} f(x_n)$.

$\underline{\lim}_{n\to\infty} f(x_n)$ denotes limit inferior of f as $n \to \infty$.

For the proof see [8].

Theorem 2.1.3: (Generalization of Theorem 2.1.1)

Let X be a normed space, $U \subseteq \mathrm{X}$ and f, h, $g_i : U \to \mathbf{R}$ and

a) f, h be lower semi-continuous and g_i, $i \in \{1, 2, \ldots, m\}$, be continuous,

b) $\mathrm{M}(p_k, \operatorname{int} S) \neq \emptyset$ for all $k \in \mathbf{N}$ and $\operatorname{int} S \neq \emptyset$.

If (x_k^*) is a sequence in U where $x_k^* \in \mathrm{M}(p_k, \operatorname{int} S)$,

then each accumulation point of the sequence (x_k^*) is a solution of (P_1).

Proof: Let $\widetilde{x}$ be an accumulation point of the sequence (x_k^*), where $x_k^* \in \mathrm{M}(p_k, \operatorname{int} S)$, then there is a subsequence $(x_{k_j}^*)$ of the sequence (x_k^*) such that $\lim_{j\to\infty} x_{k_j}^* = \widetilde{x}$.

First we show that $\tilde{x}$ is a feasible point. Since $x_{k_j}^* \in \operatorname{int} S$ we have

$$g_i(x_{k_j}^*) < 0 \text{ for all } i \in \{1, 2, \ldots, m\}, \text{ for all } j \in \mathbf{N}.$$

Since the functions g_i are continuous, we have

$$0 \geq \lim_{j\to\infty} g_i(x_{k_j}^*) = g_i(\lim_{j\to\infty} x_{k_j}^*) = g_i(\tilde{x}), \text{ for all } i \in \{1, 2, \ldots, m\}.$$

Hence $\tilde{x} \in S$.

Now since f and h are lower semi-continuous and $\lim_{j\to\infty} x_{k_j}^* = \tilde{x}$ we have

$$f(\tilde{x}) \leq \varliminf_{j\to\infty} f(x_{k_j}^*) \text{ and } h(\tilde{x}) \leq \varliminf_{j\to\infty} h(x_{k_j}^*)$$

Because $x_{k_j}^*$ is a solution of (P_{k_j}) we have

$$f(x_{k_j}^*) + d_{k_j} h(x_{k_j}^*) \leq f(x) + d_{k_j} h(x) = f(x) \text{ for all } x \in S,$$

i.e.

$$f(x_{k_j}^*) + d_{k_j} h(x_{k_j}^*) \leq f(x) \text{ for all } x \in S.$$

If we consider limit infimum on both sides, we get

$$f(\tilde{x}) \leq \varliminf_{j\to\infty} f(x_{k_j}^*) + \varliminf_{j\to\infty} d_{k_j} \varliminf_{j\to\infty} h(x_{k_j}^*) \leq f(x) \text{ for all } x \in S.$$

Therefore, $\tilde{x} \in \mathrm{M}(f, S)$. //

For convex functions we can do some propositions. In doing so we need some of their properties. In the next theorems we will give some of the most important once with respect to numerical considerations.

Def.2: Let $U \subseteq \mathrm{X}$ be a convex set and $f: U \to \mathbf{R}$. Then

a) The function f is said to be *convex* on the convex set U if and only if

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda) f(y)$$

for all $x, y \in U$ and for all $\lambda \in (0,1)$.

b) The function f is said to be *strictly convex* on U if and only if

$$f(\lambda x+(1-\lambda)y) < \lambda f(x)+(1-\lambda) f(y)$$

for all $x, y \in U, x \neq y$, and for all $\lambda \in (0, 1)$.

Theorem 2.1.4: Let $U \subseteq \mathbf{R}^n$, $f: U \rightarrow \mathbf{R}$ be convex and continuous and $S \subseteq U$ be a convex and closed set. Furthermore, let $M(f, S)$ be nonempty and bounded.

If there is a sequence $(x_k) \subseteq S$ such that

$$\lim_{k\to\infty} f(x_k) = \min_{x\in S} f(x) = f(x^*) \quad \text{for } x^* \in M(f, S),$$

then (x_k) is bounded.

Proof :(cf [9])

Since $M(f, S)$ is bounded there is an $R > 0$ such that $\|x\| \leq R$ for all $x \in M(f, S)$.

Let now $x^* \in M(f, S)$ be an arbitrary point and let $\delta_k := f(x_k) - f(x^*)$.

Cleary $\delta_k \geq 0$ because $x_k \in S$ and $x^* \in M(f, S)$.

But $\lim\limits_{k\to\infty} f(x_k) = f(x^*)$, hence

$$\lim_{k\to\infty}[f(x_k) - f(x^*)] = \lim_{k\to\infty} f(x_k) - f(x^*) = f(x^*) - f(x^*) = 0.$$

Therefore, $\lim\limits_{k\to\infty} \delta_k = \lim\limits_{k\to\infty}[f(x_k) - f(x^*)] = 0.$

Now, we have to show that (x_k) is bounded.

Suppose (x_k) is unbounded, hence there exists a subsequence (x_{k_j}) of (x_k) such that $\|x_{k_j}\| \xrightarrow{j\to\infty} \infty$. Then we have for all $j \in \mathbf{N}$ and for all $\lambda \in (0, 1)$

$$f(\lambda x_{k_j} + (1-\lambda)x^*) \le \lambda f(x_{k_j}) + (1-\lambda)f(x^*) = \lambda(f(x^*) + \delta_{k_j}) + (1-\lambda)f(x^*)$$

$$= f(x^*) + \lambda\delta_{k_j}. \tag{2.16}$$

Let now r > R be an arbitrary number. Then there is $j_0 \in \mathrm{N}$ so that

$$\left\|x_{k_j}\right\| > r > R \text{ for all } j \ge j_0.$$

For these j we choose now $\lambda := r\left\|x_{k_j}\right\|^{-1} \in (0, 1)$.

Then we get from (2.16)

$$f\left(\frac{r}{\left\|x_{k_j}\right\|}x_{k_j} + \left(1-\frac{r}{\left\|x^*_{k_j}\right\|}\right)x^*\right) \le f(x^*) + \frac{r}{\left\|x_{k_j}\right\|}\delta_{k_j}. \tag{2.17}$$

Furthermore, we have for $z_{k_j} := \frac{r}{\left\|x_{k_j}\right\|}x_{k_j} + \left(1-\frac{r}{\left\|x_{k_j}\right\|}\right)x^* \in S$ the inequality

$$\left\|z_{k_j}\right\| = \left\|\frac{r}{\left\|x_{k_j}\right\|}x_{k_j} + \left(1-\frac{r}{\left\|x_{k_j}\right\|}\right)x^*\right\| \le r + \left(1-\frac{r}{\left\|x_{k_j}\right\|}\right)\left\|x^*\right\| < r + \left\|x^*\right\|$$

$$\le r + R,$$

i.e. the sequence (z_{k_j}) is bounded. Therefore, there is a convergent subsequence ($z_{k_{ji}}$) of (z_{k_j}) with $z_{k_{ji}} \xrightarrow{i\to\infty} \tilde{z} \in S$. For this subsequence (subsequence of a subsequence) we get from (2.16)

$$f(\tilde{z}) = f(\lim_{i\to\infty} z_{k_{ji}}) = \lim_{i\to\infty} f(z_{k_{ji}}) = \lim_{i\to\infty} f\left(\frac{r}{\left\|x_{k_{ji}}\right\|}x_{k_{ji}} + \left(1-\frac{r}{\left\|x_{k_{ji}}\right\|}\right)x^*\right) \le f(x^*) + \frac{r}{\left\|x_{k_{ji}}\right\|}\lim_{i\to\infty}\delta_{k_{ji}}$$

$$= f(x^*)$$

and by the minimal property of x^* we have $f(\tilde{z}) = f(x^*)$, i.e.

$$\widetilde{z} \in \mathrm{M}(f, S). \tag{2.18}$$

Now from the definition of z_{k_j} we get

$$\left\| z_{k_j} - x^\bullet \right\|^2 = \left\| \frac{r}{\left\| x_{k_j} \right\|} x_{k_j} + \left(1 - \frac{r}{\left\| x_{k_j} \right\|}\right) x^\bullet - x^\bullet \right\|^2$$

$$= \left\| \frac{r}{\left\| x_{k_j} \right\|} x_{k_j} - \frac{r}{\left\| x_{k_j} \right\|}) x^\bullet \right\|^2$$

$$= \frac{r^2}{\left\| x_{k_j} \right\|^2} \left\| x_{k_j} - x^\bullet \right\|^2$$

$$= \frac{r^2}{\left\| x_{k_j} \right\|^2} \left[2\left(\left\| x_{k_j} \right\|^2 + \left\| x^\bullet \right\|^2 \right) - \left\| x_{k_j} + x^\bullet \right\|^2 \right] \quad \text{(parallelogram equation)}$$

$$\geq \frac{r^2}{\left\| x_{k_j} \right\|^2} \left[2\left(\left\| x_{k_j} \right\|^2 + \left\| x^\bullet \right\|^2 \right) - \left(\left\| x_{k_j} \right\| + \left\| x^\bullet \right\| \right)^2 \right] \quad \text{(Triangle inequality)}$$

$$= \frac{r^2}{\left\| x_{k_j} \right\|^2} \left[\left\| x_{k_j} \right\|^2 + \left\| x^\bullet \right\|^2 - 2\left\| x_{k_j} \right\| \left\| x^\bullet \right\| \right] = \frac{r^2}{\left\| x_{k_j} \right\|^2} \left(\left\| x_{k_j} \right\| - \left\| x^\bullet \right\| \right)^2 = r^2 \left(1 - \frac{\left\| x^\bullet \right\|}{\left\| x_{k_j} \right\|} \right)^2.$$

Hence, we have for the subsequence ($z_{k_{ji}}$)

$$\left\| z_{k_{ji}} - x^\bullet \right\|^2 \geq r^2 \left(1 - \frac{\left\| x^\bullet \right\|}{\left\| x_{k_j} \right\|} \right)^2 \qquad \text{for all } i \in \mathbf{N}.$$

Now taking limit for both sides as $i \to \infty$ we get $\left\| z_{k_{ji}} - x^\bullet \right\|^2 > r^2$. Therefore,

$$\left\| \widetilde{z} - x^\bullet \right\| \geq r > \mathrm{R}.$$

Since this is true for each $x^\bullet \in \mathrm{M}(f, S)$, we get for $x^\bullet = \widetilde{z}$ the inequality

$\|\tilde{z} - \tilde{z}\| = 0 \geq r > R$,

i.e. $R < 0$. But this is a contradiction to $R > 0$.

So the sequence cannot be unbounded. //

Partially differentiable and convex functions have some more important properties. We give some of them here below.

Theorem 2.1.5: let $S \subseteq U \subseteq \mathbf{R}^n$ be a convex set, U *be* an open and convex set and $f : U \to \mathbf{R}$ be a convex and partially differentiable function. Then

a) $\langle \nabla f(y), x - y \rangle \leq f(x) - f(y)$ for all x, y $\in U$ (subgradient inequality)

b) The point $x^* \in S$ is a minimum point of f on S if and only if

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in S.$$

c) If $x^* \in S$ and $\nabla f(x^*) = 0$, then $x^* \in$ M (f ,S).

d) If $x^* \in$ M (f , intS), then $\nabla f(x^*) = 0$.

For the proof of this theorem cf. [8].

Remark: Let Y be a vector space, $U \subseteq$ Y be convex, $f : U \to (-\infty, \infty]$ be a convex function and x_0 be an algebraical interior point of U, where $f(x_0) \in \mathbf{R}$, then

$$f'_+(x_0, x - x_0) \leq f(x) - f(x_0) \text{ for all } x \in U. \text{ (sub-gradient inequality)}$$

Proposition 2.1.6: Let $U \subseteq \mathbf{R}^n$, $g_i : U \to \mathbf{R}$, $i \in \{1,..., m\}$, be continuous functions,

$$S := \{x \in U | g_i(x) \leq 0 \, , \; i \in \{1, 2, ..., m\}\},$$

and g_i be strictly convex functions.

If $x^* \in S$ and H is a supporting hyperplane for S at x^*, then $S \cap \mathrm{H} = \{ x^* \}$.

Proof: Let $\mathrm{H} := \{ x \in \mathbf{R}^n \mid \langle u, x \rangle = \alpha \}$, $u \neq 0$, be a supporting hyperplane for S at x^*. Then by definition of a supporting hyperplane we have

$$\langle u, x \rangle \geq \alpha \quad \text{for all } x \in S,$$

$$\langle u, x^* \rangle = \alpha .$$

Now, we have to show that x^* is unique.

Suppose $\tilde{x} \in S \cap \mathrm{H}$ and $\tilde{x} \neq x^*$. Then we have (by the convexity of H and S)

$$\lambda \tilde{x} + (1-\lambda)\, x^* \in S \cap \mathrm{H} \quad \text{for all } \lambda \in (0, 1). \tag{2.19}$$

Moreover, by the strict convexity of g_i we have

$$g_i(\lambda\tilde{x} + (1-\lambda)x^*) < \lambda\, g_i(\tilde{x}) + (1-\lambda)\, g_i(x^*) \leq 0$$

for all $\lambda \in (0, 1)$, for all $i \in \{1, 2, ..., m\}$.

This implies

$$\lambda\tilde{x} + (1-\lambda)x^* \in \operatorname{int} S \text{ for all } \lambda \in (0, 1).$$

Therefore, there is an $\varepsilon > 0$ such that for the ball $\mathrm{S}\,(0,\varepsilon) := \{x \in \mathbf{R}^n \mid \|x\| \leq \varepsilon \}$ it follows

$$\lambda\tilde{x} + (1-\lambda)x^* + \mathrm{S}\,(0,\varepsilon) \subseteq S \qquad \text{for all } \lambda \in (0, 1).$$

or

$$\lambda\tilde{x} + (1-\lambda)x^* + y \in S \quad \text{for all } y, \text{ for which } \|y\| \leq \varepsilon \text{ and for all } \lambda \in (0, 1).$$

We choose $y = -\varepsilon \dfrac{u}{\|u\|}$. Then we have

$$\alpha \leq \langle u,\ \lambda\tilde{x} + (1-\lambda)x^* + y \rangle = \langle u,\ \lambda\tilde{x} + (1-\lambda)x^* \rangle + \langle u, y \rangle$$

$$= \left\langle u,\ \lambda\widetilde{x} + (1-\lambda)x^{*} \right\rangle + \left\langle u,\ -\varepsilon \frac{u}{\|u\|} \right\rangle$$

$$= \left\langle u, \lambda\widetilde{x} + (1-\lambda)x^{*} \right\rangle - \varepsilon\|u\|$$

or

$$\left\langle u,\ \lambda\widetilde{x} + (1-\lambda)x^{*} \right\rangle \geq\ \alpha\ +\ \varepsilon\|u\|\ >\ \alpha\ .$$

But this implies $\lambda\ \widetilde{x} + (1-\lambda)\,x^{*} \notin \mathrm{H}$. But this is a contradiction to (2.19).

Hence, $\widetilde{x} = x^{*}$.

Therefore, $S \cap \mathrm{H} = \{\, x^{*} \}$. //

Theorem 2.1.7: Let $U \subseteq \mathbf{R}^{n}$, $f : U \to \mathbf{R}$ be a continuous function and $\mathrm{M}(f, U) \neq \varnothing$.

Then

a) If U is a closed set, $\mathrm{M}(f, U) = \{\, x^{*} \}$ and $(\, x_{k})$ is a bounded sequence in U such that

$$\lim_{k\to\infty} f(x_k) = f(x^{*}),$$

then $x_k \xrightarrow{k\to\infty} x^{*}$.

b) Let U be a convex set and f be strictly convex.

Then $\mathrm{M}(f, U) = \{\, x^{*} \}$.

c) Let U be a closed and convex set and f be strictly convex.

If $(\, x_k)$ is a sequence in U such that

$$\lim_{k\to\infty} f(x_k) = f(x^{*}),$$

then $x_k \xrightarrow{k\to\infty} x^{*}$.

Proof: (cf.[9]) a): Since (x_k) is a bounded sequence in U, there is a subsequence (x_{k_j}) of the sequence (x_k) which is convergent, i.e. $x_{k_j} \xrightarrow{j\to\infty} \widetilde{x} \in U$.

Since f is continuous and $x^* \in \mathrm{M}(f, U)$ we have

$$f(\widetilde{x}) = f(\lim_{j\to\infty} x_{k_j}) = \lim_{j\to\infty} f(x_{k_j}) = f(x^*),$$

i.e. $\widetilde{x} \in \mathrm{M}(f, U)$.

From $\mathrm{M}(f, U) = \{x^*\}$ it follows that $\widetilde{x} = x^*$. This shows that (x_k) possesses exactly one accumulation point x^*, i.e. we get $x_k \xrightarrow{k\to\infty} x^*$.

b): Let $\widetilde{x}^*, x^* \in \mathrm{M}(f, U)$. We assume that $\widetilde{x}^* \neq x^*$.

We have

$$f(\widetilde{x}^*) = f(x^*) = \min_{x\in U} f(x)$$

and, since f is strictly convex

$$f(\lambda x^* + (1-\lambda)\widetilde{x}^*) < \lambda f(x^*) + (1-\lambda) f(\widetilde{x}^*) = f(x^*) \quad \text{for all } \lambda \in (0, 1).$$

Since U is convex, we have $\lambda x^* + (1-\lambda)\widetilde{x}^* \in U$ for all $\lambda \in (0,1)$, i.e. x^* is not a minimum point of f on U, i.e. $x^* \notin \mathrm{M}(f, U)$. But this is a contradiction to $x^* \in \mathrm{M}(f, U)$. There fore, $\widetilde{x}^* \neq x^*$ is not true, hence $\mathrm{M}(f, U) = \{x^*\}$.

c): Since U is convex and f is strictly convex by part (b), $\mathrm{M}(f, U) = \{x^*\}$.

Since f is (strictly) convex and continuous on a convex and closed set U such that $\mathrm{M}(f, U) \neq \emptyset$ and bounded (because it has exactly one element), and (x_k) is a sequence in U such that $\lim_{k\to\infty} f(x_k) = f(x^*)$, we get by Theorem 2.1.4, that (x_k) is bounded.

Hence we get from (a)

$$\lim_{k\to\infty} x_k = x^*. \quad //$$

Theorem 2.1.8: Let (P_1) be given, $f \in C(U)$ be continuously partially differentiable, convex and let g_i be strictly convex for $i \in \{1, 2, \ldots, m\}$. Moreover, let $x^* \in M(f, S)$ and $\nabla f(x^*) \neq 0$. Then

a) If $f \in C^{(1)}(U)$, then $M(f, S) = \{x^*\}$,

b) If there is a sequence (x_k) in S such that $\lim_{k \to \infty} f(x_k) = f(x^*)$ and $M(f, S)$ is bounded and S is closed, then $x_k \xrightarrow{k \to \infty} x^*$.

Proof: a): Let $\tilde{x}^*, x^* \in M(f, S)$. Suppose $\tilde{x}^* \neq x^*$.

Now, $f(\tilde{x}^*) = f(x^*) = \min_{x \in S} f(x)$ and $g_i(x^*) \leq 0$, $g_i(\tilde{x}^*) \leq 0$ for all $i \in \{1, 2, \ldots, m\}$.

By the convexity of f we have

$$
\begin{aligned}
f(x^*) &\leq f(\lambda x^* + (1-\lambda)\tilde{x}^*) \\
&\leq \lambda f(x^*) + (1-\lambda) f(\tilde{x}^*) \\
&= f(x^*),
\end{aligned}
$$

i.e. $f(x^*) = f(\lambda x^* + (1-\lambda)\tilde{x}^*)$ and, since all g_i are strictly convex, we get for all $\lambda \in (0, 1)$

$$g_i(\lambda x^* + (1-\lambda)\tilde{x}^*) < \lambda g_i(x^*) + (1-\lambda) g_i(\tilde{x}^*) \leq 0.$$

This implies $\lambda x^* + (1-\lambda)\tilde{x}^* \in \text{int } S$.

Therefore, $\lambda x^* + (1-\lambda)\tilde{x}^* \in M(f, \text{int } S)$.

By Theorem 2.1.5 d), we get

$$\nabla f(\lambda x^* + (1-\lambda)\tilde{x}^*) = 0 \text{ for all } \lambda \in (0, 1).$$

Since f is continuously differentiable we get for $\lambda \to 1$, $\nabla f(x^*) = 0$. But this is a contradiction to the supposition $\nabla f(x^*) \neq 0$.

Therefore, $\tilde{x}^* = x^*$. Hence $M(f,S) = \{x^*\}$.

b): For $x^* \in M(f,S)$ we have by Theorem 2.1.4 (since f is convex and continuous) (x_k) is bounded. From Theorem 2.1.5, a) & b), we get

$$\left\langle \nabla f(x^*), x - x^* \right\rangle \geq 0 \quad \text{for all } x \in S \tag{2.20}$$

and

$$0 \leq \left\langle \nabla f(x^*), x_k - x^* \right\rangle \leq f(x_k) - f(x^*) \xrightarrow{k \to \infty} 0,$$

i.e.
$$\lim_{k \to \infty} \left\langle \nabla f(x^*), x_k - x^* \right\rangle = 0. \tag{2.21}$$

From cosine law, $\langle a, b \rangle = \|a\| \cdot \|b\| \cdot \cos\theta,\ \theta \in [0, \pi]$, we get

$$\lim_{k \to \infty} \|\nabla f(x^*)\| \|x_k - x^*\| \cos\alpha_k = 0, \qquad 0 \leq \alpha_k \leq \pi. \tag{2.22}$$

But $\alpha_k \in \left[0, \frac{\pi}{2}\right]$ as $\left\langle \nabla f(x^*), x_k - x^* \right\rangle \geq 0$

i.e.
$$\|\nabla f(x^*)\| \|x_k - x^*\| \cos\alpha_k \geq 0 .$$

This implies $\alpha_k \in \left[0, \frac{\pi}{2}\right]$, and as $\left[0, \frac{\pi}{2}\right]$ is closed, we get (α_k) is bounded. Hence there is a convergent subsequence (α_{k_i}) converging to α, i.e. $\lim_{i \to \infty} \alpha_{k_i} = \alpha$.

Since (x_k) is bounded, there is a subsequence (x_{k_i}) of (x_k) such that $\lim_{i \to \infty} x_{k_i} = \tilde{x}$. We want to show $\tilde{x} = x^*$.

Since $\nabla f(x^*) \neq 0$ we consider two cases:

Case 1: $\alpha \neq \frac{\pi}{2}$, i.e. $\alpha \in \left[0, \frac{\pi}{2}\right)$. Then by $\nabla f(x^*) \neq 0$ we get from (2.21)

$$\lim_{i \to \infty} \|x_{k_i} - x^*\| \cdot \cos\alpha_{k_i} = 0.$$

Since $\lim_{i\to\infty}\cos\alpha_{k_i} = \cos\alpha \neq 0$ for $\alpha \in \left[0, \frac{\pi}{2}\right)$, we have

$$\lim_{i\to\infty}\left\|x_{k_i} - x^*\right\| = 0.$$

This implies, by the continuity of the norm

$$\left\|\lim_{i\to\infty} x_{k_i} - x^*\right\| = \left\|\tilde{x} - x^*\right\| = 0.$$

Hence, $\lim_{i\to\infty} x_{k_i} = x^*$.

Case 2: $\alpha = \frac{\pi}{2}$. Then since (x_k) is bounded, the subsequence (x_{k_i}) is also bounded.

Therefore, there is a convergent subsequence $(x_{k_{i_j}})$ with $\lim_{j\to\infty} x_{k_{i_j}} = \tilde{x} \in S$.

Let now

$$\mathrm{H} := \left\{x \in \mathbf{R}^n \,\middle|\, \left\langle \nabla f(x^*),\ x\right\rangle = a\right\}, \text{ where } a = \left\langle \nabla f(x^*),\ x^*\right\rangle.$$

By (2.20) we get that H is a supporting hyperplane of S at x^*.

By corollary (2.6) we have $\mathrm{H} \cap S = \{x^*\}$. Now from (2.21) we get $\tilde{x} \in \mathrm{H} \cap S$,

i.e. $\tilde{x} = x^*$.

So we have for both cases that each convergent subsequence (x_{k_i}) of (x_k) converges to x^*,

i.e. the (bounded) sequence (x_k) possesses the only accumulation point x^*,

i.e. $\lim_{k\to\infty} x_k = x^*$. //

For the next theorem we need further property of convex functions.

Theorem 2.1.9: Let g be a (strictly) convex and strictly negative function on a convex set U.

Then the function $-\frac{1}{g}$ is (strictly) convex.

Proof: Since g is convex we have

$$g(\lambda x+(1-\lambda)y) \le \lambda g(x)+(1-\lambda)g(y) < 0$$

for all $\lambda \in (0, 1)$ for all $x, y \in U$,

i.e.

$$-g(\lambda x+(1-\lambda)y) \ge -\lambda g(x)-(1-\lambda)g(y) > 0,$$

i.e.

$$\frac{1}{-g(\lambda x+(1-\lambda)y)} \le \frac{1}{-\lambda g(x)-(1-\lambda)g(y)} \quad \text{for all } \lambda \in (0,1) \quad \text{for all } x, y \in U. \quad (2.23)$$

Now we have

$$\lambda\frac{-\lambda g(x)-(1-\lambda)g(y)}{-g(x)}+(1-\lambda)\frac{-\lambda g(x)-(1-\lambda)g(y)}{-g(y)}$$

$$=\frac{-\lambda^2 g(x)-\lambda(1-\lambda)g(y)}{-g(x)}+\frac{-\lambda(1-\lambda)g(x)-(1-\lambda)^2 g(y)}{-g(y)}$$

$$=\lambda^2+\lambda(1-\lambda)\frac{g(y)}{g(x)}+\lambda(1-\lambda)\frac{g(x)}{g(y)}+(1-\lambda)^2$$

$$=\lambda^2+\lambda(1-\lambda)\underbrace{\left[\frac{g(y)}{g(x)}+\frac{g(x)}{g(y)}\right]}_{\ge 2}+(1-\lambda)^2$$

$$\ge \lambda^2+2\lambda(1-\lambda)+(1-\lambda)^2$$

$$=[\lambda+(1-\lambda)]^2=1.$$

Dividing the last inequality by $-\lambda g(x)-(1-\lambda)g(y) > 0$ we get

$$\frac{\lambda}{-g(x)}+\frac{(1-\lambda)}{-g(y)} \ge \frac{1}{-\lambda g(x)-(1-\lambda)g(y)}.$$

Together with (2.23) we have

$$\frac{1}{-g(\lambda x+(1-\lambda)y)} \le \frac{1}{-\lambda g(x)-(1-\lambda)g(y)} \le \frac{\lambda}{-g(x)} + \frac{(1-\lambda)}{-g(y)}.$$

Hence

$$\left(-\frac{1}{g}\right)(\lambda x+(1-\lambda)y) \le \lambda\left(-\frac{1}{g}\right)(x)+(1-\lambda)\left(-\frac{1}{g}\right)(y),$$

for all $\lambda \in (0, 1)$, for all $x,\ y \in U$,

i.e. $-\frac{1}{g}$ is convex.

The proof for the strict convexity is simply changing "$\le$" by "$<$". //

Theorem 2.1.10: Let $U \subseteq \mathbf{R}^n$, $f: U \to \mathbf{R}^n$ be strictly convex and continuous, let g_i be convex and continuous such that

$$S := \{x \in U \mid g_i(x) \le 0,\ i \in \{1, 2, \dots, m\}\}$$

is closed. Let $\operatorname{int} S \ne \varnothing$, $\mathrm{M}(p_k, \operatorname{int} S) \ne \varnothing$ for all $k \in \mathbf{N}$ and $\mathrm{M}(f, S) \ne \varnothing$.

Then

a) $\mathrm{M}(p_k, \operatorname{int} S) = \{x_k^*\}$,

b) The sequence (x_k^*) given by a) has at most one accumulation point.

In case of existence of the accumulation point it follows

$$\lim_{k\to\infty} x_k^* = x^* \text{ and } \mathrm{M}(f, S) = \{x^*\}.$$

Proof: (cf.[9])

a): Since f is strictly convex and $-\frac{1}{g}$ is convex on $\operatorname{int} S$, we have that p_k is a strictly convex function on $\operatorname{int} S$. By Theorem 2.1.7, b), we get $\mathrm{M}(p_k, \operatorname{int} S) = \{x_k^*\}$.

b): by Theorem 2.1.1, b), we have that each accumulation point of (x_k^*) is a solution of (P_1).

Since f is strictly convex (P_1) possesses at most one solution x^*, and hence using $M(f, S) \neq \emptyset$, we have that all accumulation point of (x_k^*) are equal to x^*. //

Remarks:

1. Theorem 2.1.10 is quite valid under the supposition that f is (only) convex and only one of the functions g_i, $i \in \{1,2,..., m\}$, is strictly convex.
2. For numerical calculations the sequence of solutions depends on the choice of the initial (starting) point x_0. Therefore, considering nonconvex functions, the sequence (x_k) generated by the procedure can be convergent to a local minimum. This is because, the function is convex in certain neighborhood of such points.

Now in the following we will consider the minimization problem:

$$\theta(d) \to \min,$$

$$\text{s.t.} \quad d \geq 0, \tag{2.24}$$

where $\theta(d) := \inf_{x \in \text{int}\, S} p(x,d)$. Here, h is a barrier function such that

a) $h(x) \geq 0$ for all $x \in \text{int}\, S$,

b) h is continuous on $\text{int}\, S$,

c) $h(x) \xrightarrow{x \to \partial S} \infty$, $x \in \text{int}\, S$ where ∂S is the boundary of S.

More specifically, a barrier function h is defined by

$$h(x) := \sum_{i=1}^{m} \phi[g_i(x)], \tag{2.25}$$

where ϕ is a function of one variable that is continuous over $\{ y: y < 0 \}$ and satisfies

$\phi(y) \geq 0$ if $y < 0$ and $\lim_{y \to 0^-} \phi(y) = \infty$.

Lemma 2.1.11: Let $f,\ g_i : \mathbf{R}^n \to \mathbf{R}$, $i \in \{1, 2, \ldots, m\}$, be continuous functions.

Let

a) $U \subseteq \mathbf{R}^n$ be nonempty and closed,

b) $\operatorname{int} S \neq \varnothing$,

c) h be a barrier function of the from (2.25) and be continuous on $\operatorname{int} S$,

d) For any given $d > 0$, if $(x_k) \subseteq \operatorname{int} S$ satisfies $f(x_k) + dh(x_k) \xrightarrow{k \to \infty} \theta(d)$, then (x_k) has a convergent subsequence.

Then,

1. For each $d > 0$, there exists an $x_d \in \operatorname{int} S$ such that

$$\theta(d) = f(x_d) + dh(x_d) = \inf_{x \in \operatorname{int} S} p(x, d),$$

2. $\inf_{x \in S} f(x) \leq \inf_{d > 0} \theta(d)$,

3. If $\omega(d) := f(x_d^{\bullet})$ and $\psi(d) := h(x_d)$, then for $d > 0$, ω and θ are non decreasing functions, and ψ is non increasing function.

Proof: 1) Let $d > 0$. By definition of θ, there exists a sequence $(x_k) \subseteq \operatorname{int} S$ such that $f(x_k) + dh(x_k) \xrightarrow{k \to \infty} \theta(d)$. By d), (x_k) has a convergent subsequence (x_{k_i}) with limit $x_d \in U$ as U is closed. By continuity of g, $g(x_d) \leq 0$. Otherwise, if $g(x_d) > 0$, since g is continuous, there exists a point $x_{k^\bullet}$ such that $g(x_{k^\bullet}) > 0$, which is not the case. Now we will show that $g(x_d) < 0$. If not, then $g_i(x_d) = 0$, for some i, and since the barrier function h satisfies (2.25), $h(x_{k_i}) \xrightarrow{i \to \infty} \infty$. Thus, $\theta(d) = \infty$, which is impossible, since $\operatorname{int} S \neq \varnothing$.

Therefore,

$$\theta(d) = \lim_{i\to\infty}[f(x_{k_i}) + dh(x_{k_i})] = f(x_d) + dh(x_d).,\text{ where } x_d \in \operatorname{int} S.$$

2) Now since $h(x) \ge 0$ we have for $d \ge 0$

$$\theta(d) = \inf_{x\in \operatorname{int} S} p(x,d)$$

$$\ge \inf_{x\in \operatorname{int} S} f(x)$$

$$\ge \inf_{x\in S} f(x).$$

Since the above inequality holds for each $d \ge 0$, we have

$$\inf_{d>0} \theta(d) \ge \inf_{x\in S} f(x).$$

3) Let $d_1 > d_2 > 0$. Since $h(x) \ge 0$,

$$p(x,d_1) \ge p(x,d_2) \quad \text{for each } x \in \operatorname{int} S.$$

Now taking infimum on both sides, we have

$$\inf_{x\in \operatorname{int} S} p(x,d_1) \ge \inf_{x\in \operatorname{int} S} p(x,d_2),$$

Thus,

$$\theta(d_1) \ge \theta(d_2).$$

By 1), there exists x_{d_1} and x_{d_2} in $\operatorname{int} S$ such that

$$f(x_{d_1}) + d_1 h(x_{d_1}) \le f(x_{d_2}) + d_1 h(x_{d_2}) \tag{2.26}$$

and $$f(x_{d_2}) + d_2 h(x_{d_2}) \le f(x_{d_1}) + d_2 h(x_{d_1}). \tag{2.27}$$

Adding (2.26) and (2.27) and rearranging, we get $(d_1 - d_2)[h(x_{d_1}) - h(x_{d_2})] \le 0$. Since $d_1 - d_2 > 0$, we get

$$h(x_{d_1}) \le h(x_{d_2}). \tag{2.28}$$

Using (2.27) and (2.28), it follows that

$$f(x_{d_2}) \le f(x_{d_2}) + \underbrace{d_2[h(x_{d_2}) - h(x_{d_1})]}_{\ge 0} \le f(x_{d_1}).$$

Hence, $f(x_{d_2}) \le f(x_{d_1})$. //

Remark:

1) From the above Lemma 2.1.11, θ is a non decreasing function of d so that

$$\inf_{d>0} \theta(d) = \lim_{d \to 0^+} \theta(d).$$

2) The assumption d) in Lemma 2.1.11 holds, if $\{x \in X : g(x) \le 0\}$ is compact.

Theorem 2.1.12: Let

a) $f : \mathbf{R}^n \to \mathbf{R}$, and $g : \mathbf{R}^n \to \mathbf{R}^m$ be continuous,

b) $U \subseteq \mathbf{R}^n$ be nonempty and closed,

c) $\operatorname{int} S \ne \varnothing$,

d) For $x^* \in M(f,S)$ it follows

$$S(x^*,\alpha) \cap \operatorname{int} S \ne \varnothing \text{ for all } \alpha > 0.$$

Then,

1) $f(x^*) = \lim_{d \to 0^+} \theta(d) = \inf_{d>0} \theta(d)$,

2) let $\theta(d) = p(x_d, d)$, where $x_d \in \operatorname{int} S$, then the limit of any convergent subsequence of (x_d) is an optimal solution to (P_1),

3) $\lim_{d \to 0^+} d\, h(x_d) = 0$.

Proof: Let $x^* \in M(f, S)$ and let $\varepsilon > 0$. For $S(x^*, \alpha)$ by d) there exists an $\tilde{x} \in \operatorname{int} S \cap S(x^*, \alpha)$, i.e. $\tilde{x}$ is feasible and $|\tilde{x} - x^*| < \alpha$.

By the continuity of f for $|\tilde{x}-x^*|<\alpha$, we get

$$|f(\tilde{x})-f(x^*)|<\varepsilon.$$

Since $f(x^*)\le f(\tilde{x})$ we have

$$f(\tilde{x})<f(x^*)+\varepsilon.$$

Then, for $d>0$

$$f(x^*)+\varepsilon+dh(\tilde{x})>f(\tilde{x})+d\,h(\tilde{x})\ge\inf_{x\in\operatorname{int}S}p(x,d)=\theta(d).$$

Taking limit for both sides as $d\to 0^+$, it follows that $f(x^*)+\varepsilon\ge\lim_{d\to 0^+}\theta(d)$.

Since this inequality holds for each $\varepsilon>0$, we get

$$f(x^*)\ge\lim_{d\to 0^+}\theta(d).$$

From Lemma 2.1.11, 2), we get

$$f(x^*)\le\inf_{d>0}\theta(d)=\lim_{d\to 0^+}\theta(d).$$

Hence, $f(x^*)=\lim_{d\to 0^+}\theta(d)=\inf_{d\to 0}\theta(d)$.

For $d\to 0^+$, since $h(x_d)\ge 0$ and $x_d\in S$, it follows that

$$\theta(d)=f(x_d)+dh(x_d)\ge f(x_d)\ge f(x^*).$$

Now taking limit for both sides as $d\to 0^+$, we get

$$f(x^*)=\lim_{d\to 0^+}f(x^*)\le\lim_{d\to 0^+}f(x_d)\le\lim_{d\to 0^+}\theta(d)=f(x^*).$$

Therefore,

$$\lim_{d\to 0^+}f(x_d)=f(x^*)\text{ and }\lim_{d\to 0^+}\theta(d)=\lim_{d\to 0^+}\left[f(x_d)+dh(x_d)\right]=f(x^*).$$

Hence,

$$\lim_{d\to 0^+}dh(x_d)=\lim_{d\to 0^+}\left[\theta(d)-f(x_d)\right]=\lim_{d\to 0^+}\theta(d)-\lim_{d\to 0^+}f(x_d)=f(x^*)-f(x^*)=0,$$

i.e. $\lim_{d \to 0^+} dh(x_d) = 0$.

Furthermore, if (x_d) has a convergent subsequence (x_{d_j}) with limit x', then

$$\lim_{j \to \infty} x_{d_j} = x'.$$

Since f is continuous and

$$f(x^*) = \lim_{j \to \infty} f(x_{d_j}) = f(\lim_{j \to \infty} x_{d_j}) = f(x'),$$

and as $x_d \in S$ for each d, we have $g_i(x_{d_j}) < 0$ for all $i \in \{1, \ldots, m\}$. In addition since the functions g_i are continuous we have

$$0 \geq \lim_{j \to \infty} g_i(x_{d_j}) = g_i\left(\lim_{j \to \infty} x_{d_j}\right) = g_i(x'), \text{ i.e. } x' \in S.$$

Therefore, $\tilde{x} \in M(f,S)$. //

Remark:

1) The optimal solution to (P_1) is indeed equal to $\lim_{d \to 0^+} \theta(d)$, so that it could be solved by a single problem of the form

$$p(x,d) \to \min, \qquad x \in U,$$

by taking limit as $d \to 0^+$, or it can be solved through a sequence of problems of the above form with decreasing values of d.

2) For each d, $x_d \in \operatorname{int} S$. It is for this reason that barrier function methods are some times also referred to as interior penalty function methods. //

2.2. Problems with equality and inequality constraints

In this section we consider a general problem of the form

(P2) $f(x) \to \min, \quad x \in S,$

$$S := \{x \in U \mid g_i(x) \le 0,\ k(x) = 0\}.$$

The present problem can be converted into an unconstrained minimization problem by constructing a function of the form

$$p_k(x, d_k) = f(x) + d_k \sum_{i=1}^{m} \phi_i[g_i(x)] + \varphi(d_k) \sum_{j=1}^{p} k_j^2(x), \qquad (2.29)$$

where ϕ_i is some function of the constraint g_i tending to infinity as the constraint boundary is approached from inside, $\varphi(d_k)$ is some function of the parameter d_k tending to infinity as $d_k \to 0$. The motivation for the third term in (2.29) is that as $\varphi(d_k) \to \infty$, the term $\sum_{j=1}^{p} k_j^2(x)$ must tend to zero. If $\sum_{j=1}^{p} k_j^2(x)$ does not tend to zero, $p_k(x, d_k)$ would tend to infinity, and this cannot happen in a sequential minimization process if the problem has a solution,

Usually (2.29) is taken as

$$p_k(x, d_k) = f(x) - d_k \sum_{i=1}^{m} \frac{1}{g_i(x)} + \frac{1}{\sqrt{d_k}} \sum_{j=1}^{p} k_j^2(x) \qquad (2.30)$$

If $p_k(x, d_k)$ is minimized for a decreasing sequence of values $d_k > 0$ the following theorem proves that the unconstrained minimum point x_k^* will converge to the solution x^* of (P2).

Theorem 2.2.1: Let $M(f, S) \neq \varnothing$ and (d_k) be a decreasing sequence.

Then each accumulation point of a sequence (x_k^*), $x_k^* \in M(p_k, U)$ is a solution of (P_2).

The proof is similar to that of Theorem 2.1.1.

Remarks:

If i) f g_i, $i=1, 2, \ldots, m$ are convex,

ii) $\sum_{j=1}^{p} k_j^2(x)$ is convex in the interior feasible domain defined by the inequality constraints,

iii) one of the functions among f, g_i, $i=1, 2, \ldots, m$ and $\sum_{j=1}^{p} k_j^2(x)$ is strictly convex,

then the solution obtained at the end of sequential minimization of p_k is guaranteed to be the global minimum of the problem (P_2).

3. The Exterior Penalty Methods

The idea of exterior penalty function method is adding a penalty term to the objective function for any violation of the constraints. This method generates a sequence of infeasible points, hence its name, whose limit is feasible and an optimal solution to the original problem.

3.1. Problems with inequality constraints

In the following we consider an optimization problem

$$(P_3) \qquad f(x) \to \min, \qquad x \in S,$$
$$S := \{x \in U \subseteq \mathbf{R}^n \mid g(x) \leq 0\}.$$

Here we choose $\widetilde{S} := U$ and $l(d) := \dfrac{1}{d} > 0$ and we use the penalty function $h : U \to \mathbf{R}$,

where

$$h(x) \begin{cases} = 0, & \text{if } x \in S \\ > 0, & \text{if } x \notin S \end{cases},$$

and for the penalty parameter we have $d \to 0$. For the choice of the penalty function there are different possibilities. The most used one are

$$\text{a)} \quad h(x) := \sum_{i=1}^{m} \{\max\{0, g_i(x)\}\}^q, \qquad q > 0, \quad x \in U$$

$$\text{b)} \quad h(x) := e^{\sum_{i=1}^{m} \{\max\{0, g_i(x)\}\}^q} - 1, \qquad q > 0 . x \in U$$

In the exterior penalty function method, the function p is often taken as

$$p(x, d) := f(x) + \frac{1}{d} \sum_{i=1}^{m} \{\max\{0, g_i(x)\}\}^q, \qquad q > 0. \tag{3.1}$$

Example 1:

Consider the following problem:

$$f(x) := x \to \min, \quad x \in S,$$
$$S := \{x \in \mathbf{R} \mid -x+2 \le 0\}.$$

In this case we choose $h(x) = [\max\{0, g(x)\}]^2$. Then,

$$h(x) = \begin{cases} 0, & \text{if } x \ge 2, \\ (-x+2)^2, & \text{if } x < 2. \end{cases}$$

Note that $\min[f(x) + \frac{1}{d}h(x)] = f(x_0) + \frac{1}{d}h(x_0)$, where $x_0 = 2 - \frac{d}{2}$,

and

$$\lim_{d \to 0}(2 - \frac{d}{2}) = 2 =: x^*.$$

Consider the following general problem with any inequality constraint

$$f(x) \to \min \quad x \in S,$$

$$S := \{x \in U \subseteq \mathbf{R}^n \mid g(x) \le 0\}.$$

It is clear that the form $f(x) + \frac{1}{d}g^2(x)$ is not appropriate, since a penalty will be incurred whether $g(x) < 0$ or $g(x) > 0$. A penalty is desired only if the point x is not feasible, that is, if $g(x) > 0$.

A suitable unconstrained problem is therefore given by

$$p(x,d) = f(x) + \frac{1}{d}\max\{0, g(x)\} \to \min, \quad x \in \mathbf{R}^n.$$

If $g(x) \le 0$, then $\max\{0, g(x)\} = 0$, and no penalty is incurred. On the other hand, if

$g(x) > 0$, then $\max\{0, g(x)\} > 0$, and the penalty term $\frac{1}{d}g(x)$ is realized. However, observe that at points x where $g(x) = 0$, even if f and g are differentiable it may be that p is not differentiable. Therefore it is more convenient to choose $\frac{1}{d}[\max\{0, g(x)\}]^2$ as penalty term.

Now we consider a sequence of optimization problem

$$(P_k) \quad p_k(x, d_k) := f(x) + \frac{1}{d_k}h(x) \rightarrow \min, \qquad x \in U.$$

and assume that $M(p_k, U) \neq \varnothing$ for all $k \in \mathbf{N}$.

We have some properties of a sequence (x_k^*) of solution of (P_k) , $k \in \mathbf{N}$.

Theorem 3.1.1: Let (d_k) be a strictly monotonically decreasing sequence and

$x_k^* \in M(p_k, U)$.

Then

(i) $p_k(x_k^*, d_k) \leq p_{k+1}(x_{k+1}^*, d_{k+1})$ for all $k \in \mathbf{N}$,

(ii) $h(x_k^*) \geq h(x_{k+1}^*)$ for all $k \in \mathbf{N}$,

(iii) $f(x_k^*) \leq f(x_{k+1}^*)$ for all $k \in \mathbf{N}$.

Proof: (cf. [9])

i): Since $k < k+1$ and (d_k) is a strictly decreasing sequence, we have $d_k > d_{k+1}$,

i.e. $\frac{1}{d_k} < \frac{1}{d_{k+1}}$ or $\frac{1}{d_{k+1}} - \frac{1}{d_k} > 0$ for all $k \in \mathbf{N}$ and $h(x) \geq 0$ for all $x \in U$.

This implies

$$\frac{1}{d_k} h(x_{k+1}^*) \le \frac{1}{d_{k+1}} h(x_{k+1}^*) \quad \text{for all } k \in \mathbf{N}.$$

Hence

$$f(x_{k+1}^*) + \frac{1}{d_k} h(x_{k+1}^*) \le f(x_{k+1}^*) + \frac{1}{d_{k+1}} h(x_{k+1}^*).$$

Furthermore, since x_k^* is a solution of (P_k) we have for all $k \in \mathbf{N}$

$$f(x_k^*) + \frac{1}{d_k} h(x_k^*) = p_k(x_k^*, d_k) \le p_k(x_{k+1}^*, d_k) = f(x_{k+1}^*) + \frac{1}{d_k} h(x_{k+1}^*)$$

$$\le f(x_{k+1}^*) + \frac{1}{d_{k+1}} h(x_{k+1}^*)$$

$$= p_{k+1}(x_{k+1}^*, d_{k+1}), \tag{3.2}$$

i.e. we have (i).

ii): Since x_{k+1}^* is a solution of (P_{k+1}) and x_k^* is a solution of (P_k) we have for all $k \in \mathbf{N}$

$$p_{k+1}(x_{k+1}^*, d_{k+1}) = f(x_{k+1}^*) + \frac{1}{d_{k+1}} h(x_{k+1}^*) \le f(x_k^*) + \frac{1}{d_{k+1}} h(x_k^*)$$

and

$$p_k(x_k^*, d_k) = f(x_k^*) + \frac{1}{d_k} h(x_k^*) \le f(x_{k+1}^*) + \frac{1}{d_k} h(x_{k+1}^*).$$

By adding these two inequalities, we obtain for all $k \in \mathbf{N}$

$$\frac{1}{d_{k+1}} [h(x_{k+1}^*) - h(x_k^*)] \le \frac{1}{d_k} [h(x_{k+1}^*) - h(x_k^*)],$$

i.e.

$$[h(x_{k+1}^*) - h(x_k^*)]\underbrace{[\frac{1}{d_k} - \frac{1}{d_{k+1}}]}_{<0} \ge 0.$$

But this implies

$$h(x_{k+1}^*) - h(x_k^*) \le 0 \quad \text{for all } k \in \mathbf{N}.$$

Therefore, $h(x_{k+1}^*) \leq h(x_k^*)$ for all $k \in \mathbf{N}$.

So we have (ii).

iii): From $p_k(x_k^*, d_k) \leq p_k(x_{k+1}^*, d_k)$, we have

$$f(x_k^*) + \frac{1}{d_k} h(x_k^*) \leq f(x_{k+1}^*) + \frac{1}{d_k} h(x_{k+1}^*),$$

this implies

$$f(x_k^*) - f(x_{k+1}^*) \leq \frac{1}{d_k} [h(x_{k+1}^*) - h(x_k^*)] \leq 0 \quad \text{for all } k \in \mathbf{N},$$

i.e.

$$f(x_k^*) \leq f(x_{k+1}^*) \quad \text{for all } k \in \mathbf{N}. \ //$$

Now we give some propositions about the solution of (P_3).

Theorem 3.1.2: Let f and h be continuous functions. Furthermore let

i) $M(f, S) \neq \emptyset$,

ii) $M(p_k, U) \neq \emptyset$,

iii) (d_k) be a decreasing sequence such that $d_k \xrightarrow{k \to \infty} 0$.

iv) (x_k^*) be a sequence, where x_k^* is a solution of (P_k), $k \in \mathbf{N}$.

Then

a) If $x^* \in M(f, S)$, then $\lim_{k \to \infty} f(x_k^*) = f(x^*)$.

b) Each accumulation point of a sequence (x_k^*) is a solution of (P_3).

Proof: (cf. [9])

Let $\tilde{x}^*$ be an accumulation point of the sequence (x_k^*), where $x_k^* \in M(p_k, U)$ then there is a subsequence $(x_{k_i}^*)$ of (x_k^*) such that $x_{k_i}^* \xrightarrow{i \to \infty} \tilde{x}^*$.

First we show that $\tilde{x}^*$ is a feasible point. Since $x_{k_i}^*$ is a solution of (p_{k_i}) we have

$$f(x_{k_i}^*) + \frac{1}{d_{k_i}} h(x_{k_i}^*) \le f(x) + \frac{1}{d_{k_i}} h(x) \qquad \text{for all } x \in U. \tag{3.3}$$

Since $S \subseteq U$ we get that (3.3) is true for all $x \in S$. But for these x we have $h(x) = 0$, so it follows from (3.3)

$$f(x_{k_i}^*) + \frac{1}{d_{k_i}} h(x_{k_i}^*) \le f(x) \quad \text{for all } x \in S$$

or

$$d_{k_i} f(x_{k_i}^*) + h(x_{k_i}^*) \le d_{k_i} f(x) \quad \text{for all } x \in S.$$

This implies (using the continuity of f and h)

$$\lim_{i\to\infty} d_{k_i} f(x_{k_i}^*) + \lim_{i\to\infty} h(x_{k_i}^*) \le \lim_{i\to\infty} d_{k_i} f(x) = 0,$$

i.e.

$$\lim_{i\to\infty} d_{k_i} \lim_{i\to\infty} f(x_{k_i}^*) + h(\lim_{i\to\infty} x_{k_i}^*) \le 0,$$

or

$$h(\tilde{x}^*) \le 0.$$

On the other hand we have (by definition of h)

$$h(x_{k_i}^*) \ge 0 \text{ for all } i \in \mathbf{N}, \text{ i.e. } \lim_{i\to\infty} h(x_{k_i}^*) = h(\lim_{i\to\infty} x_{k_i}^*) = h(\tilde{x}^*) \ge 0.$$

But this implies

$$h(\tilde{x}^*) = 0, \text{ i.e. } \tilde{x}^* \in S.$$

From

$$f(x_{k_i}^*) \le f(x_{k_i}^*) + \frac{1}{d_{k_i}} h(x_{k_i}^*) \text{ and (3.3) it follows that}$$

$$f(x_{k_i}^*) \le f(x_{k_i}^*) + \frac{1}{d_{k_i}} h(x_{k_i}^*) \le f(x) + \frac{1}{d_{k_i}} h(x) = f(x) \quad \text{for all } x \in S.$$

This implies (using again the continuity of f)

$$f(\tilde{x}^*) = \lim_{i\to\infty} f(x_{k_i}^*) \le f(x) \quad \text{for all } x \in S,$$

i.e.

$\tilde{x}^*$ is a solution of (P_3). //

Theorem 3.1.3: (Generalization of Theorem 3.1.2)

Let X be a normed space, let $U \subseteq$ X and f, h be lower semi-continuous functions. If (x_k^*) is a sequence such that $x_k^* \in$ M(p_k, X),

then each accumulation point $\tilde{x}$ of the sequence (x_k^*) is a solution of (P_3).

Proof:

Let $\tilde{x}$ be an accumulation point of (x_k^*), then there exists a subsequence $(x_{k_j}^*)$ of (x_k^*) such that $\lim_{j\to\infty} x_{k_j}^* = \tilde{x}$.

1. We prove that $\tilde{x} \in S$.

Because $x_{k_j}^*$ is a solution of (P_{k_j}) we have

$$f(x_{k_j}^*) + \frac{1}{d_{k_j}} h(x_{k_j}^*) \le f(x) + \frac{1}{d_{k_j}} h(x) = f(x) \quad \text{for all } x \in S. \tag{3.4}$$

Hence it follows

$$d_{k_j} f(x_{k_j}^*) + h(x_{k_j}^*) \le d_{k_j} f(x) \quad \text{for all } x \in S. \tag{3.5}$$

Because f and h are lower semi-continuous we have by Theorem 2.1.2

$$f(\tilde{x}) \le \varliminf_{j\to\infty} f(x_{k_j}^*) \quad \text{and} \quad h(\tilde{x}) \le \varliminf_{j\to\infty} h(x_{k_j}^*). \tag{3.6}$$

Then from (3.5) we obtain, as $d_{k_j} \to 0$,

$$0 = \varliminf_{j\to\infty} (d_{k_j} f(x)) \geq \varliminf_{j\to\infty} [d_{k_j} f(x^*_{k_j}) + h(x^*_{k_j})] \geq h(\tilde{x}) \geq 0 .$$

This implies $h(\tilde{x}) = 0$ and therefore, $\tilde{x} \in S$,

2. Now we prove that $\tilde{x} \in \mathrm{M}(f, S)$.

From (3. 4) we get

$$f(x^*_{k_j}) \leq f(x^*_{k_j}) + \frac{1}{d_{k_j}} h(x^*_{k_j}) \leq f(x) \quad \text{for all } x \in S .$$

If we consider the limit infimum on both sides, then by (3.6) we have

$$f(\tilde{x}) \leq \varliminf_{j\to\infty} f(x^*_{k_j}) \leq f(x) \quad \text{for all } x \in S ,$$

i.e. $\tilde{x} \in \mathrm{M}(f, S)$. //

Proposition 3.1.4: Let $x^*_d \in \mathrm{M}(p, U)$. If $x^*_d \in S$, then $x^*_d \in \mathrm{M}(f, S)$.

Proof: Since $x^*_d \in \mathrm{M}(p, U)$ we have

$$p(x^*_d, d) \leq p(x, d) \quad \text{for all } x \in U.$$

This implies $f(x^*_d) + \frac{1}{d} h(x^*_d) \leq f(x) + \frac{1}{d} h(x)$ for all $x \in U$,

Since $x^*_d \in S$, we have $h(x^*_d) = 0$. Hence

$$f(x^*_d) \leq f(x) + \frac{1}{d} h(x) \quad \text{for all } x \in U \text{ and so for all } x \in S .$$

Therefore, as $h(x) = 0$ for $x \in S$,

$$f(x^*_d) \leq f(x) \quad \text{for all } x \in S ,$$

i.e.

$$x_d^* \in \mathrm{M}(f, S).\ //$$

Theorem 3.1.5: If $\mathrm{M}(p, U) \neq \varnothing$ for all $d>0$ and $\mathrm{M}(f, S) \neq \varnothing$, then

i) $\lim_{d\to 0} d \cdot p(x_d^*, d) = 0.$

ii) $\lim_{d\to 0} h(x_d^*) = 0.$

Proof : i) Let $pp(d) := p(x_d^*, d)$.

First let us show that for $d_1 \geq d_2 > 0$, pp is a decreasing function.

Since

$$p(x_{d_1}^*, d_1) = \min_{x\in U} p(x, d_1) \leq p(x_{d_2}^*, d_1) \text{ and } \frac{1}{d_1} h(x_{d_2}^*) \leq \frac{1}{d_2} h(x_{d_2}^*)$$

we get

$$pp(d_1) = p(x_{d_1}^*, d_1) \leq p(x_{d_2}^*, d_1) = f(x_{d_2}^*) + \frac{1}{d_1} h(x_{d_2}^*)$$

$$\leq f(x_{d_2}^*) + \frac{1}{d_2} h(x_{d_2}^*)$$

$$= p(x_{d_2}^*, d_2) = pp(d_2).$$

Therefore, pp is a decreasing function.

Now, since $h(x^*) = 0$, we have

$$p(x_d^*, d) \leq p(x^*, d) = f(x^*) + \frac{1}{d} h(x^*)$$

$$= f(x^*),$$

i.e. $pp(d) = p(x_d^*, d) \leq f(x^*)$.

Therefore, pp is bounded, and since a bounded monotonic function is convergent, we have that $\lim_{d\to 0} p(x_d^*, d)$ exists and it is finite.

Hence ,

$$0 = \lim_{d\to 0} d \cdot \lim_{d\to 0} p(x_d^*, d) = \lim_{d\to 0} d\ p(x_d^*, d)$$

ii): Let $ff(d) := f(x_d^*)$.

Let us show that for $d_1 > d_2 > 0$ ff is a decreasing function.

The inequalities

$$p(x_{d_1}^*, d_1) \le p(x_{d_2}^*, d_1),$$

$$p(x_{d_2}^*, d_2) \le p(x_{d_1}^*, d_2)$$

imply

$$f(x_{d_1}^*) + \frac{1}{d_1} h(x_{d_1}^*) \le f(x_{d_2}^*) + \frac{1}{d_1} h(x_{d_2}^*) \tag{3.7}$$

and

$$f(x_{d_2}^*) + \frac{1}{d_2} h(x_{d_2}^*) \le f(x_{d_1}^*) + \frac{1}{d_2} h(x_{d_1}^*), \tag{3.8}$$

i.e.

$$d_1 f(x_{d_1}^*) + h(x_{d_1}^*) \le d_1 f(x_{d_2}^*) + h(x_{d_2}^*),$$

$$d_2 f(x_{d_2}^*) + h(x_{d_2}^*) \le d_2 f(x_{d_1}^*) + h(x_{d_1}^*).$$

Adding these inequalities we get

$$(d_1 - d_2)[f(x_{d_1}^*) - f(x_{d_2}^*)] \le 0,$$

and, since $d_1 - d_2 > 0$, we have

$$f(x_{d_1}^*) - f(x_{d_2}^*) \le 0,$$

i.e.

$$f(x_{d_1}^*) \le f(x_{d_2}^*).$$

Hence ff is a decreasing function.

Since $h(x) \ge 0$ for all $x \in U$, we have

$$f(x_d^*) \le f(x_d^*) + \frac{1}{d} h(x_d^*) = p(x_d^*, d) \le f(x^*).$$

Hence ff is monotone and bounded above.

This implies $\lim_{d \to 0} f(x_d^*)$ exists and it is finite.

Therefore,

$$\lim_{d \to 0} h(x_d^*) = \lim_{d \to 0} [d\, p(x_d^*, d) - d\, f(x_d^*)]$$

$$= \lim_{d \to 0} d\, p(x_d^*, d) - \lim_{d \to 0} d \lim_{d \to 0} f(x_d^*) = 0. \quad //$$

Remarks:

1. From $f(x_d^*) + \frac{1}{d} h(x_d^*) \le f(x^*)$, we get $0 \le \frac{1}{d} h(x_d^*) \le f(x^*) - f(x_d^*)$, and from the proof of Theorem 3.5, ii), for $1 \ge d > 0$ we have $f(x_1^*) \le f(x_d^*)$.

Hence, we get that

$$0 \le \frac{1}{d} h(x_d^*) \le f(x^*) - f(x_d^*) \le f(x^*) - f(x_1^*).$$

Therefore,

$$0 \le \frac{1}{d} h(x_d^*) \le f(x^*) - f(x_1^*), \text{ for } d \in (0,1],$$

i.e. we obtain uniform bounds on the penalty term $\frac{1}{d} h(x_d^*)$ for all $d \in (0,1]$.

2. Let $hh(d) := h(x_d^*)$, then by adding the inequalities (3.7) and (3.8) we find

$$\frac{1}{d_1} [h(x_{d_1}^*) - h(x_{d_2}^*)] \le \frac{1}{d_2} [h(x_{d_1}^*) - h(x_{d_2}^*)],$$

i.e.

$$\underbrace{[\frac{1}{d_1} - \frac{1}{d_2}]}_{<0} [h(x_{d_1}^*) - h(x_{d_2}^*)] \le 0.$$

Hence we have

$$h(x_{d_1}^*) \ge h(x_{d_2}^*).$$

This implies $hh(d_1) \geq hh(d_2)$ for all $d_1 \geq d_2 > 0$. Therefore, hh is increasing function.

3. $\lim_{d\to 0} d\, f(x_d^*) = \lim_{d\to 0}[d\, p(x_d^*, d) - h(x_d^*)]$

$$= \lim_{d\to 0} d\, p(x_d^*, d) - \lim_{d\to 0} h(x_d^*) = 0. \quad //$$

In the following some possibilities of approximate calculation of the point x_d^* will be provided. Assume that for each value of the parameter $d > 0$ there exists a point $\tilde{x}_d^* \in U$ such that

$$p(\tilde{x}_d^*, d) \leq pp(d) + \frac{\xi(d)}{d}, \quad \xi(d) > 0.$$

Theorem 3.1.6: If

1) $M(f,S) \neq \varnothing$ and $M(p,U) \neq \varnothing$,

2) $\lim_{d\to 0} \xi(d) = 0$,

then

$$\lim_{d\to 0} d\, p(\tilde{x}_d^*, d) = 0.$$

Moreover, if the function f is bounded from below on the set $\{\tilde{x}_d^*\}$ for all sufficiently small $d > 0$, then

$$\lim_{d\to 0} h(\tilde{x}_d^*) = 0.$$

Proof: Let $d > 0$, then by assumption, there exists a point $\tilde{x}_d^* \in U$ such that

$p(\tilde{x}_d^*, d) \leq pp(d) + \frac{\xi(d)}{d}$ and since $x_d^* \in \mathrm{M}(p, U)$, we have

$$p(x_d^*, d) \leq p(\tilde{x}_d^*, d) \leq pp(d) + \frac{\xi(d)}{d} = p(x_d^*, d) + \frac{\xi(d)}{d}.$$

But $p(x_d^*, d) \leq p(\tilde{x}_d^*, d) \leq p(x_d^*, d) + \frac{\xi(d)}{d}$ implies

$$d\, p(x_d^*, d) \leq d\, p(\tilde{x}_d^*, d) \leq d\, p(x_d^*, d) + \xi(d).$$

From Theorem 3.1.5, i), we have

$$\lim_{d\to 0} d\, p(x_d^*, d) = 0.$$

Hence by squeezing theorem

$$\lim_{d\to 0} d\, p(\tilde{x}_d^*, d) = 0.$$

Now $0 \le h(\tilde{x}_d^*) = d\, p(\tilde{x}_d^*, d) - d\, f(\tilde{x}_d^*)$ and f is bounded from below on $\{\tilde{x}_d^*\}$ for all sufficiently small $d>0$, i.e $f(\tilde{x}_d^*) \ge$ m for all sufficiently small $d>0$, $m \in \mathbf{R}$.

Hence $0 \le h(\tilde{x}_d^*) = d\, P(\tilde{x}_d^*, d) - d\, f(\tilde{x}_d^*) \le d\, p(\tilde{x}_d^*, d) - d\, m.$

Since $\lim_{d\to 0} d\, p(\tilde{x}_d^*, d) = 0$, we get by squeezing theorem

$\lim_{d\to 0} h(\tilde{x}_d^*) = 0.$ //

Def.3.1.1: $\rho(v, S) := \inf_{x\in S} \|x - v\| =: \rho$ is called the distance from the point v to the set S.

Theorem 3.1.7: Let $U \subseteq \mathbf{R}^n$, $f, h, g_i : U \to \mathbf{R}$, $i \in \{1, 2, ..., m\}$ be continuous.

If

a) for all $d > 0$ there exists $x^* \in M(f, S)$ and $x_d^* \in M(p, U)$,

b) there is a closed and bounded set $G \subseteq U$ such that $\tilde{x}_d^* \in G$ for all sufficiently small $d > 0$,

c) $\lim_{d\to 0} \frac{\xi(d)}{d} = 0$,

then

1. $\lim_{d\to 0} f(\tilde{x}_d^*) = f(x^*)$,
2. $\lim_{d\to 0} \rho(\tilde{x}_d^*, M(f, S)) = 0.$

Proof: (proof by contradiction)

1: Assume that the theorem is not true, i.e. there is a number $\varepsilon > 0$ and a sequence (d_k), with $\lim_{k\to\infty} d_k = 0$, such that

$$\left|f(\tilde{x}^*_{d_k}) - f(x^*)\right| \geq \varepsilon \text{ for all } k \in \mathbf{N}_0 := \mathbf{N} \cup \{0\} \text{ holds.}$$

Since $h(x) \geq 0$ for all $x \in U$ and $h(x^*) = 0$, we have

$$d_k f(\tilde{x}^*_{d_k}) \leq d_k f(\tilde{x}^*_{d_k}) + h(\tilde{x}^*_{d_k}) = d_k p(\tilde{x}^*_{d_k}, d_k) \leq d_k p(x^*_{d_k}, d_k) + \xi(d_k)$$

$$\leq d_k p(x^*, d_k) + \xi(d_k) = d_k f(x^*) + \xi(d_k). \tag{3.9}$$

Since G is bounded and $\tilde{x}^*_{d_k} \in G$ for all $k \geq k_0$, the sequence $(\tilde{x}^*_{d_k})$ is bounded. Without loss of generality assume $k_0 = 1$, hence there is a subsequence $(\tilde{x}^*_{d_{k_j}})$ of $(\tilde{x}^*_{d_k})$ such that

$\lim_{j\to\infty} \tilde{x}^*_{d_{k_j}} = \tilde{x} \in G$, as G is closed and each bounded sequence in $\mathbf{R}^n$ possesses a convergent subsequence.

From (3.9) we get

$$d_{k_j} f(\tilde{x}^*_{d_{k_j}}) \leq d_{k_j} f(x^*) + \xi(d_{k_j}).$$

Hence,

$$f(\tilde{x}^*_{d_{k_j}}) \leq f(x^*) + \frac{\xi(d_{k_j})}{d_{k_j}}. \tag{3.10}$$

Since f is continuous, we get

$$f(\tilde{x}) = f(\lim_{j\to\infty} \tilde{x}^*_{d_{k_j}}) = \lim_{j\to\infty} f(\tilde{x}^*_{d_{k_j}}).$$

Now, taking limit as $j \to \infty$ for both sides in (3.10) we get, using c),

$$f(\tilde{x}) = \lim_{j\to\infty} f(\tilde{x}^*_{d_{k_j}}) \leq f(x^*),$$

i.e.

$$f(\tilde{x}) \leq f(x^*).$$

Since a continuous function on a compact set has a minimum, f is bounded from below on the set G for sufficient small $d_k > 0$, and hence by Theorem 3.1.6, using $\widetilde{\xi}(d) := \frac{\xi(d)}{d} > 0$, $\lim_{d \to 0} \widetilde{\xi}(d) = 0$ and the continuity of h

$$h(\widetilde{x}) = h(\lim_{j \to \infty} \widetilde{x}^*_{d_{k_j}}) = \lim_{j \to \infty} h(\widetilde{x}^*_{d_{k_j}}) = 0.$$

Consequently, $\widetilde{x} \in S$, hence

$f(\widetilde{x}) = f(x^*)$, which contradicts the assumption.

2: Assume that $\lim_{d \to 0} \rho(\widetilde{x}^*_d, M(f, S)) = 0$ is false.

Then there are a number $\varepsilon > 0$ and a subsequence $(\widetilde{x}^*_{d_k})$ of $(\widetilde{x}^*_d)$ such that

$$\rho(\widetilde{x}^*_d, M(f,S)) > \varepsilon \text{ for all } k \in \mathbf{N}_0.$$

Now, for all $k \geq k_0$, $\widetilde{x}^*_{d_k} \in G$. Again without loss of generality assume $k_0 = 1$.

Since G is bounded and $\widetilde{x}^*_{d_k} \in G$, there is at least one subsequence $(\widetilde{x}^*_{d_{k_j}})$ of $(\widetilde{x}^*_{d_k})$ such that

$$\lim_{j \to \infty} \widetilde{x}^*_{d_{k_j}} = \widetilde{x}.$$

Hence from the proof of 1) we get $\widetilde{x} \in M(f,S)$.

Therefore, $\lim_{j \to \infty} \rho(\widetilde{x}^*_{d_{k_j}}, M(f,S)) = \lim_{j \to \infty} \rho(\widetilde{x}_{d_{k_j}}, \widetilde{x}) = 0$. But this is a contradiction

to $\rho(\widetilde{x}^*_{d_k}, M(f,S)) \geq \varepsilon$ for all $k \in \mathbf{N}_0$.

Hence $\lim_{d \to 0} \rho(\widetilde{x}^*_d, M(f,S)) = 0$. //

Remark: Any accumulation point of $(\widetilde{x}^*_d)$ belongs to $M(f,S)$.

Corollary 3.1.8: If $(\widetilde{x}^*_d) \subseteq U \setminus S$, then any accumulation point $\widetilde{x}$ belongs to $\partial M(f, S) \cap \partial S$.

Proof: Let $\tilde{x}$ be an accumulation point of $(\tilde{x}_d^*)$, hence there exists a subsequence $(\tilde{x}_{d_k}^*)$ of $(\tilde{x}_d^*)$ such that $\lim_{k\to\infty}\tilde{x}_{d_k}^* = \tilde{x}$. Now $\tilde{x} \in M(f, S) \subseteq S$ and since $\tilde{x}_d^* \in U \setminus S$ we have $\tilde{x}_{d_k}^* \notin S$ for all $k \in \mathbf{N}_0$. But $\lim_{k\to\infty}\rho(\tilde{x}_{d_k}^*, M(f,S))=0$, consequently, $\tilde{x} \notin \operatorname{int} M(f,S)$, otherwise , i.e. if $\tilde{x} \in \operatorname{int} M(f,S)$, then there exists a neighborhood $N_{\tilde{x}}$ of $\tilde{x}$ and $k_0 \in \mathbf{N}$ such that $\tilde{x}_{d_k}^* \in N_{\tilde{x}} \subseteq S$ for all $k \geq k_0$. But this contradicts to $\tilde{x}_{d_k}^* \notin S$ for all $k \in \mathbf{N}_0$.

Hence, $\tilde{x} \in \operatorname{int} M(f,S)$,

Similar, since $\tilde{x} \in S$ and $\tilde{x}_{d_k} \notin S$ for all $k \in \mathbf{N}_0$, we have $\tilde{x} \in \partial S$.

Therefore, $\tilde{x} \in \partial S \cap \partial M(f,S)$. //

Corollary 3.1.9: If $M(f,S) \subseteq \operatorname{int} S$, then there is an index $j_\circ$ such that $x_{d_{j_\circ}}^* \in M(f,S)$.

Proof :

Case 1: If there exists an index $j_\circ$ such that $x_{d_{j_\circ}}^* \in S$, then by proposition (3.4)

$x_{d_{j_\circ}}^* \in M(f,S)$. Hence the result.

Case 2: Suppose there does not exist an index $j_\circ$ such that $x_{d_{j_\circ}}^* \in S$.

This implies $(x_{d_k}^*) \subseteq U \setminus S$ for all $k \in \mathbf{N}_0$. By Corollary 3.1.8, any accumulation point of

$(x_{d_k}^*)$

belongs to $\partial S \cap \partial M(f,S)$.

This implies

$$\partial S \cap \partial M(f,S) \neq \varnothing,$$

i.e. $M(f,S)$ is not a subset of $\operatorname{int} S$. This contradicts the hypothesis. //

Remark: If f and h are convex and $\mathrm{M}(f,S) \subseteq \operatorname{int} S$, then $x_d^* \in \mathrm{M}(f,S)$ for any $d > 0$,

i.e. $\mathrm{M}(f,S) = \mathrm{M}(p, U)$. //

3.2. Problems with equality and inequality constraints

We consider the optimization problem

$$(\mathrm{P}_4) \qquad f(x) \to \min, \quad x \in S$$
$$S := \{x \in U \subseteq \mathbf{R}^n \mid g(x) \le 0,\ k(x) = 0\}$$

where $f,\ g_i, k_j : U \to \mathbf{R},\ i \in \{1, ..., m\},\ k_j,\ j \in \{1, ..., l\}$ are continuous.

The set U might typically represent simple constraints that could be easily handled explicitly, such as lower and upper bounds on the variables. In this case a suitable penalty function h is defined by

$$h(x) = \sum_{i=1}^{m} \phi[g_i(x)] + \sum_{j=1}^{l} \varphi[k_j(x)], \tag{3.11}$$

where ϕ and φ are continuous functions satisfying the following conditions:

$$\phi(y)\begin{cases} = 0, & \text{if } y \le 0 \\ > 0, & \text{if } y > 0 \end{cases},$$

$$\varphi(y)\begin{cases} = 0, & \text{if } y = 0 \\ > 0, & \text{if } y \ne 0 \end{cases}. \qquad (3.12^{`}).$$

Usually, ϕ and φ are of the forms

$$\phi(y) = [\max\{0, y\}]^q$$
$$\varphi(y) = |y|^q ,$$

where q is a positive integer. Thus, the penalty function h is usually of the form

$$h(x) = \sum_{i=1}^{m} [\max\{0, g_i(x)\}]^q + \sum_{j=1}^{l} |k_j(x)|^q . \tag{3.12}$$

Let h be a continuous function of the form (3.11) satisfying the stated properties in (3.12').

The basic penalty function approach attempts to solve the problem

$$\theta(d) \to \max,$$
$$\text{s.t. } d \geq 0,$$

where $\theta(d) := \inf_{x \in U} p(x,d)$.

Now we will prove that $\inf_{x \in S} f(x) = \sup_{d \geq 0} \theta(d) = \lim_{d \to 0} \theta(d)$.

From this result, it is clear that we can get arbitrarily close to the optimal objective value of (P_4) by computing $\theta(d)$ for a sufficiently small d. This result is established in Theorem 3.2.2. First, however, the following lemma is needed.

Lemma 3.2.1. Suppose that for each d, there is an $x_d \in U$ such that $\theta(d) = p(x_d, d)$. Then

$$\inf_{x \in S} f(x) \geq \sup_{d \geq 0} \theta(d).$$

Proof: Let $x \in S$ and $d \geq 0$. Then, as $h(x) = 0$,

$$f(x) = f(x) + \frac{1}{d} h(x) = p(x,d) \geq \inf_{y \in U} p(y,d) = \theta(d)$$

Taking supremum on both sides, and then the infimum we get

$$\inf_{x \in S} f(x) \geq \sup_{d \geq 0} \theta(d). \ //$$

Theorem 3.2.2:

Consider the problem (P_4) and assume that M(f, S) $\neq \varnothing$ and M(p, U) $\neq \varnothing$.

If

a) for each d there exists a solution $x_d^* \in U$ to the problem
$$p(x,d) \to \min, \quad x \in U,$$

b) $(x_d^*) \subseteq G \subseteq U$, G is compact.

Then

1. $\inf_{x \in S} f(x) = \sup_{d \geq 0} \theta(d) = \lim_{d \to 0} \theta(d)$ where $\theta(d) := \inf_{x \in U} p(x,d) = p(x_d^*, d)$,

2. The limit $\widetilde{x}$ of any convergent subsequence of (x_d^*) is an optimal solution to (P4),

Proof: 2) From the proof of Theorem 3.1.5, θ is monotone decreasing, hence $\sup_{d\geq 0}\theta(d) = \lim_{d\to 0}\theta(d)$ and by Theorem 3.1.5, ii) $\lim_{d\to 0} h(x_d^*)=0.$

Let $\widetilde{x}$ be an accumulation point of (x_d^*). Hence there exists a subsequence $(x_{d_k}^*)$ of (x_d^*) such that $\lim_{k\to\infty} x_{d_k} = \widetilde{x}$. Then we get

$$\sup_{d_k\geq 0}\theta(d_k) \geq \theta(d_k) = f(x_{d_k}^*)+\frac{1}{d_k}h(x_{d_k}^*) \geq f(x_{d_k}^*).$$

Since $\lim_{k\to\infty} x_{d_k} = \widetilde{x}$ and f is continuous, the above inequality implies that

$$\sup_{d_k\geq 0}\theta(d)\geq f(\widetilde{x}). \tag{3.13}$$

But $f(x^*)=\inf_{x\in S} f(x)\geq \sup_{d_k\geq 0}\theta(d_k) \geq f(\widetilde{x}).$ (3.14)

Since $\lim_{k\to\infty} h(x_{d_k}^*) = 0$, and h is continuous we get $h(\widetilde{x})=0$, i.e. $\widetilde{x}\in S$ Hence $f(x^*) = f(\widetilde{x})$.

Therefore, $\widetilde{x} \in \mathrm{M}(f, S)$.

1) From (3.14) we get

$f(x^*) = \inf_{x\in S} f(x) \geq \sup_{d_k\geq 0}\theta(d_k) \geq f(\widetilde{x})$. But $f(x^*) = f(\widetilde{x})$, hence

$$f(x^*) = \inf_{x\in S} f(x) = f(\widetilde{x}) = \sup_{d_k\geq 0}\theta(d_k) = \lim_{k\to\infty}\theta(d_k).\ //$$

Remark:

Since $\frac{1}{d_k}h(x_{d_k}^*)=\theta(d_k)-f(x_{d_k}^*)$, $\lim_{k\to\infty}\theta(d_k)=f(\widetilde{x})$, and $\lim_{k\to\infty} f(x_{d_k}^*)= f(\widetilde{x})$

we have $\lim_{k\to\infty}\frac{1}{d_k}h(x_{d_k}^*) = \lim_{k\to\infty}[\theta(d_k)-f(x_{d_k}^*)]=\lim_{k\to\infty}\theta(d_k)-\lim_{k\to\infty} f(x_{d_k}^*)=0$ //

Corollary: 3.2.3. If $h(x_d^*)=0$ for some d, then x_d^* is an optimal solution to (P_4).

Proof: If $h(x_d^*)=0$, then x_d^* is a feasible point. Furthermore, since

$$\inf_{x\in S} f(x) \geq \theta(d)= f(x_d^*)+\frac{1}{d}h(x_d^*)= f(x_d^*).$$

This implies $x_d^* \in \mathrm{M}(f,S)$. //

Remark:

1. By choosing d so small, x_d^* can be made arbitrarily close to the feasible region.

2. Furthermore, by choosing d so small $p(x_d^*,d)$ can be made arbitrarily close to $f(x^*)$.

3. The points x_d^* are generally infeasible, but as $d \to 0$, the points generated approach an optimal solution from outside the feasible region. Hence, this technique is also referred to as an exterior penalty function method.

4. Numerical approach for the penalty methods

In the numerical realization of the penalty function method there arises problems of selecting the initial value d_0 of the parameter d and the way d_k is varied. The difficulty is that, the choice of a sufficiently small d_0 allows us to hope that $x^*_{d_0}$ will be close to x^*.

Theorem 4.1: Let $f, h: \mathbf{R}^n \to \mathbf{R}$ be convex

If M(f, S) $\neq \varnothing$ and bounded, then there is a sufficiently small $d_0 > 0$ such that the sets M (p, $\mathbf{R}^n$) are uniformly bounded for all $d \in (0, d_0]$.

Proof: (cf.[11])

Since M(f, S) $\neq \varnothing$, let $x^* \in$ M(f,S) and B:= $\{x \in \mathbf{R}^n \mid d(x, x^*) \leq r\}$ so that M(f,S) $\subseteq$ intB. This is possible because M (f,S) is bounded.

First we prove that p(z, d) > p(x^*, d) for all $z \in \partial$B, $d \in (0, d_0]$. (4.1)

Case 1: Assume that the set S is bounded and r is chosen such that $S \subseteq$ intB. Let $\mu := \min_{z \in \partial B} f(z)$ and $\nu := \min_{z \in \partial B} h(z)$.

Since $S \subseteq$ intB and $z \in \partial$B we have $z \notin S$, hence $h(z) > 0$ for all $z \in \partial$B. Therefore, $\nu > 0$.

Let $d_0 \geq d > 0$, then $\frac{1}{d} h(z) \geq \frac{1}{d_0} h(z) \geq \frac{1}{d_0} \nu > \underbrace{f(x^*) - \mu}_{\leq 0}$ for all $z \in \partial$B.

Consequently,

$$\text{p}(z, d) = f(z) + \frac{1}{d} h(z) > \mu + f(x^*) - \mu = f(x^*) = f(x^*) + \frac{1}{d} \underbrace{h(x^*)}_{=0} = p(x^*, d) \text{ for}$$

all $z \in \partial$B, $d \in (0, d_0]$.

Therefore, p(z, d) > p(x^*, d) for all $z \in \partial$B, $d \in (0, d_0]$.

Case 2: Let the set S be unbounded. Since the set M(f, S) is bounded, it follows that $f(z) \geq f(x^*) + \Delta$ for all $z \in \partial B \cap S$ where $0 < \Delta \leq \min_{z \in \partial B \cap S} f(z) - f(x^*)$.

Therefore, defining

$$U_\delta(S) := \{x \in \mathbf{R}^n \mid \rho(x,S) \le \delta\}.$$

$$G_1 := \partial B \cap U_\delta(S),$$

$$G_2 := \partial B \setminus U_\delta(S),$$

we get $\partial B = G_1 \cup G_2$. We take $\delta = \delta(\Delta) > 0$ such that $f(z) > f(x^*) + \frac{1}{2}\Delta$ for all $z \in G_1$.

This is possible since each convex function on $\mathbf{R}^n$ is continuous. Thus, for all $z \in G_1$ we have

$$p(z,d) = f(z) + \frac{1}{d}h(z) \ge f(x^*) + \frac{1}{2}\Delta + \frac{1}{d}h(z) > f(x^*) = p(x^*,d). \quad (4.2)$$

Further, $\mu := \min_{z \in G_2} f(z)$ and $\nu := \min_{z \in G_2} h(z) > 0$.

Let $d_0 \ge d > 0$, then $\frac{1}{d}h(z) \ge \frac{1}{d_0} \cdot h(z) \ge \frac{1}{d_0}\nu > \underbrace{f(x^*) - \mu}_{\le 0}$ for all $z \in G_2$, $d \in (0,d_0]$.

Consequently,

$$p(z,d) = f(z) + \frac{1}{d}h(z) > \mu + f(x^*) - \mu = f(x^*)$$

$$= f(x^*) + \frac{1}{d}\underbrace{h(x^*)}_{=0} = p(x^*,d).$$

Hence, $p(z,d) > p(x^*,d)$ for all $z \in G_2$, $d \in (0,d_0]$. (4.3)

From (4.2) and (4.3) we get for all $z \in G_1 \cup G_2 = \partial B$

$$p(z,d) > p(x^*,d), \quad d \in (0,d_0].$$

Therefore, $p(z,d) > p(x^*,d)$, for all $z \in \partial B$, $d \in (0,d_0]$.

Since p is convex, as sum of two convex functions is convex, using (4.1) we get

$p(z,d) > p(x^*,d)$ for all $z \notin B$, $d \in (0,d_0]$.

If there exists $x \notin$ B such that $p(x,d) \le p(x^*, d)$, $d \in (0,d_0]$, then $[x,x^*]$ contains $z \in \partial$B such that $z = \lambda x + (1-\lambda)x^*$, $\lambda \in (0,1)$.

This implies

$$p(z, d) \le \lambda p(x, d) + (1-\lambda)p(x^*, d) \le \lambda p(x^*, d) + (1-\lambda)p(x^*, d)$$
$$= p(x^*, d),\ z \in \partial \text{B},\ d \in (0,d_0].$$

But this is a contradiction to

$$p(z, d) > p(x^*, d) \text{ for all } z \in \partial \text{B},\ d \in (0,d_0].$$

For all $x \notin$ B, since $p(x, d) > p(x^*, d) \ge p(x^*_d, d)$, we get $x^*_d \in$B , hence M(p , $\mathbf{R}^n$) $\subseteq$B.

The existence of x^*_d for all $d \in (0,d_0]$ is obvious since the function $p(x, d)$ attains minimum at a point x^*_d on the closed and bounded set B, and out side this set there holds

$$p(x, d) > f(x^*) = p(x^*, d) \ge p(x_d, d).$$

Therefore, M(p , $\mathbf{R}^n$) is uniformly bounded. //

Theorem 4.2: Let f, $h : \mathbf{R}^n \to \mathbf{R}$.

If a) f, h are convex and partialy differentiable

b) M(f , S) $\ne \varnothing$ and bounded.

Then there is a sufficiently small $d_0 > 0$ such that for all $\varepsilon > 0$ there exists $d_0 = d_0(\varepsilon) > 0$, $\delta = \delta(\varepsilon) > 0$ for all $x \notin U_\varepsilon(M(f,S))$, $d \in (0,d_0]$ such that $\|\nabla p(x, d)\| \ge \delta$.

For the proof see [11].

Theorem 4.3: Let f, $h : \mathbf{R}^n \to \mathbf{R}$

If a) f and h are convex and partially differentiable,

b) M(f , S) $\ne \varnothing$ and bounded,

c) The sequence (x_k), $x_k(d)$, is such that $\|\nabla p(x_k, d)\| \le \varepsilon_k$ for all $k \in \mathbf{N}_0$.

Then

$$\rho(x_k, \mathrm{M}(f, S)) \to 0 \text{ for } \varepsilon_k \to 0,\ k \to \infty, \text{ and } d \to 0.$$

Proof: First let us prove that (x_k) is bounded.

By Theorem 4.1, there is a sufficiently small $d_0 > 0$ such that $\mathrm{M}(p, \mathbf{R}^n)$ are uniformly bounded for all $d \in (0, d_0]$. According to Theorem 4.2, if B with radius ε such that $\mathrm{M}(p, \mathbf{R}^n) \subseteq \mathrm{int\,B}$ for all $d \in (0, d_0]$ is taken as $U_\varepsilon(M(f, S))$, then there exists $\delta = \delta(\varepsilon) > 0$ for all $x \notin \mathrm{B}$ such that $\|\nabla p(x, d)\| \geq \delta$.

Suppose (x_k) is unbounded, then there is a subsequence (x_{k_j}) of (x_k) such that $\lim_{j\to\infty} x_{k_j} = \infty$. Hence there is an index k_0 such that $x_k \notin \mathrm{B}$ and $\|\nabla p(x_k, d)\| \geq \delta$ for all $k \geq k_0$. Which is a contradiction to 3). Therefore, (x_k) is bounded.

Assume that the theorem is false, i.e. there are a number $\Delta > 0$ and a sequence (d_k), $\lim_{k\to\infty} d_k = 0$, such that $|f(x_k) - f(x^*)| \geq \Delta$ for all $x^* \in \mathrm{M}(f, S)$ and for all $k \in \mathbf{N}_0$.

Since $h(x) \geq 0$ for all $x \in \mathbf{R}^n$ and $h(x^*) = 0$, as (x_k) is bounded, we have $\|x_k - x^*_{d_k}\| \leq \eta < \infty$. Let $k := d_k$, then $\|x_k - x^*_k\| \leq \eta < \infty$, and as $p(x, d)$ is convex using Theorem 2.1.5, a) we get

$\langle \nabla p(x_k, d_k), x^*_{d_k} - x_k \rangle \leq p(x^*_{d_k}, d_k) - p(x_k, d_k)$. This implies

$$p(x_k, d_k) \leq p(x^*_{d_k}, d_k) - \langle \nabla p(x_k, d_k), x^*_{d_k} - x_k \rangle$$

$$= p(x^*_{d_k}, d_k) + \langle \nabla p(x_k, d_k), x_k - x^*_{d_k} \rangle. \tag{4.4}$$

But $x^*_{d_k} \in M(p, \mathbf{R}^n)$, hence by Theorem 2.1.5, b) we have $\langle \nabla p(x_k, d_k), x_k - x^*_{d_k} \rangle \geq 0$ for all $x_k \in \mathbf{R}^n$. Therefore,

$$0 \leq \langle \nabla p(x_k, d_k), x_k - x^*_{d_k} \rangle = \|\nabla p(x_k, d_k)\| \|x_k - x^*_{d_k}\| \cos\theta$$

$$\leq \|\nabla p(x_k, d_k)\| \|x_k - x^*_{d_k}\| \leq \varepsilon_k \eta. \tag{4.5}$$

Hence

$$f(x_k) \leq p(x^*_{d_k}, d_k) + \eta\varepsilon_k.$$

$$\leq p(x^*, d_k) + \eta\varepsilon_k$$

$$= f(x^*) + \eta\varepsilon_k. \tag{4.6}$$

Since the sequence (x_k) is bounded, there exists a subsequence (x_{k_j}) of (x_k) such that $\lim_{j\to\infty} x_{k_j} = \tilde{x}$. Therefore, by taking limit as $j \to \infty$ for both sides in (4.6) we get $\lim_{j\to\infty} f(x_{k_j}) = f(\tilde{x}) \leq f(x^*) + \eta \lim_{j\to\infty} \varepsilon_{k_j} = f(x^*)$ hence f is bounded below. By Theorem (3.1.6) $h(\tilde{x}) = \lim_{j\to\infty} h(x_{k_j}) = 0$, i.e. $\tilde{x} \in S$ and $f(\tilde{x}) \leq f(x^*)$, hence $f(\tilde{x}) = f(x^*)$, i.e. $\tilde{x} \in M(f, S)$, which is a contradiction to $|f(x_k) - f(x^*)| \geq \Delta$ for all $k \in \mathbf{N}_0$.

Hence $\lim_{k\to\infty} \rho(x_k, M(f, S)) = 0$ for $\lim_{k\to\infty} \varepsilon_k = 0$ and $d \to 0$. //

Now we give the algorithm for solving (P_1) using interior penalty method.

Let $\varepsilon > 0$ be a termination scalar, $x_0 \in \operatorname{int} S$, $d > 0$, $c \in (0,1)$. Let k=1, and go to the main step.

Main Step

1. Starting with x_k, solve the following problem:

$$p(x, d_k) \to \min, \; x \in \operatorname{int} S.$$

   Let x_{k+1} be an optimal solution, and go to step 2.

2. If $d_k h(x_{k+1}) < \varepsilon$, stop. Otherwise, let $d_{k+1} = cd_k$, k = k+1, and go to step 1.

In the Appendix we present a Program whose code is written with "Mathematica 4.1" and we used Variable Metric Method to solve the unconstrained problem in step 1.

Examples: Consider the following examples

1) $$x_1^2 + x_2^2 - 14x_1 - 6x_2 - 7 \to \min, \qquad x \in S,$$

$$S = \{x \in \mathbf{R}^2 : x_1 + x_2 - 2 \le 0, \; x_1 + 2x_2 - 3 \le 0\}.$$

The final result using $x_0 = (0, 1)$ is

$$x_9^* \approx (2.95086, \; -1.04967), \quad p_9(x_9^*, d_9) \approx -31.8037.$$

2) $$x_1^3 - 6x_1^2 + 11x_1 + x_3 \to \min, \qquad x \in S,$$

$$S = \{x \in \mathbf{R}^3 |\, x_1^2 + x_2^2 - x_3^2 \le 0, \; 4 - x_1^2 - x_2^2 - x_3^2 \le 0, \; x_3 - 5 \le 0, \; -x_i \le 0, \; i = 1,2,3.\}.$$

The final result using $x_0 = (0.1, 0.1, 3.0)$ is

$$x_{14}^* \approx (0.00332343, \; 1.4237, 1.4237), \quad p_{14}(x_{14}^*, d_{14}) \approx 1.5062.$$

3) $$\frac{1}{3}(x_1 + 1)^3 + x_2 \to \min, \qquad x \in S,$$

$$S = \{x \in \mathbf{R}^2 : -x_1 + 1 \le 0, \; -x_2 \le 0\}.$$

The final result with $x_0 = (1.1, 0.1)$ is

$$x_{13}^* \approx (1.16957,\ \ 0.0782171),\ p_{13}(x_{13}^*, d_{13}) \approx 0.136198.$$

4) $$\frac{10}{3}x_1x_2 + \frac{1}{6}x_1 \to \min, \qquad x \in S,$$

$$S := \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + \frac{5}{2}x_2^2 - \frac{19}{16} \le 0,\ -x_1 + x_2 - \frac{3}{5} \le 0\}$$

The final result with $x_0 = (1.1, 0.1)$ is

$$x_{13}^* \approx (1.16957,\ \ 0.0782171),\ p_{13}(x_{13}^*, d_{13}) \approx 0.136198.$$

5) $$(x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2 \to \min, \qquad x \in S,$$

$$S := \{(x_1, x_2) \in \mathbf{R}^2 : \frac{1}{2}x_1 - x_2 + 2 \le 0, 2x_1 - x_2 + 2 \le 0\}$$

The final result with $x_0 = (1, 5)$ is

$$x_8^* \approx (-2.8065,\ \ 3.13357),\ p_8(x_8^*, d_8) \approx 0.00452986.$$

For the corresponding table of the examples see on the appendix.

5. Result and Discussion

Let us consider the following problem

(P) $$f(x_1,x_2,x_3) = -x_1 \cdot x_2^2 \cdot e^{x_3} \rightarrow \min, \quad x \in S,$$

$$S = \{x \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + e^{x_3} - 4 \le 0, -x_1 \le 0, -x_2 \le 0, -x_3 \le 0\}$$

Using the Program given in the appendix we get the following results with $d=1$, $c=0.5$, $\delta = 0.1$, $\varepsilon=0.005$, and $x_0 = (0.5, 0.5, 1.0)$ for different barrier functions.

For $h = \sum_{i=1}^{m} - Log[-g[i]]$ the result is as follows

k	x*(k)	f(x*(k))	p(x*(k))
0	(0.5 0.5 1.)	-0.339785	1.29277
1	(0.81832 , 1.05634 , 0.524431)	-1.54273	-0.107235
2	(0.844511 , 1.13578 , 0.524096)	-1.83995	-0.907037
3	(0.864349 , 1.18644 , 0.520789)	-2.04812	-1.43613
4	(0.876192 , 1.21725 , 0.511571)	-2.16536	-1.77804
5	(0.88323 , 1.23662 , 0.500121)	-2.22714	-1.99084
6	(0.887606 , 1.24876 , 0.489632)	-2.25851	-2.11875
7	(0.890422 , 1.25613 , 0.481602)	-2.27417	-2.19348
8	(0.892226 , 1.2604 , 0.476206)	-2.28196	-2.23623
9	(0.893331 , 1.26274 , 0.47296)	-2.28584	-2.26029
10	(0.893956 , 1.26394 , 0.471209)	-2.28778	-2.27366
11	(0.894273 , 1.26452 , 0.470375)	-2.28876	-2.28102
12	(0.89441 , 1.26476 , 0.470041)	-2.28925	-2.28504

For $h = \sum_{i=1}^{m} -\frac{1}{g[i]}$ the result is as follows

k	x*(k)	f(x*(k))	p(x*(k))
--	-------------------------------	---------	---------
0	(0.5 0.5 1.)	-0.339785	5.93945
1	(0.776471 , 0.922048 , 0.615914)	-1.22214	4.21155
2	(0.789567 , 1.02012 , 0.579156)	-1.46628	1.42722
3	(0.809085 , 1.09951 , 0.548116)	-1.69213	-0.0846134
4	(0.829282 , 1.15558 , 0.521814)	-1.86605	-0.939287
5	(0.847876 , 1.19387 , 0.499526)	-1.99152	-1.43964
6	(0.863525 , 1.2199 , 0.481398)	-2.07965	-1.74155
7	(0.875617 , 1.23768 , 0.467678)	-2.14111	-1.92866
8	(0.884195 , 1.24981 , 0.458346)	-2.1842	-2.04751
9	(0.889749 , 1.2579 , 0.453047)	-2.2147	-2.12479
10	(0.892966 , 1.26298 , 0.451149)	-2.23644	-2.17613
11	(0.894555 , 1.2658 , 0.451836)	-2.252	-2.21088
12	(0.895127 , 1.267 , 0.454219)	-2.26309	-2.23472
13	(0.895143 , 1.26713 , 0.457466)	-2.27096	-2.25123
14	(0.894912 , 1.26666 , 0.460901)	-2.27651	-2.26274
15	(0.894614 , 1.26596 , 0.464058)	-2.2804	-2.27077
16	(0.894342 , 1.26525 , 0.466674)	-2.28314	-2.27639
17	(0.894137 , 1.26468 , 0.468647)	-2.28506	-2.28033

For $h = \sum_{i=1}^{m} \frac{1}{g[i]^2}$ we get the following result

k	x*(k)	f(x*(k))	p(x*(k))
0	(0.5 0.5 1.)	-0.339785	10.2967
1	(0.760822 , 0.809209 , 0.673631)	-0.977144	6.02448
2	(0.767122 , 0.857298 , 0.658082)	-1.08875	2.49545
3	(0.774759 , 0.930176 , 0.634013)	-1.2637	0.656545
4	(0.784787 , 1.02118 , 0.583722)	-1.46712	-0.362097
5	(0.79466 , 1.09464 , 0.546264)	-1.64423	-0.965882
6	(0.805404 , 1.14951 , 0.515898)	-1.78276	-1.34528
7	(0.817831 , 1.19039 , 0.489418)	-1.89058	-1.59552
8	(0.83164 , 1.21958 , 0.467558)	-1.97431	-1.76757
9	(0.845999 , 1.23906 , 0.451331)	-2.03968	-1.89008
10	(0.859881 , 1.25075 , 0.441181)	-2.09113	-1.97992
11	(0.872302 , 1.25659 , 0.436809)	-2.13187	-2.04741
12	(0.882501 , 1.25849 , 0.437226)	-2.16421	-2.09906
13	(0.890068 , 1.25819 , 0.440973)	-2.18991	-2.1391
14	(0.89498 , 1.25711 , 0.44647)	-2.21034	-2.17043
15	(0.897538 , 1.25624 , 0.452348)	-2.22663	-2.19511
16	(0.898242 , 1.25608 , 0.457643)	-2.23964	-2.21465
17	(0.897678 , 1.25674 , 0.461838)	-2.25002	-2.23016
18	(0.89642 , 1.25805 , 0.464808)	-2.25826	-2.24248
19	(0.894951 , 1.25971 , 0.466703)	-2.26478	-2.25226
20	(0.893619 , 1.26138 , 0.467806)	-2.26992	-2.26
21	(0.892629 , 1.26283 , 0.468413)	-2.27399	-2.26613
22	(0.89206 , 1.26391 , 0.468758)	-2.27722	-2.27099
23	(0.891897 , 1.26459 , 0.468992)	-2.2798	-2.27485

In the above tables we have seen that for $h = \sum_{i=1}^{m} -Log[-g[i]]$ the rate of convergence and the local minimum values of f are better than that of $h = \sum_{i=1}^{m} -\frac{1}{g[i]}$ and $h = \sum_{i=1}^{m} \frac{1}{g[i]^2}$, and $h = \sum_{i=1}^{m} -\frac{1}{g[i]}$ is better than that of $h = \sum_{i=1}^{m} \frac{1}{g[i]^2}$. The corresponding number of iterations and values of f for $h = \sum_{i=1}^{m} -Log[-g[i]]$, $h = \sum_{i=1}^{m} -\frac{1}{g[i]}$, and $h = \sum_{i=1}^{m} \frac{1}{g[i]^2}$ are respectively

12&-2.28925, 17&-2.28506, and 23&-2.2798. The possible suggestions for this difference is that as the value of $g[i]$ approaches zero, i.e. $-1 \leq g[i] \leq 0$, the values of $h=\sum_{i=1}^{m}-Log[-g[i]]$ is less than that of $h=\sum_{i=1}^{m}-\frac{1}{g[i]}$ and this is less than that of $h=\sum_{i=1}^{m}\frac{1}{g[i]^2}$ and hence as the constraint boundary is approached the value of the penalty term will be smaller for $h=\sum_{i=1}^{m}-Log[-g[i]]$ than the two with the same termination criteria, ε, δ.

Therefore, the rate of convergence and the local minimum value using $h=\sum_{i=1}^{m}-Log[-g[i]]$ is better than these penalty functions.

6. Summary and Conclusion

In chapter two and three we have seen the theoretical result of the penalty method. In chapter four computational aspects are considered and in chapter five numerical results and the effect of different choice of barrier functions are presented.

The rate of convergence and local minimum value of f is affected by the type of the barrier function provided that the same termination parameters are used. The choice of the parameters $\varepsilon, d, c, \delta$ depends on the nature of the problem and the interest of approximations for the results, i.e. if we want to have a best result we have to make ε, c, δ so small and d too large, but in this case the number of iteration will be large, hence the rate of convergence is low, but the values of the objective function will be approximately equal to the local minimum value nearest to the starting point in the descent direction.

```
Remove["Global`*"];
(* n is the dimension of the space *)
(* m is the number of constraints *)
(* f is the objective function *)
(* g[i] is the constraint function that defines the feasible set S *)
(* All constraint functions should be in the form " g[i] ≤ 0 " *)
(* d is the penalty parametre *)
(* c is the multiplier of the penalty parametre *)
(* ϵ is the termination scalar for the interior method *)
(* δ is a constant for the the termination of unconstrained problems *)
(* var and var1 is a vector of variables that defines the functions f and g[i] *)
(* x0 is an interior feasible starting point *)
(* bt is the value of the barier term *)

(*-----------------------------------------------*)

(* Inputs*)
n = 3;
m = 4;
f = -x1 * x2^2 * e^x3;
g[1] = x1^2 + x2^2 + e^x3 - 4;
g[2] = -x1;
g[3] = -x2;
g[4] = -x3;
d = 1.0 ;  c = 0.5;
ϵ = 0.005; δ = 0.1;
h = Sum[-Log[-g[i]], {i, 1, m}];
x0 = {0.5, 0.5, 1.0};

(*-----------------------------------------------------*)

f1[z_] := f /. Table[xi → z[[i]], {i, 1, n}];
h1 = IdentityMatrix[n];
If[n ≠ Length[x0], Print[" The length of the initial point is not correct"]; Abort[]];
var = Table[xi, {i, 1, n}];
var1 = Transpose[Prepend[Table[{"," xi}, {i, 2, n}], {x1}]] // MatrixForm;

(*.............................................*)

Print["The dimension of the space is : ", n];
Print["The number of constraints are : ", m];
Print["Objective function :f", var1, " = ", f];
Print["Constraints:"];
Print["Penalty parameter: d = ", d, ",", "multiplier:c = ", c];
Print["Constants for break off criterions: ϵ = ", ϵ, ",", "  δ = ", δ];
Print["Starting point: x0 = ", x0];
Print["Starting matrix : h1 = ", h1 // MatrixForm];
```

```
(*---------------------------------------------------------------------*)

(*some setings*)

xx = x0;
y[0] = x0;
x[0] = x0;
bt = ε + 1;
v = 1;
p = f + d * h;
p2[x0] = p /. Table[xi → xx[[i]], {i, 1, n}];
Print["penalty function:p", var1, " = ", p];
Print["f(x0)  = ", f /. Table[xi → xx[[i]], {i, 1, n}]];
Print["p(x0)  = ", p /. Table[xi → xx[[i]], {i, 1, n}]];

(*.......................................*)

(*+++++++++++++++++++++++++++++++++++++++++++++*)

(*iteration for the penalty steps*)

Print[""];
Print["Begin of iterations"];
While[bt > ε,
  p = f + d * h;
  grad = Table[D[p, xi], {i, 1, n}];
  norm = Sum[grad[[j]]^2, {j, 1, n}];
  q = norm /. Table[xi → xx[[i]], {i, 1, n}];
  k = 1;
  Print["The starting point is : x  = ", xx];

  (*...........................*)

  (*iteration for solving the unconstrained penalty problem (pk) *)

  While[q > δ,
   β =.; bb =.; Clear[ω]; Clear[φ]; Clear[u];
   s = grad /. Table[xi → xx[[i]], {i, 1, n}];
   s1 = -h1.s;
   s2 = Table[{s1[[i]]}, {i, 1, n}];
   s2 // MatrixForm;
   ω[β_] = xx + β * s1;
   φ[β_] = p /. Table[xi → ω[β][[i]], {i, 1, n}];
   u[β_] = D[φ[β], β];
   r = FindRoot[u[β] == 0, {β, 0}];
   bb = β /. r;
   xx = xx + bb * s1;
   q = norm /. Table[xi → xx[[i]], {i, 1, n}];
   q1 = grad /. Table[xi → xx[[i]], {i, 1, n}];
```

```
  q2 = q1 - s;
  m1 = bb * (s2.Transpose[s2]) / (s1.q2);
  m1 // MatrixForm;
  a1 = h1.q2;
  b = q2.a1;
  a2 = Table[{a1[[i]]}, {i, 1, n}];
  a2 // MatrixForm;
  n1 = - (a2.Transpose[a2]) / b;
  h1 = h1 + m1 + n1;
  k = k + 1];

 (*... ...............................*)

 (*Results*)

 kk = k;
 y[v] = xx;
 Print["The no. of iterations taken to solve the problem p[", v, "] is : ", kk];

 (*++++++++++++++++++++++++++++++++++++++++++++++*)

 x[v] = y[v];
 p1[z_] := p /. Table[x_i → z[[i]], {i, 1, n}];
 Print["x(", v, ") = ", x[v]];
 Print["f(x(", v, ")) = ", f1[x[v]]];
 Print["p(x(", v, "))=", p1[x[v]]];
 w[v] = p1[x[v]];
 x[-1] = x[0];
 cw[1, v] = {x[v][[1]], ","};
 cw[r1_, v] := Append[Append[cw[r1 - 1, v], x[v][[r1]]], ","];

 (*... .......................................*)

 bf = h /. Table[x_i → xx[[i]], {i, 1, n}];
 Print["The value of the barrier function is: ", bf];
 bt = d * bf;
 Print["bt = ", bt];
 d = c * d ;
 Print["d is ", d];
 v = v + 1;
 Print["........................"]];

(*... .............................................*)

Print["end of iterations"];
Print[""];
v1 = v;
Print["Total number of iteration steps:    v1 = ", v1 - 1];
Print[""];
Print["Approximated solution of the  optimization problem (p):"];
Print["x* = ", xx];
```

```
Print[""];
Print["minimal value :f(x*) = ", f /. Table[xi -> xx[[i]], {i, 1, n}]];
Print["minimal value :p(x*) = ", p /. Table[xi -> xx[[i]], {i, 1, n}]];
Print["the difference of p and f at the solution is ",
  ((p /. Table[xi -> xx[[i]], {i, 1, n}]) - (f /. Table[xi -> xx[[i]], {i, 1, n}]))];

(*++++++++++++++++++++++++++++++++++++++++++++++++++*)

(* Results in Table Form for n≥2  *)

Print["Results of iteration steps in table form"];
Print[""];
init = {0, {x0} // MatrixForm, f1[x0], p2[x0]};
Print[TableForm[
   Prepend[
    Prepend[
     Prepend[
      Table[
       {v, MatrixForm[{Append[cw[n - 1, v], x[v][[n]]]}], f1[x[v]], w[v]}, {v, 1, v1 - 1}],
      init],
     {"--", "--------------------------------", "---------", "---------"}],
    {"k", "                    x*(k)", " f(x*(k))", " p(x*(k))"}]]];
```

Results of iteration steps in table form

k	$x^*(k)$	$f(x^*(k))$	$p(x^*(k))$
--	--------------------------------	---------	---------
0	(0.5 0.5 1.)	-0.339785	1.29277
1	(0.81832 , 1.05634 , 0.524431)	-1.54273	-0.107235
2	(0.844511 , 1.13578 , 0.524096)	-1.83995	-0.907037
3	(0.864349 , 1.18644 , 0.520789)	-2.04812	-1.43613
4	(0.876192 , 1.21725 , 0.511571)	-2.16536	-1.77804
5	(0.88323 , 1.23662 , 0.500121)	-2.22714	-1.99084
6	(0.887606 , 1.24876 , 0.489632)	-2.25851	-2.11875
7	(0.890422 , 1.25613 , 0.481602)	-2.27417	-2.19348
8	(0.892226 , 1.2604 , 0.476206)	-2.28196	-2.23623
9	(0.893331 , 1.26274 , 0.47296)	-2.28584	-2.26029
10	(0.893956 , 1.26394 , 0.471209)	-2.28778	-2.27366
11	(0.894273 , 1.26452 , 0.470375)	-2.28876	-2.28102
12	(0.89441 , 1.26476 , 0.470041)	-2.28925	-2.28504

Appendix result

In the following we give the table for the results of the example obtained from the program

For Example 1

k	x*(k)	f(x*(k))	p(x*(k))
0	(0. 1.)	-12.	-10.
1	(2.73348 , -1.34962)	-27.8776	-25.1298
2	(2.76744 , -1.25897)	-28.9466	-26.811
3	(2.81321 , -1.20446)	-29.7933	-28.123
4	(2.84984 , -1.16115)	-30.461	-29.1508
5	(2.87937 , -1.12698)	-30.9884	-29.958
6	(2.90332 , -1.10021)	-31.4055	-30.5934
7	(2.9227 , -1.07921)	-31.7357	-31.0947
8	(2.93832 , -1.06269)	-31.9973	-31.4906
9	(2.95086 , -1.04967)	-32.2046	-31.8037

For example 2

k	x*(k)	f(x*(k))	p(x*(k))
0	(0.1 0.1 3.)	4.041	25.1849
1	(0.371195 , 1.68682 , 2.34497)	5.65255	10.3631
2	(0.248486 , 1.53049 , 2.0429)	4.42112	7.65041
3	(0.168245 , 1.43152 , 1.8276)	3.51322	5.76374
4	(0.112689 , 1.40989 , 1.73244)	2.89725	4.45524
5	(0.0781198 , 1.39097 , 1.63925)	2.46243	3.54741
6	(0.0545844 , 1.40998 , 1.56591)	2.14863	2.91066
7	(0.0390166 , 1.42178 , 1.51765)	1.93776	2.46764
8	(0.0272647 , 1.40157 , 1.49682)	1.79229	2.15823
9	(0.0190656 , 1.41422 , 1.46925)	1.67679	1.93744
10	(0.013516 , 1.41587 , 1.45096)	1.59854	1.7835
11	(0.00951906 , 1.41457 , 1.43996)	1.54412	1.67494
12	(0.00668531 , 1.41357 , 1.43294)	1.50621	1.59841
13	(0.00474842 , 1.41496 , 1.42734)	1.47944	1.5444
14	(0.00332343 , 1.41426 , 1.4237)	1.46019	1.5062

For Example 3

k	x*(k)	f(x*(k))	p(x*(k))
--	--------------------	---------	---------
0	(1.1 0.1)	0.103333	200.103
1	(2.29091 , 1.29036)	1.84585	3.04651
2	(2.15231 , 0.9737)	1.41631	2.32024
3	(2.00185 , 0.763752)	1.09832	1.77598
4	(1.88117 , 0.599083)	0.857904	1.36718
5	(1.65779 , 0.499528)	0.643758	1.03868
6	(1.54063 , 0.402629)	0.500056	0.799744
7	(1.47597 , 0.312926)	0.388442	0.616977
8	(1.41008 , 0.246259)	0.302314	0.477598
9	(1.34113 , 0.195735)	0.234525	0.370051
10	(1.28326 , 0.156207)	0.182952	0.287339
11	(1.23866 , 0.124119)	0.143105	0.223641
12	(1.20163 , 0.0984928)	0.112044	0.174389
13	(1.16957 , 0.0782171)	0.0878013	0.136198

For Example 4

k	x*(k)	f(x*(k))	p(x*(k))
--	------------------------------	---------	---------
0	(0. 0.)	0.	2.50877
1	(0.438969 , -0.269086)	-0.320574	1.67274
2	(0.501064 , -0.334573)	-0.475299	0.634493
3	(0.569706 , -0.387595)	-0.6411	0.032402
4	(0.624664 , -0.422843)	-0.776338	-0.343634
5	(0.664013 , -0.446026)	-0.876554	-0.589242
6	(0.69094 , -0.461927)	-0.948721	-0.754002
7	(0.709039 , -0.473177)	-1.00016	-0.866419
8	(0.72126 , -0.481191)	-1.03667	-0.94398
9	(0.729675 , -0.486846)	-1.06252	-0.997895
10	(0.735592 , -0.490793)	-1.08081	-1.03556
11	(0.739802 , -0.493532)	-1.09375	-1.06198
12	(0.742799 , -0.495438)	-1.1029	-1.08055
13	(0.744923 , -0.496775)	-1.10938	-1.09362
14	(0.74642 , -0.497718)	-1.11395	-1.10284
15	(0.747473 , -0.498386)	-1.11719	-1.10934

For Example 5

k	$x^*(k)$	$f(x^*(k))$	$p(x^*(k))$
0	(1 5)	386	387.4
1	(-2.8068 , 3.13357)	0.000292514	0.542673
2	(-2.8068 , 3.13357)	0.000292514	0.271483
3	(-2.8068 , 3.13357)	0.000292514	0.135888
4	(-2.8068 , 3.13357)	0.000292514	0.0680901
5	(-2.8068 , 3.13357)	0.000292514	0.0341913
6	(-2.8068 , 3.13357)	0.000292514	0.0172419
7	(-2.8068 , 3.13357)	0.000292514	0.00876722
8	(-2.8068 , 3.13357)	0.000292514	0.00452986

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DECLARATION

I hereby declare that this is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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This thesis has been submitted for examination with my approval as University advisor.

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