Discrete Dynamical Systems

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Declaration

I, Meseret Cherkos declare that this project work has been composed by me and that no part of the project work has formed the basis for the award of any Degree, Diploma, Associate ship, Fellow ship, or any other similar title to me.

Meseret Cherkos

Signature _____________

Permission

This is to certify that this project is compiled by Meseret Cherkos in the department of mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluations by examiners and eventual defense.

Dr. Tadesse Abdi

Signature ________________

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Abstract

This project is based on equations that allow us to compute the value of a function recursively from a given set of values. In the main, Hartman-Grobman theorem is proved by using different lemmas and other theorems. Population modeling with difference equations by converting to discrete version is briefly discussed. The notions of fixed and periodic points of dynamical systems are considered.

Introduction

A dynamical system is a concept of in mathematics where a fixed rule describes the time dependence of a point in geometrical space. For example the mathematical models that describe: the swinging of a clock pendulum, the flow of water in a pipe and the number of fish each spring in a lake.

At any given time a dynamical system has a state given by a set of real numbers (a vector) which can be represented by a point in appropriate state space (a geometrical manifold). Small changes in the state of the system correspond to small changes in the numbers.

The evolution rule of the dynamical system is a fixed rule that describes what future state follow from the current state .The rule is deterministic in other words, for a given time interval only one future state follows the current state.

Technically, a dynamical system is a smooth action of the real's or integers on another object (usually a manifold) .when the real's are acting the system is called continuous dynamical system, and when the integers are acting the system is called a discrete dynamical system.

Mathematical computations frequently are based on equations that allow us to compute the value of a function recursively from a given set of values. Such an equation is called a "difference equation" or "recurrence equation". These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, dynamical systems, economics, biology and other fields.

The paper consists of three chapters.

In chapter 1, some definitions and preliminary results are collected. They are systematically used in the next chapters. In particular, we recall some basic notions and results of the theory of operators in Banach spaces very briefly.

In chapter 2, we discussed the concept of difference equations with discrete time and the difference operators. And finally in the last chapter, we accordingly have studied fixed and periodic solutions of dynamical systems.

CHAPTER ONE

Preliminaries

1.1 Definitions

Banach space

In this section we recall very briefly some basic notions of the theory of banach space.

Denote the set of complex numbers by $\mathbb{C}$ and let the set of real numbers by $\mathbb{R}$. A linear space X over $\mathbb{C}$ is called (complex) linear normed space if for any $x\epsilon\chi$ a non-negative number $\|\chi\|_x = \|\chi\|$ is defined, called the norm of χ, having the following properties:

i) $\|\chi\| = 0,$ if and only if $\chi = 0$,

ii) $\|\lambda\chi\| = |\lambda|\|\chi\|$ for $\lambda\epsilon\mathbb{C}$ and $\chi\epsilon X$,

iii) $\|\chi + y\| \leq \|\chi\| + \|y\|$ for $\chi, y\epsilon X$ (triangular inequalities).

A sequence $\{h_n\}$ of elements of χ converges strongly (in the norm) to $h\epsilon\chi$ if

$$\lim_{n\to\infty}\|h_n - h\| = 0.$$

A sequence $\{h_n\}$ of elements of χ is called the fundamental (Cauchy) one if

$$\|h_n - h_m\| \to 0 \text{ as } m, n \to \infty.$$

If any fundamental sequence converges to an element of χ, then χ is called a (complex) Banach spaces.

Lipschitz conditions A map $L: E \longrightarrow E$ is lipschitz if there exists $k < \infty$ such that

$$\|L(z_1) - L(z_2)\| \leq k\|z_1 - z_2\| \text{ for all } z_1, z_2\, \epsilon E.$$

Definition of Conjugacy let M be a differentiable manifolds and $f, g: M \longrightarrow M$ is a diffeomorphism. We say that f and g topologically conjugate if there is a homeomorphism $h: M \longrightarrow M$ such that $hof = goh$.

Theorem (Intermediate value theorem) let f a continuous function defined on the interval $[a, b]$. If $f(a) > 0$ or $f(b) < 0$, then there exists c in $[a, b]$ such that $f(c) = 0$.

1.2 List of Symbols

${W^S}_{loc}(p)$: The local stable set of the periodic point p.

${W^U}_{loc}(p)$: The local unstable set of the periodic point p.

$C_b^0(\mathbb{R}^n)$: For the banach space of bounded continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

E^S : The stable eigenspaces.

E^U : The unstable eigenspaces.

$L_\mu(\chi)$: Discrete logistic equation.

$W^U(p)$: The unstable set of the periodic point p.

$W^S(p)$: The stable set of the periodic point p.

$W^S(\infty)$: The stable set of infinity.

$\|.\|$: The Euclidean norm.

δ : Central difference operator.

∇: Backward difference operator.

Δ: Forward difference operator.

E : The shift operator.

Δ_h : The difference operator.

$\mathbb{R}$ **:** Set of real numbers**.**

$\mathbb{C}$: Set of complex numbers**.**

f^n **:** The n^{th} iterate of the function of f.

CHAPTER TWO

Discrete Models

Introduction

Computing with mathematical formula sometimes involve an expression that allow evaluation of values of a function recursively from a given set of values

Example Consider a sequence of real $\{b_n\}_{n\geq 0}$.

If $\Delta b_0 = b_1 - b_0, \Delta b_1 = b_2 - b_1, \Delta b_2 = b_3 - b_2, \; . \; . \; . \; .$

Then we have $b_{n+1} = b_n + \Delta b_n \curvearrowright$ Recurrence relation

2.1 Difference equations and Difference operators

2.1.1 Difference equations

If a behavior \real world scenario occurs at discrete time lapses then the process can be modeled using difference equations.

Think of a population that has discrete generations.

Let $\chi_{n+1} \rightarrow$ the size of $(n+1)^{st}$ generation.

Suppose χ_{n+1} is a function of the n^{th} generation (assumption).

$$\chi_{n+1} = f(\chi_n), \;\; n\epsilon\{1,2,. \; . \; . \; . \; .\}. \qquad (2.1)$$

And hence, the sequence$\{\chi_n\}_{n\geq 0}$.

i.e. $\chi_0, \; f(\chi_0), \; f\big(f(\chi_0)\big), \; f\Big(f\big(f(\chi_0)\big)\Big),. \; . \; .$

Convention $f^2(\chi_0) = f\big(f(\chi_0)\big), f^3(\chi_0) = f\Big(f\big(f(\chi_0)\big)\Big), \;\; . \; . \; .$

$f^1(\chi_0) = f(\chi_0), \; \chi_0 = f^0(\chi_0)$.

The recurrence relation $\chi_{n+1} = f(\chi_n), \;\; n\epsilon\{1,2,. \; . \; . \; . \; .\}$ is called difference equation.

Definition i) A difference equation is said to be linear if f is a linear function of χ_n.

ii) A linear difference equation is said to be first order it has the form $\chi_{n+1} = a_n\chi_n + b_n$.

iii) A first order linear difference equation is said to be homogenous if

$b_n = 0$ for all n, otherwise it is non-homogenous.

iv) We say that $\chi_n = g(n)$ is a solution (analytical) of a difference equation if and only if it satisfies the difference equation.

Example $\chi_{n+1} = a\chi_n$ iteration $g(n) = a^n\chi_0$

Theorem 2.1 Let $a_n \epsilon \mathbb{R}^{mxm}$. The solution of homogenous first order linear difference equation.

$$\chi_{n+1} = a_n\chi_n \ ,\chi_0 \text{ is given} \tag{2.2}$$

for $n \geq 0$ is given by

$$\chi_n = \prod_{i=0}^{n-1} a_i \, \chi_0$$

Proof (induction)

i) For $n = 1$, $\chi_2 = a_1\chi_1 = a_1(a_0\chi_0) = (a_1a_0)\chi_0$.

ii) Assume that it holds true for $n = k$, $i.e$ $\chi_k = \left(\prod_{i=0}^{k-1} a_i\right)\chi_0$.

iii) We have to show for $n = k + 1$,

$i.e$ $\chi_{k+1} = a_k\chi_k = \chi_k a_k = \left(\prod_{i=0}^{k-1} a_i\right)\chi_0 a_k = \left(\prod_{i=0}^{k} a_i\right)\chi_0$

Intuitive approach ⟶ iteration

$$\chi_1 = a_0\chi_0 \ .$$

$$\chi_2 = a_1\chi_1 = a_1(a_0\chi_0) = (a_1a_0)\chi_0.$$

$$\chi_3 = a_2\chi_2 = a_2((a_1a_0)\chi_0) = (a_2a_1a_0)\chi_0.$$

$$\vdots$$

$$\chi_n = a_{n-1}a_{n-2}a_{n-3}. \quad . \quad . a_1 a_0 \chi_0 = (\prod_{i=0}^{n-1} a_i)\chi_0$$

Let us turn to the inhomogeneous system

$$\chi_{n+1} = a_n \chi_n + g(n), \;\; \chi_0 \text{ is given} \tag{2.3}$$

Where $a_n \epsilon \mathbb{R}^{mxm}$ and $g(n) \epsilon \mathbb{R}^m$. Since the difference of two solutions of the inhomogeneous system (2.3) satisfies the corresponding homogenous system (2.2), it suffices to find one particular solution.

Theorem 2.2 The solution of the inhomogeneous initial value problem is given by

$$\chi_n = [\prod_{i=0}^{n-1} a_i]\chi_0 + \sum_{j=0}^{n-1}[\prod_{i=j+1}^{n-1} a_i]g(j)$$

Where $\prod_{i=0}^{n-1} a_i$ is the solution of the corresponding homogenous systems.

Proof (by induction)

Problem Find an explicit formula for the Fibonacci number defined as

$$X(m) = X(m-1) + X(m-2), \;\; X(1) = X(0) = 1$$

Solution Explicitly recurrence yields the equation

$X(2) = X(1) + X(0) = 1 + 1 = 2$

$X(3) = X(2) + X(1) = 2 + 1 = 3$

$X(4) = X(3) + X(2) = 3 + 2 = 5$

$X(5) = X(4) + X(3) = 5 + 3 = 8$

$X(6) = X(5) + X(4) = 8 + 5 = 13$

$X(7) = X(6) + X(5) = 13 + 8 = 21$ Etc

We obtain the sequence of Fibonacci numbers which begins

1,1,2,3,5,8,13,21,34, ...

Example Solve the difference equation

$$\chi_{n+1} = (n+1)\chi_n + 2^n(n+1)!, \quad \chi_0 = 1$$

Solution $\chi_n = \prod_{i=0}^{n-1}(i+1)\,\chi_0 + \sum_{r=0}^{n-1}[\prod_{i=r+1}^{n-1}(i+1)]\,2^r(r+1)!$ (by theorem 2.2)

$$= \prod_{i=1}^{n} i + \sum_{r=0}^{n-1}[\prod_{i=r+1}^{n-1}(i+1)]\,(r+1)2^r!.$$

$$= n! + \sum_{r=0}^{n-1}[\prod_{i=r+1}^{n-1}(i+1)]\,(r+1)!\,2^r.$$

$$= n! + \sum_{r=0}^{n-1}[\prod_{i=r+2}^{n} i]\,(r+1)!\,2^r.$$

$$= n! + \sum_{r=0}^{n-1} n!\,2^r.$$

$$= n!\,(1 + \sum_{r=0}^{n-1} 2^r).$$

Let $S = 1 + \sum_{r=0}^{n-1} 2^r$.

$\Longrightarrow S - 1 = \sum_{r=0}^{n-1} 2^r = 1 + 2 + 2^2 + 2^3 + \ . \ . \ . \ . \ . + 2^{n-2} + 2^{n-1}.$ (*)

$\Longrightarrow 2(S-1) = \sum_{r=0}^{n-1} 2^{r+1} = 2 + 2^2 + 2^3 + \ . \ . \ . \ . \ . + 2^{n-2} + 2^{n-1} + 2^n.$ (**)

Subtract (*) from (**).

$\Longrightarrow S - 1 = 2^n - 1.$

$\Longrightarrow S = 2^n.$

$\therefore \chi_n = 2^n n!$ is the solution.

Special case

1) If $\chi_{n+1} = a\chi_n + g(n)$, a is fixed show that $X_n = a^n\chi_o + \sum_{r=0}^{n-1} a^{n-r-1}g(r)$ is the solution.
2) If $\chi_{t+1} = a\chi_t + b$ show that the solution is given by

$$\begin{cases} a^t\chi_0 + b\frac{a^t-1}{a-1}, if\ a \neq 1 \\ \chi_0 + bt, \quad if\ a = 1 \end{cases}$$

Proof 1) by induction

i) For $n = 1$, $\chi_2 = a\chi_1 + g(1)$

ii) Assume that it holds true for $n = k, i.e\ \chi_k = a^k\chi_0 + \sum_{r=0}^{k-1} a^{k-r-1}g(r)$

iii) We have to show for $n = k + 1$,

$$i.e\ \chi_{k+1} = a\chi_k + g(k) = a\left(a^k\chi_0 + \sum_{r=0}^{k-1} a^{k-r-1}g(r)\right) + g(k)$$

$$= a^{k+1}\chi_0 + \sum_{r=0}^{k-1} a^{k-r}g(r) + g(k)$$

$$= a^{k+1}\chi_0 + \sum_{r=0}^{k} a^{k-r}g(r)$$

Intuitive approach ⟶ iteration

$\chi_1 = a\chi_0 + g(0)$.

$\chi_2 = a\chi_1 + g(1) = a^2\chi_0 + ag(0) + g(1)$

$\chi_3 = a\chi_2 + g(2) = a^3\chi_0 + a^2g(0) + ag(1) + g(2)$

⋮

$\chi_n = a\chi_{n-1} + g(n-1)$

$$= a^n\chi_0 + a^{n-1}g(0) + a^{n-2}g(1) + a^{n-3}g(2) + \ldots + a^2g(n-3) + ag(n-2) + g(n-1)$$

$$= a^n\chi_o + \sum_{r=0}^{n-1} a^{n-r-1}g(r)$$

2) Consider first-order linear difference equation

$$\chi_{t+1} = a\chi_t + b \quad , t = 0, 1, 2, 3.. \tag{2.4}$$

where the state variable at time t, χ_t, is one dimensional, $\chi_t \epsilon \mathbb{R}$ the parameters a and b are constant across time (i.e., the dynamical system is autonomous), $a, b\epsilon\mathbb{R}$ and the initial value of the state variable at time 0, χ_0 is given (2.4).

A solution to the difference equation $\chi_{t+1} = a\chi_t + b$ is a trajectory (or an orbit), $\{\chi_t\}_{t=0}^{\infty}$ that satisfies this equation at any point in time. It relates the value of the state variable at time t, χ_t to the initial condition χ_0 and to the parameters a and b.

Given the value of the state variable at time 0,χ_0, the dynamical system

$\chi_{t+1} = a\chi_t + b$ Implies that the value of the value of the state variable at time 1, χ_1 is

$\chi_1 = a\chi_0 + b.$

Given the value of the state variable at time 1,χ_1, the dynamical system

$\chi_{t+1} = a\chi_t + b$ Implies that the value of the value of the state variable at time 2, χ_1 is

$\chi_2 = a\chi_1 + b = a(a\chi_0 + b) + b = a^2\chi_0 + ab + b.$

Similarly, the value of the state variable at time 3, 4. . . t is

$$\chi_3 = a\chi_2 + b = a^3\chi_0 + a^2b + ab + b.$$

$$\chi_4 = a\chi_3 + b = a^4\chi_0 + a^3b + a^2b + ab + b.$$

$$\vdots$$

$$\chi_t = a^t\chi_0 + a^{t-1}b + a^{t-2}b + \dots + ab + b.$$

Hence

$$\chi_t = a^t\chi_0 + \textstyle\sum_{i=0}^{t-1} a^i b. \tag{2.5}$$

Since $\sum_{i=0}^{t-1} a^i$ is the sum of the geometric series $\{1, a, a^2, a^3, \dots, a^{t-1}\}$ whose factor is a, it follows that

$$\sum_{i=0}^{t-1} a^i = \begin{cases} \frac{1-a^t}{1-a}, if\ a \neq 1 \\ t,\ if\ a = 1 \end{cases}$$

And therefore,

$$\chi_t = \begin{cases} a^t\chi_0 + b\frac{a^t-1}{a-1}, if\ a \neq 1 \\ \chi_0 + bt, \quad if\ a = 1 \end{cases} \tag{2.6}$$

Definition i) a linear difference equation of second order is given by

$$\chi_{n+2} = a_n \chi_{n+1} + b_n \chi_n + c_n. \qquad (2.7)$$

ii) If $c_n = 0$, then we have homogenous case, otherwise non-homogeneous.

Consider a homogeneous linear difference equation of second order

$\chi_{n+2} = a_n \chi_{n+1} + b_n \chi_n$.Where a_n and b_n are real numbers (2.8)

$\Rightarrow \chi_{n+2} - a_n \chi_{n+1} - b_n \chi_n = 0.$

Let $\alpha = -a_n, \beta = -b_n \Rightarrow \chi_{n+2} + \alpha \chi_{n+1} + \beta \chi_n = 0.$

Claim $\chi_n = \lambda^n$ is a solution for a suitable λ.

Since $\chi_n = \lambda^n \Rightarrow \chi_{n+1} = \lambda^{n+1}$, $\chi_{n+2} = \lambda^{n+2}$.

We have $\chi_{n+2} + \alpha \chi_{n+1} + \beta \chi_n = 0 \Leftrightarrow \lambda^{n+2} + \alpha \lambda^{n+1} + \beta \lambda^n = 0$

$$\lambda^n (\lambda^2 + \alpha \lambda + \beta) = 0.$$

$$\lambda^n = 0 \text{ or } \lambda^2 + \alpha \lambda + \beta = 0.$$

For if $\lambda^n = 0$, we have $\chi_n = 0 = \chi_{n+1} = \chi_{n+2}$ (trivial solution).

$\lambda^2 + \alpha \lambda + \beta = 0$ (To get non- trivial solution).

$\Longrightarrow \lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$.

Case 1 $\alpha^2 - 4\beta > 0$.

$\Longrightarrow \lambda_1 \neq \lambda_2 \epsilon \mathbb{R}$ (Distinct real roots).

$\Longrightarrow \chi_n = C_1 {\lambda_1}^n + C_2 {\lambda_2}^n$ is a solution.

Case 2 $\alpha^2 - 4\beta = 0$.

$\Longrightarrow \lambda_1 = \frac{-\alpha}{2} = \lambda_2 \epsilon \mathbb{R}$ (Double root).

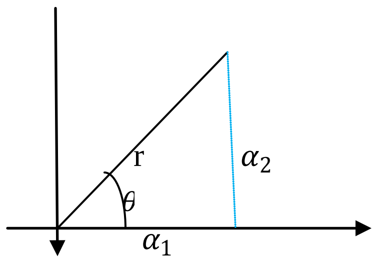
$\Longrightarrow \chi_n = (C_1 + nC_2)\lambda^n$ is a solution.

Case 3 $\alpha^2 - 4\beta < 0$.

$\Longrightarrow \lambda_1 = \frac{-\alpha \pm i\sqrt{4\beta - \alpha^2}}{2} = \overline{\lambda_2} \epsilon \mathbb{C}$ (Complex root)

Let $\alpha_1 = \frac{-\alpha}{2}$, $\alpha_2 = \frac{\sqrt{4\beta - \alpha^2}}{2}$, then $\lambda_1 = \alpha_1 + i\alpha_2 = \overline{\lambda_2}$.

Recall



$\Longrightarrow$ $\alpha_1 = r\cos(\theta)$ and $\alpha_2 = r\sin(\theta)$ then $\alpha_1 + i\alpha_2 = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ where $r = \sqrt{\alpha_1^2 + \alpha_2^2}$ and $\tan(\theta) = \frac{\alpha_2}{\alpha_1}$, $\theta = \tan^{-1}(\frac{\alpha_2}{\alpha_1})$.

$\Longrightarrow {\lambda_1}^n = (\alpha_1 + i\alpha_2)^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$ and

${\lambda_2}^n = (\alpha_1 - i\alpha_2)^n = r^n e^{-in\theta} = r^n(\cos(n\theta) - i\sin(n\theta))$.

Therefore $\chi_n = C_1{\lambda_1}^n + C_2{\lambda_2}^n = r^n(C_1\cos(n\theta) + C_2\sin(n\theta))$ is a solution.

2.1.2. Difference Operator

Let h be a non-zero real number and $f(x)$ be a function. When $f(x+h)$ and $f(x)$ are real numbers we call

$$\Delta_h f(\chi) = f(\chi + h) - f(\chi). \tag{2.9}$$

The first difference of f at χ with step h.

If $h = 1$, we simply write Δ and omit the subscript h. For example, in case of a sequence $\{\chi_n\}$ we have $\Delta\chi_n = \chi_{n+1} - \chi_{n.}$ (2.10)

Theorem2.3. Let χ_n and y_n be sequence and let C be any number. Then

i) $\Delta(\chi_n + y_n) = \Delta\chi_n + \Delta y_n$.

ii) $\Delta(C\chi_n) = C\Delta\chi_n$.

Proof i) $\Delta(\chi_n + y_n) = (\chi_{n+1} + y_{n+1}) - (\chi_n + y_n)$

$$= (\chi_{n+1} - \chi_n) + (y_{n+1} - y_n)$$

$$= \Delta\chi_n + \Delta y_n$$

ii) $\Delta(C\chi_n) = C\chi_{n+1} - C\chi_n = C(\chi_{n+1} - \chi_n)$

$$= C\Delta\chi_n.$$

Theorem2.4. Formula for the difference of particular functions and let a be a constant. Then,

a) $\Delta a^t = (a-1)a^t$ (Exponential sequence).

b) $\Delta \log(at) = \log(1 + \frac{1}{t})$ (Logarithmic sequence).

c) $\Delta \sin(at) = 2\sin(\frac{a}{2})\cos(at + \frac{a}{2})$.

d) $\Delta \cos(at) = -2\sin(\frac{a}{2})\sin\left(at + \frac{a}{2}\right)$.

Proof a) $\Delta a^t = a^{t+1} - a^t = a^t a - a^t = a^t(a-1)$

b) $\Delta \log(at) = \log\big(a(t+1)\big) - \log(at)$

$$= \log\left(\frac{a(t+1)}{at}\right) = \log(1 + \frac{1}{t})$$

c) $\Delta \sin(at) = \sin(at + a) - \sin(at) = \sin(at)\cos(a) + \sin(a)\cos(at) - \sin(at)$

$$= \sin(at)(\cos(a) - 1) + \sin(a)\cos(at)$$

But $\cos(a) = \cos(\frac{a}{2} + \frac{a}{2}) = \cos^2\left(\frac{a}{2}\right) - \sin^2(\frac{a}{2})$ and $\sin(a) = 2\sin(\frac{a}{2})\cos(\frac{a}{2})$

$$= \sin(at)\left(\cos^2\left(\frac{a}{2}\right) - \sin^2\left(\frac{a}{2}\right) - 1\right) + 2\sin(\frac{a}{2})\cos(\frac{a}{2})\cos(at)$$

$$= \sin(at)\left(-2\sin^2(\frac{a}{2})\right) + 2\sin(\frac{a}{2})\cos(\frac{a}{2})\cos(at)$$

$$= 2\sin(\frac{a}{2})\left(\cos(\frac{a}{2})\cos(at) - \sin(\frac{a}{2})\sin(at)\right)$$

$$= 2\sin(\frac{a}{2})\cos\left(a\left(t + \frac{1}{2}\right)\right)$$

d) $\Delta\cos(at) = \cos(at + a) - \cos(at) = \cos(at)\cos(a) - \sin(at)\sin(a) - \cos(at)$

$$= \cos(at)\,(\cos(a) - 1) - \sin(at)\sin(a)$$

But $\cos(a) = \cos(\frac{a}{2} + \frac{a}{2}) = \cos^2\left(\frac{a}{2}\right) - \sin^2(\frac{a}{2})$ and $\sin(a) = 2\sin(\frac{a}{2})\cos(\frac{a}{2})$

$$= \cos(at)\left(\cos^2\left(\frac{a}{2}\right) - \sin^2\left(\frac{a}{2}\right) - 1\right) - 2\sin(at)\sin(\frac{a}{2})\cos(\frac{a}{2})$$

$$= \cos(at)(-2\sin^2\left(\frac{a}{2}\right) - 2\sin(at)\sin(\frac{a}{2})\cos(\frac{a}{2})$$

$$= -2\sin(\frac{a}{2})\left(\cos(at)\sin\left(\frac{a}{2}\right) + \sin(at)\cos(\frac{a}{2})\right)$$

$$= -2\sin(\frac{a}{2})\sin(at + \frac{a}{2})$$

Example 1) the difference operator Δ can be applied to a function of several variables. Show that

a) $\Delta_n te^n = te^n(e - 1)$ and b) $\Delta_t te^n = e^n$.

2) Discrete analog of the fundamental theorem of calculus, show that

$$\sum_{i=r}^{n-1} \Delta\chi_i = \chi_n - \chi_r.$$

Solution 1.a) $\Delta_n te^n = te^{n+1} - te^n$

$$= te^n.e - te^n = te^n(e - 1)$$

b) $\Delta_t te^n = (t + 1)e^n - te^n$

$$= te^n + e^n\text{-}\, te^n = e^n$$

2) $\sum_{i=r}^{n-1} \Delta\chi_i = \Delta\chi_r + \Delta\chi_{r+1} + \Delta\chi_{r+2} + \ . \ . \ . \ . + \Delta\chi_{n-2} + \Delta\chi_{n-1}$

$$= \chi_{r+1} - \chi_r + \chi_{r+2} - \chi_{r+1} + \chi_{r+3} - \chi_{r+2} + \ . \ . \ . \ . + \chi_{n-1} - \chi_{n-2} + \chi_n - \chi_{n-1}$$

$$= -\chi_r + \chi_n = \chi_n - \chi_r.$$

2.1.2.1 Type of Difference operators

2.1.2.1.1 Forward Difference operators

Definition i) we define the forward difference operator**,** denoted by Δ, is defined as

$$\Delta f(x) = f(\chi + h) - f(\chi) \tag{2.11}$$

The expression $f(\chi + h) - f(\chi)$ gives the first forward difference of $f(\chi)$ and the operator Δ is called the first forward difference operator. Given the step size h , this formula uses the values at χ and h , the point at the next step. As it is moving in the forward direction, it is called the forward difference operator.

ii) The second forward difference operator, Δ^2, is defined as

$$\Delta^2 f(x) = \Delta(\Delta(f(\chi)) = \Delta f(\chi + h) - \Delta f(\chi). \tag{2.12}$$

We note that

$$\Delta^2 f(x) = \Delta f(\chi + h) - \Delta f(\chi).$$

$$= f(\chi + 2h) - f(\chi + h) - (f(\chi + h) - f(\chi))$$

$$= f(\chi + 2h) - 2f(\chi + h) + f(\chi)$$

In particular, for $\chi = \chi_k$ we get

$$\Delta y_k = y_{k+1} - y_k \text{ and } \Delta^2 y_k = \Delta y_{k+1} - \Delta y_k = y_{k+2} - 2y_{k+1} + y_k.$$

iii) The r^{th} forward difference operator, Δ^r, is defined as

$$\Delta^r f(\chi) = \Delta^{r-1} f(\chi + h) - \Delta^{r-1} f(\chi),\ r = 1, 2, 3,.\ .\ .\ . \tag{2.13}$$

with $\Delta^0 f(\chi) = f(\chi)$.

Thus the r^{th} forward difference at y_k uses the values at $y_k, y_{k+1}, y_{k+2}, \ . \ . \ . \ . y_{k+r}$.

2.1.2.1.2 Backward difference operator

Definition i) the first backward difference operator, denoted by ∇, is defined as

$$\nabla \mathrm{f}(\chi) = \mathrm{f}(\chi) - \mathrm{f}(\chi - h) \tag{2.14}$$

Given the step size h, note that this formula uses the values at χ and $\chi - h$, the point at the previous step. As it moves in the backward direction, it is called the backward difference operator.

ii) The r^{th} first backward difference operator, denoted by ∇^{r}, is defined as

$$\nabla^{\mathrm{r}} \mathrm{f}(\chi) = \nabla^{\mathrm{r}-1} \mathrm{f}(\chi) - \nabla^{\mathrm{r}-1} \mathrm{f}(\chi - h), \quad r = 1, 2, . \ \ . \ \ . \ \ . \tag{2.15}$$

with $\nabla^0 \mathrm{f}(\chi) = f(\chi)$

In particular, for $\chi = \chi_k$ we get

$\nabla y_k = y_k - y_{k-1}$ and $\nabla^2 y_k = y_k - 2y_{k-1} + y_{k-2}$.

Note that $\nabla^2 \mathrm{y_k} = \Delta^2 \mathrm{y_{k-2}}$.

2.1.2.1.3 Central difference operator

Definition i) the first central difference operator, denoted by δ, is defined as

$$\delta f(\chi) = f\left(\chi + \frac{h}{2}\right) - f\left(\chi - \frac{h}{2}\right). \tag{2.16}$$

ii) The r^{th} first central difference operator, denoted by δ^{r}, is defined as

$$\delta^{\mathrm{r}} \mathrm{f}(\chi) = \delta^{\mathrm{r}-1} f\left(\chi + \frac{h}{2}\right) - \delta^{\mathrm{r}-1} f\left(\chi - \frac{h}{2}\right) \tag{2.17}$$

with $\delta^0 f(\chi) = f(\chi)$.

Thus, $\delta^2 f(\chi) = f(\chi + \mathrm{h}) - 2f(\chi) + f(\chi - \mathrm{h})$. In particular, for $\chi = \chi_k$ defined $y_{k+\frac{1}{2}} = f\left(\chi_k + \frac{h}{2}\right)$ and $y_{k-\frac{1}{2}} = f\left(\chi_k - \frac{h}{2}\right)$ then $\delta y_k = y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}}$ and $\delta^2 y_k = y_{k+1} - 2y_k + y_{k-1}$.

2.1.2.1.4 Shift and identity operators

Definition (shift/identity operators)

i) The shift operator denoted by E, is the operator which shifts the value at the next point with step h i.e.

$$Ef(\chi) = f(\chi + h) \tag{2.18}$$

Thus, $Ey_i = y_{i+1}, E^2 y_i = y_{i+2}$ and $E^k y_i = y_{i+k}$

ii) The identity operator denoted by I, is defined as

$$\mathrm{I}f(\chi) = f(\chi) \tag{2.19}$$

Theorem 2.5 1) $\Delta = E - \mathrm{I}$.

2) $E(a\chi_n + by_n) = aE\chi_n + bEy_n$ for all $a, b \epsilon \mathbb{R}$

3) $E^k \chi_n = \chi_{n+k}$.

4) $\Delta\big(y(t)z(t)\big) = y(t)\Delta z(t) + Ez(t)\Delta y(t)$.

5) $\Delta\left(\frac{y(t)}{z(t)}\right) = \frac{z(t)\Delta y(t) - y(t)\Delta z(t)}{z(t)Ez(t)}$.

6) $E^m \chi_n = \sum_{k=0}^{m} \binom{m}{k} \Delta^{m-k} \chi_n$.

Proof 1) Let $Ef(\chi) = f(\chi + h)$ and $\mathrm{I}f(\chi) = f(\chi)$.

$(E - \mathrm{I})f(\chi) = Ef(\chi) - \mathrm{I}f(\chi) = f(\chi + h) - f(\chi) = \Delta f(x)$

$\therefore \Delta = E - \mathrm{I}$.

2) $E(a\chi_n + by_n) = a\chi_{n+1} + by_{n+1} = aE\chi_n + bEy_n$

$E(a\chi_n + by_n) = aE\chi_n + bEy_n$.

3) Proof by induction

i) For $k = 1,\ E\chi_n = \chi_{n+1}$.

ii) Assume that it holds for $k = m,\ i.e.\ E^m\chi_n = \chi_{n+m}$.

iii) We have to show for $k = m + 1$,

$i.e.\ E^{m+1}\chi_n = EE^m\chi_n = E(\chi_{n+m}) = \chi_{n+m+1}$.

Intuitive approach ⟶ iteration

$E\chi_n = \chi_{n+1}$.

$E^2\chi_n = \chi_{n+2}$.

$E^3\chi_n = \chi_{n+3}$.

$\vdots$

$E^m\chi_n = \chi_{n+m}$.

4) $\Delta\big(y(t)z(t)\big) = y(t+1)z(t+1) - y(t)z(t)$.

$= y(t+1)z(t+1) - y(t)z(t+1) + y(t)z(t+1) - y(t)z(t)$

$= z(t+1)\big(y(t+1) - y(t)\big) + y(t)\big(z(t+1) - z(t)\big)$

$= z(t+1)\Delta y(t) + y(t)\Delta z(t)$

$= Ez(t)\Delta y(t) + y(t)\Delta z(t)$.

5) $\Delta\left(\frac{y(t)}{z(t)}\right) = \frac{y(t+1)}{z(t+1)} - \frac{y(t)}{z(t)}$

$$= \frac{y(t+1)z(t) - y(t)z(t+1)}{z(t)z(t+1)}$$

$$= \frac{y(t+1)z(t) - y(t)z(t) + y(t)z(t) - y(t)z(t+1)}{z(t)z(t+1)}$$

$$= \frac{z(t)\big(y(t+1) - y(t)\big) - y(t)(z(t+1) - z(t)}{z(t)z(t+1)}$$

$$= \frac{z(t)\Delta y(t) - y(t)\Delta z(t)}{z(t)Ez(t)}$$

6) $E^m \chi_n = (\Delta + \mathrm{I})^m \chi_n$ by Binomial theorem

$= \sum_{k=0}^{m} \binom{m}{k} \Delta^{m-k} \chi_n$.

Non-homogeneous linear difference equation of the second order

Consider the difference equation

$$\chi_{n+2} + \alpha\chi_{n+1} + \beta\chi_n = g(n) \tag{2.20}$$

$$\Longrightarrow \chi_{n+2} + \alpha\chi_{n+1} + \beta\chi_n = 0 \quad \text{(Associated homogenous equation)} \tag{2.21}$$

Set $\chi_1(n)$ and $\chi_2(n)$ be a solution of associated homogenous equation.

Claim $\chi(n) = C_1(n)\chi_1(n) + C_2(n)\chi_2(n)$ solve (2.20) for suitable functions $C_1(n)$ and $C_2(n)$.

This is often called the method of variation of parameters.

$\Rightarrow \chi(n+1) = C_1(n+1)\chi_1(n+1) + C_2(n+1)\chi_2(n+1)$

$$= C_1(n)\chi_1(n+1) + C_2(n)\chi_2(n+1) + \underbrace{\Delta C_1(n)\chi_1(n+1) + \Delta C_2(n)\chi_1(n+1)}_{=0\ Assume}$$

$\chi(n+2) = C_1(n+1)\chi_1(n+2) + C_2(n+1)\chi_2(n+2)$

$$= C_1(n)\chi_1(n+2) + C_2(n)\chi_2(n+2) + \Delta C_1(n)\chi_1(n+2) + \Delta C_2(n)\chi_2(n+2)$$

Pulsing $\chi(n+2)$, $\chi(n+1)$ and $\chi(n)$

into (2.20) rests on the assumption that $\chi(n)$ is a solution (2.20).

Thus, for $\chi(n)$ to be a solution a solution of (2.20), we must have

$\chi_{n+2} + \alpha\chi_{n+1} + \beta\chi_n = g(n)$.

$$\Rightarrow C_1(n)\chi_1(n+2) + C_2(n)\chi_2(n+2) + \Delta C_1(n)\chi_1(n+2) + \Delta C_2(n)\chi_2(n+2) + \alpha C_1(n)\chi_1(n+1) + \alpha C_2(n)\chi_2(n+1) + \beta C_1(n)\chi_1(n) + \beta C_2(n)\chi_2(n) = g(n).$$

$$\Leftrightarrow g(n) = C_1(n)\underbrace{\left(\chi_1(n+2) + \alpha\chi_1(n+1) + \beta\chi_1(n)\right)}_{=0} + C_2(n)\underbrace{\left(\chi_2(n+2) + \alpha\chi_2(n+1) + \beta\chi_2(n)\right)}_{=0} + \Delta C_1(n)\chi_1(n+2) + \Delta C_2(n)\chi_2(n+2)$$

$$= \Delta C_1(n)\chi_1(n+2) + \Delta C_2(n)\chi_2(n+2)$$

$$\Leftrightarrow \begin{cases} \Delta C_1(n)\chi_1(n+1) + \Delta C_2(n)\chi_2(n+1) = 0 \\ \Delta C_1(n)\chi_1(n+2) + \Delta C_2(n)\chi_2(n+2) = g(n) \end{cases}$$

$$\underbrace{\begin{pmatrix} \chi_1(n+1) & \chi_2(n+1) \\ \chi_1(n+2) & \chi_2(n+2) \end{pmatrix}}_{W(n+1)=det}\begin{pmatrix} \Delta C_1(n) \\ \Delta C_2(n) \end{pmatrix} = \begin{pmatrix} 0 \\ g(n) \end{pmatrix}$$

Cramer's rule

$$\Rightarrow \begin{cases} \Delta C_1(n) = \frac{-g(n)\chi_2(n+1)}{W(n+1)} \\ \Delta C_2(n) = \frac{g(n)\chi_1(n+1)}{W(n+1)} \end{cases}$$

$$\Rightarrow C_1(n) = \sum_{k=0}^{n-1}\frac{-g(k)\chi_2(k+1)}{W(k+1)} + C_1(0) \text{ and } C_2(n) = \sum_{k=0}^{n-1}\frac{g(k)\chi_1(k+1)}{W(k+1)} + C_2(0)$$

Higher order difference equation/operator/

A) m^{th} order difference operator

$$\Delta^m\chi_n = \chi_{m+n} - m\chi_{n+m-1} + \frac{m(m-1)}{2!}\chi_{n+m-2} + \ . \ . \ . + (-1)^m\chi_n$$
$$= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\chi_{(n+m-k)}$$

B) Analog of Binomial theorem

$$\Delta^m\chi_n = (E-I)^m\chi_n$$
$$= \sum_{k=0}^{m}(-1)^k\binom{m}{k}E^{m-k}\chi_n$$
$$= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\chi_{(n+m-k)} .$$

2.2 modeling with difference equation

We consider real world scenario in which the temporal variable changes in lapses, i.e discrete time steps.

2.2.1 Population model

Let χ_n be the size of the population (n^{th} generation). If the reproductive rate is λ then we have

$\Delta\chi_n = \lambda\chi_n$ (Assuming the change is proportional to current size)

$\Longrightarrow \chi_{n+1} = (\lambda + 1)\chi_n$ linear, 1^{st} order, homogenous.

The solution is given by $\chi_n = (\lambda + 1)^n \chi_0$ for $n \epsilon \{1, 2, 3, \; . \; . \; . \; .\}$ where χ_0 is initial population.

Qualitative behavior

1) $\lambda > 0 \iff 1 + \lambda > 1.$

   $\Longrightarrow \chi_n \longrightarrow \infty$, as $n \longrightarrow \infty$.
2) $\lambda < -2 \iff 1 + \lambda < -1.$

   $\Longrightarrow \chi_n$ Oscillates between positive and negative.
3) $-2 < \lambda < 0 \iff |1 + \lambda| < 1.$

   $\Longrightarrow \chi_n \longrightarrow 0$, as $n \longrightarrow \infty$.
4) $\lambda = 0 \iff 1 + \lambda = 1.$

   $\Longrightarrow \chi_n = \chi_0$ Population is constant size.
5) $\lambda = -2 \iff 1 + \lambda = -1.$

   $\Longrightarrow \chi_n = (-1)^n \chi_0 = \begin{cases} \chi_0, n \; is \; even. \\ -\chi_0, n \; is \; odd. \end{cases}$

Note that this difference equation and its solution are useful whenever we are interested in a sequence of numbers where the $(n + 1)^{th}$ term is constant proportion of the n^{th} term.

The continuous version of the logistic model is described by differential equation

$$\dot{N(t)} = rN(t)(1 - \frac{N(t)}{L}) \qquad (2.22)$$

Where the population is limited by a maximum L for any positive initial population No, will eventually approach the limiting size $\lim_{t\mapsto\infty} N(t) = L$, where L is carrying capacity of the environment.

The discrete version reads

$$N(n+1) - N(n) = rN(n)(1 - \frac{N(n)}{L}) \tag{2.23}$$

Or equivalency

$$N(n+1) = N(n) + rN(n)\left(1 - \frac{N(n)}{L}\right)$$

$$N(n+1) = rN(n)\left(\frac{1}{r} + 1 - \frac{N(n)}{L}\right), \text{ where } \check{L} = \frac{1}{r} + 1$$

$$N(n+1) = rN(n)(\check{L} - \frac{N(n)}{L}) \tag{2.24}$$

Introducing $X_n = \frac{N(n)}{L\check{L}}$, $\mu = rL\check{L}$, we see that it suffices to consider

$$X_{n+1} = \mu X_n(1 - X_n) \tag{2.25}$$

Where X_n is a number between 0 and 1 and represents the population at year n, and hence X_0 represents the initial population (at year 0) μ is a positive number and represents a combined rate for reproduction and starvation.

This is known as logistic equation. Introducing the quadratic function

$$L_\mu(x) = \mu X(1 - X) \tag{2.26}$$

We can write the solution as n^{th} iterate of this map $X_n = L_\mu^n(Xo)$.

Equation (2.25) is a discrete logistic equation which is nonlinear.

CHAPTER THREE

Fixed and periodic points of dynamical systems

3.1 Introduction

In characterizing the notion of dynamical system, we require the concepts of motion and family of motions.

Definition1 i) Let (X, d) be a metric space, let $A \subset X$, and let $T \subset \mathbb{R}$. For any fixed $a \epsilon A,\ t_0 \epsilon T$ a mapping $p(.,a,t_0): T_{a,t_0} \to X$ is called a motion if $p(t_0, a, t_0) = a$ where $T_{a,t_0} = [t_0, t_1) \cap T,\ t_1 > t_0$ and t_1 is finite or infinite.

ii) A set S is a subset of $\bigcup_{(a,t_0)\epsilon AxT} \{T_{a,t_0} \to X\}$ is called a family of motions if for every $p(.,a,t_0)\epsilon S$, we have $p(t_0, a, t_0) = a$.

Definition2 The four - tuple $\{T, X, A, S\}$ is called a dynamical system.

In definition1 we find it useful to think of X as state space, T as time set, t_0 as initial time, a as initial condition of the motion $p(.,a,t_0)$, and A as the set of initial conditions. Note that in our definition of motion, we allow in general more than one motion to initiate from a given pair of initial data,(a, t_0).

When in definition2 $T = J \subset \mathbb{R}^+$ (with $J = \mathbb{R}^+$ allowed), we speak of a continuous-time dynamical system and when $T = J \subset \mathbb{N}$ (with $J \cap \mathbb{N} = \mathbb{N}$ allowed) we speak of a discrete-time dynamical system. Also, when in definition 2, X is a finite-dimensional vector space, we speak of finite-dimensional dynamical system, and otherwise, of an infinite-dimensional dynamical system.

3.2 Fixed points

Let X be a real vector space. A norm on χ is a map$\|.\|$: $\chi \longrightarrow [0, \infty)$ satisfies the following requirements:

1) $\|0\| = 0, \|\chi\| > 0$ for $\chi \in X\backslash\{0\}$.
2) $\|\lambda\chi\| = |\lambda|\|\chi\|$ for $\lambda\epsilon\mathbb{R}$ and $\chi\epsilon X$.
3) $\|\chi + y\| \leq \|\chi\| + \|y\|$ for $\chi, y\epsilon X$ (triangular inequalities).

The pair $(X, \|.\|)$ is called a normed vector space. Given a normed vector space X, we have the concept of convergence and of a Cauchy sequence in this space. The normed vector space is called Complete if every Cauchy sequence converges. A complete normed vector space is called a Banach space.

Definition If f is a function and $f(c) = c$, then c is a fixed point of f. A function of real numbers has a fixed at c if and only if the point (c, c) is on its graph. Thus, a function has a fixed point at c if and only if its graph intersects the line $y = \chi$ at the point (c, c) and it is denoted by $Fix(f)$.

Theorem 3.1 Let $\mathrm{I} = [a, b]$ is a closed interval and f: I $\longrightarrow$ I be a continuous function, then f has a fixed point in I.

Proof If $f(a) = a$ or $f(b) = b$ then either a or b is fixed and we are done.

Suppose $f(a) \neq a$ and $f(b) \neq b$. Let $g(\chi) = f(\chi) - \chi$ obviously, g is a continuous function.

As $f(a) \neq a$ and $f\ (a)\epsilon[a, b]$, $f(a) > a$. Similarly, $f(b) \neq b$ and $f(b)\epsilon\ [a, b]$, $f(b) < b$. Hence,

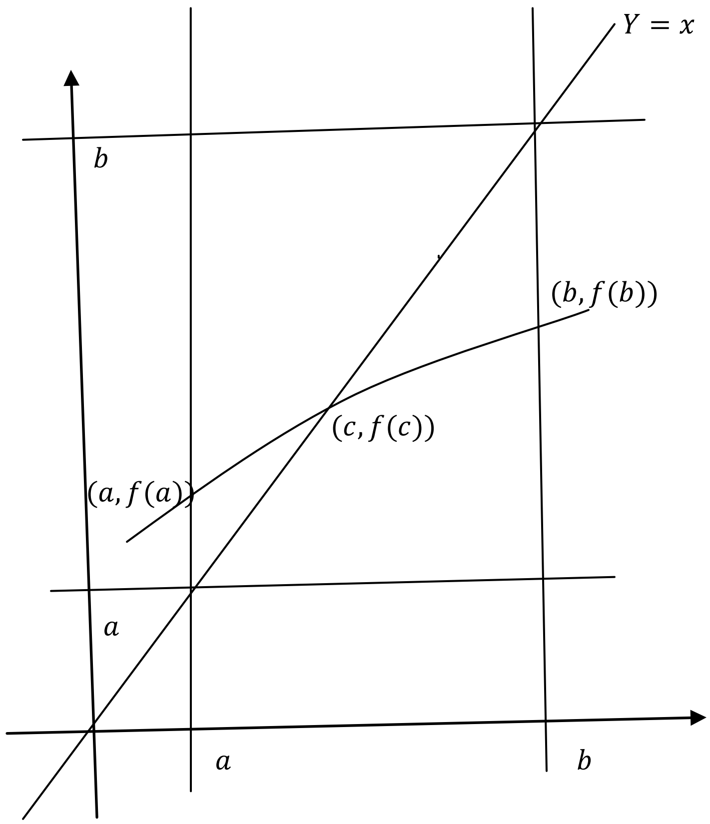


Figure 1. An illustration of the situation in theorem 3.1

Recall that f has a fixed point at C if and only if the graph of f intersects the line $Y = X$ at C we see that such an intersection is required by the conditions on f.

$g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$, since g is continuous, the intermediate value theorem implies that there is $c \in [a, b]$ such that $g(c) = 0.$ But $g(c) = f(c) - c = 0$ such that $f(c) = c$ and we are done.

Example The function $f(\chi) = 1 - \chi^2$ has a fixed point in the interval $[0,1]$; f is continuous function, the range of f over [0,1] is contained in [0,1] . By the above theorem f has a fixed point in $[0, 1]$.

Alternatively we look at the graph of f shown in figure2 below and note that it crosses the line $Y = X$ in the interval [0, 1]. Finally, find the fixed point in $[0,1]$, to get $x = \frac{-1\pm\sqrt{5}}{2}$.

But $\frac{-1-\sqrt{5}}{2} \notin [0,1]$.

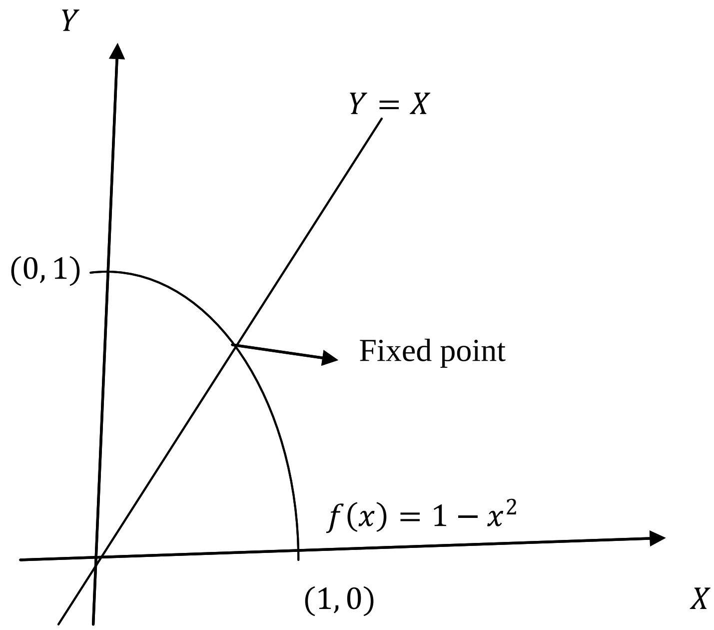


Figure2

Note that $f\left(\frac{-1+\sqrt{5}}{2}\right) = 1 - (\frac{-1+\sqrt{5}}{2})^2 = \frac{-1+\sqrt{5}}{2}$.

$\therefore$ f has a fixed point at $\frac{-1+\sqrt{5}}{2}$ in $[0,1]$.

Definition (Contraction mapping principle)

1) Let S and B be Banach space. A transformation $f: V \subseteq S \longrightarrow B$ is said to be a contraction on V if and only if there exists $\lambda(0 < \lambda < 1)$ such that $\|T_\chi - T_y\| < \lambda\|\chi - y\|$ for all $\chi, y\, \epsilon V$.
2) A fixed point of a transformation $T: U \longrightarrow U$ is a point x such that $T_\chi = \chi$.

Theorem 3.2 (Fixed point) If V is a closed subset of a Banach space B and $T: V \longrightarrow V$ is a contraction on V, then T has a unique fixed point in V.

Proof Suppose that

i) V is closed

ii) $T: V \longrightarrow V$ is a constraction.

Let $\{X_n\}_{n=0}^{\infty}$ be a sequence in V such that $X_{n+1} = T_{x_n}$ then

$$|\chi_{n+1} - \chi_n| = |T_{\chi_n} - T_{\chi_{n-1}}|$$

$$\leq \lambda|\chi_n - \chi_{n-1}|$$

$$\leq \lambda|T_{\chi_{n-1}} - T_{\chi_{n-2}}|$$

$$\leq \lambda^2|\chi_{n-1} - \chi_{n-2}|$$

$$\vdots$$

$$\leq \lambda^n|\chi_1 - \chi_0|$$

In particular for $m > n > 0$, we have

$$|\chi_m - \chi_n| = |\chi_m - \chi_{m-1} + \chi_{m-1} - \chi_n|$$

$$\leq |\chi_m - x_{m-1}| + |\chi_{m-1} - \chi_n|$$

$$\leq |\chi_m - \chi_{m-1}| + |\chi_{m-1} - \chi_{m-2}| + \ldots . |\chi_{m-n} - \chi_n|$$

$$\leq \lambda^{m-1}|\chi_1 - \chi_0| + \lambda^{m-2}|\chi_1 - \chi_0| + \ldots \lambda^n|\chi_1 - \chi_0|$$

$$\leq (\lambda^{m-1} + \lambda^{m-2} + \ldots \lambda^n)|\chi_1 - \chi_0|$$

$$\leq \lambda^n(1 + \lambda + \ldots + \lambda^{m-n-1})|\chi_1 - \chi_0|$$

(But $1 + \lambda + \ldots + \lambda^{m-n-1} = \frac{1-\lambda^{m-n}}{1-\lambda}$ by geometric series)

$$= \lambda^n\left(\frac{1-\lambda^{m-n}}{1-\lambda}\right)|\chi_1 - \chi_0|$$

$$\leq \frac{\lambda^n}{1-\lambda}|\chi_1 - \chi_0|$$

$\Longrightarrow \{\chi_n\}_{n=0}^{\infty}$ is a Cauchy sequence, and hence, it has a convergent subsequence say $\{\chi_{n_k}\}_{n=0}^{\infty}$ i.e. $\lim_{k\to\infty} \chi_{n_k} = \bar{\chi}$

$\Longrightarrow 0 = \lim_{k\to\infty} \left\|\chi_{n_{k+1}} - T_{\chi_{n_k}}\right\|$ (Continuity of the norm)

$= \left\|\lim_{k\to\infty}(\chi_{n_{k+1}} - T_{\chi_{n_k}})\right\| = \left\|\bar{\chi} - \lim_{k\to\infty} T_{\chi_{n_k}}\right\|$

$=\|\bar{\chi} - T_{\bar{\chi}}\|$

$\Longrightarrow \bar{\chi} - T_{\bar{\chi}} = 0.$

$\Longrightarrow T_{\bar{\chi}} = \bar{\chi}.$

$\Longrightarrow \bar{\chi}$ is a fixed point of T.

Since V is closed, $\bar{\chi}\epsilon V$.

It remains to show uniqueness,

Assume to the contrary that there exists $\bar{y}\epsilon V\backslash\{\bar{\chi}\}$ such that $T_{\bar{y}} = \bar{y}$

Now, $\|\bar{\chi} - \bar{y}\| = \left\|T_{\bar{\chi}} - T_{\bar{y}}\right\| \leq \lambda\|\bar{\chi} - \bar{y}\|$

$\Longrightarrow (1-\lambda)\|\bar{\chi} - \bar{y}\| \leq 0$

$\Longrightarrow \|\bar{\chi} - \bar{y}\| \leq 0$

$\Longrightarrow \|\bar{\chi} - \bar{y}\| = 0$

$\Longrightarrow \bar{\chi} - \bar{y} = 0$

$\Longrightarrow \bar{\chi} = \bar{y}$

Theorem3.3 Let I be a closed interval f: I $\longrightarrow \mathbb{R}$ be a continuous function. If f(I) $\subset$ I, then f has a fixed point in I.

Proof Let I $= [a, b]$, since f(I) $\subset$ I there are c and d points in I f(c)= a and f(d) = b . If c = a or d = b we are done. If not, then $a < c < b$ and $a < d < b$. If we define $g(\chi) = f(\chi) - \chi$, then $g(c) = f(c) - c < 0$ since $f(c) = a$ and $a < c$ likewise $g(d) = f(d) - d > 0$ since $g(c) < 0$ and $g(d) > 0$ and g is continuous the

Intermediate value theorem implies that there is e between c and d (and hence in I) satisfying $g(e) = 0$ and $f(e) = e$ which is complete the proof.

3.3 Periodic points

Definition 1) let f be a function. The point χ is periodic point of f with period K if $f^K(\chi) = \chi$. In other words, a point is a periodic point of f with period K if it is a fixed point of f^K. The periodic point χ has prime period K_o if $f^{K_o}(\chi) = \chi$ and $f^n(\chi) \neq \chi$ whenever $0 < n < K_o$. That is a periodic point has prime period K_o if it returns to its starting place for the first time after exactly K_o iterations of f. The set of all iterates of the point x is called the orbit of x and if x is a periodic point, then it and its iterates are called a periodic orbit or a periodic cycle.

2) Let f be a function, the point χ is an eventually fixed point of f if there exists N such that $f^{n+1}(\chi) = f^n(x)$ whenever $n \geq N$. The point χ is eventually periodic with period K if there exists N such that $f^{n+K}(\chi) = f^n(\chi)$ whenever $n \geq N$.

Theorem3.4 If a continuous function of the real numbers has a periodic point with prime period three, then it has periodic points of all prime periods.

Proof Let $\{a, b, c\}$ be a period three orbit of the continues function f.

Without loss of generality, we assume $a < b < c$. There are two cases, $f(a) = b$ or $f(a) = c$. We supposes $f(a) = b$. This implies $f(b) = c$ and $f(c) = a$. The proof of the case $f(a) = c$ is similar.

Let $\mathrm{I}_0 = [\mathrm{a}, \mathrm{b}]$ and let $\mathrm{I}_1 = [b, c]$. The intermediate value theorem implies that $f(\mathrm{I}_0) \supset \mathrm{I}_1$, $f(\mathrm{I}_1) \supset \mathrm{I}_1$ and $f(\mathrm{I}_1) \supset \mathrm{I}_0$. Since $f(\mathrm{I}_1) \supset \mathrm{I}_1$ theorem 3.3 implies that f has a fixed point in I_1.

Now let n be the natural number larger than 1. Suppose that there is a nested sequence of closed intervals $\mathrm{I}_1 = \mathrm{A}_0 \supset \mathrm{A}_1 \supset \mathrm{A}_2 \supset \mathrm{A}_3 \supset \mathrm{A}_4 \supset. \ .. \ ..\mathrm{A}_\mathrm{n}$, with the following properties

1) $A_0 = I_1$.
2) $f(A_k) = A_{k-1}$. for $k = 1,2,. . . . n-2$.
3) $f^k(A_k) = I_1$. for $k = 1,2,. . . . n-2$.
4) $f^{n-1}(A_{n-1}) = I_0$.
5) $f^n(A_n) = I_1$.

Since $A_n \subset I_1$, (5) and theorem 3.3 imply that f^n has a fixed point in A_n. Of course, this is identical to saying that f has a periodic point p with period n in A_n. In the next two paragraphs we use the other four conditions to show that p has prime period n.

If p is periodic point of period n in A_n, then (1) and (3) implies that $p, f(p), f^2(p)$, , . . . $f^{n-2}(p)$ are in $I_1 = [b, c]$ and (4) implies $f^{n-1}(p)$ is in $I_0 = [a, b]$. If $p = c$ then $f(p) = a$ this is not in I_1. Since the only one of the first n iterates of p which doesn't have to be in I_1 is $f^{n-1}(p)$, it must be that $n = 2$. However, this contradicts the fact that the prime period of C is three. Hence, p must be in the half – open interval $[b, c)$.

If $p = b$, then $n = 3$ since $f^2(p) = a$ which is not in I_1 and the only one of the first n iterates of p which is not in I_1 is $f^{n-1}(p)$. Assume that n is not three. It follow that p must be in the open interval (b, c). Since $f^{n-1}(p)$ is in $I_0 = [a, b]$, $f^{n-1}(p) \neq p$ and so p can't have prime period $n-1$. If the prime period of p were less than $n-1$, then (3) and the fact that p is not b or c imply that the orbit of p is contained entirely in (b, c) and this would contradict (4). Since $f^{n-1}(p) \notin (b, c)$, it must be that p has prime period n.

To complete the proof we demonstrate the existence of such sequence of closed sets for each natural number which is larger than 1. Let n be natural number $(n > 1)$. We establish each of the requirements in turn. Obviously we can choose A_0 so that $A_0 = I_1$ and (1) is satisfied. The other properties use the fact that if f is a continuous function and J and K are closed interval such that $f(J) \supset K$, then there exist an interval J_0 such that $J_0 \subset J$ and $f(J_0) = K$. This is an intuitively clear result whose proof depends on the continuity of f.

Now, since $A_0 = I_1$ and $f(I_1) \supset I_1$, we have $f(A_0) \supset A_0$ and so there is A_1 contained in A_0 such that $f(A_1) = A_0$. Since $A_1 \subset A_0$ this implies $f(A_1) \supset A_1$. Consequently, there is $A_2 \subset A_1$ such that $f(A_2) = A_1$.We continues in this manner and defines A_k for

$k = 1,2,. . . . n - 2$. In each case, we find A_k contained A_{k-1} so that $f(\mathrm{A}_k) = \mathrm{A}_{k-1}$ for $k = 1,2,. . . . n - 2$ as required by (2). Further $f(\mathrm{A}_k) \supset \mathrm{A}_k$ for each k so that process of defining the intervals A_k can continue indefinitely. Then $f(\mathrm{A}_k) = \mathrm{A}_{k-1}$ implies

$$f^2(\mathrm{A}_k) = f(f(\mathrm{A}_k)) = f(A_{k-1}) = A_{k-2}$$

$$f^3(\mathrm{A}_k) = f(f^2(\mathrm{A}_k)) = f(A_{k-2}) = A_{k-3}$$

$$\vdots$$

$f^k(\mathrm{A}_k) = A_0 = \mathrm{I}_1$ for each $k = 1,2,. . . . n - 2$ as required by (3).

To define A_{n-1} consistent with (4) we note that

$$f^{n-1}(\mathrm{A}_{n-2}) = f(f^{n-2}(A_{n-2})) = f(\mathrm{I}_1) \supset \mathrm{I}_0.$$

Hence, there is $A_{n-1} \subset A_{n-2}$ such that $f^{n-1}(A_{n-1}) = \mathrm{I}_0$. Finally,

$$f^n(A_{n-1}) = f\left(f^{n-1}(A_{n-1})\right) = f(\mathrm{I}_0) \supset \mathrm{I}_1$$

implies there is $A_n \subset A_{n-1}$ such that $f^n(\mathrm{A}_n) = \mathrm{I}_1$ as required by (5) and the proof is complete.

The function $f(\chi) = -\chi^3$ has a fixed point at 0 and a periodic cycle consisting of 1 and

-1. How should we characterize the rest of the points?

For example if we start at $\frac{1}{2}$ and iterate with f ,then we get the sequence $\frac{1}{2}, \frac{-1}{2^3}, \frac{1}{2^9}, \frac{-1}{2^{27}}$ ... so $\frac{1}{2}$ is not periodic and never reaches 0, thought it does get closer and closer to it .That is $f^n(\frac{1}{2})$ converges to zero as $n \longrightarrow \infty$ we say $\frac{1}{2}$ is forward asympptotic to 0.

Definition Let f be a function and P be a periodic point of f with period K then χ is forward asymptotic to P if the sequence $\chi,\ f^K(\chi),\ f^{2K}(\chi),\ f^{3K}(\chi)$ ...… converges to P in other words

$$\lim_{n\to\infty} f^{nk}\ (\chi) = P \tag{3.1}$$

The stable set of P denoted by $W^S(p)$ consisting of all points that are forward asymptotic to p.

If the sequence $|\chi|, |f(\chi)|, |f^2(\chi)|$ … ……grows without bound, then x is forward asymptotic to ∞. The stable set of ∞ denoted by $W^S(\infty)$ consists of all points that are forward asymptotic to ∞.

Note that when we are searching for points in the stable set of point with prime period K, we must consider the sequence $\chi, f^K(\chi), f^{2K}(\chi)$ … ….

Not the sequence $\chi, f(\chi), f^2(\chi)$ …………

Example Let $f(\chi) = \chi^3$ then the stable set of 0 consists of all points in the interval

$(-1,1)$. The stable set of ∞ consists of all points in the interval $(-\infty, -1)$ and $(1, \infty)$ the stable set of -1 contains only the point -1 and stable set of 1 contains only the point 1. Symbolically we write

$W^S(0) = (-1,1)$

$W^S(\infty) = (-\infty, -1) \cup (1, \infty)$

$W^S(1) = \{1\}$ and

$W^S(-1) = \{-1\}$

Notice that, it doesn't make sense to discuss the stable set of any other points since f has no other periodic points.

Theorem 3.5 The stable set of distinct periodic points do not intersect. In other word's if P_1 and P_2 are periodic points and $P_1 \neq P_2$ then $W^S(P_1) \cap W^S(P_2) = \phi$.

Proof Let $f(\chi)$ be a function with periodic points P_1 and P_2 of period K_1 and K_2 respectively we will show that if $W^S(P_1) \cap W^S(P_2) \neq \phi$ then $P_1 = P_2$.

Let χ be in $W^S(P_1) \cap W^S(P_2)$ or $(\chi \epsilon\ W^S(P_1) \cap W^S(P_2))$. Then for all $\varepsilon > 0$ there exists N_1 and N_2 such that $n \geq N_1$ implies

$|P_1 - f^{nk_1}(\chi)| < \frac{\varepsilon}{2}$ and $|P_2 - f^{nK_2}(\chi)| < \frac{\varepsilon}{2}$

If M is the larger of N_1 and N_2 then $n \geq M$ implies both $|P_1 - f^{nk_1}(\chi)| < \frac{\epsilon}{2}$

and $|P_2 - f^{nK_2}(\chi)| < \frac{\epsilon}{2}$. Utilizing the triangle inequality, we define that when $n \geq M$ then

$$|P_1 - P_2| = |P_1 - f^{nK_1K_2}(\chi) + f^{nK_1K_2}(\chi) - P_2|$$

$$\leq \left|P_1 - f^{(nK_2)K_1}(\chi)\right| + \left|f^{(nK_1)K_2}(\chi) - P_2\right|$$

$$< \quad \varepsilon/2+\varepsilon/2=\varepsilon$$

Since we have shown that the distance.

$\Longrightarrow P_1 = P_2$, Since $\varepsilon > 0$ is arbitrary.

We call P is attracting if there is an open neighborhood U of P such that $U \subseteq W^S(p)$. The set $W^S(p)$ is clearly positive (it is even invariant $f(W^S(p)) = W^S(p)$ if f is invariant)

Definition Let f be a function and P be a periodic point of f with period K then χ s backward asymptotic to P if the sequence $X,$ $f^{-K}(\chi),$ $f^{-2K}(\chi), f^{-3K}(\chi)$ ...… converges to $P.$ i.e.

$$\lim_{n\to\infty} f^{-nk}(\chi) = P \tag{3.2}$$

The unstable set of P denoted by $W^U(p)$ consists of all points that are backward asymptotic to P.

We call P is repelling if there is an open neighborhood U of P

such that $U \subseteq W^U(p)$.

Now let us look at the logistic map $L_\mu(\chi) = \mu\chi(1 - \chi)$ with $M = [0,1]$ we have already seen that if $\mu = 0$, then the only fixed point is 0 with $W^S(0) = [0,1]$ and all points in (0,1] are eventually periodic.

So let us next turn to the case $0 < \mu < 1$. Then we have $L_\mu(\chi) \leq \mu\chi$ and hence, $L_\mu^n(\chi) \leq \mu^n(\chi)$ show that every convergence exponentially to 0. In particular we have $W^S(p) = [0,1]$.

Note that locally this follows since $L'_{\mu}(0) = \mu < 1$. Hence L_{μ} is contracting in a neighborhood of fixed point and so all points in this neighborhood converges to the fixed point.

This result can be easily generalized to differentiable maps $f: \epsilon C^1(U, U)$, where $U \epsilon R^n$

Proposition Let P be a hyperbolic fixed point with $|f'(p)| < 1$. Then there is an open interval U about P such that if $\chi \epsilon U$, then $\lim_{n\to\infty} f^n(\chi) = P$

Proof Since f is C^1, there is $\varepsilon > 0$ such that $|f'(\chi)| < A < 1$ for $x\epsilon[p - \varepsilon, p + \varepsilon]$. By the mean value theorem $|f(\chi) - p| = |f(\chi) - f(p)| \leq A|\chi - p| < |\chi - p| \leq \varepsilon$.

Hence $f(\chi)$ is contained in $[p - \varepsilon, p + \varepsilon]$ and, in fact, is closer to P then χ is the same argument $|f^n(\chi) - p| \leq A^n|\chi - p|$ so that $f^n(\chi) \to p$ as $n \to \infty$.

Definition Let P be a hyperbolic periodic point of period n with $|(f^n)'(p)| < 1$. The point P is called an attracting periodic point or a sink.

Attracting periodic points of period n thus have neighborhoods which are mapped inside themselves by function. Such a neighborhood is called the local stable set and is denoted by W^S_{loc}.

Proposition Let P be a hyperbolic fixed points with $|f'(p)| > 1$.Then there is an open interval U of P such that if $x\epsilon U, \chi \neq p$, then there exist $k > 0$ such that $f^k(\chi) \notin U$.

Proof similar preceding preposition.

Definition A fixed point P with $|f'(p)| > 1$ is called repelling fixed point or source. The neighborhood described in the preposition is called the local unstable set and denoted W^U_{loc}.

Theorem3.6 Suppose $\epsilon C^1(U, U)$, $U \subseteq R^n$, then periodic point P with period n is attracting if all eigenvalues of $d(f^n)p$ are inside the unit circle and repelling if all eigenvalues are outside.

Proof In the first case there is a suitable norm such that $\|d(f^n)(p)\| < \theta < 1$ for any fixed θ which is larger than eigenvalues. Moreover since the norm is continuous, there is an open ball B around P such that we have $\|d(f^n)(\chi)\| \leq \theta$ for all $\chi \epsilon B$.

Hence, we have $|f^n(\chi) - P| = |f^n(\chi) - f^n(P)| \leq \theta|\chi - P|$ and the claim is obvious.

The second case can now by reduced to the first by considering the local inverse of f near P.

If none of the eigenvalue of $d(f^n)$ at a periodic point P lies on the unite circle, then P is called hyperbolic.

Note that by the chain rule the derivative is given by $d(f^n)(P) = \prod_{x\epsilon\gamma_+(p)} df_\chi = df_{f^{n-1}(P)} \ldots\ldots\ldots df_{f(p)}, df_p$ where $\gamma_+(p)$ is the forward orbit define as $\gamma_+(p) = \{f^n(p) | n\epsilon N_0\}$

A periodic orbit $\gamma_+(p)$ of $f(x)$ is called stable if for any given neighborhood $U(\gamma_+(p))$ there exists another neighborhood $V(\gamma_+(p)) \subseteq U(\gamma_+(p))$ such that any point in $V(\gamma_+(p))$ remains $U(\gamma_+(p))$ under all iterations.

Note that, this is equivalent to the fact that for any given neighborhood $U(p)$ there exists another neighborhood $V(p) \subseteq U(p)$ such that any point in $\chi\epsilon V(p)$ satisfies $f^{nm}(\chi)\epsilon U(p)$ for $m\epsilon N_0$.

Similarly, a periodic orbit $\gamma_+(p)$ of $f(\chi)$ is called asymptotically stable if it is

Stable and attracting.

Pick a periodic point p of f, $f^n(p) = p$, and an open neighborhood $U(p)$ of P. A Liapunov function is a continuous function

$$L: U(p) \to \mathbb{R} \quad (3.3)$$

Which is zero at P, positive for$\chi \neq p$, and satisfies

$$L(\chi) \geq L(f^n(\chi)), \quad \chi, f^n(\chi)\epsilon U(p)\backslash\{p\} \quad (3.4)$$

It is called a strict Liapunov function if equality in (3.4) never occurs.

As in the case of differential equations we have the following analog of Liapunov theorem.

Theorem 3.7 Suppose p is a periodic point of f. If there is a Liapunov function L, then p is stable. If, in addition, L is strict, then p is asymptotically stable.

Problem Consider the logistic map L_μ for $\mu = 1$. Show that $W^S(0) = [0,1]$.

Solution Logistic equation $L_1(\chi) = \chi(1-\chi)$ for $\mu = 1$. Then the fixed point is 0 and 1.

The $W^S(0)$ consisting of all points that are forward asymptotic to 0

That means $L^n{}_1(\chi)$ converges to zero as $n \longrightarrow \infty$ then $x\epsilon[0,1]$. therefore, $W^S(0) = [0,1]$.

3.4 Local behavior near fixed points

The local behavior of a differentiable map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ near a fixed point p. We will assume $p = 0$ without restriction and write $f(\chi) = A\chi + g(\chi)$ where $A = df_0$. The analogous results for periodic points are easily obtained by replacing f with f^n.

Theorem 3.8 (Hartman-Grobman) Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 diffeomorphism with $f(0) = 0$ hyperbolic fixed point (meaning $A = Df_0$ is hyperbolic). Then there exists a neighborhood U of 0 and a homeomorphism h from U to some other neighborhood of 0 such that $Aoh = hof$ whenever all these are defined (i.e. for all Z sufficiently close to 0). In other words

$f^n = h^{-1}oA^n oh$ near 0 so iterating a non-linear map near a hyperbolic fixed point is like just iterating the linear part.

We will begin by proving lemma and theorem

Lemma 3.9 Let E be a Banach space suppose $L: E \longrightarrow E$ is a linear with $\|L\| < a < 1$ and $G: E \longrightarrow E$ is an isomorphism satisfying $\|G^{-1}\| \leq a < 1$. Then $id + L$ and $id + G$

are isomorphism satisfying $\|(id + L)^{-1}\| \leq \frac{1}{1-a}$ and $\|(id + G)^{-1}\| \leq \frac{a}{1-a}$

Proof Take $Y\epsilon E$ and fixed $\chi\epsilon E$ such that $(id + L)\chi = Y$ $i.e. \chi = Y - L\chi$ the map $\chi \rightarrow \mu(\chi) = Y - L\chi$ is a contraction since $\|\mu(\chi_1) - \mu(\chi_2)\| = \|L(\chi_1 - \chi_2)\| \leq a\|\chi_1 - \chi_2\|$

A contraction in a banach space has a fixed point, so the required χ exists. Therefore if $\|Y\| = 1$ we have

$\|\chi\| \leq \|Y\| + \|L\chi\| \leq 1 + a\|\chi\|$, so $\|X\| \leq \frac{1}{1-a}$ and also $(id + L)^{-1}Y = \chi$ so

$\|(id + L)^{-1}\| = sup_{\|y\|=1}\{(id + L)^{-1}y\} \leq \frac{1}{1-a}$ as required

Notation write $R^n = E^S \oplus E^U$ where $E^S$and E^U the stable and unstable eigenspaces of hyperbolic matrix A respectively. Then $A^S = A \Big| E^S$ and $A^U = A \backslash E^U$ with possible change of norm we can assume $\|A^S\| \leq a < 1$ and $\|(AU)^{-1}\| \leq a < 1$. Also write $C_b^0(\mathbb{R}^n)$ for the banach space of bounded continuous maps from $\mathbb{R}^n \; to \; \mathbb{R}^n$.

Then $C_b^0(\mathbb{R}^n) = C_b^0(E^n) \oplus C_b^0(E^U)$ i.e. functions the form

$U = U^S + U^U$ where $U^S(z)\epsilon E^S$ and $U^U(z)\epsilon E^U$ for all $z\epsilon R^n$ it is convenient to define the norm as $\|U\| = \max(\|U^U\|, \|U^S\|)$.

Definition i) A map $L: E \longrightarrow E$ is lipschitz if there exists $k < \infty$ such that

$$\|L(z_1) - L(z_2)\| \leq k\|z_1 - z_2\| \text{ for all } z_1,\, z_2 \,\epsilon E.$$

ii) Definition of conjugacy let M be a differentiable manifolds and f, g: $M \longrightarrow M$ is a diffeomorphism. We say that f and g topologically conjugate if there is a homeomorphism h: $M \longrightarrow M$ such that $hof = goh$. (3.5)

Theorem.3.10 If ϕ_1 and $\phi_2 \,\epsilon\, C_b^0(\mathbb{R}^n)$ with lipschitz constant $k < \varepsilon = (1-a) \Big| \|A^{-1}\|$ for some hyperbolic matrix A, then $A + \phi_1$ and $A + \phi_2$ are conjugate on $\mathbb{R}^n$.

(Not all diffeomorphisms are of the from (linear part + something bounded) however take $f(x) = x + \log(1 + x^2)$ for example).

Proof we need to find a homeomorphism h such that $ho(A + \phi_1) = (A + \phi_2)oh$ and we will do so with the ansatz $h = id + U$ for some $U\epsilon\, C_b^0(\mathbb{R}^n)$ from this we get

$$ho(A + \phi_1) = (A + \phi_2)oh.$$

$$(id + U)o(A + \phi_1) = (A + \phi_2)o(id + U).$$

$$A + \phi_1 + Uo(A + \phi_1) = A + AoU + \phi_2 o(id + U).$$

$$\phi_1 - \phi_2 o(id + U) = AoU - Uo(A + \phi_1).$$

So write $L(u) = AU - U(A + \phi_1)$ we can rewrite this as

$L(u) = A\big(U - A^{-1}U(A+\phi_1)\big)$. We need to show that $A^{-1}L$ is invertible in which case L is invertible, once we have proved that L is invertible the above statements becomes equivalence to solving U in

$$Pu = L^{-1}(\ \phi_1 - \phi_2 o(id + U)) = U.$$

Write U in the form $U^S + U^U$ as above and let us first consider the restriction of $A^{-1}L\ to\ C_b^0\big(E^S(R^n)\big)$ and write this restriction of $A^{-1}L$ as $\mathrm{I}d - G$, where $G: C_b^0\big(E^S(R^n)\big) \longrightarrow C_b^0\big(E^S(R^n)\big)$ is defined as

$G(U^S) = A^{-1}U^S(A+\phi_1)$. Let us check G is invertible for this we first check that the map $Z \longrightarrow (A+\phi_1)(Z_1) = (A+\phi_1)(Z_2)$ then

$\varepsilon\|z_1 - z_2\| \geq \|\phi_1(z_1) - \phi_1(z_2)\| = \|Az_1 - Az_2\| \geq \|A^{-1}\|^{-1}\|z_1 - z_2\|$ Since $\varepsilon\|A^{-1}\|^{-1}$, we must have $z_1 = z_2$. Hence $Z \longrightarrow (A+\phi_1)(z)$ is injective to see that $Z \longrightarrow (A+\phi_1)(z)$ is surjective.

Note that, to find Z with $(A+\phi_1)(z) = W$ is equivalent to finding a fixed point of the map $Z \longrightarrow A^{-1}W - A^{-1}\phi_1(z)$ and since $\|A^{-1}\|\varepsilon < 1$.

This map is a constraction and therefore has a fixed point by the banach constraction theorem.

All this implies that $G: U^S \longrightarrow A^{-1}U^S(A+\phi_1)$ has an inverse namely

$G^{-1}: U^S \longrightarrow A^{-1}U^S(A+\phi_1)^{-1}$. It follows by second part of the previous lemma that

$U^S \longrightarrow A^{-1}L(U^S) = (\ \mathrm{I}d - G)(U^S)$ is invertible and $\|(\mathrm{I}d - G)^{-1}\ \| < \frac{a}{1-1} < \frac{1}{1-a}$.

Here, we used that $\|G^{-1}\| \leq a$ (to see this take U^S so that $sup_x|U^S(x)| = 1$). Then $|AU^S(A+\phi_1)^{-1}y| \leq \big|AU^S(\acute{y})\big| \leq a|U^S(\acute{y})| \leq a$ where in the last but one inequality we used that $\left\|A\ \middle|E^S\right\| \leq a$ since $L = Ao(\mathrm{I}d - G)$ and $L^{-1} = (\mathrm{I}d - G)^{-1}oA^{-1}$ this gives that $\left\|L^{-1}\ restricted\ to\ (C_b^0\big(E^S(\mathbb{R}^n)\big))\ \right\| \leq \|A^{-1}\|\ \big|(1-a)$.

Now consider the restriction to $C_b^0\big(E^U(\mathbb{R}^n)\big)E^U$ of $A^{-1}L$ i.e.

$U^U \longrightarrow U^U - A^{-1}U^U(A+\phi_1)$, by the first part of lemma this map is also invertible (because $\left\| A^{-1} \left| E^U \right\| \leq a\right)$ and has the norm less than or equal to $\frac{1}{1-a}$, putting this together gives that $\|L^{-1}\|$ is bounded by $\|A^{-1}\| \left|(1-a)\right.$.

Write as above $Pu = L^{-1}(\ \phi_1 - \phi_2(id+U))$, we have,

$$\|PU_1 - PU_2\| \leq \|L^{-1}\|\|\phi_2(id+U_1) - \phi_2(id+U_2)\| \leq \|L^{-1}\|.k.\|U_1 - U_2\|$$

Now $\|L^{-1}\|.k < \|L^{-1}\|\varepsilon \leq 1$. So this constraction and has a fixed point, so we have the required U and hence h . Note that U is unique finally we show that $h = id + U$, (where id is identity map)is a homeomorphism we have U there is

$(id+U)o(A+\phi_2) = (A+\phi_1)o(id+U)$ there exists V such that

$(id+V)o(A+\phi_2) = (A+\phi_1)o(id+V)$ Now

$(id+U)o(id+V)o(A+\phi_2) = (id+U)o(A+\phi_1)o(id+V)$

$=(A+\phi_2)$ o$(id+U)o(id+V)$ by uniqueness

$(id+U)o(id+V) = id$. By symmetry

$(id+V)o(id+U) = id$ So h is a homeomorphism

Proof theorem 3.8 (Hartman-Grobman)

Write $f(z) = Az + \varphi(z)$ where $A = Df_0$, $\varphi(0) = 0$ and $D\varphi_0 = 0$. Also $D\varphi_\chi$ is small for χ close to 0.

Let $\alpha(\chi) = 0$ for $|x| \geq 1$ and $\alpha(x) = 1$ for $|x| \leq \frac{1}{2}$ and make α some smooth. Monotonic function on interval $[-1, -\frac{1}{2}]$ and $[1/2, 1]$ so there exist $k > 1$ such that $|\dot{\alpha(x)}| \leq k$ .Fix $e > 0$ and write $g(z) = Az + \alpha(\|z\|/e)\varphi(z)$. If

$\|z\| \leq e\ /\ 2$. (Then $g(z) = f(z)$ while if $\|z\| \geq e$, then $g(z) = Az$)

It is enough to show A is conjugate to g since it deforms continuously into $e \longrightarrow \infty$

Take $\varepsilon > 0$ such that $\|D\varphi_\chi\| \leq \varepsilon/2k$ whenever $\|\chi\| \leq e$, write

$\phi(z) = \alpha(\|Z\|/e)\varphi(z)$ then

$$\|\phi(z_1) - \phi(z_2)\| \leq \|\alpha(\|z_1\|/e) - \alpha(\|z_2\|/e)\varphi(z_1)\| + \|\alpha(\|z_2\|/e)(\varphi(z_1) - \varphi(z_2))\|$$

$$\leq k\|\varphi(z_1)\| \left| {}^{\|z_1\| - \|z_2\|}/_{e} \right| + \|D\varphi_x\|\|z_1 - z_2\|$$

$$\leq k(\varepsilon/2k)\|z_1\| {}^{\|z_1 - z_2\|}/_{e} + (\varepsilon/2k)\|z_1 - z_2\|$$

$$\leq \varepsilon\|z_1 - z_2\|$$

Where the second inequality follows from the mean value theorem with χ a point on the line from z_1 to z_2 assuming those two points in the open ball $B_\varepsilon(0)$ if only $z_1 \epsilon B_\varepsilon(0)$, then the second half of the second line vanishes and the result still holds , if neither points is in the ball then $\phi(zi) = 0$ for $i = 1,2$ the results from the above theorem.

Definition (heteroclinic orbits) Heteroclinic orbits (sometimes called a heteroclinic connection) is a path space which joins two different equilllibrium points if the equilibrium points at the start and end of orbit are the same , the orbit is a homoclinic orbit .

Consider the continuous dynamical system described by the ordinary differential equation

$$\dot{\chi} = f(\chi)$$

Suppose there are equilibria at $\chi = \chi_0$ and $\chi = \chi_1$, then a solution $\varphi(t)$ is hetroclinic orbit from χ_0 to χ_1 if $\varphi(t) \longrightarrow \chi_0$ as $t \longrightarrow -\infty$ and $\varphi(t) \longrightarrow \chi_1$ as $t \longrightarrow \infty$.

This implies that the orbit is contained in stable manifold of χ_1 and unstable manifold of χ_0.

Let χ be a topological space and $f: \chi \longrightarrow \chi$ a hemeomorphism. If P is a fixed point for, the stable set P is defined by $W^S(f,p) = \{q\epsilon\chi: f^n(q) \longrightarrow P\ as\ n \longrightarrow \infty\}$ and the unstable set of P is defined $W^U(f,p) = \{q\epsilon\chi: f^{-n}(q) \longrightarrow P\ as\ n \longrightarrow \infty\}$. Here f^{-1} denotes the inverse of the function f i.e. $fof^{-1} = f^{-1}of = id_\chi$.

If P is a periodic point of least period k, then it is a fixed point of f^k, and the stable and unstable set of P are $W^S(f,p) = W^S(f^k,p)$ and $W^U(f,p) = W^U(f^k,p)$.

Given a neighborhood U of P, the local stable and unstable set of P are defined by

$W^S{}_{loc}(f,p,u) = \{q\epsilon U: f^n(q)\epsilon U \; for \; each \; n \geq 0\}$ and
$W^U{}_{loc}(f,p,u) = W^S{}_{loc}(f^{-1},p,u)$

If χ is metrizable we can define the stable and unstable sets for any point by

$W^S(f,p) = \{q\epsilon\chi: d(f^n(q), f^n(p) \longrightarrow 0 \; for \;\; n \longrightarrow \infty\}$ and $W^U(f,p) = W^S(f^{-1},p)$

where d is a metric for χ.

Corollary 3.11 If A and B are linear maps from $\mathbb{R}^n \to \mathbb{R}^n$ with A hyperbolic and

$\|B - A\|$ Sufficiently small, A and B is conjugate.

Proof Define α as in the last proof, only this time it is constant on, $(\infty, -2]$, $[-1,1]$ and $[2,\infty)$ since $B = A + (B - A)$, write$f(\chi) = A\chi + \alpha(\|\chi\|)(B - A)\chi$. Using a similar procedure to the last proof, we get $\|\alpha(\|\chi\|)(B - A)\chi - \alpha(\|y\|)(B - A)y\| \leq$

$\|(\alpha(\|\chi\|) - \alpha(\|y\|))(B - A)\chi\| + \|\alpha(\|y\|)((B - A)\chi - (B - A)y)\| \leq (2k + 1)\|B - A\|\|\chi - y\|$. Consider the cases of neither, one or both points lying inside the ball of radius 2. If neither, then the first half of the intermediate step goes to zero. This is a Lipschitz condition, and as before by theorem 3.8 this implies that f and A are conjugate. But then A and Bare conjugate on $[-\epsilon, \epsilon]^n$ for $\epsilon > 0$ small. From this one can easily construct a global conjugacy.

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