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Hierarchical Multilevel Multi-leader Multi-follower Problem: Multi-parametric Solution Approach

by

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Declaration of originality

I, Addis Belete ZEWDE, declare that this dissertation titled, "Hierarchical Multilevel Multi-leader Multi-follower Problem: Multi-parametric Solution Approach" and the work presented in it are my own original works and that it has not been presented and will not be presented to any other university for a similar or any other degree award. Any published and unpublished work used here has been cited in the text, and the list of references in the bibliography section at the end of dissertation.

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Abstract

Hierarchical multilevel multi-leader multi-follower games are non-cooperative decision problems in which multiple decision-makers of equal status in the upper-level and multiple decision-makers of equal status are involved at each of the lower-levels of the hierarchy. Much of solution methods proposed so far on the topic are either model specific which may work only for a particular sub-class of problems or are based on some strong assumptions and mainly for two level cases. In this dissertation we have proposed a multi-parametric programming based solution approach for hierarchical multilevel multi-leader multi-follower games in which the objective functions contain separable and non-separable terms (but the non-separable terms can be written as a factor of two functions, a function which depends on other level decision variables and a function which is common to all objectives across the same level) and shared constraint. The proposed solution approach transforms a hierarchical multilevel multi-leader multi-follower game into multilevel game involving a single decision maker at each level of the hierarchy. In addition, a solution algorithm for bilevel optimization problems whose lower-level problem involves convex nonlinear constraints is also developed. The solution algorithm recasts the lower-level problem as a multi-parametric problem and employs an equivalent barrier problem reformulation. The solution obtained with this method is shown to be exact if the lower-level problem and the nonlinear constraints can be expressed by a polynomial of utmost degree three with followers' variable and upto quadratic in the variable of the leader.
keywords Bilevel optimization; Hierarchical multilevel; Multi-leader Multi-follower; Multi-parametric programming; Barrier method; Nonlinear constraints; Exact solutions

Dedicated to my family.

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List of acronyms

ACQ	Abide Constraint Qualification
CQ	Constraint Qualification
$\mathcal{CR}$	Critical Region
EPEC	Equilibrium Problem with Equilibrium Constraints
GCQ	Guignard Constraint Qualification
(G)NE	(Generalized) Nash Equilibrium
IFT	Implicit Function Theorem
KKT	Karush-Kuhn-Tucker
LP	Linear Programming
LICQ	Linear Independence Constraint Qualification
MFCQ	Mangasarian-Fromovitz Constraint Qualification
MLMF	Multi-leader Multi-follower
mp	multi-parametric
MPEC	Mathematical Program with Equilibrium Constraint
NLP	Nonlinear Programming
$\mathcal{NP}$	Non-deterministic Polynomial-time
QP	Quadratic Programming
SLMF	Single-leader Multi-follower
s.t.	subject to

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Chapter 1

Introduction

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1.1 Background

Extending a single optimization problem, a non-cooperative game is a multi-agent optimization problem wherein a finite set of intelligent rational decision-makers (usually called players) who have the intention of achieving several purposes by choosing one or some strategies in their strategy set. In game theory, a solution concept is a formal rule for defining the outcome of a game. The solutions describe which strategies will be adopted by players, therefore, the results of games. An equilibrium, the most used solution concept in game theory, is the point in a game where all players have made their decisions and an outcome is reached. Among them is the Nash equilibrium which has become the foundation of non-cooperative game theory ever since its introduction. A point in the domain is called a Nash equilibrium (NE) if no player has an incentive to change his choice unilaterally [73]. It means the strategies of every player is stable and that they are satisfied with their absolute best decision and that they won't change it.

In 1951, John Nash investigated the problem of finding equilibria in situations where several competing players attempt to optimize their objective functions over strategy sets that are independent of the decisions of the remaining players, which is known as the (classical) Nash equilibrium problem. Classical games involve players whose strategies are coupled only through the dependence of utility functions on strategies of other players. In such games, it is assumed that players can only affect the utilities of the other players but not their feasible sets. However, in many practical settings there are resource limitations that binds the choices players can make. Debreu [25], Arrow and Debreu[7] and Rosen [83] extended the problem setting towards a model with coupled strategy sets, that is, strategy sets which will depend upon the decisions of the remaining players. Coupled strategy sets arise in various ways if, for instance, players share at least one constraint which could be a common budget or commonly used infrastructure. This class of games with the feasible set being a proper subset of the full Cartesian product of the individual players' strategy sets has been given several names in the literature: social equilibrium games (Debreu [25]), pseudo-Nash equilibrium games, generalized Nash equilibrium games, normalized equilibrium games and coupled constraint equilibrium games (Rosen [83]).

The other type of equilibrium is called a Stackelberg equilibrium, which is a solution for a Stackelberg game. A Stackelberg (single-leader-follower) game arises when one player, the leader, commits to a strategy, while the remaining players, the followers, react to the strategy selected by competing among themselves. That is, the reaction of the followers is a Nash equilibrium parameterized by the decision variables for the leader. The leader chooses an optimal strategy anticipating how the followers will react. Related to the above two types of games is the multi-leader-follower game in which multiple competitive firms commit to their decisions prior to a number of competitive followers that react to the decisions made by the leaders.

1.1.1 Multilevel MLMF problems

Multilevel multi-leader multi-follower (MLMF) game is a hierarchical decentralized decision system in which there are multiple higher-level decision-makers (who are referred to as *leaders*) and many lower-level decision-makers (who are referred to as *followers*). In this formulation, the role of leaders and followers may not refer to the level of hierarchy in a management, rather it indicates the sequence of actions in the decision process among themselves. The leaders compete in a Nash game constrained by the equilibrium conditions of another Nash game among the followers. In the sequential part of the game Stackelberg behavior is assumed, from which the leader makes their decision first by competing in a Nash game constrained by the equilibrium conditions of another Nash game among the followers and the followers react by optimizing their objective functions conditioned on the leader's decision. Since originally appeared in a 1973 paper by Bracken and McGill [18] multilevel decision-making (some authors designated it as multilevel programming or multilevel optimization) often appears in many decentralized management problems in the real world and has motivated a number of researches on decision models, solution approaches and applications.

At each levels of decision hierarchy in multilevel MLMF problems, one needs to solve parametric generalized Nash Equilibrium problems, where the variables from upper levels are considered as parameters. It is well known that solving such problems can be a tedious and error-prone task [92]. Many of the solution approaches for solving problems with MLMF nature apply a reformulation of the Nash equilibrium problem and Stackelberg equilibrium problem as an 'equivalent' variational inequality (VI) problem and as a mathematical problem with equilibrium constraints (MPECs), respectively. But such reformulations have several limitations as mentioned in [65]. In addition, a standard approach in MPECs requires ascertaining when the reaction map admits fixed points. But this is difficult due to the lack of continuity in the solution set associated with the equilibrium constraints capturing the follower equilibrium. Because of these challenges most of the solution methods work only for particular subclasses of multilevel MLMF games which satisfy some strict conditions (such as, strict convexity, separability, etc) for two levels.

Under this approach, Okuguchi [75] presented models for multi-leader-follower games in a Cournot regime. With the assumption that each leader can exactly anticipate the aggregate reaction curve of the followers Okuguchi proved existence of an equilibrium solution for such problems. Moreover, uniqueness of such a solution is asserted for a special case of the problem where all leaders share an identical cost function and make identical decisions.

Sherali [87] also studied a multi-leader-follower game, referring it as a Stackelberg model, by associating it with equilibrium problem with equilibrium constraints (EPECs). In forward market equilibrium model, Su [91] extended the existence result of [75] and [87] under some weaker conditions. The approach is related to the relaxation method used in mathematical program with equilibrium constraints [85] that relaxes the complementarity condition of each leader and drives the relaxation parameter to zero. The works of Pang and Fukushima [77] and Leyffer and Munson [65] are also categorized under this reformulation.

By defining a new equilibrium concept called remedial leader-follower Nash equilibrium, Pang and Fukushima [77] formulated a class of MLMF games as a generalized Nash equilibrium problem with convexified strategy sets and presented an existence result with this equilibrium concept. On the other hand, based on the strong stationarity conditions of each leader in a two-level MLMF game and the equivalence between the Karush-Kuhn-Tucker conditions of the individual mathematical program with equilibrium constraints and strong stationarity assumption, Leyffer and Munson [65] derived a family of nonlinear complementarity problem to solve some class of bilevel MLMF games.

Although there are several instances of EPECs for which equilibria have been shown to exist, there are also fairly simple EPECs which admit no equilibria as shown in [77]. In addition, to replace the lower-level problems by their optimality conditions, this reformulation requires convexity of the lower level problems and some strict regularity conditions on the constraints. Under such condition it can be shown that the reformulated problem is equivalent to the original one. However, if the level of decision hierarchy is more than two then the equilibrium problem of the lower levels become highly non-convex and it will be difficult to establish continuity of the set-valued solution (equilibrium) function. Moreover, for problems with more than three hierarchical levels with multiple leaders and multiple followers at each level, it is not possible also to apply the principle of variational inequality to solve them.

An alternative solution approach in solving multilevel MLMF problems is reformulation of the problem as an equivalent hierarchical multilevel problem involving a single decision maker at all levels of the hierarchy. In such a case, once it is equivalently reformulated as a hierarchical multilevel problem, one can apply existing solution approaches for multilevel optimization models to solve the problem, though such methods themselves are limited. Wang et al. [100] used this reformulation for a convex bilevel problem with multiple followers and separable second-level objective functions. Under some weaker conditions, this reformulation is also employed in the works of Kulkarni and Shanbhag [63] and Kassa and Kassa [61]. The type of multilevel MLMF games considered in [63] is a problem in which the objective function of each player consists of separable terms and common non-separable terms across all the followers. Extending this result, Kassa and Kassa [61] reformulated a class of multilevel single-leader multiple-follower games, that consist of separable terms and non-separable terms across all the followers parameterized by constant positive weights. Tharakunnel and Bhattacharyya [93] used the so called "reinforcement learning" approach with Q-learning scheme to propose an algorithm to solve bilevel single-leader multi-follower (bilevel SLMF) games.

Multilevel MLMF games have been increasingly appearing in decentralized management situations in the current age of integrated economic developments where business firms work in a decentralized manner in a complex commercial networks comprised of suppliers, manufacturers, sales and logistics companies, customers and other specialized service functions [50]. In a multilevel multi-leader multi-follower games various relationships among multiple leaders in the upper-level and multiple followers at the lower-levels would generate different decision processes. The relationships could be expressed in terms of the objective functions, constraint sets and whether the constraints are shard or not. Due to its computational difficulties and mathematical complexities such as non-convexity and $\mathcal{NP}$-hardness[1] such optimization problems are lacking efficient algorithms.

The solution approaches for multilevel hierarchical and multilevel MLMF problems that are proposed so far are sensitive to the way the criteria functions at each of the levels are formulated. The existence of equilibria have been obtained mainly for multi-leader-follower games with specific structure (such as bilevel case, single leader case) and with constraint sets and/or objective functions assumed to have a nice property (such as linearity, convexity, differentiability, separability). Moreover, the development of implementable solution algorithms for multilevel MLMF games is at its infancy, and researchers are still working on this direction.

1.1.2 Bilevel optimization problems

Optimization problems involving two decision makers at two different decision levels are referred to as bilevel programming problems. The upper-level decision

[1] A problem is $\mathcal{NP}$-hard if it is polynomial-time reducible to an NP-complete problem.

maker (leader) solves an optimization problem which includes in its constraint set another optimization problem solved by the lower-level (follower) decision maker. Each decision maker (leader or follower) tries to optimize his/her own objective function without considering the objective of the other level, but the decision of each level affects the choice of values in the optimization of the other level [9]. Therefore, the leader may be able to influence the behavior of the follower without completely controlling the follower's action. At the same time the leader may be simultaneously affected by the follower's behavior.

Bilevel programming techniques was first introduced by Bracken and McGill [18] mainly to model a decentralized non-cooperative decision system in a hierarchical organization with one leader and multiple followers, but the term bilevel and multilevel programming were coined to such problems later by Candler and Norton [22].

Bilevel programming problems can be used to describe models for decision making situations in which decisions are carried out in sequential orders. Interest in bilevel problems has been growing both from the theoretical as well as from the practical points of view. Bilevel programming problems arise in different fields of applications, such as, management, economics, engineering and transportation (see [9, 24, 26, 27] and the references therein).

Bilevel programming problems are generally non-convex and non-differentiable, even in the linear case in which all objectives and constraint functions are linear. In fact, bilevel programming problems have been shown to be strongly nondeterministic polynomial-time hard ($\mathcal{NP}$-hard) [52], and especially, nonlinear bilevel programming problems are very difficult to solve.

Several algorithms have been proposed to solve bilevel programming problems. The existing analytic solution methods proposed for solving bilevel optimization problem can be roughly divided into the following categories: methods based on Karush-Kuhn-Tucker (KKT) conditions, interior point methods and multi-parametric approach. In KKT based methods, the second level problem is replaced by the KKT optimality conditions, yielding a one-level optimization problem with complementarity constraints. Hence, the problem transfers into a single objective with complementary constraints. Extreme point algorithms [21], kth best algorithm [16, 17, 102, 108], branch-and-bound algorithm [4, 8–12, 52, 67], descent method [79, 84, 96] and penalty functions method [2, 3, 66, 99, 104, 105] are KKT based solution approaches. Extreme point algorithms, k-th best algorithm, branch-and-bound algorithm and descent method restricted to linear problems. Whereas penalty functions methods are proposed to solve nonlinear bilevel programming problems.

The interior point methods formulate many large linear programs as nonlinear problems and solve them with various modifications of nonlinear algorithms. These methods require all iterates to satisfy the inequality constraints in the problem strictly. The primal-dual method [103] is a class of these methods which is the most efficient practical approach.

In heuristic methods bilevel programming problems are transformed into a single level problem by using transformation methods and then heuristic methods are utilized to find out the optimal solution. These approaches are generally appropriate to search global optimal solutions in very large space whenever convex or non-convex feasible domain is allowed. But there is no guarantee to find global optimal solutions or even bounded solutions. In heuristic methods there are no limitations on the differentiability of the functions, so several heuristic approaches are proposed in the literature, including: particle swarm optimization algorithm [50], simulated annealing [6], genetic algorithm [68], tabu search [46], systematic evolutionary algorithm [106], evolutionary algorithm [88, 101, 107].

Multi-parametric based algorithms for nonlinear bilevel programming problems are proposed in [36, 37, 58, 60]. The ability of multi-parametric programming to provide the solution of a problem across the entire parameter space gave rise to novel global optimization algorithms for the exact solution of different classes of bilevel programming problems [36]. The core idea of multi-parametric based algorithms for bilevel problems is to recast the lower-level problem as a multi-parametric programming problem, in which the optimization variables of the upper-level problem are considered as parameters for the lower-level. The resulting exact parametric solutions are then substituted into the upper-level problem, which can be solved using global optimization solver as a set of single-level programming problems. However, certain limitations are recognized in most of the algorithms developed so far due to convergence to a local optimal solution, specific structures of the problem (such as, the constraints of the lower-level problem is usually assumed to be affine or quadratic).

For linear and quadratic multi-parametric problems with polyhedral constraints, exact solutions can be established as shown in [15, 28, 32]. It was also shown that the optimal solution can be expressed as a set of linear functions and the critical regions as a set of linear inequalities. The linearity of the optimal solution results in linear and quadratic expression of the optimal value function respectively for linear and quadratic multi-parametric problems. On the other hand, for general nonlinear multi-parametric problems obtaining the exact expressions of optimal solution, optimal value function and the characterization of the critical regions is a very difficult task. That could be one of the main reasons for all major efforts in this area have focused on providing approximate solutions.

For the convex case, the first algorithm for the solution of problem multi-parametric problems was presented by Dua and Pistikopoulos in [33], which is based on the linearization of the objective and constraint functions and the solution of the resulting multi-parametric linear program (mp-LP). This algorithm has been developed further by Acevedo and Salgueiro in [1], and modified by using convex quadratic approximations for objective function (i.e., quadratic approximation of the objective function and linear approximation of the constraints) in [30, 55]. A distinctly different approach to Dua and Pistikopoulos [33] for the solution of multi-parametric problems was proposed by Johansen [56].

Pistikopoulos et al. [80] proposed a solution strategy for special non-convex multi-parametric programming problems based on a branch-and-bound algorithm to locate the global parametric solution. The key procedures of this algorithm was modified by Kassa and Kassa [59] and proposed a global optimization technique for solving a more general multi-parametric non-convex programming problems using sensitivity analysis. In particular, the solution approach effectively solves parametric problems with any twice continuously differentiable nonlinear objective function and having only polyhedral constraints. Domínguez and Pistikopoulos [29] proposed a multi-parametric quadratic approximation algorithm for non-linear multi-parametric problems to address the case where the feasible parameter space is defined by a set of nonlinear equations. A quadratic approximation solution strategy is also proposed by Pappas et al. in [78] to find exact solutions for quadratically constrained multi-parametric quadratic problems and used an active set-based approach for the exploration of the parameter space.

1.2 Mathematical formulation

This dissertation mainly focuses on the solution procedures of decentralized multi-level games, which allow all decision-makers to have different objective functions, and to play a non-cooperative game among themselves. Decentralized multilevel games can be classified based on the number of hierarchical levels in the decision process and the number of players assuming the role of the leader and that of the follower in those levels. Three cases are modeled: (i) multilevel hierarchical, where there are many levels, with one decision-maker in each level; (ii) hierarchical multilevel single-leader multi-follower game, where there is one higher-level decision-maker (who is referred to as the leader) and many lower-level decision-makers (who are referred to as followers) and (iii) hierarchical multilevel multi-leader multi-follower game, where two or more Stackelberg players with identical followers compete non-cooperatively. In all cases, Stackelberg behavior is assumed, whence the leader makes its decision first and the divisions react by optimizing their objective functions conditioned on the leader's decision.

1.2.1 Multilevel hierarchical games

Multilevel hierarchical game is a mathematical programming with many levels that models decentralized planning problems with decision makers arranged hierarchically and there is only one decision maker at each level. The execution of decisions is sequential, from top to the bottom level. Each decision maker independently minimizes their own objective but is affected by the actions of other decision makers at the different levels. Multilevel hierarchical games are nested optimization problems in which part of the constraint set is defined by the solution sets of another optimization problems, so-called follower's problems [61].

A k-level hierarchical game where leaders and followers at all levels have their own decision variables, objective functions and constraints can be formulated mathematically as follows.

$$\begin{cases} \min\limits_{y_1 \in Y_1} f_1(y_1, y_2, \ldots, y_k) \\ \text{s.t.} \begin{cases} g_1(y_1, y_2, \ldots, y_k) \leq 0, \text{ and} \\ (y_2, y_3, \ldots, y_k) \text{ solves } \begin{cases} \min\limits_{y_2 \in Y_2} f_2(y_1, y_2, \ldots, y_k) \\ \text{s.t.} \begin{cases} g_2(y_1, y_2, \ldots, y_k) \leq 0, \text{ and} \\ \qquad \ddots \\ y_k \text{ solves } \begin{cases} \min\limits_{y_k \in Y_k} f_k(y_1, y_2, \ldots, y_k) \\ \text{s.t. } g_k(y_1, y_2, \ldots, y_k) \leq 0, \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{1.2.1}$$

where $y_i \in Y_i \subset \mathbb{R}^{n_i}$, the functions $f_i : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}$ are the i^{th}*-level objective functions* and the vector-valued functions $g_i : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \to \mathbb{R}^{m_i}$ are called the i^{th}*-level constraints* respectively.

For $k = 2$, problem (1.2.1) becomes a bilevel hierarchical game and can be described as,

$$\begin{cases} \min\limits_{y \in Y} F(x,y) \\ s.t. \begin{cases} G(x,y) \leq 0, \text{ and} \\ x \text{ solves } \begin{cases} \min\limits_{x \in X} f(x,y) \\ s.t. \;\; g(x,y) \leq 0, \end{cases} \end{cases} \end{cases} \tag{1.2.2}$$

where $x \in X \subseteq \mathbb{R}^{n_1}$ and $y \in Y \subseteq \mathbb{R}^{n_2}$. The variables of problem (1.2.2) are divided into two classes, namely the *upper-level variables* y and the *lower-level variables* x. Similarly, the functions $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ are the *upper-level* and *lower-level objective functions* respectively, while the vector-valued functions $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1}$ and $g : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_2}$ are called the *upper-level* and *lower-level constraints* respectively. The *relaxed* problem associated with (1.2.2) is

$$\begin{cases} \min\limits_{x,y} F(x,y) \\ s.t. \begin{cases} G(x,y) \leq 0, \\ g(x,y) \leq 0, \end{cases} \end{cases} \tag{1.2.3}$$

and its optimal value is a lower bound for the optimal value of (1.2.2).

Solution concepts related to bilevel program (1.2.2) are defined as follows,

(i) The *relaxed feasible region* (or *constraint region*) is

$$\Omega = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : x \in X, y \in Y, G(x,y) \leq 0 \text{ and } g(x,y) \leq 0\}.$$

(ii) For a given (fixed) vector $\bar{y} \in Y$, the *lower-level feasible set* is defined by

$$\Omega(\bar{y}) = \{x \in X \subset \mathbb{R}^{n_1} : g(x,\bar{y}) \leq 0\}.$$

(iii) The *lower-level reaction set* (or *rational reaction set*) is

$$\mathcal{R}(\bar{y}) = \{x \in X \subset \mathbb{R}^{n_1} : x \in \text{argmin}\{f(\hat{x},\bar{y}) : \hat{x} \in \Omega(\bar{y})\}\}.$$

Every $x \in \mathcal{R}(\bar{y})$ is a *rational response*. For a given y, $\mathcal{R}(y)$ is an implicitly defined multi-valued function of y that may be empty for some values of its argument.

(iv) The set

$$\mathcal{IR} = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : y \in Y, G(x,y) \leq 0, x \in \mathcal{R}(y)\},$$

that regroups the feasible points of problem (1.2.2), corresponds to the feasible set of the leader, and is known as the *induced region* (or *inducible region*). This set is usually non-convex and it can even be disconnected or empty in presence of upper-level constraints.

Using lower level reaction set problem (1.2.2) can be expressed as constrained optimization problem as follows,

$$\begin{aligned} &\min_{y \in Y} \ F(x,y) \\ &s.t. \ \begin{cases} G(x,y) \leq 0, \\ x \in \mathcal{R}(y), \end{cases} \end{aligned} \tag{1.2.4}$$

where $\mathcal{R}(y) = \{x \in X \subset \mathbb{R}^{n_1} : x \in \text{argmin}\{f(\hat{x}, y) : \hat{x} \in \Omega(y)\}\}$.
Also using the induced region of the leader problem of (1.2.2) can be expressed as

$$\begin{aligned} &\min_{x,y} \ F(x,y) \\ &\quad s.t. \ (x,y) \in \mathcal{IR}. \end{aligned} \tag{1.2.5}$$

The expressions (1.2.4) and (1.2.5) are well-defined if the lower level problem has a unique optimal solution. But an ambiguity arises in case of multiple lower level optimal solutions for any given upper level decision vector.

1.2.2 Hierarchical multilevel multi-leader multi-follower games

Hierarchical multilevel multi-leader multi-follower (in short, MLMF) games are class of hierarchical games in which several players take the position as leaders and the rest of players who serve as followers. All leaders compete with each other in a non-cooperative (generalized) Nash game in the upper-level and make their decisions first by anticipating the responses of followers. At the same time, all followers select their own optimal responses by competing with each other in a (generalized) Nash game in the lower-level parameterized by the leader's decision.

Consider a k-level hierarchical multi-leader multi-follower game involving N_1 decision makers at the first-level, N_2 decision makers at the second-level, ..., and N_k decision makers at the k^{th}-level. Under the assumptions that (i) there is a shared constraint common to all problems at the same level, (ii) the reaction of all followers are consistent across all the leaders at all hierarchical levels, and (iii) leaders and followers at all levels have their own decision variables, objective functions and constraints, the problem can be formulated mathematically as follows.

For $n = 1, \dots, N_1$, the vector $(y_1^n, y_2, \dots, y_k)$ solves an optimization problem,

$$\begin{cases} \min\limits_{y_1^n \in Y_1^n} F_1^n(y_1, y_2, \dots, y_k) \\ \text{s.t.} \begin{cases} G_1^n(y_1^n, y_2, \dots, y_k) \le 0, \\ H_1(y_1, y_2, \dots, y_k) \le 0, \text{ and for all } i = 1, \dots, N_2, \\ (y_2^i, y_3, \dots, y_k) \text{ solves} \begin{cases} \min\limits_{y_2^i \in Y_2^i} f_2^i(y_1, y_2, \dots, y_k) \\ \text{s.t.} \begin{cases} g_2^i(y_1, y_2, \dots, y_k) \le 0, \\ h_2(y_1, y_2, \dots, y_k) \le 0, \text{ and} \\ \ddots \\ \text{for all } l = 1, \dots, N_k, \\ y_k^l \text{ solves} \begin{cases} \min\limits_{y_k^l \in Y_k^l} f_k^l(y_1, y_2, \dots, y_k) \\ \text{s.t.} \begin{cases} g_k^l(y_1, y_2, \dots, y_k) \le 0, \\ h_k(y_1, y_2, \dots, y_k) \le 0, \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{1.2.6}$$

where $y_1 = (y_1^1, \dots, y_1^{N_1})$ is a decision vector for the leader's problem and $(\forall n)\ y_1^n \in Y_1^n$ is a decision vector of the n^{th}-leader and $y_1^{-n} = (y_1^1, \dots, y_1^{n-1}, y_1^{n+1}, \dots, y_1^{N_1})$ is a vector of the decision variables for all leaders without y_1^n; note that as customary we may write $y_1 = (y_1^n; y_1^{-n})$. The shared constraint H_1 is common to all leaders whereas, the constraint G_1^n is only for the n^{th}-leader. Similarly, at the second-level, $y_2 = (y_2^1, \dots, y_2^{N_2})$ is a decision vector for the second-level problem and $\forall i \in \{1, 2, \dots, N_2\},\ y_2^i \in Y_2^i$ is a decision vector of the i^{th}-player at the second-level optimization problem and $y_2^{-i} = (y_2^1, \dots, y_2^{i-1}, y_2^{i+1}, \dots, y_2^{N_2})$ is a vector of the decision variables of all second-level players without the vector y_2^i; and $y_2 = (y_2^i; y_2^{-i})$. The shared constraint h_2 is common to all second-level players whereas, the constraint G_2^i affects only the i^{th} player of the second-level problem.

Finally, at k^{th}-level, $y_k = (y_k^1, \dots, y_k^{N_k})$ is a decision vector for the k^{th}-level problem and $\forall l \in \{1, 2, \dots, N_k\},\ y_k^l \in Y_k^l$ is a decision vector of the l^{th}-player at the k^{th}-level problem and $y_k^{-l} = (y_k^1, \dots, y_k^{l-1}, y_k^{l+1}, \dots, y_k^{N_k})$ is a vector of the decision variables of all k^{th}-level players without the vector y_k^l; and $y_k = (y_k^l; y_k^{-l})$. The shared constraint h_k is common to all k^{th}-level players whereas, the constraint G_k^l is assumed to affect only the l^{th}-player of the k^{th}-level problem.

Remark 1. *In problem* (1.2.6), *when there is only one higher-level decision maker (leader) the problem is called a hierarchical multilevel single-leader multi-follower (in short, SLMF) problem.*

In problem (1.2.6), if $k = 2$ the problem is called a bilevel MLMF game. As a bilevel problem, all leaders in the upper-level compete with each other in a non-cooperative Nash game and make their decisions first by anticipating the responses of all the followers. Upon receipt of the leaders' decisions, all followers compete with each other in a parametric non-cooperative Nash game in the lower level with the strategies of leaders as exogenous parameters [54]. This makes the problem of solving a multilevel problem having multiple decision makers at each decision level quite difficult and complex, particularly when there is information exchange between followers. In may applications, for instance in supply chain management models, however there are more than two decision agents, like manufacturers, distributors, retailers (or vendors), etc. If we follow the traditional approach, there will appear non-convex terms and shared variables across the followers at the $(k-1)^{th}$-level due to complementarity conditions from the k^{th}-level followers. This may result in challenging task in the process of solving multilevel MLMF problem having shared resources and information.

1.3 Statement of the problem

Solution approaches for solving multilevel problems with SLMF nature apply a reformulation of the Nash equilibrium problem and Stackelberg equilibrium problem as an 'equivalent' variational inequality (VI) problems and mathematical problems with equilibrium constraints (MPECs), respectively. On the other hand, the multilevel MLMF game can be reformulated as an equilibrium problem with equilibrium constraints (EPEC), one of the major approaches for MLMF game. An EPEC is an equilibrium problem consisting of several parametric MPECs which contain the players' strategies as parameters. However, finding an equilibrium point of an EPECs is much more difficult than solving a single MPECs, because the constraints of each leader's problem depend on the other rival leaders' strategies, and all leaders share decision variables of the followers. Thus, the equilibria of an EPEC can be achieved when all MPECs are solved simultaneously. But such reformulations have several limitations as mentioned in [65]. Due to these challenges most of the solution methods work only for particular subclasses of multilevel MLMF games which satisfy some strict conditions.

An alternative solution approach in solving multilevel MLMF problems is reformulation of the problem as an equivalent hierarchical multilevel problem involving a single decision maker at all levels of the hierarchy. In such a case, once it is equivalently reformulated as a hierarchical multilevel problem, one can apply existing solution approaches for multilevel optimization models to solve the problem. However, certain limitations are recognized in most of the algorithms developed so far, to solve multilevel optimization problems, due to convergence to a local optimal solution (e.g. heuristic methods), specific structure of the problem (such as bilevel case, single leader case) and with constraint sets and/or objective functions assumed to have a nice property (such as linearity, separability, assuming potential function).

Therefore, this dissertation is intended to study the solution approaches for general multilevel problems with multiple-leaders and multiple-followers setting and with shared constraints.

1.4 Objectives

The objective of this dissertation is to develop solution method for hierarchical multilevel multi-leader multi-follower problems in which multiple leaders of equal status in the upper-level and multiple followers of equal status are involved at each lower-level of the hierarchy. The specific objectives of this dissertation are summarized as follows.

1. To develop a new solution method that can solve multi-parametric nonlinear problems with convex nonlinear constraints.
2. To develop an algorithmic approach that can be applied to find the solution of a bilevel optimization problem whose lower-level problem is a convex optimization involving nonlinear constraints.
3. To develop a solution algorithm that can solve some classes of multilevel MLMF games.

1.5 Contributions

A summary of the contributions of this dissertation is as follows:

- a novel solution approach for solving multi-parametric problems with nonlinear constraints,

– multi-parametric based solution algorithm for bilevel optimization problems whose lower-level optimization involves nonlinear convex constraints,

– method to transform classes of hierarchical multilevel MLMF games into equivalent multilevel hierarchical games having a single decision maker at each levels, and

– a solution algorithm for hierarchical multilevel MLMF games.

Publications from this dissertation
The work from this dissertation that has been published as journal articles or book chapters are listed below.
Peer Reviewed

- Zewde, A. B., & Kassa, S. M. (2021). Multilevel multi-leader multiple-follower games with nonseparable objectives and shared constraints. *Computational Management Science*, 1-21. https://doi.org/10.1007/s10287-021-00398-5

- Zewde, A. B., & Kassa, S. M. (2021). Hierarchical multilevel optimization with multiple-leaders multiple-followers setting and nonseparable objectives. *RAIRO-Oper. Res.*, 55, 2915-2939. https://doi.org/10.1051/ro/2021146

- Zewde, A. B., & Kassa, S. M. (2021). Multi-parametric approach for multilevel multi-leader-multi-follower games using equivalent reformulations. *J. Math. Comput. Sci.*, 11(3), 2955-2980. https://doi.org/10.28919/jmcs/5641

- Zewde A.B., Kassa S.M. (2020) A Method for Solving Some Class of Multilevel Multi-leader Multi-follower Programming Problems. In: Le Thi H., Le H., Pham Dinh T. (eds) Optimization of Complex Systems: Theory, Models, Algorithms and Applications. WCGO 2019. *Advances in Intelligent Systems and Computing*, vol 991. Springer, Cham.

  https://doi.org/10.1007/978-3-030-21803-4_59

1.6 Organization of the study

The remaining chapters of this dissertation are organized as follows. Chapter 2 presents preliminary concepts in optimization and game theory that are necessary for the remaining parts of this dissertation.

Chapter 3 introduces multi-parametric programming problems and presents a novel solution algorithm for solving multi-parametric problems with nonlinear constraints.

Chapter 4 proposes a global solution strategy for the general classes of nonlinear bilevel programming problems whose lower-level problem involve a convex non-linear constraints.

Chapter 5 presents a solution procedure for two classes of multilevel MLMF games: (i) hierarchical multilevel MLMF weighted potential games with the assumption that at each level there is a shared constraint common to all problems of same level, and (ii) hierarchical multilevel MLMF games with the assumptions that at each level there is a shared constraint common to all problems of same level and that the objective functions at all levels contain separable and non-separable terms. The non-separable terms are assumed to be written as a factor of two functions where the first one is a function of other level decision variables and the second factor is common to all objectives across the same level.

The reformulations of the above mentioned two classes of games into equivalent multilevel hierarchical games involving a single decision maker at each level over the hierarchy is proposed, and equivalence between the original problems and the reformulated ones is established. Using these equivalent reformulations, a multi-parametric based solution procedure is also proposed for such games.

Finally, in Chapter 6 concluding remarks and recommendations for future research directions are indicated.

Chapter 2

Preliminaries

Contents

In this chapter, we give some basic definitions and preliminary mathematical background on optimization problems and game theory. In the next section we follow mainly the presentation given in [5, 13, 82].

2.1 Optimization problems

Optimization or mathematical programming is concerned with minimizing or maximizing some quantity, represented by an objective function. Optimization problems can be divided into two large classes, namely *Constrained* and *Unconstrained* problems. The basic unconstrained optimization problem can be stated in its standard form as

$$\begin{aligned} &\min f(x) \\ &s.t.\ \ x \in \mathbb{R}^n, \end{aligned} \tag{2.1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the *objective function*. On the other hand, constrained optimization problems can be written as

$$\min f(x) \tag{2.1.2a}$$

$$s.t.\ \ x \in \mathbb{R}^n, \tag{2.1.2b}$$

$$g_i(x) \leq 0, \ \ i = 1, \dots, m \tag{2.1.2c}$$

$$h_j(x) = 0, \ \ j = 1, \dots, l. \tag{2.1.2d}$$

Equations (2.1.2b–2.1.2d) indicate the *constraints*. The inequality and equality constraints functions, respectively, are defined by the functions $g_i, h_j : \mathbb{R}^n \to \mathbb{R},\ i = 1, \dots, m,\ j = 1, \dots, l$. A point $x \in X$ is said to be *feasible* if it satisfies all the constraints, and the set of all feasible points is called the *feasible set*, and denoted by $\mathcal{F} = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i = 1, \dots, m,\ h_j(x) = 0,\ \ j = 1, \dots, l\}$.

Since any maximization problem can be rewritten as a minimization problem,

$$\max f(x) = -\min(-f(x)),$$

the formulations (2.1.1) and (2.1.2) are called standard formulations. Because of this, all problems in this dissertation will be described as minimization problems.

2.1.1 Solutions of optimization problems

In a minimization problem, if we are looking for a point x^* in the domain $\mathcal{D}$ of f such that

$$f(x^*) \leq f(x),\ \text{for all } x \in \mathcal{D},$$

then x^* is called the *global minimizer* and $f(x^*)$ the *global minimum* of f. Similarly, in a constrained problem, the solution must lie inside the feasible set $\mathcal{F}$, and thus a global constrained minimizer satisfies

$$f(x^*) \leq f(x),\ \text{for all } x \in \mathcal{F}.$$

If a solution x^* in a neighborhood $\mathcal{N} \subset \mathcal{D}$ of x^*, satisfies

$$f(x^*) \leq f(x),\ \text{for all (feasible) } x \in \mathcal{N}, \tag{2.1.3}$$

then x^* is called a *local minimizer* and $f(x^*)$ a *local minimum* of f in $\mathcal{N}$. If x^* is such that

$$f(x^*) < f(x),\ \text{for all (feasible) } x \in \mathcal{N}, \tag{2.1.4}$$

then x^* is called a *strict local minimizer* and $f(x^*)$ a *strict local minimum* of f in $\mathcal{N}$.

There are some optimality conditions, which allow us to determine if we are at the solution or not. So, let us state a few definitions which will be useful for the rest part of this dissertation.

2.1.2 Derivatives and Taylor's Theorem

Consider a real function $f : \mathbb{R} \to \mathbb{R}$. Then, f is *differentiable* if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

exists. If this is the case, then

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

is called the *derivative* of f.

Definition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a multivariate function. We say that f is differentiable at a point $x_0 \in \mathbb{R}^n$ if all its partial derivatives*

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{f(x + h\boldsymbol{e}_i) - f(x)}{h}, \; i = 1, \ldots, n$$

where $\boldsymbol{e}_i$ is the i-th coordinate vector in $\mathbb{R}^n$, exist and if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)}{\|x - x_0\|} = 0.$$

If this is the case, then we define the *gradient* of f as the vector that groups all its partial derivatives, and we denote it by

$$\nabla f(x) = \begin{pmatrix} \dfrac{\partial f(x)}{\partial x_1} \\ \vdots \\ \dfrac{\partial f(x)}{\partial x_n} \end{pmatrix}.$$

If f is differentiable, and all derivatives of f are continuous with respect to x, then we say that f is *continuously differentiable*, and this is denoted by $f \in \mathcal{C}^1$.

The second partial derivatives of f are defined by

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial f(x)}{\partial x_j} \right), \; 1 \le i, j \le n.$$

If all second partial derivatives of f exist, then f is said to be *twice differentiable*; if, furthermore, all second partial derivatives of f are continuous, we say that f is *twice continuously differentiable*, and denote this by $f \in \mathcal{C}^2$.

The $n \times n$ matrix defined as

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{pmatrix}$$

is called the *Hessian matrix* of f.

For the function, $h : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ which depends on several vectors, we will use the following notation: $\nabla_x h(x, y) \in \mathbb{R}^n$ and $\nabla^2_{xx} h(x, y) \in \mathbb{R}^{n \times n}$ will be used to denote the gradient and the Hessian matrix of h with respect to x; $\nabla_y h(x, y) \in \mathbb{R}^q$ and $\nabla^2_y h(x, y) \in \mathbb{R}^{q \times q}$ will be used to denote the gradient and the Hessian matrix of h with respect to y; and $\nabla h(x, y) \in \mathbb{R}^{n+q}$ and $\nabla^2 h(x, y) \in \mathbb{R}^{(n+q)\times(n+q)}$ will denote the complete first-and second-order derivatives of h with respect to both x and y.

Theorem 2.1 (Taylor's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and $d \in \mathbb{R}^n$. Then, there exists some $t \in (0, 1)$ such that*

$$f(x + d) = f(x) + \nabla f(x + td)^T d.$$

Moreover, if f is twice continuously differentiable, we can find, some $t \in (0, 1)$,

$$f(x + d) = f(x) + \nabla f(x)^T d + \frac{1}{2} d^T \nabla^2 f(x + td) d.$$

Definition 2.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. We say that F is differentiable at a point $x_0 \in \mathbb{R}^n$ if the partial derivatives*

$$\frac{\partial F_j}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{F_j(x_0 + h\boldsymbol{e}_i) - f_j(x_0)}{h},$$

all exist and if

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - \sum_{j=1}^n \nabla F_j(x_0) \cdot (x - x_0)\boldsymbol{e}_j\|}{\|x - x_0\|} = 0.$$

In particular, for a differentiable vector function $F : \mathbb{R}^n \to \mathbb{R}^m$ where $F(x) = (F_1(X), \ldots, F_m(X))^t$, the *Jacobian* of F, denoted by the gradient notation $\nabla F(x)$, is given by the $m \times n$ matrix

$$\nabla F(x) = \begin{pmatrix} \nabla F_1(x)^t \\ \vdots \\ \nabla F_m(x)^t \end{pmatrix}_{m \times n} = \begin{pmatrix} \frac{\partial F_1(x)}{\partial x_1} & \frac{\partial F_1(x)}{\partial x_2} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\ \frac{\partial F_2(x)}{\partial x_1} & \frac{\partial F_2(x)}{\partial x_2} & \cdots & \frac{\partial F_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m(x)}{\partial x_1} & \frac{\partial F_m(x)}{\partial x_2} & \cdots & \frac{\partial F_m(x)}{\partial x_n} \end{pmatrix}_{m \times n}$$

whose rows correspond to the transpose of the gradients of $F_1, \ldots, F_m$, respectively.

A subset $\mathcal{K} \subset \mathbb{R}^n$ is a *cone* if for any point $x \in \mathcal{K}$ and any $\alpha \in [0, \infty)$, the point αx lies in $\mathcal{K}$. The point $\{0\}$ belongs to any cone. A subset $\mathcal{D} \subset \mathbb{R}^n$ is *convex* if for any points $x, y \in \mathcal{D}$ and $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y \in \mathcal{D}$.

Definition 2.3. *A function $f : X \to Y$ is convex (resp. concave) if and only if $\forall x, y \in X, \forall \lambda \in [0, 1]$,*

$$f(\lambda x + (1 - \lambda)y) \leq (resp. \geq)\lambda f(x) + (1 - \lambda)f(y). \tag{2.1.5}$$

Strict convexity/concavity is obtained when inequality (2.1.5) *is strict.*

If we define epigraph ($epi(f)$) and hypgraph ($hyp(f)$) of f by $epi(f) = \{(x, y), f(x) \leq y\}$ and $hyp(f) = \{(x, y), f(x) \geq y\}$, then the convexity (resp. concavity) of a function f is equivalent to the convexity of $epi(f)$ (resp. $hyp(f)$).

A function $f = (f_1, f_2, \ldots, f_n)$ is convex if every component of f is convex.

A function $h(x) : \mathbb{R}^n \to \mathbb{R}^l$ is *affine* if it can be written as $h(x) = Ax - b$ for some $A \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$.

The following operations preserves convexity:

(i) Non-negative weighted sums: let $w_1, \ldots, w_n \geq 0$ and $f_1, \ldots, f_n$ are convex functions, then $f(x) = w_1 f_1(x) + \ldots + w_n f_n(x)$ is convex.

(ii) Composition with an affine mapping: let $f : \mathbb{R}^n \to \mathbb{R}, A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(x) = f(Ax + b)$ with $dom(g) = \{x | Ax + b \in dom(f)\}$. Then g is convex (resp. concave) if f is convex (resp. concave).

(iii) Point-wise maximum: if $f_1, \ldots, f_n$ are convex, then

$$f(x) = \max\{f_1(x), f_2(x), \ldots, f_m(x)\},$$

with $dom(f) = dom(f_1) \cap dom(f_2) \cap \ldots \cap dom(f_n)$ is also convex.

(iv) Restriction to a line: let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and fix $x, y \in \mathbb{R}$, then $g : \mathbb{R} \to \mathbb{R}$ defined by $g(\alpha) = f(x + \alpha y)$ is convex.

(v) Composition rule: $f(x) = h(g(x))$ is convex if g is convex and h is convex non-decreasing (or g is concave and h is convex non-increasing). For example, $f(x) = e^{g(x)}$ is convex if g is convex, $f(x) = \frac{1}{g(x)}$ is convex if g is concave and positive, $f(x) = g(x)^p, p \geq 1$ is convex if g is convex and positive.

Definition 2.4. *A function $f : X \to Y$ is quasi-convex (resp. quasi-concave) if and only if $\forall x, y \in X, \forall \lambda \in (0,1)$,*

$$f(\lambda x+(1-\lambda)y) \leq \max(f(x), f(y)),\ (resp.)\ f(\lambda x+(1-\lambda)y) \geq \min(f(x), f(y)). \quad (2.1.6)$$

Strict quasi-convexity/concavity is obtained when inequality (2.1.6) *is strict.*

Definition 2.5. *A correspondence F from a set X to a set Y is assigns to each x in X a subset $F(x)$ of Y.*

The terms *multifunction* or *set-valued function* are sometimes used to mean a correspondence. A correspondence is also denoted by $F : X \to 2^Y$ or $F : X \twoheadrightarrow Y$ to distinguish a correspondence from a function from X to a set Y.

Let $F : X \twoheadrightarrow Y$ be a correspondence. As with functions, X is referred as *domain* of F, and Y as the *range space* (or *codomain*). The *image* of a set $A \subset X$ under F is the set $F(A) = \bigcup_{x \in A} F(x)$. The *range* of F is the image of X. The *graph* of F is given by $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$.

The inverse image of a set A under a function f is the set $\{x : f(x) \subset A\}$. Just as functions have inverses, so do correspondences.

Definition 2.6. *For a correspondence $F : X \twoheadrightarrow Y$*

(i) the upper inverse F^+ (also called the strong inverse) of a subset A of Y is defined by $F^+(A) = \{x : F(x) \subset A\}$.

(ii) the lower inverse F^- (also called the weak inverse) of a subset A of Y is defined by $F^-(A) = \{x : F(x) \cap A \neq \emptyset\}$.

Note that if F is singleton-valued (that is, if F is a function), then both $F^+(A)$ and $F^-(A)$ coincide with the inverse of A viewing F as a function.

Definition 2.7. *Let $F : X \to 2^Y$ be a set valued map, then*

(i) F is upper semi-continuous at x if for every neighborhood U of $F(x)$ the upper inverse image $F^+(U)$ is also a neighborhood of x in X.

(ii) F is lower semi-continuous at x if for every neighborhood U of $F(x)$ the lower inverse image $F^-(U)$ is also a neighborhood of x in X.

Definition 2.8. *(i) A bounded polyhedron is a finite union of convex hulls of finite-point sets.*

(ii) (Redundant Constraints). Let a polyhedron Θ be represented by $A\theta \leq b$. We say that $A_i\theta \leq b_i$ is redundant if $A_j\theta \leq b_j \; \forall j \neq i \Rightarrow A_i\theta \leq b_i$ (i.e., it can be removed from the description of the polyhedron).

(iii) (Minimal Representation). A representation of a polyhedron is minimal if there are no redundant constraints.

(iv) A polyhedron is a subset S of R^n homeomorphic to a bounded polyhedron P, i.e. there exists a bijective function between S and P.

2.1.3 Conditions for the solutions of unconstrained problems

Consider the unconstrained optimization problem as defined in (2.1.1). Then the following theorems helps us to identify the solution to such problems.

Theorem 2.2 (First-Order Necessary Conditions)**.** *If x^* is a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$, where f is continuously differentiable in an open neighborhood $\mathcal{N}$ of x^*, then*

$$\nabla f(x^*) = 0. \tag{2.1.7}$$

Any x^* that satisfies (2.1.7) is called a *stationary point* of f.

If $\nabla^2 f$ exists and is continuous in a neighborhood of x^*, we can state another necessary condition satisfied by a local minimizer.

Theorem 2.3 (Second-Order Necessary Conditions)**.** *If x^* is a local minimizer of f, and f is twice continuously differentiable in an open neighborhood $\mathcal{N}$ of x^*, then*

$$\nabla f(x^*) = 0 \text{ and } \nabla^2 f(x^*) \text{ is positive semidefinite.} \tag{2.1.8}$$

From the above theorem we can see that any local minimizer must be a stationary point; however, the opposite is not necessarily true. If the next conditions, called *sufficient conditions*, are satisfied by a stationary point x^*, they guarantee that it is a local minimizer.

Theorem 2.4 (Second-Order Sufficient Conditions)**.** *Let f be twice continuously differentiable on an open neighborhood $\mathcal{N}$ of x^*. If x^* satisfies*

$$\nabla f(x^*) = 0 \text{ and } \nabla^2 f(x^*) \text{ is positive definite,} \tag{2.1.9}$$

then x^* is a strict local minimizer of f.

It is important to note here that the second-order sufficient conditions are not necessary: a point can be a strict local minimizer and fail to satisfy the sufficient conditions.

2.1.4 Conditions for the solutions of constrained problems

Now, to characterize a constrained strict local minimizer of a function f, we define some additional concepts related to constrained problems. Unless otherwise stated, throughout this subsection and the rest of this dissertation, $X = \mathbb{R}^n$.

Let the index sets of the inequality and equality constraints, respectively, be $\mathcal{I} = \{1, 2, \ldots, m\}$ $\mathcal{E} = \{1, 2, \ldots, l\}$, we say that an inequality constraint g_i, for some index $i \in \mathcal{I}$ is *binding* (or *active*) at a feasible point x if $g_i(x) = 0$. At any feasible point x, the *active index set* is defined as,

$$\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} : g_i(x) = 0\}.$$

The *Lagrangian* (or *Lagrange function*) for problem (2.1.2) is defined by

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i \in \mathcal{I}} \lambda_i g_i(x) + \sum_{j \in \mathcal{E}} \mu_j h_j(x),$$

and the scalars $\lambda_i, i \in \mathcal{I}$ and $\mu_j, j \in \mathcal{E}$ are called the *Lagrange multipliers*.

The vectors of Lagrangian multipliers $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ and $\mu = (\mu_1, \ldots, \mu_l) \in \mathbb{R}^l$ are called the *dual variables* for the problem (2.1.2).

In constrained optimization, a solution is not only characterized by conditions on the objective function at the solution, but also by conditions on the constraints. These conditions are called *constraint qualifications*(CQ). So constraint qualification is an assumption made about the functions that define the constraints of the problem that, when satisfied by a local minimizer, ensures that it is stationary.

A subset $C \subseteq \mathbb{R}^n$ is a *cone* when $td \in C$, for all $t \geq 0$ and $d \in C$. For any cone C, we use C^* to denote its *dual cone* as $C^* = \{d \in \mathbb{R}^n \mid d^T x \geq 0, \forall x \in C\}$. Given a set $S \subseteq \mathbb{R}^n$, the polar of S, is given by $\mathcal{P}(S) := \{p \in \mathbb{R}^n \mid p^T x \leq 0 \text{ for all } x \in S\}$.

Consider the constrained problem (2.1.2) with continuously differentiable functions f, g_i, h_j. Then for a feasible point $\bar{x} \in \mathcal{F}$, the *(Bouligand) tangent cone* and the *(Fréchet) normal cone* for (2.1.2) at $\bar{x}$ is respectively defined as,

$$\mathcal{T}(\bar{x}; \mathcal{F}) = \left\{d \in \mathbb{R}^n \mid \text{ there exist } t_k \downarrow 0 \text{ and } d^k \to d \text{ with } x + t_k d^k \in \mathcal{F}\right\},$$

$$\mathcal{N}(\bar{x}; \mathcal{F}) = \left\{x^* \in \mathbb{R}^n \mid d^T x^* \leq 0, \text{ for all } d \in \mathcal{T}(\bar{x}; \mathcal{F})\right\}$$

and the *linearized cone* for (2.1.2) at $\bar{x}$ is

$$\mathfrak{L}(\bar{x}; \mathcal{F}) = \{d \in \mathbb{R}^n \mid \nabla g_i(\bar{x})^T d \leq 0, \text{ for all } i \text{ such that } g_i(\bar{x}) = 0\}.$$

Remark 2. *(i) Both the tangent and the linearized cone are closed cones and the latter is polyhedral convex, where that is not necessarily true for the tangent cone. (ii)* $\mathcal{T}(\bar{x};\mathcal{F}) \subset \mathfrak{L}(\bar{x};\mathcal{F})$ *holds for all* $\bar{x} \in \mathcal{F}$*. (iii) The dual cone is always a closed and convex cone. (iv) For points* $\bar{x}$ *outside* $\mathcal{F}$*, the tangent and normal cones are taken to be empty. (v) For* $\bar{x}$ *in the interior of* X*, the tangent cone is* $\mathbb{R}^n$ *and the normal cone is* $\{0\}$*. (vi) For a convex set* S *and any point* $\bar{x} \in S$*, we have* $\mathcal{T}(\bar{x};S)^* = -\mathcal{N}(\bar{x};S)$*.*

Let $\bar{x}$ be feasible for (2.1.2), we say that the Abide Constraint Qualification (ACQ) holds at $\bar{x}$ (and write $ACQ(\bar{x})$) if

$$\mathcal{T}(\bar{x};\mathcal{F}) = \mathfrak{L}(\bar{x};\mathcal{F}).$$

Let $\bar{x}$ be feasible for (2.1.2), we say that the Guignard Constraint Qualification (GCQ) holds at $\bar{x}$ (and write $GCQ(\bar{x})$) if

$$\mathcal{P}(\mathcal{T}(\bar{x};\mathcal{F})) = \mathcal{P}(\mathfrak{L}(\bar{x};\mathcal{F})),$$

i.e., if the polar of the tangent equals the polar of the linearized cone.

Let $\bar{x}$ be feasible for (2.1.2). We say that the linear independence constraint qualification (LICQ) holds at $\bar{x}$ (and write $LICQ(\bar{x})$) if the gradients

$$\{\nabla g_i(\bar{x})\ (i \in \mathcal{A}(\bar{x})),\ \ \nabla h_j(\bar{x})\ (j \in \mathcal{E})\}$$

are linearly independent. And we say that the Mangasarian-Fromovitz independence constraint qualification (MFCQ) holds at $\bar{x}$ (and write $MFCQ(\bar{x})$) if the gradients

$$\{\nabla h_j(\bar{x})\ (j \in \mathcal{E})\}$$

are linearly independent and there exists a vector $d \in \mathbb{R}^n$ such that

$$\left\{\nabla g_i(\bar{x})^T d < 0\ (i \in \mathcal{A}(\bar{x})),\ \ \nabla h_j(\bar{x})^T d = 0\ (j \in \mathcal{E})\right\},$$

i.e., if the gradients of the active constraints are positively linearly independent.

Note that the Karush-Kuhn-Tucker conditions are satisfied at local optimal solutions of differentiable optimization problems.

Theorem 2.5. *Let* $\bar{x}$ *be feasible for* (2.1.2)*. Then the following implications hold:*

$$LICQ(\bar{x}) \Rightarrow MFCQ(\bar{x}) \Rightarrow ACQ(\bar{x}) \Rightarrow GCQ(\bar{x}).$$

Proof. See Proposition 2.8 in [43]. □

Now we can state the optimality conditions for constrained problems. First we define the first order optimality conditions.

Theorem 2.6 (First Order Necessary Conditions). *Let x^* is a local solution of the constrained optimization problem* (2.1.2) *and that the LICQ(x^*) holds at x^*. Then there exists a Lagrange multiplier vectors λ^* and μ^*, such that the following conditions are satisfied at (x^*, λ^*, μ^*):*

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0, \tag{2.1.10a}$$

$$h_j(x^*) = 0 \ \textit{for all} \ \ j \in \mathcal{E}, \tag{2.1.10b}$$

$$g_i(x^*) = 0 \ \textit{for all} \ \ i \in \mathcal{I}, \tag{2.1.10c}$$

$$\lambda_i^* \geq 0 \ \textit{for all} \ \ i \in \mathcal{I}, \tag{2.1.10d}$$

$$\lambda_i^* g_i(x^*) = 0 \ \textit{for all} \ \ i \in \mathcal{I}. \tag{2.1.10e}$$

Conditions (2.1.10) are known as the *Karush-Kuhn-Tucker (KKT) conditions*. Equation (2.1.10a) is called the *stationarity condition*, (2.1.10b) and (2.1.10c) are called the *feasibility conditions*, (2.1.10d) states the *non-negativity of the multipliers* and (2.1.10e) is the *complementarity condition*. A point x^* satisfying (2.1.10) is called a *first-order critical point* or *KKT point* for problem (2.1.2).

The constrained optimization problem (2.1.2) is a *convex problem* if $f, g_i, i = 1, \ldots, m$ are convex functions and $h_j, j = 1, \ldots, l$ are affine functions.

Theorem 2.7 (First Order Sufficient Conditions). *Suppose x^* be a feasible solution of* (2.1.2)*, and suppose it satisfies the KKT conditions* (2.1.10)*. If* (2.1.2) *is a convex problem, then the KKT point x^* is a global minimizer of* (2.1.2).

Theorem 2.8 (Slater's condition). *Suppose the problem* (2.1.2) *satisfies Slater condition, i.e., $g_i, i = 1, \ldots, m$ are convex, $h_j, j = 1, \ldots, l$ are linearly independent, and there exists a point $x^0 \in X$ which satisfies $h(x^0) = 0$ and $g(x^0) < 0$. Then the KKT conditions are necessary to characterize an optimal solution.*

To state the second order optimality conditions, first we need to define the *critical cone* $\mathcal{N}_+(x^*, \lambda^*, \mu^*)$ as

$$\mathcal{N}_+(x^*, \lambda^*, \mu^*) = \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{l} g_i(x^*)^T d = 0, \ \forall i \in (\mathcal{A}(x^*) \cap \mathcal{I} \ \text{with} \ \lambda_i^* > 0), \\ g_i(x^*)^T d \geq 0, \ \forall i \in (\mathcal{A}(x^*) \cap \mathcal{I} \ \text{with} \ \lambda_i^* = 0) \\ \quad \text{and} \ h_j(x^*)^T d = 0, \ \forall j \in \mathcal{E} \end{array} \right\}.$$

Theorem 2.9 (Second-order Necessary Conditions)**.** *Suppose that x^* is a local minimizer of* (2.1.2) *and that the LICQ(x^*) holds. Let (λ^*, μ^*) be the Lagrange multipliers vector for which the KKT conditions* (2.1.10) *are satisfied. Then,*

$$d^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*, \mu^*) d \geq 0, \; \forall d \in \mathcal{N}_+(x^*, \lambda^*, \mu^*). \tag{2.1.11}$$

A point x^* that satisfies (2.1.11) is called a *second-order critical point* for problem (2.1.2).

Theorem 2.10 (Second-order Sufficient Conditions)**.** *Suppose that for some $x^* \in \mathcal{F}$ there exists a Lagrange multipliers vector (λ^*, μ^*) such that the KKT conditions* (2.1.10) *are satisfied. Assume also that*

$$d^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*, \mu^*) d > 0, \; \forall d \in \mathcal{N}_+(x^*, \lambda^*, \mu^*), d \neq 0, \tag{2.1.12}$$

then x^* is a strict local solution for problem (2.1.2).

2.2 Game theory

In this section we follow the presentation given in [94] for common definitions and the discussion given in [20, 53, 62] for mathematical formulations and analysis of game theory.

Game theory studies the strategic decision-making, where an individual makes a choice by taking into account the others' choices. It is the study of problem of conflict and cooperation among independent decision-makers. Game theoretic concepts apply when the actions of several agents are interdependent. These agents could also be individual, groups, firms, or any combination of those. The concepts of theory of games offer a language to formulate, structure, analyze, and understand strategic scenarios.

In a typical game, the subsequent three elements should be specified: the players of the game, the strategies available to each player, and the payoffs for every outcome. *Player* is an agent who makes decisions in a game. *Strategy* is one of the given possible actions of a player. *Payoff* (also called *utility*) is a number that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities and expected outcome values. Therefore, a game defines an interaction between some agents that has a series of available strategies, where a strategy determines an action of the agent in the game. Games appear in *strategic*, *extensive* and *coalitional* form. The first two constitute the basic model of non-cooperative game theory and the coalitional form is the basic model of cooperative game theory [72].

A *strategic form game* is a model of interactive decision making used to analyze situations in which two or more individuals, called players, make decisions (or choose actions) that will influence one another's welfare. A game in strategic form lists each player's strategies, and the outcomes that result from each possible combination of choices. An outcome is represented by a separate *payoff* for each player, that measures how much the player likes the outcome.

Definition 2.9 (Strategic game). *A strategic game is a triplet $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ where $N = \{1, \ldots, n\}$ is the finite set of players, X_i is the set of actions of player $i \in N$ and $f_i : X = \prod_{j\in N} X_j \to \mathbb{R}$ is a utility function for player i.*

In the above definition, N is the set of players in the game Γ. For each player $i \in N$, X_i is the set of actions available to player i. When the strategic form game Γ is played, each player i must choose one of the actions in the set X_i. For each combination of actions, or *pure strategy profile* $x = (x_j)_{j\in N} \in X$ (specifying one action for each player), the number $f_i(x)$ represents the *payoff* that player i would get in this game if x were the combination of actions implemented by the players. In a strategic form game it is assumed that all the players choose their actions simultaneously.

For each $i \in N$, let $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$ be a profile of actions taken by all players in the game other than i and write $x = (x_i, x_{-i})$. We denote the set of all such profiles conveniently as $X_{-i} = \prod_{j\in N, j\neq i} X_j$.

A strategy $x_i \in X_i$ *weakly dominates* action $x^{*,i} \in X_i$ for player i if $f_i(x_i, x_{-i}) \leq f_i(x_i^*, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $f_i(x_i, x_{-i}) < f_i(x_i^*, x_{-i})$ for some $x_{-i} \in X_{-i}$. It *strictly dominates* x_i^* if $f_i(x_i, x_{-i}) < f_i(x_i^*, x_{-i})$ for all $x_{-i} \in X_{-i}$.

A strategy $x_i \in X_i$ is *weakly dominant* if it weakly dominates every action in X_i. It is called *strictly dominant* if it strictly dominates every action in X_i. Take a game in strategic form and consider any two actions $x_i, x_i^* \in X_i$ for any player i. We say that x_i is *strictly dominated* by x_i^* if $f_i(x_i, x_{-i}) > f_i(x_i^*, x_{-i})$ for all $x_{-i} \in X_{-i}$. We say that x_i is *weakly dominated* by x_i^* if $f_i(x_i, x_{-i}) \geq f_i(x_i^*, x_{-i})$ for all $x_{-i} \in X_{-i}$ while $f_i(x_i, x_{-i}) > f_i(x_i^*, x_{-i})$ for some $x_{-i} \in X_{-i}$.

Rational players make choices which end in the result they like most, given what their opponents do. In the extreme case, a player may have two strategies A and B

for, given any combination of strategies of the other players, the outcome resulting from A is better than the outcome resulting from B. Then strategy A is said to *dominate* strategy B. A rational player will never prefer to play a dominated strategy.

2.2.1 Nash game

John Nash introduced the well-known concept of Nash equilibrium [71, 72] in non-cooperative games involving two or more players. In such a game, called the Nash game or Nash equilibrium problem (NEP), all players are assumed to know the objective functions of other players and make decisions to choose their own strategies at the same time by taking into account the strategies of other players. When each player can get no more benefit by changing his current strategy unilaterally (i.e., all players have no incentive to change their current strategies), the strategy tuple composed of the current strategies of all players forms a Nash equilibrium.

Consider a strategic game denoted by $\Gamma = (N, (f_i)_{i\in N}, (X_i)_{i\in N})$. If we denote the set of players by N, then for $i \in N$, player i's strategy is denoted by vector $x_i \in \mathbb{R}^{n_i}$ and the payoff function $f_i(x)$ depends on all players' strategies, which are collectively denoted by the vector $x \in \mathbb{R}^n$ consisting of sub-vectors $x_i \in \mathbb{R}^{n_i}$, $i \in N$, $n = n_1 + ... + n_N$, and $n_{-i} = n - n_i$. Player i's (pure) strategy is denoted by $x_i \in X_i \subset \mathbb{R}^{n_i}$ which is independent of the other players' strategies which are denoted collectively as $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \in X_{-i} \subset \mathbb{R}^{n-i}$, which consists of all the other players' strategies, player i solves the following optimization problem for his own variable x_i,

$$\begin{aligned} &\min_{x_i} f_i(x_i, x_{-i}) \\ &s.t. \;\; x_i \in X_i, \end{aligned} \tag{2.2.1}$$

where $f_i(x_i, x_{-i})$ is player i's payoff function.

Definition 2.10 (Nash equilibrium). *A (pure strategy) Nash Equilibrium of a strategic game $\Gamma = (N, (f_i)_{i\in N}, (X_i)_{i\in N})$ is a strategy profile $x^* \in X$ such that for all $i \in N$,*

$$f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*) \;\; \text{for all} \;\; x_i \in X_i. \tag{2.2.2}$$

In a two player game, for example, an action profile (x_1^*, x_2^*) is a Nash equilibrium if the following two conditions hold

$$\begin{aligned} x_1^* &\in \operatorname*{argmin}_{x_1 \in X_1} f_1(x_1, x_2^*), \\ x_2^* &\in \operatorname*{argmin}_{x_2 \in X_2} f_2(x_1^*, x_2). \end{aligned}$$

Therefore, we may say that, in a Nash equilibrium, each player's choice of action is a best response to the actions actually taken by his opponents.

Define the best response correspondence of player i in a strategic form game as the correspondence $Y_i : X_{-i} \twoheadrightarrow X_i$ given by

$$Y_i(x_{-i}) = \{x_i \in X_i : f_i(x_i, x_{-i}) \leq f_i(x_i^*, x_{-i}) \text{ for all } x_i^* \in X_i\}. \tag{2.2.3}$$

Note that, for each $x_{-i} \in X_{-i}$, the set (2.2.3) may or may not be a singleton.

It is also possible to define Nash equilibrium based on the notion of the best response correspondence (2.2.3). For example, in a $2-$person game, if player 2 plays x_2, player 1's best choice is to play some action in $Y_1(x_2)$,

$$Y_1(x_2) = \{x_1 \in X_1 : f_1(x_1, x_2) \leq f_2(x_1^*, x_2) \text{ for all } x_1^* \in X_1\},$$

or any $2-$person game in strategic form Γ, we have (x_1^*, x_2^*) is a Nash equilibrium if and only if

$$x_1^* \in Y_1(x_2^*) \text{ and } x_2^* \in Y_2(x_1^*),$$

where

$$Y_2(x_1) = \{x_2 \in X_2 : f_2(x_1, x_2) \leq f_2(x_1, x_2^*) \text{ for all } x_2^* \in X_2\}.$$

For each i define the best-response correspondences by the mapping $\beta_i(x_{-i}) : X_{-1} \twoheadrightarrow X_i$

$$\beta_i(x_{-i}) = \underset{x_i \in X_i}{\operatorname{argmin}} f_i(x_i, x_{-i}).$$

An action profile x^* is a Nash equilibrium if and only if

$$x_i^* \in \beta_i(x_{-i}^*) \quad \text{for all } \; i \in N.$$

Theorem 2.11 (Debreu, Glicksberg, Fan, 1952)**.** *[25, 38, 47] Let $X_i \subseteq \mathbb{R}^{n_i}$, $(i = 1, \ldots, N)$ be nonempty, convex and compact. If $\forall i$, $f_i : X \to \mathbb{R}$ is continuous in x and quasi-convex in x_i, then there exists a pure strategy Nash equilibrium.*

The convexity assumption of the objective function f_i with respect to x_i is sometimes called player-convexity.

Let 2^{X_i} be the family of subsets of X_i. Let $C_i : X_{-i} \to 2^{X_i}$ be a constraint correspondence of Player i, i.e. a function mapping a point in X_{-i} to a subset of X_i. Thus, $C_i(x_{-i})$ defines the i^{th} player action space given other players' action x_{-i}. Typically, the constraint correspondence C_i is defined by a parametrized action space as $C_i(x_{-i}) = \{x_i \in X_i, g_i(x_i, x_{-i}) \leq 0\}$, where $g_i : X \to \mathbb{R}^{m_i}$ is a constraint function.

When g_i does not depend on x_{-i}, we get back to the standard game. A generalized game is described by $(N, X_i, C_i(\cdot), f_i(\cdot))$ and is also called an abstract economy (this is due to Debreu's work in [7, 25]).

Definition 2.11. *The generalized Nash equilibrium for a generalized game (N, X_i, C_i, f_i) is defined as a point x^* solving for all $i \in N$,*

$$x_i^* \in \underset{x_i \in C_i(x_{-i}^*)}{\operatorname{argmin}} \; f_i(x_i, x_{-i}^*). \tag{2.2.4}$$

2.2.2 Stackelberg game

The Stackelberg game, which is also called a leader-follower game, was first proposed by Stackelberg [97]. It is based on economic monopolization phenomena. In a Stackelberg game, one player acts as a leader and the rest as followers. The problem is then to find an optimal strategy for the leader, assuming that the followers react in a rational way which will optimize their objective functions, given the leader's actions. Stackelberg used a hierarchical model to describe a market situation in which decision makers try to optimize their decisions based on individually different objectives but are affected by a certain hierarchy.

The classical Stackelberg leadership model considers the case of a single leader and single follower. Let X and Y be the strategy sets for the leader and follower respectively. Denote their objective function by $F(x, y)$ and $f(x, y)$ respectively. Knowing the selection x of the leader, the follower can select his best strategy $y(x)$ such that his objective function $f(x, y)$ is minimized, i.e.,

$$y(x) \in \phi(x) = \underset{y \in Y}{\operatorname{argmin}} \, f(x, y). \tag{2.2.5}$$

The leader then obtains the best strategy $x \in X$ as

$$x \in \underset{x \in X}{\operatorname{argmin}} \{F(x, y) : y \in \phi(x)\}. \tag{2.2.6}$$

Formulae (2.2.5) and (2.2.6) can be combined to express the Stackelberg game as follows:

$$\begin{aligned} &\min_x F(x, y) \\ &s.t \;\; x \in X, \\ &\qquad y \in \underset{y \in Y}{\operatorname{argmin}} \, f(x, y). \end{aligned}$$

Bilevel programming is more general than Stackelberg game in the sense that the strategy sets (also called the admissible sets) depend on both x and y. This leads to a general bilevel single follower programming as follows [22]:

$$\begin{aligned} &\min_{x} F(x,y) \\ &s.t\ \ G(x,y) \leq 0, \\ &\quad\ y \in \operatorname{argmin}\{f(x,y) : g(x,y) \leq 0\}. \end{aligned} \tag{2.2.7}$$

Bilevel single follower programming problem (2.2.7) is a generalization of several well-known optimization problem. For example, if $F(x, y) = -f(x, y)$, then it is a classical min-max problem; if $F(x, y) = f(x, y)$, we have a realization of the decomposition approach to optimization problem; if the dependence of both the leader's and the follower's problem on y is dropped, the problem is reduced to a bi-criteria optimization problem [26].

The Stackelberg game can be considered as an extension of the well-known Nash game. In the Nash game, we assume that there are N players, and the i^{th} player has a strategy set X_i, and his objective function is $f_i(x)$ for $i = 1, 2, \dots, n$, where $x = (x_1, x_2, \dots, x_N)$. Each player chooses a strategy based on the choices of the other players and there is no hierarchy. The unstructured problem is modeled as follows: for $i = 1, 2, \dots, N$, we have

$$\min_{x_i \in X_i} f_i(x).$$

This is a Nash game in which all players aim to minimize their corresponding objective functions.

In contrast, there is a hierarchy between the leader and followers in the Stackelberg game. The leader is aware of the choices of the followers, thus the leader, being in a superior position with regard to everyone else, can achieve the best objective while forcing the followers to respond to this choice of strategy by solving the Stackelberg game. Without loss of generality, we now assume that the first player is the leader, and the rest of the players are followers. Let $X_{-1} = X_2 \times X_3 \times \cdots \times X_k$, $f_{-1}(x) = (f_2(x), \dots, f_k(x))$ and $x_{-1} = (x_2, \dots, x_k) \in X_{-1}$. The above Nash game is accordingly transformed into a Stackelberg game, which is given as follows:

$$\begin{aligned} &\min_{x_1 \in X_1} f_1(x) \\ &s.t\ \ x_{-1} \in \operatorname{argmin}\{f_{-1}(x) : x_{-1} \in X_{-1}\}. \end{aligned}$$

This is a Stackelberg game or leader-follower game.

2.2.3 Potential games

A strategic game might be a potential game if it admits a potential. Potential functions quantify the difference within the payoff due to unilateral deviation of each player either exactly (exact potential games), in sign (ordinal potential games), or deviation to the best response (best-response potential games). Potential function are often interpreted as a measure of the disagreement among players, or, equivalently as the drift towards the Nash equilibrium (NE). Potential function can replace the utility function of different players while preserves some of the game's structure like NE and best response. If a finite game (i.e., the player set and actions sets are finite) admits a potential function, then the potential function achieves a global optimum, which is also a local optimum, and hence the game has a minimum of one equilibrium in pure strategies.

Since it was introduced by Monderer and Shapley [69] in 1996 potential games received increasing attention and various notions of potential games are introduced and studied in the literature. Generalized ordinal, ordinal, exact, and weighted potential games were introduced in [69]. Voorneveld et al. [98] studied ordinal potential games and characterized several properties of these games. The notions of best-response potential games, pseudo-potential games and quasi-potential games were studied in [98], [35] and [63], respectively.

Consider a strategic game $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ with $N = \{1, \ldots, n\}$ being the finite set of players, X_i is the set of actions of player $i \in N$ and $f_i : X = \prod_{j\in N} X_j \to \mathbb{R}$ is a utility function for player i.

(**E**) A game Γ is an *exact potential game* or simply a *potential game* if there exists a (potential) function $P : X \to \mathbb{R}$ such that for each $i \in N$ and $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$

$$f_i(x_i, x_{-i}) - f_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i}). \tag{2.2.8}$$

A finite potential game always possesses a Nash equilibrium. Also, if each X_i is compact and f_i a continuous function, such an exact potential game also possesses a Nash equilibrium.

If for each $i \in N, X_i \subset \mathbb{R}$ is an open set and f_i is a continuously differentiable function on $\prod_{i\in N} X_i \subset \mathbb{R}^n$, then the following is an equivalent condition for a game to be a potential game. Thus the problem of finding pure Nash equilibria of a potential game Γ is equivalent to finding local optima for the optimization problem with state space the pure strategy space X of the game and objective P the potential function of the game.

Lemma 2.12. *Let Γ be a game as described above with for each $X_i \subset \mathbb{R}$ for each $i \in N$ and each f_i is a continuously differentiable function on $\mathbb{R}^n$. Then, the function P is a potential function for the game Γ if and only if*

$$\frac{\partial f_i}{\partial x_i} = \frac{\partial P}{\partial x_i}.$$

Proof. In equation (2.2.8), let $x_i' = x_i + h$ for some $h \in \mathbb{R}$ with $h \neq 0$. Then,

$$f_i(x_i, x_{-i}) - f_i(x_i + h, x_{-i}) = P(x_i, x_{-i}) - P(x_i + h, x_{-i}).$$

Divide both sides by h and take the limit of $h \to 0$. □

Theorem 2.13. *Suppose in addition that each f_i is twice continuously differentiable. Then, a game Γ is a potential game if and only if*

$$\frac{\partial^2 f_i}{\partial x_i \partial x_j} = \frac{\partial^2 f_j}{\partial x_i \partial x_j}.$$

Proof. See Theorem 4.5 in [69]. □

Proposition 2.14. *If a game $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is a potential game , there exist functions $P : \prod_{i\in N} X_i \to \mathbb{R}$ and $Q_i : X_{-i} \to \mathbb{R}$ for each $i \in N$ such that for all $x \in \prod_{i\in N} X_i$,*

$$f_i(x) = P(x) + Q_i(x_{-i}).$$

Proof. See Proposition 1 in [89] □

The above proposition implies that an exact potential game can be written as a sum of common payoff game and a dummy game. A game $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is a dummy game if for each $i \in N$ and $x_{-i} \in X_{-i}$,

$$f_i(x_i, x_{-i}) = f_i(x_i', x_{-i}) \ \forall x_i', x_i \in X_i.$$

That is, f_i does not depend on the strategy choice of player i. Therefore, f_i is a function of x_{-i} only.

Proposition 2.15. *If P and P' are potential functions corresponding to a potential game Γ, then for each $x \in \prod_{i\in N} X_i,\ P(x) - P'(x)$ is a constant.*

Proof. See Lemma 2.7 in [69]. □

Weaker notions of potential games are weighted potential game defined in [69], best response potential game defined in [98], pseudo potential game defined in [35], iterated potential game defined in [76], and nested potential game defined in [95].

(**W**) A game $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ is a *weighted potential game* if there exists a (weighted potential) function $P : X \to \mathbb{R}$ and a positive weight vector $(w_i)_{i \in N}$ such that for each $i \in N$ and $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$

$$f_i(x_i, x_{-i}) - f_i(x_i', x_{-i}) = w_i(P(x_i, x_{-i}) - P(x_i', x_{-i})).$$

(**O**) A game $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ is an *ordinal potential game* if there exists an (ordinal potential) function $P : X \to \mathbb{R}$ such that for each $i \in N$ and $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$

$$f_i(x_i, x_{-i}) - f_i(x_i', x_{-i}) > 0 \Leftrightarrow P(x_i, x_{-i}) - P(x_i', x_{-i}) > 0.$$

The next set of potential games are defined using the best response correspondence β_i for each player $i \in N$.

(**G**) A game $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ is a *generalized ordinal potential game* if there exists a (generalized ordinal potential) function $P : X \to \mathbb{R}$ such that for each $i \in N$ and $x_i, x_i' \in X_i$ and $x_{-i} \in X_{-i}$

$$f_i(x_i, x_{-i}) - f_i(x_i', x_{-i}) > 0 \Rightarrow P(x_i, x_{-i}) - P(x_i', x_{-i}) > 0,$$

(**B**) A game $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ is a *best-response potential game* if there exists a (best-response potential) function $P : X \to \mathbb{R}$ such that for each $i \in N$ and $x_{-i} \in X_{-i}$

$$\beta_i(x_{-i}) = \operatorname*{argmin}_{x_i \in X_i} P(x_i, x_{-i}),$$

where $\beta_i(x_{-i}) = \operatorname*{argmin}_{x_i \in X_i} f_i(x_i, x_{-i})$.

Note that $x = (x_1, x_2, \ldots, x_n) \in X$ is a Nash equilibrium (NE) if $x_i \in \beta_i(x_{-i})$ for all $i \in N$.

(**P**) A game $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ is a *pseudo potential game* if there exists a (pseudo-potential) function $P : X \to \mathbb{R}$ such that for each $i \in N$ and $x_{-i} \in X_{-i}$

$$\beta_i(x_{-i}) \supseteq \operatorname*{argmin}_{x_i \in X_i} P(x_i, x_{-i}),$$

where $\operatorname*{argmin}_{x_i \in X_i} P(x_i, x_{-i}) = \{x_i^* \in X_i : P(x_i^*, x_{-i}) \leq P(x_i', x_{-i}) \; \forall x_i' \in X_i\}$.

(**Q**) A game $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ is a *quasi potential game* if there exists a (quasi-potential) function $P : X \to \mathbb{R}$ such that

$$x \in \beta(x) \Rightarrow x \in \operatorname*{argmin}_{x \in X} P(x),$$

where $\beta(x) = \prod_{i \in N} \beta_i(x_{-i})$ is the joint best-response correspondence.

Theorem 2.16. *Consider the class of finite strategic games of type* $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$. *Suppose that the classes of finite exact, weighted, ordinal, generalized ordinal, best-response, pseudo-potential, and quasi-potential games are denoted by* E, W, O, G, B, P, *and* Q *respectively, then*

(i) $\emptyset \neq E \subset W \subset O \subset G \subset P \subset Q$,

(ii) $\emptyset \neq E \subset W \subset O \subset B \subset P \subset Q$,

(i) $G \cap B \neq \emptyset, G \setminus B \neq \emptyset, B \setminus G \neq \emptyset$.

Proof. See Proposition 4 in [86]. □

In determining Nash equilibrium for a finite strategic games it is helpful to find a potential functions or weaker versions of potential functions. Because if we find that, the problem reduces to determining the optimal solutions of the potential functions.

Chapter 3

Multi-parametric programming

Contents

Multi-parametric programming problems considers optimization problems where the data depend upon one or more parameters. Multi-parametric programming techniques systematically subdivide the parameter space into characteristic regions where the optimal value and an optimizer are given as explicit functions of the parameters [14].

3.1 Multi-parametric unconstrained problems

Let $f(x,\theta) \in \mathcal{C}^2$, $f : \mathbb{R}^n \times \Theta \to \mathbb{R}$, where the parameter set $\Theta \subset \mathbb{R}^p$ is open. By considering a *parametric problem* with parameter $\theta \in \Theta$ we will characterize a local or global minimizer $x = x(\theta)$ of

$$\min_{x \in \mathbb{R}^n} f(x,\theta). \tag{3.1.1}$$

For a fixed $\theta = \theta_0$, problem (3.1.1) will be a standard nonlinear problem,

$$\min_{x \in \mathbb{R}^n} f(x,\theta_0). \tag{3.1.2}$$

If for any $\theta \in \Theta$ the function $f(x,\theta)$ is convex in x then a (local) minimizer $x(\theta)$ is a global minimizer of (3.1.1). In this case we can define the (global) value function of (3.1.1) by

$$v(\theta) = \inf_{x \in \mathbb{R}^n} f(x,\theta).$$

In the general nonlinear case, $x(\theta)$ is thought to be a local minimizer of (3.1.1) and the value function is defined locally by

$$v(\theta) = f(x(\theta),\theta).$$

To solve this problem (3.1.1), for $\theta \in \Theta$, we have to find solutions x of the critical point equation

$$H(x,\theta) := \nabla_x f(x,\theta) = 0. \tag{3.1.3}$$

Theorem 3.1 (Implicit Function Theorem (IFT))**.** *[90] Let $H : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ be a $\mathcal{C}^1$-function $H(x,\theta)$. Suppose for $(x,\theta) \in \mathbb{R}^n \times \mathbb{R}^p$ we have $H(x_0,\theta_0) = 0$ and the matrix $\nabla_x H(x_0,\theta_0)$ is non-singular. Then there is a neighborhood $B_\varepsilon(\theta_0), \varepsilon > 0$, of θ and a $\mathcal{C}^1$-function $x : B_\varepsilon(\theta_0) \to \mathbb{R}^n$ satisfying $x(\theta_0) = x_0$ such that near (x_0,θ_0) the solution set $S(H) := \{(x,\theta)|H(x,\theta) = 0\}$ is described by*

$$\{(x(\theta),\theta) \in \mathbb{R}^n \times \mathbb{R}^p | \theta \in B_\varepsilon(\theta_0)\}, \tag{3.1.4}$$

i.e., $H(x(\theta),\theta) = 0$ for $\theta \in B_\varepsilon(\theta_0)$. So, locally near (x_0,θ_0), the set $S(H)$ is a p dimensional $\mathcal{C}^1$-manifold. Moreover, the gradient $\nabla x(\theta)$ is given by

$$\nabla x(\theta) = -[\nabla_x H(x(\theta),\theta)]^{-1} \nabla_\theta H(x(\theta),\theta) \text{ for } \theta \in B_\varepsilon(\theta_0).$$

Proof. Note that if $x(\theta)$ is a $\mathcal{C}^1$-function satisfying $H(x(\theta),\theta) = 0$, then differentiation with respect to θ yields by applying the chain rule,

$$\nabla_x H(x(\theta),\theta) \nabla_\theta x(\theta) + \nabla_\theta H(x(\theta),\theta) = 0.$$

□

Theorem 3.2 (Local stability result based on IFT)**.** *Let $f(x,\theta)$ be a $\mathcal{C}^2$-function. Suppose, x_0 is a (local) minimizer of* (3.1.2)*, $\theta_0 \in \Theta$, such that*

$$\nabla_x f(x_0,\theta_0) = 0 \ \textit{and} \ \nabla_x^2 f(x_0,\theta_0) > 0 \ \textit{(positive definite)}.$$

(i.e., x_0 is an isolated strict local minimizer of order 2.)

Then there exists a neighborhood $B_\epsilon(\theta_0),\ \epsilon > 0$, of θ_0 and a $\mathcal{C}^1$-function $x : B_\epsilon(\theta_0) \to \mathbb{R}^n$ such that $x(\theta_0) = x_0$ and for any $\theta \in B_\epsilon(\theta_0)$, $x(\theta)$ is an (isolated) strict local minimizer of (3.1.1)*. Moreover for $\theta \in B_\epsilon(\theta_0)$,*

$$\nabla x(\theta) = -[\nabla_x^2 f(x(\theta),\theta)]^{-1} \nabla_{x\theta}^2 f(x(\theta),\theta),$$

and the value function $v(\theta) = f(x(\theta),\theta)$ is a $\mathcal{C}^2$-function with

$$\nabla v(\theta) = \nabla_\theta f(x(\theta),\theta) \ \textit{and} \ \nabla^2 v(\theta) = \nabla_{\theta x}^2 f(x(\theta),\theta)\nabla x(\theta) + \nabla_\theta^2 f(x(\theta),\theta).$$

Proof. Apply the IFT to the critical point equation $\nabla_x f(x,\theta) = 0$. □

3.2 Multi-parametric constrained problems

3.2.1 Multi-parametric problem with inequality constraints

Consider nonlinear parametric problems of the form:

$$\begin{aligned} &\min_x f(x,\theta) \\ &\text{s.t. } x \in \mathcal{F}(\theta) = \{x \in \mathbb{R}^n | g_j(x,\theta) \le 0, j \in J := \{1,\dots,m\}\}, \end{aligned} \tag{3.2.1}$$

where $f,\ g_j \in \mathcal{C}^2$ and $f,\ g_j : \mathbb{R}^n \times \Theta \to \mathbb{R}$.

If for any $\theta \in \Theta \subset \mathbb{R}^p$ the functions $f(x,\theta), g_j(x,\theta),\ j \in J$, are convex in x then for any θ the problem (3.2.1) is convex and any (local) minimizer $x(\theta)$ is a global one. In this case we can define the (global) value function of (3.2.1) by

$$v(\theta) = \inf_{x \in \mathcal{F}(\theta)} f(x,\theta),$$

($v(\theta) = \infty,$ if $\mathcal{F}(\theta) = \emptyset$) and the set $S(\theta)$ of global minimizers:

$$S(\theta) = \{x \in \mathcal{F}(\theta) | f(x,\theta) = v(\theta)\}.$$

In the general nonlinear case, we consider local minimizer $x(\theta)$ of (3.2.1) and the corresponding local minimum value function,

$$v(\theta) = f(x(\theta),\theta).$$

For $\theta_0 \in \Theta$ and feasible $x_0 \in \mathcal{F}(\theta_0)$ we denote by $J_0(x_0, \theta_0)$ the active index set,

$$J_0(x_0, \theta_0) = \{j \in J | g_j(x_0, \theta_0) = 0\},$$

and by $\mathcal{L}(x, \theta, \mu)$ the Lagrangian function (near (x_0, θ_0)),

$$\mathcal{L}(x, \theta, \mu) = f(x, \theta) + \sum_{j \in J_0(x_0, \theta_0)} \mu_j g_j(x, \theta).$$

To find, near (x_0, θ_0), local minimizers x of (3.2.1) we are looking for solutions $(x, \theta, \mu) = (x(\theta), \theta, \mu(\theta))$ of the KKT equations with $\mu_j \geq 0, j \in J_0(x_0, \theta_0)$,

$$H(x, \theta, \mu) := \begin{array}{r} \nabla_x f(x, \theta) + \sum_{j \in J_0(x_0, \theta_0)} \mu_j \nabla_x g_j(x, \theta) = 0, \\ g_j(x, \theta) = 0, j \in J_0(x_0, \theta_0). \end{array} \tag{3.2.2}$$

Lemma 3.3. *[90] Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $B \in \mathbb{R}^{n \times m} (n \geq m)$. If matrix B has full rank m and $d^T A d \neq 0 \; \forall d \in \mathbb{R}^n \setminus \{0\}$ such that $B^T d = 0$, then matrix $\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$ is non-singular.*

Theorem 3.4 (Local stability result based on IFT)**.** *[90] Let $x_0 \in \mathcal{F}(\theta_0)$. Suppose that with multipliers μ_j the KKT condition $\nabla_x \mathcal{L}(x_0, \theta_0, \mu_0) = 0$ is satisfied such that*

(1) LICQ holds at (x_0, θ_0)

(2) $\mu_{0j} > 0, \forall j \in J_0(x_0, \theta_0)$ (strict complementary), and either

(3a) (order one condition) $|J_0(x_0, \theta_0)| = n$, or

(3b) (order two condition)

$$d^T \nabla_x^2 L(x_0, \theta_0, \mu_0) d > 0,$$

where T_{x_0, θ_0} is the tangent space $T_{x_0, \theta_0} = \{d | \nabla_x g_j(x_0, \theta_0) = 0, \; j \in J_0(x_0, \theta_0)\}$.

(x is a local minimizer of (3.2.1) *of order $s = 1$ in case (3a) and of order $s = 2$ in case (3b)).*

Then there exist a neighborhood $B_\varepsilon(\theta_0), \varepsilon > 0$, of θ_0 and $\mathcal{C}^1$-functions $x : B_\varepsilon(\theta_0) \to \mathbb{R}^n, \mu : B_\varepsilon(\theta_0) \to \mathbb{R}^{|J_0(x_0, \theta_0)|}$ such that $x(\theta_0) = x_0, \mu(\theta_0) = \mu_0$ and for any $\theta \in B_\varepsilon(\theta_0)$ the point $x(\theta)$ is a strict local minimizer of (3.2.1) *(of order 1 in case (3a) and of order 2 in case (3b)) with corresponding multiplier vector $\mu(\theta)$. Moreover, for $\theta \in B_\varepsilon(\theta_0)$ the derivative of the value function $v(\theta) = f(x(\theta), \theta)$ is*

$$\nabla v(\theta) = \nabla_\theta \mathcal{L}(x(\theta), \theta, \mu(\theta)).$$

The derivative of the solution function $x(\theta)$ is given in case (3a) by

$$\nabla x(\theta) = -\left[\nabla_x g_j(x(\theta),\theta), j \in J_0(x_0,\theta_0)\right]^{-1}\left[\nabla_\theta g_j(x(\theta),\theta), j \in J_0(x_0,\theta_0)\right]$$

and in case (3b) by

$$\begin{pmatrix} \nabla_x x(\theta) \\ \nabla_x \mu(\theta) \end{pmatrix} = -[\nabla_{(x,\mu)} H(x(\theta),\theta,\mu(\theta))]^{-1}\nabla_\theta H(x(\theta),\theta,\mu(\theta)).$$

Proof. [90] In case (3a), by LICQ and $|J_0(x_0,\theta_0)| = n$ the system (3.2.2) aligned into the system of n equations in n variables,

$$g_j(x,\theta) = 0, \;\; j \in J_0(x_0,\theta_0),$$

for $x = x(\theta)$ with $\nabla x(\theta)$ given as solution of (derivatives with respect to θ)

$$\nabla_x g_j(x(\theta),\theta)\nabla x(\theta) + \nabla_\theta g_j(x(\theta),\theta) = 0, \;\; j \in J_0(x_0,\theta_0), \tag{3.2.3}$$

and the equations

$$\nabla_x f(x(\theta),\theta)\nabla x(\theta) + \sum_{j\in J_0(x_0,\theta_0)} \mu_j \nabla_x g_j(x(\theta),\theta) = 0, \tag{3.2.4}$$

for the components $\mu_j = \mu_j(\theta), j \in J_0(x_0,\theta_0)$ $(\mu_j(\theta), j \in J \setminus J_0(x_0,\theta_0))$. For the value function $v(\theta) = f(x(\theta),\theta)$, by differentiation with respect to θ, we find using (3.2.3) and (3.2.4)

$$\begin{aligned} \nabla v(\theta) &= \nabla_x f(x(\theta),\theta)\nabla x(\theta) + \nabla_\theta f(x(\theta),\theta) \\ &= -\sum_{j\in J_0(x_0,\theta_0)} \mu_j \nabla_x g_j(x(\theta),\theta)\nabla x(\theta) + \nabla_\theta f(x(\theta),\theta) \\ &= \sum_{j\in J_0(x_0,\theta_0)} \mu_j \nabla_\theta g_j(x(\theta),\theta) + \nabla_\theta f(x(\theta),\theta) \\ &= \nabla_\theta L(x(\theta),\theta,\mu(\theta)). \end{aligned}$$

In case (3b) we have to apply the IFT to the coupled system (3.2.2) and by Lemma 3.3 the matrix $\nabla_{(x,\mu)} H(x_0,\theta_0,\mu_0)$ is non-singular, where

$$\nabla_{(x,\mu)} H(x_0,\theta_0,\mu_0) = \begin{pmatrix} \nabla_x^2 H(x_0,\theta_0,\mu_0) & B \\ B^T & 0 \end{pmatrix}$$

with $B = (\nabla_x g_j(x_0,\theta_0)^T, \; j \in J_0(x_0,\theta_0))$. □

3.2.2 Multi-parametric problem with mixed constraints

A general form of multi-parametric nonlinear problem (mp-NLP), as it was stated in [39], is defined as

$$\min_x f(x,\theta)$$
$$s.t. \begin{cases} g_i(x,\theta) \le 0, \ i=1,\dots,p, \\ h_j(x,\theta) = 0, \ j=1,\dots,q, \end{cases} \tag{3.2.5}$$

where $\theta \in \Theta \subseteq \mathbf{R}^m$ is the parameters vector, $x \in X \subseteq \mathbf{R}^n$ is the vector of the decision variables, and $f, g_i, h_j : \mathbf{R}^n \times \mathbf{R}^m \longrightarrow \mathbf{R}$ are parametric nonlinear functions.

In multi-parametric programming we are in quest of a dynamic solution which is dependent on the parameter (i.e., an optimal solution, $x(\theta)$, and an optimal value function, $v(\theta)$, that can be expressed as a function of the parameters, θ) and the parametric regions, which are called critical regions ($\mathcal{CR}$), on which the expressions $x(\theta)$ and $v(\theta)$ are valid [28].

Solution algorithms in multi-parametric programming has been based on the *basic sensitivity theorem* of Fiacco [39]. We shall discuss the general sensitivity analysis of a nonlinear programming problem here below.

The Lagrangian function associated with (3.2.5) is defined by:

$$\mathcal{L}(x,\theta,\lambda,\mu) = f(x,\theta) + \sum_{i=1}^{p} \lambda_i g_i(x,\theta) + \sum_{j=1}^{q} \mu_j^T h_j(x,\theta), \tag{3.2.6}$$

where $\lambda = (\lambda_1,\dots,\lambda_p)^T$ and $\mu = (\mu_1,\dots,\mu_q)^T$ are the Lagrange multiplier vectors associated with inequality and equality constraints g_i and h_j, respectively.

If $\theta = \theta_0$ is fixed, we will have a classical nonlinear problem,

$$\min_x f(x,\theta_0)$$
$$s.t. \begin{cases} g_i(x,\theta_0) \le 0, \ i=1,\dots,p, \\ h_j(x,\theta_0) = 0, \ j=1,\dots,q. \end{cases} \tag{3.2.7}$$

The first-order sensitivity results for a parametric nonlinear programming problem (3.2.5) was presented by Fiacco [39], based on the following assumptions:

Assumption 1. *The functions defining problem* (3.2.5) *are twice continuously differentiable in* (x,θ) *in a neighborhood of* (x_0,θ_0)*, where* x_0 *is the solution of problem* (3.2.7).

Assumption 2. *The second order sufficient conditions for a local minimum of problem* (3.2.5) *hold at* x_0 *with associated Lagrange multipliers* λ_0 *and* μ_0.

Assumption 3. *The gradients* $\nabla_x g_i(x_0, \theta_0)$*, for all* i *such that* $g_i(x_0, \theta_0) = 0$ *and* $\nabla_x h_j(x_0, \theta_0)$*,* $j = 1, \dots, q$ *are linearly independent.*

Assumption 4. *Strict complementary slackness holds at* (x_0, θ_0)*, i.e.* $(\lambda_0)_i > 0$ *for all* i *such that* $g_i(x_0, \theta_0) = 0$*.*

Theorem 3.5 (Fiacco, 1976). *If Assumptions 1, 2, 3 and 4 hold for problem* (3.2.5) *at* (x_0, θ_0)*, then:*

(a) x_0 *is a local isolated minimizing point of problem* (3.2.7) *and the associated multipliers* λ_0 *and* μ_0 *are unique.*

(b) For θ *in a neighborhood of* θ_0*, there exists a unique, once continuously differentiable vector function* $z(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]^T$ *satisfying the second order sufficient conditions for a local minimum of problem* (3.2.5) *such that* $z(\theta_0) = z_0 = [x_0, \lambda_0, \mu_0]^T$ *and hence,* $x(\theta)$ *is a locally unique local minimum of problem* (3.2.5) *with associated unique multipliers* $\lambda(\theta)$ *and* $\mu(\theta)$*.*

(c) For θ *near* θ_0*, the set of active inequalities is unchanged, strict complementary slackness holds for* $\lambda_i(\theta)$ *for* i *such that* $g_i(x(\theta), \theta) = 0$*, and the active constraint gradients are linearly independent at* $x(\theta)$*.*

Remark 3. *The non-degeneracy assumptions (Assumptions 1, 2, 3 and 4) in Theorem 3.5 seem rather restrictive specially for parametric problems. However, the assumptions are required to be satisfied at each of the chosen parameter values* θ_0 *in the feasible parameter region. Therefore, the strict complementarity assumption should not necessarily be satisfied for the parametric problem in general.*

The derivative of $z(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]^T$ can be calculated near to the point θ_0 by the expression

$$\nabla_\theta z(\theta) = [M(\theta)]^{-1} N(\theta),$$

where $M(\theta)$ is the Jacobian with respect to (x, λ, μ) of the following Karush-Kuhn-Tucker system (satisfied by $z(\theta)$ near $\theta = \theta_0$):

$$\begin{aligned} \nabla_x \mathcal{L}[x(\theta), \lambda(\theta), \mu(\theta), \theta] &= 0 \\ \lambda_i g_i[x(\theta), \theta] &= 0, \ i = 1, \dots, p \\ h_j[x(\theta), \theta] &= 0, \ j = 1, \dots, q, \end{aligned}$$

and $N(\theta)$ is the negative of the Jacobian of the Karush-Kuhn-Tucker system with respect to θ.

Corollary 1 (Fiacco, 1976). *Under the Assumptions 1, 2, 3 and 4 a first-order estimation of the optimal solution* $[x(\theta), \lambda(\theta), \mu(\theta)]$ *in a neighborhood of* θ_0 *is*

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - M_0^{-1} \cdot N_0 \cdot (\theta - \theta_0) + o(\|\theta\|), \tag{3.2.8}$$

where $[x_0, \lambda_0, \mu_0] = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)], M_0 = M(\theta_0), N_0 = N(\theta_0)$*, and* $M(\theta)$ *and* $N(\theta)$ *are defined as*

$$M(\theta) = \left[\begin{array}{c|c|c} \nabla^2_{xx}\mathcal{L} & \nabla_x g & \nabla_x h \\ \hline -\lambda \nabla_x^T g & diag(-g_i) & 0 \\ \hline \nabla_x^T h & 0 & 0 \end{array}\right],$$

$$N(\theta) = [\nabla^2_{\theta x}\mathcal{L}, -\lambda_1 \nabla_\theta^T g_1, \ldots, -\lambda_p \nabla_\theta^T g_p, \nabla_\theta^T h_1, \ldots, \nabla_\theta^T h_q]^T$$

and $\phi(\theta) = o(\|\theta\|)$ *means that* $\phi(\theta)/\|\theta\| \to 0$ *as* $\theta \to \theta_0$.

The Assumptions 1, 2, 3 and 4 ensure that the inverse of M_0 exists and hence for problems involving convex f, g and h, the parametric solutions within the corresponding critical regions, are necessary and sufficient.

In the absence of nonlinear constraints, based on Corollary 1, Dua et al.[32] proposed an algorithm to solve (3.2.5) in the entire range of the varying parameters for general convex problems. The space of θ where solution (3.2.8) remains optimal to (3.2.5) (the *critical region*, $\mathcal{CR}$) can be obtained by using feasibility and optimality conditions [32]. Each piecewise linear approximation is confined to regions defined by feasibility and optimality conditions. If $\breve{g}$ corresponds to the inactive polyhedral constraints and $\breve{\lambda}$ to the Lagrangian multipliers of the active constraints, then the critical regions can be defined as,

$$\mathcal{CR} = \begin{cases} \breve{g}(x(\theta), \theta) \leq 0, & \text{Feasibility conditions,} \\ \breve{\lambda}(\theta) \geq 0, & \text{Optimality conditions.} \end{cases}$$

And, the optimal solution $x(\theta)$ can be approximated explicitly as a conditional piecewise linear function [32]:

$$\begin{cases} x = C_1 + K_1\theta, & \text{if } \theta \in \mathcal{CR}_1 \\ x = C_2 + K_2\theta, & \text{if } \theta \in \mathcal{CR}_2 \\ \quad\vdots & \\ x = C_p + K_p\theta, & \text{if } \theta \in \mathcal{CR}_p \end{cases}$$

where C_i are column vectors and K_i are real matrices, whereas $\mathcal{CR}_i \subseteq \mathbb{R}^m$ are critical regions and note that $\mathcal{CR}_i$ denotes the i^{th} critical region.

After defining the critical region $\mathcal{CR}$ in which the parametric solution is valid, if $\mathcal{CR}$ has not covered the entire parametric region, we repeat again the same mathematical procedure as in above with any new feasible parameter ($\theta = \theta_0$) taken from the rest of the parametric regions until the parametric region has been explored successfully as described in [59]. To define the rest of the parametric region, let $\Theta = \{\theta \in \mathbb{R}^m : \theta^L \leq \theta \leq \theta^U\}$ be the overall parametric region (where θ^L and θ^U represent the lower and upper bounds of the parametric region) and let the inequalities, labeled by $c_1 \leq 0, c_2 \leq 0, c_3 \leq 0$ define $\mathcal{CR}_0$. Now the rest of the parameter region $\mathcal{CR}^{Rest} = \Theta \setminus \mathcal{CR}_0$ can be characterized by considering each of the inequalities which comprise $\mathcal{CR}_0$, reversing their signs one by one and removing redundant constraints [34]. For example, consider inequality $c_1 \leq 0$, the rest of the region can be addressed by taking the complement of the inequality $c_1 \leq 0$ and removing redundant constraints in Θ, which is $\mathcal{CR}_1^{Rest} = c_1 > 0, \theta_1 \geq \theta_1^L, \theta_2 \leq \theta_2^U$ where, $\theta = (\theta_1, \theta_2)$. Thus by considering the rest of the inequalities, the total of the rest region is given by, $\mathcal{CR}^{Rest} = \mathcal{CR}_1^{Rest} \cup \mathcal{CR}_2^{Rest} \cup \mathcal{CR}_3^{Rest}$, where $\mathcal{CR}_1^{Rest}, \mathcal{CR}_2^{Rest}$ and $\mathcal{CR}_3^{Rest}$ are given in Table 3.1 for two dimensional case.

Table 3.1: Definition of the rest regions

Region	Inequalities	
$\mathcal{CR}_1^{Rest}$	$c_1 > 0,$	$\theta_1 \geq \theta_1^L, \theta_2 \leq \theta_2^U$
$\mathcal{CR}_2^{Rest}$	$c_1 \leq 0, c_2 > 0,$	$\theta_1 \leq \theta_1^U, \theta_2 \leq \theta_2^U$
$\mathcal{CR}_3^{Rest}$	$c_1 \leq 0, c_2 \leq 0, c_3 > 0,$	$\theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2$

Theorem 3.6. *Let $\Theta \subseteq \mathbb{R}^m$ be a polyhedron and $\mathcal{CR}_0 = \{\theta \in \Theta\colon A\theta - b \leq 0\} \subseteq \Theta$, be a critical region. Assume $\mathcal{CR}_0 \neq \emptyset$. Also let $\mathcal{CR}_j = \{\theta \in \Theta\colon A_j\theta - b_j > 0,\ A_i\theta - b_i \leq 0,\ \forall i < j,\ j = 1, \ldots, K\}$ where $K = size(b)$, and let $\mathcal{CR}^{Rest} = \bigcup_{j=1}^{K} \mathcal{CR}_j$. Then (i) $\mathcal{CR}^{Rest} \bigcup \mathcal{CR}_0 = \Theta$, (ii) $\mathcal{CR}_0 \bigcap \mathcal{CR}_j = \emptyset$, and (iii) $\mathcal{CR}_j \bigcap \mathcal{CR}_i = \emptyset,\ j \neq i$, i.e. $\{\mathcal{CR}_0, \mathcal{CR}_1, \ldots, \mathcal{CR}_K\}$ is a partition of Θ.*

Proof. See the proof of Theorem 3 in [15]. □

Theorem 3.6 shows that the parametric region Θ can be partitioned within a finite feasible choice of the parameter $\theta = \theta_0$.

Remark 4. *If the multi-parametric problem contains some special non-convexity formulation (as described in [60]) in the objective function and the constraints are polyhedral, we can apply the branch-and-bound multi-parametric programming procedure proposed in [60]. This approach works by convexifying the objective function to underestimate them by* *convex functions. Then, the resulting convex parametric under-estimator problem is solved using the multi-parametric problem approach.*

For multi-parametric linear problems (mp-LP) and multi-parametric quadratic problems (mp-QP) with linear constraints, exact solutions can be computed using the first-order estimation and we can easily characterize the critical region and perform a cutting plane procedure in exploring the rest of the critical regions. On the other hand, for general multi-parametric nonlinear programs obtaining the exact solution (i.e., expressions of $x(\theta)$, $v(\theta)$ and the characterization of the critical regions) is a very difficult task [40, 41, 64]. Especially when the constraints are non-linear use of the cutting plane method to explore the critical regions can not be employed. That could be one of the reasons for almost all major efforts in this area have focused on providing approximate solutions for (3.2.5).

Methods for general multi-parametric nonlinear problems (mp-NLP) produce approximate solutions and can be broadly categorized into three areas: path-following or homotopy methods [45, 48], parameter space partition methods [14, 70], and problem approximation methods [30, 33]. Homotopy methods are only used in the single parameter case. They determine a continuous "path" of KKT points created as the parameter moves along an interval (i.e., the parameter space). Partition methods solve mp-NLPs by dividing the parameter space into smaller sets to approximate the critical regions of the optimal solution. The problem is then solved as a standard NLP at the vertices of each set and optimal value and decision functions are interpolated from the results. Approximation methods replace the mp-NLP with a series of mp-LP or mp-QP problems that can then be solved for exact solutions; additional approximations can be made until the desired accuracy is achieved [64].

For the convex case, the first algorithm for approximate solution of problem (3.2.5) was presented by Dua and Pistikopoulos in [33], which is based on the linearization of the objective and constraint functions and the solution of the resulting multi-parametric linear program. This algorithm has been developed further by Acevedo and Salgueiro in [1], and modified by using convex quadratic approximations for objective functions (i.e., quadratic approximation of the objective function and linear approximation of the constraints) in [30, 55]. A distinctly different approach to Dua and Pistikopoulos [33] for the solution of problem (3.2.5) was proposed by Johansen [56].

Pistikopoulos et al. [80] proposed a solution strategy for special non-convex multi-parametric programming problems based on a branch-and-bound algorithm to locate the global parametric solution. The key procedures of this algorithm was modified by Kassa and Kassa [59] and proposed a global optimization technique for solving a more general multi-parametric non-convex programming problems using sensitivity analysis. In particular, the solution approach effectively solves parametric problems with any twice continuously differentiable nonlinear objective function and having only polyhedral constraints. Domínguez and Pistikopoulos [29] proposed a multi-parametric quadratic approximation algorithm for multi-parametric nonlinear programs to address the case where the feasible parameter space is defined by a set of nonlinear equations.

The critical region, the space of θ where the parametric solution remains optimal to (3.2.5), is determined based on the feasibility and optimality conditions [34, 37, 80]. When a multi-parametric program contains only polyhedral constraints, as discussed above, we can easily characterize the critical region and perform a cutting plane procedure in exploring the rest of the critical regions. However, in solving multi-parametric problems with nonlinear constraints there are two main challenges we need to address. (i) How to find parametric solutions in the nonlinear constrained region; and (ii) how to explore the nonlinear parameter space through the procedures of the sensitivity theory. In the next section, these challenges will be addressed by the solution method proposed for non-linearly constrained multi-parametric optimization.

3.3 Algorithm for solving multi-parametric problems with nonlinear constraints

This section presents a novel algorithmic approach to find solutions of a multi-parametric problem with convex nonlinear constraints that can address those challenges raised above. To tackle the first challenge (*i.e.*, Challenge *(i)* of the above two), the algorithm employs a barrier function reformulation technique to construct a multi-parametric programs with polyhedral constraints based on the basic sensitivity analysis in [39]. Even if, the nonlinear constraint is transformed into the objective using the barrier method, each solution still needs to satisfy the constraints as the critical regions are determined by the feasibility and optimality conditions. This invokes challenge *(ii)* of the above mentioned issues. To address this challenge we will compute analytically the nonlinear solutions on the feasible boundary of the nonlinear constraints.

3.3.1 Equivalent multi-parametric barrier problem

In this subsection we present the reformulation of a multi-parametric problem with convex nonlinear constraints and bounded parametric region as an equivalent multi-parametric barrier problem with polyhedral constraints. The reformulation uses a barrier function method to incorporate the nonlinear constraints into the objective, so that the problem is equivalently reformulated as a multi-parametric problem having only polyhedral constraints.

Consider the following multi-parametric problem whose polyhedral constraints and the nonlinear constraints are separately represented as shown below

$$\begin{cases} \min\limits_{x} f(x,\theta) \\ s.t. \begin{cases} g(x,\theta) \leq 0, \\ h(x,\theta) \leq 0, \end{cases} \end{cases} \tag{3.3.1}$$

where $x \in \mathbb{R}^n$ is the decision variable, $\theta \in \mathbb{R}^m$ is the parameter vector, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the objective function, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is polyhedral inequality constraints and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$ is vector of nonlinear inequality constraints.

We make the following assumptions on the structure of (3.3.1) that will enable us to reformulate the problem as an equivalent barrier problem:

Assumption 5. *The polyhedral inequality constraint is given by* $g(x,\theta) = Ax - B\theta - b \leq 0$ *and it is assumed that* $rank(A) = p$. *If* A_i, B_i *and* b_i*, respectively, represent the* i^{th}*-row of* $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{p\times m}$ *and* $b \in \mathbb{R}^p$*, then* $g = (g_1, \ldots, g_p)$ *where* $g_i(x,\theta) = A_i x - B_i\theta - b_i \leq 0,\ i = 1, \ldots, p$.

Assumption 6. *The vector of the nonlinear inequality constraints* $h = (h_1, \ldots, h_q)$ *and the objective function* f *are twice continuously differentiable and are jointly convex in both the variables.*

Assumption 7. *The parameters* θ *are bounded and there exists vector values* θ^L *and* θ^U *such that* $\theta^L \leq \theta \leq \theta^U$.

For $\theta \in \mathbb{R}^m$, the feasible set of (3.3.1) is defined by,

$$X(\theta) = \{x\colon\, g(x,\theta) = Ax - B\theta - b \leq 0,\ \ h(x,\theta) \leq 0\}.$$

The Lagrangian function associated with (3.3.1) is defined by:

$$\mathcal{L}(x,\theta,\lambda,\mu) = f(x,\theta) + \lambda^T g(x,\theta) + \mu^T h(x,\theta), \tag{3.3.2}$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)^T$ and $\mu = (\mu_1, \ldots, \mu_q)^T$ are the Lagrange multiplier vectors associated with linear and nonlinear constraints g_i and h_j, respectively.

Define the penalty function for problem (3.3.1) as,

$$W(x, \theta, t) = f(x, \theta) + t\psi(x, \theta), \tag{3.3.3}$$

where t is a positive real parameter. Here, $\psi(x, \theta)$ is the so called *barrier function*, which is assumed to be non-negative and continuous over the region $\{x \in \mathbb{R}^n : h(x, \theta) < 0\}$ and tends to ∞ as the boundary of the feasible region $\{x \in \mathbb{R}^n : h(x, \theta) \leq 0\}$ is approached from the interior. Thus, the function $\psi(x, \theta)$ sets a barrier against leaving the interior of the feasible region and it can be defined as,

$$\psi(x, \theta) = -\sum_{j=1}^{q} \ln(-h_j(x, \theta)), \tag{3.3.4}$$

with domain $\{x \in \mathbb{R}^n : h(x, \theta) < 0\}$.

Since $h(x, \theta)$ is assumed to be convex, the logarithmic barrier function $\psi(x, \theta)$ has the following key properties:

i) $\psi(x, \theta)$ is twice continuously differentiable and

$$\nabla_x \psi(x, \theta) = \sum_{j=1}^{q} \frac{1}{(-h_j(x, \theta))} \nabla_x h_j(x, \theta),$$

$$\nabla^2_{xx} \psi(x, \theta) = \sum_{j=1}^{q} \frac{1}{h_j^2(x, \theta)} \nabla_x h_j(x, \theta) \nabla_x h_j(x, \theta)^T + \sum_{j=1}^{q} \frac{1}{(-h_j(x, \theta))} \nabla^2_{xx} h_j(x, \theta).$$

ii) $\psi(x, \theta)$ is convex: this is because, for an appropriate size non-zero vector d,

$$\begin{aligned} d^T \nabla^2_{xx} \psi(x, \theta) d &= \sum_{j=1}^{q} \left[\frac{1}{h_j^2(x, \theta)} d^T \nabla_x h_j(x, \theta) \nabla_x h_j(x, \theta)^T d + \frac{1}{(-h_j(x, \theta))} d^T \nabla^2_{xx} h_j(x, \theta) d \right] \\ &= \sum_{j=1}^{q} \left[\frac{1}{h_j^2(x, \theta)} \|\nabla_x h_j(x, \theta)^T d\|^2 + \frac{1}{(-h_j(x, \theta))} d^T \nabla^2_{xx} h_j(x, \theta) d \right] \geq 0 \end{aligned}$$

since $\frac{1}{-h_j(x,\theta)} > 0$ and $d^T \nabla^2_{xx} h_j(x, \theta) d \geq 0$ (as each h_j is assumed to be convex, and hence $\nabla^2_{xx} h_j(x, \theta)$ is positive semi-definite). This implies that, at all points $\nabla^2_{xx} \psi(x, \theta)$ is positive semi-definite. Therefore, $\psi(x, \theta)$ is convex.

By assuming that the domain of the logarithmic barrier (3.3.4) intersects the polyhedral set defined by the inequality $g(x, \theta) \leq 0$ define the following barrier problem,

$$\begin{aligned} &\min_x \{W(x, \theta, t) = f(x, \theta) + t\psi(x, \theta)\} \\ &s.t.\ g(x, \theta) = Ax - B\theta - b \leq 0. \end{aligned} \tag{3.3.5}$$

Remark 5. *If the polyhedral constraints $g_i(x,\theta)$ are not given explicitly in* (3.3.1) *then we will consider the parametric bounds $\theta^L \leq \theta \leq \theta^U$ as the polyhedral constraints instead.*

For $\theta \in \mathbb{R}^m$, the feasible set of (3.3.5) is defined by,

$$X^{\#}(\theta) = \{x\colon g(x,\theta) = Ax - B\theta - b \leq 0,\ \ h(x,\theta) < 0\}\,.$$

The Lagrangian function associated with (3.3.5) is defined by:

$$\mathcal{L}^{\#}(x,\theta,\lambda,t) = W(x,\theta,t) + \lambda^T g(x,\theta) = f(x,\theta) + t\psi(x,\theta) + \lambda^T g(x,\theta), \tag{3.3.6}$$

where $\lambda = (\lambda_1,\ldots,\lambda_p)^T$ is the Lagrange multiplier vector associated with polyhedral inequality constraint g.

For a given $\theta = \theta_0$ the multi-parametric problem (3.3.5) can be reduced to a classical nonlinear problem,

$$\begin{aligned} &\min_x\{W(x,\theta_0,t) = f(x,\theta_0) + t\psi(x,\theta_0)\} \\ &s.t.\ g(x,\theta_0) = Ax - B\theta_0 - b \leq 0. \end{aligned} \tag{3.3.7}$$

Theorem 3.7. *If $x^*(\theta,t), t > 0$, is an optimal solution of* (3.3.5)*, then*

$$x^*(\theta) = \lim_{t\to 0^+} x^*(\theta,t)$$

is an optimal solution of the problem (3.3.1).

Proof. Suppose that $x^*(\theta,t),\ t > 0$ is an optimal solution of (3.3.5) for a given $\theta \in \mathbb{R}^m$. This implies that there exists a Lagrangian multiplier vector $\lambda^*(\theta,t)$ such that $x^*(\theta,t)$ satisfies

$$\begin{cases} \nabla_x\mathcal{L}^{\#}(x^*(\theta,t),\lambda^*(\theta,t),\theta,t) = \nabla_x f(x^*(\theta,t),\theta) + t\nabla_x\psi(x^*(\theta,t),\theta) + A^T\lambda^*(\theta,t) \\ \qquad = \nabla_x f(x^*(\theta,t),\theta) + t\displaystyle\sum_{j=1}^{q}\frac{1}{(-h_j(x^*(\theta,t),\theta))}\nabla_x h_j(x^*(\theta,t),\theta) \\ \qquad + A^T\lambda^*(\theta,t) \\ \qquad = 0 \\ \lambda_i^*(\theta,t)g_i(x^*(\theta,t),\theta) = 0\ \ (i = 1,\ldots,p) \\ g_i(x^*(\theta,t),\theta) \leq 0\ \ (i = 1,\ldots,p) \\ \lambda_i^*(\theta,t) \geq 0\ \ (i = 1,\ldots,p) \\ h_j(x^*(\theta,t),\theta) < 0\ \ (j = 1,\ldots,q). \end{cases} \tag{3.3.8}$$

For a given $x^*(\theta,t),\ t > 0$, define the vector $\mu^*(\theta,t) = (\mu_1^*(\theta,t),\ldots,\mu_q^*(\theta,t))^T$ by

$$\mu_j^*(\theta,t)\begin{cases} = \frac{t}{-h_j(x^*(\theta,t),\theta)} & , x^*(\theta,t) \in X^{\#}(\theta) \\ \geq 0 & , x^*(\theta,t) \notin X^{\#}(\theta) \end{cases},\ j = 1,\ldots,q \tag{3.3.9}$$

and the vectors

$$x^*(\theta) = \lim_{t \to 0^+} x^*(\theta, t), \;\; \lambda^*(\theta) = \lim_{t \to 0^+} \lambda^*(\theta, t) \;\text{ and }\; \mu^*(\theta) = \lim_{t \to 0^+} \mu^*(\theta, t). \tag{3.3.10}$$

We claim that $x^*(\theta)$ is a minimizer and $(\lambda^*(\theta), \mu^*(\theta))$ is a dual feasible pair for problem (3.3.1).

Indeed, using (3.3.10) in the optimality conditions (3.3.8) and letting $t \to 0^+$ we have,

$$\begin{cases} \nabla_x f(x^*(\theta), \theta) + \sum_{j=1}^{q} \mu_j^*(\theta) \nabla_x h_j(x^*(\theta), \theta) + A^T \lambda^*(\theta) = 0 \\ \lambda_i^* g_i(x^*(\theta), \theta) = 0, \; i = 1, \dots, p \\ g_i(x^*(\theta), \theta) \leq 0, \; i = 1, \dots, p \\ \lambda_i^*(\theta) \geq 0, \; i = 1, \dots, p \\ \mu_j^*(\theta) h_j(x^*(\theta), \theta) = 0, \; j = 1, \dots, q, \; x^*(\theta) \in X^{\#}(\theta) \\ h_j(x^*(\theta), \theta) < 0, \; x^*(\theta) \in X^{\#}(\theta) \\ h_j(x^*(\theta), \theta) = 0, \; x^*(\theta) \notin X^{\#}(\theta) \\ \mu_j^*(\theta) \geq 0, \; j = 1, \dots, q. \end{cases} \tag{3.3.11}$$

From (3.3.11) we see that $x = x^*(\theta)$ minimizes (3.3.2) for $\lambda = \lambda^*(\theta)$ and $\mu = \mu^*(\theta)$, which means that $(\lambda^*(\theta), \mu^*(\theta))$ is a dual feasible pair for problem (3.3.1). Therefore, $x^*(\theta)$ is an optimal solution of (3.3.1). □

Remark 6. *In the proof of Theorem 3.7, we have assumed that $\lim_{t \to 0^+} x^*(\theta, t)$ exists for Equation* (3.3.10) *to hold. But this is justified by Theorem 25 in [42] if Assumptions 1, 2, 3 and 4 are satisfied. Then the existence of $\lambda^*(\theta)$ and $\mu^*(\theta)$ follows from the existence of $x^*(\theta)$ and continuity of the involved functions.*

Theorem 3.8. *The relation between an optimal value function $f^*(\theta)$ of the original problem* (3.3.1) *and optimal value function $W^*(\theta, t)$ of the barrier problem* (3.3.5) *is given by*

$$f^*(\theta) = \lim_{t \to 0^+} W^*(\theta, t). \tag{3.3.12}$$

Proof. Suppose that $x^*(\theta, t), \; t > 0$ is an optimal solution of problem (3.3.5) for a given $\theta \in \mathbb{R}^m$. This implies that $W^*(\theta, t) = W(x^*(\theta, t), \theta, t) = f(x^*(\theta, t), \theta) + t\psi(x^*(\theta, t), \theta)$, then taking the limit on both sides we have

$$\lim_{t \to 0^+} W^*(\theta, t) = \lim_{t \to 0^+} f(x^*(\theta, t), \theta) = f(x^*(\theta), \theta), \tag{3.3.13}$$

which results from the continuity of f.

Consider the multiplier $\lambda^*(\theta, t)$ corresponding to $x^*(\theta, t)$ and multiplier $\mu^*(\theta, t)$ defined by (3.3.9). Then the dual function, π of problem (3.3.1) evaluated at $(\lambda^*(\theta, t), \mu^*(\theta, t))$ is equal to:

$$
\begin{aligned}
\pi(\lambda^*(\theta,t), \mu^*(\theta)) &= f(x^*(\theta,t),\theta) + \sum_{i=1}^{p} \lambda_i^*(\theta,t) g_i(x^*(\theta,t),\theta) + \sum_{j=1}^{q} \mu_j^*(\theta,t) h_j(x^*(\theta),\theta) \\
&= f(x^*(\theta,t),\theta) + \sum_{i=1}^{p} [\underbrace{\lambda_i^*(\theta,t) g_i(x^*(\theta,t),\theta)}_{=0}] + \sum_{j=1}^{q} [\quad \underbrace{\mu_j^*(\theta,t) h_j(x^*(\theta,t),\theta)}_{=\begin{cases} -t & , x^*(\theta,t) \in X^{\#}(\theta) \\ 0 & , x^*(\theta,t) \notin X^{\#}(\theta) \end{cases}} \quad] \\
&\geq f(x^*(\theta,t),\theta) + \sum_{j=1}^{q} [-t] \\
&\geq f(x^*(\theta,t),\theta) - qt.
\end{aligned}
$$

In particular, the duality gap associated with $x^*(\theta, t) \in X^{\#}(\theta)$ and the dual feasible pair $(\lambda^*(\theta, t), \mu^*(\theta, t))$ is simply qt. Using the weak duality, $f^*(\theta) \geq \pi(\lambda^*(\theta, t), \mu^*(\theta, t))$, in the above inequality we have

$$0 \leq f(x^*(\theta,t),\theta) - f^*(\theta) \leq qt.$$

Implying that

$$f^*(\theta) = \lim_{t \to 0^+} f(x^*(\theta,t),\theta). \tag{3.3.14}$$

Then the assertion of the theorem follows from (3.3.13) and (3.3.14). □

3.3.2 Sensitivity analysis of the barrier problem

Suppose that problem (3.3.1) satisfies the Assumptions 5, 6 and 7 with y considered as the parameter vector. Now, for a given $\theta_0 \in \mathbb{R}^m$ and $t > 0$, if $x_0(t) = x(\theta_0, t)$ is the local minimizer of (3.3.5), then there exists a Lagrangian multiplier $\lambda_0(t) = \lambda(\theta_0, t)$ such that $x_0(t)$ satisfies the KKT conditions:

$$
\begin{aligned}
\nabla_x \mathcal{L}^{\#}(x(\theta_0,t), \lambda(\theta_0,t), \theta_0, t) &= 0 \\
\lambda_i g_i(x(\theta_0,t),\theta_0) &= 0.
\end{aligned}
\tag{3.3.15}
$$

Let $z = [x^T(\theta,t) \mid \lambda^T(\theta,t)]^T$ and $a = [z(\theta,t)^T \mid \theta^T]^T = [x^T(\theta,t) \mid \lambda^T(\theta,t) \mid \theta^T]^T$, define

$$H(a) = \begin{bmatrix} \nabla_x \mathcal{L}^{\#}(x(\theta,t), \lambda(\theta,t), \theta, t) \\ \lambda_i g_i(x(\theta,t),\theta) \end{bmatrix}. \tag{3.3.16}$$

Note that for $a_0 = [z_0^T \mid \theta_0^T]^T = [z^T(\theta_0,t) \mid \theta_0^T]^T$, equation (3.3.15) can be written as $H(a_0) = 0$. That means, the local minimizer of (3.3.5) is the zero of (3.3.16).

If $H(a)$ is continuously differentiable in the neighborhood of a_0, then the Taylor's series expansion of H is given by

$$H(a) = H(a_0)+\nabla_a H(a_0)(a-a_0)+\frac{1}{2}(a-a_0)^T\nabla_{aa}H(a_0)(a-a_0)+o(\|a-a_0\|^2), \quad (3.3.17)$$

where $o(\|a-a_0\|^2) \to 0$ as $a \to a_0$.

Now we consider the following two cases.

Case 1. *If the second and higher order derivatives of function* (3.3.16) *are zero (i.e., when* (3.3.16) *consist of functions of up to a quadratic polynomial order, or when the objective function in* (3.3.5) *is up to second-degree polynomial, or when the objective function and constraints in* (3.3.1) *are up to second-degree polynomial in x and affine linear in θ), then the Taylor's expansion* (3.3.17) *is exact and is given by*

$$H(a) = \underbrace{H(a_0)}_{=0} + \nabla_a H(a_0)(a-a_0). \quad (3.3.18)$$

Then, we will find a solution $z(\theta,t)$ such that (3.3.18) *is zero in a neighborhood of a_0,*

$$\nabla_a H(a_0)(a-a_0) = 0,$$

which can also be written as

$$[\nabla_z H(a_0)|\nabla_\theta H(a_0)]\left[(z-z_0)^T|(\theta-\theta_0)^T\right]^T = 0.$$

Therefore, $z(\theta,t)$ can be computed from the following system of linear equations

$$\nabla_z H(a_0)(z-z_0) + \nabla_\theta H(a_0)(\theta-\theta_0) = 0. \quad (3.3.19)$$

This is an exact solution for quadratically constrained quadratic problems, where the parameter vector θ appears only in linear form in the problem.

Case 2. *If the third and higher order derivatives of function* (3.3.16) *is zero (i.e., when* (3.3.16) *consist of functions of up to a quadratic polynomial order both in x and θ, or when the objective function and constraints in* (3.3.1) *are up to third-degree polynomial in x and up to quadratic in θ), then the Taylor's expansion* (3.3.17) *is exact and is given by*

$$H(a) = H(a_0) + \nabla_a H(a_0)(a-a_0) + \frac{1}{2}(a-a_0)^T\nabla_{aa}H(a_0)(a-a_0). \quad (3.3.20)$$

Then, we can find a solution $z(\theta,t)$ such that (3.3.20) *is zero in a neighborhood of a_0,*

$$H(a_0) + \nabla_a H(a_0)(a-a_0) + \frac{1}{2}(a-a_0)^T\nabla_{aa}H(a_0)(a-a_0) = 0.$$

Since $H(a_0)=0$, $z(\theta,t)$ will be computed from the following system of quadratic equations

$$(a-a_0)^T\nabla_{aa}H(a_0)(a-a_0) + 2\nabla_a H(a_0)(a-a_0) = 0. \quad (3.3.21)$$

From the above two cases, we can observe that when problem (3.3.1) contains objective and constraint functions up to third-order polynomials in x and quadratic in θ, the function $H(a)$ defined by (3.3.17) consists of equations of up to second order polynomial in all its variables. As the result the solutions of the system (3.3.21) are exact to (3.3.5).

A quadratic approximation solution strategy is also proposed by Pappas et al. in [78] to find exact solutions for quadratically constrained multi-parametric quadratic problems and used an active set-based approach for the exploration of the parameter space. Their approach requires the determination of analytic solutions of a system of quadratic equations which is a very challenging task especially for large scale optimization problems. To see this, if we let $(a - a_0)^T = [(z - z_0)^T \mid (\theta - \theta_0)^T]$ equation (3.3.21) can be written as

$$\begin{bmatrix} (z - z_0) \\ (\theta - \theta_0) \end{bmatrix}^T \left[\begin{array}{c|c} \nabla_{zz} H(a_0) & \nabla_{\theta z} H(a_0) \\ \hline \nabla_{z\theta} H(a_0) & \nabla_{\theta\theta} H(a_0) \end{array}\right] \begin{bmatrix} (z - z_0) \\ (\theta - \theta_0) \end{bmatrix} + 2 \left[\nabla_z H(a_0) \mid \nabla_\theta H(a_0)\right] \begin{bmatrix} (z - z_0) \\ (\theta - \theta_0) \end{bmatrix} = 0$$

resulting in the following $2 \times (n + p)$ system of quadratic equations in z and θ

$$\begin{cases} (z - z_0)^T \nabla_{zz} H(a_0)(z - z_0) + (\theta - \theta_0)^T \nabla_{z\theta} H(a_0)(z - z_0) + 2\nabla_z H(a_0)(z - z_0) = 0 \\ (z - z_0)^T \nabla_{\theta z} H(a_0)(\theta - \theta_0) + (\theta - \theta_0)^T \nabla_{\theta\theta} H(a_0)(\theta - \theta_0) + 2\nabla_\theta H(a_0)(\theta - \theta_0) = 0 \end{cases} \tag{3.3.22}$$

where $\nabla_z H(a_0)$ and $\nabla_\theta H(a_0)$ are matrices of dimension $(n+p) \times (n+p)$ and $(n+p) \times m$, respectively. Whereas $\nabla_{zz} H(a_0)$, $\nabla_{z\theta} H(a_0)$, $\nabla_{\theta z} H(a_0)$ and $\nabla_{\theta\theta} H(a_0)$, respectively are $(n + p) \times (n + p)$ by $(n + p)$, $(n + p) \times (n + p)$ by m, $(n + p) \times m$ by $(n + p)$ and $(n + p) \times m$ by m tensors of rank three.

Similarly, if we let $(a - a_0)^T = [(x - x_0)^T \mid (\lambda - \lambda_0)^T \mid (\theta - \theta_0)^T]$ equation (3.3.21) can also be written as

$$\begin{bmatrix} (x - x_0) \\ (\lambda - \lambda_0) \\ (\theta - \theta_0) \end{bmatrix}^T \left[\begin{array}{c|c|c} \nabla_{xx} H(a_0) & \nabla_{\lambda x} H(a_0) & \nabla_{\theta x} H(a_0) \\ \hline \nabla_{x\lambda} H(a_0) & \nabla_{\lambda\lambda} H(a_0) & \nabla_{\theta\lambda} H(a_0) \\ \hline \nabla_{x\theta} H(a_0) & \nabla_{\lambda\theta} H(a_0) & \nabla_{\theta\theta} H(a_0) \end{array}\right] \begin{bmatrix} (x - x_0) \\ (\lambda - \lambda_0) \\ (\theta - \theta_0) \end{bmatrix} + 2 \begin{bmatrix} \nabla_x H(a_0) \\ \nabla_\lambda H(a_0) \\ \nabla_\theta H(a_0) \end{bmatrix}^T \begin{bmatrix} (x - x_0) \\ (\lambda - \lambda_0) \\ (\theta - \theta_0) \end{bmatrix} = 0$$

resulting in the following $(n + p) \times (n + p + m)$ system of quadratic equations in x,

λ and θ,

$$\begin{cases} \left[(x-x_0)^T\nabla_{xx}H(a_0)+(\lambda-\lambda_0)^T\nabla_{x\lambda}H(a_0)+(\theta-\theta_0)^T\nabla_{x\theta}H(a_0)+2\nabla_x H(a_0)\right](x-x_0)=0 \\ \left[(x-x_0)^T\nabla_{\lambda x}H(a_0)+(\lambda-\lambda_0)^T\nabla_{\lambda\lambda}H(a_0)+(\theta-\theta_0)^T\nabla_{\lambda\theta}H(a_0)+2\nabla_\lambda H(a_0)\right](\lambda-\lambda_0)=0 \\ \left[(x-x_0)^T\nabla_{\theta x}H(a_0)+(\lambda-\lambda_0)^T\nabla_{\theta\lambda}H(a_0)+(\theta-\theta_0)^T\nabla_{\theta\theta}H(a_0)+2\nabla_\theta H(a_0)\right](\theta-\theta_0)=0 \end{cases} \tag{3.3.23}$$

where $\nabla_x H(a_0)$, $\nabla_\lambda H(a_0)$ and $\nabla_\theta H(a_0)$ are matrices of dimension $(n+p)\times n$, $(n+p)\times p$ and $(n+p)\times m$, respectively. Whereas $\nabla_{xx}H(a_0)$, $\nabla_{x\lambda}H(a_0)$, $\nabla_{x\theta}H(a_0)$, $\nabla_{\lambda x}H(a_0)$, $\nabla_{\lambda\lambda}H(a_0)$, $\nabla_{\lambda\theta}H(a_0)$, $\nabla_{\theta x}H(a_0)$, $\nabla_{\theta\lambda}H(a_0)$ and $\nabla_{\theta\theta}H(a_0)$ respectively are $(n+p)\times n$ by n, $(n+p)\times n$ by p, $(n+p)\times n$ by m, $(n+p)\times p$ by n, $(n+p)\times p$ by p, $(n+p)\times p$ by m, $(n+p)\times m$ by n, $(n+p)\times m$ by p and $(n+p)\times m$ by m tensors of rank three.

Even though a quadratic approximation provides an exact solution, it is evident from the systems of quadratic equations (3.3.22) and (3.3.23) that it is computationally challenging to explore the solutions in the given parameter space. Therefore, in the next subsections we will concentrate on a solution strategy for multi-parametric problems with nonlinear constraints, based on a linear approximation in the interior and analytic solutions on the boundary of the nonlinear constraints. The proposed method can also provide exact solutions for multi-parametric problems whose objective and the constraint functions are polynomials of up to cubic degree in the optimization variable and quadratic in the parameters vector if we use quadratic approximation of $H(a)$ in stead.

For multi-parametric polynomial programming problem, where the objective and constraint functions are polynomial, an exact solution method is proposed in [23, 31], the method solves the system of equality constraints that arise from the first-order KKT conditions of the problem. Fotiou et al. [44] also proposed an exact solution method for multi-parametric polynomial programming problem by exact multi-parametric nonlinear inversion of the optimality conditions. The methods proposed in [23, 31, 44] work by making use of the theory of Gröbner Bases where the Buchberger algorithm can be used to transform the set of polynomial equations into a triangular system of equations. The triangular system is the nonlinear polynomial equivalent of the triangular system obtained by Gaussian elimination for a linear system of equations. Computational complexity of this method grows exponentially with the number of variables.

3.3.3 Exact and approximate solutions of the barrier problem

The following proposition establishes the first-order approximation of the local minimizer $x(\theta, t)$ of the multi-parametric barrier problem (3.3.5).

Theorem 3.9. *Let x_0 be an optimal solution of* (3.3.7) *with associated Lagrange multiplier λ_0 and barrier parameter $t > 0$ corresponding to θ_0. If functions f, g and h are twice continuously differentiable and convex in (x, θ) in a neighborhood of (x_0, θ_0), then the local minimizer $x(\theta, t)$ of* (3.3.5) *and the associated Lagrange multiplier $\lambda(\theta, t)$, in a neighborhood of (x_0, θ_0), can be approximated by*

$$\begin{bmatrix} x(\theta,t) \\ \lambda(\theta,t) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} + [M_0(t)]^{-1} \cdot N_0(t) \cdot (\theta - \theta_0), \tag{3.3.24}$$

where $x_0 = x(\theta_0)$, $\lambda_0 = \lambda(\theta_0)$, $M_0(t) = M(\theta_0, t)$ and $N_0(t) = N(\theta_0, t)$. $M(\theta, t)$ and $N(\theta, t)$ are given by

$$\begin{cases} M(\theta,t) = \left[\begin{array}{c|c} \nabla^2_{xx}\mathcal{L}^{\#} & \nabla_x g \\ \hline -\lambda \nabla^T_x g & diag(-g_i) \end{array} \right], \\ N(\theta,t) = [\nabla^2_{\theta x}\mathcal{L}^{\#}, -\lambda_1 \nabla_\theta g_1^T, \ldots, -\lambda_p \nabla_\theta g_p^T]^T \end{cases} \tag{3.3.25}$$

and $\mathcal{L}^{\#}(x, \theta, \lambda, t)$ is defined by equation (3.3.6).

Proof. If f, g and h in (3.3.5) are twice continuously differentiable in (x, θ) in a neighborhood of (x_0, θ_0), then as shown in Section 3.3.1 $\psi(x, \theta)$ is also twice continuously differentiable in (x, θ) in a neighborhood of (x_0, θ_0). Therefore, for $t > 0$, $W(x, \theta, t)$ is twice continuously differentiable in (x, θ) in a neighborhood of (x_0, θ_0). Consequently (3.3.5) will satisfy the Assumptions 1, 2, 3 and 4.

Now, if we let $M(\theta, t) = \nabla_z H(z(\theta, t), \theta)$ and $N(\theta, t) = -\nabla_\theta H(z(\theta, t), \theta)$ the system of linear equations (3.3.19) is written as

$$M(\theta_0, t)(z - z_0) = N(\theta_0, t)(\theta - \theta_0). \tag{3.3.26}$$

Assumptions 1, 2, 3 and 4, which is shown to be satisfied by (3.3.5), ensure that $M(\theta_0, t)$ is non-singular, as a result the first order estimate of $z(\theta, t)$ is explicitly given by,

$$z(\theta, t) = z(\theta_0, t) + [M(\theta_0, t)]^{-1} N(\theta_0, t)(\theta - \theta_0). \tag{3.3.27}$$

Therefore, an approximation of $[x(\theta, t), \lambda(\theta, t)]$ in a neighborhood of (x_0, θ_0), is given by (3.3.24). □

Remark 7. *Multi-parametric problems whose objective and constraint functions are quadratic in terms of x and linear in terms of θ, have a common property that the function $H(a)$ in* (3.3.16) *consists of only linear equations. Therefore, the exact solution of multi-parametric programs that consists of quadratic and/or linear constraints and objective functions in terms of x and linear in θ can be established using the above reformulation approximation.*

Moreover, exact parametric solutions for problems which are cubic in x and quadratic in θ can also be established using system (3.3.23)*.*

Now, we consider the sensitivity analysis to describe the structure of the critical region ($\mathcal{CR}$) for the reformulated problem. The multi-parametric problem (3.3.1) have p linear and q nonlinear constraints; and its corresponding multi-parametric barrier problem (3.3.5) have p linear constraints. Let $\mathcal{P} = \{1, 2, \dots, p\}$, $\mathcal{Q} = \{1, 2, \dots, q\}$, and define the active and inactive sets of the barrier problem by $\mathcal{A}(x, \theta) = \{i \in \mathcal{P} : g_i(x, \theta) = 0\}$ and $\mathcal{I}(x, \theta) = \mathcal{P} \setminus \mathcal{A}(x, \theta)$ respectively.

We used the first order approximation (3.3.24) given in Theorem 3.9 to find the local minimizer $x(\theta, t)$ of (3.3.5) and its corresponding Lagrangian multiplier $\lambda(\theta, t)$. Then, evaluate the limits,

$$x^L(\theta) = \lim_{t \to 0^+} x(\theta, t), \tag{3.3.28a}$$

$$\lambda(\theta) = \lim_{t \to 0^+} \lambda(\theta, t), \tag{3.3.28b}$$

Then, the critical regions in which the parametric solution (3.3.28a) remains a valid solution to (3.3.5) is defined by the following feasibility and optimality conditions,

$$\begin{cases} g_{\mathcal{I}(x,\theta)}(x(\theta), \theta) \leq 0 & \text{Feasibility conditions} \\ \lambda_{\mathcal{A}(x,\theta)}(\theta) \geq 0 & \text{Optimality conditions} \end{cases}$$

where $g_{\mathcal{I}(x,\theta)}(x(\theta), \theta)$ are the inactive polyhedral constraints and $\lambda_{\mathcal{A}(x,\theta)}(\theta)$ are the Lagrangian multipliers corresponding to the active constraints.

That means, for a given initial exploration region defined by a polyhedral set of parameters $\Theta = \{\theta \in \mathbb{R}^m : \theta^L \leq \theta \leq \theta^U\}$, the critical region is defined as

$$\mathcal{CR}_0 = \{\theta \in \Theta : g_{\mathcal{I}(x,\theta)}(x^L(\theta), \theta) \leq 0,\ \lambda_{\mathcal{A}(x,\theta)}(\theta) \geq 0\}. \tag{3.3.29}$$

Since problem (3.3.5) is assumed to be a convex parametric problem with linear constraints we follow the same procedures proposed by Dua et al. [32] and Kassa & Kassa [59] to explore the entire range of the varying parameters.

– If $\Theta = \mathcal{CR}_0$, we are done exploring the parameter space and (3.3.5) will have a single parametric solution which is valid in the entire parameter space.

– If $\mathcal{CR}_0$ has not covered the initial parameter space Θ (i.e., if $\Theta \setminus \mathcal{CR}_0 \neq \emptyset$), we repeat the same mathematical procedure as in above by considering any new feasible parameter ($\theta = \theta_0$) taken from the rest of parametric regions until the initial parametric space has been explored successfully.

For example, let $\Theta = \{\theta \in \mathbb{R}^2 : \theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2 \leq \theta_2^U\}$ be an initial exploration region and let the inequalities defining the critical region (3.3.29) are labeled by $c_1 \leq 0, c_2 \leq 0, c_3 \leq 0$. That means, $\mathcal{CR} = \{\theta \in \Theta : c_1(\theta) \leq 0, c_2(\theta) \leq 0, c_3(\theta) \leq 0\}$. The rest of the parameter region $\mathcal{CR}^{Rest} = \Theta \setminus \mathcal{CR}$ can be characterized by considering each of the inequalities which comprise $\mathcal{CR}_0$, reversing their signs one by one and removing redundant constraints. Then, the rest regions will be $\mathcal{CR}^{Rest} = \mathcal{CR}_1^{Rest} \cup \mathcal{CR}_2^{Rest} \cup \mathcal{CR}_3^{Rest}$ where $\mathcal{CR}_1^{Rest} = \{\theta : c_1(\theta) > 0, \theta_1^L \leq \theta_1 \leq \theta_1^U\}$, $\mathcal{CR}_2^{Rest} = \{\theta : c_1(\theta) \leq 0, c_2(\theta) > 0, \theta_1 \geq \theta_1^L, \theta_2 \leq \theta_2^U\}$ and $\mathcal{CR}_3^{Rest} = \{\theta : c_1(\theta) \leq 0, c_2(\theta) \leq 0, c_3(\theta) > 0, \theta_1 \leq \theta_1^U, \theta_2 \leq \theta_2^U\}$ are as shown in Fig. 3.1.

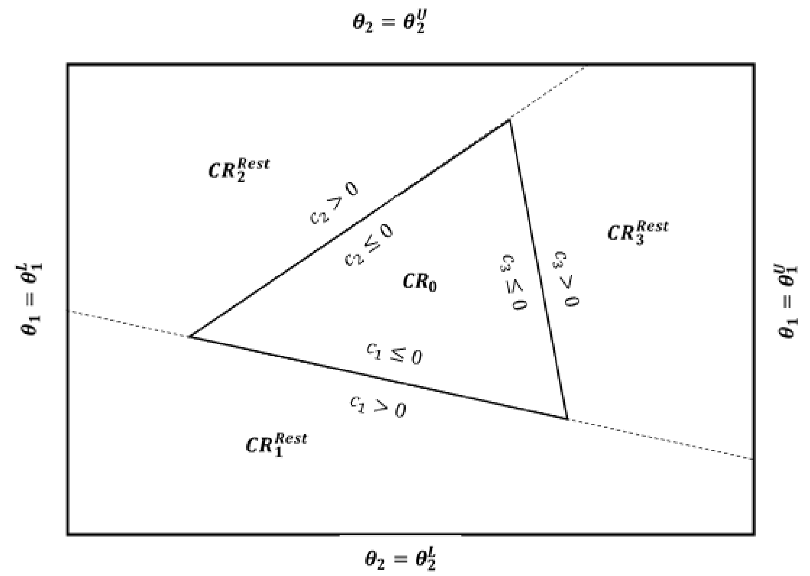

Figure 3.1: Rest region determination

This procedure has been employed in literature to solve multi-parametric optimization and multi-level problems with polyhedral constraints (see for instance, [32, 33, 59–61]).

3.3.4 Solutions of a general convex multi-parametric problems

This subsections presents methods to find solutions of the nonlinear multi-parametric problem (3.3.1) in the interior and on the boundary of the nonlinear constraints.

Exact and approximate solutions in the interior

After solving the barrier problem (3.3.5), we get a set of linear solutions, say,

$$x^L(\theta) = \Big\{ x^{L_k}(\theta) \quad \text{on } \mathcal{CR}_k,\ k = 1, \ldots, r, \tag{3.3.30}$$

in r critical regions, $\mathcal{CR}_1, \dots, \mathcal{CR}_r$. The critical regions $\mathcal{CR}_1, \dots, \mathcal{CR}_r$ are partitions of the initial parameter space Θ, so we have $\Theta = \bigcup_{k=1}^{r} \mathcal{CR}_k$.

For any $k \in \mathcal{K} = \{1, 2, \dots, r\}$ the critical region $\mathcal{CR}_k$ is determined by

$$\mathcal{CR}_k = \{\theta \in \mathcal{CR}_I : g_{\mathcal{I}_k}(x^{L_k}(\theta), \theta) \leq 0,\ \lambda_{\mathcal{A}_k}(\theta) \geq 0\},$$

where $\mathcal{I}_k = \mathcal{I}(x^{L_k}(\theta), \theta)$ and $\mathcal{A}_k = \mathcal{A}(x^{L_k}(\theta), \theta)$.

Since problem (3.3.1) and (3.3.5) are equivalent in the interior region $X^{\#}(\theta)$, the linear solution (3.3.30) is also a solution to (3.3.1) provided that it satisfies the non-linear feasibility condition $h(x(\theta), \theta) < 0$. Therefore, (3.3.30) is a valid solution to (3.3.1) in a modified corresponding critical regions

$$\mathcal{CR}_k^{\#} = \{\theta \in \mathcal{CR}_k : h(x^{L_k}(\theta), \theta) < 0\}. \tag{3.3.31}$$

Let $\Theta^{\#} = \bigcup_{k=1}^{r} \mathcal{CR}_r^{\#}$, then the rest of the parameter region $\mathcal{CR}^{Rest} = \Theta \setminus \Theta^{\#}$ can be characterized by considering each of the nonlinear inequalities which comprise $\mathcal{CR}_k^{\#}$, $(k = 1 : r)$, by reversing the signs of each of the nonlinear constraints one by one and removing redundant parts.

That means, for $k \in \mathcal{K}$, define the rest of the corresponding region as

$$\begin{aligned}
\mathcal{CR}_{k1}^{Rest} &= \left\{\theta \in \mathcal{CR}_k : h_1(x^{L_k}(\theta), \theta) \geq 0\right\}, \\
\mathcal{CR}_{k2}^{Rest} &= \left\{\theta \in \mathcal{CR}_k : h_1(x^{L_k}(\theta), \theta) < 0,\ h_2(x^{L_k}(\theta), \theta) \geq 0\right\}, \\
\mathcal{CR}_{k3}^{Rest} &= \left\{\theta \in \mathcal{CR}_k : h_1(x^{L_k}(\theta), \theta) < 0,\ h_2(x^{L_k}(\theta), \theta) < 0,\ h_3(x^{L_k}(\theta), \theta) \geq 0\right\},
\end{aligned}$$

and so on. Generally, for $k \in \mathcal{K},\ j \in \mathcal{Q}$, the rest of the critical region is defined as

$$\mathcal{CR}_{kj}^{Rest} = \left\{\theta \in \mathcal{CR}_k : h_1(x^{L_k}(\theta), \theta) < 0, \dots, h_{j-1}(x^{L_k}(\theta), \theta) < 0,\ h_j(x^{L_k}(\theta), \theta) \geq 0\right\}. \tag{3.3.32}$$

– If $\mathcal{CR}^{Rest} = \left\{\mathcal{CR}_{kj}^{Rest}\right\}_{k \in \mathcal{R},\ j \in \mathcal{Q}} = \emptyset$, there are no regions cut out by the nonlinear constraints, we are done exploring the parameter space Θ and problem (3.3.1) has linear solutions only,

$$x^L(\theta) = \left\{x^{L_k}(\theta) \quad \text{on } \mathcal{CR}_k^{\#},\ k = 1, \dots, r.\right. \tag{3.3.33}$$

– But, in the case $\mathcal{CR}^{Rest} \neq \emptyset$, we need to further explore this region for optimal solutions that may exist on the boundary of the nonlinear constraints which are valid in the rest of the regions.

Analytic solutions on the boundary of the nonlinear region

The next step is finding an analytic solution of (3.3.1) on the boundary of nonlinear constraints, i.e. in the region $\{X(\theta) \setminus X^{\#}(\theta)\} = \{x : g(x,\theta) \leq 0, h(x,\theta) = 0, \theta \in \mathcal{CR}^{Rest}\}$ whenever $\mathcal{CR}^{Rest} \neq \emptyset$.

In the region $\mathcal{CR}_{kj}^{Rest}$ we will determine the nonlinear solution $x^{NL_{kj}}(\theta)$ as a function of θ from the system of equations containing a nonlinear equation $h_j(x,\theta) = 0$ and linear equations of active constraints $g_{\mathcal{A}_k}(x,\theta) = 0$ of $\mathcal{CR}_k^{\#}$. That means, in each region $\mathcal{CR}_{kj}^{Rest}$, $(k \in \mathcal{R},\ j \in \mathcal{Q})$ the solution is computed from the following systems of nonlinear parametric equations,

$$\begin{cases} h_j(x,\theta) = 0, \\ g_{\mathcal{A}_k}(x,\theta) = 0. \end{cases} \tag{3.3.34}$$

In the critical region $\mathcal{CR}_k$,

- if $|\mathcal{A}_k| = n - 1$, the system (3.3.34) is consistent and it may have a unique solution.
- if $|\mathcal{A}_k| < n - 1$, the system (3.3.34) is inconsistent and hence for all $j \in \mathcal{Q}$ the region $\mathcal{CR}_{kj}^{Rest}$ is infeasible.
- if $|\mathcal{A}_k| > n - 1$, the system (3.3.34) is over-determined and hence no solution or the solution is valid only at a single point.

For each $k \in \mathcal{R}$ and $j \in \mathcal{Q}$, we will solve the system (3.3.34) to find the nonlinear solution,

$$x^{NL}(y) = \left\{ x^{NL_{kj}}(y) \quad \text{on } \mathcal{CR}_{kj}^{\#\#},\ k \in \mathcal{K},\ j \in \mathcal{Q}, \right. \tag{3.3.35}$$

where the critical region $\mathcal{CR}_{kj}^{\#\#}$ in which the solution $x^{NL_{kj}}(y)$ is valid is determined by

$$\mathcal{CR}_{kj}^{\#\#} = \{\theta \in \mathcal{CR}_{kj}^{Rest} :\ g_{\mathcal{I}_k}(x^{NL_{kj}}(\theta),\theta) \leq 0\}. \tag{3.3.36}$$

Note that, the problem is infeasible in the region

$$\mathcal{CR}_{kj}^{Rest} - \mathcal{CR}_{kj}^{\#\#} = \mathcal{CR}_k \setminus (\mathcal{CR}_k^{\#} \cup \mathcal{CR}_{kj}^{\#\#}),\ \forall k \in \mathcal{K}, \forall j \in \mathcal{Q}.$$

Remark 8. *In obtaining the analytic solution at the boundary, our approach requires solving a system of equations* (3.3.34) *having only one nonlinear equation in it. That means it is required to solve a set of linear equations appended with only one nonlinear equation. This makes the approach less computationally costly.*

Finally, the optimal solution of (3.3.1) for the given initial parameter space Θ is given by,

$$x(\theta) = \begin{cases} x^{L_k}(\theta) & \text{on } \mathcal{CR}_k^{\#}, k = 1, \dots, r \\ x^{NL_{kj}}(\theta) & \text{on } \mathcal{CR}^{\#\#}, k = 1, \dots, r\ , j = 1, \dots, q \\ \text{Infeasible} & \text{on } \mathcal{CR}_k \setminus (\mathcal{CR}_k^{\#} \cup \mathcal{CR}_{kj}^{\#\#}), k = 1, \dots, r\ , j = 1, \dots, q. \end{cases} \tag{3.3.37}$$

Remark 9. *It is helpful to note here that problem (3.3.1) has a linear solution in the region* $\mathcal{CR}^{\#} = \bigcup_{k \in \mathcal{R}} \mathcal{CR}_k^{\#}$*, a nonlinear solution in the region* $\mathcal{CR}^{\#\#} = \bigcup_{k \in \mathcal{R}, j \in \mathcal{Q}} \mathcal{CR}_{kj}^{\#\#}$ *and infeasible in the region* $\Theta \setminus (\mathcal{CR}^{\#} \cup \mathcal{CR}^{\#\#})$.

Remark 10. *In this subsection, to find the solution* $x^*(\theta)$ *of the barrier problem we have used a linear approximation* (3.3.18) *to obtain the solution* (3.3.24)*, but one may use a quadratic approximation to find the solution* $x^*(\theta)$ *to obtain exact solution for quadratically constrained nonlinear problems.*

3.3.5 Algorithm to find an exact solution for multi-parametric nonlinear problems

This subsection presents an algorithmic approach to find solutions of a general convex multi-parametric problems of the form (3.3.1) and that satisfy the assumptions 5, 6 and 7. The proposed solution strategy is partitioned into two phases and utilizes both geometric approach and the active set-based algebraic approach for the exploration of the parameter space and to find an optimal parametric solution. In the first phase, since the constraints of the barrier problem (3.3.5) are linear we used a geometric approach to explore the critical regions in finding an optimal solution to the barrier problem (also an optimal solution to the nonlinear multi-parametric problem (3.3.1)).

PHASE I: REFORMULATION AND FINDING SOLUTION IN THE INTERIOR OF THE FEASIBLE REGION

Reformulate problem (3.3.1) as a barrier multi-parametric problem (3.3.5) as discussed in Subsection 3.3.1. Use **Algorithm 1** to find a linear approximate solutions of the barrier problem and the corresponding polyhedral critical regions.

Define: $\mathcal{R}-$ a list of regions to be partitioned, x^L - a list of optimal linear solutions, $\mathcal{CR}^*-$ a list of optimal critical regions, and $\Theta-$ an initial parameter space.

Remark 11. *Theorem 3.6 shows that the parametric region* Θ *can be partitioned within a finite feasible choice of the parameter* $\theta = \theta_0$ *(and with such choices, the complete map*

Table 3.2: Algorithm to find a linear solution in the interior

ALGORITHM 1

INITIALIZATION: Set $\mathcal{R} = \{\Theta\}$, $x^L = \{\ \}$ and $\mathcal{CR}^* = \{\ \}$;

ITERATION: **while** $\mathcal{R} \neq \emptyset$ **do**

STEP 1: Select a $\mathcal{CR}$ from $\mathcal{R}$ and remove it from the list;

STEP 2: Select a $\theta_0 \in \mathcal{CR}$ and solve (3.3.7) at θ_0;

STEP 3: Record the solution $x_0(t)$ of (3.3.7);

STEP 4: Compute $M_0(t)$ and $N_0(t)$ from (3.3.25) at $(x_0(t), \theta_0)$;

STEP 5: Compute the first-order approximation of $[x(\theta,t), \lambda(\theta,t)]$ using (3.3.24);

STEP 6: Use (3.3.28) to find $x(\theta)$ and $\lambda(\theta)$; and add $x(\theta)$ to x^L;

STEP 7: Use (3.3.29) to define $\mathcal{CR}_0$ and remove all redundant constraints;

STEP 8: Partition the rest of the parameter space $\mathcal{CR}_0^{Rest} = \Theta \setminus \mathcal{CR}_0$ using the procedure discussed in Subsection 3.2.2. Say, the partitioning generated $\mathcal{CR}_j,\ j = 1, \ldots, J$ regions;

STEP 9: Add $\mathcal{CR}_0$ to $\mathcal{CR}^*$, and append the set $\{\mathcal{CR}_j,\ \forall j =, \ldots, J\}$ to $\mathcal{R}$,

TERMINATION: $\mathcal{R} = \emptyset$.

exploration of the parameter space is guaranteed) when the multi-parametric region has polyhedral constraints. Therefore, ***Algorithm 1*** *terminates within finite steps.*

Once a linear parametric solution is obtained in the interior of the nonlinear constraints, we will check the solution whether it satisfies the nonlinear feasibility condition. If it does we will stop, otherwise we will go to **Phase II**.

PHASE II: FINDING SOLUTION ON THE BOUNDARY OF THE FEASIBLE REGION

In this phase we will compute the nonlinear solution at the boundary of the nonlinear constraints. The procedures in this phase will be summarized as follows.

From **Phase II**, say, we get a set of linear parametric solutions $x^L(y)$ from (3.3.30). For $k \in \mathcal{K}$ and $j \in \mathcal{Q}$, determine $\mathcal{CR}_k^{\#}$ from (3.3.31) and $\mathcal{CR}_{kj}^{Rest}$ from (3.3.32). Then, use **Algorithm 2** to find an optimal solution to the problem (3.3.1) on the boundary.

After fully exploring the initial parametric space Θ, say, the barrier problem (3.3.5) has r parametric solutions in r critical regions, then in **Algorithm 2** we will solve solutions of the nonlinear parts for at most $r \times q$ times. Therefore, **Algorithm 2** terminates within finite steps.

Remark 12. *In the second phase, we employed an active set-based approach to find an optimal solution to the problem* (3.3.1) *that may exist at the boundary of the active nonlinear*

Table 3.3: Algorithm to find solution on the boundary

ALGORITHM 2
INITIALIZATION: $LIST = \{\mathcal{CR}_{kj}^{Rest}\}_{k\in\mathcal{K}, j\in\mathcal{Q}}$, $x^L = \{x^{L_k}(\theta)\}_{k\in\mathcal{K}}$, $x^{NL} = \{\ \}$ & $\mathcal{CR}^{\#\#} = \{\ \}$;
ITERATION: $k = 1 : r,\ j = 1 : q$; **while** $LIST \neq \emptyset$ **do**
STEP 1: Remove the solution $x^{L_k}(\theta)$ from x^L whenever $\mathcal{CR}_k^{\#} = \emptyset$;
STEP 2: Find the nonlinear solution $x^{NL_{kj}}(\theta)$ in $\mathcal{CR}_{kj}^{Rest}$ from (3.3.34);
STEP 3: Use (3.3.36) to determine the critical region, $\mathcal{CR}_{kj}^{\#\#}$ of $x^{NL_{kj}}(\theta)$;
STEP 4: Add $x^{NL_{kj}}(\theta)$ to x^{NL} and add $\mathcal{CR}_{kj}^{\#\#}$ to $\mathcal{CR}^{\#\#}$;
STEP 5: Remove $\mathcal{CR}_{kj}^{Rest}$ from $LIST$,
TERMINATION: $LIST = \emptyset$.

constraints. After fully exploring the initial parametric space Θ, the barrier problem (3.3.5) has r parametric solutions in r critical regions, then in the second phase we will solve solutions of the nonlinear parts for at most $r \times q$ combinations (where q represents the number of nonlinear constraints). Therefore, the exploration of the active nonlinear constraints will terminate within finite steps.

Finally, the optimal solution of (3.3.1) is given by,

$$x(\theta) = \begin{cases} x^L(\theta) & \text{on } \mathcal{CR}_k^{\#},\ k \in \mathcal{K}, \\ x^{NL}(\theta) & \text{on } \mathcal{CR}_{kj}^{\#\#},\ k \in \mathcal{K},\ j \in \mathcal{Q}, \\ \text{Infeasible} & \text{on } \mathcal{CR}_k \setminus (\mathcal{CR}_k^{\#} \cup \mathcal{CR}_{kj}^{\#\#}),\ k \in \mathcal{K},\ j \in \mathcal{Q}.. \end{cases}$$

Note that in particular case, when the objective function and constraints in (3.3.1) are up to second-degree polynomial in x and linear in θ, the above solution procedure produces an exact solution for the multi-parametric programming problem with quadratic constraints.

The proposed solution strategy is based on the assumption that the active set of the problem (3.3.7) is unique. This uniqueness can only be guaranteed if the solution of (3.3.7) is non-degenerate. Primal degeneracy is caused by the presence of weakly redundant constraints (i.e. constraints that are redundant yet intersect with the feasible parameter space) [81]. In particular, the space where the weakly redundant constraints hold as equality is lower-dimensional with respect to the overall feasible parameter space. Thus, if any weakly redundant constraint is chosen as an element of the active set, then the resulting critical region will be lower-dimensional. Since problem (3.3.5) is linearly constrained, the presence of a lower-dimensional critical region can be detected by calculating the radius of the Chebyshev ball [81].

Redundant linear inequality constraints can be removed by using approaches for the removal of redundant constraints discussed in [74, 81].

3.3.6 Illustrative examples

Example 3.1. *Consider the following problem taken from [78],*

$$\min_x f(x,\theta) = x^2 + 4x + 5$$
$$s.t. \begin{cases} x^2 + 2x - \theta_2 + 1 \leq 0,\ x - \theta_1 + \theta_2 \leq 0, \\ -5 \leq x \leq 3, \\ -2 \leq \theta_1 \leq 2,\ 0 \leq \theta_2 \leq 3. \end{cases} \tag{3.3.38}$$

Problem (3.3.38) *is a multi-parametric problem with parameter* $\theta = (\theta_1, \theta_2)$ *and consists of nonlinear,* $h(x,\theta) = x^2 + 2x - \theta_2 + 1$ *and linear,* $g(x,\theta) = x - \theta_1 + \theta_2$ *constraints. To implement the proposed algorithm we need to define an equivalent barrier problem.*

For $t > 0$ *we can define the barrier reformulation of* (3.3.38) *as*

$$\min_x W(x,\theta,t) = x^2 + 4x + 5 + t\psi(x,\theta)$$
$$s.t. \begin{cases} x - \theta_1 + \theta_2 \leq 0,\ \ -5 \leq x \leq 3, \\ -2 \leq \theta_1 \leq 2,\ 0 \leq \theta_2 \leq 3, \end{cases} \tag{3.3.39}$$

where $\psi(x,\theta) = -\ln(-h(x,\theta))$ *the barrier function with domain* $\{x \in X : h(x,\theta) < 0\}$.

The domain and Lagrangian function of (3.3.39) *are* $X^{\#}(\theta) = \{x : h(x,\theta) < 0,\ g(x,\theta) \leq 0, -5 \leq x \leq 3, -2 \leq \theta_1 \leq 2,\ 0 \leq \theta_2 \leq 3\}$ *and* $\mathcal{L}(x,\theta,\lambda,t) = x^2 + 4x + 5 + t\psi(x,\theta) + \lambda g(x,\theta)$*, respectively.*

By solving the barrier problem (3.3.39) *using the method outlined in Subsection 3.3.3, we obtain a parametric solution,*

$$x(\theta,t) = \begin{cases} \frac{-4t}{14t+9}(\theta_2 - 2.5) - 2 & \text{on } \mathcal{CR}_1(t) = \begin{cases} -\theta_1 + \theta_2 - \frac{4t}{14t+9}(\theta_2 - 2.5) - 2 \leq 0, \\ \theta_1 \leq 2,\ 0 \leq \theta_2 \leq 3,\ t > 0; \end{cases} \\ \theta_1 - \theta_2 & \text{on } \mathcal{CR}_2(t) = \begin{cases} (\frac{56t}{3} + 2)(\theta_1 - 0.5) - (\frac{40t}{3} + 2)(\theta_2 - 3) \leq 1, \\ \theta_1 \geq -2,\ \theta_2 \leq 3,\ t > 0. \end{cases} \end{cases}$$

As $t \to 0^+$, the solutions and the critical regions becomes,

$$x^L(\theta) = \begin{cases} -2 & \text{on } \mathcal{CR}_1 = \begin{cases} \theta_1 - \theta_2 + 2 \geq 0, \\ \theta_1 \leq 2, \ 0 \leq \theta_2 \leq 3; \end{cases} \\ \theta_1 - \theta_2 & \text{on } \mathcal{CR}_2 = \begin{cases} \theta_1 - \theta_2 + 2 \leq 0, \\ \theta_1 \geq -2, \ \theta_2 \leq 3, \end{cases} \end{cases} \tag{3.3.40}$$

where $\mathcal{CR}_1$ and $\mathcal{CR}_2$ are as shown in Fig.3.2.

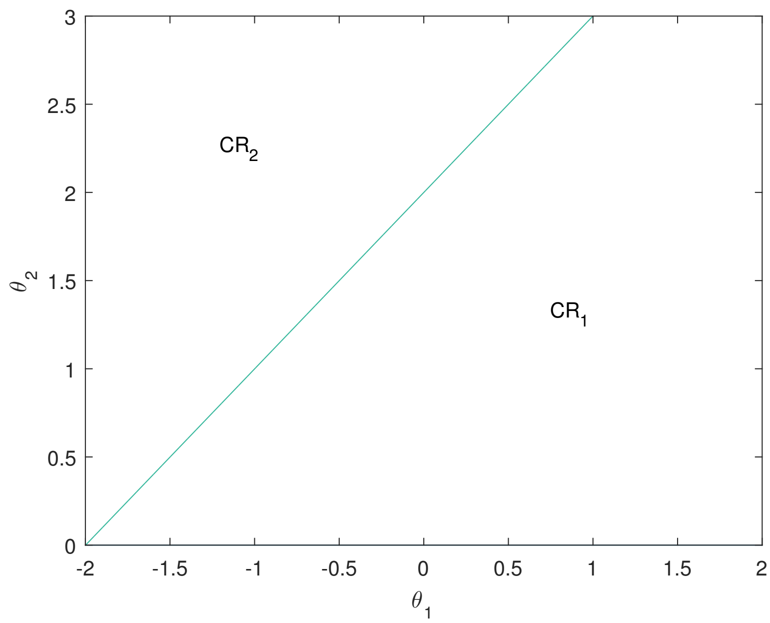

Figure 3.2: Critical regions for problem (3.3.39)

The linear solution given by (3.3.40) *is also a solution of* (3.3.38) *in the interior of the nonlinear constraints, but with different critical regions that can be defined by $\mathcal{CR}^{\#} = \{\theta \in \mathcal{CR} : \ h(x^L(\theta), \theta) < 0\}$,*

$$x^L(\theta) = \begin{cases} -2 & \text{on } \mathcal{CR}_1^{\#} = \begin{cases} \theta_1 - \theta_2 + 2 \geq 0, \\ 1 - \theta_2 < 0, \\ \theta_1 \leq 2, \ 0 \leq \theta_2 \leq 3; \end{cases} \\ \theta_1 - \theta_2 & \text{on } \mathcal{CR}_2^{\#} = \begin{cases} \theta_1 - \theta_2 + 2 \leq 0, \\ (\theta_1 - \theta_2)^2 + 2\theta_1 - 3\theta_2 + 1 < 0, \\ \theta_1 \geq -2, \ \theta_2 \leq 3. \end{cases} \end{cases}$$

Next, we will define the rest of the critical regions and explore the nonlinear solutions on the boundary of the nonlinear constraints. The rest of the regions will be determined by $\mathcal{CR}_i^{Rest} = \mathcal{CR}_i \setminus \mathcal{CR}_i^{\#}, \ i = 1, 2,$

$$\mathcal{CR}_1^{Rest} = \begin{cases} \theta_1 - \theta_2 + 2 \geq 0, \\ 1 - \theta_2 \geq 0, \\ \theta_1 \leq 2,\ 0 \leq \theta_2 \leq 3 \end{cases} \quad \text{and } \mathcal{CR}_2^{Rest} = \begin{cases} \theta_1 - \theta_2 + 2 \leq 0, \\ (\theta_1 - \theta_2)^2 + 2\theta_1 - 3\theta_2 \geq -1, \\ \theta_1 \geq -2,\ \theta_2 \leq 3. \end{cases}$$

The nonlinear solutions in the rest of the regions is determined from $x^{NL}(\theta) = \arg\{h(x,\theta) = 0,\ g_{\mathcal{A}}(x,\theta) = 0,\ g_{\mathcal{I}}(x,\theta) \leq 0\}$.

In $\mathcal{CR}_1^{\#}$ *the linear constraint* $g(x(\theta),\theta)$ *is inactive. Therefore, the nonlinear solution in* $\mathcal{CR}_1^{Rest}$ *is determined by*

$$\begin{aligned} x^{NL_1}(\theta) &= \arg_x\{h(x(\theta),\theta) = 0, g(x(\theta),\theta) \leq 0\} \\ &= \arg_x\{x^2 + 2x - \theta_2 + 1 = 0, x - \theta_1 + \theta_2 \leq 0\}. \end{aligned}$$

After solving for x *in terms of* θ *we have* $x^{NL_1}(\theta) = \pm\sqrt{\theta_2} - 1$.

Since $x^{L_1}(\theta) = -2$ *at* $\theta_2 = 1$ *which is a common point for* $\mathcal{CR}_1^{\#}$ *and* $\mathcal{CR}_1^{Rest}$ *the only parametric solution is* $x^{NL_1}(\theta) = -\sqrt{\theta_2} - 1$ *which is valid in*

$$\mathcal{CR}_1^{\#\#} = \left\{\theta \in \mathcal{CR}_1^{Rest} : g_{\mathcal{I}}(x^{NL_1}(\theta),\theta) \leq 0\right\} = \begin{cases} \theta_1 - \theta_2 + 2 \geq 0, \\ 1 - \theta_2 \geq 0, \\ -\sqrt{\theta_2} + \theta_2 - \theta_1 - 1 \leq 0, \\ \theta_1 \leq 2,\ 0 \leq \theta_2 \leq 3. \end{cases}$$

In $\mathcal{CR}_2^{\#}$ *the linear constraint* $g(x(\theta),\theta)$ *is active. Therefore, the nonlinear solution in* $\mathcal{CR}_2^{Rest}$ *is determined by*

$$\begin{aligned} x^{NL_2}(\theta) &= \arg_x\{h(x(\theta),\theta) = 0, g(x(\theta),\theta) = 0\} \\ &= \arg_x\{x^2 + 2x - \theta_2 + 1 = 0, \theta_1 - \theta_2 + 2 = 0\}, \end{aligned}$$

which has no solution, implying that $\mathcal{CR}_2^{\#\#} = \emptyset$ *and hence* $\mathcal{CR}_2^{Rest}$ *is infeasible.*

Therefore, the optimal solution is given by,

$$x(\theta) = \begin{cases} -2 & \text{on } \mathcal{CR}_1^{\#} \\ \theta_1 - \theta_2 & \text{on } \mathcal{CR}_2^{\#} \\ -\sqrt{\theta_2} - 1 & \text{on } \mathcal{CR}_1^{\#\#} \\ \text{Infeasible} & \text{on } \mathcal{CR}_2^{Rest} \cup \{\mathcal{CR}_1^{Rest} \setminus \mathcal{CR}_1^{\#}\} \end{cases}$$

where $\mathcal{CR}_1^{\#}, \mathcal{CR}_2^{\#}$ *and* $\mathcal{CR}_1^{\#\#}$ *are as shown in Fig.3.3. The same solution as it was reported in [78].* □

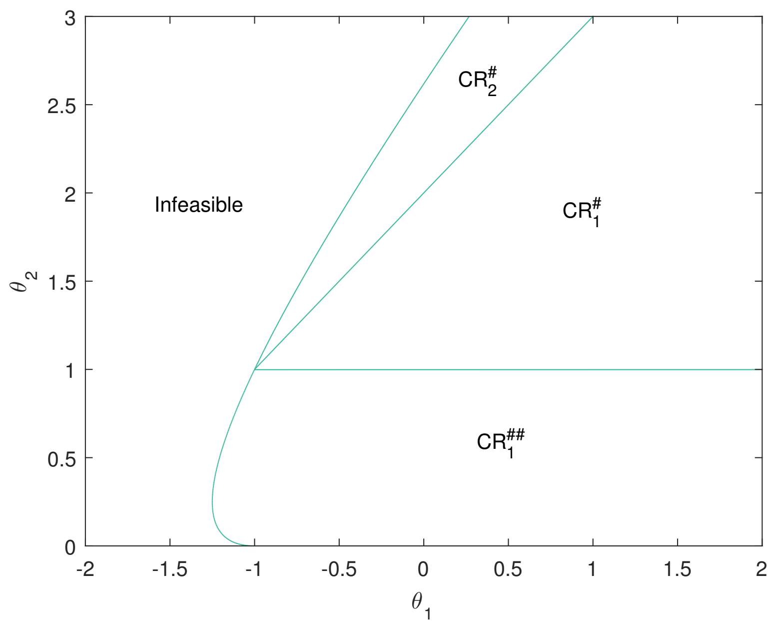

Figure 3.3: Critical regions for the problem (3.3.38)

Example 3.2. *Consider the following multi-parametric problem,*

$$\min_x f(x,\theta) = x_1^2 + 3x_2^2 - 6x_2 + (x_1 - x_2)^2$$
$$s.t. \begin{cases} x_2^2 + 5x_2 - 10\theta_1 - 15 \le 0, \\ x_1 - x_2 - \theta_1 + 2\theta_2 \le 0, \\ x_1 + x_2 + 5\theta_2 - 12 \le 0, \\ -3 \le \theta_1 \le 3, -4 \le \theta_2 \le 5, \\ -4 \le x_1, x_2 \le 4. \end{cases} \tag{3.3.41}$$

Here, $h(x,\theta) = x_2^2 + 5x_2 - 10\theta_1 - 15$ *and* $g(x,\theta) = [x_1 - x_2 - \theta_1 + 2\theta_2, x_1 + x_2 + 5\theta_2 - 12]$. *The barrier function is given by* $\psi(x,\theta) = -\ln(-h(x,\theta))$ *with domain* $\{x \in X : h(x,\theta) < 0\}$. *For* $t > 0$ *we can define the barrier reformulation for* (3.3.41) *as*

$$\min_x W(x,\theta,t) = x_1^2 + 3x_2^2 - 6x_2 + (x_1 - x_2)^2 + t\psi(x,\theta)$$
$$s.t. \begin{cases} x_1 - x_2 - \theta_1 + 2\theta_2 \le 0, \\ x_1 + x_2 + 5\theta_2 - 12 \le 0, \\ -3 \le \theta_1 \le 3, -4 \le \theta_2 \le 5, \\ -4 \le x_1, x_2 \le 4, \end{cases} \tag{3.3.42}$$

with domain $X^\#(\theta) = \{x : h(x,\theta) < 0,\ g(x,\theta) \le 0, -3 \le \theta_1 \le 3, -4 \le \theta_2 \le 5, -4 \le x_1, x_2 \le 4\}$.

Problem (3.3.42) *is a multi-parametric problem with parameter* $\theta = (\theta_1, \theta_2)$ *and Lagrangian function* $\mathcal{L}(x, \theta, \lambda, t) = x_1^2 + 3x_2^2 - 6x_2 + (x_1 - x_2)^2 + t\psi(x, \theta) + \lambda^T g(x, \theta)$.

By solving the barrier problem (3.3.42)*, as* $t \to 0^+$*, the solutions and the critical regions will be,*

$$x(\theta) = \begin{cases} \begin{bmatrix} 0.75\theta_1 - 1.5\theta_2 + 0.75 \\ -0.25\theta_1 + 0.5\theta_2 + 0.75 \end{bmatrix} & \text{on } \mathcal{CR}_1 = \begin{cases} 0.1240\theta_1 + 0.9923\theta_2 \leq 2.6047, \\ -3 \leq \theta_1 \leq 3,\ \theta_2 \geq -4 \end{cases} \\ \begin{bmatrix} 0.5\theta_1 - 3.5\theta_2 + 6 \\ -0.5\theta_1 - 1.5\theta_2 + 6 \end{bmatrix} & \text{on } \mathcal{CR}_2 = \begin{cases} 0.1240\theta_1 + 0.9923\theta_2 \geq 2.6047, \\ -3 \leq \theta_1 \leq 3,\ \theta_2 \leq 5 \end{cases} \end{cases} \tag{3.3.43}$$

where $\mathcal{CR}_1$ *and* $\mathcal{CR}_2$ *are as shown in Fig.3.4*

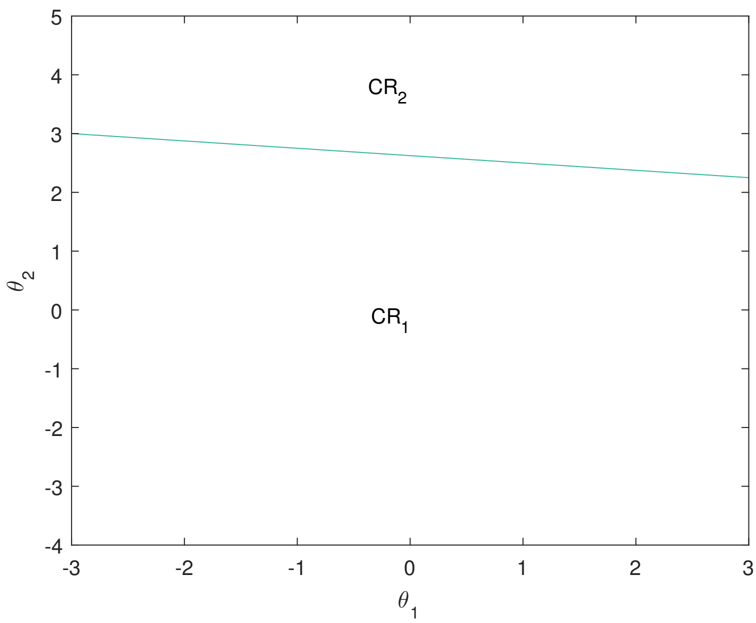

Figure 3.4: Critical regions for the problem (3.3.42)

The linear solution (3.3.43) *is also a solution of* (3.3.41) *in the interior of the nonlinear constraints, but with different critical regions* $\mathcal{CR}^\# = \{\theta \in \mathcal{CR} : h(x^L(\theta), \theta) < 0\}$. *Therefore, the linear solutions and the corresponding critical regions of* (3.3.41) *will be:*

$$x^L(\theta) = \begin{cases} \begin{bmatrix} 0.75\theta_1 - 1.5\theta_2 + 0.75 \\ -0.25\theta_1 + 0.5\theta_2 + 0.75 \end{bmatrix} & \text{on } \mathcal{CR}_1^\#, \\ \begin{bmatrix} 0.5\theta_1 - 3.5\theta_2 + 6 \\ -0.5\theta_1 - 1.5\theta_2 + 6 \end{bmatrix} & \text{on } \mathcal{CR}_2^\#; \end{cases}$$

where

$$\mathcal{CR}_1^{\#} = \begin{cases} 0.0625\theta_1^2 - 0.25\theta_1\theta_2 - 11.625\theta_1 + 0.25\theta_2^2 + 3.25\theta_2 - 10.6875 \leq 0, \\ 0.1240\theta_1 + 0.9923\theta_2 - 2.6047 \leq 0, \\ -3 \leq \theta_1 \leq 3,\ \theta_2 \geq -4 \end{cases}$$

and

$$\mathcal{CR}_2^{\#} = \begin{cases} 0.25\theta_1^2 + 1.5\theta_1\theta_2 - 18.5\theta_1 + 2.25\theta_2^2 - 25.5\theta_2 + 51 \leq 0, \\ 0.1240\theta_1 + 0.9923\theta_2 - 2.6047 \geq 0, \\ \theta_1 \leq 3,\ \theta_2 \leq 5. \end{cases}$$

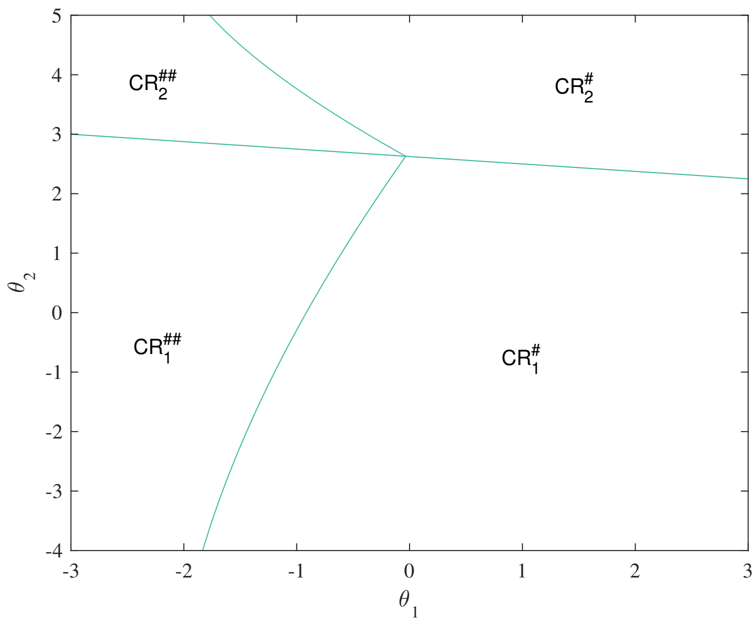

Figure 3.5: Critical regions for problem (3.3.42)

The rest of the spaces will be determined by $\mathcal{CR}_i^{Rest} = \mathcal{CR}_i \setminus \mathcal{CR}_i^{\#},\ i = 1, 2,$

$$\mathcal{CR}_1^{Rest} = \begin{cases} 0.0625\theta_1^2 - 0.25\theta_1\theta_2 - 11.625\theta_1 + 0.25\theta_2^2 + 3.25\theta_2 - 10.6875 \geq 0, \\ 0.1240\theta_1 + 0.9923\theta_2 - 2.6047 \leq 0, \\ \theta_1 \geq -3,\ \theta_2 \geq -4; \end{cases}$$

$$\mathcal{CR}_2^{Rest} = \begin{cases} 0.25\theta_1^2 + 1.5\theta_1\theta_2 - 18.5\theta_1 + 2.25\theta_2^2 - 25.5\theta_2 + 51 \geq 0, \\ 0.1240\theta_1 + 0.9923\theta_2 - 2.6047 \geq 0, \\ \theta_1 \leq 3,\ \theta_2 \geq -4. \end{cases}$$

Next we will determine the nonlinear solution in the rest of the critical regions.

Since both $g_1(x,\theta)$ *and* $g_2(x,\theta)$ *are inactive in the region* $\mathcal{CR}_1^{\#}$*, the nonlinear solution in* $\mathcal{CR}_1^{Rest}$ *is determined from* $x^{NL}(\theta) = \arg_x\{h(x,\theta) = 0,\ g_1(x,\theta) \leq 0,\ g_2(x,\theta) \leq 0\}$.

$$x_{NL}(\theta) = \arg\{x_2^2 + 5x_2 - 10\theta_1 = 15,\ x_1 - x_2 - \theta_1 + 2\theta_2 \leq 0,\ x_1 + x_2 + 5\theta_2 \leq 12\},$$

which is infeasible for $\theta_1 \geq -3, \theta_2 \geq -4$.

Since $g_1(x,\theta)$ *is inactive and* $g_2(x,\theta)$ *is active in the region* $\mathcal{CR}_2^{\#}$, *the nonlinear solution in* $\mathcal{CR}_2^{Rest}$ *is determined from* $x^{NL}(\theta) = \arg\{h(x,\theta) = 0,\ g_2(x,\theta) = 0\ g_1(x,\theta) \leq 0\}$. *Thus,*

$$x^{NL}(\theta) = \arg_x\{x_2^2 + 5x_2 - 10\theta_1 = 15,\ x_1 + x_2 + 5\theta_2 = 12,\ x_1 - x_2 - \theta_1 + 2\theta_2 \leq 0\},$$

which results in

$$x^{NL}(\theta) = \begin{bmatrix} \mp 0.5\sqrt{5(8\theta_1 + 17)} - 5\theta_2 + 14.5 \\ \pm 0.5\sqrt{5(8\theta_1 + 17)} - 2.5 \end{bmatrix}.$$

Since $x = (-3.2191, 2.0731)$ *at* $\theta = (-0.0337, 2.6292)$ *which is a common point for* $\mathcal{CR}_1^{\#}$, $\mathcal{CR}_2^{\#}$ *and* $\mathcal{CR}_2^{Rest}$, *the only nonlinear parametric solution which is valid in* $\mathcal{CR}_2^{\#\#} = \mathcal{CR}_2^{Rest}$ *is*

$$x^{NL}(\theta) = \begin{bmatrix} -0.5\sqrt{5(8\theta_1 + 17)} - 5x_2 + 14.5 \\ 0.5\sqrt{5(8\theta_1 + 17)} - 2.5 \end{bmatrix}.$$

Therefore, the optimal solution is given by,

$$x(\theta) = \begin{cases} \begin{bmatrix} 0.75\theta_1 - 1.5\theta_2 + 0.75 \\ -0.25\theta_1 + 0.5\theta_2 + 0.75 \end{bmatrix} & \text{on } \mathcal{CR}_1^{\#} \\ \begin{bmatrix} 0.5\theta_1 - 3.5\theta_2 + 6 \\ -0.5\theta_1 - 1.5\theta_2 + 6 \end{bmatrix} & \text{on } \mathcal{CR}_2^{\#} \\ \text{Infeasible} & \text{on } \mathcal{CR}_1^{\#\#} \\ \begin{bmatrix} -0.5\sqrt{5(8\theta_1 + 17)} - 5x_2 + 14.5 \\ 0.5\sqrt{5(8\theta_1 + 17)} - 2.5 \end{bmatrix} & \text{on } \mathcal{CR}_2^{\#\#} \end{cases}$$

where $\mathcal{CR}_1^{\#}, \mathcal{CR}_2^{\#}, \mathcal{CR}_1^{\#\#}$ *and* $\mathcal{CR}_2^{\#\#}$ *are as indicated in Fig.3.5.* □

Example 3.3. *Consider the following multi-parametric problem of nonlinear objective with a cubic constraint,*

$$\min_x f(x,\theta) = (x_1 + 2)^3 + x_2^2 - x_1 + 2x_2$$
$$s.t. \begin{cases} 5.5x_1^2 + (x_2 + 3.5)^3 - 5\theta_2 \leq 25 \\ -x_1^3 + 12x_1^2 + 6x_2 + 1.5\theta_2^2 \leq 86 \\ 2x_1 + x_2 - \theta_1 - \theta_2 \leq 4 \\ 3x_2 + \theta_1 - 2\theta_2 \leq -1 \\ -10 \leq \theta_1, \theta_2 \leq 10, \\ -2 \leq x_1 \leq 4, -3 \leq x_2 \leq 5. \end{cases} \tag{3.3.44}$$

Here, $G(x,\theta) = [5.5x_1^2 + (x_2+3.5)^3 - 5\theta_2 - 25, -x_1^3 + 12x_1^2 + 6x_2 + 1.5\theta_2^2 - 86]$ *and* $g(x,\theta) = [2x_1 + x_2 - \theta_1 - \theta_2 - 4, 3x_2 + \theta_1 - 2\theta_2 + 1]$. *The barrier function is given by* $\psi(x,\theta) = -\ln(-G_1(x,\theta) - G_2(x,\theta))$ *with domain* $\{x \in X : G(x,\theta) < 0\}$.

For $t > 0$ *we can define the barrier reformulation for (3.3.44) as*

$$\min_x W(x,\theta,t) = (x_1+2)^3 + x_2^2 - x_1 + 2x_2 + t\psi(x,\theta)$$
$$s.t. \begin{cases} 2x_1 + x_2 - \theta_1 - \theta_2 \le 4 \\ 3x_2 + \theta_1 - 2\theta_2 \le -1 \\ -10 \le \theta_1, \theta_2 \le 10, \\ -2 \le x_1 \le 4, -3 \le x_2 \le 5. \end{cases} \tag{3.3.45}$$

with domain $X^{\#}(\theta) = \{x : G(x,\theta) < 0,\ g(x,\theta) \le 0, -10 \le \theta_1, \theta_2 \le 10, -2 \le x_1 \le 4, -3 \le x_2 \le 5\}$.

Problem (3.3.45) is a multi-parametric problem with parameter $\theta = (\theta_1, \theta_2)$ *and Lagrangian function* $\mathcal{L}(x,\theta,\lambda,t) = (x_1+2)^3 + x_2^2 - x_1 + 2x_2 + t\psi(x,\theta) + \lambda^T g(x,\theta)$.

After solving the barrier problem (3.3.45), the optimal solutions and the corresponding critical regions will be,

$$x(\theta) = \begin{cases} \begin{bmatrix} 0.0000\theta_1 - 0.0000\theta_2 - 1.0000 \\ -0.0001\theta_1 + 0.0002\theta_2 - 1.0000 \end{bmatrix} & \text{on } \mathcal{CR}_1 \\ \begin{bmatrix} 0.6667\theta_1 + 0.1667\theta_2 + 2.1667 \\ -0.3333\theta_1 + 0.6667\theta_2 - 0.3333 \end{bmatrix} & \text{on } \mathcal{CR}_2 \\ \begin{bmatrix} 0.0000\theta_1 - 0.0000\theta_2 - 1.0001 \\ -0.3333\theta_1 + 0.6667\theta_2 - 0.3333 \end{bmatrix} & \text{on } \mathcal{CR}_3 \\ \begin{bmatrix} 0.0000\theta_1 - 0.0000\theta_2 - 1.0000 \\ -0.3333\theta_1 + 0.6667\theta_2 - 0.3333 \end{bmatrix} & \text{on } \mathcal{CR}_4 \end{cases} \tag{3.3.46}$$

$$\mathcal{CR}_1 = \begin{cases} 0.44720\theta_1 - 0.8945\theta_2 \le 0.8947 \\ 0.7072\theta_1 + 0.70706\theta_2 \ge 4.9498 \\ -10 \le \theta_1 \le 10,\ \theta_2 \le 10 \end{cases}, \mathcal{CR}_2 = \begin{cases} 1.33337\theta_1 + 0.3335\theta_2 \le -6.1692 \\ 0.7072\theta_1 + 0.7070\theta_2 \le 4.9498 \\ \theta_1 \ge -10,\ \theta_2 \ge -10 \end{cases}$$

$$\mathcal{CR}_3 = \begin{cases} 1.33337\theta_1 + 0.3335\theta_2 \ge -6.1692 \\ 0.7072\theta_1 + 0.7070\theta_2 \le 4.9498 \\ \theta_2 \ge -10 \end{cases}, \mathcal{CR}_4 = \begin{cases} 0.44720\theta_1 - 0.8945\theta_2 \ge 0.8947 \\ 0.7072\theta_1 + 0.7070\theta_2 \ge 4.9498 \\ \theta_1 \le 10,\ \theta_2 \ge -10 \end{cases}$$

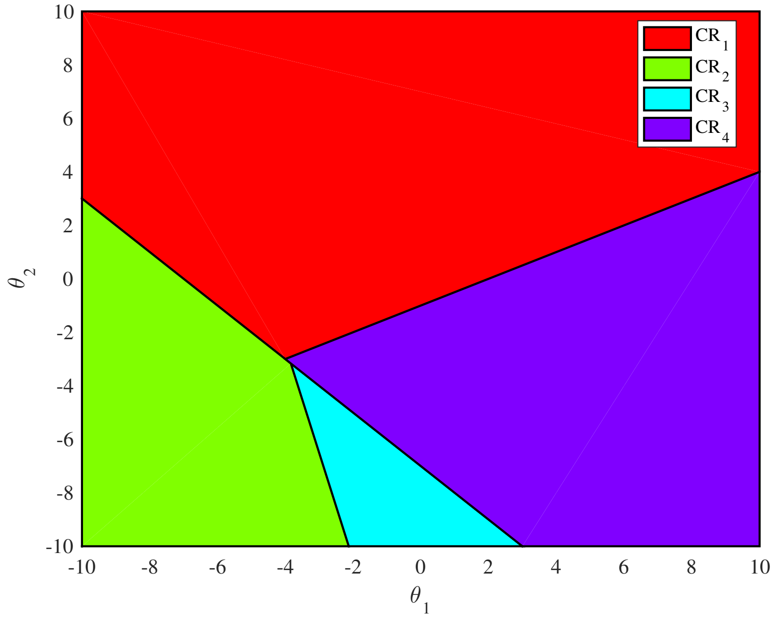

Figure 3.6: Critical regions for the problem (3.3.45)

where $\mathcal{CR}_1$, $\mathcal{CR}_2$, $\mathcal{CR}_3$ and $\mathcal{CR}_4$ are as shown in Fig.3.6

The linear solution (3.3.46) is also a solution of (3.3.44) in the interior of the nonlinear constraints, but with different critical regions $\mathcal{CR}^{\#} = \{\theta \in \mathcal{CR} : G(x^L(\theta),\theta) < 0\}$. Therefore, the linear solutions and the corresponding critical regions of (3.3.44) will be:

$$x^L(\theta) = \begin{cases} \begin{bmatrix} 0.0000\theta_1 - 0.0000\theta_2 - 1.0000 \\ -0.0001\theta_1 - 0.0002\theta_2 - 1.0001 \end{bmatrix} & \text{on } \mathcal{CR}_1^{\#} \\ \begin{bmatrix} 0.5714\theta_1 + 0.1429\theta_2 + 1.7143 \\ -0.1429\theta_1 + 0.7143\theta_2 + 0.5714 \end{bmatrix} & \text{on } \mathcal{CR}_2^{\#} \\ \begin{bmatrix} 0.0308\theta_1 - 0.0616\theta_2 - 1.0502 \\ -0.3231\theta_1 + 0.6461\theta_2 - 0.3501 \end{bmatrix} & \text{on } \mathcal{CR}_3^{\#} \\ \begin{bmatrix} 0.0316\theta_1 - 0.0633\theta_2 - 1.0559 \\ -0.3228\theta_1 + 0.6456\theta_2 - 0.3520 \end{bmatrix} & \text{on } \mathcal{CR}_4^{\#} \end{cases}$$

where

$$\mathcal{CR}_1^{\#} = \begin{cases} G_1(x_1^L(\theta),\theta) < 0 \\ G_2(x_1^L(\theta),\theta) < 0 \\ 0.44472\theta_1 - 0.8945\theta_2 \leq 0.8947 \\ 0.7072\theta_1 + 0.70706\theta_2 \geq 4.9498 \\ -10 \leq \theta_1 \leq 10,\ \theta_2 \leq 10 \end{cases} ; \mathcal{CR}_2^{\#} = \begin{cases} G_1(x_2^L(\theta),\theta) < 0 \\ 1.3337\theta_1 + 0.3335\theta_2 \leq -6.1692 \\ 0.7072\theta_1 + 0.7070\theta_2 \leq 4.9498 \\ \theta_1 \geq -10,\ \theta_2 \geq -10 \end{cases} ;$$

$$\mathcal{CR}_3^{\#}=\begin{cases}G_1(x_3^L(\theta),\theta)<0\\1.33337\theta_1+0.33335\theta_2\geq -6.1692\\0.70727\theta_1+0.70700\theta_2\leq 4.9498\\\theta_2\geq -10\end{cases};\mathcal{CR}_4^{\#}=\begin{cases}G_1(x_4^L(\theta),\theta)<0\\G_2(x_4^L(\theta),\theta)<0\\0.44727\theta_1-0.89445\theta_2\geq 0.8947\\0.70727\theta_1+0.70700\theta_2\geq 4.9498\\\theta_1\leq 10,\ \theta_2\geq -10\end{cases}$$

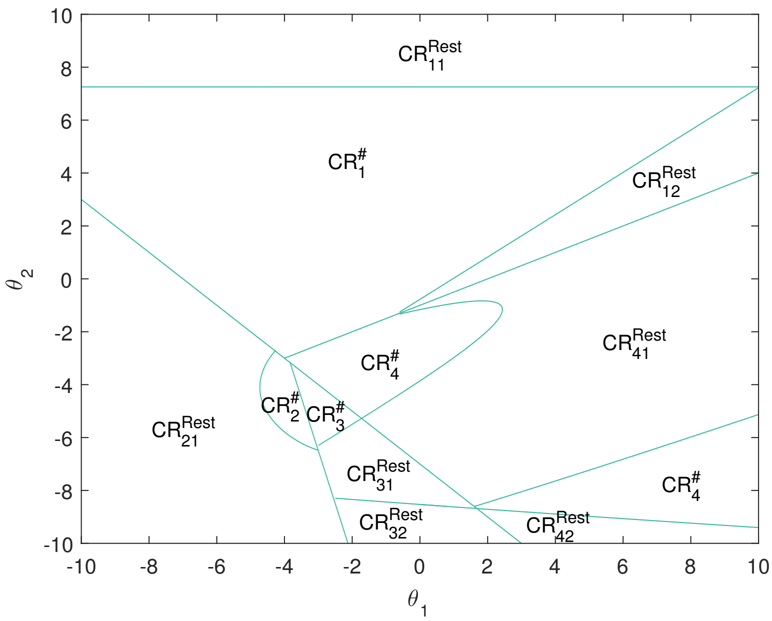

Figure 3.7: Critical regions of the linear solutions of problem (3.3.44)

The rest of the spaces will be determined by $\mathcal{CR}_{kj}^{Rest}=\mathcal{CR}_k\setminus\mathcal{CR}_k^{\#}$ $(k=1:4)$,

$$\mathcal{CR}_{11}^{Rest}=\begin{cases}G_1(x_1^L(\theta),\theta)<0\\G_2(x_1^L(\theta),\theta)\geq 0\\0.44727\theta_1-0.89445\theta_2\leq 0.8947\\0.70727\theta_1+0.707066\theta_2\geq 4.9498\\-10\leq\theta_1\leq 10,\ \theta_2\leq 10\end{cases};\mathcal{CR}_{12}^{Rest}=\begin{cases}G_1(x_1^L(\theta),\theta)\geq 0\\G_2(x_1^L(\theta),\theta)<0\\0.44727\theta_1-0.89445\theta_2\leq 0.8947\\0.70727\theta_1+0.707066\theta_2\geq 4.9498\\-10\leq\theta_1\leq 10,\ \theta_2\leq 10\end{cases};$$

$$\mathcal{CR}_{21}^{Rest}=\begin{cases}G_1(x_2^L(\theta),\theta)\geq 0\\1.33337\theta_1+0.33335\theta_2\leq -6.1692\\0.70727\theta_1+0.70700\theta_2\leq 4.9498\\\theta_1\geq -10,\ \theta_2\geq -10\end{cases};$$

$$\mathcal{CR}_{31}^{Rest} = \begin{cases} G_1(x_3^L(\theta), \theta) \geq 0 \\ G_2(x_3^L(\theta), \theta) < 0 \\ 1.3337\theta_1 + 0.3335\theta_2 \geq -6.1692 \\ 0.7072\theta_1 + 0.7070\theta_2 \leq 4.9498 \\ \theta_2 \geq -10 \end{cases} ; \mathcal{CR}_{32}^{Rest} = \begin{cases} G_2(x_3^L(\theta), \theta) \geq 0 \\ 1.3337\theta_1 + 0.3335\theta_2 \geq -6.1692 \\ 0.7072\theta_1 + 0.7070\theta_2 \leq 4.9498 \\ \theta_2 \geq -10 \end{cases};$$

$$\mathcal{CR}_{41}^{Rest} = \begin{cases} G_1(x_4^L(\theta), \theta) \geq 0 \\ G_2(x_4^L(\theta), \theta) < 0 \\ 0.4472\theta_1 - 0.8945\theta_2 \geq 0.8947 \\ 0.7072\theta_1 + 0.7070\theta_2 \geq 4.9498 \\ \theta_1 \leq 10, \ \theta_2 \geq -10 \end{cases} ; \mathcal{CR}_{42}^{Rest} = \begin{cases} G_1(x_4^L(\theta), \theta) < 0 \\ G_2(x_4^L(\theta), \theta) \geq 0 \\ 0.4472\theta_1 - 0.8945\theta_2 \geq 0.8947 \\ 0.7072\theta_1 + 0.7070\theta_2 \geq 4.9498 \\ \theta_1 \leq 10, \ \theta_2 \geq -10 \end{cases};$$

Next we will determine the nonlinear solution in the rest of the critical regions.

- *In the region $\mathcal{CR}_1^{\#}$ both $g_1(x, \theta)$ and $g_2(x, \theta)$ are inactive (i.e., $|\mathcal{A}_1| = 0 < 1$), therefore the rest regions $\mathcal{CR}_{11}^{Rest}$ and $\mathcal{CR}_{12}^{Rest}$ are infeasible.*
- *In the region $\mathcal{CR}_2^{\#}$ both $g_1(x, \theta)$ and $g_2(x, \theta)$ are active (i.e., $|\mathcal{A}_2| = 2 > 1$), therefore the rest region $\mathcal{CR}_{21}^{Rest}$ is infeasible.*
- *In the regions $\mathcal{CR}_3^{\#}$ and $\mathcal{CR}_4^{\#}$, $g_1(x, \theta)$ is inactive and $g_2(x, \theta)$ is active (i.e., $|\mathcal{A}_3| = 1$), then the nonlinear solutions are determined as follows:*

– *In $\mathcal{CR}_{31}^{Rest}$, from the system of equation,*

$$\begin{cases} G_1(x, \theta) = 5.5x_1^2 + (x_2 + 3.5)^3 - 5\theta_2 - 25 = 0 \\ g_2(x, \theta) = 3x_2 + \theta_1 - 2\theta_2 + 1 = 0 \end{cases} \tag{3.3.47}$$

Solving (3.3.47) *we have*

$$x_{31}^{NL}(\theta) = \begin{bmatrix} \sqrt{\begin{array}{l} 0.0067\theta_1^3 - 0.0404\theta_1^2\theta_2 - 0.1919\theta_1^2 + 0.0808\theta_1\theta_2^2 + 0.7677\theta_1\theta_2 \\ + 1.0960\theta_1 - 0.0539\theta_2^3 - 0.7677\theta_2^2 - 2.7374\theta_2 - 1.2281 \end{array}} \\ \frac{1}{3}(2\theta_2 - \theta_1 - 1) \end{bmatrix}$$

which is valid in the region $\mathcal{CR}_{31}^{\#\#} = \{\mathcal{CR}_{31}^{Rest}\} \cap \{\theta : g_1(x_{31}^{NL}(\theta), \theta) \leq 0, \theta_1^2\theta_2 + 4.75\theta_1^2 + 1.3333\theta_2^3 + 19\theta_2^2 - 0.1667\theta_1^3 - 2\theta_1\theta_2^2 - 19\theta_1\theta_2 - 27.125\theta_1 + 67.75\theta_2 + 30.3958 \leq 0\}$

– *In $\mathcal{CR}_{32}^{Rest}$, we have the system of equations,*

$$\begin{cases} G_2(x, \theta) = -x_1^3 + 12x_1^2 + 6x_2 + 1.5\theta_2^2 - 86 = 0 \\ g_2(x, \theta) = 3x_2 + \theta_1 - 2\theta_2 + 1 = 0. \end{cases} \tag{3.3.48}$$

The system (3.3.48) *has no unique solution, therefore $\mathcal{CR}_{32}^{Rest}$ is infeasible.*

– *In $\mathcal{CR}_{41}^{Rest}$, we have the system of equations,*

$$\begin{cases} G_1(x,\theta) = 5.5x_1^2 + (x_2+3.5)^3 - 5\theta_2 - 25 = 0 \\ g_2(x,\theta) = 3x_2 + \theta_1 - 2\theta_2 + 1 = 0. \end{cases} \tag{3.3.49}$$

Solving (3.3.49) *we have*

$$x_{41}^{NL}(\theta) = \begin{bmatrix} -\sqrt{\begin{array}{l} 0.0067\theta_1^3 - 0.0404\theta_1^2\theta_2 - 0.1919\theta_1^2 + 0.0808\theta_1\theta_2^2 + 0.7677\theta_1\theta_2 \\ + 1.0960\theta_1 - 0.0539\theta_2^3 - 0.7677\theta_2^2 - 2.7374\theta_2 - 1.2281 \end{array}} \\ \frac{1}{3}(2\theta_2 - \theta_1 - 1) \end{bmatrix}$$

which is valid in the region $\mathcal{CR}_{41}^{\#\#} = \{\mathcal{CR}_{41}^{Rest}\} \cap \{\theta : g_1(x_{41}^{NL}(\theta),\theta) \le 0, \theta_1^2\theta_2 + 4.75\theta_1^2 + 1.3333\theta_2^3 + 19\theta_2^2 - 0.1667\theta_1^3 - 2\theta_1\theta_2^2 - 19\theta_1\theta_2 - 27.125\theta_1 + 67.75\theta_2 + 30.3958 \le 0\}$

– *In $\mathcal{CR}_{42}^{Rest}$, we get the system of equations,*

$$\begin{cases} G_2(x,\theta) = -x_1^3 + 12x_1^2 + 6x_2 + 1.5\theta_2^2 - 86 = 0 \\ g_2(x,\theta) = 3x_2 + \theta_1 - 2\theta_2 + 1 = 0. \end{cases} \tag{3.3.50}$$

The system (3.3.50) *has no unique solution, therefore $\mathcal{CR}_{42}^{Rest}$ is infeasible.*

Therefore, the optimal solution is given by,

$$x(\theta) = \begin{cases} \begin{bmatrix} 0.0000\theta_1 - 0.0000\theta_2 - 1.0000 \\ -0.0001\theta_1 - 0.0002\theta_2 - 1.0001 \end{bmatrix} & \textit{on } \mathcal{CR}_1^{\#} \\ \begin{bmatrix} 0.5714\theta_1 + 0.1429\theta_2 + 1.7143 \\ -0.1429\theta_1 + 0.7143\theta_2 + 0.5714 \end{bmatrix} & \textit{on } \mathcal{CR}_2^{\#} \\ \begin{bmatrix} 0.0308\theta_1 - 0.0616\theta_2 - 1.0502 \\ -0.3231\theta_1 + 0.6461\theta_2 - 0.3501 \end{bmatrix} & \textit{on } \mathcal{CR}_3^{\#} \\ \begin{bmatrix} 0.0316\theta_1 - 0.0633\theta_2 - 1.0559 \\ -0.3228\theta_1 + 0.6456\theta_2 - 0.3520 \end{bmatrix} & \textit{on } \mathcal{CR}_4^{\#} \\ \textit{Infeasible} & \textit{on } \mathcal{CR}_{11}^{Rest} \cup \mathcal{CR}_{12}^{Rest} \cup \mathcal{CR}_{21}^{Rest} \\ x_{31}^{NL}(\theta) & \textit{on } \mathcal{CR}_3^{\#\#} \\ \textit{Infeasible} & \textit{on } \mathcal{CR}_{31}^{Rest} \setminus \mathcal{CR}_3^{\#\#} \\ x_{41}^{NL}(\theta) & \textit{on } \mathcal{CR}_4^{\#\#} \\ \textit{Infeasible} & \textit{on } \mathcal{CR}_{41}^{Rest} \setminus \mathcal{CR}_4^{\#\#}. \end{cases}$$

where $\mathcal{CR}_3^{\#\#}$ and $\mathcal{CR}_4^{\#\#}$ are as indicated in Fig.3.8.

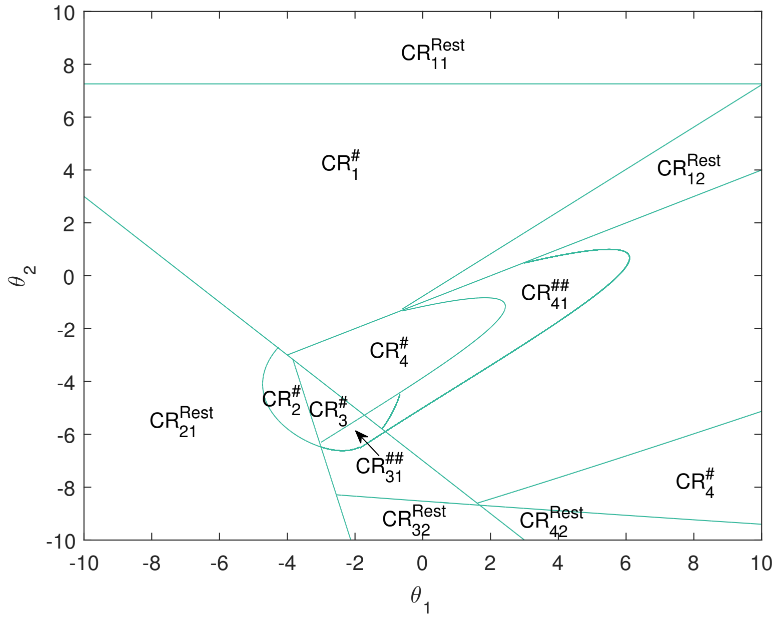

Figure 3.8: Critical regions for both linear and nonlinear solutions of problem (3.3.44)

3.4 Chapter summary

The proposed method gives exact solutions for problems whose objective and constraints are quadratic with respect to the optimization variable and linear with respect to the parameters vectors. Moreover, if the parametric solutions are approximated by quadratic functions (rather than linear ones) the same method can be employed to find exact parametric solutions for convex problems with objectives and constraints are cubic-polynomials in the optimization variable, and quadratic in the parameters vector. However, the procedure requires solving a large system of quadratic equations.

Chapter 4

Multi-parametric method for bilevel nonlinear problems

Contents

This chapter presents a novel algorithmic approach to find solutions of bilevel programming problems whose lower-level problem involves a convex nonlinear constraints. The solution strategy starts by recasting the lower-level problem of the bilevel optimization as multi-parametric programming problem where the variable from upper-level problem is considered as parameter. Next, a barrier function reformulation technique is employed to construct a multi-parametric barrier problem. Then, solutions of the barrier problem is computed in the interior and on the boundary of the nonlinear constraints. Finally, the optimal parametric solution is substituted into the upper-level problems and standard nonlinear optimization algorithms are used to solve the resulting nonlinear optimization problem.

In this chapter, we separately consider and categorize the polyhedral constraints and the nonlinear constraints of the lower-level problem of the general bilevel optimization (1.2.2) for proposing a solution procedure. The following problem structure is to be considered.

$$\begin{cases} \min\limits_{y \in Y} F(x,y) \\ \text{subject to} \begin{cases} G(x,y) \leq 0, \text{ and} \\ x \text{ solves} \boxed{\begin{cases} \min\limits_{x \in X} f(x,y) \\ \text{subject to} \begin{cases} g(x,y) \leq 0, \\ h(x,y) \leq 0, \end{cases} \end{cases}} \end{cases} \end{cases} \tag{4.0.1}$$

where $g(x,y)$ represent polyhedral inequality constraints and $h(x,y)$ represents the nonlinear inequality constraints.

A multi-parametric solution approach for bilevel optimization is a global solution strategy that works by rewriting the lower-level optimization problem as a multi-parametric problem. The resulting problem can be solved globally and the parametric solutions can be substituted into the upper-level optimization problem.

The lower-level problem of (4.0.1) can be formulated as a multi-parametric problem

$$\boxed{\begin{cases} \min\limits_{x \in X} f(x,y) \\ \text{subject to} \begin{cases} g(x,y) \leq 0, \\ h(x,y) \leq 0, \end{cases} \end{cases}} \tag{4.0.2}$$

where the upper-level optimization variables vector y is considered as a parameter.

Problem (4.0.2) is solved to obtain the explicit optimal reaction function $x(y)$ with its corresponding critical regions where $x(y)$ remains optimal. This explicit functional form can be used in the upper-level problem to convert the bilevel optimization into a single-level problem with additional constraints corresponding to each critical region,

$$\begin{cases} \min\limits_{y \in Y} F(x(y),y) \\ \text{subject to} \begin{cases} G(x(y),y) \leq 0, \\ x(y) \text{ solves (4.0.2), and} \\ y \in \{\text{the critical regions where } x(y) \text{ remains optimal}\}. \end{cases} \end{cases} \tag{4.0.3}$$

When the lower-level problem (4.0.2) contains only polyhedral constraints and convex objective function, it will be solved using algorithm discussed in Subsection 3.2.2.

If the lower-level problem (4.0.2) contains some special non-convexity formulation (as described in [61]) in the objective function and the constraints are polyhedral, we can apply the branch-and-bound multi-parametric programming procedure proposed in [61]. This approach works by convexifying the objective function to underestimate them by convex functions. Then, the resulting convex parametric underestimator problem is solved using the multi-parametric problem approach.

4.1 The proposed solution method

This section proposes solution method for nonlinear bilevel optimization problems of the form (4.0.1). Suppose that problem (4.0.2) satisfy the Assumptions 5, 6 and 7. The solution strategy recasts the lower-level problem of (4.0.1) as multi-parametric problem (4.0.2) and employs an equivalent barrier problem reformulation. Then, the resulting barrier problem is solved to obtain an explicit parametric solution $x^*(y)$ for the rational reaction set of the lower-level problem in the corresponding critical (stability) region of the parameter space. This explicit functional form can be used in the upper-level problem to obtain the optimal solution of the leader in each critical region.

4.1.1 Equivalent multi-parametric barrier problem

An equivalent multi-parametric barrier problem for the lower-level problem (4.0.2) is given by,

$$\begin{aligned} &\min_x \{W(x, y, t) = f(x, y) + t\psi(x, y)\} \\ &s.t.\ g(x, y) = Ax - By - b \leq 0, \end{aligned} \tag{4.1.1}$$

where t is a positive real parameter, $\psi(x, y) = -\sum_{j=1}^{q} \ln(-h_j(x, y))$ is the barrier function with domain $\{x \in \mathbb{R}^n : h(x, y) < 0\}$ and $W(x, y, t) = f(x, y) + t\psi(x, y)$ is the penalty function.

Remark 13. *If the polyhedral constraints $g_i(x, y)$ are not given explicitly in* (4.0.2) *then we will consider the parametric bounds $y^L \leq y \leq y^U$ as the polyhedral constraints instead.*

In Section 3.3, it is shown that, if $x^*(y, t), t > 0$ is an optimal solution of (4.1.1), then

$$x^*(y) = \lim_{t \to 0^+} x^*(y, t)$$

is an optimal solution of the problem (4.0.2).

Find the solution of the barrier problem (4.1.1) using the procedures discussed in Subsection 3.3.3.

4.1.2 Solutions of the lower-level problem

Employ procedures of an algorithm proposed in Section 3.3.5 to find solutions of the nonlinear lower-level problem (4.0.2) in the interior and on the boundary of the nonlinear constraints. After solving problem (4.0.2) for the given initial parameter space Y, we get an optimal solution, say,

$$x(y) = \begin{cases} x^L & \text{on } \mathcal{CR}^{\#}, \\ x^{NL} & \text{on } \mathcal{CR}^{\#\#}, \\ \text{Infeasible} & \text{on } Y \setminus (\mathcal{CR}^{\#} \cup \mathcal{CR}^{\#\#}) \end{cases} \tag{4.1.2}$$

Note that problem (4.0.2) has a linear solution in the region $\mathcal{CR}^{\#} = \bigcup_{k\in\mathcal{K}} \mathcal{CR}_k^{\#}$, a nonlinear solution in the region $\mathcal{CR}^{\#\#} = \bigcup_{k\in\mathcal{K}, j\in\mathcal{Q}} \mathcal{CR}_{kj}^{\#\#}$ and infeasible in the region $Y \setminus (\mathcal{CR}^{\#} \cup \mathcal{CR}^{\#\#})$.

4.1.3 Solutions of the upper-level problem

The optimal solution obtained in Section 4.1.2 can be used in the leader problem to obtain the optimal solution of the upper-level problem in each critical region. Each parametric optimal solution $x^*(y)$ in (4.1.2) is substituted into the upper-level problem of (4.0.1), to obtain a one-level nonlinear optimization problem with additional constraints corresponding to each critical region,

$$\begin{cases} \min\limits_{y\in Y} F(x(y), y) \\ \text{subject to} \begin{cases} G(x(y), y) \leq 0, \\ x(y) \text{ given by (4.1.2)}, \\ y \in \mathcal{CR}^{\#} \quad \text{if } x(y) = x^L(y), \\ y \in \mathcal{CR}^{\#\#} \text{ if } x(y) = x^{NL}(y). \end{cases} \end{cases} \tag{4.1.3}$$

In each of the critical regions problem (4.1.3) is solved using any of the standard global optimization techniques. Choose the best solution y^* out of the solutions obtained from (4.1.3) for each finite number of critical regions.

Finally, the set of optimal solutions for the bilevel optimization (4.0.1) is given by

$$\{(x(y^*), y^*) : x(y) \text{ is given by (4.1.2)},\ y^* \text{ solves (4.1.3)}\}. \tag{4.1.4}$$

4.2 Algorithm of the proposed solution strategy

This section presents an algorithm of the proposed solution strategy discussed in Section 4.1 that can be applied to find the solution of a bilevel optimization problem of the form (4.0.1) whose lower-level problem satisfy the Assumptions 5, 6 and 7. Procedures of the solution approach are partitioned into four phases.

PHASE I: RECASTING THE LOWER-LEVEL PROBLEM AS MULTI-PARAMETRIC PROBLEM

Cast the lower-level problem of (4.0.1) as a multi-parametric problem (4.0.2) with parameter y. Then, GO TO Phase II.

PHASE II: REFORMULATION AND FINDING SOLUTION OF THE BARRIER PROBLEM
Reformulate problem (4.0.2) as a barrier multi-parametric problem (4.1.1) as discussed in Section 4.1.1. Define: $\mathcal{R}-$ a list of regions to be partitioned, $\mathcal{CR}^*-$ a list of optimal critical regions, and $Y-$ an initial parameter space. Use **Algorithm 1** (Table 3.2) and the procedures discussed in Subsection 3.3.5 to find the solutions of the barrier problem (4.1.1).

Then, GO TO Phase III.
PHASE III: FINDING SOLUTION OF THE LOWER-LEVEL PROBLEM

Use **Algorithm 2** (Table 3.3) and the procedures discussed in Subsection 3.3.5 to find an optimal solution to the lower-level problem (4.0.2) on the boundary.

Then, after getting the optimal solution,

$$x(y) = \begin{cases} x^{L_k}(y) & \text{on } \mathcal{CR}_k^{\#},\ k \in \mathcal{K}, \\ x^{NL_{kj}}(y) & \text{on } \mathcal{CR}_{kj}^{\#\#},\ k \in \mathcal{K},\ j \in \mathcal{Q}, \\ \text{Infeasible} & \text{on } \mathcal{CR}_k \setminus (\mathcal{CR}_k^{\#} \cup \mathcal{CR}_{kj}^{\#\#}),\ k \in \mathcal{K},\ j \in \mathcal{Q}. \end{cases} \tag{4.2.1}$$

of (4.0.2) GO TO Phase IV.

PHASE IV: FINDING SOLUTION OF THE UPPER-LEVEL PROBLEM

In the final phase of the algorithm, the optimal solution obtained in Phase IV can be used in the leader problem to obtain the optimal solution of the upper-level problem in each critical region. Substitute in each case the optimal parametric solution (4.2.1) into the upper-level problem of (4.0.1), to obtain a set of one level nonlinear programming problems, for each of the indices k and j

$$\begin{cases} \min\limits_{y \in Y} F(x(y), y) \\ \text{subject to} \begin{cases} G(x(y), y) \leq 0, \\ x(y) \text{ given by (4.2.1)}, \\ y \in \mathcal{CR}_k^{\#} \quad \text{if } x(y) = x^{L_k}(y), \\ y \in \mathcal{CR}_{kj}^{\#\#} \text{ if } x(y) = x^{NL_{kj}}(y). \end{cases} \end{cases} \tag{4.2.2}$$

Use any of the standard nonlinear programming algorithms available to solve problem (4.2.2) in each critical region $\mathcal{CR}_k^{\#}(k = 1 : r)$ and $\mathcal{CR}_{kj}^{\#\#}(k = 1 : r, j = 1 : q)$. Choose the best solution (which provides the minimum objective value) y^* out of the solutions obtained from (4.2.2) for each finite number of critical regions. Then, the set of optimal solutions for the bilevel optimization problem (4.0.1) is given by

$$\{(x(y^*), y^*) : x(y) \text{ is given by (4.2.1)}, \ y^* \text{ solves (4.2.2)}\}. \tag{4.2.3}$$

We have used a Multi-Parametric Toolbox in MATLAB for the numerical implementation of the proposed solution algorithm in computing x as a function of the parameter y.

4.3 Illustrative examples

Example 4.1. *Consider the following bilevel optimization problem,*

$$\begin{cases} \min\limits_{y} F(x, y) = (y_1 - x_1 + 1)^2 + (y_2 - x_2 + 1)^2 \\ \textit{subject to} \begin{cases} \min\limits_{x} f(x, y) = -x_1^2 + x_2^2 + x_1 + 2x_2 + y_1^2 + y_2^2 \\ \textit{subject to} \begin{cases} x_1 + x_2^2 + y_1 - 1.5 \leq 0, \\ x_1^2 - x_2 + y_2 - 3 \leq 0, \\ x_1 - 3x_2 + y_1 - 2y_2 - 2 \leq 0, \\ -3x_1 + x_2 + y_1 + y_2 - 1 \leq 0, \\ -2 \leq x_1, x_2 \leq 2, \\ -1 \leq y_1, y_2 \leq 1. \end{cases} \end{cases} \end{cases} \tag{4.3.1}$$

The lower-level optimization problem of (4.3.1) *is a multi-parametric problem,*

$$\begin{cases} \min_x f(x,y) = -x_1^2 + x_2^2 + x_1 + 2x_2 + y_1^2 + y_2^2 \\ \text{subject to} \begin{cases} x_1 + x_2^2 + y_1 - 1.5 \leq 0, \\ x_1^2 - x_2 + y_2 - 3 \leq 0, \\ x_1 - 3x_2 + y_1 - 2y_2 - 2 \leq 0, \\ -3x_1 + x_2 + y_1 + y_2 - 1 \leq 0, \\ -2 \leq x_1, x_2 \leq 2, \\ -1 \leq y_1, y_2 \leq 1. \end{cases} \end{cases} \tag{4.3.2}$$

Here, $h_1(x,y) = x_1 + x_2^2 + y_1 - 1.5$, $h_2(x,y) = x_1^2 - x_2 + y_2 - 3$, $g_1(x,y) = x_1 - 3x_2 + y_1 - 2y_2 - 2$ *and* $g_2(x,y) = -3x_1 + x_2 + y_1 + y_2 - 1$. *So the barrier function is defined as* $\psi(x,y) = -\ln(-h_1(x,y)) - \ln(-h_2(x,y))$ *with domain* $\{x : h_1(x,y) < 0,\ h_2(x,y) < 0\}$.

For $t > 0$ *we can define the equivalent barrier problem for* (4.3.2) *as*

$$\begin{cases} \min_x \{W(x,y,t) = -x_1^2 + x_2^2 + x_1 + 2x_2 + y_1^2 + y_2^2 + t\psi(x,y)\} \\ \text{subject to} \begin{cases} x_1 - 3x_2 + y_1 - 2y_2 - 2 \leq 0, \\ -3x_1 + x_2 + y_1 + y_2 - 1 \leq 0, \\ -2 \leq x_1, x_2 \leq 2, \\ -1 \leq y_1, y_2 \leq 1, \end{cases} \end{cases} \tag{4.3.3}$$

with domain $X^{\#}(y) = \{x : h_1(x,y) < 0, h_2(x,y) < 0, g_1(x,y) \leq 0, g_2(x,y) \leq 0, -2 \leq x_1, x_2 \leq 2, -1 y_1, y_2 \leq 1\}$.

Problem (4.3.3) *is a multi-parametric problem with parameter* $y = (y_1, y_2)$ *and Lagrangian function* $\mathcal{L}(x,y,\lambda,t) = -x_1^2 + x_2^2 + x_1 + 2x_2 + y_1^2 + y_2^2 + t\psi(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y)$.

By solving the barrier problem (4.3.3), *we obtain the following parametric solutions in the corresponding critical regions,*

$$x(y) = \begin{cases} \begin{bmatrix} 0.5000y_1 + 0.1250y_2 - 0.6250 \\ 0.5000y_1 - 0.6250y_2 - 1.4376 \end{bmatrix} & \text{on } \mathcal{CR}_1 \\ \begin{bmatrix} 0.3750y_1 + 0.3750y_2 - 0.8125 \\ 0.1250y_1 + 0.1250y_2 - 1.4376 \end{bmatrix} & \text{on } \mathcal{CR}_2 \end{cases} \tag{4.3.4}$$

where

$$\mathcal{CR}_1 = \begin{cases} -0.4473y_1 + 0.8944y_2 \leq 0.6707, \\ -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1 \end{cases}$$

and

$$\mathcal{CR}_2 = \begin{cases} -0.4473y_1 + 0.8944y_2 \geq 0.6707, \\ y_1 \geq -1, \ y_2 \leq 1, \end{cases}$$

are as shown in Fig.4.1. The linear solution (4.3.4) *is also a solution of* (4.3.2) *in the*

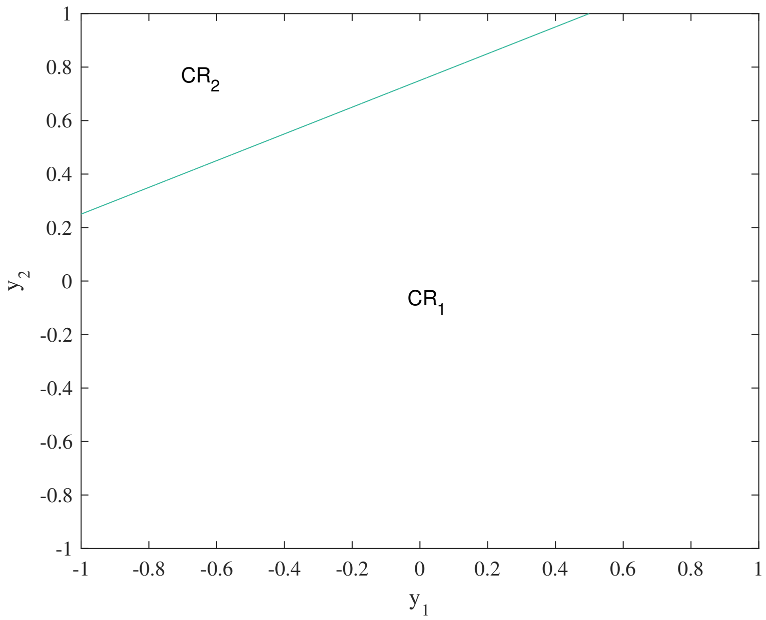

Figure 4.1: Critical regions for the problem (4.3.3)

interior of the nonlinear constraints, but with different critical regions $\mathcal{CR}^{\#} = \{\theta \in \mathcal{CR} : h(x(\theta), \theta) < 0\}$*. Therefore, the linear solutions and the corresponding critical regions of* (4.3.2) *will be:*

$$x^L(y) = \begin{cases} \begin{bmatrix} 0.5000y_1 + 0.1250y_2 - 0.6250 \\ 0.5000y_1 - 0.6250y_2 - 1.4376 \end{bmatrix} & \text{on } \mathcal{CR}_1^{\#} \\ \begin{bmatrix} 0.3750y_1 + 0.3750y_2 - 0.8125 \\ 0.1250y_1 + 0.1250y_2 - 1.4376 \end{bmatrix} & \text{on } \mathcal{CR}_2^{\#} \end{cases}$$

where

$$\mathcal{CR}_1^{\#} = \begin{cases} 0.2500y_1^2 - 0.6250y_1y_2 + 0.6250y_1 + 0.3906y_2^2 + 1.2188y_2 < 1.3594, \\ y_1^2 + 0.2500y_1y_2 - 1.7500y_1 + 0.0313y_2^2 + 1.3125y_2 < 1.3438, \\ -0.4473y_1 + 0.8944y_2 \leq 0.6707, \\ -1 \leq y_1 \leq 1, y_2 \leq 1 \end{cases}$$

and

$$\mathcal{CR}_2^{\#} = \begin{cases} 0.0156y_1^2 + 0.0315y_1y_2 + 1.0155y_1 + 0.0156y_2^2 + 0.0156y_2 < 0.2457, \\ 0.2813y_1^2 + 0.5625y_1y_2 - 1.3439y_1 + 0.2813y_2^2 - 0.03438y_2 < 0.2419, \\ -0.4473y_1 + 0.8944y_2 \geq 0.6707, \\ y_2 \leq 1. \end{cases}$$

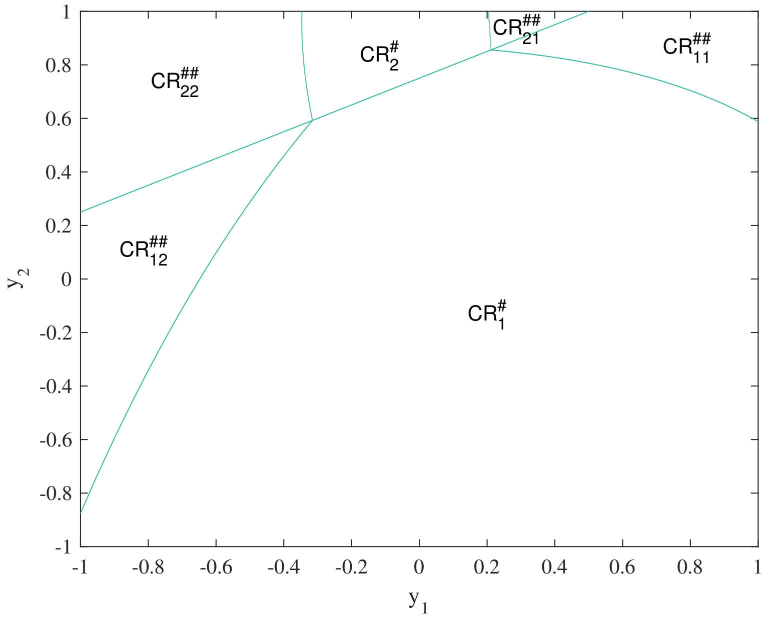

Figure 4.2: Critical regions for the problem (4.3.2)

The rest of the spaces will be determined by $\mathcal{CR}^{Rest} = \mathcal{CR} \setminus \mathcal{CR}^{\#}$*,*

$$\mathcal{CR}_{11}^{Rest} = \begin{cases} 0.2500y_1^2 - 0.6250y_1y_2 + 0.6250y_1 + 0.3906y_2^2 + 1.2188y_2 \geq 1.3594, \\ y_1^2 + 0.2500y_1y_2 - 1.7500y_1 + 0.0313y_2^2 + 1.3125y_2 < 1.3438, \\ -0.4473y_1 + 0.8944y_2 \leq 0.6707, \\ y_1 \leq 1, y_2 \leq 1; \end{cases}$$

$$\mathcal{CR}_{12}^{Rest} = \begin{cases} 0.2500y_1^2 - 0.6250y_1y_2 + 0.6250y_1 + 0.3906y_2^2 + 1.2188y_2 < 1.3594, \\ y_1^2 + 0.2500y_1y_2 - 1.7500y_1 + 0.0313y_2^2 + 1.3125y_2 \geq 1.3438, \\ -0.4473y_1 + 0.8944y_2 \leq 0.6707, \\ y_1 \geq -1; \end{cases}$$

$$\mathcal{CR}_{21}^{Rest} = \begin{cases} 0.0156y_1^2 + 0.0315y_1y_2 + 1.0155y_1 + 0.0156y_2^2 + 0.0156y_2 \geq 0.2457, \\ 0.2813y_1^2 + 0.5625y_1y_2 - 1.3439y_1 + 0.2813y_2^2 - 0.03438y_2 < 0.2419, \\ -0.4473y_1 + 0.8944y_2 \geq 0.6707, \\ y_2 \leq 1; \end{cases}$$

$$\mathcal{CR}_{22}^{Rest} = \begin{cases} 0.0156y_1^2 + 0.0315y_1y_2 + 1.0155y_1 + 0.0156y_2^2 + 0.0156y_2 < 0.2457, \\ 0.2813y_1^2 + 0.5625y_1y_2 - 1.3439y_1 + 0.2813y_2^2 - 0.03438y_2 \geq 0.2419, \\ -0.4473y_1 + 0.8944y_2 \geq 0.6707, \\ y_1 \geq -1, y_2 \leq 1. \end{cases}$$

The linear constraint $g_1(x, y)$ *is active and* $g_2(x, y)$ *is inactive in the region* $\mathcal{CR}_1^{\#}$*, hence the nonlinear solutions in the rest regions* $\mathcal{CR}_{11}^{Rest}$ *and* $\mathcal{CR}_{12}^{Rest}$ *is determined as follows.*

The nonlinear solution in $\mathcal{CR}_{11}^{Rest}$ *is determined from* $x^{NL_{11}}(y) = \arg\{h_1(x,y) = 0,\ g_1(x,y) = 0,\ g_2(x,y) \leq 0\}$. *Thus*

$$x^{NL_{11}}(y) = \arg\{x_1 + x_2^2 + y_1 = 1.5, x_1 - 3x_2 + y_1 - 2y_2 = 2, -3x_1 + x_2 + y_1 + y_2 \leq 1\},$$

which results in

$$x^{NL_{11}}(y) = \begin{bmatrix} 2y_2 - y_1 + \frac{1}{2}(\pm\sqrt{3(7-8y_2)} - 5) \\ \frac{1}{2}(\pm\sqrt{3(7-8y_2)} - 3) \end{bmatrix}.$$

Since $x = (-0.4121, -1.3039)$ *at* $y = (0.2119, 0.8558)$ *which is a common point for* $\mathcal{CR}_1^{\#}$ *and* $\mathcal{CR}_{11}^{Rest}$*, the only nonlinear parametric solution is*

$$x^{NL_{11}}(y) = \begin{bmatrix} 2y_2 - y_1 + \frac{1}{2}(\sqrt{3(7-8y_2)} - 5) \\ \frac{1}{2}(\sqrt{3(7-8y_2)} - 3) \end{bmatrix}$$

which is valid in $\mathcal{CR}_{11}^{\#\#} = \mathcal{CR}_{11}^{Rest}$. *In* $\mathcal{CR}_{12}^{Rest}$ *the nonlinear solution is determined from* $x^{NL_{12}}(y) = \arg\{h_2(x,y) = 0,\ g_1(x,y) = 0,\ g_2(x,y) \leq 0\}$. *Thus*

$$x^{NL_{12}}(y) = \arg\{x_1^2 - x_2 + y_2 = 3, x_1 - 3x_2 + y_1 - 2y_2 = 2, -3x_1 + x_2 + y_1 + y_2 \leq 1\},$$

which results in

$$x^{NL_{12}}(y) = \begin{bmatrix} \frac{1}{12}(1 \pm \sqrt{24y_1 - 120y_2 + 169}) \\ \frac{1}{3}(y_1 - 2y_1) + \frac{1}{36}(\pm\sqrt{24y_1 - 120y_2 + 169} - 23) \end{bmatrix}.$$

Since $x = (-0.7090, -1.4029)$ *at* $y = (-0.3159, 0.5918)$ *which is a common point for* $\mathcal{CR}_1^{\#}$ *and* $\mathcal{CR}_{12}^{Rest}$*, the only nonlinear parametric solution is*

$$x^{NL_{12}}(y) = \begin{bmatrix} \frac{1}{12}(1 - \sqrt{24y_1 - 120y_2 + 169}) \\ \frac{1}{3}(y_1 - 2y_1) - \frac{1}{36}(\sqrt{24y_1 - 120y_2 + 169} + 23) \end{bmatrix}$$

which is valid in $\mathcal{CR}_{12}^{\#\#} = \mathcal{CR}_{12}^{Rest}$.

The linear constraint $g_1(x,y)$ *is inactive and* $g_2(x,y)$ *is active in the region* $\mathcal{CR}_2^{\#}$*, hence the nonlinear solutions in the rest regions* $\mathcal{CR}_{21}^{Rest}$ *and* $\mathcal{CR}_{22}^{Rest}$ *is determined as follows.*
The nonlinear solution in $\mathcal{CR}_{21}^{Rest}$ *is determined from* $x^{NL_{21}}(y) = \arg\{h_1(x,y) = 0,\ g_2(x,y) = 0,\ g_1(x,y) \leq 0\}$. *Thus*

$$x^{NL_{11}}(y) = \arg\{x_1 + x_2^2 + y_1 = 1.5, -3x_1 + x_2 + y_1 + y_2 = 1, x_1 - 3x_2 + y_1 - 2y_2 \leq 2\},$$

which results in

$$x^{NL_{21}}(y) = \begin{bmatrix} \frac{1}{3}(y_1 + y_2) + \frac{1}{36}(\pm\sqrt{268 - 48y_2 - 192y_1} - 14) \\ \frac{1}{12}(\pm\sqrt{268 - 48y_2 - 192y_1} - 2) \end{bmatrix}$$

Since $x = (-0.4121, -1.3039)$ *at* $y = (0.2119, 0.8558)$ *which is a common point for* $\mathcal{CR}_2^{\#}$ *and* $\mathcal{CR}_{21}^{Rest}$*, the only nonlinear parametric solution is*

$$x^{NL_{21}}(y) = \begin{bmatrix} \frac{1}{3}(y_1 + y_2) - \frac{1}{36}(\sqrt{268 - 48y_2 - 192y_1} + 14) \\ -\frac{1}{12}(\sqrt{268 - 48y_2 - 192y_1} + 2) \end{bmatrix}$$

which is valid in $\mathcal{CR}_{21}^{\#\#} = \mathcal{CR}_{21}^{Rest}$*. In* $\mathcal{CR}_{22}^{Rest}$ *the nonlinear solution is determined from* $x^{NL_{22}}(y) = \arg\{h_2(x, y) = 0,\ g_2(x, y) = 0,\ g_1(x, y) \leq 0\}$*. Thus*

$$x^{NL_{22}}(y) = \arg\{x_1^2 - x_2 + y_2 = 3, -3x_1 + x_2 + y_1 + y_2 = 1, x_1 - 3x_2 + y_1 - 2y_2 \leq 2\},$$

which results in

$$x^{NL_{22}}(y) = \begin{bmatrix} \frac{1}{4}(3 \pm \sqrt{41 - 16y_2 - 8y_1}) \\ -y_1 - y_1 - \frac{1}{4}(13 \pm 3\sqrt{41 - 16y_2 - 8y_1}) \end{bmatrix}.$$

Since $x = (-0.7090, -1.4029)$ *at* $y = (-0.3159, 0.5918)$ *which is a common point for* $\mathcal{CR}_2^{\#}$ *and* $\mathcal{CR}_{22}^{Rest}$*, the only nonlinear parametric solution is*

$$x^{NL_{22}}(y) = \begin{bmatrix} \frac{1}{4}(3 - \sqrt{41 - 16y_2 - 8y_1}) \\ -y_1 - y_1 - \frac{1}{4}(13 - 3\sqrt{41 - 16y_2 - 8y_1}) \end{bmatrix}$$

which is valid in $\mathcal{CR}_{22}^{\#\#} = \mathcal{CR}_{22}^{Rest}$*.*

Therefore, the optimal solution is given by,

$$x(y) = \begin{cases} \begin{bmatrix} 0.5000y_1 + 0.1250y_2 - 0.6250 \\ 0.5000y_1 - 0.6250y_2 - 1.4376 \end{bmatrix} & \text{on } \mathcal{CR}_1^{\#} \\ \begin{bmatrix} 0.3750y_1 + 0.3750y_2 - 0.8125 \\ 0.1250y_1 + 0.1250y_2 - 1.4376 \end{bmatrix} & \text{on } \mathcal{CR}_2^{\#} \\ \begin{bmatrix} 2y_2 - y_1 + \frac{1}{2}(\sqrt{3(7 - 8y_2)} - 5) \\ \frac{1}{2}(\sqrt{3(7 - 8y_2)} - 3) \end{bmatrix} & \text{on } \mathcal{CR}_{11}^{\#\#} \\ \begin{bmatrix} \frac{1}{12}(1 - \sqrt{24y_1 - 120y_2 + 169}) \\ \frac{1}{3}(y_1 - 2y_1) - \frac{1}{36}(\sqrt{24y_1 - 120y_2 + 169} + 23) \end{bmatrix} & \text{on } \mathcal{CR}_{12}^{\#\#} \\ \begin{bmatrix} \frac{1}{3}(y_1 + y_2) - \frac{1}{36}(\sqrt{268 - 48y_2 - 192y_1} + 14) \\ -\frac{1}{12}(\sqrt{268 - 48y_2 - 192y_1} + 2) \end{bmatrix} & \text{on } \mathcal{CR}_{21}^{\#\#} \\ \begin{bmatrix} \frac{1}{4}(3 - \sqrt{41 - 16y_2 - 8y_1}) \\ -y_1 - y_1 - \frac{1}{4}(13 - 3\sqrt{41 - 16y_2 - 8y_1}) \end{bmatrix} & \text{on } \mathcal{CR}_{22}^{\#\#} \end{cases}$$

where $\mathcal{CR}_1^{\#}$*,* $\mathcal{CR}_2^{\#}$*,* $\mathcal{CR}_{11}^{\#\#}$*,* $\mathcal{CR}_{12}^{\#\#}$*,* $\mathcal{CR}_{21}^{\#\#}$ *and* $\mathcal{CR}_{22}^{\#\#}$ *are as indicated in Fig.4.2.*

Incorporating the solution $x(y)$ into the upper-level problem of (4.3.1) *and solving the resulting nonlinear problems in each critical regions, we get following optimal solutions:*
In $\mathcal{CR}_1^{\#}$, $(x, y) = (-1.2500, -0.7500, -1.0000, -1.0000)$ *and objective value* $F = 2.1250$.
In $\mathcal{CR}_2^{\#}$, $(x, y) = (-0.7087, -1.4030, -0.3155, 0.5924)$ *and objective value* $F = 10.9138$.
In $\mathcal{CR}_{11}^{\#\#}$, $(x, y) = (-0.0515, -0.7426, 1.0000, 0.5882)$ *and objective value* $F = 9.6412$.
In $\mathcal{CR}_{12}^{\#\#}$, $(x, y) = (-1.2344, -0.8278, -1.0000, -0.8755)$ *and objective value* $F = 2.4308$.
In $\mathcal{CR}_{21}^{\#\#}$, $(x, y) = (-0.1984, -1.0949, 0.4996, 1.0000)$ *and objective value* $F = 12.4617$.
In $\mathcal{CR}_{22}^{\#\#}$, $(x, y) = (-0.9270, -1.0312, -1.0000, 0.2502)$ *and objective value* $F = 6.0640$.

Now, comparing all the values of the upper-level objective in each of the critical regions, we can see that the objective value obtained in $\mathcal{CR}_1^{\#}$ *gives a better result. Hence we take the solution in* $\mathcal{CR}_1^{\#}$ *as an optimal solution to the upper-level problem of* (4.3.1).

Therefore, the optimal solution to the bilevel problem (4.3.1) *is* $(-1.25, -0.75, -1, -1)$ *with optimal leader objective* $F = 2.1250$*; and optimal follower objective* $f = -1.7500$. □

4.4 Chapter summary

In this chapter we have proposed a multi-parametric programming approach algorithm for the solution of bilevel programming problems whose lower-level problem involve convex nonlinear constraints. The method starts by recasting the lower-level problem of the bilevel optimization as a multi-parametric programming problem where the variable from upper-level problem is considered as parameter. The algorithm employed a barrier function reformulation to transform a given multi-parametric problem with convex nonlinear constraints and bounded parametric region into an equivalent multi-parametric programming problem with polyhedral constraints. The resulting problem is solved by a global solution strategy for general convex multi-parametric problems to obtain a set of parametric solutions and the corresponding polyhedral critical regions. Methods to find solutions of the nonlinear multi-parametric problem in the interior and on the boundary of the nonlinear constraints and methods of exploration of the parameter space are also provided. The multi-parametric solutions obtained for the lower-level problem are incorporated in the upper-level problem to create a set of single-level optimization problems which are solved using standard global optimization techniques.

The solution obtained with this method is shown to be exact if the lower-level problem and the nonlinear constraints can be expressed by a polynomial of utmost degree three with followers' variable and upto quadratic in the variable of the leader.

Chapter 5

Hierarchical multilevel multi-leader multi-follower game

Contents

This chapter, presents the reformulation of two classes of multilevel MLMF games as multilevel hierarchical games. A multi-parametric based solution algorithm is also provided.

5.1 Transformation of a multilevel MLMF weighted potential game into a multilevel SLMF game

In this section, a hierarchical multilevel MLMF weighted potential game with shared linear constraint is equivalently reformulated as a hierarchical multilevel SLMF game. Then the resulting problem can be solved by the existing solution algorithms proposed for such problems. Particularly, in this section we have used the procedures proposed in [61] to reformulate the resulting multilevel SLMF game as a multilevel hierarchical game involving a single decision maker at each level of the hierarchy. Finally, the multi-parametric solution algorithms proposed in [36, 37] (for convex case) and [60] (for non-convex case) are used to solve the multilevel hierarchical problem. For the sake of clarity in presentation, the method is described using a bilevel MLMF game, however it can be extended to a general k-level case.

5.1.1 Trilevel SLMF formulation of a bilevel MLMF game

Consider a bilevel multi-leader multi-follower (bilevel MLMF) problem in which N leaders compete in a non-cooperative game subject to the equilibrium conditions of M followers competing in a lower-level game given leader level decisions. If we denote x^i, $i \in \{1, \ldots, N\}$, the decision variables vector for leader i and y^j, $j \in \{1, \ldots, M\}$, the decision variables for follower j. The leaders' and followers' variables are abbreviated, respectively by $x = (x^1, \ldots, x^N)$ and $y = (y^1, \ldots, y^M)$. The Stackelberg game played by leader i is given by:

$$\begin{cases} \min\limits_{x^i \in X^i} F_i(x, y) \\ \text{subject to} \begin{cases} G_i(x^i, y) \leq 0, \\ H(x, y) \leq 0, \text{ and for all } j = 1, \ldots, M, \\ y^j \text{ solves} \begin{cases} \min\limits_{y^j \in Y^j} f_j(x, y) \\ \text{subject to} \begin{cases} g_j(x, y^j) \leq 0, \\ h(x, y) \leq 0. \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.1.1}$$

Define the following sets that can be used to characterize solutions of the problem (5.1.1).

(i) The feasible set of problem (5.1.1) is given by

$$\mathcal{F} = \left\{(x, y) : H(x, y) \leq 0,\ h(x, y) \leq 0,\ (\forall i)\ G_i(x^i, y) \leq 0,\ (\forall j)\ g_j(x, y^j) \leq 0\right\}.$$

(ii) For any given leaders' strategy x, the feasible set for the j^{th}-follower is defined as $\mathcal{F}_j(x, y^{-j}) = \{y^j \in Y^j : g_j(x, y^j) \leq 0,\ h(x, y) \leq 0\}$.

(iii) The Nash rational reaction set for the j^{th}-follower, is defined by,

$$\mathcal{R}_j(x, y^{-j}) = \left\{ \bar{y}^j \in Y^j : \bar{y}^j \in \underset{y^j}{\operatorname{argmin}} \left\{ f_j(x, y) \text{ subject to } y^j \in \mathcal{F}_j(x, y^{-j}) \right\} \right\}.$$

(iv) The feasible set for the i^{th}-leader, is defined as

$$\mathcal{F}_i(x^{-i}) = \left\{ (x^i, y) : (x, y) \in \mathcal{F},\ \forall (y^j) \in \mathcal{R}_j(x, y^{-j}) \right\}.$$

(v) The Nash rational reaction set for the i^{th}-leader is given by

$$\mathcal{R}_i(x^{-i}) = \left\{ (\bar{x}^i, y) : \bar{x}^i \in \underset{x^i}{\operatorname{argmin}} \left\{ F_i(x, y) \text{ subject to } (x^i, y) \in \mathcal{F}_i(x^{-i}) \right\} \right\}.$$

(vi) The set of Stackelberg-Nash equilibrium points of problem (5.1.1) is given by

$$\mathcal{E} = \left\{ (x, y) \in \mathcal{F} : (\forall i)(x^i, y) \in \mathcal{R}_i(x^{-i}),\ (\forall j)(y^j) \in \mathcal{R}_j(x, y^{*,-j}) \right\}.$$

Remark 14. *If* $(x^*, y^*) \in \mathcal{E}$*, for* $\forall (x, y) \in \mathcal{F}$*,* $(\forall i)$ *and* $(\forall j)$ *we have*

$$F_i(x^*, y^*) \leq F_i(x, y^*) \leq F_i(x, y) \text{ and } f_j(x^*, y^*) \leq f_j(x^*, y) \leq f_j(x, y).$$

Using the reaction sets, the leaders Nash problem of (5.1.1) is given by

$$\forall i\ (x^i, y) \in \begin{cases} \underset{x^i}{\operatorname{argmin}}\ F_i(x, y) \\ \text{s.t. } (x^i, y) \in \mathcal{R}_i(x^{-i}). \end{cases} \tag{5.1.2}$$

Definition 5.1. *A tuple* $(x^*, y) = (x^{*,i}, x^{*,-i}, y)$ *is called an optimal Nash equilibrium solution to the problem* (5.1.2)*, if it satisfies the following conditions:*

$$F_i(x^{*,i}, x^{*,-i}, y) \leq F_i(x^i, x^{*,-i}, y),\ (\forall i)(x^i, y) \in \mathcal{R}_i(x^{*,-i}).$$

Given a strategy x of the leaders, the M followers play the Nash game:

$$\forall j\ (y^j) \in \begin{cases} \underset{y^j}{\operatorname{argmin}}\ f_j(x, y) \\ \text{s.t. } y^j \in \mathcal{R}_j(x, y^{-j}). \end{cases} \tag{5.1.3}$$

Definition 5.2. *A tuple* $y^* = (y^{*,j}, y^{*,-j})$ *is called an optimal Nash equilibrium solution to the problem* (5.1.3)*, if it satisfies the following conditions:*

$$f_j(x, y^{*,j}, y^{*,-j}) \leq f_j(x, y^j, y^{*,-j}),\ (\forall j)\ (y^j) \in \mathcal{R}_j(x, y^{*,-j}).$$

Now we shall formulate an equivalent trilevel SLMF programming problem for (5.1.1) and we will show their equivalence. Let us add an upper level decision maker, a suppositional (or dummy) leader, to the problem (5.1.1) with the corresponding decision variable z, where $z = (x, y)$, and objective function equal to a constant α. Then the multiple leaders in the upper level problem of (5.1.1) become middle-level followers and the multiple followers in the lower level problem of (5.1.1) become bottom-level followers in the second level and we will get the following trilevel SLMF programming:

$$\begin{cases} \min\limits_{z} \alpha \\ \text{s.t.} \begin{cases} z = (x, y), \text{ and for all } i = 1, \ldots, N, \\ (x^i, y) \text{ solves } \begin{cases} \min\limits_{x^i \in X^i} F_i(x, y) \\ \text{s.t.} \begin{cases} G_i(x^i, y) \leq 0, \\ H(x, y) \leq 0, \text{ and for all } j = 1, \ldots, M, \\ y^j \text{ solves } \begin{cases} \min\limits_{y^j \in Y^j} f_j(x, y) \\ \text{s.t.} \begin{cases} g_j(x, y^j) \leq 0, \\ h(x, y) \leq 0. \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.1.4}$$

Let us assume that, for all i and for each j, the functions $F_i,\ G_i,\ H,\ h,\ f_j,\ g_j$ in (5.1.1) are twice continuously differentiable, and that the followers' constraint functions satisfy the Guignard constraint qualifications conditions [49]. Define some relevant sets related to problem (5.1.4) as follows:

(i) The feasible set for the third level followers problem is defined as:

$$\Omega_3(x, y^{-j}) = \left\{ y^j \in Y^j : g_j(x, y^j) \leq 0,\ h(x, y) \leq 0 \right\}.$$

(ii) The rational reaction set for the third level followers problem is given by a set of parametric solutions,

$$\Psi_3(x, y^{-j}) = \left\{ \bar{y}^j \in Y^j : \bar{y}^j \in \operatorname{argmin} \left\{ f_j(x, y) : y^j \in \Omega_3(x, y^{-j}) \right\} \right\}.$$

(iii) The feasible set for the second level problem is given by

$$\begin{aligned} \Omega_2(x^{-i}) = \big\{ (x^i, y) \in X^i \times Y : G_i(x^i, y) \leq 0,\ H(x, y) \leq 0, \\ g_j(x, y^j) \leq 0,\ h(x, y) \leq 0,\ y^j \in \Psi_3(x, y^{-j}) \big\}. \end{aligned}$$

(iv) The rational reaction set for the second level followers problem is defined as:

$$\Psi_2(x^{-i}) = \left\{ (x^i, y) \in X^i \times Y : x^i \in \arg\min \left\{ F_i(x, y) : (x^i, y) \in \Omega_2(x^{-i}) \right\} \right\}$$

(v) The feasible set of problem (5.1.4) is given by:

$$\Phi = \big\{(z, x, y) : z = (x, y),\ g_j(x, y^j) \leq 0,\ h(x, y) \leq 0,\ j = 1, \ldots, M,\\ G_i(x^i, y) \leq 0,\ H(x, y) \leq 0,\ i = 1, \ldots, N\big\}.$$

(vi) The inducible region of problem (5.1.4) is given by:

$$\mathcal{IR} = \big\{(z, x, y) : (z, x, y) \in \Phi,\ (x^i, y) \in \Psi_2(x^{-i})\big\}.$$

With these definitions of sets, problem (5.1.4) could be equivalently rewritten as:

$$\begin{aligned} &\min_z \alpha \\ &s.t.\ (z, x, y) \in \mathcal{IR}. \end{aligned} \tag{5.1.5}$$

Since every feasible point of (5.1.5) is an optimal point, the optimal set, $\mathcal{S}^*$, of (5.1.5) is equal to $\mathcal{IR}$.

Once we have established relations between the bilevel SLMF problem (5.1.1) and the trilevel SLMF problem (5.1.4). We will describe their equivalence with the following theorem.

Theorem 5.1. *A point* (x^*, y^*) *is an optimal solution to* (5.1.1) *if and only if* (z^*, x^*, y^*) *is an optimal solution to* (5.1.5).

Proof. Suppose that (z^*, x^*, y^*) is an optimal solution to (5.1.1), i.e., $(z^*, x^*, y^*) \in \mathcal{E}$ which implies that, $(z^*, x^*, y^*) \in \mathcal{F}$, $(x^{*,i}, y^*) \in \mathcal{R}_i(x^{*,-i})$, $i = 1, \ldots, N$.

That means,

$$\begin{aligned} &(x^{*,i}, y^*) \in \Psi_2(x^{*,-i}),\ h(x^*, y^*) \leq 0,\ g_j(x^*, y^{*,j}) \leq 0,\ j = 1, \ldots, M, \\ &H(x^*, y^*) \leq 0,\ G_i(x^{*,i}, y^*) \leq 0,\ i = 1, \ldots, N. \end{aligned}$$

Then for any point (z^*, x^*, y^*), $z^* = (x^*, y^*)$, and $(z^*, x^*, y^*) \in \mathcal{E}$, we have

$$\begin{aligned} &(x^{*,i}, y^*) \in \Psi_2(x^{*,-i}),\ z^* = (x^*, y^*),\ h(x^*, y^*) \leq 0,\ g_j(x^*, y^{*,j}) \leq 0,\ j = 1, \ldots, M, \\ &H(x^*, y^*) \leq 0,\ G_i(x^{*,i}, y^*) \leq 0,\ i = 1, \ldots, N. \end{aligned}$$

This implies that $(z^*, x^*, y^*) \in \Phi$ and $(x^{*,i}, y^*) \in \Psi_2(x^{*,-i})$. Therefore $(z^*, x^*, y^*) \in \mathcal{IR} = \mathcal{E}^*$ and hence (z^*, x^*, y^*) is an optimal solution to (5.1.5).

Conversely, suppose that (z^*, x^*, y^*) is an optimal solution to (5.1.5), i.e., $(z^*, x^*, y^*) \in \mathcal{E}^*$, then we have $(z^*, x^*, y^*) \in \Phi$ and $(x^{*,i}, y^*) \in \Psi_2(x^{*,-i})$. This implies the following

$$\begin{aligned} &h(x^*, y^*) \leq 0,\ (x^{*,i}, y^*) \in \mathcal{R}_i(x^{*,-i}),\ g_j(x^*, y^{*,j}) \leq 0,\ j = 1, \ldots, M, \\ &H(x^*, y^*) \leq 0,\ G_i(x^{*,i}, y^*) \leq 0,\ i = 1, \ldots, N. \end{aligned}$$

That is, $(x^*, y^*) \in \mathcal{F}$, $(x^{*,i}, y^*) \in \mathcal{R}_i(x^{*,-i})$, $i = 1, \ldots, N$. Therefore $(x^*, y^*) \in \mathcal{E}$ and hence (x^*, y^*) is an optimal solution to (5.1.1). □

Then the resulting problem can be solved by the existing solution algorithms proposed for hierarchical multilevel problems with single leader and multiple followers. In this section, we make the following assumptions on the structure of the objective functions of (5.1.1) that will enable us to use the procedures proposed in [61] to reformulate (5.1.4) as a hierarchical trilevel game involving a single decision maker at both levels.

Assumption 8. *The leaders objective can be written as*

$$F_i(x,y) = \hat{F}_i(x^i,y) + \bar{F}_i(x^{-i},y) + \rho_i \tilde{F}(x,y) = \breve{F}_i(x^i,x^{-i},y) + \rho_i \tilde{F}(x,y)$$

where, for any $y \in Y$, $(\forall i)(0 < \rho_i < \infty)$.

Assumption 9. *The followers objective can be written as*

$$f_j(x,y) = \hat{f}_j(x,y^j) + \bar{f}_j(x,y^{-j}) + \delta_j \tilde{f}(x,y) = \breve{f}_j(x,y^j,y^{-j}) + \delta_j \tilde{f}(x,y)$$

where, for any $x \in X$, $(\forall j)(0 < \delta_j < \infty)$.

As it was shown in [61] , under the Assumptions 8 and 9, problem (5.1.4) can be reformulated as a trilevel hierarchical game with single decision maker at all levels of the hierarchy is given by

$$\begin{cases} \min\limits_{z} \alpha \\ \text{s.t.} \begin{cases} z = (x,y), \text{ and} \\ (x,y) \text{ solves } \begin{cases} \min\limits_{x \in X} \left\{ \left[\sum\limits_{j=1}^{M} \dfrac{1}{\delta_j(x)} \hat{f}_j(x,y^j) \right] + \tilde{f}(x,y) \right\} \\ \text{s.t.} \begin{cases} G_i(x^i,y) \le 0 \ (\forall i), \\ H(x,y) \le 0, \text{ and} \\ y \text{ solves } \begin{cases} \min\limits_{y \in Y} \left\{ \left[\sum\limits_{j=1}^{M} \dfrac{1}{\delta_j(x)} \hat{f}_j(x,y^j) \right] + \tilde{f}(x,y) \right\} \\ \text{s.t.} \begin{cases} g_j(x,y^j) \le 0 \ (\forall j), \\ h(x,y) \le 0. \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.1.6}$$

Proposition 5.2. *Suppose that* (5.1.1) *satisfies the Assumptions 8 and 9. A point* (x^*,y^*) *is an optimal solution to* (5.1.1) *if and only if* (z^*,x^*,y^*) *is an optimal solution to* (5.1.6).

Proof. Follows from Theorem 5.1 and the equivalence of (5.1.6) and (5.1.4). □

Remark 15. *The idea described above can be extended to any finite k-level multi-leader multi-follower programming problem. By adding a dummy upper decision maker, problem* (1.2.6) *can be equivalently reformulated as $(k+1)$-level SLMF game. As a result, leaders in the upper level problem of* (1.2.6) *become followers at the second-level and followers at m^{th}-level problem of* (1.2.6) *become followers at $(m+1)^{th}$-level, where $m \in \{2, \ldots, k\}$.*

5.1.2 Multilevel SLMF formulation of a multilevel MLMF game

In this subsection we will generalize the transformation discussed above into a multilevel MLMF game.

Based on the Remark 15, the k-level MLMF game (1.2.6) can be reformulated as $(k+1)$-level SLMF game having a single leader and multiple followers at all lower-levels,

$$
\begin{cases}
\min\limits_{x} \alpha \\
s.t. \begin{cases}
x = (y_1, y_2, \ldots, y_k), \text{ and for all } n = 1, \ldots, N_1, (y_1^n, y_2, y_3, \ldots, y_k) \text{ solves} \\
\min\limits_{y_1^n \in Y_1^n} F_1^n(y_1, y_2, \ldots, y_k) \\
\text{s.t.} \begin{cases}
G_1^n(y_1^n, y_2, \ldots, y_k) \leq 0, \\
H_1(y_1, y_2, \ldots, y_k) \leq 0, \text{ and for all } i = 1, \ldots, N_2, (y_2^i, y_3, \ldots, y_k) \text{ solves} \\
\min\limits_{y_2^i \in Y_2^i} f_2^i(y_1, y_2, \ldots, y_k) \\
\text{s.t.} \begin{cases}
g_2^i(y_1, y_2, \ldots, y_k) \leq 0, \\
h_2(y_1, y_2, \ldots, y_k) \leq 0, \text{ and} \\
\qquad \ddots \\
\text{for all } l = 1, \ldots, N_k, \\
y_k^l \text{ solves} \begin{cases}
\min\limits_{y_k^l \in Y_k^l} f_k^l(y_1, y_2, \ldots, y_k) \\
\text{s.t.} \begin{cases}
g_k^l(y_1, y_2, \ldots, y_k) \leq 0, \\
h_k(y_1, y_2, \ldots, y_k) \leq 0.
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\tag{5.1.7}
$$

For the problem (1.2.6) if we make an assumption that each leaders' objective function consisting of separable terms and parameterized common terms across all leaders with positive weights; and that at all levels in the hierarchy each followers' objective function consisting of separable terms and parameterized common terms across all followers of the same level with positive weights. That means, (1.2.6) satisfies the following assumptions:

Assumption 10. *The leaders' objective functions can be written as*

$$F_1^n(y_1, y_2, \dots, y_k) = F_1^n(y_1^n, y_2, \dots, y_k) + F_1^n(y_1^{-n}, y_2, \dots, y_k) + \rho_1^n \tilde{F}_1(y_1, y_2, \dots, y_k),$$

where $(\forall n)\ (0 < \rho_1^n < \infty)$.

Assumption 11. *At all levels over the hierarchy, c^{th}-follower's objective function at m^{th}-level , $m \in 2, 3, \dots, k$, can be written as*

$$\begin{aligned} f_m^c(y_1, y_2, \dots, y_k) &= f_m^c(y_1, \dots, y_{m-1}, y_m^c, y_{m+1} \dots, y_k) \\ &\quad + f_m^c(y_1, \dots, y_{m-1}, y_m^{-c}, y_{m+1} \dots, y_k) + \rho_m^c \tilde{f}_m^c(y_1, y_2, \dots, y_k), \end{aligned}$$

where $0 < \rho_m^c < \infty$ for each $m = 2, 3, \dots, k$ and $c = i, j, \dots, l$.

As it was shown in [61], if problem (5.1.7) satisfy Assumptions 10 and 11, problem (5.1.7) can be reformulated into its equivalent multilevel hierarchical problem having single decision maker at each decision level:

$$\begin{cases} \min_x \alpha \\ s.t. \begin{cases} x = (y_1, y_2, \dots, y_k), \text{ and } (y_1, y_2, y_3, \dots, y_k) \text{ solves} \\ \min\limits_{y_1} \left\{ \left[\sum\limits_{n=1}^{N} \frac{1}{\rho_1^n} \hat{F}_1^n(y_1^n, y_2, y_3, \dots, y_k) \right] + \tilde{F}_1(y) \right\} \\ s.t. \begin{cases} G_1^n(y_1^n, y_2, \dots, y_k) \le 0\ (\forall n), \\ H_1(y_1, y_2, \dots, y_k) \le 0, \text{ and } (y_2, y_3, \dots, y_k) \text{ solves} \\ \min\limits_{y_2} \left\{ \left[\sum\limits_{i=1}^{I} \frac{1}{\rho_2^i} \hat{f}_2^i(y_1, y_2^i, y_3, \dots, y_k) \right] + \tilde{f}_2^i(y) \right\} \\ s.t. \begin{cases} g_2^i(y_1, y_2, \dots, y_k) \le 0\ (\forall i), \\ h_2(y_1, y_2, \dots, y_k) \le 0, \text{ and} \\ \quad \ddots \\ y_k \text{ solves} \begin{cases} \min\limits_{y_k} \left\{ \left[\sum\limits_{l=1}^{L} \frac{1}{\rho_k^l} \hat{f}_k^l(y_1, y_2, y_3, \dots, y_k^l) \right] + \tilde{f}_k^l(y) \right\} \\ s.t. \begin{cases} g_k^l(y_1, y_2, \dots, y_k) \le 0\ (\forall l), \\ h_k(y_1, y_2, \dots, y_k) \le 0, \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.1.8}$$

where $y = (y_1, y_2, y_3, \dots, y_k)$.

Proposition 5.3. *Suppose that* (1.2.6) *satisfies Assumptions 10 and 11. A point $(y_1^*, y_2^*, \dots, y_k^*)$ is an optimal solution to* (1.2.6) *if and only if $(x^*, y_1^*, y_2^*, \dots, y_k^*)$ is an optimal solution to* (5.1.8).

The proposed solution approach can solve multilevel multi-leader multi-follower problems whose objective values at all levels of the decision hierarchy have common, but having different positive weights of, non-separable terms and with the constraints at each level are polyhedral.

5.2 Reformulation of a multilevel MLMF game as a hierarchical multilevel game

This section considers hierarchical multi-leader multi-follower games in which multiple leaders of equal status in the upper-level and multiple followers of equal status are involved at each lower-level of the hierarchy. The leader can only influence the reactions of followers through their decision variables, while the followers optimize their objective functions in view of the decisions of the leader and other middle level decision agents. We have assumed that (i) the objective at all levels have separable and non-separable terms (but the non-separable terms can be written as a factor of two functions where the first one is a function of other level decision variables and the second factor is common to all objectives across the same level), (ii) at each level there is a shared constraint common to all problems of same level, and (iii) the leader and followers have their own decision variables and objective functions.

The proposed solution procedure for the above considered class of multilevel MLMF game transforms the given problem into an equivalent multilevel hierarchical problems having single decision maker at each level of the hierarchy and without increasing the level of hierarchy in the problem. Then, the resulting multilevel hierarchical problem is solved by multi-parametric solution algorithms proposed in [36, 37] (for convex case) and [60] (for non-convex case).

5.2.1 Equivalent bilevel optimization problem for a bilevel MLMF game

This section presents a reformulation of some classes of a bilevel MLMF problem (5.1.1) into an equivalent bilevel optimization problem involving only a single decision maker at both levels of the hierarchy, which can be extended later to any finite hierarchical levels.

When the Nash problem (5.1.2) has objective functions with separable terms and a non-separable term which is common to all leaders (i.e., for each i if the objective functions are written as $F_i(x^i, x^{-i}, y) = \hat{F}_i(x^i) + \breve{F}_i(x^{-i}) + \tilde{F}(x, y)$), by defining a *quasi-potential* function

$$\mathbf{F}(x, y) = \left[\sum_{i=1}^{N} \hat{F}_i(x^i)\right] + \tilde{F}(x, y),$$

Kulkarni and Shanbhag [63] equivalently reformulated problem (5.1.2) as a single optimization problem

$$\begin{aligned} \min_x \mathbf{F}(x, y) &= \left[\sum_{i=1}^{N} \hat{F}_i(x^i)\right] + \tilde{F}(x, y) \\ \text{subject to } & (x^i, y) \in \mathcal{R}_i(x^{-i}) \ (\forall i). \end{aligned} \tag{5.2.1}$$

And they have shown that, the global minimizers of problem (5.2.1) are global equilibria of the problem (5.1.2). In general, it has been shown in [63] that if the objectives of the leaders and the followers admit a quasi-potential function formulation, then the global minimizers of this quasi-potential function problem are global equilibria of the multilevel MLMF game.

The quasi-potential function problem of Kulkarni and Shanbhag considered in [63] covers only a class of games whose objective functions are written as separable terms and a non-separable term which is common to all decision makers at that level. In this work we have considered a game in which the objective functions contain a separable and non-separable terms; but the non-separable terms can be written as a factor of two functions where the first one is a function of other level decision variables and the second factor is common to all objectives across the same level.

We make the following assumptions on the structure of the objective functions of (5.1.1) that will enable us to reformulate a bilevel MLMF game as a hierarchical bilevel game involving a single decision maker at both levels:

Assumption 12. *The leaders objective in* (5.1.1) *can be written as*

$$F_i(x, y) = \hat{F}_i(x^i, y) + \bar{F}_i(x^{-i}, y) + \rho_i(y)\tilde{F}(x, y) = \breve{F}_i(x^i, x^{-i}, y) + \rho_i(y)\tilde{F}(x, y)$$

where, for any $y \in Y$, $(\forall i)(0 < \rho_i(y) < \infty)$.

Assumption 13. *The followers objective in* (5.1.1) *can be written as*

$$f_j(x, y) = \hat{f}_j(x, y^j) + \bar{f}_j(x, y^{-j}) + \delta_j(x)\tilde{f}(x, y) = \breve{f}_j(x, y^j, y^{-j}) + \delta_j(x)\tilde{f}(x, y)$$

where, for any $x \in X$, $(\forall j)(0 < \delta_j(x) < \infty)$.

Assumption 14. *$\rho_i(\cdot)$ and $\delta_j(\cdot)$ are twice continuously differentiable functions and uniformly bounded away from zero.*

Lemma 5.4. *If Assumptions 12, 13 and 14 hold for any $(x, y) \in X \times Y$, then there exist functions $\pi_F(x, y)$ and $\pi_f(x, y)$ such that*

$$\breve{F}_i(x^{*,i}, x^{*,-i}, y) - \breve{F}_i(x^i, x^{*,-i}, y) = \rho_i(y)\left[\pi_F(x^{*,i}, x^{*,-i}, y) - \pi_F(x^i, x^{*,-i}, y)\right], \ (\forall i),$$

$$\breve{f}_j(x, y^{*,j}, y^{*,-j}) - \breve{f}_j(x, y^j, y^{*,-j}) = \delta_j(x)\left[\pi_f(x, y^{*,j}, y^{*,-j}) - \pi_f(x, y^j, y^{*,-j})\right], \ (\forall j).$$

Proof. For $(x, y) \in X \times Y$ and $0 < \rho_i(y) < \infty$, define the function

$$\pi_F(x, y) = \sum_{i=1}^{N} \frac{1}{\rho_i(y)} \hat{F}_i(x^i, y).$$

Note that, because of Assumption 12, the function π_F is well defined. Then, for any i we have

$$\begin{aligned}
&\rho_i(y)\left[\pi_F(x^{*,i}, x^{*,-i}, y) - \pi_F(x^i, x^{*,-i}, y)\right] \\
&= \rho_i(y)\left[\sum_{k=1,k\neq i}^{N} \frac{1}{\rho_k(y)} \hat{F}_k(x^{*,k}, y) - \sum_{k=1,k\neq i}^{N} \frac{1}{\rho_k(y)} \hat{F}_k(x^{*,k}, y)\right] \\
&\quad + \rho_i(y)\left[\frac{1}{\rho_i(y)} \hat{F}_i(x^{*,i}, y) - \frac{1}{\rho_i(y)} \hat{F}_i(x^i, y)\right] \\
&= \rho_i(y)\left[\frac{1}{\rho_i(y)} \hat{F}_i(x^{*,i}, y) - \frac{1}{\rho_i(y)} \hat{F}_i(x^i, y)\right] \\
&= \hat{F}_i(x^{*,i}, y) - \hat{F}_i(x^i, y) \\
&= [\hat{F}_i(x^{*,i}, y) + \bar{F}_i(x^{*,-i}, y)] - [\hat{F}_i(x^i, y) + \bar{F}_i(x^{*,-i}, y)] \\
&= \breve{F}_i(x^{*,i}, x^{*,-i}, y) - \breve{F}_i(x^i, x^{*,-i}, y).
\end{aligned}$$

Implying that

$$\breve{F}_i(x^{*,i}, x^{*,-i}, y) - \breve{F}_i(x^i, x^{*,-i}, y) = \rho_i(y)\left[\pi_F(x^{*,i}, x^{*,-i}, y) - \pi_F(x^i, x^{*,-i}, y)\right], \ (\forall i).$$

Similarly, for $(x, y) \in X \times Y$ and $0 < \delta_i(x) < \infty$ by defining

$$\pi_f(x, y) = \sum_{j=1}^{M} \frac{1}{\delta_j(x)} \hat{f}_j(x, y^j)$$

we will have

$$\breve{f}_j(x, y^{*,j}, y^{*,-j}) - \breve{f}_j(x, y^j, y^{*,-j}) = \delta_j(x)\left[\pi_f(x, y^{*,j}, y^{*,-j}) - \pi_f(x, y^j, y^{*,-j})\right], \ (\forall j).$$

Hence, the conclusion of the Lemma follows. □

Define the following optimization problem,

$$\begin{cases} \min_x \mathbf{F}(x,y) = \left[\sum_{i=1}^{N} \frac{1}{\rho_i(y)} \hat{F}_i(x^i,y)\right] + \tilde{F}(x,y) \\ \text{subject to } \ (x^i,y) \in \mathcal{R}_i(x^{-i}) \ (\forall i). \end{cases} \tag{5.2.2}$$

Equivalence between optimal solution of (5.2.2) and equilibrium point of the Nash problem (5.1.2) can be established based on the following Theorem.

Theorem 5.5. *Suppose that problem* (5.1.2) *satisfies Assumption 12. If* (x^*,y) *is a global optimal solution of* (5.2.2)*, then* (x^*,y) *is a global Nash equilibrium point of* (5.1.2).

Proof. Let $(x^*,y) = (x^{*,i},x^{*,-i},y)$ be an optimal solution to (5.2.2), then $\forall (x^i,y) \in \mathcal{R}_i(x^{-i})$,

$$\sum_{i=1}^{N} \frac{1}{\rho_i(y)} \hat{F}_i(x^{*,i},y) + \tilde{F}(x^{*,i},x^{*,-i},y) \leq \sum_{i=1}^{N} \frac{1}{\rho_i(y)} \hat{F}_i(x^i,y) + \tilde{F}(x^i,x^{-i},y).$$

If we let $\pi_F(x,y) = \sum_{i=1}^{N} \frac{1}{\rho_i(y)} \hat{F}_i(x^i,y)$, it will follow that

$$\pi_F(x^{*,i},x^{*,-i},y) + \tilde{F}(x^{*,i},x^{*,-i},y) \leq \pi_F(x^i,x^{-i},y) + \tilde{F}(x^i,x^{-i},y), \ \forall (x^i,y) \in \mathcal{R}_i(x^{-i}).$$

Particularly for $x^{-i} = x^{*,-i}$ and $\forall (x^i,y) \in \mathcal{R}_i(x^{*,-i})$, we have

$$\pi_F(x^{*,i},x^{*,-i},y) + \tilde{F}(x^{*,i},x^{*,-i},y) \leq \pi_F(x^i,x^{*,-i},y) + \tilde{F}(x^i,x^{*,-i},y).$$

By rearranging the above expression, we have

$$\pi_F(x^{*,i},x^{*,-i},y) - \pi_F(x^i,x^{*,-i},y) + \tilde{F}(x^{*,i},x^{*,-i},y) \leq \tilde{F}(x^i,x^{*,-i},y).$$

Multiplying both sides of the last inequality by $\rho_i(y) > 0$, results in $\forall (x^i,y) \in \mathcal{R}_i(x^{*,-i})$,

$$\rho_i(y)\left[\pi_F(x^{*,i},x^{*,-i},y) - \pi_F(x^i,x^{*,-i},y)\right] + \rho_i(y)\tilde{F}(x^{*,i},x^{*,-i},y) \leq \rho_i(y)\tilde{F}(x^i,x^{*,-i},y).$$

Using the results from Lemma 5.4,

$$\rho_i(y)\left[\pi_F(x^{*,i},x^{*,-i},y) - \pi_F(x^i,x^{*,-i},y)\right] = \breve{F}_i(x^{*,i},x^{*,-i},y) - \breve{F}_i(x^i,x^{*,-i},y).$$

That means, in the above inequality, $\forall (x^i,y) \in \mathcal{R}_i(x^{*,-i})$ we have

$$\breve{F}_i(x^{*,i},x^{*,-i},y) - \breve{F}_i(x^i,x^{*,-i},y) + \rho_i(y)\tilde{F}(x^{*,i},x^{*,-i},y) \leq \rho_i(y)\tilde{F}(x^i,x^{*,-i},y).$$

Implying, $\forall (x^i,y) \in \mathcal{R}_i(x^{*,-i})$

$$\breve{F}_i(x^{*,i},x^{*,-i},y) + \rho_i(y)\tilde{F}(x^{*,i},x^{*,-i}y) \leq \breve{F}_i(x^i,x^{*,-i},y) + \rho_i(y)\tilde{F}(x^i,x^{*,-i},y).$$

This is equivalent to

$$F_i(x^{*,i}, x^{*,-i}, y) \leq F_i(x^i, x^{*,-i}, y), \ (\forall i)(x^i, y) \in \mathcal{R}_i(x^{*,-i}). \tag{5.2.3}$$

From inequality (5.2.3) we can see that the tuple (x^*, y), satisfies Definition 5.1. Therefore, (x^*, y) is a global Nash equilibrium point of (5.1.2). □

For a given strategy x, define the following optimization problem,

$$\begin{cases} \min\limits_{y} \mathbf{f}(x, y) = \left[\sum\limits_{j=1}^{M} \dfrac{1}{\delta_j(x)} \hat{f}_j(x, y^j)\right] + \tilde{f}(x, y) \\ \text{subject to } \ y^j \in \mathcal{R}_j(x, y^{-j}) \ (\forall j). \end{cases} \tag{5.2.4}$$

Then the relation between (5.2.4) and (5.1.3) can be established as follows.

Theorem 5.6. *Suppose that problem* (5.1.3) *satisfies Assumption 13. If y^* is an optimal solution of* (5.2.4), *then y^* is a Nash reaction point of* (5.1.3).

Proof. Let $y^* = (y^{*,j}, y^{*,-j})$ be an optimal solution for (5.2.4), then $\forall y^j \in \mathcal{R}_j(x, y^{-j})$

$$\sum_{j=1}^{M} \frac{1}{\delta_j(x)} \hat{f}_j(x, y^{*,j}) + \tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \sum_{j=1}^{M} \frac{1}{\delta_j(x)} \hat{f}_j(x, y^j) + \tilde{f}(x, y^j, y^{-j}).$$

If we let $\pi_f(x, y) = \sum_{j=1}^{M} \frac{1}{\delta_j(x)} \hat{f}_j(x, y^j)$, the above inequality becomes

$$\pi_f(x, y^{*,j}, y^{*,-j}) + \tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \pi_f(x, y^j, y^{-j}) + \tilde{f}(x, y^j, y^{-j}), \ \forall y^j \in \mathcal{R}_j(x, y^{-j}).$$

Particularly for $y^{-j} = y^{*,-j}$ and $\forall y^j \in \mathcal{R}_j(x, y^{*,-j})$, we have

$$\pi_f(x, y^{*,j}, y^{*,-j}) + \tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \pi_f(x, y^j, y^{*,-j}) + \tilde{f}(x, y^j, y^{*,-j}).$$

By rearranging the last inequality, we have an expression, $\forall y^j \in \mathcal{R}_j(x, y^{*,-j})$,

$$\pi_f(x, y^{*,j}, y^{*,-j}) - \pi_f(x, y^j, y^{*,-j}) + \tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \tilde{f}(x, y^j, y^{*,-j}) \tag{5.2.5}$$

Multiply both sides of (5.2.5) by $\delta_j(x) > 0$ to get,

$$\delta_j(x) \left[\pi_f(x, y^{*,j}, y^{*,-j}) - \pi_f(x, y^j, y^{*,-j})\right] + \delta_j(x)\tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \delta_j(x)\tilde{f}(x, y^j, y^{*,-j}).$$

Using the results from Lemma 5.4,

$$\delta_j(x) \left[\pi_f(x, y^{*,j}) - \pi_f(x, y^j)\right] = \breve{f}_i(x, y^{*,j}, y^{*,-j}) - \breve{f}_i(x, y^j, y^{*,-j});$$

and using this in the last inequality, $\forall y^j \in \mathcal{R}_j(x, y^{*,-j})$ we have

$$\breve{f}_j(x, y^{*,j}, y^{*,-j}) - \breve{f}_j(x, y^j, y^{*,-j}) + \delta_j(x)\tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \delta_j(x)\tilde{f}(x, y^j, y^{*,-j}).$$

Which implies, $\forall y^j \in \mathcal{R}_j(x, y^{*,-j})$

$$\breve{f}_j(x, y^{*,j}, y^{*,-j}) + \delta_j(x)\tilde{f}(x, y^{*,j}, y^{*,-j}) \leq \breve{f}_j(x, y^j, y^{*,-j}) + \delta_j(x)\tilde{f}(x, y^j, y^{*,-j}).$$

Or equivalently,

$$f_j(x, y^{*,j}, y^{*,-j}) \leq f_j(x, y^j, y^{*,-j}), \ \forall y^j \in \mathcal{R}_j(x, y^{*,-j}). \tag{5.2.6}$$

From inequality (5.2.6) we can see that the tuple y^*, satisfies Definition 5.2. Therefore, y^* is a global Nash reaction point of problem (5.1.3). □

Under the Assumptions 12, 13 and 14, an equivalent reformulation of bilevel MLMF game (5.1.1) as a bilevel game with single decision maker at both levels of the hierarchy is given by

$$\begin{cases} \min_{x \in X} \mathbf{F}(x, y) = \left[\sum_{i=1}^{N} \frac{1}{\rho_i(y)} \hat{F}_i(x^i, y)\right] + \tilde{F}(x, y) \\ \text{subject to} \begin{cases} G_i(x^i, y) \leq 0 \ (\forall i), \\ H(x, y) \leq 0, \ \text{and} \\ y \text{ solves} \begin{cases} \min_{y \in Y} \mathbf{f}(x, y) = \left[\sum_{j=1}^{M} \frac{1}{\delta_j(x)} \hat{f}_j(x, y^j)\right] + \tilde{f}(x, y) \\ \text{subject to} \begin{cases} g_j(x, y^j) \leq 0 \ (\forall j), \\ h(x, y) \leq 0. \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.2.7}$$

Proposition 5.7. *Suppose that* (5.1.1) *satisfies the Assumptions 12, 13 and 14. If* (x^*, y^*) *is a Stackelberg equilibrium point of* (5.2.7)*, then* (x^*, y^*) *is a Stackelberg-Nash equilibrium point of* (5.1.1).

Proof. Follows from Theorems 5.5 and 5.6. □

Note that when all the functions $\rho_i(y)$ and $\delta_j(x)$ are always equal to a constant 1, then problem (5.1.1) reduces to the quasi-potential game considered in [63]. Therefore, our result improves the one given in [63].

Remark 16. *The idea described above can be extended to any finite k-level multi-leader multi-follower game by reformulating the problems on the same level with a single objective as stated above. That is, any multilevel multi-leader multi-follower game with a property that every objective function in the problem consists of separable terms and non-separable terms (but each of the non-separable terms can be written as a factor of two functions one of* *the factor being common across all players of the same level) can be equivalently reformulated as a multilevel single-leader single-follower problem without increasing the vertical hierarchical levels. However the resulting multilevel optimization problem requires a solution approach that is different from the traditional KKT reformulation to avoid the effect of the complementarity conditions in the middle level problems.*

5.2.2 Equivalent hierarchical trilevel game formulation of a trilevel MLMF game

Consider a trilevel MLMF game involving N decision makers at the first-level, M decision makers at the second-level and L decision makers at the third-level which can be formulated mathematically as follows. For all $n \in \{1, \ldots, N\}$, (y_1, y_2, y_3) solves an optimization problem,

$$\begin{cases} \min\limits_{y_1^n \in Y_1^n} F_1^n(y_1, y_2, y_3) \\ \text{s.t.} \begin{cases} G_1^n(y_1^n, y_2, y_3) \leq 0, \\ H_1(y_1, y_2, y_3) \leq 0, \text{ and for all } i = 1, \ldots, M \\ (y_2^i, y_3) \text{ solves} \begin{cases} \min\limits_{y_2^i \in Y_2^i} f_2^i(y_1, y_2, y_3) \\ \text{s.t.} \begin{cases} g_2^i(y_1, y_2^i, y_3) \leq 0, \\ h_2(y_1, y_2, y_3) \leq 0, \text{ and for all } l = 1, \ldots, L \\ y_3^l \text{ solves} \begin{cases} \min\limits_{y_3^l \in Y_3^l} f_3^l(y_1, y_2, y_3) \\ \text{s.t.} \begin{cases} g_3^l(y_1, y_2, y_3^l) \leq 0, \\ h_3(y_1, y_2, y_3) \leq 0. \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.2.8}$$

Assume that each of the objective functions of the third and second level followers is convex with respect to its own decision variable vector and the Guignard constraint qualifications [49] hold for the followers constraints.

We make the following assumptions on the structure of the objective functions of (5.2.8):

Assumption 15. *The objective functions at the first-level can be written as*

$$F_1^n(y_1, y_2, y_3) = \hat{F}_1^n(y_1^n, y_2, y_3) + \bar{F}_1^n(y_1^{-n}, y_2, y_3) + \rho_1^n(y_2, y_3)\tilde{F}_1(y_1, y_2, y_3),$$

where $(\forall n)$ $(0 < \rho_1^n(y_2, y_3) < \infty)$.

Assumption 16. *The objective functions at the second-level can be written as*

$$f_2^i(y_1, y_2, y_3) = \hat{f}_2^i(y_1, y_2^i, y_3) + \bar{f}_2^i(y_1, y_2^{-i}, y_3) + \rho_2^i(y_1, y_3)\tilde{f}_2(y_1, y_2, y_3),$$

where $(\forall i)\ \ (0 < \rho_2^i(y_1, y_3) < \infty)$.

Assumption 17. *The objective functions at the third-level can be written as*

$$f_3^j(y_1, y_2, y_3) = \hat{f}_3^j(y_1, y_2, y_3^j) + \bar{f}_3^j(y_1, y_2, y_3^{-j}) + \rho_3^j(y_1, y_2)\tilde{f}_3(y_1, y_2, y_3),$$

where $(\forall j)\ \ (0 < \rho_3^j(y_1, y_2) < \infty)$.

Assumption 18. $\rho_1^n(\cdot)$, $\rho_2^i(\cdot)$ *and* $\rho_3^j(\cdot)$ *are twice continuously differentiable functions and uniformly bounded away from zero.*

If (5.2.8) satisfies the Assumptions 15, 16, 17 and 18, then it can be equivalently formulated as a trilevel optimization problem having a single decision maker at all levels as follows:

$$\begin{cases} \min\limits_{y_1} \left[\sum\limits_{n=1}^{N} \dfrac{\hat{F}_1^n(y_1^n, y_2, y_3)}{\rho_1^n(y_2, y_3)} \right] + \tilde{F}_1(y_1, y_2, y_3) \\ \text{s.t.} \begin{cases} G_1^n(y_1^n, y_2, y_3) \le 0 \ (\forall n), \\ H_1(y_1, y_2, y_3) \le 0, \text{ and} \\ (y_2, y_3) \text{ solves} \begin{cases} \min\limits_{y_2} \left[\sum\limits_{i=1}^{M} \dfrac{\hat{f}_2^i(y_1, y_2^i, y_3)}{\rho_2^i(y_1, y_3)} \right] + \tilde{f}_2(y_1, y_2, y_3) \\ \text{s.t.} \begin{cases} g_2^i(y_1, y_2^i, y_3) \le 0 \ (\forall i), \\ h_2(y_1, y_2, y_3) \le 0, \text{ and} \\ y_3 \text{ solves} \begin{cases} \min\limits_{y_3} \left[\sum\limits_{j=1}^{L} \dfrac{\hat{f}_3^j(y_1, y_2, y_3^j)}{\rho_3^j(y_1, y_2)} \right] + \tilde{f}_3(y_1, y_2, y_3) \\ \text{s.t.} \begin{cases} g_3^l(y_1, y_2, y_3^l) \le 0 \ (\forall l), \\ h_3(y_1, y_2, y_3) \le 0. \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.2.9}$$

Then the following statement is a direct consequence of Proposition 5.7. Hence we state it here without proof.

Proposition 5.8. *Suppose that* (5.2.8) *satisfies the Assumptions 15, 16, 17 and 18. If* (y_1^*, y_2^*, y_3^*) *is a Stackelberg equilibrium point of* (5.2.9)*, then* (y_1^*, y_2^*, y_3^*) *is a Stackelberg-Nash equilibrium point of* (5.2.8).

Note that the approach by Kulkarni and Shanbhag [63] leads to an MPEC for two level problems whose global solution provides an equilibrium to a bilevel MLMF game. When we have more than two hierarchical levels, the non-convex expression resulted from the lower-level problems makes it difficult to solve. For example, in trilevel programs the complementarity condition of the third-level KKT conditions makes the second-level problem non-convex. Therefore, using the same transformation will result in a problem which is very difficult to solve (if it is tractable at all). To avoid this situation, we apply multi-parametric procedures for transforming the lower level problems. In this procedure, instead of embedding lower level problems into the middle level through the KKT conditions, we will equivalently transform the trilevel problem into a single-level problem by sequentially substituting the parametric solutions in problems of the middle and upper levels.

5.2.3 Equivalent hierarchical multilevel formulation of a multilevel MLMF game

Consider a class of multilevel MLMF problem (1.2.6) with the following assumptions on the structure of the objective functions:

Assumption 19. *The objective functions at the first-level can be written as*

$$F_1^n(y) = \hat{F}_1^n(y_1^n, y_2, \ldots, y_k) + \bar{F}_1^n(y_1^{-n}, y_2, \ldots, y_k) + \rho_1^n(y_2, \ldots, y_k)\tilde{F}_1(y),$$

where $(\forall n)\ \ (0 < \rho_1^n(y_2, \ldots, y_k) < \infty)$.

Assumption 20. *The objective functions at the second-level can be written as*

$$f_2^i(y) = \hat{f}_2^i(y_1, y_2^i, \ldots, y_k) + \bar{f}_2^i(y_1, y_2^{-i}, \ldots, y_k) + \rho_2^i(y_1, y_3, \ldots, y_k)\tilde{f}_2(y),$$

where $(\forall i)\ \ (0 < \rho_2^i(y_1, y_3, \ldots, y_k) < \infty)$.

Assumption 21. *The objective functions at the* k^{th}*-level can be written as*

$$f_k^l(y) = \hat{f}_k^l(y_1, y_2, \ldots, y_k^l) + \bar{f}_k^l(y_1, y_2, \ldots, y_k^{-l}) + \rho_3^j(y_1, y_2, \ldots, y_{k-l})\tilde{f}_k(y),$$

where $(\forall l)\ \ (0 < \rho_k^l(y_1, y_2, \ldots, y_{k-l}) < \infty)$.

Assumption 22. $\rho_1^n(\cdot)$, $\rho_2^i(\cdot)$ *and* $\rho_k^l(\cdot)$ *are twice continuously differentiable functions and uniformly bounded away from zero.*

Problem (1.2.6) satisfying the Assumptions 19, 20, 21 and 22 can be equivalently reformulated as a hierarchical multilevel programming problem having single combined objective at each decision level:

$$\begin{cases} \min\limits_{y_1} \left\{ \left[\sum_{n=1}^{N} \frac{1}{\rho_1^n(\cdot)} \hat{F}_1^n(y_1^n, y_2, y_3, \ldots, y_k) \right] + \tilde{F}_1(y_1, y_2, y_3, \ldots, y_k) \right\} \\ s.t. \begin{cases} G_1^n(y_1^n, y_2, y_3, \ldots, y_k) \leq 0, \; n = 1, \ldots, N, \\ H_1(y_1, y_2, y_3, \ldots, y_k) \leq 0, \text{ and} \\ \min\limits_{y_2} \left\{ \left[\sum_{i=1}^{I} \frac{1}{\rho_2^i(\cdot)} \hat{f}_2^i(y_1, y_2^i, y_3, \ldots, y_k) \right] + \tilde{f}_2^i(y_1, y_2, y_3, \ldots, y_k) \right\} \\ s.t. \begin{cases} g_2^i(y_1, y_2^i, y_3, \ldots, y_k) \leq 0, \; i = 1, \ldots, I, \\ h_2(y_1, y_2, y_3, \ldots, y_k) \leq 0, \text{ and} \\ \qquad \ddots \\ \min\limits_{y_k} \left\{ \left[\sum_{l=1}^{L} \frac{1}{\rho_k^l(\cdot)} \hat{f}_k^l(y_1, y_2, y_3, \ldots, y_k^l) \right] + \tilde{f}_k^l(y_1, y_2, y_3, \ldots, y_k) \right\} \\ s.t. \begin{cases} g_k^l(y_1, y_2, y_3, \ldots, y_k^l) \leq 0, \; l = 1, \ldots, L, \\ h_k(y_1, y_2, y_3, \ldots, y_k) \leq 0. \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.2.10}$$

Proposition 5.9. *Suppose that problem* (5.2.10) *satisfies Assumptions 19, 20, 21 and 22. If* $(y_1^*, y_2^*, \ldots, y_k^*)$ *is a Stackelberg equilibrium point of* (5.2.10)*, then* $(y_1^*, y_2^*, \ldots, y_k^*)$ *is a Stackelberg-Nash equilibrium point of* (1.2.6).

The reformulation presented above generalizes the reformulation proposed by Kassa and Kassa [61] in which they considered a multilevel single-leader multiple-follower game involving a single decision maker in the upper-level and in the case when the objective function of all decision maker consists of a separable and non-separable term weighted by constant terms.

5.3 Multi-parametric solution method for special classes of multilevel MLMF games

In this section we suggest a pseudo algorithmic approach to solve two classes of hierarchical multilevel MLMF games; namely,

(1) a hierarchical multilevel MLMF weighted potential game whose objective values at all levels of the decision hierarchy have common, but having different positive weights of, non-separable terms and with the constraints at each level

are polyhedral. (i.e., class of a hierarchical multilevel MLMF games satisfying Assumptions 10 and 11).

(2) a hierarchical multilevel MLMF game whose objective functions at each level have non-separable terms. The non-separable terms assumed to be written as a factor of two functions, a function which depends on other level decision variables and a function which is common to all objectives across the same level (i.e., class of a hierarchical multilevel MLMF games satisfying Assumptions 19, 20 and 21).

The algorithm starts by transforming hierarchical multilevel multi-leader multi-follower games of the forms (1) and (2) above into multilevel hierarchical games involving a single decision maker over the hierarchy as discussed in Section 5.1 and Section 5.2, respectively. The resulting problem can be solved by the existing solution algorithms proposed for multilevel hierarchical problems. Particularly, in this chapter we have assumed that the all constraints are polyhedral and we have used multi-parametric solution algorithms proposed in [36, 37] (for convex case) and [60] (for non-convex case). The multi-parametric solution method recasts each of the optimization problems in lower levels as a multi-parametric programming problem where the variables from upper level problems are considered as parameters. Then, obtain an analytical parametric solution for the rational reaction set for each of the sub problems in the corresponding critical (stability) region of the parameter space. The basic steps of the proposed algorithm are described in the table below.

Remark 17. *Since the number of partitions of the critical regions are finite as shown in [32, 59], the algorithmic procedures described in Steps 2 and 3 terminate after a finite number of iterations. Hence, the above algorithm requires only a finite number of iterative procedures to arrive at the required solution.*

To apply the proposed solution algorithm, it is to be noted that each of the transformed problems at the lower levels must satisfy the required four Assumptions 1, 2, 3 and 4. These conditions seem to be very strong and restrictive. However, many practical application problems satisfy these conditions.

For example, the oligopoly market problem with divisible homogeneous products, that is described and analyzed in [57], satisfies all of the assumptions in our reformulation in Section 5.2. Therefore, it can be considered as a particular example of our proposed formulation.

Table 5.1: Algorithm to solve multilevel MLMF problems

ALGORITHM 3

STEP 1: Reformulate a given hierarchical k-level MLMF problem as a k-level hierarchical problem as discussed in Section 5.1 and 5.2

INITIALIZE: $i = k : -1 : 1$ and GO TO **Step 2**

STEP 2: The i^{th}-level problem of the problem resulting from **Step 2** is treated as a multi-parametric problem with y_i being the optimization variable and the rest of variables other than i^{th}-level decision variable (*i.e.* $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k)$ the parameters; and solved by multi-parametric approach proposed in [36, 37] for convex case and branch-and-bound multi-parametric approach proposed in [60] for non-convex case. GO TO **Step 3**

STEP 3: Substitute the parametric solution from STEP 2 in the $(i-1)^{th}$-level problem of the given hierarchical k-level MLMF problem.

If $i - 1 = 1$ GO TO **Step 4**; otherwise, put $i = i - 1$ and GO TO **Step 2**.

STEP 4: Use an existing standard nonlinear optimization algorithms to solve the single-level optimization problem with decision variable y_1, resulting from **Step 3**.

TERMINATE

The second example is the supply chain management problem. Supply chains are systems with multiple components such as supplier, manufacturer, distributer, retailer and customer, that exchange information with one another [19]. Mathematical models formulated to analyse the overall economic process of the supply chain management are usually described by MLMF games with at least three hierarchical levels (see for instance [51], where all the involved functions are assumed to be linear). Most of the deterministic versions of such problems satisfy the conditions required in our model formulation and solution algorithm.

The method proposed in Section 5.1 will be illustrated in the three numerical examples (**Example 5.1**, **5.2** and **5.3**) provided below.

5.4 Illustrative examples

Example 5.1. *Consider the following bilevel MLMF programming problem:*

$$\begin{cases} \min_{x_1} F_1(x_1,x_2,y_1,y_2) = \frac{1}{2}x_1 - y_1 \\ \min_{x_2} F_2(x_1,x_2,y_1,y_2) = -\frac{1}{2}x_2 - y_2 \end{cases}$$
$$s.t. \begin{cases} \begin{cases} \min_{y_1} f_1(x_1,x_2,y_1,y_2) = y_1(-1+x_1+x_2) + \frac{1}{2}y_1^2 \\ \min_{y_2} f_2(x_1,x_2,y_1,y_2) = y_2(-1+x_1+x_2) + \frac{1}{2}y_2^2 \end{cases} \\ s.t. \begin{cases} 0 \le x_1,\ x_2 \le 1, \\ y_1 \ge 0,\ y_2 \ge 0. \end{cases} \end{cases} \tag{5.4.1}$$

An equivalent trilevel single-leader multi-follower problem for (5.4.1) *is given by:*

$$\min_z \alpha$$
$$s.t. \begin{cases} z = (x,y), \\ \begin{cases} \min_{x_1} F_1(x_1,x_2,y_1,y_2) = \frac{1}{2}x_1 - y_1 \\ \min_{x_2} F_2(x_1,x_2,y_1,y_2) = -\frac{1}{2}x_2 - y_2 \end{cases} \\ s.t. \begin{cases} \begin{cases} \min_{y_1} f_1(x_1,x_2,y_1,y_2) = y_1(-1+x_1+x_2) + \frac{1}{2}y_1^2 \\ \min_{y_2} f_2(x_1,x_2,y_1,y_2) = y_2(-1+x_1+x_2) + \frac{1}{2}y_2^2 \end{cases} \\ s.t. \begin{cases} 0 \le x_1,\ x_2 \le 1, \\ y_1 \ge 0,\ y_2 \ge 0. \end{cases} \end{cases} \end{cases} \tag{5.4.2}$$

Then (5.4.2) *is transformed into the following trilevel hierarchical problem:*

$$\min_z \alpha$$
$$s.t. \begin{cases} z = (x,y), \\ \min_x F(x,y) = \frac{1}{2}x_1 - \frac{1}{2}x_2 - y_1 - y_2 \\ s.t. \begin{cases} 0 \le x_1,\ x_2 \le 1, \\ \min_y f(x,y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + y_1(-1+x_1+x_2) + y_2(-1+x_1+x_2) \\ s.t. \begin{cases} 0 \le x_1,\ x_2 \le 1, \\ y_1 \ge 0,\ y_2 \ge 0. \end{cases} \end{cases} \end{cases} \tag{5.4.3}$$

Then the third level problem in (5.4.3) *can be considered as a MPP problem with parameter* $x = (x_1, x_2)$*:*

$$\min_{y} f(x,y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + y_1(-1+x_1+x_2) + y_2(-1+x_1+x_2)$$
$$s.t. \begin{cases} 0 \leq x_1,\ x_2 \leq 1, \\ y_1 \geq 0,\ y_2 \geq 0. \end{cases} \tag{5.4.4}$$

The Lagrangian of the problem is given by, $\mathcal{L}(x,y,\lambda) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + y_1(-1+x_1+x_2) + y_2(-1+x_1+x_2)$ *and the KKT points are given by*

$$\begin{cases} y_1\dfrac{\partial \mathcal{L}}{\partial y_1} = y_1(-1+x_1+x_2) = 0,\ \dfrac{\partial \mathcal{L}}{\partial y_1} = -1+x_1+x_2 \geq 0,\ y_1 \geq 0, \\ y_2\dfrac{\partial \mathcal{L}}{\partial y_2} = y_2(-1+x_1+x_2) = 0,\ \dfrac{\partial \mathcal{L}}{\partial y_2} = -1+x_1+x_2 \geq 0,\ y_2 \geq 0. \end{cases}$$

Therefore, the parametric solution with the corresponding critical regions are given by:

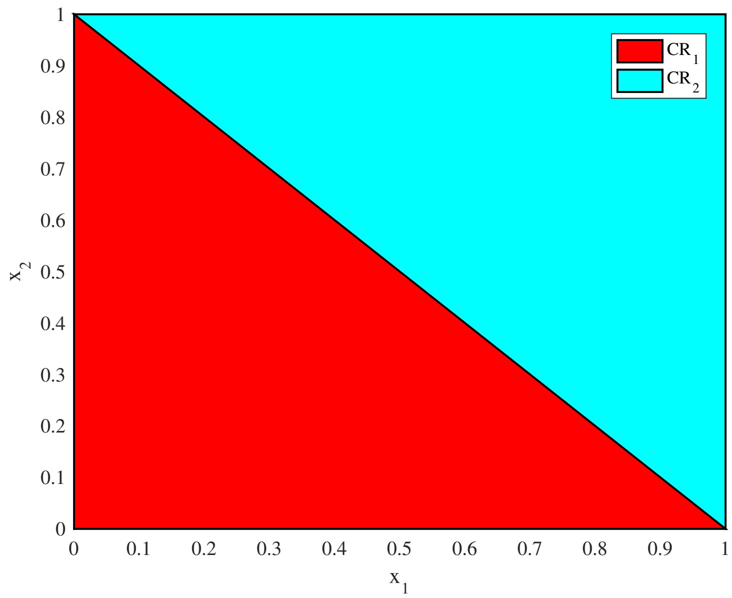

Figure 5.1: Critical regions for the second level problem of (5.4.3)

$$\mathcal{CR}_1 = \begin{cases} y^*(x) = \begin{bmatrix} 1-x_1-x_2 \\ 1-x_1-x_2 \end{bmatrix}, \\ x_1+x_2 \leq 1, \\ 0 \leq x_1, x_2 \leq 1 \end{cases} \quad \text{and} \quad \mathcal{CR}_2 = \begin{cases} y^*(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ x_1+x_2 \geq 1, \\ 0 \leq x_1, x_2 \leq 1 \end{cases}$$

which can be incorporated into the second level followers problem of (5.4.3) *and after solving the resulting problems in each critical region we have the following solutions: In* $\mathcal{CR}_1$*,* *the optimal solution is* $(x_1, x_2, y_1, y_2) = (0, 0, 1, 1)$ *with the corresponding second level follower problem objective value* $F = -2$. *In* $\mathcal{CR}_2$, *the optimal solution is* $(x_1, x_2, y_1, y_2) = (0, 1, 0, 0)$ *with the corresponding second level follower problem objective value* $F = 0$.

Since the objective value obtained in $\mathcal{CR}_1$ *is better we can take* $(x_1, x_2, y_1, y_2) = (0, 0, 1, 1)$ *as an optimal solution to the second level followers problem of* (5.4.3). *Therefore, the optimal solution to the bilevel MLMF programming problem* (5.4.1) *is* $(x_1, x_2, y_1, y_2) = (0, 0, 1, 1)$ *with the corresponding objective values* $F_1 = -1,\ F_2 = -1,\ f_2 = -0.5$ *and* $f_2 = -0.5$. □

Example 5.2. *Consider the following nonlinear bilevel MLMF programming problem:*

$$\begin{cases} \min\limits_{x_1} F_1(x,y) = x_1^2 - x_1x_2 - x_1 + x_1y_1 \\ \min\limits_{x_2} F_2(x,y) = x_2^2 - \frac{1}{2}x_1x_2 - 2x_2 + y_2 \end{cases}$$
$$s.t. \begin{cases} x_1 + x_2 \leq 1.5, \\ \begin{cases} \min\limits_{y_1} f_1(x,y) = \frac{1}{2}y_1^2 + y_1y_2 + y_1 - x_1y_1 \\ \min\limits_{y_2} f_2(x,y) = \frac{1}{2}y_2^2 + y_1y_2 + y_2 - x_2y_2 \end{cases} \\ s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \leq 3,\ y_1 + y_2 \leq 2.5, \\ 0 \leq x_1, x_2 \leq 1,\ 0 \leq y_1 \leq 2,\ 0 \leq y_2 \leq 1. \end{cases} \end{cases} \tag{5.4.5}$$

An equivalent trilevel single-leader multi-follower problem for (5.4.5) *is given by:*

$$\min_{z} \alpha$$
$$s.t. \begin{cases} z = (x,y), \\ \begin{cases} \min\limits_{x_1} F_1(x,y) = x_1^2 - x_1x_2 - x_1 + x_1y_1 \\ \min\limits_{x_2} F_2(x,y) = x_2^2 - \frac{1}{2}x_1x_2 - 2x_2 + y_2 \end{cases} \\ s.t. \begin{cases} x_1 + x_2 \leq 1.5, \\ \begin{cases} \min\limits_{y_1} f_1(x,y) = \frac{1}{2}y_1^2 + y_1y_2 + y_1 - x_1y_1 \\ \min\limits_{y_2} f_2(x,y) = \frac{1}{2}y_2^2 + y_1y_2 + y_2 - x_2y_2 \end{cases} \\ s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \leq 3,\ y_1 + y_2 \leq 2.5, \\ 0 \leq x_1, x_2 \leq 1,\ 0 \leq y_1 \leq 2,\ 0 \leq y_2 \leq 1. \end{cases} \end{cases} \end{cases} \tag{5.4.6}$$

Then (5.4.6) *is transformed into the following trilevel hierarchical problem:*

$$\min_{z} \alpha$$
$$s.t. \begin{cases} z = (x, y), \\ \min_{x} F(x, y) = x_1^2 - x_1 + x_1 y_1 + 2x_2^2 - 4x_2 - x_1 x_2 \\ s.t. \begin{cases} x_1 + x_2 \leq 1.5, \\ \min_{y} f(x, y) = \frac{1}{2} y_1^2 + y_1 - x_1 y_1 + \frac{1}{2} y_2^2 + y_2 - x_2 y_2 + y_1 y_2 \\ s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \leq 3, \; y_1 + y_2 \leq 2.5, \\ 0 \leq x_1, x_2 \leq 1, \; 0 \leq y_1 \leq 2, \; 0 \leq y_2 \leq 1. \end{cases} \end{cases} \end{cases} \tag{5.4.7}$$

Then the third level problem in (5.4.7) *can be considered as a MPP problem with parameter* $x = (x_1, x_2)$*:*

$$\min_{y} f(x, y) = \frac{1}{2} y_1^2 + y_1 - x_1 y_1 + \frac{1}{2} y_2^2 + y_2 - x_2 y_2 + y_1 y_2$$
$$s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \leq 3, \; y_1 + y_2 \leq 2.5, \\ 0 \leq x_1, x_2 \leq 1, \; 0 \leq y_1 \leq 2, \; 0 \leq y_2 \leq 1. \end{cases} \tag{5.4.8}$$

Problem (5.4.8) *have a bilinear term* $b_{12} y_1 y_2 = y_1 y_2$*, a concave function* $c(y) = 0$*,* $h_1(x) = -x_1 y_1 - x_2 y_2$ *and* $h_2(y) = y_1 + y_2$ *at the objective function. This can result in multiple Nash equilibrium reaction for at least one feasible choice of the leader's problem. So we should apply a mathematical procedure described in [60]. The convex envelope of the bilinear terms* $y_1 y_2$*, denoted by* $VexR[b_{12} y_1 y_2]$*, taken over the rectangle*

$$R = \{(y_1, y_2) : 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1\}$$

is be obtained as follows: $b_{12} = 1 > 0 \Rightarrow l_{12}^1(y_1, y_2) = 0, \; l_{12}^2(y_1, y_2) = y_1 + 2y_2 - 2,$

$VexR[b_{12} y_1 y_2] = \max\{0, y_1 + 2y_2 - 2\} = y_1 + 2y_2 - 2.$

Therefore the under-estimator of the objective function in (5.4.8) *is equal to:*

$$\begin{aligned} &\frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + VexR[b_{12} y_1 y_2] + Vex_c + h_1(x) + h_2(y) \\ &\quad = \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + 2y_1 + 3y_2 - x_1 y_1 - x_2 y_2 - 2. \end{aligned}$$

Thus, the under-estimator problem of (5.4.8) *is formulated as:*

$$\min_{y} f(x, y) = \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + 2y_1 + 3y_2 - x_1 y_1 - x_2 y_2 - 2$$
$$s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \leq 3, \; y_1 + y_2 \leq 2.5, \\ 0 \leq x_1, x_2 \leq 1, \; 0 \leq y_1 \leq 2, \; 0 \leq y_2 \leq 1. \end{cases} \tag{5.4.9}$$

The Lagrangian of the problem is given by, $\mathcal{L}(x, y, \lambda) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + 2y_1 + 3y_2 - x_1y_1 - x_2y_2 - 2 + \lambda_1(2y_1 + y_2 + x_1 - 2x_2 - 3) + \lambda_2(y_1 + y_2 - 2.5)$.
Apply the multi-parametric programming approach to solve (5.4.9),
$\begin{pmatrix} y(x) \\ \lambda(x) \end{pmatrix} = \begin{pmatrix} y_0 \\ \lambda_0 \end{pmatrix} - M_0^{-1} \cdot N_0 \cdot (x - x_0)$, *where* $(y_0, \lambda_0) = [y(x_0), \lambda(x_0)]$, $M_0 = M(x_0)$ *and* $N_0 = N(x_0)$, *we have got the following parametric solutions,*
$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} x_1 - 0.5 \\ x_2 - 0.5 \end{bmatrix}$ *in the critical region* $\mathcal{CR} = \mathcal{CR}_I = \{0 \leq x_1, x_2 \leq 1\}$.
Incorporating the solution into the second level followers problem of (5.4.7) *and we obtain:*

$$\begin{aligned} &\min_{x} F(x, y) = 2x_1^2 + 2x_2^2 - x_1x_2 - 1.5x_1 - 4x_2 \\ &s.t. \begin{cases} x_1 + x_2 \leq 1.5, \\ 0 \leq x_1, x_2 \leq 1. \end{cases} \end{aligned} \tag{5.4.10}$$

Solving (5.4.10) *we obtain the solution* $(x_1, x_2) = (0.5, 1)$. *Then it is incorporated into the leader's problem of* (5.4.7) *and solved to obtain the solution* $z = (x_1, x_2, y_1, y_2) = (0.5, 1, 0, 0.5)$. *Therefore, the optimal solution to the bilevel multi-leader multi-follower programming problem* (5.4.5) *is* $(x_1, x_2, y_1, y_2) = (0.5, 1, 0, 0.5)$ *with the corresponding objective values* $F_1 = -0.75,\ F_2 = -0.75,\ f_1 = 0$ *and* $f_2 = 0.125$. □

Example 5.3. *Consider the following trilevel MLMF programming problem:*

$$\begin{cases} \min\limits_{x_1} F_1(x, y, z) = x_1^2 + y_1x_2 + 2z_1 \\ \min\limits_{x_2} F_2(x, y, z) = e^{x_2} - 3x_1y_2^2 - z_2 \end{cases}$$
$$s.t. \begin{cases} \begin{cases} \min\limits_{y_1} f_1^2(x, y, z) = (y_1 - x_1)^2 + z_1^2 \\ \min\limits_{y_2} f_2^2(x, y, z) = x_2^2 + (y_2 - 2)^2 + z_2 \end{cases} \\ s.t. \begin{cases} \begin{cases} \min\limits_{z_1} f_1^3(x, y, z) = x_1 - y_1z_2 + 2z_1^2 \\ \min\limits_{z_2} f_2^3(x, y, z) = x_2 + y_2z_1 + 3z_2^2 \end{cases} \\ s.t. \begin{cases} 2x_1 + 2y_1 + 3z_1 \geq 6,\ x_2 + y_2 + z_2 \geq 1, \\ 2x_1 + x_2 + 3y_1 + z_1 \geq 3,\ x_1 + x_2 + y_2 + z_2 \geq 1, \\ x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\ 0 \leq x_1,\ x_2 \leq 1,\ 0 \leq y_1,\ y_2 \leq 1,\ 0 \leq z_1,\ z_2 \leq 3. \end{cases} \end{cases} \end{cases} \tag{5.4.11}$$

An equivalent four-level single-leader multi-follower problem for (5.4.11) *is given by:*

$$
\begin{aligned}
&\min_{u} \alpha \\
&s.t. \begin{cases}
u = (x, y, z) \\
\begin{cases}
\min\limits_{x_1} F_1(x, y, z) = x_1^2 + y_1 x_2 + 2z_1 \\
\min\limits_{x_2} F_2(x, y, z) = e^{x_2} - 3x_1 y_2^2 - z_2
\end{cases} \\
s.t. \begin{cases}
\begin{cases}
\min\limits_{y_1} f_1^2(x, y, z) = (y_1 - x_1)^2 + z_1^2 \\
\min\limits_{y_2} f_2^2(x, y, z) = x_2^2 + (y_2 - 2)^2 + z_2
\end{cases} \\
s.t. \begin{cases}
\begin{cases}
\min\limits_{z_1} f_1^3(x, y, z) = x_1 - y_1 z_2 + 2z_1^2 \\
\min\limits_{z_2} f_2^3(x, y, z) = x_2 + y_2 z_1 + 3z_2^2
\end{cases} \\
s.t. \begin{cases}
2x_1 + 2y_1 + 3z_1 \geq 6,\ x_2 + y_2 + z_2 \geq 1, \\
2x_1 + x_2 + 3y_1 + z_1 \geq 3,\ x_1 + x_2 + y_2 + z_2 \geq 1, \\
x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\
0 \leq x_1,\ x_2 \leq 1,\ 0 \leq y_1,\ y_2 \leq 1,\ 0 \leq z_1,\ z_2 \leq 3.
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{aligned}
\tag{5.4.12}
$$

Then (5.4.12) *is transformed into the following trilevel programming problem:*

$$
\begin{aligned}
&\min_{u} \alpha \\
&s.t. \begin{cases}
u = (x, y, z) \\
\min\limits_{x} F(x, y, z) = x_1^2 + e^{x_2} + x_2 y_1 - 3x_1 y_2^2 \\
s.t. \begin{cases}
\min\limits_{y} f^2(x, y, z) = (y_1 - x_1)^2 + (y_2 - 2)^2 \\
s.t. \begin{cases}
\min\limits_{z} f^3(x, y, z) = 2z_1^2 + 3z_2^2 - y_1 z_2 + y_2 z_1 \\
s.t. \begin{cases}
2x_1 + 2y_1 + 3z_1 \geq 6,\ x_2 + y_2 + z_2 \geq 1, \\
2x_1 + x_2 + 3y_1 + z_1 \geq 3,\ x_1 + x_2 + y_2 + z_2 \geq 1, \\
x_1 + 5x_2 + y_1 + y_2 + z_1 + 2z_2 \geq 4, \\
0 \leq x_1,\ x_2 \leq 1,\ 0 \leq y_1,\ y_2 \leq 1,\ 0 \leq z_1,\ z_2 \leq 3.
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\end{aligned}
\tag{5.4.13}
$$

Then the fourth level problem in (5.4.13) *can be considered as a MPP problem with param-*

eter $(x,y)=(x_1,x_2,y_1,y_2)$*:*

$$\min_{z} f^3(x,y,z)=2z_1^2+3z_2^2-y_1z_2+y_2z_1$$

$$s.t. \begin{cases} 2x_1+2y_1+3z_1\geq 6,\ x_2+y_2+z_2\geq 1, \\ 2x_1+x_2+3y_1+z_1\geq 3,\ x_1+x_2+y_2+z_2\geq 1, \\ x_1+5x_2+y_1+y_2+z_1+2z_2\geq 4, \\ 0\leq x_1,\ x_2\leq 1,\ 0\leq y_1,\ y_2\leq 1,\ 0\leq z_1,\ z_2\leq 3. \end{cases} \tag{5.4.14}$$

Applying multi-parametric programming approach to solve (5.4.14).
$\begin{pmatrix} z(x,y) \\ \lambda(x,y) \end{pmatrix} = \begin{pmatrix} z_0 \\ \lambda_0 \end{pmatrix} - M_0^{-1}\cdot N_0 \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$*, where* $\begin{pmatrix} z_0 \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} z(x_0,y_0) \\ \lambda(x_0,y_0) \end{pmatrix}$, $M_0=M(x_0,y_0)$ *and* $N_0=N(x_0,y_0)$ *, we have got the following parametric solutions,*

$$\begin{bmatrix} z_1(x,y) \\ z_2(x,y) \end{bmatrix} = \begin{bmatrix} 2-0.6667x_1-0.6667y_1 \\ 0.1667y_1 \end{bmatrix} \textit{in } \mathcal{CR}_1 \textit{ where}$$

$$\mathcal{CR}_1=\begin{cases} -0.7023x_2-0.1170y_1-0.7023y_2\leq -0.7023 \\ -0.4650x_1-0.3487x_2-0.8137y_1\leq -0.3487 \\ -0.0647x_1-0.9703x_2-0.1294y_1+y_2\leq -0.3881 \\ 0\leq x_1\leq 1,\ \ x_2\leq 1 \\ 0\leq y_1,y_2\leq 1 \end{cases}$$

$$\begin{bmatrix} z_1(x,y) \\ z_2(x,y) \end{bmatrix} = \begin{bmatrix} 2-0.6667x_1-0.6667y_1 \\ 1-y_1-y_2 \end{bmatrix} \textit{in } \mathcal{CR}_2 \textit{ where}$$

$$\mathcal{CR}_2=\begin{cases} 0.7023x_2+0.1170y_1+0.7023y_2\leq 0.7023 \\ -0.4650x_1-0.3487x_2-0.8137y_1\leq -0.3487 \\ -0.1043x_1-0.9383x_2-0.1043y_1+0.3128y_2\leq 0 \\ 0\leq x_1\leq 1,\ \ x_2\geq 0 \\ 0\leq y_1\leq 1,\ \ y_2\geq 0 \end{cases}$$

$$\begin{bmatrix} z_1(x,y) \\ z_2(x,y) \end{bmatrix} = \begin{bmatrix} 3-2x_1-x_2-3y_1 \\ 1-x_2-y_2 \end{bmatrix} \textit{in } \mathcal{CR}_3 \textit{ where}$$

$$\mathcal{CR}_3=\begin{cases} 0.7023x_2+0.1170y_1+0.7023y_2\leq 0.7023 \\ 0.4650x_1+0.3487x_2+0.8137y_1\leq 0.3487 \\ 0.3162x_1-0.6325x_2+0.6325y_1+0.3162y_1\leq 0.3162 \\ x_1\geq 0,\ \ x_2\geq 0 \\ y_1\geq 0,\ \ y_2\geq 0 \end{cases}$$

Corresponding to the first critical set of fourth level-problem, we have got the following

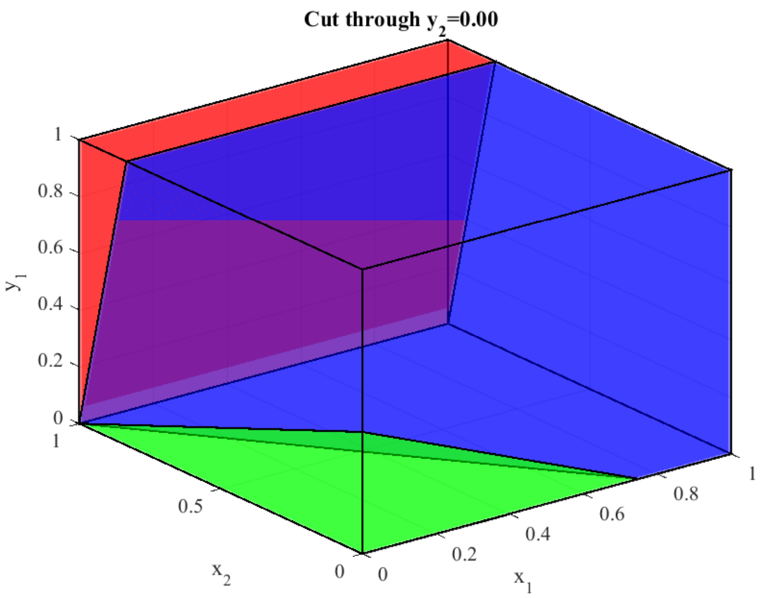

Figure 5.2: Critical regions for the problem (5.4.14)

parametric solutions with parameter $x = (x_1, x_2)$ *to the third-level followers problem of* (5.4.13),

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \mathcal{CR}_{1-1} = \begin{cases} -0.9648x_1 - 0.2631x_2 \leq -0.2631 \\ -0.1962x_1 - 0.9806x_2 \leq -0.1961 \\ x_1 \leq 1,\ 0 \leq x_2 \leq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.5714x_1 - 0.4285x_2 + 0.4285 \\ -0.0001x_2 + 1 \end{bmatrix}, \mathcal{CR}_{1-2} = \begin{cases} 0.0101x_1 - 0.9999x_2 \leq -0.1514 \\ 0.9648x_1 + 0.2613x_2 \leq 0.2631 \\ x_1 \geq 0 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.5385x_1 - 2.3072x_2 + 0.6686 \\ -0.6923x_1 - 3.4608x_2 + 1.5538 \end{bmatrix}, \mathcal{CR}_{1-3} = \begin{cases} 0.1962x_1 + 0.9806x_2 \leq 0.1961 \\ -0.9648x_1 - 0.2613x_2 \leq -0.2631 \\ x_2 \geq 0 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.5385x_1 - 2.3072x_2 + 0.7724 \\ -0.6923x_1 - 3.4608x_2 + 1.4845 \end{bmatrix}, \mathcal{CR}_{1-4} = \begin{cases} -0.0101x_1 + 0.9999x_2 \leq 0.1514 \\ 0.9648x_1 + 0.2613x_2 \leq 0.2631 \\ x_1 \geq 0 \end{cases}$$

Corresponding to the second critical set of fourth level-problem, we have got the following parametric solutions with parameter $x = (x_1, x_2)$ *to the third-level followers problem of* (5.4.13),

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.9730x_1 - 0.1621x_2 - 0.1621 \\ -0.1621x_1 - 0.9730x_2 + 1.0270 \end{bmatrix}, \mathcal{CR}_{2-1} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3769 \\ -0.2048x_1 - 0.9788x_2 \leq -0.2700 \\ x_1 \leq 1,\ x_2 \leq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.5715x_1 - 0.4285x_2 + 0.4285 \\ 0.0952x_1 - 0.9286x_2 + 0.9286 \end{bmatrix}, \mathcal{CR}_{2-2} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ -0.0126x_1 - 0.9999x_2 \leq -0.2076 \\ 0 \leq x_1,\ x_2 \leq 1 \end{cases}$$

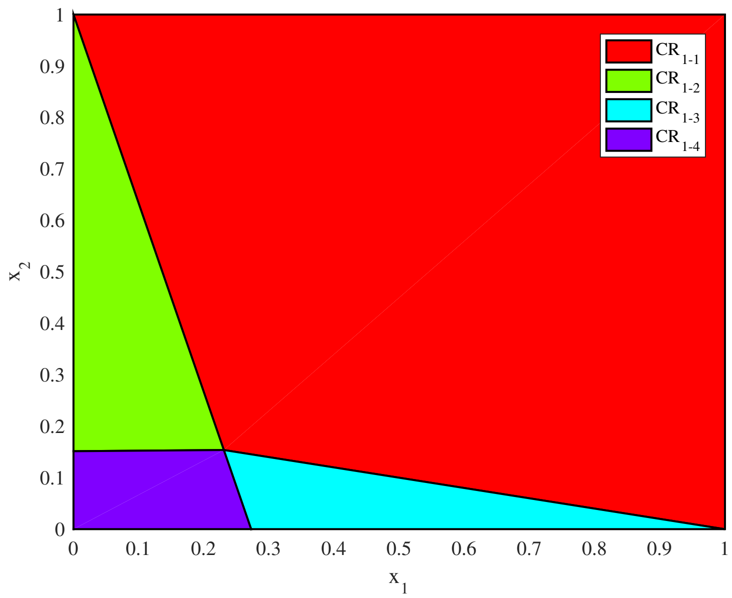

Figure 5.3: Critical regions for the third-level problem of (5.4.13) that corresponds to $\mathcal{CR}_1$ (red region in Fig.5.2)

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.7999x_1 - 0.9002x_2 + 0.2001 \\ 0.6001x_1 + 2.6995x_2 + 0.0667 \end{bmatrix}, \mathcal{CR}_{2-3} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3769 \\ 0.2023x_1 + 0.9793x_2 \leq 0.2483 \\ x_1 \leq 1,\ 0 \leq x_2 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.7999x_1 - 0.9001x_2 + 0.6002 \\ 0.6002x_1 + 2.6995x_2 + 0.2001 \end{bmatrix}, \mathcal{CR}_{2-4} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ 0.2023x_1 + 0.9793x_2 \leq 0.1931 \\ 0 \leq x_1,\ 0 \leq x_2 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.6667x_1 - 7.9979x_2 + 1.9996 \\ 0.1111x_1 + 0.3324x_2 + 0.6669 \end{bmatrix}, \mathcal{CR}_{2-5} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3769 \\ -0.2023x_1 - 0.9793x_2 \leq -0.2483 \\ 0.2048x_1 + 0.9788x_2 \leq 0.2700 \\ x_1 \geq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.6668x_1 - 7.9988x_2 + 1.9999 \\ 0.1111x_1 + 0.3326x_2 + 0.6668 \end{bmatrix}, \mathcal{CR}_{2-6} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ 0.2023x_1 + 0.9793x_2 \leq -0.2221 \\ 0.0126x_1 + 0.9999x_2 \leq 0.2076 \\ -0.2023x_1 - 0.9793x_2 \leq -0.1931 \\ 0 \leq x_1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.6668x_1 - 7.9986x_2 + 1.9998 \\ 0.1111x_1 + 0.3326x_2 + 0.6668 \end{bmatrix}, \mathcal{CR}_{2-7} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3769 \\ -0.2023x_1 - 0.9793x_2 \leq -0.2221 \\ 0.0126x_1 + 0.9999x_2 \leq 0.2076 \end{cases}$$

Corresponding to the third critical set of fourth level-problem we have got the following parametric solutions with parameter $x = (x_1, x_2)$ to the third-level followers problem of

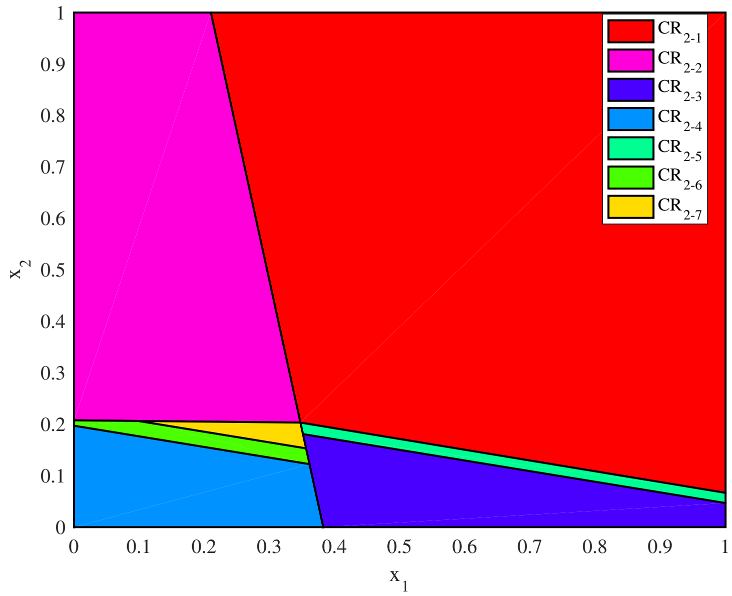

Figure 5.4: Critical regions for the third-level problem of (5.4.13) that corresponds to $\mathcal{CR}_2$ (blue region in Fig.5.2)

(5.4.13),

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0.9730x_1 - 0.1622x_2 - 0.1614 \\ -0.1621x_1 - 0.9730x_2 + 1.0269 \end{bmatrix}, \mathcal{CR}_{3-1} = \begin{cases} 0.9854x_1 + 0.1700x_2 \leq 0.3764 \\ 0.6451x_1 - 0.7641x_2 \leq 0.0686 \\ x_1 \geq 0,\ 0 \leq x_2 \leq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.2000x_1 + 0.8001x_2 + 0.0504 \\ -0.5999x_1 + 0.4000x_2 + 0.8992 \end{bmatrix}, \mathcal{CR}_{3-2} = \begin{cases} 0.2894x_1 + 0.9572x_2 \leq 0.2902 \\ 0.9854x_1 + 0.1700x_2 \leq 0.3764 \\ x_2 \geq 0 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.5712x_1 - 0.4285x_2 + 0.4284 \\ 0.0952x_1 - 0.9286x_2 + 0.9286 \end{bmatrix}, \mathcal{CR}_{3-3} = \begin{cases} -0.0125x_1 - 0.9999x_2 \leq -0.2075 \\ 0.9854x_1 + 0.1700x_2 \leq 0.4660 \\ -0.9854x_1 - 0.1700x_2 \leq -0.3764 \\ x_2 \leq 1 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.5715x_1 - 0.4285x_2 + 0.4285 \\ 0.0952x_1 - 0.9286x_2 + 0.9286 \end{bmatrix}, \mathcal{CR}_{3-4} = \begin{cases} 0.9847x_1 + 0.1743x_2 \leq 0.8362 \\ -0.0125x_1 - 0.9999x_2 \leq -0.2075 \\ -0.9854x_1 - 0.1700x_2 \leq -0.4660 \\ x_1 \leq 1,\ x_2 \geq 0 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.2000x_1 + 0.8000x_2 + 0.0560 \\ -0.5999x_1 + 0.4000x_2 + 0.8880 \end{bmatrix}, \mathcal{CR}_{3-5} = \begin{cases} 0.2894x_1 + 0.9572x_2 \leq 0.2902 \\ -0.9854x_1 - 0.1700x_2 \leq -0.3764 \\ x_1 \leq 1,\ x_2 \geq 0 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.5453x_1 + 1.6362x_2 \\ 0.0909x_1 - 1.2726x_2 + 1 \end{bmatrix}, \mathcal{CR}_{3-6} = \begin{cases} -0.9854x_1 - 0.1700x_2 \leq -0.3764 \\ 0.0125x_1 + 0.9999x_2 \leq 0.2075 \\ -0.2894x_1 - 0.9572x_2 \leq -0.2902 \end{cases}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -0.2008x_1 + 0.7973x_2 + 0.0185 \\ -0.5983x_1 + 0.4054x_2 + 0.9630 \end{bmatrix}, \mathcal{CR}_{3-7} = \begin{cases} 0.2894x_1 + 0.9572x_2 \leq 0.3202 \\ -0.2894x_1 - 0.9572x_2 \leq -0.2902 \\ -0.6436x_1 + 0.7654x_2 \leq -0.3080 \\ x_1 \leq 1 \end{cases}$$

and no feasible solution in $\mathcal{CR}_{3-8}$ and $\mathcal{CR}_{3-9}$.

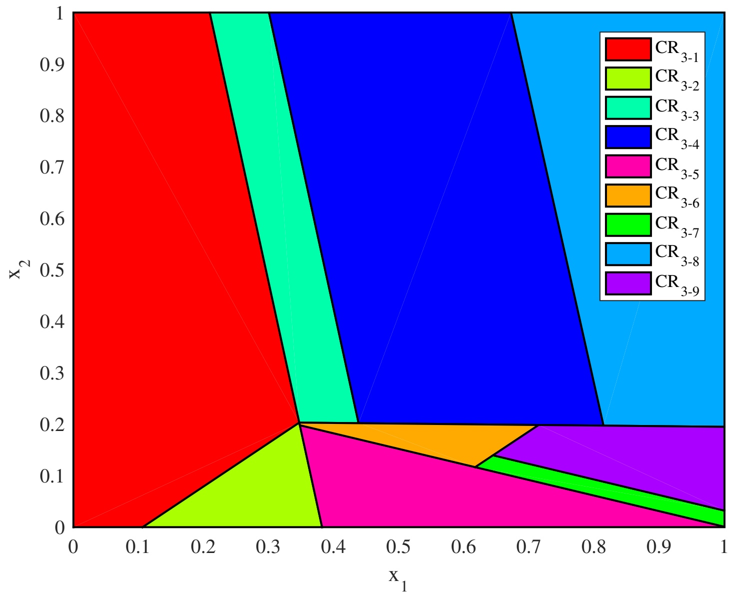

Figure 5.5: Critical regions for the third-level problem of (5.4.13) that corresponds to $\mathcal{CR}_3$ (green region in Fig.5.2)

Substituting the above parametric solutions into the second-level followers problem of (5.4.13) *and solving the resulting problems in each critical regions we obtain*

$$(x_1, x_2) = \begin{cases} (1,0), (0.2308, 0.1537), (0.7006, 0.0199), (0.2481, 0.0904), \\ (0.2475, 0.1864), (0.3474, 0.2032), (0.3606, 0.1269), (0.1426, 0), \\ (0.0918, 0.2103), (0.4274, 0.1864), (0.3474, 0.2032), (0.2033, 0.0819), \\ (0.3115, 0.0149), (0.4380, 0.2020), (0.5998, 0.2), (0.3778, 0.0244), \\ (0.4768, 0.1590), (0.6438, 0.1389), \end{cases}$$

in the critical region $\mathcal{CR}_{1-1}$, $\mathcal{CR}_{1-2}$, $\mathcal{CR}_{1-3}$, $\mathcal{CR}_{1-4}$, $\mathcal{CR}_{2-1}$, $\mathcal{CR}_{2-2}$, $\mathcal{CR}_{2-3}$, $\mathcal{CR}_{2-4}$, $\mathcal{CR}_{2-5}$, $\mathcal{CR}_{2-6}$, $\mathcal{CR}_{2-7}$, $\mathcal{CR}_{3-1}$, $\mathcal{CR}_{3-2}$, $\mathcal{CR}_{2-3}$, $\mathcal{CR}_{3-4}$, $\mathcal{CR}_{3-5}$, $\mathcal{CR}_{3-6}$, $\mathcal{CR}_{3-7}$, respectively. But the point $(x_1, x_2) = (1, 0)$ provides a best solution with respect to the second-level problem of (5.4.13), *as it was decided by the upper-level decision makers we take this point as an optimal solution. Therefore, the optimal solution to the trilevel MLMF problem* (5.4.11) *is $u = (x_1, x_2, y_1, y_2, z_1, z_2) = (1, 0, 1, 1, 0.6666, 0.1667)$ on $\mathcal{CR}_{1-1}$ with optimal objective values $F_1 = 2.3332$, $F_2 = -2.1667$, $f_1^2 = 0.4444$, $f_2^2 = 1.1667$, $f_1^3 = 1.7220$ and $f_2^3 = 0.7500$.* □

The method proposed in Section 5.2 will be illustrated in the six numerical examples (**Example 5.4 – Example 5.9**) provided below.

Example 5.4. *Consider the following bilevel two-leader two-follower game:*

$$\begin{cases} \begin{cases} \min\limits_{x_1} F_1(x,y) = e^{-x_1+1} + (y_1^2+1)e^{x_1+2x_2} - x_1 - x_2 - 2y_1 + y_2 \\ \min\limits_{x_2} F_2(x,y) = e^{x_1+2x_2-y_2+1} + x_1 - 3x_2 + y_1 + y_2 \end{cases} \\ \text{s.t.} \begin{cases} \begin{cases} \min\limits_{y_1} f_1(x,y) = x_1 + x_2 - 3y_1 + y_2 + (x_1+1)(y_1+y_2^2)^2 \\ \min\limits_{y_2} f_2(x,y) = x_1 + x_2 + y_1 - 3y_2 + (x_2+1)(y_1+y_2^2)^2 \end{cases} \\ \text{s.t.} \begin{cases} 2x_1 + x_2 + 5y_1 + y_2 \le 16, \\ 6x_1 - 3x_2 + y_1 + 2y_2 \le 24, \\ 0 \le y_1, y_2 \le 10,\ 0 \le x_1, x_2 \le 5. \end{cases} \end{cases} \end{cases} \tag{5.4.15}$$

Using the procedures discussed in Subsection 5.2.1, problem (5.4.15) *can be reformulated as a bilevel single-leader single-follower problem as follows:*

$$\begin{cases} \min\limits_{x_1,x_2} F(x,y) = \dfrac{e^{-x_1+1} - x_1}{y_1^2+1} - 3x_2 e^{y_1-1} + e^{x_1+2x_2} \\ \text{s.t.} \begin{cases} \min\limits_{y_1,y_2} f(x,y) = -\dfrac{3y_1}{x_1+1} - \dfrac{3y_2}{x_2+1} + (y_1+y_2^2)^2 \\ \text{s.t.} \begin{cases} 2x_1 + x_2 + 5y_1 + y_2 \le 16, \\ 6x_1 - 3x_2 + y_1 + 2y_2 \le 24, \\ 0 \le y_1, y_2 \le 10,\ 0 \le x_1, x_2 \le 5. \end{cases} \end{cases} \end{cases} \tag{5.4.16}$$

The follower's problem in (5.4.16) *can be considered as a multi-parametric problem with parameter* $x = (x_1, x_2)$ *and solved using the algorithm proposed in [37]. Then the parametric solutions with their corresponding critical regions (see Fig. 5.6) are given by,*

$$\mathcal{CR}_1 = \begin{cases} y^*(x) = \begin{bmatrix} -0.3822x_1 - 0.1920x_2 + 3.1220 \\ -0.0888x_1 - 0.0399x_2 + 0.3902 \end{bmatrix}, \\ 0.8570x_1 - 0.5154x_2 \le 3.1659, \\ 0 \le x_1 \le 5,\ 0 \le x_2 \le 5; \end{cases}$$

$$\mathcal{CR}_2 = \begin{cases} y^*(x) = \begin{bmatrix} 0.2222x_1 - 0.5556x_2 + 0.8889 \\ -3.1111x_1 + 1.7778x_2 + 11.5556 \end{bmatrix}, \\ -0.8570x_1 + 0.5154x_2 \le -3.1659, \\ x_1 \le 5,\ x_2 \ge 0. \end{cases}$$

which can be incorporated into the leader's problem of (5.4.16) *and the resulting nonlinear problem is solved in each critical regions. The optimal solution to the leader's problem of*

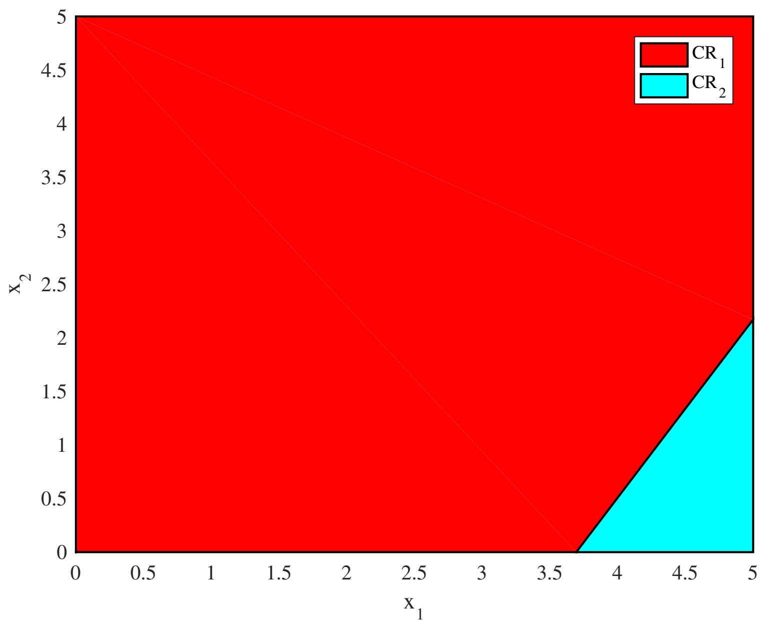

Figure 5.6: Critical regions for the second-level problem of (5.4.16)

(5.4.16) *is* $(x_1, x_2, y_1, y_2) = (0, 0, 3.1220, 0.3902)$ *with the corresponding objective value* $F = 1.2529$ *in* $\mathcal{CR}_1$ *and* $(x_1, x_2, y_1, y_2) = (3.6949, 0, 1.7099, 0.0620)$ *with the corresponding objective value* $F = 39.2957$ *in* $\mathcal{CR}_2$*. Since the objective value obtained in* $\mathcal{CR}_1$ *is better we can take* $(0, 0, 3.1220, 0.3902)$ *as an optimal solution to the leader's problem of* (5.4.16).

Therefore, the optimal solution to the bilevel multi-leader multi-follower problem (5.4.15) *is* $(0, 0, 3.1220, 0.3902)$ *with the corresponding objective values* $F_1 = 7.6114$*,* $F_2 = 5.3523$*,* $f_1 = 1.7450$ *and* $f_2 = 12.6722$*.* □

Example 5.5. *Consider the following bilevel two-leader two-follower game:*

$$\begin{cases} \min\limits_{x_1} F_1(x, y) = -x_1 y_2^2 - x_2 y_1^2 + x_1 e^{x_2} e^{y_1}, \\ \min\limits_{x_2} F_2(x, y) = x_1 - 3x_2 y_1 + (y_2 + 1) x_1 e^{x_2} \end{cases}$$
$$s.t. \begin{cases} \begin{cases} \min\limits_{y_1} f_1(x, y) = y_1^2 + (1 - x_1) y_2 - (x_2^2 + 2) \ln(y_1 + y_2 + 4), \\ \min\limits_{y_2} f_2(x, y) = -y_2^2 + (1 - x_2) y_1 - (x_1^2 + 3) \ln(y_1 + y_2 + 4) \end{cases} \\ s.t. \begin{cases} x_1 + 2y_1 - y_2 - 2 \leq 0, \\ x_1 - x_2 - y_1 + y_2 - 1 \leq 0, \\ 0 \leq x_1, x_2 \leq 2, \; 0 \leq y_1, y_2 \leq 2. \end{cases} \end{cases} \tag{5.4.17}$$

As it was discussed in Subsection 5.2.1 problem (5.4.17) *can be equivalently reformulated* *as a bilevel single-leader single-follower problem,*

$$\min_x F(x,y) = x_1(e^{x_2} - y_2^2 e^{-y_1}) - \frac{3y_1x_2}{y_2+1}$$
$$s.t. \begin{cases} \min_y f(x,y) = \dfrac{y_1^2}{x_2^2+2} - \dfrac{y_2^2}{x_1^2+3} - \ln(y_1+y_2+4) \\ s.t. \begin{cases} x_1 + 2y_1 - y_2 - 2 \leq 0, \\ x_1 - x_2 - y_1 + y_2 - 1 \leq 0, \\ 0 \leq x_1, x_2 \leq 2,\ 0 \leq y_1, y_2 \leq 2. \end{cases} \end{cases} \tag{5.4.18}$$

By treating the leaders decision variable, $x = (x_1, x_2)$ as a parameter, the inner problem in (5.4.18) *can be considered as a multi-parametric problem:*

$$\begin{cases} \min_y f(x,y) = \dfrac{y_1^2}{x_2^2+2} - \dfrac{y_2^2}{x_1^2+3} - \ln(y_1+y_2+4) \\ s.t. \begin{cases} x_1 + 2y_1 - y_2 - 2 \leq 0, \\ x_1 - x_2 - y_1 + y_2 - 1 \leq 0, \\ 0 \leq x_1, x_2 \leq 2,\ 0 \leq y_1, y_2 \leq 2. \end{cases} \end{cases} \tag{5.4.19}$$

So we use multi-parametric approach discussed in Section 3 to solve (5.4.19)*. The Lagrangian of* (5.4.19) *is given by, $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y)$, where $g_1(x,y) = x_1 + 2y_1 - y_2 - 2$ and $g_2(x,y) = x_1 - x_2 - y_1 + y_2 - 1$.*

After exploring all the parameter spaces, the optimal solution to the problem (5.4.19) *with their corresponding critical regions (see Fig. 5.7) are*

$$y^*(x) = \begin{cases} \begin{bmatrix} -3.8630x_1 + 3.6393x_2 + 1.2237 \\ -4.8630x_1 + 4.6393x_2 + 2.2237 \end{bmatrix} & in\ \mathcal{CR}_1 \\ \begin{bmatrix} -6.5148x_1 + 4.5813x_2 + 3.3640 \\ -7.5148x_1 + 5.5813x_2 + 4.3640 \end{bmatrix} & in\ \mathcal{CR}_2 \\ \begin{bmatrix} 0.2870x_2 - 0.1114 \\ 0.0107x_2 + 1.9786 \end{bmatrix} & in\ \mathcal{CR}_3 \\ \begin{bmatrix} -14.7758x_1 + 10.1658x_2 - 2.4725 \\ -15.7758x_1 + 11.1658x_2 - 1.4725 \end{bmatrix} & in\ \mathcal{CR}_4 \\ \begin{bmatrix} -1.6379x_1 + 1.0111x_2 + 2.2657 \\ -2.6379x_1 2.0111x_2 + 3.2657 \end{bmatrix} & in\ \mathcal{CR}_5 \end{cases}$$

where $\mathcal{CR}_1 = \begin{cases} 0.7881x_1 - 0.6156x_2 \leq 0.3224, \\ -0.5767x_1 + 0.8170x_2 \leq 0.5498, \\ x_1 \geq 0,\ 0 \leq x_2 \leq 2; \end{cases}$

$$\mathcal{CR}_2 = \begin{cases} -0.7881x_1 + 0.6156x_2 \leq -0.3224, \\ x_1 \leq 2,\ 0 \leq x_2 \leq 2; \end{cases}$$

$$\mathcal{CR}_3 = \begin{cases} 0.6167x_1 - 0.7872x_2 \leq -0.6723, \\ x_1 \geq 0,\ x_2 \leq 2; \end{cases}$$

$$\mathcal{CR}_4 = \begin{cases} -0.8125x_1 + 0.5829x_2 \leq 0.3480, \\ -0.6167x_1 + 0.7872x_2 \leq 0.6723, \\ 0.5767x_1 - 0.8170x_2 \leq -0.5498, \\ x_2 \leq 2; \end{cases} \quad \textit{and}$$

$$\mathcal{CR}_5 = \begin{cases} 0.8125x_1 - 0.5829x_2 \leq -0.3480, \\ -0.6167x_1 + 0.7872x_2 \leq 0.6723, \\ 0.5767x_1 - 0.8170x_2 \leq -0.5498, \\ x_1 \geq 0. \end{cases}$$

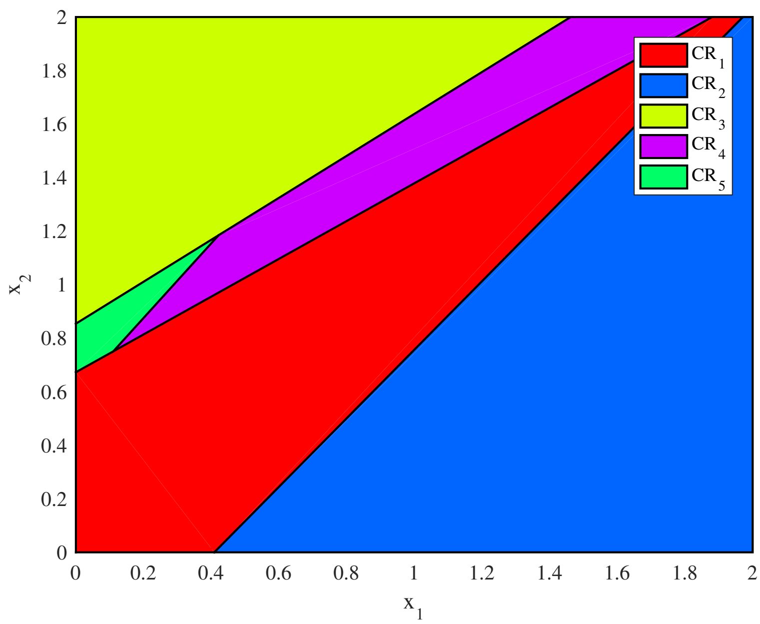

Figure 5.7: Critical regions for parametric problem (5.4.19)

This can be incorporated into the upper-level problem of (5.4.18) *and the resulting nonlinear problems are solved in each critical regions. The following are the optimal solutions:* $(x_1, x_2, y_1, y_2) = (0.5157, 0.4923, 1.0234, 2.0000)$ *with objective value* $F = -0.4014$*,* $(x_1, x_2, y_1, y_2) = (0.5035, 0.1208, 0.6372, 1.2545)$ *with objective value* $F = 0.0467$*,* $(x_1, x_2, y_1, y_2) = (0.2610, 1.0586, 0.1924, 1.9899)$ *with objective value* $F = -0.3047$*,* $(x_1, x_2, y_1, y_2) = (0.5121, 1.0345, 0.4776, 2.0000)$ *with objective value* $F = -0.3238$

and $(x_1, x_2, y_1, y_2) = (0.3773, 0.4820, 2.1349, 3.2395)$ *with objective value* $F = -0.5854$*, respectively in* $\mathcal{CR}_1$*,* $\mathcal{CR}_2$*,* $\mathcal{CR}_3$*,* $\mathcal{CR}_4$ *and* $\mathcal{CR}_5$*.*

Now, comparing all the values of the objective of the leader in each of the critical regions, we can see that the objective value obtained in $\mathcal{CR}_5$ *gives a better result. Hence we take* $(0.3773, 0.4820, 2.1349, 3.2395)$ *as an optimal solution to the upper-problem of* (5.4.18)*.*

Therefore, the optimal solution to the bilevel multi-leader multi-follower problem (5.4.17) *is* $(x^*,\ y^*) = (0.3773,\ 0.4820,\ 2.1349,\ 3.2395)$ *with optimal leaders objective* $F_1 = -0.9899$ *and* $F_2 = -0.1191$*; and optimal followers objective* $f_1 = 1.5792$ *and* $f_2 = -16.4213$*.* □

Example 5.6. *Consider the following nonlinear bilevel two-leader two-follower problem involving bilinear terms at the lower-level:*

$$\begin{cases} \min\limits_{x_1} F_1(x,y) = (x_1 - y_1)^2 + (x_2 - 2)^2 \\ \min\limits_{x_2} F_2(x,y) = \frac{1}{2}x_1^2 - x_1 y_2 - \frac{1}{2}x_2^2 \\ s.t. \begin{cases} \begin{cases} \min\limits_{y_1} f_1(x,y) = \frac{1}{2}y_1^2 + y_1 - x_1 y_1 + (x_1^2+1) y_1 y_2 \\ \min\limits_{y_2} f_2(x,y) = \frac{1}{2}y_2^2 + y_2 - x_2 y_2 + (2 - x_2) y_1 y_2 \end{cases} \\ s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \le 3, y_1 + y_2 \le 2.5, \\ 0 \le y_1 \le 2, 0 \le y_2 \le 1, 0 \le x_1, x_2 \le 1. \end{cases} \end{cases} \end{cases} \tag{5.4.20}$$

Using the procedures discussed in Subsection 5.2.1, problem (5.4.20) *can be reformulated as bilevel optimization problem as follows:*

$$\begin{aligned} &\min_{x_1,x_2} F(x,y) = (x_1 - y_1)^2 - \frac{1}{2}x_2^2 \\ &s.t. \begin{cases} \min\limits_{y_1,y_2} f(x,y) = \dfrac{y_1^2 + 2(1-x_1)y_1}{2(x_1^2+1)} + \dfrac{y_2^2 + 2(1-x_2)y_2}{2(2-x_2)} + y_1 y_2 \\ s.t. \begin{cases} 2y_1 + y_2 + x_1 - 2x_2 \le 3,\ y_1 + y_2 \le 2.5, \\ 0 \le y_1 \le 2,\ 0 \le y_2 \le 1,\ 0 \le x_1,\ x_2 \le 1. \end{cases} \end{cases} \end{aligned} \tag{5.4.21}$$

The follower's problem in (5.4.21) *can be considered as a multi-parametric problem with parameter* $x = (x_1, x_2)$ *and solved using the algorithm proposed [59]. But due to the bilinear term* $b_{12}y_1y_2 = y_1y_2$ *in the objective function the problem may result in multiple Nash equilibrium reaction for at least one feasible choice of the leader's problem. So we should apply a mathematical procedure described in [61] to under-estimate the problem. The convex envelope of the bilinear term* y_1y_2 *taken over the rectangle* $R = \{(y_1, y_2) : 0 \le y_1 \le 2,\ 0 \le y_2 \le 1\}$ *is denoted by* $VexR[b_{12}y_1y_2]$ *and can be obtained as follows,* $b_{12} = 1 > 0 \Rightarrow l_{12}^1(y_1, y_2) = 0$*,* $l_{12}^2(y_1, y_2) = y_1 + 2y_2 - 2$*,*

$$VexR[b_{12}y_1y_2] = \max\left\{l_{12}^1(y_1,y_2), l_{12}^2(y_1,y_2)\right\} = \max\left\{0, y_1 + 2y_2 - 2\right\} = y_1 + 2y_2 - 2.$$

Therefore the under-estimator of the second-level objective function in (5.4.21) *is equal to*

$$\frac{y_1^2+2(1-x_1)y_1}{2(x_1^2+1)}+\frac{y_2^2+2(1-x_2)y_2}{2(2-x_2)}+y_1y_2+V_{ex}R[b_{12}y_1y_2].$$

Thus, the follower's problem in (5.4.21) *is under-estimated by,*

$$\begin{aligned}&\min_{y_1,y_2} f(x,y)=\frac{y_1^2}{2(x_1^2+1)}+\frac{(x_1^2-x_1+2)y_1}{x_1^2+1}+\frac{y_2^2}{2(2-x_2)}+\frac{(5-3x_2)y_2}{2-x_2}-2\\&s.t.\ \begin{cases}2y_1+y_2+x_1-2x_2\le 3,\ y_1+y_2\le 2.5,\\0\le y_1\le 2,\ 0\le y_2\le 1,\ 0\le x_1,\ x_2\le 1.\end{cases}\end{aligned}\tag{5.4.22}$$

Then (5.4.22) *is solved using the proposed algorithm in [37] and we have got the following parametric solutions with the corresponding critical region,*

$$\mathcal{CR}=\begin{cases}y^*(x)=\begin{bmatrix}x_1\\\frac{1}{2}x_2\end{bmatrix},\\0\le x_1\le 1,\ \ 0\le x_2\le 1.\end{cases}$$

Incorporating the solution into the leader's problem of (5.4.21) *and solving the resulting nonlinear optimization problem we obtain two optimal solutions* $(x_1,x_2)=(1,1)$ *and* $(x_1,x_2)=(0,1)$. *These result in two optimal solutions for problem* (5.4.21), *namely*

$(x_1,x_2,y_1,y_2)=(1,1,1,0.5)$ *with objective values* $F=-0.5,\ f=\frac{7}{8}$ *and*

$(x_1,x_2,y_1,y_2)=(0,1,0,0.5)$ *with objective values* $F=-0.5,\ f=\frac{1}{8}$.

The two solutions are indifferent for the leader's problem, but the optimal solution $(x_1,x_2,y_1,y_2)=(0,1,0,0.5)$ *results in a better objective value for the follower's problem.*

Therefore, the optimal solution to the bilevel MLMF problem (5.4.20) *is* $(x_1,x_2,y_1,y_2)=(0,1,0,0.5)$ *with the corresponding objective values* $F_1=1$, $F_2=-\frac{1}{2}$, $f_1=0$ *and* $f_2=\frac{1}{8}$. □

Example 5.7. *Consider the following nonlinear trilevel MLMF programming problem:*

$$
\begin{cases}
\begin{cases}
\min\limits_{x_1} F_1(x,y,z) = (x_1-y_1)^2 + y_2^2 + x_2 e^{-(x_1+z_1z_2)} + y_2 e^{-(x_1x_2+z_1z_2)} \\
\min\limits_{x_2} F_2(x,y,z) = (x_2-1)^2 + x_1y_2z_2 + \dfrac{1}{1+y_1+y_2}(x_2e^{-x_1} + y_2e^{-z_1z_2})
\end{cases} \\
s.t. \begin{cases}
\begin{cases}
\min\limits_{y_1} f_2^1(x,y,z) = (y_1-z_2)^2 + (y_2-2)^2 + (x(1)+1)(y_1^2+y_2^2) \\
\min\limits_{y_2} f_2^2(x,y,z) = \dfrac{1}{2}(y_1^2-y_2^2) - y_1z_1 + (x(2)+2)(y_1^2+y_2^2)
\end{cases} \\
s.t. \begin{cases}
\begin{cases}
\min\limits_{z_1} f_3^1(x,y,z) = z_1^2 + (1-y_1)z_1 - (-x_1+y_2^2+4)(\ln(z_1+z_2+1)) \\
\min\limits_{z_2} f_3^2(x,y,z) = z_2^2 + (1-y_2)z_2 - (-x_2+y_1^2+5)(\ln(z_1+z_2+1))
\end{cases} \\
s.t. \begin{cases}
x_1+y_1+y_2+z_1 \le 4, -x_2+y_1+2y_2+z_2 \le 3, \\
2x_1-x_2+z_1+z_2 \le 4, 0 \le z_1,\ z_2 \le 2, \\
0 \le y_1,\ y_2 \le 0.5,\ 1 \le x_1,\ x_2 \le 3.
\end{cases}
\end{cases}
\end{cases}
\end{cases}
\tag{5.4.23}
$$

Using the procedures discussed in Section 5.2.2, problem (5.4.23) *can be reformulated as trilevel hierarchical problem as follows:*

$$
\begin{aligned}
&\min_x F_1(x,y,z) = (x_1-y_1)^2 e^{z_1z_2} + (1+y_1+y_2)(x_2-1)^2 + x_2e^{-x_1} + y_2e^{-z_1z_2} \\
&s.t. \begin{cases}
\min\limits_{y_1,y_2} f_2(x,y,z) = \dfrac{(y_1-z_2)^2}{(x(1)+1)} - \dfrac{y_2^2}{2(x(2)+2)} + y_1^2 + y_2^2 \\
s.t. \begin{cases}
\min\limits_{z_1,z_2} f_3(x,y,z) = \dfrac{z_1^2+(1-y_1)z_1}{(-x_1+y_2^2+4)} + \dfrac{z_2^2+(1-y_2)z_2}{(-x_2+y_1^2+5)} - \ln(z_1+z_2+1) \\
s.t. \begin{cases}
x_1+y_1+y_2+z_1 \le 4, \quad -x_2+y_1+2y_2+z_2 \le 3, \\
2x_1-x_2+z_1+z_2 \le 4,\ 0 \le z_1,\ z_2 \le 2, \\
0 \le y_1,\ y_2 \le 0.5,\ 1 \le x_1,\ x_2 \le 3.
\end{cases}
\end{cases}
\end{cases}
\end{aligned}
\tag{5.4.24}
$$

Then the third-level follower problem in (5.4.24) *can be considered as a multi-parametric problem with parameters* $\theta = (x_1, x_2, y_1, y_2)$*. Implementing a multi-parametric programming procedures, we obtain the following parametric solutions with their corresponding critical regions (see Fig. 5.8),*

$$
\mathcal{CR}_1 = \begin{cases}
z^*(x,y) = \begin{bmatrix} -0.3422x_1 + 0.0439x_2 + 0.4137y_1 - 0.0863y_2 + 0.5616 \\ 0.6132x_1 - 0.1795x_2 - 0.1467y_1 + 0.3533y_2 + 0.2717 \end{bmatrix}, \\
0.8847x_1 - 0.4424x_2 + 0.1040y_1 + 0.1040y_2 \le 1.2337, \\
1 \le x_1, \quad 1 \le x_2 \le 3, \quad 0 \le y_1,\ y_2 \le 0.5;
\end{cases}
$$

$$\mathcal{CR}_2 = \begin{cases} z^*(x,y) = \begin{bmatrix} -0.6073x_1 + 0.1622x_2 + 0.4302y_1 - 0.0698y_2 + 1.0864 \\ = 1.3927x_1 + 0.8378x_2 - 0.4302y_1 + 0.0698y_2 + 2.9136 \end{bmatrix}, \\ -0.8847x_1 + 0.4424x_2 - 0.1040y_1 - 0.1040y_2 \leq -1.2337, \\ x_1 \leq 3, \quad 1 \leq x_2 \leq 3, \quad 0 \leq y_1, \ y_2 \leq 0.5. \end{cases}$$

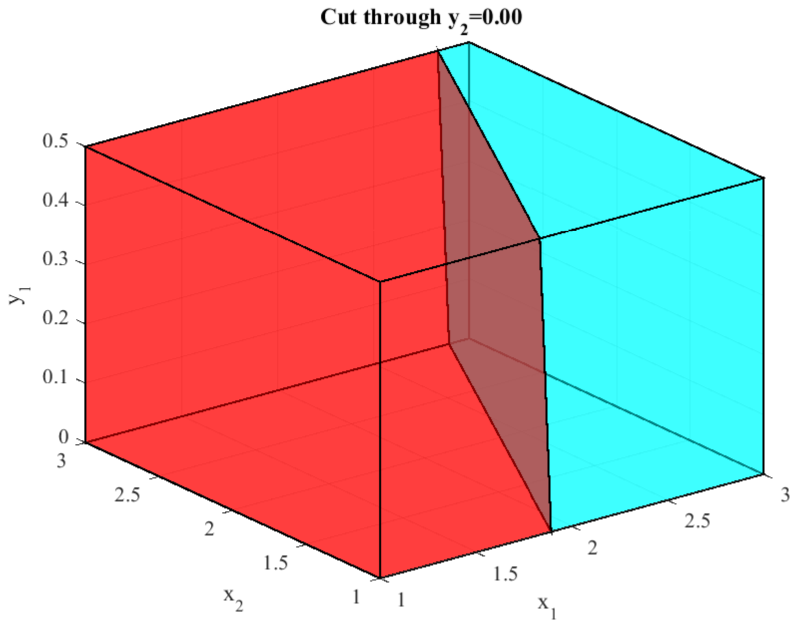

Figure 5.8: Critical regions for the third-level problem of (5.4.24)

Incorporating these solutions into the second-level problem of (5.4.24) *and solving the resulting problems we get the following parametric solutions with their corresponding critical regions (see Fig. 5.9),*

$$\mathcal{CR}_1 = \begin{cases} y^*(x) = \begin{bmatrix} 0.0733x_1 + 0.1707 \\ -0.0251x_1 + 0.0251 \end{bmatrix}, \\ 0.8954x_1 - 0.4452x_2 \leq 1.2211, \\ 1 \leq x_1, \quad 1 \leq x_2 \leq 3; \end{cases}$$

$$\mathcal{CR}_2 = \begin{cases} y^*(x) = \begin{bmatrix} -0.1800x_1 + 0.8354 \\ 0.0102x_1 - 0.0307 \end{bmatrix}, \\ -0.8954x_1 + 0.4452x_2 \leq -1.2211, \\ x_1 \leq 3, \quad 1 \leq x_2 \leq 3. \end{cases}$$

Using these parametric solution in the leader's problem of (5.4.24) *and solving the resulting problems in each of the critical regions we obtain an optimal solution:*
$(x_1, x_2, y_1, y_2, z_1, z_2) = (1, 1, 0.2440, 0, 0.3642, 0.6695)$ *with objective value* $F = 1.0973$

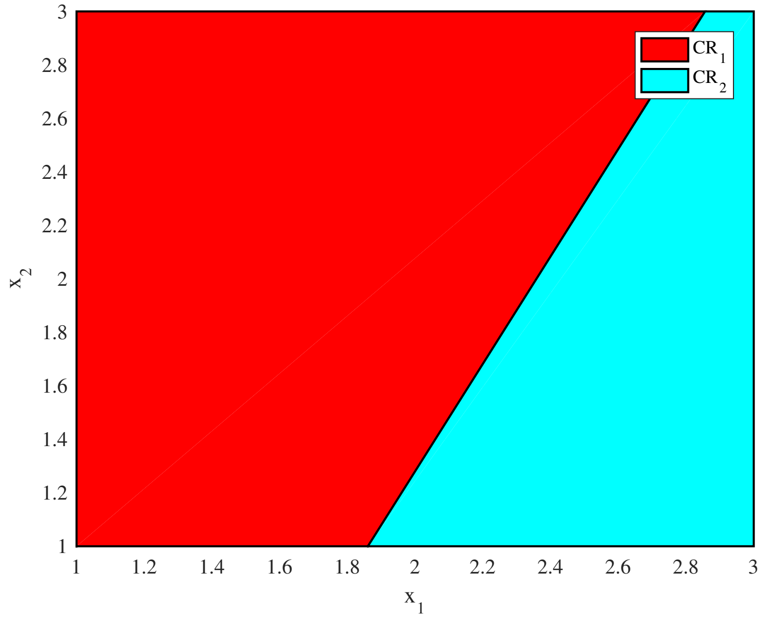

Figure 5.9: Critical regions for the second-level problem of (5.4.24)

and $(x_1, x_2, y_1, y_2, z_1, z_2) = (2.8836, 1, 0.3164, 0, 0, 1.1773)$ *with objective value* $F = 6.1905$*, respectively in* $\mathcal{CR}_1$ *and* $\mathcal{CR}_2$*.*

Since the objective value obtained in $\mathcal{CR}_1$ *is better, we take* $(1, 1, 0.2440, 0, 0.3642, 0.6695)$ *as an optimal solution to the problem* (5.4.24)*.*

Hence, the optimal solution to the trilevel multi-leader multi-follower problem (5.4.23) *is* $(x_1, x_2, y_1, y_2, z_1, z_2) = (1, 1, 0.2440, 0, 0.3642, 0.6695)$ *with objective values* $F_1 = 0.8598$*,* $F_2 = 0.2957$*,* $f_2^1 = 4.3001$*,* $f_2^2 = 0.1195$*,* $f_3^1 = -1.7216$ *and* $f_3^2 = -1.7640$*.* $\square$

Example 5.8. *Consider the following nonlinear trilevel MLMF programming problem:*

$$\begin{cases}\min\limits_{x_1} F_1^1(x,y,z) = (x_1 - z_1)^2 - (y_1 + 4)\cos(\frac{\pi}{2}(x_2 - x_1)),\\ \min\limits_{x_2} F_1^2(x,y,z) = (x_2 - 1)^2 - (-y_2 + z_3 + 3)\cos(\frac{\pi}{2}(x_2 - x_1))\end{cases}$$
$$s.t.\begin{cases}\begin{cases}\min\limits_{y_1} f_2^1(x,y,z) = (y_1 - 1)^2 + (y_2 - 2)^2 + (x_2 + z_2 + 5)(y_1^2 + y_2)^2,\\ \min\limits_{y_2} f_2^2(x,y,z) = -\frac{1}{2}y_2^2 - y_1 z_1 + (x_1 - z_1 + 6)(y_1^2 + y_2)^2,\end{cases}\\ s.t.\begin{cases}\begin{cases}\min\limits_{z_1} f_3^1(x,y,z) = z_1^2 + z_1 + (y_1^2 + 1)(y_2 x_1^2 - z_1^2 - z_2^2),\\ \min\limits_{z_2} f_3^2(x,y,z) = z_2^2 - y_2 z_2 + (2 - y_2)(y_2 x_1^2 - z_1^2 - z_2^2),\\ \min\limits_{z_3} f_3^3(x,y,z) = -z_3^2 + x_2(z_1 - z_2) + (1 + x_2)(y_2 x_1^2 - z_1^2 - z_2^2),\end{cases}\\ s.t.\begin{cases}-y_1 + 2y_2 - 2z_1 - z_3 - 2 \le 0,\\ x_1 - x_2 + y_1 - z_2 + z_3 - 5 \le 0,\\ 4x_1 - 2y_2 - z_1 + z_2 + z_3 - 3 \le 0,\\ 0 \le x_1, x_2 \le 1, 0 \le y_1, y_2 \le 1,\\ 0 \le z_1, z_2, z_3 \le 2.\end{cases}\end{cases}\end{cases} \tag{5.4.25}$$

Problem (5.4.25) *is equivalently reformulated as trilevel hierarchical problem having a single decision maker at each levels of the hierarchy:*

$$\min_x F_1(x,y,z) = \frac{(x_1 - z_1)^2}{(y_1 + 4)} + \frac{(x_2 - 1)^2}{(y_2 - z_3 + 3)} - \cos(\frac{\pi}{2}(y_2 - z_1))$$
$$s.t.\begin{cases}\min\limits_{y} f_2(x,y,z) = \frac{(y_1 - 1)^2}{(x_2 + z_2 + 5)} - \frac{y_2^2}{2(x_1 - z_1 + 6)} + (y_1^2 + y_2)^2,\\ s.t.\begin{cases}\min\limits_{z} f_3(x,y,z) = \frac{z_1^2 + z_1}{y_1^2 + 1} + \frac{z_2^2 - y_2 z_2}{2 - y_2} - \frac{z_3^2}{x_2 + y_1 + 1} + y_2 x_1^2 - z_1^2 - z_2^2,\\ s.t.\begin{cases}-y_1 + 2y_2 - 2z_1 - z_3 - 2 \le 0,\\ x_1 - x_2 + y_1 - z_2 + z_3 - 5 \le 0,\\ 4x_1 - 2y_2 - z_1 + z_2 + z_3 - 3 \le 0,\\ 0 \le x_1, x_2 \le 1, 0 \le y_1, y_2 \le 1,\\ 0 \le z_1, z_2, z_3 \le 2.\end{cases}\end{cases}\end{cases} \tag{5.4.26}$$

By treating the first and second level decision variables, (x, y) as parameters, the third-level problem in (5.4.26) *can be considered as a multi-parametric problem (with the parameter* *vector $\theta = (x, y)$):*

$$\begin{cases} \min_{z} f_3(x,y,z) = \dfrac{z_1^2+z_1}{y_1^2+1} + \dfrac{z_2^2-y_2z_2}{2-y_2} - \dfrac{z_3^2}{x_2+y_1+1} + y_2x_1^2 - z_1^2 - z_2^2 \\ s.t. \begin{cases} -y_1+2y_2-2z_1-z_3-2 \le 0, \\ x_1-x_2+y_1-z_2+z_3-5 \le 0, \\ 4x_1-2y_2-z_1+z_2+z_3-3 \le 0, \\ 0 \le x_1, x_2 \le 1, 0 \le y_1, y_2 \le 1, \\ 0 \le z_1, z_2, z_3 \le 2. \end{cases} \end{cases} \tag{5.4.27}$$

The Lagrangian of (5.4.27) *is given by* $\mathcal{L}(x,y,z,\lambda) = f_3(x,y,z)+\lambda_1g_1+\lambda_2g_2+\lambda_3g_3$*, where* $g_1 = -y_1+2y_2-2z_1-z_3-2$, $g_2 = x_1-x_2+y_1-z_2+z_3-5$ *and* $g_3 = 4x_1-2y_2-z_1+z_2+z_3-3$. *After exploring all the parameter spaces, the optimal solution to the problem* (5.4.27) *with their corresponding critical regions (see Fig. 5.10) are:*

$$\mathcal{CR}_1 = \begin{cases} z^*(x,y) = \begin{bmatrix} -0.7111y_1 - 0.9028y_2 + 1.7361x_1 - 0.0611 \\ -0.3556y_1 + 0.7361y_2 - 1.5694x_1 + 2.5944 \\ -0.3556y_1 + 0.3611y_2 - 0.6944x_1 + 0.3444 \end{bmatrix}, \textit{with} \\ 0.1731y_1 + 0.7667y_2 - 0.6183x_1 - 0.4946 \le 0, \\ 0 \le y_1, y_2 \le 1,\ 0 \le x_1, x_2 \le 1; \end{cases}$$

$$\textit{and } \mathcal{CR}_2 = \begin{cases} z^*(x,y) = \begin{bmatrix} -1.2255y_1 + 0.7353 \\ 1.6071y_2 - 2.8571x_1 + 1.2857 \\ 0 \end{bmatrix}, \textit{with} \\ -0.1731y_1 - 0.7667y_2 + 0.6183x_1 + 0.4946 \le 0, \\ 0 \le y_1, y_2 \le 1,\ 0 \le x_1, x_2 \le 1. \end{cases}$$

Using these solutions into the second-level problem of (5.4.26) *results in a multi-parametric problems of parameter x. Again by employing a multi-parametric approach, in $\mathcal{CR}_1$ we have,*

$$\begin{cases} y^*(x) = \begin{bmatrix} 0.0561x_1 - 0.0358x_2 + 0.3433 \\ -0.0424x_1 + 0.0270x_2 + 0.0077 \end{bmatrix}, \textit{with} \\ 0 \le x_1, x_2 \le 1 \end{cases}$$

and in $\mathcal{CR}_2$ we have,

$$\begin{cases} y^*(x) = \begin{bmatrix} -0.0838x_1 - 0.0105x_2 + 0.1864 \\ 0.8254x_1 + 0.0024x_2 + 0.6030 \end{bmatrix}, \textit{with} \\ 0 \le x_1, x_2 \le 1. \end{cases}$$

Incorporating these solutions into the leader problem of (5.4.26) *and solving the resulting*

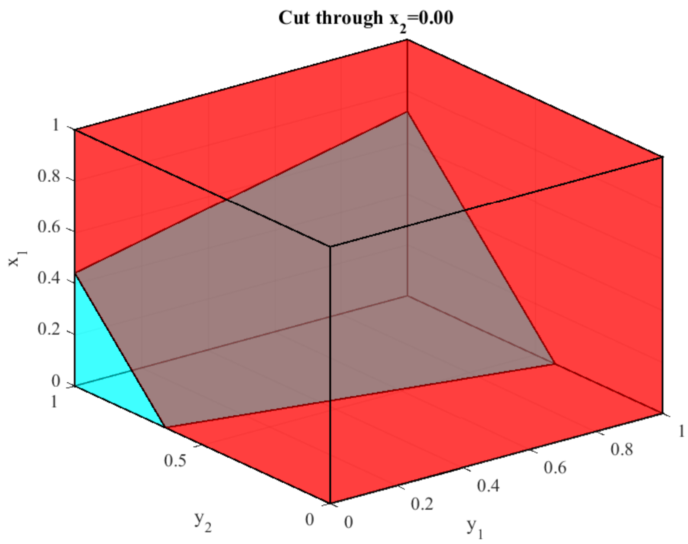

Figure 5.10: Critical regions for parametric problem (5.4.19)

problem, we obtain: $(x, y, z) = (0.3230, 0.4720, 0.3446, 0.0068, 0.2486, 1.9699, 0)$ *with objective value* $F_1 = -0.8782$ *and* $(x, y, z) = (0.4792, 0.6299, 0.1397, 1, 0.5641, 1.5237, 0)$ *with objective value* $F_1 = -0.9022$ *respectively in* $\mathcal{CR}_1$ *and* $\mathcal{CR}_2$*. Since the objective value obtained in* $\mathcal{CR}_2$ *is better we can take it as an optimal solution to the upper-problem of* (5.4.26).

Therefore, the optimal solution to the trilevel MLMF problem (5.4.25) *is* $(x^*, y^*, z^*) = (0.4792, 0.6299, 0.1397, 1, 0.5641, 1.5237, 0)$ *with optimal leaders objective* $F_1^1 = -4.8533$ *and* $F_1^2 = -1.8072$*; optimal second-level objectives* $f_2^1 = 9.1757$ *and* $f_2^2 = 5.0693$*; and optimal third-level objectives* $f_3^1 = -1.5750$*,* $f_3^2 = -1.6124$ *and* $f_3^3 = -4.5332$*.* □

Example 5.9. *Consider the following bilevel two-leader two-follower problem with nonlinear constraints:*

$$\begin{cases} \min\limits_{x_1} F_1(x,y) = x_1^2 + x_2 y_1 + y_2^2 + x_1 x_2, \\ \min\limits_{x_2} F_2(x,y) = x_2^2 - 4x_2 + x_1^2 - y_1 + x_1 x_2 \\ s.t. \begin{cases} \begin{cases} \min\limits_{y_1} f_1(x,y) = y_1^2 + x_1 y_2 + 4(y_1 - y_2)^2, \\ \min\limits_{y_2} f_2(x,y) = y_2^2 - 5y_2 + x_2 y_1 + 2(y_1 - y_2)^2 \end{cases} \\ s.t. \begin{cases} y_2^2 + 5y_2 - 10x_1 - 15 \le 0, y_1 - y_2 - x_1 + 2x_2 \le 0, \\ y_1 + y_2 + 5x_2 - 12 \le 0, -3 \le x_1, x_2 \le 3, \\ -5 \le y_1, y_2 \le 5. \end{cases} \end{cases} \end{cases} \tag{5.4.28}$$

Problem (5.4.28) *can be equivalently reformulated as a bilevel optimization problem,*

$$\min_{x} F(x,y) = x_1^2 + x_2y_1 - 4x_2 + x_1x_2$$
$$s.t. \begin{cases} \min_{y} f(x,y) = 0.25y_1^2 + 0.5y_2^2 - 2.5y_2 + (y_1 - y_2)^2 \\ s.t. \begin{cases} y_2^2 + 5y_2 - 10x_1 - 15 \le 0, y_1 - y_2 - x_1 + 2x_2 \le 0, \\ y_1 + y_2 + 5x_2 - 12 \le 0, -3 \le x_1, x_2 \le 3, \\ -5 \le y_1, y_2 \le 5. \end{cases} \end{cases} \tag{5.4.29}$$

By treating the leaders decision variable, $x = (x_1, x_2)$ *as a parameter, the inner problem in* (5.4.29) *can be considered as a multi-parametric problem:*

$$\min_{y} f(x,y) = 0.25y_1^2 + 0.5y_2^2 - 2.5y_2 + (y_1 - y_2)^2$$
$$s.t. \begin{cases} y_2^2 + 5y_2 - 10x_1 - 15 \le 0, y_1 - y_2 - x_1 + 2x_2 \le 0, \\ y_1 + y_2 + 5x_2 - 12 \le 0, -3 \le x_1, x_2 \le 3, \\ -5 \le y_1, y_2 \le 5. \end{cases} \tag{5.4.30}$$

So we use multi-parametric approach discussed in Section 3.3 to solve (5.4.30). *Here,* $G(x,y) = y_2^2 + 5y_2 - 10x_1 - 15$ *and* $g(x,y) = [y_1 - y_2 - x_1 + 2x_2, y_1 + y_2 + 5x_2 - 12]$. *So for* $x \in X^*$ *the barrier function,* $\psi(x,y)$*, is define as* $\psi(x,y) = -\ln(-G(x,y))$ *with domain* $\{y : G(x,y) < 0\}$.

Assuming the intersection of the domain of the logarithmic barrier and the polyhedral sets is nonempty, i.e., $Y_B(x) = \{y : G(x,y) < 0,\ g(x,y) \le 0, -3 \le x_1, x_2 \le 3, -5 \le y_1, y_2 \le 5\} \neq \emptyset$*, and for* $t > 0$ *we can define the barrier approximation for* (3.3.38) *as*

$$\min_{y}\{W(x,y,t) = 0.25y_1^2 + 0.5y_2^2 - 2.5y_2 + (y_1 - y_2)^2 + t\psi(x,y)\}$$
$$s.t. \begin{cases} y_1 - y_2 - x_1 + 2x_2 \le 0, y_1 + y_2 + 5x_2 - 12 \le 0, \\ -5 \le y_1, y_2 \le 5. \end{cases} \tag{5.4.31}$$

Problem (5.4.31)*) is a MPP with parameter* $x = (x_1, x_2)$ *and its Lagrangian becomes,*

$$\mathcal{L}(x,y,\lambda,t) = 0.25y_1^2 + 0.5y_2^2 - 2.5y_2 + (y_1 - y_2)^2 + t\psi(x,y) + \lambda^T g(x,y),$$

$$M(x,t) = \left[\begin{array}{c|c} \nabla^2_{yy}\mathcal{L} & \nabla_y g \\ \hline -\lambda\nabla_y^T g & diag(-g) \end{array}\right],\ N(x,t) = \left[\ \nabla^2_{xy}\mathcal{L}, -\lambda\nabla_x g\ \right]^T,$$

$$\begin{bmatrix} y(x,t) \\ \lambda(x,t) \end{bmatrix} = \begin{bmatrix} y_0 \\ \lambda_0 \end{bmatrix} - [M_0(t)]^{-1} \cdot N_0(t) \cdot [x - x_0],$$

where $(y_0, \lambda_0) = (y(x_0), \lambda(x_0))$, $M_0(t) = M(x_0, t)$ *and* $N_0(t) = N(x_0, t)$.

By solving the barrier problem (5.4.31), as $t \to 0^+$, the solutions and the critical regions will be,

$$y(x) = \begin{cases} \begin{bmatrix} 0.6681x_1 - 1.3340x_2 + 1.6663 \\ -0.3319x_1 + 0.6660x_2 + 1.6663 \end{bmatrix} & on\ \mathcal{CR}_1, \\ \begin{bmatrix} 0.5x_1 - 3.5x_2 + 6 \\ -0.5x_1 - 1.5x_2 + 6 \end{bmatrix} & on\ \mathcal{CR}_2, \end{cases}$$

$$\mathcal{CR}_1 = \begin{cases} -0.0774x_1 + 0.9970x_2 - 1.9948 \leq 0, \\ -3 \leq x_1 \leq 3,\ x_2 \geq -3; \end{cases}$$

$$\mathcal{CR}_2 = \begin{cases} 0.0774x_1 - 0.9970x_2 + 1.9948 \leq 0, \\ -3 \leq x_1 \leq 3,\ x_2 \leq 3, \end{cases}$$

where $\mathcal{CR}_1$ and $\mathcal{CR}_2$ are as shown in Fig.5.11.

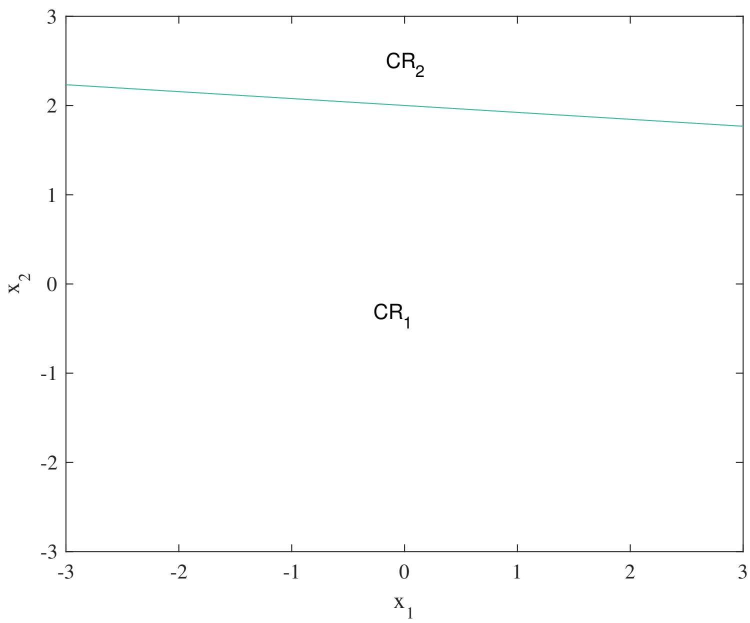

Figure 5.11: Critical regions for the problem (5.4.31)

Then, the critical regions of (5.4.30) is determined as follows:

$$\mathcal{CR}^L = \{\mathcal{CR}\} \cap \{x :\ G(y(x), x) \leq 0\},$$

$$y_L(x) = \begin{cases} \begin{bmatrix} 0.6681x_1 - 1.3340x_2 + 1.6663 \\ -0.3319x_1 + 0.6660x_2 + 1.6663 \end{bmatrix} & on\ \mathcal{CR}_1^L, \\ \begin{bmatrix} 0.5x_1 - 3.5x_2 + 6 \\ -0.5x_1 - 1.5x_2 + 6 \end{bmatrix} & on\ \mathcal{CR}_2^L, \end{cases}$$

$$\mathcal{CR}_1^L = \begin{cases} 0.0774x_1 + 0.9970x_2 - 1.9948 \leq 0, \\ 0.1102x_1^2 - 0.4421x_1x_2 - 12.7660x_1 + 0.4435x_2^2 + 5.5491x_2 \leq 3.8917, \\ -3 \leq x_1 \leq 3, \ x_2 \geq -3; \end{cases}$$

$$\mathcal{CR}_2^L = \begin{cases} -0.0774x_1 - 0.9970x_2 + 1.9948 \leq 0, \\ 0.25x_1^2 + 1.5x_1x_2 - 18.5x_1 + 2.25x_2^2 - 25.5x_2 \leq -51, \\ -3 \leq x_1 \leq 3, \ x_2 \leq 3, \end{cases}$$

where $\mathcal{CR}_1^L$ and $\mathcal{CR}_2^L$ are as shown in Fig.5.12.

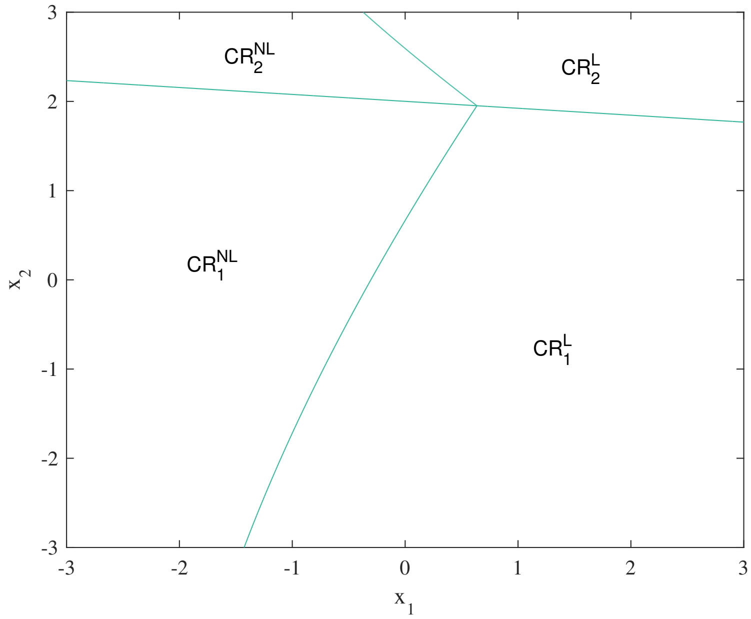

Figure 5.12: Critical regions for the problem (5.4.31)

The rest of the spaces will be,

$$\mathcal{CR}_1^{NL} = \begin{cases} 0.0774x_1 + 0.9970x_2 - 1.9948 \leq 0, \\ 0.1102x_1^2 - 0.4421x_1x_2 - 12.7660x_1 + 0.4435x_2^2 + 5.5491x_2 \geq 3.8917, \\ -3 \leq x_1 \leq 3, \ x_2 \geq -3; \end{cases}$$

$$\mathcal{CR}_2^{NL} = \begin{cases} -0.0774x_1 - 0.9970x_2 + 1.9948 \leq 0, \\ 0.25x_1^2 + 1.5x_1x_2 - 18.5x_1 + 2.25x_2^2 - 25.5x_2 \geq -51, \\ -3 \leq x_1 \leq 3, \ x_2 \leq 3. \end{cases}$$

Next we will determine the nonlinear solution as follows:

In $\mathcal{CR}_1^{NL}$ the solution is determined from $Y_{NL}(x) = \arg\{G(x,y) = 0, g_2(x,y) < 0\}$

$$y_{NL}(x) = \arg\{y_2^2 + 5y_2 - 10x_1 - 15 = 0, y_1 + y_2 + 5x_2 - 12 < 0\},$$

which is infeasible.

In $\mathcal{CR}_2^{NL}$ *the solution is determined from* $Y_{NL}(x) = \arg\{G(x,y)=0, g_2(x,y)=0\}$

$$y_{NL}(x) = \arg\{y_2^2 + 5y_2 - 10x_1 - 15 = 0, y_1 + y_2 + 5x_2 - 12 = 0\},$$

which results in

$$Y_{NL}(x) = \begin{bmatrix} \mp 0.5\sqrt{5(8x_1+17)} - 5x_2 + 14.5 \\ \pm 0.5\sqrt{5(8x_1+17)} - 2.5 \end{bmatrix}.$$

Since $y = (-0.5120, 2.7547)$ *at* $x = (0.6362, 1.9515)$ *which is a common point for* $\mathcal{CR}_1^L$, $\mathcal{CR}_2^L$ *and* $\mathcal{CR}_2^L$*, the only valid nonlinear parametric solution in* $\mathcal{CR}_2^{NL}$ *is*

$$Y_{NL}(x) = \begin{bmatrix} -0.5\sqrt{5(8x_1+17)} - 5x_2 + 14.5 \\ 0.5\sqrt{5(8x_1+17)} - 2.5 \end{bmatrix}.$$

Therefore, the optimal solution is given by,

$$y(x) = \begin{cases} \begin{bmatrix} 0.6681x_1 - 1.3340x_2 + 1.6663 \\ -0.3319x_1 + 0.6660x_2 + 1.6663 \end{bmatrix} & \text{on } \mathcal{CR}_1^L, \\ \begin{bmatrix} 0.5x_1 - 3.5x_2 + 6 \\ -0.5x_1 - 1.5x_2 + 6 \end{bmatrix} & \text{on } \mathcal{CR}_2^L, \\ \text{Infeasible} & \text{on } \mathcal{CR}_1^{NL}, \\ \begin{bmatrix} -0.5\sqrt{5(8x_1+17)} - 5x_2 + 14.5 \\ 0.5\sqrt{5(8x_1+17)} - 2.5 \end{bmatrix} & \text{on } \mathcal{CR}_2^{NL}, \end{cases}$$

where $\mathcal{CR}_1^L, \mathcal{CR}_2^L, \mathcal{CR}_1^{NL}$ *and* $\mathcal{CR}_2^{NL}$ *are as indicated in Fig.5.12.*

Incorporating the solution $y(x)$ *into the upper-level problem of* (5.4.29) *and solving the resulting nonlinear problems in each critical regions, we get following optimal solutions:*
In $\mathcal{CR}_1^L$, $(x,y) = (-0.4365, 1.0305, -0.0000, 2.4975)$ *and objective value* $F = -2.2574$.
In $\mathcal{CR}_2^L$: $(x,y) = (1.2998, 1.9000, -0.0000, 2.5002)$ *and objective value* $F = 3.7788$.
In $\mathcal{CR}_2^{NL}$: $(x,y) = (-0.5710, 1.1428, 4.8437, 1.4421)$ *and objective value* $F = -2.2857$.

Now, comparing all the values of the objective of the leader in each of the critical regions, we can see that the objective value obtained in $\mathcal{CR}_2^{NL}$ *gives a better result. Hence we take the solution in* $\mathcal{CR}_2^{NL}$ *as an optimal solution to the upper-problem of* (5.4.29).

Therefore, the optimal solution to the bilevel MLMF problem (5.4.28) *is*
$(x^*,\ y^*) = (-0.5710,\ 1.1428,\ 4.8437,\ 1.4421)$ *with optimal leaders objective* $F_1 = 7.2880$ *and* $F_2 = -8.4355$*; and optimal followers objective* $f_1 = 68.9219$ *and* $f_2 = 41.1532$. □

Remark 18. ***(i)*** *After the reformulation of multilevel multi-leader multi-follower problems into multilevel hierarchical programs having a single objective function at each level of the hierarchy, we applied the multi-parametric problem approach for convex case and branch-and-bound multi-parametric problem approach for non-convex case. Hence, in addition to the assumption that* $0 < \rho_k^c(\cdot) < \infty$, *we also require to assume that* $\rho_k^c(\cdot)$ *is twice continuously differentiable for all* $y_k \in Y_k, k = 1, \ldots, K$, $c = n, i, j$, $\forall n,\ n = 1, \ldots, N$, $\forall i,\ i = 1, \ldots, M$ *and* $\forall j,\ j = 1, \ldots, L$. ***(ii)*** *When the values of* $\rho_k^c(\cdot)$ *is very large the numerical methods implemented in the actual procedure of solving* (1.2.1) *may approximate* $\frac{1}{\rho_k^c(\cdot)}$ *by zero. But this does not have a significant effect on the optimality of the solution. Therefore, the proposed solution approach is still valid even if the values of any of the varying factor function, (*$\rho_k^c(\cdot)$ *for all* k, c*), of the non-separable terms are very large.*

5.5 Chapter summary

The equivalence reformulation proposed in Sections 5.1 and 5.2 maintains the equilibrium points of the original problem and does not require the smoothness of the involved functions, so that we can use any of the existing methods to solve the resulting multilevel problem with one player at each level. However, due to the requirements of the multi-parametric solution methods (that is used to solve multilevel hierarchical problems in this work), we additionally imposed second order smoothness conditions as well as convexity of the lower level problems. These conditions may not be necessary if one uses other methods (like the heuristic methods) to solve the resulting equivalent multilevel hierarchical optimization problems. Moreover, if the non-separable terms are non-convex, the problem will be non-convex multilevel and can still be solved applying the convexification approach.

Chapter 6

Conclusions and recommendations for future research

Contents

The results presented in this dissertation are related to hierarchical multilevel MLMF games in which the objective functions contain separable and non-separable terms (but the non-separable terms can be written as a factor of two functions, a function which depends on other level decision variables and a function which is common to all objectives across the same level) and shared constraint; and related to bilevel programming problems whose lower-level problem involve convex nonlinear constraints.

6.1 Conclusive remarks

The solution algorithms proposed in this dissertation are based on multi-parametric optimization, but existing exact solution techniques for multi-parametric optimization are limited to multi-parametric problems with polyhedral constraints. So, in Section 3.3 a novel solution strategy for multi-parametric problems with nonlinear constraints and bounded regions is developed; the proposed solution strategy serve as a base in the developments of solution algorithm for bilevel nonlinear optimization problems. The method uses a barrier function to reformulate a multi-parametric problem with convex nonlinear constraints and bounded parametric region as a barrier multi-parametric problems with polyhedral constraints. Methods to find solutions of the nonlinear multi-parametric problem in the interior and on the boundary of the nonlinear constraints and methods of exploration of the parameter space are also provided. The method employs a barrier function reformulation technique to construct a barrier multi-parametric problem with polyhedral constraints. The method gives exact solutions for problems whose objective and constraints are quadratic with respect to the optimization variable and linear with respect to the parameters vectors.

A multi-parametric optimization based algorithm for the solution of bilevel programming problems whose lower-level problem involve convex nonlinear constraints is developed in Chapter 4. The solution strategy starts by recasting the lower-level problem of the bilevel optimization as multi-parametric programming problem where the variable from upper-level problem is considered as parameter. Next, a barrier function reformulation technique is employed to construct a multi-parametric barrier problem with only polyhedral constraints. Then, solutions of the barrier problem is computed in the interior and on the boundary of the nonlinear constraints by a global solution strategy developed in Section 3.3 to obtain a set of parametric solutions and the corresponding critical regions. Finally, the optimal parametric solution is substituted into the upper-level problems and standard nonlinear optimization algorithms are used to solve the resulting nonlinear optimization problem.

Solution procedure for hierarchical multilevel multi-leader multi-follower games is proposed in Chapter 5. The class of multilevel MLMF games considered in this chapter are those games satisfying the assumptions that: (i) the objective at all levels have separable and non-separable terms (but the non-separable terms can be written as a factor of two functions where the first one is a function of other level decision variables and the second factor is common to all objectives across the same level), (ii) at each level there is a shared constraint common to all problems of same level, and (iii) the leader and followers have their own decision variables and objective functions .

The solution procedure transforms the given multilevel MLMF game into an equivalent hierarchical multilevel game having a single decision maker at each level of the hierarchy. The transformation maintains the equilibrium points of the original problem, so that we can use existing methods to solve the resulting hierarchical multilevel problem with one player at each level. The proposed equivalence reformulation does not require the smoothness of the involved functions. However, due to the requirements of the multi-parametric solution methods (that is used to solve multilevel hierarchical problems in this dissertation), we additionally imposed second order smoothness conditions as well as convexity of the lower level problems. These conditions may not be necessary if one uses other methods (like the heuristic methods) to solve the resulting equivalent multilevel hierarchical optimization problems. Moreover, if the non-separable terms are non-convex, the problem will be non-convex multilevel and can still be solved applying the convexification approach.

6.2 Challenges and future research

The contributions presented in this dissertation provided a method to transform some class of hierarchical multilevel MLMF games into hierarchical multilevel games with a single decision maker at each level and proposed a solution algorithm to solve multilevel MLMF games. The existence of equilibria have been obtained for hierarchical multilevel multi-leader multi-follower games with non-separable objective functions. We have transformed such problems into equivalent multilevel hierarchical problems having a single decision maker at each level of the hierarchy. The transformation is limited to problems with the property that the objective functions at all levels have separable and non-separable terms. The non-separable terms are assumed to be written as a factor of two functions where the first one is a function of other level decision variables and the second factor is common to all objectives across the same level. However, in some practical problems the factor of the non-separable terms which varies among objectives of the same level, could vary not only with the variation in the choice of the variables of other level decision makers but also in the choice of variables of other decision makers of the same level. This situation is not addressed by the solution proposed in this dissertation.

Another limitation is related to the solution technique used, multi-parametric optimization, to solve the multilevel hierarchical problem. Existing exact solution techniques for parametric optimization are limited to problems with specific mathematical structures (linear, quadratic, etc). We have developed a novel multi-parametric based solution procedure that can solve a bilevel optimization problem with convex nonlinear constraints, but it is difficult to extend the proposed solution procedure to hierarchical problems with three or higher levels.

Despite the limitations discussed above, many of the presented ideas and concepts of the current study are extendable to other problem domains and optimization algorithms. In the future, I would like to continue the research on multilevel hierarchical and hierarchical multilevel multi-leader multi-follower problems by extending the developed solution schemes to different setting for the objective functions and constraint sets.

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