

SOFT X-RAY PRODUCTION BY PHOTON SCATTERING FROM AN ACCRETING NEUTRON STARS

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To my parents.

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Abstract

We have calculated Compton scattering in strong magnetic fields and obtained a full expression of the scattering cross-section(σ) of a particle, or the total area of the scatterer as seen by the incident photons. This Scattering Cross-section is valid only for the scattering of a low-frequency photon with non-relativistic electrons. We have made a plot of scattering cross-section versus incident photon frequency and tried to see the relationship the two parameters have got with respect to each other. The comparison of the incident and the scattered photon frequencies and the relationship between the two is also shown. That is, we have tried to show that as our mechanism of soft x-ray production is compton scattering, the photon frequency after scattering is less than its incident value. This tells us the fact that low photon frequencies are obtained and this inturn implies the production of soft X-rays at the surface of the neutron stars. Therefore, there exists a mechanism as a source of soft($E_s < 10kev$) radiation in binary neutron stars, in the form of photon scattering with both the initial and final electrons on the ground magnetic states. This mechanism provides a copious source of soft photons which can be redistributed to higher frequencies by inverse Comptonization.

Table of Contents

Table of Contents	vi
List of Figures	1
Introduction	2
1 Neutron Stars	5
1.1 Introduction	5
1.2 Accretion in Neutron Stars	6
1.3 The Efficiency of the Accretion Process	7
1.4 X-ray Binaries	8
2 Thermal Emission of soft X-rays from Cooling Neutron stars	13
3 SOFT X-RAY PRODUCTION BY COMPTON SCATTERING	17
3.1 Introduction	17
3.2 Cross-section in laboratory frame	17
3.2.1 Derivation of the magnetic Feynman propagator	18
3.2.2 Calculation of the scattering amplitude	23
3.2.3 Cross-section in the Electron rest frame	37
4 Results and Discussion	50
4.1 The scattering cross-section and incident photon frequency	50
4.2 The incident and the scattered photon frequencies	53
5 Conclusion	55
Bibliography	56
Declaration	58

List of Figures

1.1	Binary system of a compact object and a companion star. Accreting gas forms a disk around the compact object.	10
1.2	a LMXB system and its typical length length scales.	11
2.1	Cooling of neutron stars	14
2.2	Cooling curves for neutron star models with 1.1, 1.2..., 1.7 M_{\odot}	15
4.1	Cross-section versus $\frac{\omega_i}{\omega_i}$	52
4.2	Scattered photon frequency versus Incident photon frequency	54

Introduction

Soft X-rays from neutron stars is of different origin. These origins can be grossly divided into thermal and non-thermal sources. Most X-ray emitting Isolated Neutron stars are radio pulsars. Their X-ray radiation may be generated by relativistic particles in the pulsar magnetosphere (non thermal radiation) or it may be emitted from the NS surface if its temperature is high enough [1]. So far, more than 30 radio pulsars have been detected in X-rays. It is believed that at least 4 of them (PSR B0656+14, PSR B1055-52, the Vela pulsar PSR 0833-45, and Geminga, whose radio pulsations have been discovered only recently) are thermal emitters (their radiation comes from the NS surface layers or atmospheres). In general, X-rays from isolated (non-accreting) neutron stars are not entirely pulsed, and sometimes no pulsed fraction has been detected at all, i.e there is a significant fraction in the X-ray luminosity, which is not of magnetospheric origin. The spectrum of this radiation is of thermal nature. During the stellar collapse phase at a supernova explosion, the temperature of the forming compact remnant reaches several 100 Billion degrees, which then rapidly (within hours and a few days) cools down by a factor of more than 100 through neutrino emission. Depending on neutron star cooling models the neutron star surface will stay above temperatures of a few million degrees for about 10^5 years during which it cools down by mostly emitting X-rays. Cooling via neutrino emission is restricted

mostly to the first 10^3 years. The four above-mentioned pulsars are in their adolescent age. The youngest, Vela pulsar, is only 10,000 years old, and the oldest, PSR B1055-52, is 500,000 years old. Most radio pulsars are much older and colder objects, with temperatures less than 100,000 K, so that their thermal radiation is extremely faint in the X-ray range, although they can be observed in the UV/optical range. However, a small fraction of their surfaces near the magnetic poles, so-called polar caps, may be hot enough to emit detectable X-rays. Even very old millisecond pulsars, whose typical ages are of the order of a few billion years, may have hot polar caps.

There exist a class of NSs which are not truly isolated but may display themselves as INSs during long time intervals. These NSs are components of binary systems which show transient behavior in which long periods of quiescence are interrupted by powerful X-ray bursts caused by accretion of stellar material from the secondary companion. The energy released during these accretion episodes heats the NS crust, and this heat is further radiated from the NS surface during quiescence. Soft X-rays can also be produced by photon scattering at the surface of the neutron stars. In this paper our main focus is on the production of soft X-rays by photon scattering at the surface of the neutron stars.

In binary systems, the radiation from strong X-ray sources is generally believed to be due to accretion onto a neutron star from a binary companion star [2]. The occurrence of pulsation in some X-ray sources is attributed to a strong magnetic field associated with the neutron star which channels the captured matter onto the polar regions of the star. Near the surface of the neutron star, photon densities can arise that are greater than the atmospheric particle densities. Such a situation is conducive to emission by Compton scattering, which becomes an important source of soft radiation [3]. The

process involves Compton scattering in which the electron's initial and final states are both in the magnetic ground state. Charged particles on higher Landau levels have a very short lifetime $\tau \leq 10^{-15} s$ due to the cyclotron radiation in strong magnetic fields [4]. It is therefore reasonable to assume that electrons either before or after the magnetic scattering should be on the ground Landau level. That is to say an electron can only occupy a higher Landau level in its intermediate states. This mechanism provides a new source of soft photons which can be redistributed to higher frequencies by inverse Comptonization. Some general idea about Neutron stars, X-ray binaries and accretion is briefly discussed in the first chapter of this paper. As is the case in almost all isolated neutron stars, the only mechanism of soft X-ray production is through thermal processes, thermal emission of soft X-rays from cooling neutron stars is outlined in the second chapter of this thesis. The full expression for the scattering cross-section is calculated in the third chapter, which is in fact Herold's scattering cross-section. The interpretation of the results obtained from production of soft X-rays by Compton scattering is discussed.

Chapter 1

Neutron Stars

1.1 Introduction

A cloud in space collapses due to gravity. In a normal star the particles 'feel' gravity as well but do not collapse because gravity is balanced by the radiation pressure and the temperature dependent gas pressure. For a massive star, the radiation pressure is produced by successive fusion of the elements of iron. If the temperature is below the fusion temperature of an available element, little contraction can raise the temperature to ignition level. A star which has no fuel left for fusion in its core, collapses to either a white dwarf, a neutron star or a black hole depending on its mass. In a white dwarf, the gravity is balanced by the degenerate-electron pressure which prevents further gravitational collapse. For a massive predecessor, the degenerate electron pressure is not sufficient to prevent a further collapse. Such a massive star will collapse to a neutron star in which degenerate-neutron pressure balances gravity. If the star is too massive for that, the star collapses to become a black hole. This event, the collapse of a massive star to a neutron star produces a supernova explosion. A

typical neutron star has a mass of $1.4 - 10M_{\odot}$ and a radius of about $10km$ (The predecessor could have a radius upto 10^5) [5]. White dwarfs, neutron stars and black holes are in general referred to as compact objects, and they are extremely dense.

1.2 Accretion in Neutron Stars

By accretion, we mean the accumulation of diffuse gas or matter onto some object under the influence of gravity. The extraction of gravitational potential energy from material which accrete onto a gravitating body is known to be the principal source of power in several types of close binary systems (as will be treated in the following section), and is widely believed to provide the power supply in active galactic nuclei, and quasars. Thus, the new role for gravity arises because accretion on to compact objects is natural and powerful mechanism for the production of high energy radiation. Some simple order of magnitude estimate will show how this works. For a body of mass M_0 and radius R_0 , the gravitational potential energy released by the accretion of a mass ΔM on its surface is $\Delta E = \frac{GM_0\Delta M}{R_0}$. If the accreting body is neutron star with radius $R_0 = 10km$ and $M_0 = M_{\odot}$, then the yield ΔE is about $1.3 \times 10^{20}erg$ per accreted gram. This radiation is expected to be released mainly in the form of electromagnetic radiation [6]. For comparison, we refer to the energy released to the Hydrogen burning in which Hydrogen atoms fuse to produce Helium atoms: $\Delta E_{nuc} = 6 \times 10^8 ergg^{-1}$. Therefore, we can see that $\Delta E_{nuc} \ll \Delta E$. At high luminosity, the accretion rate may itself be controlled by an outward momentum transferred by scattering and absorption. It might seem as though we could generate arbitrarily large luminosities by allowing material to fall at a sufficiently great rate onto a star. However, there is a limit to this luminosity, since if the luminosity is too great,

radiation pressure will blowaway the infalling matter. This limiting luminosity, which is known as the Eddington luminosity, is found by balancing inward force of gravity against the outward pressure of the radiation [7].

1.3 The Efficiency of the Accretion Process

Consider the accretion of matter onto a star of mass M and radius R . If matter of mass ΔM falls onto the star in free-fall from infinity, it acquires kinetic energy as its gravitational potential energy becomes more and more negative. Considering a proton falling in from infinity, we can write $\frac{1}{2}m_p\nu_{ff}^2 = \frac{GMm_p}{r}$, where ν_{ff} is the velocity of free-fall. Assuming all the matter accumulates on the surface of the star, the kinetic energy of the free-fall has to be radiated away as heat which is available to power X-ray source. If the rate at which mass is accreted on to the star is \dot{m} , then the rate at which kinetic energy is dissipated at the surface of the star is $\frac{1}{2}\dot{m}\nu_{ff}^2$, and hence the luminosity of the source is

$L = \frac{1}{2}\dot{m}\nu_{ff}^2 = \frac{GM\dot{m}}{R}$. Introducing the Schwarzschild radius of a star of mass M as $r_g = \frac{2GM}{c^2}$, we find $GM = \frac{c^2 r_g}{2}$. Thus, $L = \frac{1}{2}\dot{m}c^2(\frac{r_g}{R})$. Therefore, the luminosity can be written as $L = \xi\dot{m}c^2$, where ξ is the efficiency of conversion of the rest mass energy of the accreted matter into heat. $\xi = \frac{1}{2}(\frac{r_g}{R})$. For a neutron star of mass $M = M_\odot$ and $R = 15km$, $\xi = 0.1$. Thus, accretion on to a neutron stars is a remarkably powerful source of energy. The largest release of nuclear binding energy occurs in the conversion of hydrogen into helium for which $\xi \simeq 7 \times 10^{-3}$. Thus, accretion on to neutron stars is an order of magnitude more efficient as an energy source as compared with nuclear energy generation.

1.4 X-ray Binaries

X-ray binaries belong to the system in which a compact object and a stellar companion orbit each other at a distance small enough to enable mass transfer from the companion star (secondary or donor) onto the compact object (primary or accretor) [8]. X-ray binaries containing a neutron star or a black hole can be divided into two classes based on the mass of the companion star: high mass X-ray binaries (HMXBs) having a stellar companion usually more massive than $10M_{\odot}$, and low mass X-ray binaries (LMXBs) having a stellar companion with a mass around $1M_{\odot}$ or less. In HMXBs mass transfer is caused by the stellar wind of the compact star (see Fig.), while in LMXBs mass is accreted by Roche lobe overflow from a low mass star to a neutron star (see below). Neutron stars in X-ray binaries are relatively young and theoretically estimated to have a strong magnetic field of the strength $10^{12} - 10^{13}G$ [9]. The Roche model makes use of gravitational equipotential surfaces. The equipotential surface through L_1 , the point where the gravitational forces of the companion and the compact object and the centrifugal force cancel, consists of two lobes called the Roche lobes. When, due to its evolutionary phase of expansion, the companion fills its Roche lobe, matter will stream through L_1 to the compact object. This process is called Roche lobe overflow. Because both binary components orbit each other, and because of conservation of angular momentum, the transferred mass is not able to fall straight to the compact object and therefore forms a disk from which matter spirals inwards following the magnetic field lines. The gas in the accretion disk also transports angular momentum outwards by friction and viscosity. The flow from the companion star causes a "hot spot" at the p-point where it reaches the accretion disk. Due to the large amounts of gravitational energy (up to several 10^{38}erg/s) that

are released by this accretion, matter is heated up to $\sim 10^7 K$ which is high enough for thermal X-ray emission. X-ray binaries are the brightest X-ray sources in the sky (after the sun). In the case of neutron stars, the X-ray emission originates from both the accretion disk and the neutron star surface, whereas in black holes the X-ray emission only comes from the accretion disk because black holes do not have a solid surface.

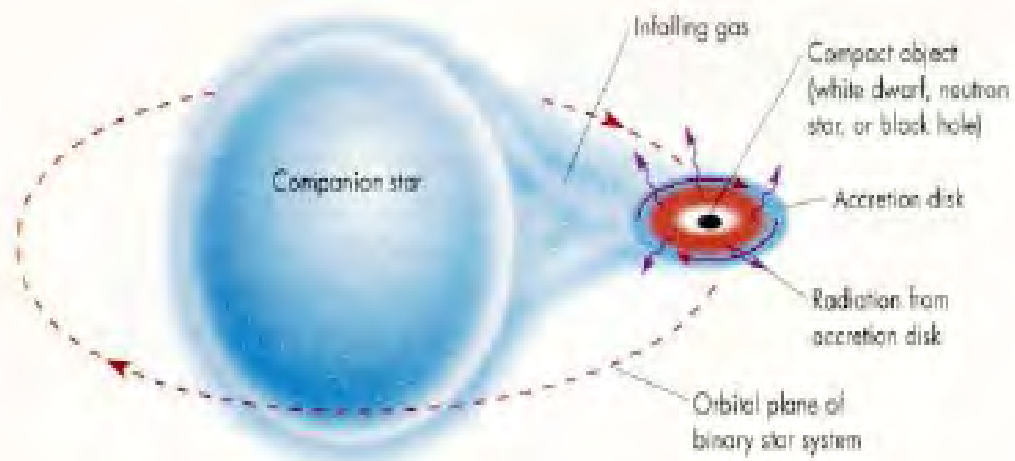


Figure 1.1: Binary system of a compact object and a companion star. Accreting gas forms a disk around the compact object.

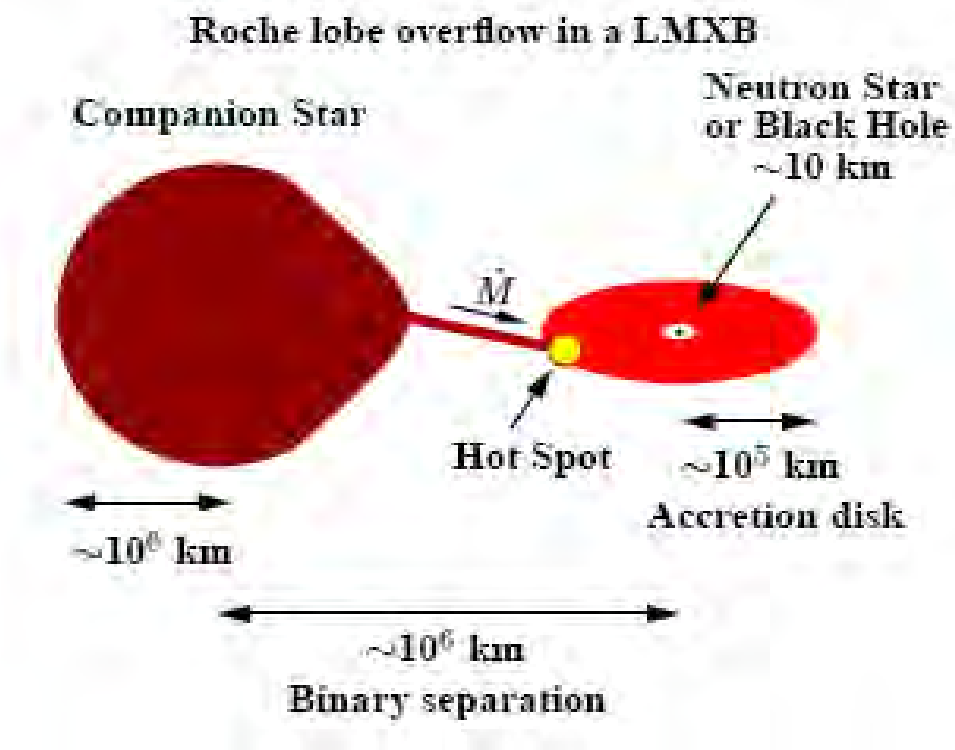


Figure 1.2: a LMXB system and its typical length length scales.

Chapter 2

Thermal Emission of soft X-rays from Cooling Neutron stars

A neutron star is created in the aftermath of the gravitational collapse of the core of a massive star ($\sim 8 - 20M_{\odot}$) at the end of its life. Such stars have core temperatures of $T_c = 10^9 K$ [10] prior to collapse. Following collapse, it is expected that the neutron star forms with central temperature $T_c \sim 10^{11} K$ [11]. At the moment of a neutron star's birth, the star cools off very quickly by neutrino emission, so that within a couple of seconds the temperature is below $10^{11} K$ and falling fast [12]. The cooling is realized via two channels, by neutrino emission from the entire stellar body and by heat conduction from internal layers to the surface resulting in **thermal emission of photons (i.e., soft X-rays)** [13]. While the powerful neutrino quickly cools the core, the crust stays hot. The heat gradually flows inward on a conduction time-scale, and the whole process can be thought as a cooling wave propagation from the centre towards the surface. During this thermal relaxation, the effective temperature stays almost constant at about $250 eV$. When the cooling wave reaches the surface, the effective temperature drops sharply by as much as an order of magnitude in the fast-cooling scenario, and by a factor of 2-3 in the slow cooling scenario. The duration

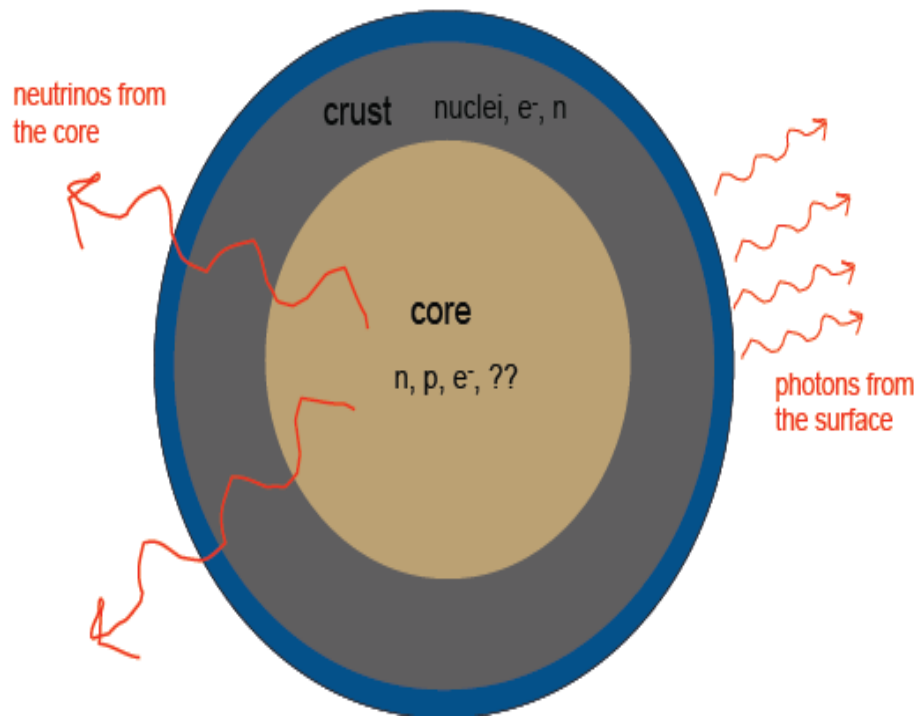
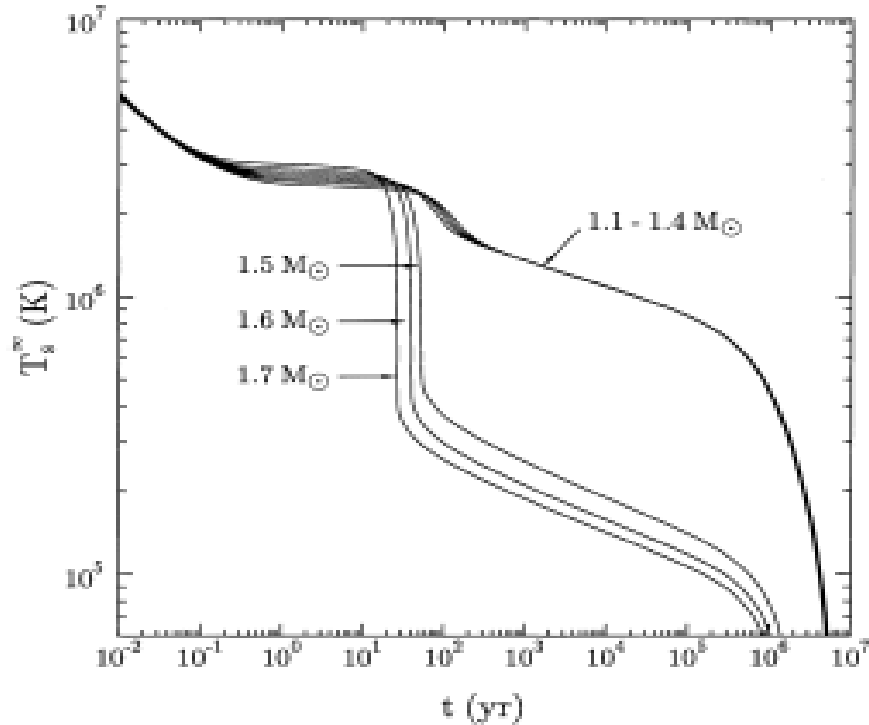


Figure 2.1: Cooling of neutron stars



cv.png

Figure 2.2: Cooling curves for neutron star models with 1.1, 1.2, ..., $1.7M_{\odot}$

of the relaxation epoch depends on the heat capacity and thermal conductivity of the inner crust [14]. Fig 2.1 above shows some cooling models of neutron stars with different masses. In the low-mass models, $M < 1.44M_{\odot}$, the stars follow the standard cooling scenario and the cooling curves are almost independent of M . The higher-mass models go through the fast-cooling scenario and demonstrate a spectacular drop in surface temperature at the ends of the thermal relaxation epoch, $t \sim 50yr$, as a result of the emergence of the cooling wave on the surface. The same, although much less pronounced, effect takes place in the case of slow cooling. For high-mass stars ($M > 1.44M_{\odot}$), the cooling curves again depend weakly on the mass. The change

of the slope of the cooling curves at $t \sim 10^5 - 10^6 yr$ manifests the transition from the neutrino to the photon cooling stage. From the above cooling curve, we see that at a temperature of about 2×10^6 , corresponding to a neutron star of mass $1.4M_{\odot}$, the energy is below the soft X-ray range. That is, neutron stars of age around the order of $10^4 yr$ are even do not produce soft X-rays thermally.

Chapter 3

SOFT X-RAY PRODUCTION BY COMPTON SCATTERING

3.1 Introduction

Since Herold's study of the Compton scattering in strong magnetic fields in the electron rest frame (ERF) [15], his expression of cross-section has been widely used in astrophysical calculations. This cross-section is valid only for the scattering of a low-frequency photon with nonrelativistic electrons, and is applicable for the production of Soft X-rays by Compton scattering at the surface of the neutron stars. We will first derive the cross-section in laboratory frame and then arrive at the desired cross-section in the electron's rest frame.

3.2 Cross-section in laboratory frame

Taking into account the fact that the ground Landau level is not degenerate due to the spinor $u_{1,0}(x_1, k) = 0$, the initial and final states of an electron-photon scattering

system can be represented by:

$$|i, t_i\rangle = c_0^+(p_i, t_i)a_{\lambda_i}^+(k_i, t_i)|0\rangle \quad (3.2.1)$$

$$|f, t_f\rangle = c_0^+(p_f, t_f)a_{\lambda_f}^+(k_f, t_f)|0\rangle \quad (3.2.2)$$

where, $c_0^+ = c_{2,0}^+$ is the electron creation operator, a_{λ}^+ is the photon creation operator and $p_{i(f)}$ denotes the incident(outgoing) electron momentum. We stress here that the operators in these two expressions are Heisenberg ones, not the free ones, meaning that interaction have been considered.

3.2.1 Derivation of the magnetic Feynman propagator

Consider the Dirac equation in the form,

$$[i\partial - eA(x) - m]\psi(x, t) = 0 \quad (3.2.3)$$

in which $e > 0$ is assumed and the asymmetry gauge is taken, i.e. $A(x) = (0, Bx_1, 0)$.

The solution of(2.1.5)with positive energy is:

$$\psi_{s,n}^+(x, t) = Nu_{s,n}(x_1, p)e^{[-iE_n(p_3)t + ip_2x_2 + ip_3x_3]} \quad s = 1, 2 \quad (3.2.4)$$

where E_n indicates the Landau energy of an electron and N is a normalization constant.

$$E_n(p_3) = (m^2 + p_3^2 + 2neB)^{1/2}, N = \left[\frac{E_n(p_3) + m}{2E_n(p_3)}\right]^{1/2} \quad (3.2.5)$$

The spinors $u_{1,n}(x_1, p), u_{2,n}(x_1, p)$ are given by:

$$u_{1,n}(x_1, p) = \begin{bmatrix} I_{n-1}(x_1, p_2) \\ 0 \\ \frac{p_3}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_n(x_1, p_2) \end{bmatrix} \quad (3.2.6)$$

$$u_{2,n}(x_1, p) = \begin{bmatrix} 0 \\ I_n(x_1, p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{-p_3}{E_n(p_3)+m} I_n(x_1, p_2) \end{bmatrix} \quad (3.2.7)$$

in which $I_n(x_1, p_2)$ is the harmonic oscillator wave function,

$$I_n(x_1, p_2) = (\lambda\sqrt{\pi}2^n n!)^{-1/2} e^{[-1/2(\frac{x_1}{\lambda} + \lambda p_2)^2]} H_n(\frac{x_1}{\lambda} + \lambda p_2) \quad (3.2.8)$$

obeying the orthogonal condition

$$\int dx_1 I_n(x_1, p_2) I_m(x_1, p_2) = \delta_{nm} \quad (3.2.9)$$

where $\lambda^{-1} = \sqrt{eB}$. The solution of (2.1.5) with negative energy is

$$\psi_{s,n}^{(-)}(x) = N v_{s,n}(x_1, p) e^{[iE_n(p_3)t + ip_2 x_2 + ip_3 x_3]} \quad s = 1, 2 \quad (3.2.10)$$

with the spinors $v_{1,n}(x_1, p) v_{2,n}(x_1, p)$ given by

$$v_{1,n}(x_1, p) = \begin{bmatrix} \frac{-p_3}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_n(x_1, p_2) \\ I_{n-1}(x_1, p_2) \\ 0 \end{bmatrix} \quad (3.2.11)$$

$$v_{2,n}(x_1, p) = \begin{bmatrix} \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{p_3}{E_n(p_3)+m} I_n(x_1, p_2) \\ 0 \\ I_n(x_1, p_2) \end{bmatrix} \quad (3.2.12)$$

It is clear that the spinors $u_{s,n}(x_1, p), v_{s,n}(x_1, p), s=1,2$ form a complete and orthogonalized basis in the four-dimensional vector space. Therefore the Dirac field operators can be expanded as:

$$\psi(x) = \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L} \sum_{p_2, p_3} [c_{s,n}(p, t) u_{s,n}(x_1, p) + d_{s,n}^+(p, t) v_{s,n}(x_1, p)] e^{(ip_2 x_2 + ip_3 x_3)} \quad (3.2.13)$$

$$\psi^+(x) = \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L} \sum_{p_2, p_3} [c_{s,n}^+(p, t) u_{s,n}^+(x_1, p) + d_{s,n}(p, t) v_{s,n}^+(x_1, p)] e^{(-ip_2 x_2 - ip_3 x_3)} \quad (3.2.14)$$

We know that the Feynman propagator of an electron is given by:

$$S_F(x, y) = -i \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = -i \langle 0 | \psi(x) \psi^+(y) \gamma^0 | 0 \rangle, \quad (3.2.15)$$

for $t_x > t_y$

$$\begin{aligned} \Rightarrow S_F(x, y) &= -i \langle 0 | \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L^2} \sum_{p_2, p_3} \{ [c_{s,n}(p, t) u_{s,n}(x_1, p) + d_{s,n}^+(p, t) v_{s,n}(x_1, p)] \\ &\quad \times [c_{s,n}^+(p, t) u_{s,n}^+(y_1, p) + d_{s,n}(p, t) v_{s,n}^+(y_1, p)] \} \\ &\quad \times e^{ip_2(x_2 - y_2) + ip_3(x_3 - y_3)} \gamma^0 | 0 \rangle \end{aligned} \quad (3.2.16)$$

$$\begin{aligned} \Rightarrow S_F(x, y) &= -i \langle 0 | \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L^2} \sum_{p_2, p_3} \{ [c_{s,n}(p, t) u_{s,n}(x_1, p) c_{s,n}^+(p, t) u_{s,n}^+(y_1, p)] \\ &\quad + [c_{s,n}(p, t) u_{s,n}(x_1, p) d_{s,n}^+(p, t) v_{s,n}^+(y_1, p)] \\ &\quad + [d_{s,n}^+(p, t) v_{s,n}(x_1, p) c_{s,n}^+(p, t) u_{s,n}^+(y_1, p)] \\ &\quad + [d_{s,n}^+(p, t) v_{s,n}(x_1, p) d_{s,n}(p, t) v_{s,n}^+(y_1, p)] \} \times e^{ip_2(x_2 - y_2) + ip_3(x_3 - y_3)} \gamma^0 | 0 \rangle \end{aligned} \quad (3.2.17)$$

From the anti-commutation relations between the annihilation and destruction operators, we have

$$c_{s,n}(p_1, t) c_{r,m}^+(p_2, t) + c_{r,m}^+(p_2, t) c_{s,n}(p_1, t) = \delta_{sr} \delta_{nm} \delta_{p_1 p_2} \quad (3.2.18)$$

$$d_{s,n}(p_1, t) d_{r,m}^+(p_2, t) + d_{r,m}^+(p_2, t) d_{s,n}(p_1, t) = \delta_{sr} \delta_{nm} \delta_{p_1 p_2} \quad (3.2.19)$$

Therefore,

$$\langle 0 | c_{s,n}(p_1, t) c_{r,m}^+(p_2, t) | 0 \rangle = \delta_{sr} \delta_{nm} \delta_{p_1 p_2}$$

and

$$\langle 0|d_{s,n}(p_1, t)d_{r,m}^+(p_2, t)|0\rangle = \delta_{sr}\delta_{nm}\delta_{p_1p_2},$$

from the properties of raising and lowering operators.

Using these two identities in(2.1.19), we will have

$$\begin{aligned} S_F(x, y) &= -i\langle 0|\sum_{n=0}^{\infty}\sum_{s=1}^2\frac{1}{L^2}\sum_{p_2, p_3}[c_{s,n}(p, t)u_{s,n}(x_1, p)c_{s,n}^+(p, t)u_{s,n}^+(y_1, p)] \\ &\quad \times e^{ip_2(x_2-y_2)+ip_3(x_3-y_3)}\gamma^0|0\rangle \end{aligned} \quad (3.2.20)$$

$$\Rightarrow S_F(x, y) = \frac{-i}{L^2}\sum_{n=0}^{\infty}\sum_{s=1}^2\sum_{p_2, p_3}u_{s,n}(x_1, p)u_{s,n}^+(y_1, p)e^{ip_2(x_2-y_2)+ip_3(x_3-y_3)}\gamma^0 \quad (3.2.21)$$

$$\begin{aligned} &= \frac{-i}{L^2}\sum_{n=0}^{\infty}\sum_{p_2, p_3}\{u_{1,n}(x_1, p)u_{1,n}^+(y_1, p) + u_{2,n}(x_1, p)u_{2,n}^+(y_1, p)\}e^{ip_2(x_2-y_2)+ip_3(x_3-y_3)}\gamma^0\} \\ & \quad (3.2.22) \end{aligned}$$

$$\begin{aligned} u_{1,n}(x_1, p)u_{1,n}^+(y_1, p) &= \begin{bmatrix} I_{n-1}(x_1, p_2) \\ 0 \\ \frac{p_3}{E_n(p_3)+m}I_{n-1}(x_1, p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m}I_n(x_1, p_2) \end{bmatrix} \\ &\quad \times \begin{bmatrix} I_{n-1}(y_1, p_2) & 0 & \frac{p_3}{E_n(p_3)+m}I_{n-1}(y_1, p_2) & i\frac{\sqrt{2neB}}{E_n(p_3)+m}I_n(y_1, p_2) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} I_{n-1}I_{n-1} & 0 & \frac{p_3}{E_n(p_3)+m}I_{n-1}I_{n-1} & \frac{i\sqrt{2neB}}{E_n(p_3)+m}I_{n-1}I_n \\ 0 & 0 & 0 & 0 \\ \frac{p_3}{E_n(p_3)+m}I_{n-1}I_{n-1} & 0 & \frac{p_3^2}{(E_n(p_3)+m)^2}I_{n-1}I_{n-1} & -ip_3\frac{\sqrt{2neB}}{(E_n(p_3)+m)^2}I_{n-1}I_{n-1} \\ \frac{i\sqrt{2neB}}{E_n(p_3)+m}I_nI_{n-1} & 0 & \frac{I\sqrt{2neB}}{(E_n(p_3)+m)^2}I_nI_{n-1} & \frac{2neB}{(E_n(p_3)+m)^2}I_nI_n \end{bmatrix} \quad (3.2.23) \end{aligned}$$

$$\begin{aligned}
u_{2,n}(x_1, p)u_{2,n}^+(y_1, p) &= \begin{bmatrix} 0 \\ I_n(x_1, p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{-p_3}{E_n(p_3)+m} I_n(x_1, p_2) \end{bmatrix} \\
&\times \begin{bmatrix} 0 & I_n(y_1, p_2) & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(y_1, p_2) & \frac{-p_3}{E_n(p_3)+m} I_n(y_1, p_2) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n I_n & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_n I_{n-1} & \frac{-p_3}{E_n(p_3)+m} I_n I_n \\ 0 & \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1} I_n & \frac{2neB}{(E_n(p_3)+m)^2} I_{n-1} I_{n-1} & \frac{i\sqrt{2neB}}{(E_n(p_3)+m)^2} I_{n-1} I_n \\ 0 & \frac{-p_3}{E_n(p_3)+m} I_n I_n & \frac{-ip_3\sqrt{2neB}}{(E_n(p_3)+m)^2} I_n I_{n-1} & \frac{p_3^2}{(E_n(p_3)+m)^2} I_n I_n \end{bmatrix} \quad (3.2.24)
\end{aligned}$$

Summing (2.1.26) and (2.1.28), we will obtain:

$$\begin{bmatrix} I_{n-1} I_{n-1} & 0 & \frac{p_3}{E_n(p_3)+m} I_{n-1} I_{n-1} & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1} I_n \\ 0 & I_n I_n & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_n I_{n-1} & \frac{-p_3}{E_n(p_3)+m} I_n I_n \\ \frac{p_3}{E_n(p_3)+m} I_{n-1} I_{n-1} & \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1} I_n & \frac{(p_3^2+2neB)}{(E_n(p_3)+m)^2} I_{n-1} I_{n-1} & 0 \\ \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_n I_{n-1} & \frac{-p_3}{E_n(p_3)+m} I_n I_n & 0 & \frac{(p_3^2+2neB)}{(E_n(p_3)+m)^2} I_n I_n \end{bmatrix} \quad (3.2.25)$$

We know that

$$E_n(p_3) = (m^2 + p_3^2 + 2neB)^{\frac{1}{2}} \Rightarrow p_3^2 + 2neB = E_n^2(p_3) - m^2 = (E_n(p_3) - m)(E_n(p_3) + m)$$

After inserting this relation into (2.1.29) and multiplying by γ^0 , the propagator will have the form:

$$S_F(x, y) = \frac{1}{L^2} \sum_{p_2, p_3} \int \frac{d\omega}{2\Pi} \sum_{n=0}^{\infty} \frac{S_n(x_1, y_1, p)}{(\omega + i\epsilon)(E_n + m)} \times e^{[-i\omega(t_x - t_y) + ip_2(x_2 - y_2) + ip_3(x_3 - y_3)]} \quad (3.2.26)$$

where,

$$S_n(x_1, y_1, p) = \begin{bmatrix} (E_n + m)I_{n-1}I_{n-1} & 0 & -p_3I_{n-1}I_{n-1} & i\sqrt{2neB}I_{n-1}I_n \\ 0 & (E_n(p_3) + m) & -i\sqrt{2neB}I_nI_{n-1} & p_3I_nI_n \\ p_3I_nI_n & -i\sqrt{2neB}I_{n-1}I_n & -(E_n - m)I_{n-1}I_{n-1} & 0 \\ i\sqrt{2neB}I_nI_{n-1} & -p_3I_nI_n & 0 & -(E_n - m)I_nI_n \end{bmatrix}$$

3.2.2 Calculation of the scattering amplitude

$$\psi^+(x) = \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L} \sum_{p_2, p_3} [c_{s,n}^+(p, t)u_{s,n}^+(x_1, p) + d_{s,n}(p, t)v_{s,n}^+(x_1, p)] e^{-ip_2x_2 - ip_3x_3} \quad (3.2.27)$$

$$\begin{aligned} \langle 0|Tc_0(p_f, t_f)\bar{\psi}(r_1, t)|0\rangle &= \langle 0|c_0(p_f, t_f)\bar{\psi}(r_1, t)|0\rangle, t_f \rightarrow \infty \\ &= \langle 0|c_0(p_f, t_f)\psi^+(r_1, t)\gamma^0|0\rangle \\ &= \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L} \sum_{p_2, p_3} [\langle 0|c_0(p_f, t_f)c_{s,n}^+(p, t)u_{s,n}^+(x_1, p)\gamma^0|0\rangle \\ &\quad + \langle 0|c_0(p_f, t_f)d_{s,n}(p, t)v_{s,n}^+(x_1, p)\gamma^0|0\rangle] \times e^{-i(p_2x_2 + p_3x_3)} \\ &= \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{L} \sum_{p_2, p_3} [\langle 0|c_0(p_f, t_f)c_{s,n}^+(p, t)u_{s,n}^+(x_1, p)\gamma^0|0\rangle] \times e^{-i(p_2x_2 + p_3x_3)} \\ &= \frac{1}{L} \sum_{p_2, p_3} [\langle 0|c_0(p_f, t_f)c_{2,0}^+(p, t)u_{2,0}^+(x_1, p)\gamma^0|0\rangle] \times e^{-i(p_2x_2 + p_3x_3)} \\ &= \frac{1}{L} \sum_{p_2, p_3} \bar{u}_0(x_1, p_f) e^{-i(p_2x_2 + p_3x_3)} \end{aligned} \quad (3.2.28)$$

The incident(outgoing) electron momentum can be expressed as $p_{i(f)} = (0, a_{i(f)}, p_{i(f)})$, meaning the incident(outgoing) electron momentum along the direction of the magnetic field(taken as the Z direction) is $p_{i(f)}$ and the centre of the Landau orbit is $-\lambda^2 a_{i(f)}$

$$\Rightarrow \langle 0|Tc_0(p_f, t_f)\bar{\psi}(r_1, t)|0\rangle = \frac{1}{L} \bar{u}_0(x_1, p_f) e^{(-iE_f t_f - ip_f z_1 - ia_f y_1 + iE_f t_1)} \quad (3.2.29)$$

similarly,

$$\langle 0|Ta_{\lambda_f}(k_f, t_f)A_{\mu}(\vec{r}_1, t)|0\rangle = \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2\omega_f}} \epsilon_{\mu}^{(\lambda_f)} e^{(-i\omega_f t_f + i\omega_f t_1 - i\vec{k}_f \cdot \vec{r}_1)} \quad (3.2.30)$$

but, $i\vec{k}_f \cdot \vec{r}_1 = ik_{fx}x_1 + ik_{fy}y_1 + ik_{fz}z_1$,

where, $k_{fz} = k_f \cos\theta_f$

$$\langle 0|Ta_{\lambda_f}(k_f, t_f)A_\mu(\vec{r}_1, t)|0\rangle = \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2\omega_f}} \epsilon_\mu^{(\lambda_f)} e^{(-i\omega_f t_f + i\omega_f t_1 - ik_{fx}x_1 - ik_{fy}y_1 - ik_{fz}z_1)} \quad (3.2.31)$$

$$\langle 0|T\psi(\vec{r}_2, t)c_0^+(p_i, t_i)|0\rangle = \frac{1}{L} u_0(x_2, p_i) e^{(iE_i t_i + ip_i z_2 + ia_i y_2 - iE_i t_2)} \quad (3.2.32)$$

$$\langle 0|Ta_{\lambda_i}^+(k_i, t_i)A_\mu(\vec{r}_1, t)|0\rangle = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_i}} \epsilon_\mu^{(\lambda_i)} e^{(i\omega_i t_i - i\omega_i t_2 + i\vec{k}_i \cdot \vec{r}_2)} \quad (3.2.33)$$

$$\langle 0|Ta_{\lambda_i}^+(k_i, t_i)A_\mu(\vec{r}_1, t)|0\rangle = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_i}} \epsilon_\mu^{(\lambda_i)} e^{(i\omega_i t_i - i\omega_i t_2 + ik_{ix}x_2 + ik_{iy}y_2 + ik_i \cos\theta_i z_2)} \quad (3.2.34)$$

The scattering matrix can be generally expressed by:

$$S_{fi} = \lim_{t_i \rightarrow -\infty, t_f \rightarrow \infty} \langle 0|T[c_0(p_f, t_f)a_{\lambda_f}(k_f, t_f)c_0^+(p_i, t_i)a_{\lambda_i}^+(k_i, t_i)]|0\rangle \quad (3.2.35)$$

Introducing the incoming interaction picture and making perturbation expansions, one then obtains under the Born approximation:

$$\begin{aligned} S_{fi} &= \lim_{t_i \rightarrow -\infty, t_f \rightarrow \infty} e^2 \int d^4x_1 d^4x_2 \langle 0|Tc_0(p_f, t_f)\bar{\psi}(x_1)|0\rangle \\ &\times \gamma_\mu \langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \gamma_\nu \\ &\times [\langle 0|Ta_{\lambda_f}(k_f, t_f)A_\mu(x_1)|0\rangle \langle 0|Ta_{\lambda_i}^+(k_i, t_i)A_\nu(x_2)|0\rangle \\ &+ \langle 0|Ta_{\lambda_f}(k_f, t_f)A_\nu(x_2)|0\rangle \langle 0|Ta_{\lambda_i}^+(k_i, t_i)A_\mu(x_1)|0\rangle] \\ &\times \langle 0|T\psi(x_2)c_0^+(p_i, t_i)|0\rangle \end{aligned} \quad (3.2.36)$$

$$\begin{aligned}
S_{fi} &= \lim_{t_i \rightarrow -\infty, t_f \rightarrow \infty} e^2 \int d^4x_1 d^4x_2 \frac{1}{L} \bar{u}_0(x_1, p_f) e^{-i(E_f t_f - E_f t_1) - i(p_f z_1 + a_f y_1)} \\
&\times \gamma_\mu \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle \gamma_\nu \\
&\times \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2\omega_f}} \epsilon_\mu^{(\lambda_f)} e^{(-i\omega_f t_f + i\omega_f t_1 - ik_{fx} x_1 - ik_{fy} y_1 - ik_f \cos \theta_f z_1)} \\
&\times \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_i}} \epsilon_\mu^{(\lambda_i)} e^{(i\omega_i t_i - i\omega_i t_2 + ik_{ix} x_2 + ik_{iy} y_2 + ik_i \cos \theta_i z_2)} \\
&\times \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2\omega_f}} \epsilon_\nu^{(\lambda_f)} e^{-i(\omega_f t_f - \omega_f t_2) - ik_{fx} x_2 - ik_{fy} y_2 - ik_f \cos \theta_f z_2} \\
&\times \left(\frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_i}} \epsilon_\mu^{(\lambda_i)} e^{i(\omega_i t_i - i\omega_i t_1) + ik_{ix} x_1 + ik_{iy} y_1 + ik_i \cos \theta_i z_1} \right) \\
&\times \frac{1}{L} u_0(x_2, p_i) e^{i(E_i t_i - E_i t_2) + i(p_i z_2 + a_i y_2)} \tag{3.2.37}
\end{aligned}$$

$$\begin{aligned}
S_{fi} &= \lim_{t_i \rightarrow -\infty, t_f \rightarrow \infty} \frac{1}{L^2 V} \frac{1}{\sqrt{4\omega_i \omega_f}} \exp(2) \int d^4x_1 d^4x_2 \bar{u}_0(x_1, p_f) \gamma_\mu \\
&\times \frac{1}{L^2} \sum_{q_y, q_z} \int \frac{dq_0}{2\pi} \sum_{n=0}^{\infty} \frac{S_n(x_1, x_2, q)}{q_0^2 - E_n^2(q_z) + i\epsilon} \\
&\times \exp[iq_0(t_2 - t_1) + iq_y(y_1 - y_2) + iq_z(z_1 - z_2)] \gamma_\nu \\
&\times \{ [\exp[-i(E_f t_f - E_f t_1) - i(p_f z_1 + a_f y_1)]] \epsilon_\mu^{(\lambda_f)} \\
&\times \exp[-i(\omega_f t_f - \omega_f t_1) - ik_{fx} x_1 - ik_{fy} y_1 - ik_f \cos \theta_f z_1] \\
&\times \epsilon_\nu^{(\lambda_i)} \exp[i(\omega_i t_i - \omega_i t_2) + ik_{ix} x_2 + ik_{iy} y_2 + ik_i \cos \theta_i z_2] \\
&+ [\exp[-i(E_f t_f - E_f t_1) - i(p_f z_1 + a_f y_1)]] \epsilon_\nu^{(\lambda_f)} \\
&\times \exp[-i(\omega_f t_f - \omega_f t_2) - ik_{fx} x_2 - ik_{fy} y_2 - ik_f \cos \theta_f z_2] \\
&\times \epsilon_\mu^{(\lambda_i)} \exp[i(\omega_i t_i - \omega_i t_1) + ik_{ix} x_1 + ik_{iy} y_1 + ik_i \cos \theta_i z_1] \} \\
&\times u_0(x_2, p_i) \exp[i(E_i t_i - E_i t_2) + i(p_i z_2 + a_i y_2)] \tag{3.2.38}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow S_{fi} &= \frac{(2\Pi)^5}{L^2V} * \frac{e^2}{2\sqrt{\omega_i\omega_f}} \sum_{n=0}^{\infty} \frac{1}{L^2} \sum_{q_y, q_z} \int d^4x_1 d^4x_2 \int dq_0 \bar{u}_0(x_1, p_f) \\
&\times \left[\hat{\epsilon}_f \frac{S_n(x_1, x_2, q)}{(E_i + \omega_i)^2 - E_n^2(q_z) + i\epsilon} \right. \\
&\times \hat{\epsilon}_i \delta(\omega_f + E_f - q_0) \delta(q_0 - \omega_i - E_i) \\
&\times \delta(k_{iy} + a_i - q_y) \delta(q_y - k_{fy} - a_f) \delta(q_z - k_f \cos \theta_f - p_f) \\
&\times \delta(k_i \cos \theta_i + p_i - q_z) \exp(-ik_{fx}x_1 + ik_{ix}x_2) + \hat{\epsilon}_i \frac{S_n(x_1, x_2, q)}{(E_i - \omega_f)^2 - E_n^2(q_z) + i\epsilon} \\
&\times \hat{\epsilon}_f \delta(\omega_f - E_i + q_0) \delta(E_f - \omega_i - q_0) \\
&\times \delta(k_{iy} - a_f + q_y) \delta(a_i - k_{fy} - q_y) \delta(q_z + k_i \cos \theta_i - p_f) \\
&\times \delta(p_i - k_f \cos \theta_f - q_z) \exp(ik_{ix}x_1 - ik_{fx}x_2) \left. \right] u_0(x_2, p_i) \\
&\times \lim_{t_i \rightarrow -\infty, t_f \rightarrow \infty} \exp[-i(E_f + \omega_f)t_f + i(E_i - \omega_i)t_i] \tag{3.2.39}
\end{aligned}$$

after substituting all the expressions given from(2.1.33)to(2.1.39) into(2.1.40), where $\hat{\epsilon}_f = \epsilon_\mu^{(\lambda_f)} \gamma_\mu, \hat{\epsilon}_i = \epsilon_\mu^{(\lambda_i)} \gamma_\mu$.

The phase factor at the end of(2.1.42)can be ignored, since the cross-section $\sim |S_{fi}|^2$. The spinors $\bar{u}_0(x_1, p_f), u_0(x_2, p_i)$ can be expressed as:

$$\bar{u}_0(x_1, p_f) = \left[\frac{E_f + m}{2E_f} \right]^{\frac{1}{2}} I_0(x_1, a_f) \bar{u}_f(p_f) \tag{3.2.40}$$

$$u_0^T(x_2, p_i) = \left[\frac{E_i + m}{2E_i} \right]^{\frac{1}{2}} I_0(x_2, a_i) u_i^T(p_i), \tag{3.2.41}$$

where

$$\bar{u}_f(p_f) = \left(0 \quad 1 \quad 0 \quad \frac{p_f}{E_f + m} \right) \tag{3.2.42}$$

$$u_i^T(p_i) = \left(0 \quad 1 \quad 0 \quad \frac{-p_i}{E_i + m} \right) \tag{3.2.43}$$

$$\begin{aligned}
S_{fi} &= \frac{(2\Pi)^5 \exp(2)}{L^2 V \sqrt{4\omega_i \omega_f}} \sum_{n=0}^{\infty} \frac{1}{L^2} \sum_{q_y, q_z} \left[\frac{(E_f + m)(E_i + m)}{4E_i E_f} \right]^{\frac{1}{2}} \\
&\times \int d^4 x_1 d^4 x_2 \int dq_0 I_0(x_1, a_f) \bar{u}_f(p_f) I_0(x_2, a_i) u_i(p_i) \\
&\times \left\{ \hat{\epsilon}_f \frac{S_n(x_1, x_2, q)}{(E_i + \omega_i)^2 - E_n^2(q_z) + i\epsilon} \hat{\epsilon}_i \delta(\omega_f + E_f - q_0) \delta(q_0 - \omega_i - E_i) \right. \\
&\times \delta(k_{iy} + a_i - q_y) \delta(q_y - k_{fy} - a_f) \delta(q_z - k_f \cos \theta_f - p_f) \\
&\times \delta(k_i \cos \theta_i + p_i - q_z) \exp(-ik_{fx} x_1 + ik_{ix} x_2) \\
&+ \hat{\epsilon}_i \frac{S_n(x_1, x_2, q)}{(E_i + \omega_i)^2 - E_n^2(q_z) + i\epsilon} \\
&\times \hat{\epsilon}_f \delta(\omega_f - E_i + q_0) \delta(E_f - \omega_i - q_0) \\
&\times \delta(k_{iy} - a_f + q_y) \delta(a_i - k_{fy} - q_y) \delta(q_z + k_i \cos \theta_i - p_f) \\
&\times \left. \delta(p_i + k_f \cos \theta_f - q_z) \exp(ik_{ix} x_1 - ik_{fx} x_2) \right\} \tag{3.2.44}
\end{aligned}$$

Let $I_1 = \int d^4 x_1 d^4 x_2 \exp(-ik_{fx} x_1) I_0(x_1, a_f) S_n(x_1, x_2, q) \exp(ik_{ix} x_2) I_0(x_2, a_i)$, and

$$I_2 = \int d^4 x_1 d^4 x_2 \exp(-ik_{fx} x_1) I_0(x_1, a_f) S_n(x_1, x_2, q) \exp(ik_{ix} x_2) I_0(x_2, a_i)$$

With the help of the integral:

$$\int_{-\infty}^{\infty} d\xi H_n(\xi) \exp[-(\xi - \alpha)^2] = \sqrt{\Pi}(2\alpha)^n, \text{ these two integrals yield values,}$$

$$\begin{aligned}
I_1 &= \frac{(\lambda^2 k_f^+ k_i^-)^{(n-1)}}{2^n n!} A_n \exp\left[-\frac{\lambda^2}{4}(k_i^2 \sin^2 \theta_i + k_f^2 \sin^2 \theta_f)\right] \\
&\times \exp\left[i\lambda^2(a_f k_{fx} + \frac{1}{2} k_{fx} k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2} k_{ix} k_{iy})\right], \tag{3.2.45}
\end{aligned}$$

where $k_i^{\pm} = k_{ix} \pm ik_{iy}$ and $k_f^{\pm} = k_{fx} \pm ik_{fy}$ and A_n is defined by

$$A_n = \begin{pmatrix} 2n(q_0 + m) & 0 & -2nq_z & -2nk_i^- \\ 0 & (q_0 + m)\lambda^2 k_f^+ k_i^- & -2nk_f^+ & q_z \lambda^2 k_f^+ k_i^- \\ 2nq_z & -2nk_i^- & -2n(q_0 - m) & 0 \\ 2nk_f^+ & -q_z \lambda^2 k_f^+ k_i^- & 0 & -(q_0 - m)\lambda^2 k_f^+ k_i^- \end{pmatrix} \tag{3.2.46}$$

Similarly,

$$\begin{aligned}
I_2 &= \frac{(\lambda^2 k_f^+ k_i^-)^{(n-1)}}{2^n n!} B_n \exp\left[-\frac{\lambda^2}{4}(k_i^2 \sin^2 \theta_i + k_f^2 \sin^2 \theta_f)\right] \\
&\times \exp\left[-i\lambda^2(a_f k_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} + \frac{1}{2}k_{fx}k_{fy})\right], \quad (3.2.47)
\end{aligned}$$

with the matrix B_n given by

$$B_n = \begin{pmatrix} 2n(q_0 + m) & 0 & -2nq_z & 2nk_f^- \\ 0 & (q_0 + m)\lambda^2 k_f^- k_i^+ & 2nk_i^+ & q_z \lambda^2 k_f^- k_i^+ \\ 2nq_z & -2nk_f^- & -2n(q_0 - m) & 0 \\ -2nk_i^+ & -q_z \lambda^2 k_f^- k_i^+ & 0 & -(q_0 - m)\lambda^2 k_f^- k_i^+ \end{pmatrix} \quad (3.2.48)$$

$$\begin{aligned}
S_{fi} &= \frac{(2\Pi)^5 \exp(2)}{L^2 V \sqrt{4\omega_i \omega_f}} \sum_{n=0}^{\infty} \frac{1}{L^2} \sum_{q_y, q_z} \left[\frac{(E_i + m)(E_f + m)}{4E_i E_f} \right]^{\frac{1}{2}} \\
&\times \frac{(\lambda^2 k_f^+ k_i^-)^{(n-1)}}{2^n n!} \exp\left[-\frac{\lambda^2}{4}(k_i^2 \sin^2 \theta_i + k_f^2 \sin^2 \theta_f)\right] \\
&\times \left\{ \frac{\bar{u}_f(p_f) \hat{\epsilon}_f A_n \hat{\epsilon}_i u_i(p_i)}{(E_i + \omega_i)^2 - E_n^2(q_z) + i\epsilon} \right. \\
&\times \exp\left[i\lambda^2(a_f k_{fx} + \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2}k_{ix}k_{iy})\right] \\
&\times \int dq_0 \delta(\omega_f + E_f - q_0) \delta(q_0 - \omega_i - E_i) \delta(k_{iy} + a_i - q_y) \\
&\times \delta(q_y - k_{fy} - a_f) \delta(q_z - k_f \cos \theta_f - p_f) \delta(k_i \cos \theta_i - p_i - q_z) \\
&+ \frac{\bar{u}_f(p_f) \hat{\epsilon}_i B_n \hat{\epsilon}_f u_i(p_i)}{(E_i - \omega_f)^2 - E_n^2(q_z) + i\epsilon} \\
&\times \exp\left[-i\lambda^2(a_f k_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} + \frac{1}{2}k_{fx}k_{fy})\right] \\
&\times \int dq_0 \delta(\omega_f - E_i + q_0) \delta(E_f - \omega_i - q_0) \delta(k_{iy} - a_f + q_y) \\
&\times \left. \delta(a_i - k_{fy} - q_y) \delta(p_i - k_f \cos \theta_f - q_z) \delta(q_z + k_i \cos \theta_i - p_f) \right\} \quad (3.2.49)
\end{aligned}$$

But,

$$\sum_{q_y, q_z} \int dq_0 \delta(\omega_f + E_f - q_0) \delta(q_0 - \omega_i - E_i) \delta(k_{iy} + a_i - q_y) \delta(q_y - k_{fy} - a_f) \delta(q_z - k_f \cos \theta_f - p_f)$$

$$*\delta(k_i \cos \theta_i - p_i - q_z) = \delta(\omega_f + E_f - \omega_i - E_i) \delta(k_{iy} + a_i - k_{fy} - a_f) \delta(k_i \cos \theta_i + p_i - k_f \cos \theta_f - p_f)$$

Similarly,

$$\sum_{q_y, q_z} \int dq_0 \delta(\omega_f - E_i + q_0) \delta(E_f - \omega_i - q_0) \delta(k_{iy} - a_f + q_y) \times \delta(a_i - k_{fy} - q_y) \delta(p_i - k_f \cos \theta_f - q_z)$$

$$*\delta(k_i \cos \theta_i + q_z - p_f) = \delta(\omega_f + E_f - \omega_i - E_i) \delta(k_{iy} + a_i - k_{fy} - a_f) \delta(k_i \cos \theta_i + p_i - k_f \cos \theta_f - p_f)$$

$$\begin{aligned} \Rightarrow S_{fi} &= \frac{(2\Pi)^3 \exp(2)}{L^2 V \sqrt{4\omega_i \omega_f}} \sum_{n=0}^{\infty} \frac{1}{L^2} \left[\frac{(E_i + m)(E_f + m)}{4E_i E_f} \right]^{\frac{1}{2}} \\ &\times \frac{(\lambda^2 k_f^+ k_i^-)^{(n-1)}}{2^n n!} \exp\left[-\frac{\lambda^2}{4} (k_i^2 \sin^2 \theta_i + k_f^2 \sin^2 \theta_f)\right] \\ &\times \left\{ \frac{P_1}{(E_i + \omega_i)^2 - E_n^2(q_z) + i\epsilon} \right. \\ &\times \exp\left[i\lambda^2 (a_f k_{fx} + \frac{1}{2} k_{fx} k_{fy}) - i\lambda^2 (a_i k_{ix} + \frac{1}{2} k_{ix} k_{iy})\right] \\ &+ \frac{P_2}{(E_i - \omega_f)^2 - E_n^2(q_z) + i\epsilon} \\ &\times \exp\left[-i\lambda^2 (a_f k_{ix} - \frac{1}{2} k_{ix} k_{iy}) + i\lambda^2 (a_i k_{fx} + \frac{1}{2} k_{fx} k_{fy})\right] \left. \right\} \\ &\times \delta(\omega_f + E_f - \omega_i - E_i) \delta(k_{iy} + a_i - k_{fy} - a_f) \\ &\times \delta(k_i \cos \theta_i + p_i - k_f \cos \theta_f - p_f), \end{aligned} \quad (3.2.50)$$

where $P_1 = \bar{u}_f(p_f) \hat{\epsilon}_f A_n \hat{\epsilon}_i u_i(p_i)$ and $P_2 = \bar{u}_f(p_f) \hat{\epsilon}_i B_n \hat{\epsilon}_f u_i(p_i)$, where $\hat{\epsilon}_f = \epsilon_\mu^{\lambda_f} \gamma_\mu$, $\hat{\epsilon}_i = \epsilon_\mu^{\lambda_i} \gamma_\mu$

These two scalar matrix products (i.e. P_1 and P_2) for all combinations of $\gamma_1, \gamma_2, \gamma_3$ will have the form:

$$\begin{aligned} P_1 &= 2n \left[\frac{(q_0 + m) P_i P_f}{(E_i + m)(E_f + m)} + (q_0 - m) \right] \epsilon_f^+ \epsilon_i^- \\ &+ 2n \left[\frac{P_f}{E_f + m} + \frac{P_i}{E_i + m} \right] (k_i \epsilon_f^+ \epsilon_{iz} + k_f^+ \epsilon_{fz} \epsilon_i^- + q_z \epsilon_f^+ \epsilon_i^-) \\ &+ \lambda^2 k_f^+ k_i^- \left[\frac{(q_0 + m) P_i P_f}{(E_i + m)(E_f + m)} + (q_0 - m) + \frac{q_z P_f}{E_f + m} + \frac{q_z P_i}{E_i + m} \right] \epsilon_{fz} \epsilon_{iz} \end{aligned} \quad (3.2.51)$$

$$\begin{aligned}
P_2 &= 2n \left[\frac{(q_0 + m)P_i P_f}{(E_i + m)(E_f + m)} + (q_0 - m) \right] \epsilon_i^+ \epsilon_f^- \\
&- 2n \left[\frac{P_f}{E_f + m} + \frac{P_i}{E_i + m} \right] (k_i^+ \epsilon_f^- \epsilon_{iz} + k_f^- \epsilon_i^+ \epsilon_{fz} + q_z \epsilon_i^+ \epsilon_f^-) \\
&+ \lambda^2 k_f^- k_i^+ \left[\frac{(q_0 + m)P_i P_f}{(E_i + m)(E_f + m)} + (q_0 - m) + \frac{q_z P_f}{E_f + m} + \frac{q_z P_i}{E_i + m} \right] \epsilon_{fz} \epsilon_{iz}
\end{aligned} \tag{3.2.52}$$

Evaluating $\exp[i\lambda^2(a_f k_{fx} + \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2}k_{ix}k_{iy})]$ and $\exp[-i\lambda^2(a_f k_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} - \frac{1}{2}k_{fx}k_{fy})]$

at the value $a_f = k_{iy} + a_i - k_{fy}$:

$$\begin{aligned}
&\exp[i\lambda^2(a_f k_{fx} + \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2}k_{ix}k_{iy})] \\
&= \exp[i\lambda^2(k_{iy}k_{fx} + a_i k_{fx} - k_{fx}k_{fy} + \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2}k_{ix}k_{iy})] \\
&= \exp[i\lambda^2(k_{iy}k_{fx} + a_i k_{fx} - \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2}k_{ix}k_{iy})] \\
&= \exp(i\lambda^2(k_{iy}k_{fx})) \exp[i\lambda^2(-\frac{1}{2}k_{fx}k_{fy} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} - a_i k_{ix})] \\
&= \exp(i\lambda^2(k_{iy}k_{fx})) \exp[-\frac{i\lambda^2}{2}(k_{ix}k_{iy} + k_{fx}k_{fy}) - i\lambda^2 a_i (k_{ix} - k_{fx})]
\end{aligned}$$

Similarly

$$\begin{aligned}
&\exp[-i\lambda^2(a_f k_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} - \frac{1}{2}k_{fx}k_{fy})] \\
&= \exp(i\lambda^2(k_{ix}k_{fy})) \exp[-\frac{i\lambda^2}{2}(k_{ix}k_{iy} + k_{fx}k_{fy}) - i\lambda^2 a_i (k_{ix} - k_{fx})]
\end{aligned}$$

and substituting it back into the original equation together with P_1 and P_2 , the full expression for the S_{fi} will be:

$$\begin{aligned}
S_{fi} &= \frac{(2\Pi)^3 \exp(2)}{L^2 V \sqrt{4\omega_i \omega_f}} \left[\frac{(E_i + m)(E_f + m)}{4E_i E_f} \right]^{\frac{1}{2}} \exp[-\frac{\lambda^2}{4}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)] \\
&\times \exp[-i\lambda^2 a_i (k_{ix} - k_{fx}) - i\frac{\lambda^2}{2}(k_{ix}k_{iy} + k_{fx}k_{fy})] X \\
&\times \delta(E_i + \omega_i - E_f - \omega_f) \delta(p_i + k_i \cos \theta_i - p_f - k_f \cos \theta_f) \\
&\times \delta(a_i - k_{iy} - a_f - k_{fy})
\end{aligned} \tag{3.2.53}$$

where ω_i and ω_f are frequencies of the incident and scattered photons, E_i and E_f represent the energies of the incident and scattered electrons, respectively, $\theta_i(\theta_f)$ denotes the angle between the incoming(outgoing)photon. X is given by the following expression as follows:

$$X = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\frac{\lambda^2 k_i^- k_f^+}{2} \right)^n \exp(i\lambda^2 k_{iy} k_{fx}) \left(\frac{X_1}{(E_i + \omega_i)^2 - E_{i,n+1}^2} + \frac{X_2}{(E_i + \omega_i)^2 - E_{i,n}^2} \right) \right. \\ \left. + \left(\frac{\lambda^2 k_i^- k_f^+}{2} \right)^n \exp(i\lambda^2 k_{ix} k_{fy}) \left(\frac{X_1'}{(E_i - \omega_f)^2 - E_{f,n+1}^2} + \frac{X_2'}{(E_i - \omega_f)^2 - E_{f,n}^2} \right) \right]$$

in which $k_i^\pm = k_{ix} \pm ik_{iy}$, $k_f^\pm = k_{fx} \pm ik_{fy}$,

$$E_{i,n}^2 = m^2 + (p_i + \omega_i \cos \theta_i)^2 + 2neB \quad (3.2.54)$$

$$E_{f,n}^2 = m^2 + (p_i - \omega_f \cos \theta_f)^2 + 2neB \quad (3.2.55)$$

and $X_i, X_i', i = 1, 2$, are given by

$$X_1 = \left[\frac{(E_i + \omega_i + m)p_i(p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f)}{(E_i + m)(E_f + m)} + (E_i + \omega_i - m) \right] e_i^- e_f^+ \\ + \left(\frac{p_i}{E_i + m} + \frac{p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f}{E_f + m} \right) \\ \times [k_f^+ e_{fz} e_i^- + k_i^- e_{iz} e_f^+ - (p_i + \omega_i \cos \theta_i) e_f^+ e_i^-] \quad (3.2.56)$$

$$X_2 = \left[\frac{(E_i + \omega_i + m)p_i(p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f)}{(E_i + m)(E_f + m)} \right. \\ \left. + \left(\frac{p_i}{E_i + m} + \frac{p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f}{E_f + m} \right) \right. \\ \left. \times (p_i + \omega_i \cos \theta_i) + (E_i + \omega_i - m) \right] e_{fz} e_{iz} \quad (3.2.57)$$

$$\begin{aligned}
X'_1 &= \left[\frac{(E_i - \omega_f + m)p_i(p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f)}{(E_i + m)(E_f + m)} + (E_i - \omega_f - m) \right] e_i^+ e_f^- \\
&+ \left(\frac{p_i}{E_i + m} + \frac{p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f}{E_f + m} \right) \\
&\times [k_i^+ e_{iz} e_f^- + k_f^- e_{fz} e_i^+ + (p_i - \omega_f \cos \theta_f) e_i^+ e_f^-] \quad (3.2.58)
\end{aligned}$$

$$\begin{aligned}
X'_2 &= \left[\frac{(E_i - \omega_f + m)p_i(p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f)}{(E_i + m)(E_f + m)} + (E_i - \omega_f - m) \right. \\
&\left. + \left(\frac{p_i}{E_i + m} + \frac{p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f}{E_f + m} \right) (p_i - \omega_f \cos \theta_f) \right] e_{iz} e_{fz} \quad (3.2.59)
\end{aligned}$$

where e_i and e_f are polarizations of the incident and scattered photons and $e_i^\pm = e_{ix} \pm ie_{iy}$, $e_f^\pm = e_{fx} \pm \pm ie_{fy}$.

$$\begin{aligned}
|S_{fi}|^2 &= \frac{(2\Pi)^6 \exp(4)}{L^4 V^2 4\omega_i \omega_f} \left[\frac{(E_i + m)(E_f + m)}{4E_i E_f} \right] \exp\left[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)\right] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})] |X|^2 \\
&\times \delta^2(E_i + \omega_i - E_f - \omega_f) \delta^2(p_i + k_i \cos \theta_i - p_f - k_f \cos \theta_f) \\
&\times \delta^2(a_i - k_{iy} - a_f - k_{fy}) \quad (3.2.60)
\end{aligned}$$

In the expressions for X_1, X'_1, X_2 , and X'_2 putting $p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f = p_f$, we get the following.

$$\begin{aligned}
X_1 &= \{[A - B(p_i + \omega_i \cos \theta_i) e_f^+ e_i^- + B(k_f^+ e_{fz} e_i^- + k_i^- e_{iz} e_f^+)](E_f + m)^{-1} (E_i + m)^{-1}, \\
X_2 &= \{[A + B(p_i + \omega_i \cos \theta_i) e_{fz} e_{iz}]\}(E_f + m)^{-1} (E_i + m)^{-1}, \\
X'_1 &= \{[A' - B(p_i - \omega_f \cos \theta_f) e_f^- e_i^+ - B(k_f^- e_{fz} e_i^+ + k_i^+ e_{iz} e_f^-)](E_f + m)^{-1} (E_i + m)^{-1} \text{ and} \\
X'_2 &= \{[A' + B(p_i - \omega_f \cos \theta_f) e_{fz} e_{iz}]\}(E_f + m)^{-1} (E_i + m)^{-1}
\end{aligned}$$

In the above two expressions the coefficients A, A', B are defined by

$$A = (E_i + \omega_i + m)p_i p_f + (E_i + \omega_i - m)(E_i + m)(E_f + m) \quad (3.2.61)$$

$$A' = (E_i - \omega_f + m)p_i p_f + (E_i - \omega_f - m)(E_i + m)(E_f + m) \quad (3.2.62)$$

$$B = p_i(E_f + m) + p_f(E_i + m) \quad (3.2.63)$$

respectively.

Substituting these results back into the expression for X :

$$\begin{aligned} X &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\frac{\lambda^2 k_i^- k_f^+}{2} \right)^n \exp(i\lambda^2 k_{iy} k_{fx}) \right. \\ &\times \left(\frac{[A - B(p_i + \omega_i \cos \theta_i)] e_f^+ e_i^- + B(k_f^+ e_{fz} e_i^- + k_i^- e_{iz} e_f^+)(E_f + m)^{-1}(E_i + m)^{-1}}{(E_i + \omega_i)^2 - E_{i,n+1}^2} \right. \\ &+ \left. \frac{[A + B(p_i + \omega_i \cos \theta_i)] e_{fz} e_{iz} (E_f + m)^{-1}(E_i + m)^{-1}}{(E_i + \omega_i)^2 - E_{i,n}^2} \right) \\ &+ \left(\frac{\lambda^2 k_i^+ k_f^-}{2} \right)^n \exp(i\lambda^2 k_{ix} k_{fy}) \\ &\times \left(\frac{[A' - B(p_i - \omega_f \cos \theta_f)] e_f^- e_i^+ - B(k_f^- e_{fz} e_i^+ + k_i^+ e_{iz} e_f^-)(E_f + m)^{-1}(E_i + m)^{-1}}{(E_i - \omega_f)^2 - E_{f,n+1}^2} \right. \\ &+ \left. \frac{[A' + B(p_i - \omega_f \cos \theta_f)] e_{fz} e_{iz} (E_f + m)^{-1}(E_i + m)^{-1}}{(E_i - \omega_f)^2 - E_{f,n}^2} \right) \Big] \quad (3.2.64) \end{aligned}$$

$$\begin{aligned}
X &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\frac{\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f}{2} e^{-i(\phi_i - \phi_f)} \right)^n e^{i\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f \cos \phi_f \sin \phi_i} \right. \\
&\times \left(\frac{[A - B(p_i + \omega_i \cos \theta_i)] e_f^+ e_i^- + B(k_f^+ e_{fz} e_i^- + k_i^- e_{iz} e_f^+)}{(E_i + \omega_i)^2 - E_{i,n+1}^2} (E_f + m)^{-1} (E_i + m)^{-1} \right. \\
&+ \left. \frac{[A + B(p_i + \omega_i \cos \theta_i)] e_{fz} e_{iz}}{(E_i + \omega_i)^2 - E_{i,n}^2} (E_f + m)^{-1} (E_i + m)^{-1} \right) \\
&+ \left(\frac{\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f}{2} e^{i(\phi_i - \phi_f)} \right)^n e^{i\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f \sin \phi_f \cos \phi_i} \\
&\times \left(\frac{[A' - B(p_i - \omega_f \cos \theta_f)] e_f^- e_i^+ - B(k_f^- e_{fz} e_i^+ + k_i^+ e_{iz} e_f^-)}{(E_i - \omega_f)^2 - E_{f,n+1}^2} (E_f + m)^{-1} (E_i + m)^{-1} \right. \\
&+ \left. \left. \frac{[A' + B(p_i - \omega_f \cos \theta_f)] e_{fz} e_{iz}}{(E_i - \omega_f)^2 - E_{f,n}^2} (E_f + m)^{-1} (E_i + m)^{-1} \right) \right] \quad (3.2.65)
\end{aligned}$$

Letting

$$\begin{aligned}
Y_1 &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f}{2} e^{-i(\phi_i - \phi_f)} \right)^n \exp(i\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f \cos \phi_f \sin \phi_i) \\
&\times \left[\frac{[A - B(p_i + \omega_i \cos \theta_i)] e_f^+ e_i^- + B(k_f^+ e_{fz} e_i^- + k_i^- e_{iz} e_f^+)}{(E_i + \omega_i)^2 - E_{i,n+1}^2} \right. \\
&+ \left. \frac{[A + B(p_i + \omega_i \cos \theta_i)] e_{fz} e_{iz}}{(E_i + \omega_i)^2 - E_{i,n}^2} \right]
\end{aligned}$$

$$\begin{aligned}
Y_2 &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f}{2} e^{i(\phi_i - \phi_f)} \right)^n \exp(i\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f \sin \phi_f \cos \phi_i) \\
&\times \left[\frac{[A' - B(p_i - \omega_f \cos \theta_f)] e_f^- e_i^+ - B(k_f^- e_{fz} e_i^+ + k_i^+ e_{iz} e_f^-)}{(E_i - \omega_f)^2 - E_{f,n+1}^2} \right. \\
&+ \left. \frac{[A' + B(p_i - \omega_f \cos \theta_f)] e_{fz} e_{iz}}{(E_i - \omega_f)^2 - E_{f,n}^2} \right]
\end{aligned}$$

$$\Rightarrow Y = (Y_1 + Y_2) = X(E_f + m)(E_i + m)$$

$$\Rightarrow X = \frac{Y}{(E_f + m)(E_i + m)}$$

$$\text{Substituting } |X|^2 = \frac{|Y|^2}{(E_f + m)^2 (E_i + m)^2} \text{ in } |S_{fi}|^2 :$$

$$\begin{aligned}
|S_{fi}|^2 &= \frac{(2\Pi)^6 \exp(4) [(E_i + m)(E_f + m)]}{L^4 V^2 4\omega_i \omega_f} \exp[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})] \frac{|Y|^2}{(E_f + m)^2 (E_i + m)^2} \\
&\times \delta^2(E_i + \omega_i - E_f - \omega_f) \delta^2(p_i + k_i \cos \theta_i - p_f - k_f \cos \theta_f) \\
&\times \delta^2(a_i - k_{iy} - a_f - k_{fy}) \tag{3.2.66}
\end{aligned}$$

Taking the final states of the scattered photon and electron to be $\frac{V d^3 \omega_f}{(2\Pi)^3}$ and $\frac{V d^3 p_f}{(2\Pi)^3}$ and $\frac{V d^3 \omega_f}{(2\Pi)^3} = \frac{V \omega_f^2 d\Omega_f}{(2\Pi)^3}$, and also using $e^4 = 16\Pi^2 m^2 r_0^2$

$$\begin{aligned}
\frac{|S_{fi}|^2}{T} &= \frac{(2\Pi)^6 16\Pi^2 m^2 r_0^2}{L^4 V^2 4\omega_i \omega_f} [\frac{1}{4E_i E_f}] \exp[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})] \\
&\times \frac{|Y|^2}{(E_f + m)(E_i + m)} \frac{T}{2\Pi} \frac{L}{2\Pi} \frac{1}{T} \delta^2(a_i - k_{iy} - a_f - k_{fy}) \tag{3.2.67}
\end{aligned}$$

where we have used

$$\begin{aligned}
\delta^2(E_i + \omega_i - E_f - \omega_f) &= \frac{T}{2\Pi} \delta(E_i + \omega_i - E_f - \omega_f), \\
\delta^2(p_i + k_i \cos \theta_i - p_f - k_f \cos \theta_f) &= \frac{L}{2\Pi} \delta(p_i + k_i \cos \theta_i - p_f - k_f \cos \theta_f). \text{ Dividing this} \\
\text{result by the relative incident flux density, } &\frac{(E_i - p_i \cos \theta_i)}{V E_i}:
\end{aligned}$$

$$\begin{aligned}
\frac{|S_{fi}|^2}{T} \frac{V E_i}{(E_i - p_i \cos \theta_i)} &= \frac{(2\Pi)^6 m^2 r_0^2}{L^3 V^2 4\omega_i \omega_f} [\frac{1}{E_i E_f}] \exp[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})] \\
&\times \frac{|Y|^2}{(E_f + m)(E_i + m)} \frac{V E_i}{(E_i - p_i \cos \theta_i)} \\
&\times \delta^2(a_i - k_{iy} - a_f - k_{fy}) \tag{3.2.68}
\end{aligned}$$

Summing over the final states of the scattered photon and electron:

$$\begin{aligned}
d\sigma &= \frac{(2\Pi)^6}{V^2} \frac{m^2 r_0^2}{4\omega_i \omega_f} \left[\frac{1}{E_f} \right] \exp\left[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)\right] \\
&\times \exp\left[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})\right] \\
&\times \frac{|Y|^2}{(E_f + m)(E_i + m)} \frac{1}{(E_i - p_i \cos \theta_i)} \frac{V\omega_f^2 d\Omega_f}{(2\Pi)^3} \frac{V d^3 p_f}{(2\Pi)^3} \\
&\times \delta^2(a_i - k_{iy} - a_f - k_{fy}) \tag{3.2.69}
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega_f} &= \frac{r_0^2 \omega_f}{4 \omega_i} \frac{m^2}{(E_f + m)(E_i + m)(E_i - p_i \cos \theta_i)} \\
&\times \exp\left[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)\right] |Y|^2 \\
&\times \left\{ \frac{1}{E_f} \exp\left[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})\right] \right. \\
&\times \left. \delta^2(a_i - k_{iy} - a_f - k_{fy}) d^3 p_f \right\} \tag{3.2.70}
\end{aligned}$$

But,

$$\begin{aligned}
\frac{1}{[E_f - (p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f) \cos \theta_f]} &= \left\{ \frac{1}{E_f} \exp\left[-i2\lambda^2 a_i(k_{ix} - k_{fx}) - i\lambda^2(k_{ix}k_{iy} + k_{fx}k_{fy})\right] \right. \\
&\times \left. \delta^2(a_i - k_{iy} - a_f - k_{fy}) d^3 p_f \right\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d\sigma}{d\Omega_f} &= \frac{r_0^2 \omega_f}{4 \omega_i} \frac{m^2}{(E_i + m)(E_f + m)} \\
&\times \frac{\exp\left[-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)\right] |Y|^2}{(E_i - p_i \cos \theta_i)[E_f - (p_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f) \cos \theta_f]}, \tag{3.2.71}
\end{aligned}$$

in which r_0 is the classical electron radius

The above result is performed by letting:

$$\vec{k}_i = \omega_i(\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i), \text{ and } \vec{k}_f = \omega_f(\sin \theta_f \cos \phi_f, \sin \theta_f \sin \phi_f, \cos \theta_f)$$

Therefore,

$$\left(\frac{\lambda^2}{2} k_i^- k_f^+\right)^n = \left[\frac{\lambda^2}{2} \omega_i \omega_f \sin \theta_i \sin \theta_f e^{-i(\phi_i - \phi_f)}\right]^n$$

and

$$\left(\frac{\lambda^2}{2} k_i^+ k_f^-\right)^n = \left[\frac{\lambda^2}{2} \omega_i \omega_f \sin \theta_i \sin \theta_f e^{i(\phi_i - \phi_f)}\right]^n$$

$$\text{using } k_i^- k_f^+ = (k_{ix} - ik_{iy})(k_{fx} + ik_{fy})$$

3.2.3 Cross-section in the Electron rest frame

To simplify calculations, we choose the coordinate system with $\phi_i = 0$. Denoting $\phi_f = \phi$, we can write

$$k_i = \omega_i(\sin \theta_i, 0, \cos \theta_i), k_f = \omega_f(\sin \theta_f \cos \phi, \sin \theta_f \sin \phi, \cos \phi) \quad (3.2.72)$$

Taking into account the fact that photons have only two transversal polarizations, the polarizations of incident and scattered photons can be chosen as

$$e_i^{(1)} = (-\cos \theta_i, 0, \sin \theta_i); e_i^{(2)} = (0, -1, 0) \quad (3.2.73)$$

$$e_f^{(1)} = (-\cos \theta_f \cos \phi, -\cos \theta_f \sin \phi, \sin \theta_f); e_f^{(2)} = (\sin \phi, -\cos \phi, 0). \quad (3.2.74)$$

The above choice is not unique but is convenient for calculations. Now we define reduced quantities

$$\Delta_i = \frac{\omega_i}{m}; \Delta_f = \frac{\omega_f}{m}; \Delta_0 = \frac{\omega_0}{m} \quad (3.2.75)$$

where $\omega_0 = \frac{eB}{m}$ is the cyclotron frequency. We denote the reduced Doppler frequencies by

$$\Delta_{ir} = \gamma(1 - \beta \cos \theta_i)\Delta_i; \Delta_{fr} = \gamma(1 - \beta \cos \theta_f)\Delta_f \quad (3.2.76)$$

Averaging over the polarizations of incident photons and summing over those of scattered photons, the total differential cross section will be:

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{2} \sum_{e_i^{(1)}, e_f^{(1)}, e_i^{(2)}, e_f^{(2)}} \frac{r_0^2 \omega_f}{4\omega_i} \frac{m^2}{E_i + m} \frac{e^{-\frac{\lambda^2}{2}(\omega_i^2 \sin^2 \theta_i + \omega_f^2 \sin^2 \theta_f)} |Y|^2}{(E_f + m)(E_i - P_i \cos \theta_i)[E_f - (P_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f) \cos \theta_f]} \quad (3.2.77)$$

where $Y = Y_1 + Y_2$

Note that: $\omega_f = \Delta_f m, \omega_i = \Delta_i m$ and $E_i + m = m(\frac{E_i}{m} + 1) = m(\gamma + 1)$ and in similar way $E_f + m = E_i + \omega_i - \omega_f + m = m(\frac{E_i}{m} + \frac{\omega_i}{m} - \frac{\omega_f}{m} + 1) = m(1 + \gamma + \Delta_i - \Delta_f)$ and $\omega_i^2 = \Delta_i^2 m^2 = \lambda^2 \omega_i^2 = \lambda^2 \Delta_i^2 m^2 = \frac{B_c}{B} \Delta_i^2$ Similarly:

$$\lambda^2 \omega_f^2 = \lambda^2 \Delta_f^2 m^2 = \frac{B_c}{B} \Delta_f^2 \text{ and } E_i - P_i \cos \theta_i = E_i(1 - \frac{P_i}{E_i} \cos \theta_i) = E_i(1 - \beta \cos \theta_i) = m\gamma(1 - \beta \cos \theta_i)$$

$$E_f - (P_i + \omega_i \cos \theta_i - \omega_f \cos \theta_f) \cos \theta_f = E_f - P_i \cos \theta_f + \omega_i \cos \theta_i \cos \theta_f + \omega_f \cos^2 \theta_f \quad (3.2.78)$$

Using the usual trigonometric relation $\cos^2 \theta_f = 1 - \sin^2 \theta_f$ and $E_f = E_i - \omega_i - \omega_f$

$$E_i - \omega_i - \omega_f - P_i \cos \theta_f + \omega_i \cos \theta_i \cos \theta_f + \omega_f \cos^2 \theta_f \quad (3.2.79)$$

$$E_i + \omega_i(1 - \cos \theta_i \cos \theta_f) - P_i \cos \theta_f + \omega_f(\cos^2 \theta_f - 1) \quad (3.2.80)$$

$$E_i - P_i \cos \theta_f + \omega_i(1 - \cos \theta_i \cos \theta_f) - \omega_f \sin^2 \theta_f \quad (3.2.81)$$

This gives $= E_i(1 - \beta \cos \theta_f) + m\Delta_i(1 - \cos \theta_i \cos \theta_f) - m\Delta_f \sin^2 \theta_f$

and this will reduce to: $m[\gamma(1 - \beta \cos \theta_f) + \Delta_i(1 - \cos \theta_i \cos \theta_f) - \Delta_f \sin^2 \theta_f]$

Using all this back in the expression $\frac{d\sigma}{d\Omega_f}$ gives

$$\begin{aligned} \frac{d\sigma}{d\Omega_f} &= \frac{1}{2} \sum_{e_i^{(1)}, e_f^{(1)}, e_i^{(2)}, e_f^{(2)}} \frac{r_0^2 \Delta_f}{4 m^2 \Delta_{ir}} \frac{e^{\frac{-B_c}{2B}(\Delta_f^2 \sin^2 \theta_f + \Delta_i^2 \sin^2 \theta_i)} |Y|^2}{(\gamma + 1)(1 + \gamma + \Delta_i - \Delta_f)} \\ &\times \frac{1}{[\gamma(1 - \beta \cos \theta_f) + \Delta_i(1 - \cos \theta_i \cos \theta_f) - \Delta_f \sin^2 \theta_f]} \end{aligned} \quad (3.2.82)$$

Where $\Delta_{ir} = \gamma(1 - \beta \cos \theta_i) \Delta_i$ Now

$$\begin{aligned} \frac{d\sigma}{d\Omega_f} &= \frac{r_0^2}{8} \frac{\Delta_f}{\Delta_{ir}(1 + \gamma)(1 + \gamma + \Delta_i - \Delta_f)} \times \frac{e^{\frac{-B_c}{2B}(\Delta_f^2 \sin^2 \theta_f + \Delta_i^2 \sin^2 \theta_i)}}{[\gamma(1 - \beta \cos \theta_f) + \Delta_i(1 - \cos \theta_i \cos \theta_f) - \Delta_f \sin^2 \theta_f]} \\ &\times \sum_{e_i^{(1)}, e_f^{(1)}, e_i^{(2)}, e_f^{(2)}} \frac{|Y|^2}{m^2} \\ &\sum_{e_i^{(1)}, e_f^{(1)}, e_i^{(2)}, e_f^{(2)}} \frac{|Y|^2}{m^2} \text{ where } Y = Y_1 + Y_2 \text{ is given by:} \end{aligned}$$

$$\frac{1}{m^2} |Y(1i \rightarrow 1f)|^2 + \frac{1}{m^2} |Y(1i \rightarrow 2f)|^2 + \frac{1}{m^2} |Y(2i \rightarrow 1f)|^2 + \frac{1}{m^2} |Y(2i \rightarrow 2f)|^2$$

Where $\lambda_i \rightarrow \lambda_f$, $\lambda_{i(f)} = 1_{i(f)}, 2_{i(f)}$ represents the scattering from the polarization λ_i to λ_f

$$|Y(1_i \rightarrow 1_f)| = |Y_1(1_i \rightarrow 1_f) + Y_2(1_i \rightarrow 1_f)|$$

$\phi_i = 0, \phi_f = \phi$ and

$$\frac{\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f}{2} = \frac{B_c}{2B} \Delta_i \Delta_f \sin \theta_i \sin \theta_f = \xi$$

and $e^{-i(\phi_i - \phi_f)} = e^{i\phi}$ and $e^{[i\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f \cos \phi_f \sin \phi_i]} = e^0 = 1$

$$\frac{1}{(E_i + \omega_i)^2 - E_{i,n+1}^2} = \frac{1}{E_i^2 + \omega_i^2 + 2E_i \omega_i - m^2 - P_i^2 - \omega_i^2 \cos^2 \theta_i - 2P_i \omega_i \cos \theta_i - 2(n+1)eB} \quad (3.2.83)$$

using $E_{i,n+1}^2 = m^2 + (P_i + \omega_i \cos \theta_i)^2 + 2(n+1)eB$

$$\frac{1}{E_i^2 + 2E_i\omega_i - m^2 - P_i^2 - 2P_i\omega_i \cos \theta_i - 2(n+1)eB + \omega_i^2 \sin^2 \theta_i} \quad (3.2.84)$$

$$= \frac{1}{2E_i\omega_i - 2P_i\omega_i \cos \theta_i - 2neB + \omega_i^2 \sin^2 \theta_i} \quad (3.2.85)$$

$$= \frac{1}{2m\gamma m\Delta_i - 2m\gamma\beta m\Delta_i \cos \theta_i - 2nm^2\Delta_0 + m^2\Delta_i^2 \sin^2 \theta_i} \quad (3.2.86)$$

$$= \frac{1}{m^2[2(\Delta_{ir} - n\Delta_0) + \Delta_i^2 \sin^2 \theta_i]} \quad (3.2.87)$$

$$= \frac{1}{m^2} S_{i,n+1} \quad (3.2.88)$$

Similarly in Y_2 :

$$\frac{1}{(E_i - \omega_f)^2 - E_{f,n+1}^2} = \frac{1}{m^2[2(\Delta_{fr} + n\Delta_0) - \Delta_f^2 \sin^2 \theta_f]} \quad (3.2.89)$$

$$= \frac{1}{m^2} S_{f,n+1} \text{ and } \frac{\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f}{2} = \frac{B_c}{2B} \Delta_i \Delta_f \sin \theta_i \sin \theta_f = \xi \text{ and}$$

$$e^{i(\phi_i - \phi_f)} = e^{-i\phi}, e^{[i\lambda^2 \omega_i \omega_f \sin \theta_i \sin \theta_f \sin \phi_f \cos \phi_i]} = e^{i\frac{B_c}{B} \Delta_i \Delta_f \sin \theta_i \sin \theta_f \sin \phi} = e^{i2\xi \sin \phi} = e^{i\eta}$$

where $\eta = 2\xi \sin \phi$

Therefore, $Y = Y_1 + Y_2$ will be:

$$\begin{aligned} |Y| &= \{[A - B(P_i + \omega_i \cos \theta_i)]e_f^+ e_i^- + B(K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+)\} \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{1}{m^2} S_{i,n+1} e^{in\phi} \\ &+ [A + B(P_i + \omega_i \cos \theta_i)]e_{fz} e_{iz} \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{1}{m^2} S_{i,n} e^{in\phi} \\ &+ \{[A' - B(P_i - \omega_f \cos \theta_f)]e_f^- e_i^+ - B(K_f^- e_{fz} e_i^+ + K_i^+ e_{iz} e_f^-)\} \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{1}{m^2} S_{f,n+1} e^{-in\phi + i\eta} \\ &+ [A' + B(P_i - \omega_f \cos \theta_f)]e_{fz} e_{iz} \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{1}{m^2} S_{f,n} e^{-in\phi + i\eta} \end{aligned} \quad (3.2.90)$$

To calculate $|Y(1_i \rightarrow 1_f)|$, we use the polarization components: $e_i^{(1)} = (\cos \theta_i, 0, \sin \theta_i)$, $e_f^{(1)} = (-\cos \theta_f \cos \phi, -\cos \theta_f \sin \phi, \sin \theta_f)$ and

$$K_i = \omega_i(\sin \theta_i, 0, \cos \theta_i), K_f = \omega_f(\sin \theta_f \cos \phi, \sin \theta_f \sin \phi, \cos \phi)$$

Therefore:

$$e_f^- e_i^+ = e_{fx} e_{ix} + e_{fy} e_{iy} - i e_{fy} e_{ix} + i e_{fx} e_{iy} = \cos \theta_i \cos \theta_f \cos \phi - i \cos \theta_i \cos \theta_f \sin \phi$$

$$e_f^+ e_i^- = \cos \theta_i \cos \theta_f \cos \phi + i \cos \theta_i \cos \theta_f \sin \phi$$

$$\begin{aligned} K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+ &= (K_{fx} e_{ix} + K_{fy} e_{iy} + i K_{fy} e_{ix} - i K_{fx} e_{iy}) e_{fz} \\ &+ (K_{ix} e_{fx} + K_{iy} e_{fy} + i K_{ix} e_{fy} - i K_{iy} e_{fx}) e_{iz} \quad (3.2.91) \end{aligned}$$

$$\begin{aligned} K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+ &= (-\omega_f \sin \theta_f \cos \phi \cos \theta_i - i \omega_f \sin \theta_f \sin \phi \cos \theta_i) \sin \theta_f \\ &+ (-\omega_i \sin \theta_i \cos \phi \cos \theta_f - i \omega_i \sin \theta_i \sin \phi \cos \theta_f) \sin \theta_i \end{aligned}$$

$$e_{fz} e_{iz} = \sin \theta_i \sin \theta_f$$

and

$$\begin{aligned} K_f^- e_{fz} e_i^+ + K_i^+ e_{iz} e_f^- &= (-\omega_f \sin \theta_f \cos \phi \cos \theta_i + i \omega_f \sin \theta_f \sin \phi \cos \theta_i) \sin \theta_f \\ &+ (-\omega_i \sin \theta_i \cos \phi \cos \theta_f + i \omega_i \sin \theta_i \sin \phi \cos \theta_f) \sin \theta_i \end{aligned}$$

The first term in the curly bracket of (3.2.94),

$$\begin{aligned} &\{[A - B(P_i + \omega_i \cos \theta_i)] e_f^+ e_i^- + B(K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+)\} \\ &= \{[(E_i + \omega_i + m) P_i P_f + (E_i + \omega_i - m)(E_i + m)(E_f + m)] \\ &- [(P_i(E_f + m) + P_f(E_i + m))(P_i + \omega_i \cos \theta_i)] [\cos \theta_i \cos \theta_f \cos \phi + i \cos \theta_i \cos \theta_f \sin \phi] \\ &+ [P_i(E_f + m) + P_f(E_i + m)] [-\omega_f \sin^2 \theta_f \cos \phi \cos \theta_i - i \omega_f \sin^2 \theta_f \sin \phi \cos \theta_i \\ &- \omega_i \sin^2 \theta_i \cos \phi \cos \theta_f - i \omega_i \sin^2 \theta_i \sin \phi \cos \theta_f] \quad (3.2.92) \end{aligned}$$

$$\begin{aligned}
&= e^{i\phi} m^3 [(\beta\gamma(1 + \gamma + \Delta_i)(\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f) + (\gamma - 1 + \Delta_f)(1 + \gamma)(1 + \Delta_i - \Delta_f)) \\
&- (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f)(1 + \gamma))(\beta\gamma + \Delta_i \cos \theta_i)] \cos \theta_f \\
&- (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_i)(1 + \gamma)\Delta_f \sin^2 \theta_f) \cos \theta_i \\
&- (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f)(1 + \gamma)\Delta_i \sin^2 \theta_i) \cos \theta_f] \quad (3.2.93)
\end{aligned}$$

The second square bracket, $[A + B(P_i + \omega_i \cos \theta_i)]e_{fz}e_{iz}$

$$\begin{aligned}
&= \{[(E_i + \omega_i + m)P_i P_f + (E_i + \omega_i - m)(E_i + m)(E_f + m)] \\
&+ [(P_i(E_f + m) + P_f(E_i + m))(P_i + \omega_i \cos \theta_i)]\}(\sin \theta_i \sin \theta_f \theta_i) \\
&= m^3[a + b(\beta\gamma + \Delta_i \cos \theta_i)] \sin \theta_i \sin \theta_f
\end{aligned}$$

The third curly bracket, $\{[A' - B(P_i - \omega_f \cos \theta_f)]e_f^- e_i^+ - B(K_f^- e_{fz} e_i^+ + K_i^+ e_{iz} e_f^-)\}$

$$\begin{aligned}
&= \{[(E_i - \omega_f + m)P_i P_f + (E_i - \omega_f - m)(E_i + m)(E_f + m)] \\
&- [P_i(E_i + m) + P_f(E_i + m)](P_i - \omega_f \cos \theta_f)\}(\cos \theta_i \cos \theta_f \cos \phi - i \cos \theta_i \cos \theta_f \sin \phi) \\
&- [P_i(E_f + m) + P_f(E_i + m)](-\omega_f \sin^2 \theta_f \cos \phi \cos \theta_i + i\omega_f \sin^2 \theta_f \sin \phi \cos \theta_i) \\
&- \omega_i \sin^2 \theta_i \cos \phi \cos \theta_f + i\omega_i \sin^2 \theta_i \sin \phi \cos \theta_f] \quad (3.2.94)
\end{aligned}$$

$$\begin{aligned}
&= e^{-i\phi} m^3 [(\beta\gamma(1 + \gamma - \Delta_f)(\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f) + (\gamma - 1 - \Delta_f)(1 + \gamma)(1 + \Delta_i - \Delta_f)) \\
&- (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f)(1 + \gamma))(\beta\gamma - \Delta_f \sin \theta_f)] \Delta_f \sin^2 \theta_f \cos \theta_i \\
&+ (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_i)(1 + \gamma)\Delta_i \sin^2 \theta_i \cos \theta_f)] \quad (3.2.95)
\end{aligned}$$

and the fourth square bracket can be solved as:

$$\begin{aligned}
[A' + B(P_i - \omega_f \cos \theta_f)]e_{fz}e_{iz} &= [(E_i - \omega_f + m)P_i P_f + (E_i - \omega_f - m)(E_i + m)(E_f + m)] \\
&+ (P_i(E_f + m) + P_f(E_i + m))(P_i - \omega_f \cos \theta_i)] \sin \theta_i \sin \theta_f
\end{aligned}$$

$$=m^3[a' + b(\beta\gamma - \Delta_f \cos \theta_f)] \sin \theta_i \sin \theta_f$$

Thus substituting this back in to $|Y|$ gives $Y(1_i \rightarrow 1_f)$ thus:

$$\begin{aligned} Y(1_i \rightarrow 1_f) &= m^3 \{ [(A_- \cos \theta_f - B_1) \cos \theta_i - B_2 \cos \theta_f] \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{S_{i,n+1}}{m^2} e^{i(n+1)\phi} \\ &\quad - [(A'_- \cos \theta_f + B_1) \cos \theta_i + B_2 \cos \theta_f] \sum_{n=0}^{\infty} \frac{1}{n!} \frac{S_{f,n+1}}{m^2} e^{-i[(n+1)\phi - \eta]} \\ &\quad + \sin \theta_i \sin \theta_f [A_+ \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{S_{i,n}}{m^2} e^{in\phi} - A'_+ \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \frac{S_{f,n}}{m^2} e^{-i(n\phi - \eta)}] \} \end{aligned}$$

Where $A_{\pm} = a \pm b(\beta\gamma + \Delta\Delta_i \cos \theta_i)$, $A'_{\pm} = a' \pm b(\beta\gamma - \Delta_f \cos \theta_f)$, $b_1 = b\Delta_f \sin^2 \theta_i$

in which

$$a = \beta\gamma(1 + \gamma + \Delta_i)(\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f) + (\gamma - 1 + \Delta_i)(1 + \gamma)(1 + \gamma + \Delta_i - \Delta_f)$$

$$a' = \beta\gamma(1 + \gamma - \Delta_f)(\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f) + (\gamma - 1 + \Delta_f)(1 + \gamma)(1 + \gamma + \Delta_i - \Delta_f)$$

$$b = \beta\gamma(1 + \gamma + \Delta_i - \Delta_f)(\beta\gamma + \Delta_i \cos \theta_i - \Delta_f \cos \theta_f)(1 + \gamma)$$

which can be written again as:

$$\begin{aligned} Y(1_i \rightarrow 1_f) &= mD_1 \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n S_{i,n+1} e^{i(n+1)\phi} \\ &\quad - mD_2 \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n S_{f,n+1} e^{-i[(n+1)\phi - \eta]} \\ &\quad + m \sin \theta_i \sin \theta_f [A_+ \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n S_{i,n} e^{in\phi} \\ &\quad - mA'_+ \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n S_{f,n} e^{-i(n\phi - \eta)}] \end{aligned} \quad (3.2.96)$$

Following similar procedures for $Y(1_i \rightarrow 2_f)$, $Y(2_i \rightarrow 1_f)$ and $Y(2_i \rightarrow 2_f)$, that is using the respective polarization components:

$$\begin{aligned}
Y(1_i \rightarrow 2_f) &= mi(A_- \cos \theta_i - B_2) \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{i,n+1} e^{i(n+1)\phi} \\
&+ mi[A'_- \cos \theta_i + B_2] \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{f,n+1} e^{-i(n+1)\phi + i\eta} \quad (3.2.97)
\end{aligned}$$

$$\begin{aligned}
Y(2_i \rightarrow 1_f) &= -mi(A_- \cos \theta_f - B_1) \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{i,n+1} e^{i(n+1)\phi} \\
&+ -mi[A'_- \cos \theta_f + B_1] \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{f,n+1} e^{-i(n+1)\phi + i\eta} \quad (3.2.98)
\end{aligned}$$

$$\begin{aligned}
Y(2_i \rightarrow 2_f) &= mA_- \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{i,n+1} e^{i(n+1)\phi} \\
&- mA'_- \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{f,n+1} e^{-i(n+1)\phi + i\eta} \quad (3.2.99)
\end{aligned}$$

For $n = 0$:

$$\begin{aligned}
Y(1_i \rightarrow 1_f) &= mD_1 S_{i,1} e^{i\phi} - mD_2 S_{f,1} e^{-i(\phi-\eta)} + m \sin \theta_i \sin \theta_f [A_f S_{i,0} - A'_f S_{f,0} e^{i\eta}] \\
Y(1_i \rightarrow 2_f) &= mi(A_- \cos \theta_i - B_2) S_{i,1} e^{i\phi} + mi(A'_- \cos \theta_i + B_2) S_{f,1} e^{-i(\phi-\eta)} \\
Y(2_i \rightarrow 1_f) &= -mi(A_- \cos \theta_f - B_1) S_{i,1} e^{i\phi} - mi(A'_- \cos \theta_f + B_1) S_{f,1} e^{-i(\phi-\eta)} \\
Y'(2_i \rightarrow 2_f) &= mA_- S_{i,1} e^{i\phi} - A'_- S_{f,1} e^{-i(\phi-\eta)} \quad (3.2.100)
\end{aligned}$$

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= \{D_1^2 S_{i,1}^2 e^{2i\phi} + D_2^2 S_{f,1}^2 e^{-2i(\phi-\eta)} + \sin^2 \theta_i \sin^2 \theta_f \\
&\times [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}]^2 \\
&+ 2(D_1 S_{i,1} e^{i\phi} - D_2 S_{f,1} e^{-i(\phi-\eta)}) \sin \theta_i \sin \theta_f \\
&\times [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}] \\
&- 2D_1 D_2 S_{i,1} e^{i\phi} S_{f,1}^2 e^{-i(\phi-\eta)}\} + |-(A_- \cos \theta_i - B_2)^2 S_{i,1}^2 e^{2i\phi} \\
&- (A'_- \cos \theta_i + B_2)^2 S_{f,1}^2 e^{-2i(\phi-\eta)} - 2(A_- \cos \theta_i - B_2) \\
&\times (A'_- \cos \theta_i + B_2) S_{i,1} S_{f,1} e^{i\phi} e^{i(\phi-\eta)}| \\
&+ |-(A_- \cos \theta_f - B_1)^2 S_{i,1}^2 e^{2i\phi} - (A'_- \cos \theta_f + B_1)^2 S_{f,1}^2 e^{-2i(\phi-\eta)} \\
&- 2(A_- \cos \theta_f - B_1)(A'_- \cos \theta_f + B_1) S_{i,1} S_{f,1} e^{i\phi} e^{-i(\phi-\eta)}| \\
&+ |A_-^2 S_{i,1}^2 e^{2i\phi} + A_-'^2 S_{f,1}^2 e^{-2i(\phi-\eta)} - 2A_- A'_- S_{i,1} e^{i\phi} S_{f,1} e^{-i(\phi-\eta)}| \quad (3.2.101)
\end{aligned}$$

taking the magnitude of $|Y|^2$ leads to:

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= [D_1^2 + (A_- \cos \theta_i - B_2)^2 + (A_- \cos \theta_f - B_1)^2 + A_-^2] S_{i,1}^2 e^{2i\phi} \\
&+ [D_2^2 + (A_- \cos \theta_i + B_2)^2 + (A'_- \cos \theta_f + B_1)^2 + A_-'^2] S_{f,1}^2 e^{-2i(\phi-\eta)} \\
&+ \sin^2 \theta_i \sin^2 \theta_f [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}]^2 \\
&+ 2[(A_- \cos \theta_i - B_2)(A'_- \cos \theta_i + B_2) + (A_- \cos \theta_f - B_1)(A'_- \cos \theta_f + B_1) \\
&- D_1 D_2 - A_- A'_-] e^{i\phi} e^{-i(\phi-\eta)} S_{i,1} S_{f,1} \\
&+ 2(D_1 S_{i,1} e^{i\phi} - D_2 S_{f,1} e^{-i(\phi-\eta)}) \sin \theta_i \sin \theta_f [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}] \quad (3.2.102)
\end{aligned}$$

Letting $[D_1^2 + (A_- \cos \theta_i - B_2)^2 + (A_- \cos \theta_f - B_1)^2 + A_-^2] = C_1$, and $[D_2^2 + (A_- \cos \theta_i +$

$$B_2)^2 + (A'_- \cos \theta_f + B_1)^2 + A'^2_-] = C_2$$

$$\begin{aligned} \frac{|Y|^2}{m^2} &= C_1 S_{i,1}^2 e^{-2i\phi} + C_2 S_{f,1}^2 e^{-2i(\phi-\eta)} \sin^2 \theta_i \sin^2 \theta_f \\ &\times [A_+^2 S_{i,0}^2 + A_+'^2 S_{f,0}^2 e^{2i\eta} - 2A_+ A'_+ S_{i,0} S_{f,0} e^{i\eta}] \\ &+ 2[(A_- \cos \theta_i)(A'_- \cos \theta_f + B_2) \\ &+ (A_- \cos \theta_f - B_1)(A'_- \cos \theta_f - B_1) \\ &- D_1 D_2 - A_- A'_-] e^{i\phi} e^{-i(\phi-\eta)} S_{i,1} S_{f,1} \\ &+ 2[(-D_1 A'_+ S_{i,1} S_{f,0} e^{i(\phi+\eta)} - D_2 A_+ S_{i,0} S_{f,1} e^{-i(\phi-\eta)}) \\ &+ D_1 A_+ S_{i,1} S_{i,0} e^{i\phi} + D_2 A'_+ S_{f,1} S_{f,0} e^{-i(\phi-\eta)} e^{i\eta}] \end{aligned} \quad (3.2.103)$$

Now integrating over ϕ the above equation gives the following:

$$\begin{aligned} \frac{|Y|^2}{m^2} &= C_1 S_{i,1}^2 \int e^{2i\phi} d\phi + C_2 S_{f,1}^2 \int e^{-2i(\phi-\eta)} + \sin^2 \theta_i \sin^2 \theta_f [A_+^2 S_{i,0}^2 + A_+'^2 S_{f,0}^2 \int e^{2i\eta} d\phi \\ &- 2A_+ A'_+ S_{i,0} S_{f,0} \int e^{i\eta} d\phi] + 2[(A_- \cos \theta_i - B_2)(A'_- \cos \theta_i + B_2) + (A_- \cos \theta_f - B_1) \\ &\times (A'_- \cos \theta_f + B_1) - D_1 D_2 - A_- A'_-] S_{i,1} S_{f,1} \int e^{i\phi} e^{-i(\phi-\eta)} d\phi - 2(D_1 A'_+ S_{i,1} S_{f,0} \int e^{i(\phi+\eta)} d\phi \\ &+ D_2 A_+ S_{i,0} S_{f,1} \int e^{-i(\phi-\eta)} \sin \theta_i \sin \theta_f + 2(D_1 A_+ S_{i,1} S_{i,0} \int e^{i\phi} d\phi + D_2 A'_+ S_{f,1} S_{f,0} \int e^{-i(\phi-\eta)} \end{aligned}$$

Since $\int e^{2i\phi} d\phi = \int e^{-2i(\phi-\eta)} d\phi = \int e^{2i\eta} d\phi = 2\Pi$ and $\int e^{i\eta} = 2\Pi$

Introducing the Bessel functions $J_n(-\xi) = \frac{1}{2\pi} \int_0^{2\Pi} d\phi \cos[n\phi - \xi \sin(\phi)]$, $n = 0, 1, 2$

$$J_0(-\xi) = 2\pi \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos(-\xi \sin \pi)$$

$$\int e^{i\phi} e^{-i(\phi-\eta)} d\phi = 2\pi J_2(-\xi)$$

Let $Q = [(A_- \cos \theta_i - B_2)(A'_- \cos \theta_i + B_2) + (A_- \cos \theta_f - B_1)(A'_- \cos \theta_f + B_1) -$

$$D_1 D_2 - A_- A'_-]$$

$$\begin{aligned} \frac{|Y|^2}{m^2} &= 2\pi C_1 S_{i,1}^2 + 2\pi C_2 S_{f,1}^2 + \sin^2 \theta_i \sin^2 \theta_f [2\pi A_+^2 S_{i,0}^2 - 2A_+ A'_+ S_{i,0} S_{f,0} 2\pi J_0(\xi)] \\ &+ 2A_+ A'_+ S_{i,0} S_{f,0} 2\pi - 2A_+ A'_+ S_{i,0} S_{f,0} 2\Pi] \\ &= 2[Q S_{i,1} S_{f,1} 2\pi J_2(-\xi)] \\ &- 2[D_1 A'_+ S_{i,1} S_{f,0} 2\pi J_1(-\xi) + D_2 A'_+ S_{i,0} S_{f,1} 2\pi J_1(-\xi)] \sin \theta_i \sin \theta_f \\ &+ 2[D_1 A_+ S_{i,1} S_{i,0} 2\pi + D_2 A'_+ S_{f,1} S_{f,0} 2\pi] \end{aligned} \quad (3.2.104)$$

$$\begin{aligned} \frac{|Y|^2}{2\pi m^2} &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + \sin^2 \theta_i \sin^2 \theta_f [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2A_+ A'_+ S_{i,0} S_{f,0} (1 - J_0(\xi))] \\ &+ 2Q S_{i,1} S_{f,1} J_2(\xi) - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \sin \theta_i \sin \theta_f J_1(-\xi) \\ &+ 2[D_1 A_+ S_{i,1} S_{i,0} + D_2 A'_+ S_{f,1} S_{f,0}] \end{aligned} \quad (3.2.105)$$

$$\begin{aligned} &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2(1 - J_0(\xi)) A_+ A'_+ S_{i,0} S_{f,0}] (\sin \theta_i \sin \theta_f)^2 \\ &+ 2(Q) J_2(-\xi) S_{i,1} S_{f,1} - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \sin \theta_i \sin \theta_f J_1(-\xi) \end{aligned} \quad (3.2.106)$$

Now substituting back the value of Q we will get

$$\begin{aligned} Y_r = \frac{|Y|^2}{2\pi m^2} &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2(1 - J_0(\xi)) A_+ A'_+ S_{i,0} S_{f,0}] (\sin \theta_i \sin \theta_f)^2 \\ &+ 2[(A_- \cos \theta_i - B_2)(A'_- \cos \theta_i + B_2) + (A_- \cos \theta_f - B_1)(A'_- \cos \theta_f + B_1)] \\ &- D_1 D_2 - A_- A'_-] J_2(-\xi) S_{i,1} S_{f,1} - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \sin \theta_i \sin \theta_f J_1(-\xi) \end{aligned}$$

The differential cross-section(3.2.83) can be reduced to

$$d\sigma = \sigma(\Delta_i, \theta_i, \gamma, \theta_f) \sin \theta_f d\theta_f, \quad (3.2.107)$$

where

$$\begin{aligned} \sigma(\Delta_i, \theta_i, \gamma, \theta_f) &= \frac{\Pi r_0^2}{4} \frac{\Delta_f}{\Delta_{ir}(1 + \gamma)(1 + \gamma + \Delta_i - \Delta_f)} \\ &\times \frac{\exp(\frac{-B_c}{2B} \Delta_f^2 \sin^2 \theta_f) Y_r}{[\gamma(1 - \beta \cos \theta_f) + \Delta_i(1 - \cos \theta_i \cos \theta_f) - \Delta_f \sin^2 \theta_f]} \end{aligned} \quad (3.2.108)$$

By noting that $J_1(-\xi)$ and $J_2(-\xi)$ are negligible for $\Delta_i \ll 1$, the above expression can be simplified to:

$$\begin{aligned} Y_r &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})]^2 (\sin \theta_i \sin \theta_f)^2 \\ &+ [2(1 - J_0(\xi)) A_+ A'_+ S_{i,0} S_{f,0}] (\sin \theta_i \sin \theta_f)^2 \end{aligned} \quad (3.2.109)$$

The above approximation can be justified by considering the nonrelativistic limit in the EFF($\gamma = 1, \beta = 0$). Under the Thomson limit ($\Delta_f \simeq \Delta_i \ll 1$) and $A_- = -A'_- \simeq 4\Delta_i, B_1 = B_2 \simeq 0, A_+ = -A'_+ \simeq 4\Delta_i$ and $J_0(\xi) \simeq 1$ Therefore,

$$\begin{aligned} D_1 &= (A_- \cos \theta_f - B_1) \cos \theta_i - B_2 \cos \theta_f \\ &= 4\Delta_i \cos \theta_i \cos \theta_f \end{aligned}$$

$$\begin{aligned} C_1 &= D_1^2 + (A_- \cos \theta_i - B_2)^2 + (A_- \cos \theta_f - B_1)^2 + A_-^2 \\ &= D_1^2 + (4\Delta_i \cos \theta_i)^2 + (4\Delta_i \cos \theta_f)^2 + (4\Delta_i)^2 \\ &= (4\Delta_i \cos \theta_i \cos \theta_f)^2 + (4\Delta_i \cos \theta_i)^2 + (4\Delta_i \cos \theta_f)^2 + (4\Delta_i)^2 \end{aligned}$$

$$\begin{aligned} D_2 &= (A'_- \cos \theta_f + B_1) \cos \theta_i + B_2 \cos \theta_f \\ &= -4\Delta_i \cos \theta_i \cos \theta_f \end{aligned}$$

$$\begin{aligned} C_2 &= D_2^2 + (A'_- \cos \theta_i + B_2)^2 + (A'_- \cos \theta_f + B_1)^2 + A_-'^2 \\ &= (-4\Delta_i \cos \theta_i \cos \theta_f)^2 + (-4\Delta_i \cos \theta_i)^2 + (-4\Delta_i \cos \theta_f)^2 + (-4\Delta_i)^2 \end{aligned}$$

$$\begin{aligned}
Y_r &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})]^2 (\sin \theta_i \sin \theta_f)^2 \\
&+ [2(1 - J_0(\zeta)) A_+ A'_+ S_{i,0} S_{f,0}] (\sin \theta_i \sin \theta_f)^2 \\
&= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})]^2 (\sin \theta_i \sin \theta_f)^2 \\
&= [(4\Delta_i \cos \theta_i \cos \theta_f)^2 (4\Delta_i \cos \theta_i)^2 + (4\Delta_i \cos \theta_f)^2 + (4\Delta_i)^2] \left(\frac{1}{2(\Delta_i - \Delta_0)}\right)^2 \\
&+ [(-4\Delta_i \cos \theta_i \cos \theta_f)^2 + (-4\Delta_i \cos \theta_i)^2 + (-4\Delta_i \cos \theta_f)^2 + (-4\Delta_i)^2] \left(\frac{1}{2(\Delta_i + \Delta_0)}\right)^2 \\
&+ [(4\Delta_i) \left(\frac{1}{2\Delta_i}\right) - (-4\Delta_i) \left(\frac{1}{2\Delta_i}\right)]^2 (\sin \theta_i \sin \theta_f)^2 \\
&= 4\Delta_i^2 [\cos^2 \theta_i \cos^2 \theta_f + \cos^2 \theta_i + \cos^2 \theta_f + 1] \left[\frac{1}{(\Delta_i - \Delta_0)^2} + \frac{1}{(\Delta_i + \Delta_0)^2}\right] \\
&+ 16 \sin^2 \theta_i \sin^2 \theta_f \\
&= 4[\cos^2 \theta_f (\cos^2 \theta_i + 1) + 1(\cos^2 \theta_i + 1)] \left[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}\right] + 16 \sin^2 \theta_i \sin^2 \theta_f \\
&= 16 \sin^2 \theta_i \sin^2 \theta_f + 4(1 + \cos^2 \theta_i)(1 + \cos^2 \theta_f) \left[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}\right]
\end{aligned}$$

$$\begin{aligned}
\sigma(\Delta_i, \theta_i, \theta_f) &= \frac{\Pi r_0^2 Y_r}{4 \cdot 4} \\
&= \frac{\Pi r_0^2}{16} \{16 \sin^2 \theta_i \sin^2 \theta_f + 4(1 + \cos^2 \theta_i)(1 + \cos^2 \theta_f) \left[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}\right]\}
\end{aligned}$$

$$\frac{\sigma(\Delta_i, \theta_i, \theta_f)}{\Pi r_0^2} = \sin^2 \theta_i \sin^2 \theta_f + \frac{1}{4}(1 + \cos^2 \theta_i)(1 + \cos^2 \theta_f) \left[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}\right],$$

which is just Herold's nonrelativistic result.

Chapter 4

Results and Discussion

In binary systems, the radiation from strong X-ray sources is generally believed to be due to accretion onto a neutron star from a binary companion star. Due to this, near the surface of the neutron star, photon densities can arise that are greater than the atmospheric particle densities. Such a situation is conducive to emission by compton scattering. That is, these photons undergo scattering at the surface of the neutron stars and the net result is the production of soft X-rays. We have calculated the scattering cross-section of compton scattering in strong magnetic fields, which is Herold's nonrelativistic result. This cross-section is obtained under the consideration of the non relativistic limit in the ERF($\gamma = 1$, or $\beta = 0$). This cross-section is applicable for the particular case cited above.

4.1 The scattering cross-section and incident photon frequency

The scattering cross-section we derived earlier,

$$\frac{\sigma(\Delta_i, \theta_i, \theta_f)}{\Pi r_0^2} = \sin^2 \theta_i \sin^2 \theta_f + \frac{1}{4}(1 + \cos^2 \theta_i)(1 + \cos^2 \theta_f) \left[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2} \right]$$

can be written in units of Thomson's cross-section [$\sigma_T = (\frac{8\pi}{3})r_0^2$] as

$$\frac{\sigma}{\sigma_T} = \frac{3}{8} \left\{ \sin^2 \theta_i \sin^2 \theta_f + \frac{1}{4} (1 + \cos^2 \theta_i) (1 + \cos^2 \theta_f) \left(\frac{\Delta_i}{\Delta_0} \right)^2 \left[\frac{1}{(\frac{\Delta_i}{\Delta_0} - 1)^2} + \frac{1}{(\frac{\Delta_i}{\Delta_0} + 1)^2} \right] \right\}$$

$$\text{or} = \frac{3}{8} \left\{ \sin^2 \theta_i \sin^2 \theta_f + \frac{1}{4} (1 + \cos^2 \theta_i) (1 + \cos^2 \theta_f) \left(\frac{\omega_i}{\omega_0} \right)^2 \left[\frac{1}{(\frac{\omega_i}{\omega_0} - 1)^2} + \frac{1}{(\frac{\omega_i}{\omega_0} + 1)^2} \right] \right\}$$

Since $\frac{\omega_i}{\omega_0} \ll 1$, the total cross-section is a function of ω_i^2 (in units of the resonance frequency). The plot of total cross-section versus incident photon frequency is given below.

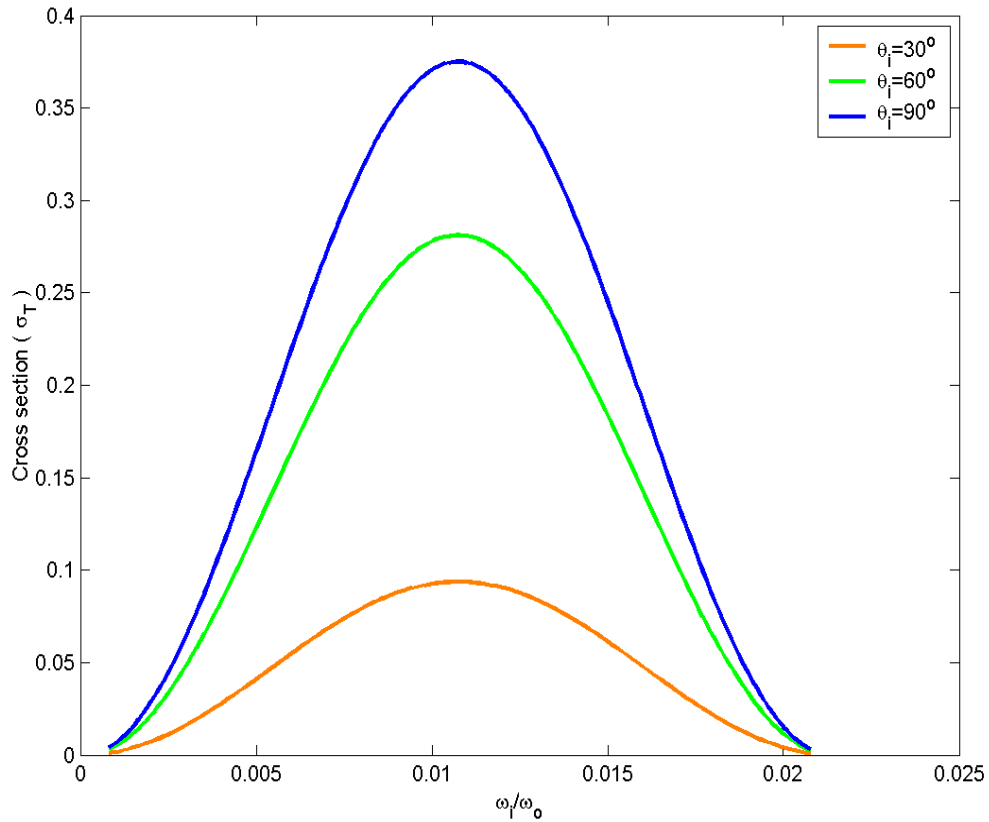


Figure 4.1: Cross-section versus $\frac{\omega_i}{\omega_0}$

As can be seen from diagram (4.1) above, the scattering cross-section increases as the incident photon frequency increases for a given incident photon angle to some maximum value, and then it starts to decrease for some values of the incident photon frequency. This means that as the incident photon frequency approaches a resonance frequency, the cross-section become very low.

4.2 The incident and the scattered photon frequencies

From the conservation of energy and momentum along the z -direction:

$$p_{iz} + k_{iz} = p_{fz} + k_{fz} \Rightarrow p_i + k_i \cos \theta_i = p_f + k_f \cos \theta_f, \text{ and}$$

$E_f + \omega_f = E_i + \omega_i \Rightarrow \omega_i + (m^2 + p_i^2 + 2NeB)^{\frac{1}{2}} = \omega_f + (m^2 + p_f^2 + 2N'eB)^{\frac{1}{2}}$. Thus, for $N = N' = 0$, the energy of the scattered photon is given by:

$$\omega_f = \frac{\{m + \omega_i(1 - \cos \theta_i \cos \theta_f) - [m^2 + 2m\omega_i \cos \theta_f(\cos \theta_f - \cos \theta_i) + \omega_i^2(\cos \theta_f - \cos \theta_i)^2]^{\frac{1}{2}}\}}{\sin^2 \theta_f} \quad (4.2.1)$$

This replaces the well-known Compton formula without magnetic field where the electron motion is not restricted to "one dimension". From the above expression for the final photon frequency, we can draw the relationship between the incident and the scattered photon frequencies.

From the plot of ω_f versus ω_i , we can draw some conclusions:

The final photon frequency decreases as the the incident photon frequency of a given range increases. This is in a good agreement with the fact that the final photon frequency should be less than the initial photon frequency in compton scattering, which is a good indication of the production of Soft X-rays at the surface of the neutron stars.

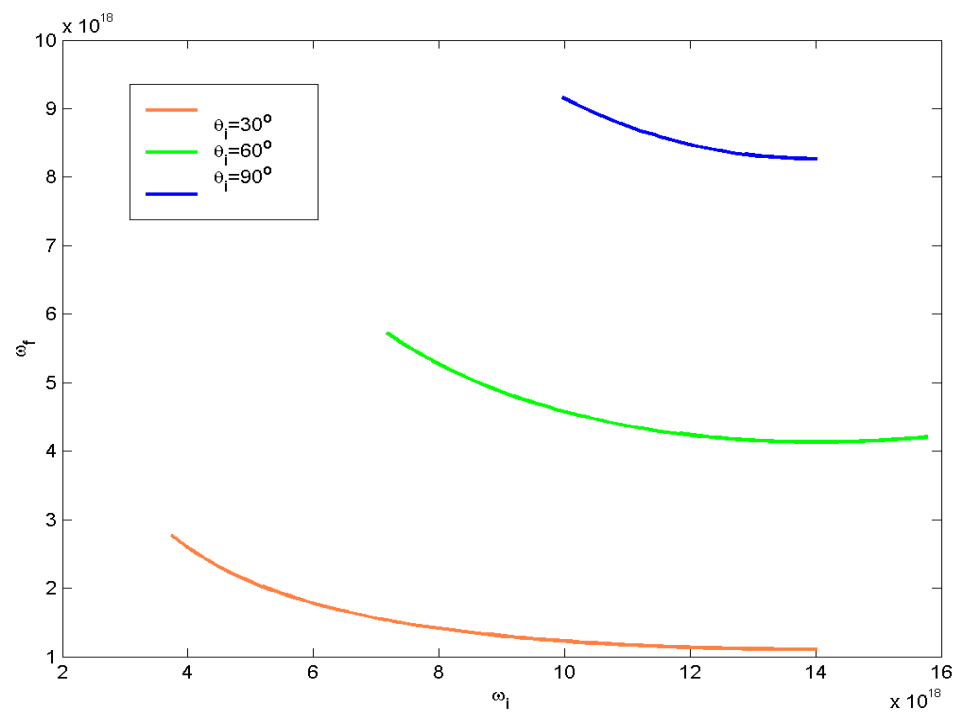


Figure 4.2: Scattered photon frequency versus Incident photon frequency

Chapter 5

Conclusion

From the plot of the scattered photon frequency versus the incident photon frequency, it is evident that the final photon frequency is less than the initial. The source for the relatively hard X-ray photons is expected to be infalling accreted matter. Therefore, after scattering, low photon frequencies are obtained which is obviously the indication of the production of soft x-rays at the surface of the neutron stars, this being due to photon densities at the surface of the neutron stars undergoing scattering. Therefore, there exists a mechanism as a source of soft ($E_s < 10kev$) radiation in binary neutron stars, in the form of photon scattering with both the initial and final electrons on the ground magnetic states. This mechanism provides a source of soft photons which can be redistributed to higher frequencies by inverse Comptonization. We know that in young neutron stars, soft x-ray emission is found to be of thermal origin and being a relic of the initial heat content from their birth. Since the surface temperatures of the neutron stars decrease as their ages increase, old neutron stars are no more be able to produce soft x-rays by thermal emissions. For such neutron stars, compton scattering is the dominant mechanism of the production of soft x-rays at their surfaces. Therefore, soft X-rays are are not the inherent properties of the neutron stars.

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Declaration

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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