LIE GROUPS - A PHYSICAL APPROACH

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ABSTRACT

Symmetries play a vital role in physics. If interactions are not known precisely, the underlying symmetries reflected in the phenomenology, provide valuable information on the interactions, even when interactions are known symmetries continue to remain a great asset.

We concentrate on continuous symmetries. Associated with these are their Lie groups and algebras. We take up the study of such algebras following the very elegant approach due to Schwinger starting from the bilinear products of fermion or boson creation operators a wide variety of Lie algebras can be generated. That, such algebras are relevant to physics follows from the simple fact that such bilinear products figure frequently in physical problems. Our aim is:

a) to study the general classification of such algebras,

b) to study their broad general characterstics,

c) to apply them to physical problems.

The applications we choose to study are principally from elementary particle and nuclear physics and many body theories. The accidental degeneracies encountered in quantum mechanics are easily understood in this algebraic framework. Our principal objective is to grasp the essentials without recourse to unwarranted mathematics and to learn to use these techniques in physical problems.

C O N T E N T S

CHAPTER 1

INTRODUCTION

In Physics, the lack of the exact knowledge of the interaction Hamiltonian is a common experience. Well known examples are the strong and weak nuclear forces. However, phenomenology often suggests underlying symmetries holding exactly or approximately in the real world. A lot can be learnt by studying these symmetries.

The present thesis considers continuous symmetries. Associated with these symmetries are their Lie groups and Lie algebras. We study here, these Lie algebras using a method employed by Schwinger[1] for the treatment of angular momentum and extended to various other groups by Lipkin[2]. The method is originally due to Wigner[3]. The attractive feature of this line of approach is its intuitive physical character avoiding unwarranted mathematical complexities. We shall find that in these studies an angular momentum like algebra (quasi-spin algebra) figures again and again and hence the angular-momentum machinery can be exploited to great advantage. Though we are principally interested in high energy physics phenomena we shall take this opportunity to point out where and how such algebras become useful in a variety of branches of physics. Thus, in summary, we aim to acquire a good working knowledge of the rich variety of algebras whose classification we shall obtain in our studies.

Let us first talk briefly about symmetry groups in quantum mechanics, and introduce the terminology we shall need constantly in what follows in this study.

We have noted that a satisfactory theory of strong interactions is not yet known. Hence, the study of the general symmetry properties of strong interactions assumes a special role. In fact, it allows one to obtain a satisfactory classification of hadrons and to derive a number of verifiable quantitative relations.

We know very well that all hadrons are divided into small families-isomultiplets, which can be assigned definite values of the isospin T. The members of a given multiplet differ in the isospin projection T_3 which determines the value of the electric charge and when the electromagnetic interaction is "switched off" (a sensible approximation under certain conditions) they have strictly the same mass. In strong interactions the quantum numbers T and T_3 are conserved.

A parallel situation is a common place experience in non-relativistic quantum mechanics. For example, when a non-relativistic particle moves in a central field its allowed states are also grouped into different sets characterized by different values of the total angular momentum J. The wave functions belonging to one given set are distinguished by the values of the angular momentum component J_3 and correspond to one and the same energy i.e. they form a degenerate level. The particle motion is such that J and J_3 are constants of motion.

The invariance of a theory with respect to a definite class of transformations in a real or in a certain abstract space is characteristic of all similar cases. The set of transformations is closed, there is an identity or unit transformation and to each transformation there corresponds an inverse transformation. Such invariance (symmetry) transformations are said to form a group. Its elements will be written as g and the successive application of two transformations g_1 followed by g_2 will be written as $g_2 g_1$. As an example, all space rotations form a group- a three dimensional rotation group denoted by O(3) or R_3.

The invariance of a theory with respect to a given group of transformations involves two aspects: definite transformation properties of the wave function Ψ and definite transformation properties of the Hamiltonian H. With respect to a group of transformation g the entire Hilbert space of the wave functions breaks up into subspaces, i.e. there are sets of wavefunctions which transform under the action of the group elements only into each other as

$$\Psi' = U(g)\Psi \quad , \tag{1.1}$$

To the product $g_2 g_1$ there corresponds the product of the operators U(g):

$$U(g_2 g_1) = U(g_2)U(g_1) \quad . \tag{1.2}$$

We say that U(g) form a representation of the given group. The dimensionality of space (the maximum number of linearly

independent wavefunctions) in which these operators act is said to be the dimension of the given representation. If the invariant subspace does not contain any invariant subspaces of lower dimension then one says that the representation is irreducible. We shall call the set of wavefunctions transforming according to an irreducible representation of a symmetry group a multiplet. Thus, the spherical harmonics $Y_{\ell m}(\theta,\phi)$ form a $(2\ell+1)$ dimensional multiplet of the rotation group in three dimensions.

It is easily established that the invariance of a theory under the transformations of a group implies,

$$\left[H,\ U(g)\right] = 0 \ . \qquad (1.3)$$

We now come to Lie Groups[4]. Their elements are single valued differentiable functions of a finite number of real parameters. These are chosen such that the identity element corresponds to all parameters having a value zero. Thus, for a Lie group

$$g = g(\alpha_1, \ldots, \alpha_n),\ g(0,\ldots,0) = I \ . \qquad (1.4)$$

The number n of all independent real parameters of the Lie Group is said to be its dimension. Any group element can be connected continuously to the identity element. The group characteristics can be obtained by studying elements close to the identity, i.e, for which the group parameters are infinitesimal so that,

$$g(\alpha_1,\ldots,\alpha_n) \approx I + \sum_{k=1}^{n} \alpha_k \frac{\partial g(\alpha_1,\ldots,\alpha_n)}{\partial \alpha_k}\Big|_{\alpha_1=\ldots=\alpha_n=0}$$

$$\equiv I + i \sum \alpha_k \ell_{-k} \quad . \tag{1.5}$$

The quantities,

$$\ell_k = -i \frac{\partial g}{\partial \alpha_k}\Big|_{\alpha_1=\ldots\alpha_n=0} \quad , \tag{1.6}$$

called group generators, represent square matrices whose dimension equals the dimension of the space in which the group transformations act. Thus, the rotation group generators are 3 x 3 matrices.

Clearly we also have,

$$U(g) = U(\alpha_1,\ldots,\alpha_n), \quad U(0,\ldots,0) = I \ . \tag{1.7}$$

The N dimensional representation operators corresponding to infinitesimal transformations have the form,

$$U(g) = I + \sum \alpha_k \frac{\partial U(\alpha_1,\ldots,\alpha_n)}{\partial \alpha_k}\Big|_{\alpha_1=\ldots=\alpha_n=0} \equiv I + i \sum_{k=1}^{n} \alpha_k L_k . \tag{1.8}$$

With $$L_k = -i \frac{\partial U}{\partial \alpha_k}\Big|_{\alpha_1=\ldots=\alpha_n=0} \quad , \tag{1.9}$$

L_k, are called representation generators and are N X N matrices. For the rotation group these are, to within multiplicative factors, the operators J_x, J_y and J_z of the angular momentum components. It is obvious that for a finite transformation,

$$U(g) = \exp(i\Sigma\alpha_k L_k). \qquad (1.10)$$

Physics demands that the U's be unitary. This means that the L_k must be hermitian. The condition (eqn. 1.3) transforms into

$$[H, L_k] = 0. \qquad (1.3a)$$

The closure property of the group forces the representation generators to satisfy,

$$[L_i, L_j] = \sum_k C_{ijk} L_k . \qquad (1.11)$$

This is known as the Lie algebra of the group. The entire structure of the group is determined by the constants C_{ijk} which are appropriately called the structure constants of the group.

We shall be specifically concerned with SU(N) groups. SU(N) is a group of unitary unimodular NXN matrices. The unimodularity condition is immediately seen to imply (from equation 1.10) that

$$\text{Trace } L_k = 0 , \quad k = 1, \ldots, n, \qquad (1.12)$$

Before we come to the end of this very short summary let us note down the following facts which underline the importance of studying symmetry groups in quantum mechanics:

1. There exist mutually commuting functions of the representation generators which commute with them. They are called invariant or Casimir operators. They have the same value for a given multiplet and are thus like the

multiplet of the identity operator acting on a given multiplet (Shur's Lemma). This means that the wave functions of one multiplet are eigen-functions of any Casimir operator with the same eigen value. Thus, a natural set of quantum numbers arises equal in number to the invariant operators characterizing the multiplet as a whole. The rotation group has one Casimir operator $J^2 \equiv J_x^2 + J_y^2 + J_z^2$ so that each multiplet is characterized by the value of the angular momentum J.

2. Among the representation generators there may exist several mutually commuting generators. Their number is determined by the properties of the group and is said to be its rank. The basis functions of a multiplet can be chosen to be simultaneous eigenfunctions of these generators. The corresponding eigenvalues are the quantum numbers classifying the wave functions belonging to a given multiplet. In the rotation group there are no mutually commuting generators, hence it is a group of rank one. Thus, the wave functions with a given J can be characterized by just one more quantum number which we usually take to be the value of J_z.

3. The generators and invariant operators are Hermitian and commute with the Hamiltonian. Hence, to them correspond conserved and physically measurable quantities.

4. It follows that if the wave function of the initial state of a system belongs to a certain multiplet then as a result of a reaction the system will make a transition to a new state belonging to the same multiplet. This establishes definite selection rules for the reactions. Thus, under the action of a rotationally invariant interaction a state of angular momentum J will go into a state of total angular momentum J only.

5. From eqn. 1.3a and Shur's Lemma it follows that the eigenvalues of the Hamiltonian (the values of energy or mass of the elementary particles) are the same for the wave functions of a given multiplet. This accounts for the presence of degeneracy and allows one to establish the multiplicity which is equal to the dimension of the multiplet.

6. Turning the argument around, if the observed multiplicity is higher than that expected from the symmetry group we are motivated to examine more closely the structure of the Hamiltonian to see if it has more symmetry than is apparent. The well known problem of the excess degeneracy of the hydrogen atom and the isotropic oscillator were understood this way.

Armed with the relevant fundamentals of symmetry groups we move on to the Schwinger approach to the theory of angular momentum. After getting equipped with this method we shall see how it can be applied to the various SU(N) groups and how it naturally suggests some entirely new algebras which do find applications in physics.

CHAPTER 2

THE HARMONIC OSCILLATOR MODEL FOR ANGULAR MOMENTUM

Schwinger[1] has given a fascinating formalism of the quantum theory of angular momentum using two independent one dimensional harmonic oscillators. Various quantities associated with the angular momentum theory are obtainable with great ease using this approach. We present below the essentials of this approach and record the derivation of rotation matrices so as to illustrate the simplicity and elegance of this method.

A one dimensional oscillator with mass μ and angular frequency ω and described by the canonical variables p,q has the Hamiltonian (with $\hbar = 1$)

$$H = \frac{p^2}{2\mu} + \tfrac{1}{2}\mu\omega^2 q^2 \quad , \tag{2.1}$$

$$= \omega(a^+a + \tfrac{1}{2}) \quad , \tag{2.2}$$

where,
$$a = \sqrt{\frac{\mu\omega}{2}}\left(q + \frac{ip}{\mu\omega}\right) \text{ and } a^+ = \sqrt{\frac{\mu\omega}{2}}\left(q - \frac{ip}{\mu\omega}\right) \quad . \tag{2.3}$$

The operators a and a^+ satisfy,

$$[a,\ a^+] = 1 \quad . \tag{2.4}$$

Clearly,
$$[H,\ a^+a] = 0. \tag{2.5}$$

Thus, H and a^+a share common eigen vectors. The eigen values of a^+a are integers $n(=0,1,2...)$. Thus, the oscillator eigenfunctions ψ_n obey,

$$a^+a\psi_n = n\psi_n \quad . \tag{2.6}$$

It is easily checked that $a^+\psi_n$ and $a\psi_n$ are also eigenstates of the number operator $N = a^+a$ with eigenvalues $(n+1)$ and $(n-1)$ respectively. Thus, a^+ and a can be identified as number raising and lowering operators, respectively.

The normalized eigenstate of N with eigenvalue n is,

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle \quad , \tag{2.7}$$

where $|0\rangle$ represents the ground state of the oscillator with no quanta,

Now consider two independent oscillators described by the pairs (a_+, a_+^+) and (a_-, a_-^+) respectively. The "+" operators commute with the "-" operators and

$$[a_+, a_+^+] = [a_-, a_-^+] = 1 \ , \tag{2.8}$$

The normalized eigenstates of the combined system can be written in self evident notation as,

$$|n_+, n_-\rangle = \frac{(a_+^+)^{n_+} (a_-^+)^{n_-}}{\sqrt{n_+!}\ \sqrt{n_-!}} |0\rangle, \tag{2.9}$$

Where $|0\rangle$ is the state with $n_+ = n_- = 0$.

We now introduce the following hermitian operators,

$$J_x \equiv \tfrac{1}{2} \quad (a_+^+ a_- + a_-^+ a_+) \quad , \tag{2.10a}$$

$$J_y \equiv \tfrac{1}{2i} \quad (a_+^+ a_- - a_-^+ a_+) \quad , \tag{2.10b}$$

$$J_z \equiv \tfrac{1}{2} \quad (a_+^+ a_+ - a_-^+ a_-) \quad . \tag{2.10c}$$

It is trival to check that the J's satisfy,

$$[J_x, J_y] = iJ_z \quad , \quad [J_y, J_z] = iJ_x \quad , \quad [J_z, J_x] = iJ_y \tag{2.11}$$

Thus, J_x, J_y and J_z obey the angular momentum commutation rules. All the results that follow from these commutation relations would also follow from the two oscillator model embodied in eqns. 2.10(a,b,c).

Defining $J^2 \equiv J_x^2 + J_y^2 + J_z^2$ one finds in a straight forward way that,

$$J^2 = \frac{N}{2} \left(\frac{N}{2} + 1\right) \quad , \tag{2.12}$$

where, $N = a_+^+ a_+ + a_-^+ a_-$ is the total number operator. Thus, the eigenvalues of J^2 are $\frac{n}{2}\left(\frac{n}{2} + 1\right)$ or $j(j+1)$ with $j = \frac{n}{2} = 0, \frac{1}{2}, 1 \ldots$. Further,

$$J_z |n_+ n_-\rangle = \left(\frac{n_+ - n_-}{2}\right) |n_+ n_-\rangle \quad . \tag{2.13}$$

Introducing $m = \dfrac{n_+ - n_-}{2}$, the state $|n_+ n_-\rangle$ can be alternatively denoted as $|jm\rangle$ with $J^2|jm\rangle = j(j+1)|jm\rangle$ and $J_z|jm\rangle = m|jm\rangle$. If

$j = \frac{n_+ + n_-}{2}$ is fixed then the allowed values of m are

$\frac{2j - 2n_-}{2} = j - n_-$, Where n_- runs from 0 to 2j in unit steps. Thus, the allowed values of m are +j, j-1, ... -j , as desired.

We can also introduce the usual raising and lowering operators, $J_\pm \equiv J_x \pm i J_y$ with,

$$J_+ = a_+^+ a_- \quad \text{and} \quad J_- = a_-^+ a_+ \quad . \tag{2.14}$$

Eqn.(2.13) makes it clear that each "+" quantum contributes $+\frac{1}{2}$ to the m value and each "-" quantum contributes $-\frac{1}{2}$. Eqn.(2.14) tells us that J_+ is an operator which destroys $-\frac{1}{2}$ unit of m value and creates $+\frac{1}{2}$ unit there by increasing the m value by one unit without changing the j value, since $n_+ \to n_+ + 1$ and $n_- \to n_- - 1$ so that $n_+ + n_-$ remain unchanged. In other words J_+ commutes with the total number operator and hence with J^2. In the same manner the expected action of J_- can be understood. Eqn.(2.9) can be rewritten as,

$$|jm\rangle = \frac{(a_+^+)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^+)^{j-m}}{\sqrt{(j-m)!}} |0\rangle \quad , \tag{2.15}$$

This, then, is the explicit form of the angular momentum eigenstates in this formalism. Let us calculate the matrix elements of J_+ using this formalism.

$$J_+ |jm\rangle = a_+^+ a_- |jm\rangle$$

$$= a_+^+ a_- \frac{(a_+^+)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^+)^{j-m}}{\sqrt{(j-m)!}} |0\rangle$$

$$= \frac{(a_+^+)^{j+m+1}\, a_-(a_-^+)^{j-m}\,|0\rangle}{\sqrt{(j+m)!\,(j-m)!}}$$

Using the identity $[A, B^n] = n[A, B]B^{n-1}$ one gets,

$$a_-(a_-^+)^{j-m} = (j-m)(a_-^+)^{j-m-1} + (a_-^+)^{j-m}a_-$$

since $a_-|0\rangle = 0$, we have,

$$J_+|jm\rangle = \frac{(a_+^+)^{j+m+1}}{\sqrt{(j+m)!}} \left| \frac{(j-m)(a_-^+)^{j-m-1}}{\sqrt{(j-m)!}} \right| |0\rangle$$

$$= \sqrt{\frac{(j-m)(j+m+1)}{(j+m+1)!\,(j-m-1)!}}\,(a_+^+)^{j+m+1}(a_-^+)^{j-m-1}\,|0\rangle \quad (2.16)$$

From eqn.(2.15) we have,

$$|jm+1\rangle = \frac{(a_+^+)^{j+m+1}\,(a_-^+)^{j-m-1}}{\sqrt{(j+m+1)!\,(j-m-1)!}}\,|0\rangle \quad (2.17)$$

Combining eqns. (2.16) and (2.17) we get,

$$J_+|jm\rangle = \sqrt{(j-m)(j+m+1)}\;|jm+1\rangle \quad (2.18)$$

Similarily we can show that

$$J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\;|jm-1\rangle \quad (2.19)$$

So far we have considered bilinear products of operators describing the oscillators, that do not change the number of quanta (that is how J_x, J_y, J_z commute with J^2 in this model).

However, the model naturally suggests number changing (and hence j changing) bilinear products. These are,

$$k_+ = a_+^\dagger a_-^\dagger \quad \text{and} \quad k_- = a_+ a_- \ . \tag{2.20}$$

From our previous discussion it follows immediately that k_+ increases the j value by one unit and k_- decreases the j value by one unit. They do not affect the m value. Proceeding exactly as above we can check that,

$$k_+ |jm\rangle = \sqrt{(j+m+1)(j-m+1)}\ |j+1,m\rangle \ , \tag{2.21a}$$

$$k_- |jm\rangle = \sqrt{(j+m)(j-m)}\ |j-1,m\rangle \ . \tag{2.21b}$$

We shall come back to these number changing operators in a latter chapter. Finally, we record the derivation of rotation matrices using this approach. To appreciate the elegance afforded by this method one should compare this with the standard derivation of these matrices[6].

The operator representing the rotations ψ,θ,ϕ (Euler angles) is:

$$R(\phi,\theta,\psi) = e^{-i\phi J_z}\ e^{-i\theta J_y}\ e^{-i\psi J_z} \ .$$

The matrix elements of the rotation matrix are,

$$D^j_{mm'}(\phi,\theta,\psi) \equiv \langle jm| e^{-i\phi J_z}\ e^{-i\theta J_y}\ e^{-i\psi J_z} |jm'\rangle . \tag{2.22}$$

$$= e^{-im\phi}\ e^{-im'\psi} \langle jm|\ e^{-i\theta J_y} |jm'\rangle . \tag{2.23}$$

Thus, the non-trivial part that need be evaluated is,

$$d^j_{mm'}(\theta) \equiv \langle jm|e^{-i\theta J_y}|jm'\rangle \ . \qquad (2.24)$$

Using eqn.(2.15) and its conjugate we have,

$$d^j_{mm'}(\theta) = \frac{\langle 0|a_+^{j+m}a_-^{j-m}\ e^{-\frac{\theta}{2}(a_+^\dagger a_- - a_-^\dagger a_+)}\ (a_+^\dagger)^{j+m'}(a_-^\dagger)^{j-m'}|0\rangle}{\sqrt{(j+m)!(j-m)!}\ \sqrt{(j+m')!(j-m')!}} \qquad (2.25)$$

Eqn.(2.25) is the expectation value of a product of lots of $a_\pm$ and $a_\pm^\dagger$ in the ground state $|0\rangle$. It is evaluated easily by introducing a generating function. Given the parameters X_+ and X_- let us define,

$$G(X_+, X_-) \equiv \sum_{m'} \frac{X_+^{j+m'}\ X_-^{j-m'}\ d^j_{mm'}(\theta)}{(j+m')!(j-m')!}$$

$$= \sum_{m'} \frac{X_+^{j+m'}\ X_-^{j-m'}\ \langle jm|\ e^{-i\theta J_y}\ |jm'\rangle}{\sqrt{(j+m')!(j-m')!}}$$

$$= \langle jm|e^{-i\theta J_y} \sum_{m'} \frac{(X_+a_+^\dagger)^{j+m'}(X_-a_-^\dagger)^{j-m'}}{(j+m')!(j-m')!}\ |0\rangle \qquad (2.26)$$

$$= \langle jm|e^{-i\theta J_y}\ \frac{(X_+a_+^\dagger + X_-a_-^\dagger)^{2j}}{(2j)!}\ |0\rangle \ . \qquad (2.27)$$

In the last step we have used the binomial theorem exploiting the commutativity of $a_+^\dagger$ and $a_-^\dagger$. Now,

$$e^{-i\theta J_y}(X_+a_+^+ + X_-a_-^+)^{2j} = |e^{-iJ_y\theta}(X_+a_+^+ + X_-a_-^+)e^{iJ_y\theta}|^{2j} e^{-iJ_y\theta} \qquad (2.28)$$

and $$e^{-iJ_y\theta}\ |0\rangle = |0\rangle \ , \qquad (2.29)$$

Hence, to evaluate $G(X_+, X_-)$ we need the behaviour of $(X_+a_+^+ + X_-a_-^+)$ under rotation around the y-axis. We exploit the identity,

$$e^{\lambda A}\ B\ e^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \ldots$$

to get,

$$e^{-i\theta J_y}\ a_+^+\ e^{iJ_y\theta} = a_+^+ \cos\left(\frac{\theta}{2}\right) \pm a_-^+ \sin\left(\frac{\theta}{2}\right) \qquad (2.30)$$

Using eqns. (28-30) we get,

$$G(X_+,X_-) = \frac{1}{(2j)!}\langle jm|\,|X_+(a_+^+\cos\tfrac{\theta}{2} + a_-^+\sin\tfrac{\theta}{2}) + X_-(a_-^+\cos\tfrac{\theta}{2} - a_-^+\sin\tfrac{\theta}{2})|^{2j}|0\rangle .$$

(2.31)

Using the binomial theorem in reverse order we get,

$$G(X_+,X_-) = \sum_{m'=-j}^{m'=+j} \langle jm|\,|a_+^+\ (X_+\cos\tfrac{\theta}{2} - X_-\sin\tfrac{\theta}{2})|^{j+m'} \times$$

$$\frac{|a_-^+\ (X_-\cos\frac{\theta}{2} + X_+\sin\frac{\theta}{2}|^{\,j-m'}}{(j-m')!}\ |0\rangle \qquad (2.32)$$

$$(j+m')!$$

Using the orthogonality of $|jm\rangle$ it is clear that only one term on the right hand side contributes. Thus,

$$G(X_+X_-) = \frac{(X_+\cos\frac{\theta}{2} - X_-\sin\frac{\theta}{2})^{j+m}(X_-\cos\frac{\theta}{2} + X_+\sin\frac{\theta}{2})^{j-m}}{\sqrt{(j+m)!}\quad\sqrt{(j-m)!}} . \qquad (2.33)$$

Using the defining equation for $G(X_+,X_-)$ and taking

$X_+ = -\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $X_- = t - \cos^2\frac{\theta}{2}$ we get upon equating the two results for $G(X_+,X_-)$

$$\sum_{m''} \frac{(-)^{j+m''}(\sin\frac{\theta}{2}\cos\frac{\theta}{2})^{j+m''}(t-\cos^2\frac{\theta}{2})^{j-m''}d^j_{mm'}(\theta)}{\sqrt{(j+m'')!\quad(j-m'')!}}$$

$$= \frac{(\sin\frac{\theta}{2})^{j+m}(\cos\frac{\theta}{2})^{j-m}\ t^{j+m}\ (1-t)^{j-m}}{\sqrt{(j+m)!\quad(j-m)!}} . \qquad (2.34)$$

To extract $d^j_{mm'}(\theta)$ from the above equation we should arrnage matters so that only one term with $m''=m$ survives on the left hand side. For this purpose we differentiate eqn.(2.34) $(j-m')$ times with respect to t and finally set $t = \cos\frac{\theta}{2}$, we then find that

$$d^j_{mm'}(\theta) = (-)^{j+m'}\sqrt{\frac{(j+m')!}{(j+m)!\,(j-m)!\,(j-m')!}}(\sin\tfrac{\theta}{2})^{m-m'}(\cos\tfrac{\theta}{2})^{-m-m'} \quad X$$

$$\{\frac{d}{dt}\,|\,t^{j+m}\,(1-t)^{j-m}|\}_{t=\cos^2\frac{\theta}{2}}$$

$$= (-)^{j+m'}(\sin\tfrac{\theta}{2})^{m'-m}(\cos\tfrac{\theta}{2})^{m+m'}\,x|\,(1-t)^{m-m'}t^{-m-m'}\frac{d}{dt}\{t^{j+m}(1-t)^{j-m}\}|_{t=\cos^2\frac{\theta}{2}} .$$

This is the desired result. (2.35)

In a later chapter we shall see how the spherical tensors can also be written in a convenient form in the Schwinger formulation. Before we conclude this chapter let us note that using the number conserving bilinear products of boson creation and annihilation operators one has been able to generate a Lie algebra. It is easily established that a similar algebra can be generated by using bilinear products of fermion operators even though the single operators in that case satisfy anticommutation relations.

The fermion operators will find a natural inlet when we talk about isospin and unitary symmetry in the report to follow.

CHAPTER 3

THE ISOSPIN SYMMETRY (SU(2))

1 Motivation

We now come to the first example where an angular momentum like algebra becomes relevant to physics. This is the example of isospin symmetry which figures in nuclear physics and hadron spectroscopy.

Nuclear physics phenomenology tells us that the proton (p) and neutron (n) are essentially the same particles. Their masses are nearly equal and they have the same spin and parity. The strong interactions between two protons, two neutrons and a proton and a neutron are the same in the same space-spin states. Such similarity in behaviour holds for mesons and hyperons also. It is electromagnetism that distinguishes between members of a given group of particles behaving identically under strong forces. Electromagnetism corresponds to a much weaker force compared to the strong force (within the domain it is operational). It is, therefore, sensible to ignore electromagnetism compared to strong interactions in the first approximation. Under such a situation, no distinction seems to be left between p and n. They correspond to degenerate levels of a quantum mechanical system. Experience tells us that degeneracy signals an underlying symmetry of the associated (strong interaction) Hamiltonian. Since it is the n-p degeneracy we want to understand, the

simplest symmetry one can think of should correspond to transformations in a 2-dimensional complex vector space $V^2(C)$ which leaves the Hamiltonian unchanged. We can then think of a single entity-the Nucleon which appears in two states n and p.

The parallelism with a spin-½ state in a rotationally symmetric world is obvious. When electromagnetism is "switched on" it defines a direction in this $V^2(C)$ and breaks the symmetry just as a magnetic field singles out a direction in ordinary space and breaks the rotational symmetry, Zeeman splitting the otherwise doubly degenerate spin-½ sates. In our case the doubly degenerate nucleon state splits into a proton and a neutron state. This symmetry is the SU(2) symmetry which we shall discuss here. Recalling the discussion of Chapter one we know that the hadrons should fall into multiplets of particles degenerate in mass in the absence of electromagnetism. This expectation is fully borne out in practice. Hadrons do divide into multiplets of SU(2) group. The members of each multiplet have small mass-differences which of course have to be there since there is no "switching off" of electromagnetism in the real world.

Before we take up the SU(2) group along the lines of the previous chapter let us note down a few general facts about this group.

.2 Rudiments of SU(2)

By the group SU(2) is meant the set of all unitary unimodular 2x2 matrices,

$$g^+g = gg^+ = I \quad , \quad \det g = 1 \quad . \tag{3.1}$$

For a simple geometrical interpretation we consider the space $V^2(C)$ of vectors $\underline{x}$ written in column form,

$$\underline{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with} \quad \underline{x}^+ = (x_1^*, x_2^*) \equiv (x^1, x^2) \quad . \tag{3.2}$$

In this space we consider linear transformations,

$$\underline{x}' = gx \quad \text{or} \quad x_i' = g_i^j x_j \tag{3.3}$$

(Superscript numbers the column and the subscript the rows, they run over the values 1,2 and repeated indices imply symmation). 2x2 unitary matrices correspond to transformation matrices (eqn.3.3) which do not change the quadratic form (length of x),

$$\underline{x}^+\underline{x} = x^1 x_1 = x_1^* x_1 + x_2^* x_2 \quad . \tag{3.4}$$

Following Chapter 1 we have, for infinitesimal transformations,

$$g = 1 + i\,\varepsilon_\alpha \tau_\alpha \quad , \tag{3.5}$$

where τ_α are the generators of SU(2) and ε_α are its parametrs whose number n defines the dimension of the group. τ_α are traceless hermitian 2x2 matrices. The eight real elements of τ_α obey five conditions,

$$\tau_\alpha^+ = \tau_\alpha \quad \text{and} \quad \mathrm{Tr}\,\tau_\alpha = 0 \quad . \tag{3.6}$$

Thus, there are three independent such matrices. Hence, $n = 3$ for SU(2). A well known set of matrices fulfilling the bill are the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.7}$$

These determine the algebra of SU(2) to be,

$$[\tau_i, \tau_j] = i \varepsilon_{ijk} \tau_k, \; i,j,k = 1,2,3 \quad . \tag{3.8}$$

(Repeated indices are to be summed over and ε_{ijk} is the usual Levi-Cevita symbol).

This is a group of rank one and the Casimir operator is

$$\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2 \quad . \tag{3.9}$$

Generalizing from here the SU(2) algebra is,

$$[F_i, F_j] = i \varepsilon_{ijk} F_k \quad . \tag{3.10}$$

Here, the F's are the general representation generators. The fundamental representationi is two-dimensional for which the generators are Pauli matrices. The regular representation is three-dimensional (for $n = 3$) and the F's for these are calculable in terms of the structure constnats.

Finally, the algebras of SU(2) is clearly identical to the angular momentum algebra. However, the operattons are performed in $V^2(\mathbb{C})$ and not in $V^3(R)$.

.3 Generation of the Lie Algebra Using Schwinger Approach.

To obtain the Lie algebra in the present context we start with the (n,p) multiplet, which should transform according to a two dimensional irreducible representation of the Lie group. We ignore the space-spin part which can be trivially included as we shall see later. Thus, our system has simply two quantum states available to it. We introduce two pairs of operators (a_p^+, a_p) and (a_n^+, a_n) which create and destroy a proton and a neutron respectively. We can form four number conserving bilinear products $a_p^+ a_n$, $a_n^+ a_p$, $a_p^+ a_p$ and $a_n^+ a_n$. The first two seem to have the actions of $\tau_\pm \equiv \tau_x \pm i\tau_y$ and the remaining are number operators. We also introduce a total number operator $B \equiv a_p^+ a_p + a_n^+ a_n$ which counts the number of particles and hence the baryon number in the present context.

We group these operators as follows:

$$\begin{aligned} B &= a_p^+ a_p + a_n^+ a_n , \\ T_+ &= a_p^+ a_n \ , \ T_- = a_n^+ a_p , \\ T_o (\equiv T_3) &= \tfrac{1}{2}(a_p^+ a_p - a_n^+ a_n) . \end{aligned} \tag{3.11}$$

Incidently, $T_o = Q - \frac{1}{2} B$, where Q is the charge operator $(a_p^+ a_p)$ for the (n,p) system.

Naturally, B commutes with $T_{\pm,o}$, these being number conserving operators. This can be explicitly checked quite easily from the definitions, (eqn.3.11). The operators T_+, T_-, T_o satisfy the expected algebra of SU(2),

$$[T_o, T_+] = \pm T_\pm \ , \ [T_+, T_-] = 2T_o \ . \qquad (3.12)$$

As an illustration let us check the relation $[T_+ , T_-] = 2T$

$$\text{L.H.S.} = [a_p^+ a_n, a_n^+ a_p] = a_p^+ a_n a_n^+ a_p - a_n^+ a_p a_p^+ a_n$$

$$= a_p^+ a_p (-a_n^+ a_n + 1) - a_n^+ a_n (1 - a_p^+ a_p)$$

$$= a_p^+ a_p - a_n^+ a_n = 2T_o \ .$$

The four operators (eqn.3.11) generate the algebra of the full U(2) group with $T_{\pm,o}$ generating a sub algebra (eqn.3,13) of the group SU(2). The group transformations $\left(\text{e.g. } g = 1+\varepsilon(T_+ + T_-)\right)$ mix the n and the p states, producing rotations in $V^2(C)$. Clearly, there is no physically meaningful $V^3(R)$ we could associate with these transformations. We have obtained an angular momentum algebra and hence all angular momentum results which follow from the commutation relations also apply to this case. The isospin operators defined by eqn.(3.11) can be easily generalized by adding the index K everywhere and summing over K. The index K describes the space and spin degrees of freedom. The commutation rules (eqn.3.12) remain valid, for bilinea products corresponding to two different quantum states K

and K^1, commute. Thus, each K acts independently in the commutator. Obviously, we can continue to use the simple notation of eqn. (3.11) and take care of K at any stage we choose to.

In analogy to angular momentum we can introduce the total isospin operator T with $T^2 = T(T+1)$. Any state can be written as $|T,T_3\rangle$ and to a given T, a multiplet would contain 2T+1 members. Thus, $T = 0, \frac{1}{2}, 1, \ldots$ for singlets, doublets, triplets,.... The proton and neutron belong to the doublet with $T=\frac{1}{2}$. Using the defining eqn. for Q it is clear that, $Q|p\rangle = |p\rangle$ and $Q|n\rangle = 0$, as required, with $|p\rangle \equiv |\frac{1}{2},\frac{1}{2}\rangle$ and $|n\rangle \equiv |\frac{1}{2},-\frac{1}{2}\rangle$. The pions can be grouped into a triplet with $T = 1$, with $T_3^{\pi}|\pi^+\rangle = |\pi^+\rangle$, $T_3^{\pi}|\pi^0\rangle = 0$, $T_3^{\pi}|\pi^-\rangle = -|\pi^-\rangle$.

Since $B = 0$ for pions, the charge operator here is the same as T_3^{π}.

Other multiplets e.g., (K^+, K^0), $(\bar{K}^-, \bar{K}^0)$, $(\Sigma^+,\Sigma^-,\Sigma^0)$ etc. can be similarly formed.

In what follows we shall have use for the isospin states for a pion-nucleon system. The problem is mathematically identical to that of constructing the total angular momentum states of a system consisting of states of angular momentum $j_1=\frac{1}{2}$ and $j_2=1$. We introduce the notation $T \equiv T^{\pi}+T^{N}$. Our aim is to construct the states $|T,T_3\rangle$ from the states $|T^{(\pi)}, T_3^{(\pi)}, T^{(N)}, T_3^{(N)}\rangle \equiv |T_3^{(\pi)}\, T_3^{(N)}\rangle$.

Angular momentum theory tells us that,

$$|T,T_3\rangle = \sum_{T_3^\pi T_3^N} |T_3^\pi T_3^N\rangle \langle 1,\tfrac{1}{2},T_3^\pi,T_3^N|1,\tfrac{1}{2},T,T_3\rangle \ , \qquad (3.13)$$

where the clebsch Gordon co-efficients are given by (note that $T = \frac{3}{2}$ or $\frac{1}{2}$),

$$\langle 1,\tfrac{1}{2},T_3\mp\tfrac{1}{2},\pm\tfrac{1}{2} \mid 1,\tfrac{1}{2},\tfrac{3}{2},T_3\rangle = \sqrt{\tfrac{1}{2}\pm\frac{T_3}{3}} \quad \text{and}$$

$$\langle 1,\tfrac{1}{2},T_3\mp\tfrac{1}{2},\pm\tfrac{1}{2} \mid 1,\tfrac{1}{2},\tfrac{1}{2},T_3\rangle = \pm\sqrt{\tfrac{1}{2}\mp\frac{T_3}{3}} \ . \qquad (3.14)$$

Using eqn.(3.14) repeatedly we find,

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |\pi^+p\rangle, |\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\pi^+n\rangle + \sqrt{\tfrac{2}{3}}|\pi^0p\rangle \ ,$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\pi^-p\rangle + \sqrt{\tfrac{2}{3}}|\pi^0n\rangle \ , \ |\tfrac{3}{2},-\tfrac{3}{2}\rangle = |\pi^-n\rangle \qquad (3.15)$$

and

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\pi^+n\rangle - \sqrt{\tfrac{1}{3}}|\pi^0p\rangle \ ,$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\pi^0n\rangle - \sqrt{\tfrac{2}{3}}|\pi^-p\rangle \ . \qquad (3.16)$$

The above relations can be inverted directly or by using the properties of Clebsch Gordon co-efficients to get,

$$|\pi^+p\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle \ , \ |\pi^0p\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$|\pi^-p\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle \ , \qquad (3.17)$$

$$|\pi^+n\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\pi^0n\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle \ , \ |\pi^-n\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle \ .$$

Physically if the pion and nucleon are in a state $|\frac{1}{2},\frac{1}{2}\rangle$, for example, then we do not know whether we have a π^+ or π^0 or a neutron or proton. The probability amplitude for having a $\pi^+ n$ is $\sqrt{\frac{2}{3}}$ and that for $\pi^0 p$ is $-\sqrt{\frac{1}{3}}$. All other amplitudes, for example for a $\pi^+ p$, are zero. All our results can be interpreted in a similar way.

In the next section we come to the relevance of all this formalism to physics.

3.4 The SU(2) Group and Physics

We talked about the identical behaviour of the protons and neutrons when nuclear forces alone are considered. Conventionally this is described as the charge-independence of nuclear forces. Isospin gives us the simple and elegant description of this property. Charge-independence says that the proton-proton, neutron-neutron, and the neutron-proton forces in the same spatial-spin states are identical. Thus, if the two nucleons are in an isotriplet state (symmetric) then the space-spin function is antisymmetric. If the p-n system is an isosinglet then the remaining function is symmetric. This means that the interaction between two nucleons in states $|T,T_3\rangle = |11\rangle$, $|10\rangle$ and $|1,-1\rangle$ are the same, other things being equal. The interaction in the state $|00\rangle$ is different.

The proton and the neutron form a bound state called the deuteron whose ground state coordinate function is symme-

tric and so is the spin part (the ground state is $^3S_1$). Thus, it must be an iso-singlet. If the deuteron were in the $^1S_0$ state it would be an iso-vector. No such bound state exists.

Thus, under the assumption of charge-independence nuclear forces are determined by the total isospin T rather than by T_3. Hence, the nuclear Hamiltonian must be a function only of the Casimir operator T^2 which is

$$T^2 = (\underline{T}^1+\underline{T}^2)^2 = T^{(1)2} + T^{(2)2} + 2\underline{T}^{(1)}.\underline{T}^2 \quad , \tag{3.18}$$

where $\underline{T}^1$ and $\underline{T}^2$ are the relevant isospin operators for the two nucleons. Clearly, the isospin dependence of the interaction Hamiltonian is of the form,

$$H_{int} = A + B(T^1 \cdot T^2) \quad , \tag{3.19}$$

It is easily checked that,

$$[H_{int}, T_{\pm,0}] = 0 = [H_{int}, T^2] \quad . \tag{3.20}$$

Thus, not only the third component of isospin is conserved (charge conservation) but the total isospin is also conserved. The interaction hamiltonian is an isoscalar. It is here, that the physics is brought in. Thus, in a strong interaction process an initial state belonging to a given multiplet can go over into a final state belonging to the same multiplet with the same value of T_3. We have the selection rule $\Delta T = 0$, for a purely strong interaction process. This result has profound physical consequences

for scattering and decay processes going via strong interactions. We will illustrate this with several examples. Let us consider nucleon-nucleon and nucleon-pion scattering. Other hadronic scattering processes can be considered along the same lines.

Consider a general reaction,

$$a_i + b_i \to c_i + d_i \quad , \tag{3.21}$$

where all particles of the type a,b,c and d belong to one and the same multiplet. The scattering amplitude f^i is proportional to the matrix element,

$$M^i \equiv \langle c_i d_i \mid a_i b_i \rangle \quad . \tag{3.22}$$

$|f^i|^2$ defines the differential scattering cross-section for the process. Assume that the initial state has the wave function $|T,T_3\rangle$. Ignoring the Coulomb effect, charge-independence tells us that isospin is conserved in the process, i.e.,

$$\langle T'T_3 | T\ T_3\rangle = 0 \quad \text{for } T' \neq T \ , \tag{3.23}$$

i.e., the final state wave function is of the type $|T\ T_3\rangle$. Furthermore, the strong interaction matrix element could depend only on T and not on T_3, so that,

$$\langle T\ T_3 \mid T\ T_3\rangle \equiv M^{(T)} \quad . \tag{3.24}$$

We will now see that the matrix element of any real process of the type (3.21) can be expressed in terms of a few $M^{(T)}$

(often two). For this we expand $|a_1 b_1\rangle$ and $|c_1 d_1\rangle$ in terms of wavefunctions $|T\ T_3\rangle$, substitute these expansions in eqn.(3.22) and use eqns.(3.23) and (3.24). We can then establish a number of relations between the cross-sections for different processes corresponding to the same initial and same final spatial-spin states of the particles involved in the scattering.

Examples:

1. Consider pp and pn scattering. Following the procedure of the last paragraph we write,

$$|p'p''\rangle = |1,+1\rangle \quad , \quad |n'p''\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |00\rangle), \qquad (3.25)$$
$$|p'n''\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |00\rangle).$$

Thus for $p'+p'' \rightarrow p'+p''$ we have,

$$M^{pp} = \langle p'p'' \mid p'p''\rangle = \langle 1,+1 \mid 1,+1\rangle = M^1. \qquad (3.26)$$

In the np scattering two processes are possible,

The ordinary elastic scattering,

$$n'+p'' \rightarrow n'+p'' \quad ,$$

and charge-exchange scattering,

$$n'+p'' \rightarrow p'+n'' \quad .$$

For the first of these,

$$M^{el} = \langle n'p'' \mid n'p''\rangle = \tfrac{1}{2}(M^{(1)}+M^{(0)}), \qquad (3.27)$$

and for the second,

$$M^{ch.ex} = \langle p'n'' \mid n'p''\rangle = \tfrac{1}{2}(M^{(1)}-M^{(0)}) \qquad (3.28)$$

From these we have,

$$\frac{d\sigma^{pp}}{d\Omega} \sim |M^1|^2 , \quad \frac{d\sigma^{el}}{d\Omega} \sim \frac{1}{4}|M^{(1)}+M^{(0)}|^2 , \tag{3.29}$$

$$\frac{d\sigma^{ch.ex}}{d\Omega} \sim \frac{1}{4}|M^{(1)}-M^{(0)}|^2 .$$

Thus,

$$\frac{d\sigma^{pp}}{d\Omega} \sim |M^{(1)}|^2 ,$$

$$\frac{d\sigma^{np}}{d\Omega} = \frac{d\sigma^{el}}{d\Omega} + \frac{d\sigma^{ch.ex}}{d\Omega} \sim \frac{1}{2}|M^1|^2 + \frac{1}{2}|M^0|^2 \tag{3.30}$$

Since the angular dependence of $M^{(1)}$ and $M^{(0)}$ can be different, then, inspite of charge-independence the behaviour of the proton-proton and proton-neutron scattering cross-sections as functions of the angular variables can be different. This is borne out experimentally. In the energy range 300-500 MeV in the centre-of-mass system the first cross-section is almost indpendent of the scattering-angle whereas the second has a minimum at $\theta = \frac{\pi}{2}$ increasing sharply in the backward direction and more slowly in the forward direction.

2. Consider $p + p \rightarrow d + \pi^+$,

$n + p \rightarrow d + \pi^0$.

We noted that deuteron is an isoscalar.

$$\therefore \quad |\pi^+ d\rangle = |1,+1\rangle , \; |\pi^0 d\rangle = |1,0\rangle , \tag{3.31}$$

and $M^{pp} = |\pi^+ d|pp\rangle = \langle 1,+1|1,+1\rangle = M^{(1)}$

$$M^{np} = \langle \Pi^0 d | np \rangle = \frac{1}{\sqrt{2}} | \langle 10 | 10 \rangle - | 00 \rangle | = \frac{1}{\sqrt{2}} M^1 \qquad (3.32)$$

$$\therefore \quad \frac{d\sigma^{pp}/d\Omega}{d\sigma^{np}/d\Omega} = 2 \qquad (3.33)$$

This checks experimentally.

3. Consider the reactions:

$$\Pi^+ + p \to \Pi^+ + p \ , \ \Pi^- + p \to \Pi^- + p \ , \ \Pi^- + p \to \Pi^0 + n$$

Now, $M^{(+)} \equiv \langle \Pi^+ p | \Pi^+ p \rangle = \left\langle \frac{3}{2}\ \frac{3}{2} \middle| \frac{3}{2}\ \frac{3}{2} \right\rangle = M^{3/2}$,

$$M^{(-)} \equiv \langle \Pi^- p | \Pi^- p \rangle = \sqrt{\tfrac{1}{3}} \left\langle \tfrac{3}{2}\ -\tfrac{1}{2} \middle| \sqrt{\tfrac{2}{3}} \left\langle \tfrac{1}{2}\ -\tfrac{1}{2} \middle| \sqrt{\tfrac{2}{3}} \left| \tfrac{3}{2}\ -\tfrac{1}{2} \right\rangle - \sqrt{\tfrac{1}{3}} \left| \tfrac{1}{2}\ -\tfrac{1}{2} \right\rangle \right.\right.$$

$$= \frac{1}{3} M^{(3/2)} + \frac{2}{3} M^{(1/2)} \qquad (3.34)$$

$$M^{(0)} \equiv \langle \Pi^0 p | \Pi^- p \rangle = \sqrt{\tfrac{2}{3}} \left\langle \tfrac{3}{2}\ \tfrac{1}{2} \middle| -\sqrt{\tfrac{1}{3}} \left\langle \tfrac{1}{2}\ \tfrac{1}{2} \middle| \sqrt{\tfrac{1}{3}} \left| \tfrac{3}{2}\ -\tfrac{1}{2} \right\rangle - \sqrt{\tfrac{2}{3}} \left| \tfrac{1}{2}\ -\tfrac{1}{2} \right\rangle \right.\right.$$

$$= \sqrt{\frac{2}{3}} M^{(3/2)} - \frac{\sqrt{2}}{3} M^{(1/2)} .$$

Thus, $$\frac{d\sigma^+}{d\Omega} : \frac{d\sigma^-}{d\Omega} : \frac{d\sigma^0}{d\Omega} = |M^{(3/2)}|^2 : \frac{1}{9} |M^{(3/2)} + 2M^{(1/2)}|^2 : \frac{2}{9} |M^{(3/2)} - 2M^{(1/2)}|^2 . \qquad (3.35)$$

At high energies the scattering takes place predominantly in the $T = 3/2$, $\ell = 1$, $j = 3/2$ channel. So we can neglect $M^{1/2}$ compared to $M^{3/2}$.

In this case the ratios are 9:1:2.

For an energy of 120 MeV of the incident pion the experimental cross-sections are in the ratio 93:11:22 which is roughly 9:1:2 . Thus, the scattering in the state $T = 3/2$ is dominating in this range. Other interesting results follow in other limits such as if $M^{(3/2)} \simeq M^{(1/2)}$ or if $M^{(3/2)} \simeq 0$. Comparison with experiments tells us under what conditions such situations may obtain.

4. We have to often deal with vertices involving the virtual production and absorption of pions and nucleons. If these are strong interaction vertices they are isospin conserving. They can be related to each other by using the C.G. machinery. Thus, for example, we can relate the various pion-nucleon coupling constants as,

$$g_{pn\pi^-} : g_{pp\pi^0} : g_{nn\pi^0} = 1 : \frac{1}{\sqrt{2}} : \left(-\frac{1}{\sqrt{2}}\right).$$

We shall omit the details.

Having discussed the consequences of exact isospin symmetry, let us finally not forget, that isospin is after all a broken symmetry. The neutron and the proton are not exactly the same in the real world. We do this in the final section.

3.5 Symmetry Breaking Effects

Isospin is a broken symmetry. However, if the symmetry breaking part has well defined transformation properties and can be considered a small correction then perturbation theory can be applied to deduce meaningful results. Looking at the properties of physical protons and neutrons or members of any given isomultiplet of hadrons one can conclude that the symmetry breaking corrections should be small (see comments later). Hence, perturbation theory should work. Further, the interaction Hamiltonian for the

electromagnetic case also seems to have very simple tensorial behaviour under isospin. We know that $H_{int}^{e.m} \propto j_{\mu}^{e.m} A_{\mu}$ where A_{μ} is the four-potential of the electromagnetic field and $j_{\mu}^{e.m}$ is the hadronic electromagnetic current. Now the space integral of j_o is the charge operator $Q(=T_o + \frac{1}{2}B$ for nucleons). B is an isoscalar and T_o is the third component of an isovector. Thus, $H_{int}^{e.m.}$ behaves as the sum of an isoscalar (S) and the third component of an isovector (V). With this assumption we can make definite predictions and compare them with experiments. We give two examples.

1. Consider the magnetic moment operator μ which is linear in the current and hence should be of the form $S+V_o$. Thus, exploiting the Wigner-Eckart theorem for angular momentum we can write,

$$\begin{aligned} \mu(\Sigma^+) &\equiv \langle \Sigma^+ | \mu | \Sigma^+ \rangle = S + v \ , \\ \mu(\Sigma^-) &\equiv \langle \Sigma^- | \mu | \Sigma^- \rangle = S - v \ , \\ \mu(\Sigma^0) &\equiv \langle \Sigma^0 | \mu | \Sigma^0 \rangle = S \ , \end{aligned} \qquad (3.36)$$

where the matrix elements of V_o have been calculated interms of the those of T_o in accordance with the Wigner-Eckart theorem.

Eliminating the two unknowns, S and v, we get the sum rule,

$$\mu(\Sigma^0) = \frac{\mu(\Sigma^+) + \mu(\Sigma^-)}{2} \ . \qquad (3.37)$$

2. As a second example we can consider the photo-production of pions from nucleons $\gamma + N \rightarrow \pi + N$. Treating the electromagnetic interaction to lowest order gives a matrix element of the form, $\langle \pi N | S+V_o | N \rangle$. The scalar part conserves isospin while the vector part can change the isospin. We thus have three reduced matrix elements (unknowns) .

$$\langle \tfrac{1}{2} || S || \tfrac{1}{2} \rangle \; ; \quad \langle \tfrac{3}{2} || V_o || \tfrac{1}{2} \rangle \; , \; \langle \tfrac{1}{2} || V || \tfrac{1}{2} \rangle \; .$$

But these describe four charge state reactions, $\gamma+p\rightarrow p+\pi^0$, $\gamma+p\rightarrow n+\pi^+$, $\gamma+n\rightarrow p+\pi^-$ and $\gamma+n\rightarrow n+\pi^0$. We thus have one relationship between the four processes. Now it is known that in the low energy region the scattering is dominated by the 3-3 resonance with $T = j = \frac{3}{2}$. In this case only one amplitude is effective and we need to keep only $\langle \frac{3}{2} || V || \frac{1}{2} \rangle$. Hence, the four processes are expressed in terms of only one unknown. Their ratios are fixed.

Finally, intuitively, it is clear that at energies much larger than the mass differences within isomultiplets isospin symmetry should be a good symmetry. Let us also note down the reason for assuming that the mass-differences within isomultiplets should be electromagnetic in origin. The fractional deviations $\Delta M/M$ from the average M of a multiplet are of the same order of magnitude as the fine-structure constant (which enters into simple theoretical

estimates of $(\Delta M/M)$. Whence the belief that departures from perfect SU(2) can be completely ascribed to electromagnetism. There are, however, no universally acceptable calculations of these effects but no better ideas are available either.

Summary

In this example of isospin we have seen how a Lie algebra can be generated from bilinear products of creation and annihilation operators. We have noted further how physics can be incorporated by demanding the invariance of the strong interaction Hamiltonian under isospin rotations and by ascribing simple tensorial behaviour to electromagnetic interaction under isospin. Further, we have profited greatly from the fact that the associated algebra is identical to the angular momentum algebra. However, as the high energy physics phenomenology grew richer it became gradually clear that the isospin group was too restricted to accommodate all the data and the need for bigger groups became a physical necessity. In the next chapter we move on to this new higher symmetry, namely SU(3) or unitary symmetry.

A SU(3) multiplet groups together several isospin multiplets of different strangeness (or hypercharge). However, unlike SU(2), SU(3) is not an exact symmetry of strong interactions because the particles in a multiplet have large mass-differences. Here, we have an example where a badly broken symmetry provides meaningful results. Since two quantum numbers T_3 and Y are needed to specify the location of a particle within a multiplet we need a two-dimensional plot (weight diagram) with T_3 and Y as the coordinate axes.

4.2 Rudiments of SU(3)

SU(3) is the group of all unitary unimodular 3x3 matrices which correspond to linear transformations in $V^3(C)$ conserving the quadratic form $\underset{\sim}{x}^{+}\underset{\sim}{x} = x^{1}x_1 = x_1^{*}x_1 + x_2^{*}x_2 + x_3^{*}x_3$.

The generators λ_α are 3x3 traceless hermitian matrices,

$$\lambda_\alpha^{+} = \lambda_\alpha \quad \text{and} \quad \mathrm{Tr}\,\lambda_\alpha = 0. \tag{4.1}$$

The eighteen real parameters obey nine conditions of hermiticity and one of tracelessness. Thus, there are eight independent matrices with the above properties. This specifies the dimension of the group. Among λ_α there are two mutually commuting ones (SU(3) is a rank 2 group) . These can be simultaneously diagonalized. Conventionally λ_3 and λ_8 are chosen diagonal. A commonly used set is[8]:

CHAPTER 4

THE UNITARY SYMMETRY AND ELEMENTARY PARTICLES*

4.1 Motivation

After the discovery of strange particles the framework of isospin symmetry proved to be too narrow to describe the emerging phenomenology. SU(2) is a group of rank one and it gives only one conserved additive quantum number (T_3) by which members of an isomultiplet can be classified. It was found that hadrons are characterized by an additional additive quantum number called strangeness (S) or equivalently by the hypercharge (Y). So, one needed a group of rank 2. The group SU(3) was found to be the most suitable for this purpose. The hadrons of a given spin, parity and baryon number can be grouped into multiplets transforming as the irreducible representations of SU(3). This was called the eightfold way by Gell-Mann[7] because the $\frac{1}{2}^+$ baryons and 0^- pseudoscalar mesons belonged to the eight-dimensional representation of this group.

At the time Gell-Mann and Neeman proposed the SU(3) symmetry there were several gaps in the multiplets. These were obvious predictions of the theory. In particular a particle Ω^- with $S = -3$ and spin $= \frac{3}{2}$ was predicted at 1675 MeV and was subsequently discovered at the expected mass[8]. It was this evidence that ultimately settled the battle in favour of the group SU(3).

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* Only a limited account will be presented here.

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\ \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\ \lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

(4.2)

$$\lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\ \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\ \lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

If all parameters ε_α of SU(3) except for the first three are set equal to zero then we obtain the group SU(2).

A typical SU(3) transformation is given by,

$$U = \exp(\tfrac{1}{2}\, i\, \varepsilon_\alpha\, \lambda_\alpha). \tag{4.3}$$

The SU(3) algebra is obtained by working out the commutators of the λ matrices. It is,

$$[\tfrac{1}{2}\lambda_i,\ \tfrac{1}{2}\lambda_j] = i\, f_{ijk}(\tfrac{1}{2}\lambda_k) \quad , \tag{4.4}$$

where f_{ijk} are the totally antisymmetric structure constants which we shall not list here. The matrices also obey anti-commutation relations,

$$[\tfrac{1}{2}\lambda_i,\ \tfrac{1}{2}\lambda_j]_+ = \frac{1}{3}\,\delta_{ij} + d_{ijk}(\tfrac{1}{2}\lambda_k) \quad , \tag{4.5}$$

where d_{ijk} are symmetric under the interchange of i,j,k.

Finally, the general SU(3) algebra is,

$$[F_i,\ F_j] = i\, f_{ijk} F_k \quad . \tag{4.6}$$

A full study of SU(3) then consists of finding NxN matrices F_i which trnsform the N-dimensional states by,

$$\phi \rightarrow \phi' \simeq (I + i\varepsilon_\alpha F_\alpha)\phi \quad , \tag{4.7}$$

These states form N dimensional multiplets of SU(3). The SU(3) Casimir operators can be written down but we shall not find any direct use for them here. The multiplets can be characterised by two numbers as we shall see later. A Clebsch - Gordon machinery can be set up for the group SU(3)[9] but we shall exploit the subgroups of SU(3) so that the angular momentum machinery would suffice here.

4.3 The Lie Algebra

To get the Lie algebra using the method agreed upon, we start with three quantum states. A convenient example (though obsolete) is the Sakata model, which is the natural generalization of the SU(2) model used previously, in which the (n,p) multiplet is enlarged to include the λ hyperon with zero isospin and strangeness = -1. We add two more operators a_λ^+ , a_λ to the earlier four. We can construct nine number conserving bilinear products. As before these break up into (1+8) operators with the baryon number operator $B \equiv a_p^+ a_p + a_n^+ a_n + a_\lambda^+ a_\lambda$ commuting with the rest which do not change the baryon number. We write these nine operators in the following convenient form:

$$
\begin{aligned}
B &\equiv a_p^+ a_p + a_n^+ a_n + a_\lambda^+ a_\lambda \quad , \\
T_+ &\equiv a_p^+ a_n \quad , \quad T_- \equiv a_n^+ a_p \quad , \\
T_o &\equiv \tfrac{1}{2}(a_p^+ a_p - a_n^+ a_n) \quad , \\
B_+ &\equiv a_p^+ a_\lambda \quad , \quad B_- \equiv a_n^+ a_\lambda \quad , \\
C_+ &\equiv a_\lambda^+ a_n \quad , \quad C_- \equiv a_\lambda^+ a_p \quad , \\
N &\equiv \tfrac{1}{3}(a_p^+ a_p + a_n^+ a_n - 2a_\lambda^+ a_\lambda) = \tfrac{1}{3}B + S \quad .
\end{aligned}
\tag{4.8}
$$

These operators generate linear unitary transformations in a proton-neutron-lambda Hilbert space. The baryon number operator generates trivial phase transformation, and the remaining eight generate the algebra of SU(3). Inspection of the operators(eqn. 4.8) shows that the eight operators form a Lie algebra of rank 2. The two operators T_o and N commute with one another and there is none which commutes with both of them. The action of these operators is manifest in the defining equations. The remaining six operators are typical step operators shifting the eigenvalues of T_o and N. Many of the commutators can be read off directly from the manifest action of the operators. For example B_+ creates a proton and annihilates a lambda, thus raising T_o by $+\frac{1}{2}$ so that $[T_o, B_+] = \frac{1}{2} B_+$. In this way we can directly write down the following commutation relations.

$$
\begin{aligned}
&[T_o, T_{\pm}] = \pm T_{\pm}, \quad [N, T_{\pm}] = 0, \quad [T_o, B_{\pm}] = \pm \tfrac{1}{2} B_{\pm} \\
&[T_o, C_{\pm}] = \pm \tfrac{1}{2} C_{\pm}, \quad [N, B_{\pm}] = B_{\pm}, \quad [N, C_{\pm}] = - C_{\pm} \quad .
\end{aligned}
\tag{4.9}
$$

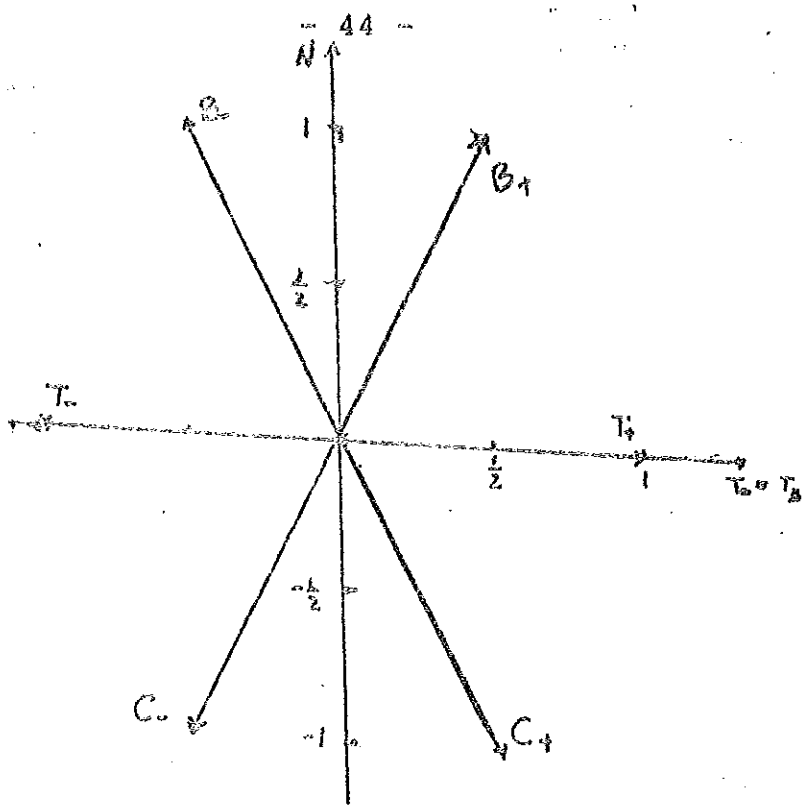

N
1
$\frac{1}{2}$
$-\frac{1}{2}$
-1
B_-
B_+
C_-
C_+
T_-
T_+
$\frac{1}{2}$
1
$T_0 = T_3$

Fig 4.1

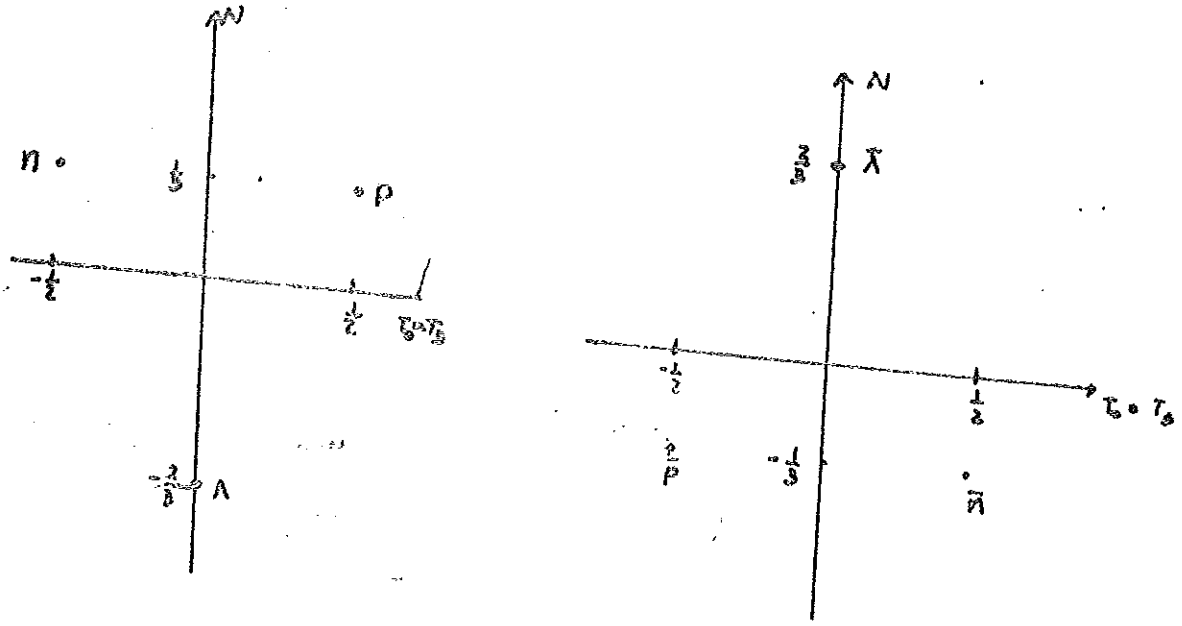

N
n
$\frac{1}{3}$
p
$-\frac{1}{2}$
$\frac{1}{2}$
$T_0 = T_3$
$-\frac{2}{3}$
A
N
$\frac{2}{3}$
X
$-\frac{1}{2}$
$\frac{1}{2}$
$T_0 = T_3$
$\frac{2}{P}$
$-\frac{1}{3}$
$\bar{n}$

Fig 4.2

should occur in the same SU(3) multiplet. They should appear alternately with increasing values of N.

Let us consider some simple examples of SU(3) multiplets. The n-p-λ and $\bar{n}$-$\bar{p}$-$\bar{\lambda}$ triplets form multiplets, since the eight SU(3) operators simply transform these into each other. These are shown in Fig.(4.2). Consider now the states resulting from a combination of one sakaton (p or n or λ) and one antisakaton. These states are generated from the vacuum by a product of a sakaton creation and a sakaton annihilation operator, their SU(3) properties are identical to those of the operators of eqn.4.8. Thus, the states will break up into a singlet and octet. They are shown in Fig.4.3 (incidently we have proved the rule $3 \times 3^* = 1 + 8$). Naturally, the octet looks very much like the diagram (Fig.4.1) of the group generators.

The octet reveals a new element compared with an angular momentum multiplet. At N = 0, T_3 = 0 we have two points. This shows that the eigenvalues of N and T_3 may not uniquely specify a state within a multiplet. An additional quantum number is necessary which is not determined by the SU(3) algebra. For our problem the value of the Casimir operator of the isospin subgroup namely that of T^2 does the job. The four N = 0 states split into an isotriplet and an isosinglet. We now record some important facts mostly without proof:

1. The weight diagrams have a convex boundary.

2. The shape and the size of the multiplets can be fixed in terms of two parameters λ and μ with $\lambda = 2T$ at $N=N_{max}$

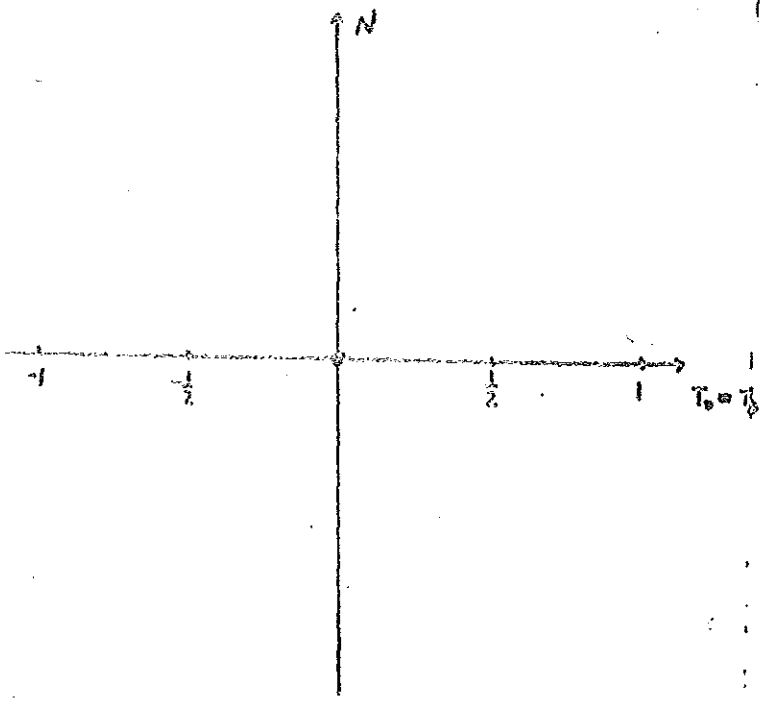


Fig 4.3a singlet.

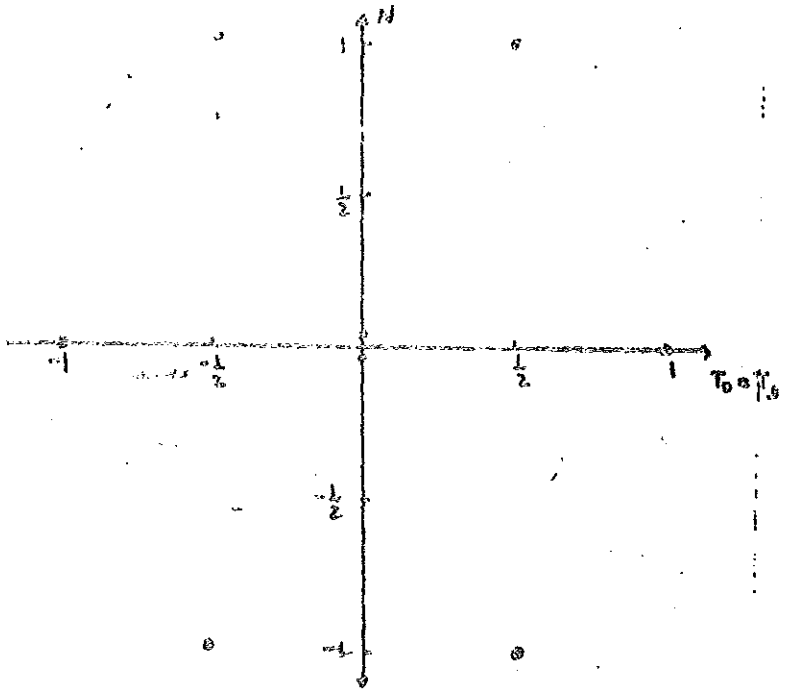


Fig 4.3b Octet.

and $\mu = 2T$ at $N = N_{min}$.

Thus, λ and μ are integers. The two triplets are (1,0) and (0,1) respectively and the octet is (1,1). The number of states in a (λ,μ) multiplet is $\frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$. Thus, the octet is made up of eight states.

3. The weight diagrams have to be symmetric abbut the N-axis and to the lines inclined at 120^0 to the horizontal. This follows from the equivalence of p,n and λ states in the sakata model.

4. The conjugate representation of (λ,μ) is (μ,λ). This corresponds to the fact that charge conjugation amounts to inversion through the origin.

5. Casimir operators are functions of λ and μ. Thus, labelling with λ, μ is simpler. This happens for angular momentum also where the Casimir operator has values $j(j+1)$ but labelling index is j.

6. All integral values of λ and μ give multiplets with the following rules for deciding how many states there are for each lattice point:

   i) The outer ring of lattice points is always single.

   ii) Each successive inner ring has one state more at each lattice point than the previous one until we reach a triangle or a point.

   iii) The number of states within the triangle is the same as on the perimeter of the triangle. The diagram for the octet illustrates these features.

7. The eigenvalues of the operator $N(=\frac{1}{3}B+S)$ are of the form n, $n+\frac{1}{3}$, $n-\frac{1}{3}$. A given multiplet is obviously of one of these types, for SU(3) operations change N by ± 1 or zero. The three types of multiplets correspond to $\frac{1}{3}(\lambda-\mu)$ integral, integral $\pm\frac{1}{3}$ respectively. Thus, the octet contains $N = 1, 0, -1$ and is n-type.

8. Combining SU(3) multiplets: since N is additive, for a system of several parts, the total N is the sum of individual N's. So, when multiplets are combined one has,

$$(n)+(n) \to n, \quad (n\pm\tfrac{1}{3})+n \to n\pm\tfrac{1}{3}, \quad (n+\tfrac{1}{3})+(n-\tfrac{1}{3}) \to n \ .$$

Rules analogous to the case of angular momentum can be developed which are not easy to remember unlike the angular momentum rule. A coupling of great significance is the coupling of two octets.

$$(1,1) \otimes (1,1) = (0,0)+(1,1)+(2,2)+(3,0)+(0,3)+(1,1),$$

Like with the coupling of $j_1=1$ and $j_2=1$ in angular momentum for example, we can ask for the symmetry properties of the multiplets with respect to the interchange of the two parts. A simple and quick recipe for such couplings is the Young Tableaux technique[2,4].

4.5 Generalization to any SU(3) Algebra

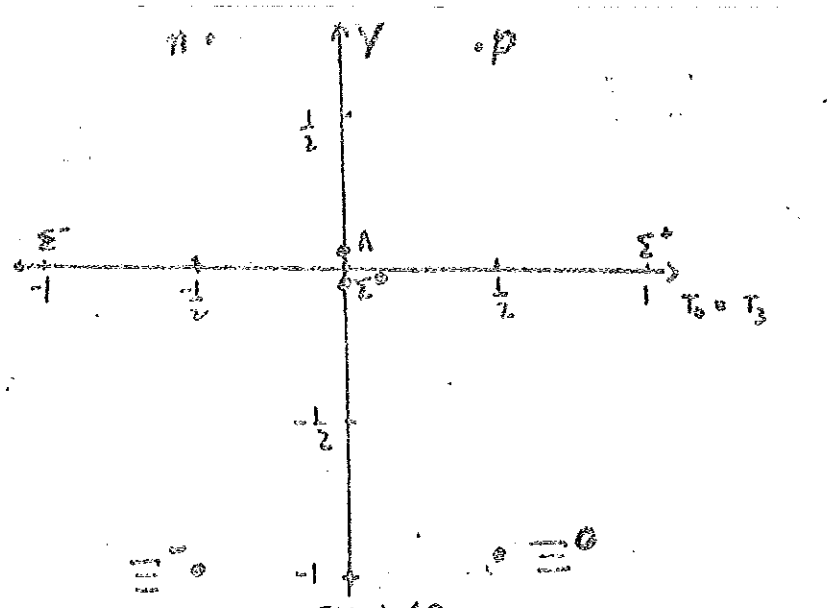
We have obtained the Lie algebra of SU(3) using the Sakata model. Irrespective of the validity of the Sakata model the algebra remains valid. And, in fact, the Sakata model is not supported by experimental evidence. Phenomenology suggests that the eight $\frac{1}{2}^+$ baryons should be grouped into an octet but Sakata model groups p,n and λ already into a triplet leaving the other five members for other multiplets. It turns out that the Σ hyperon must be placed at least in a sextet which includes a nucleon like particle and a particle with Y = 2, T = 0 which must have spin $\frac{1}{2}^+$. Such particles have never been found. The Ξ goes at least in a 15-plet with large number of unoccupied places and the Ω cannot be included in any lower multiplets. Such a classification is obviously most unsatisfactory.

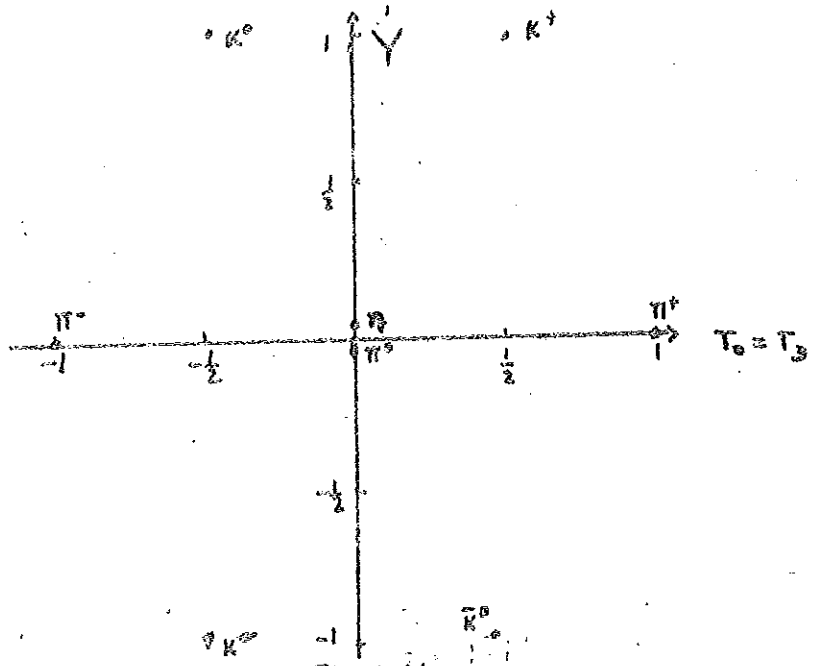
Whenever a set of eight operators satisfying commutation relations like those of the operators (eqns.4.9 and 4.10) arise in any physical problem we know they constitute the Lie algebra of SU(3). These operators can be used to characterize the states of the associated quantum mechanical system and group them into multiplets which are already known from the Sakata model. Of course, a given model may not lead to some multiplets. For example, the angular momentum algebra can be obtained from orbital angular momentum but then the multiplets with half integral j would have been missed.

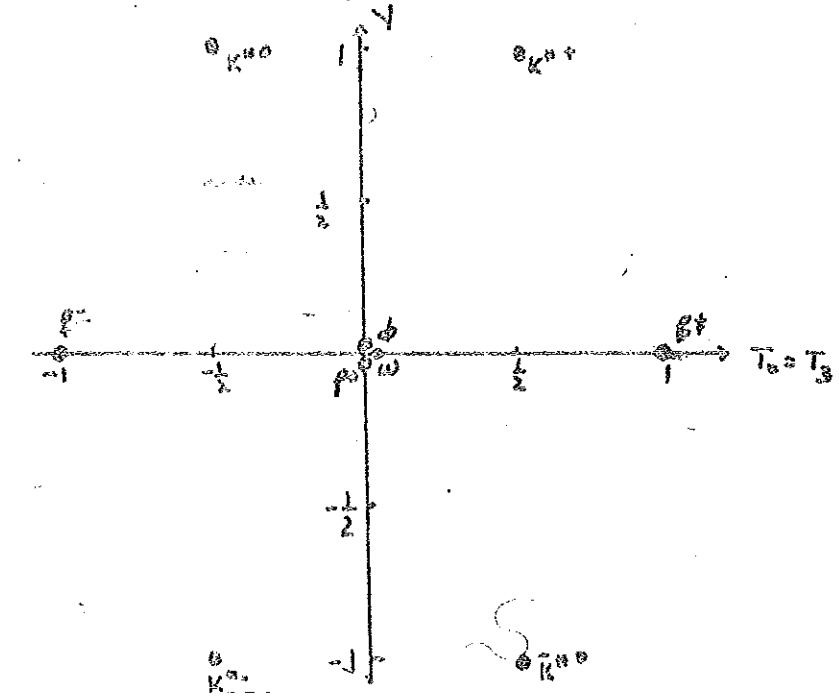
4.6 The Eightfold Way[7]

Consider the two dimensional (T_3-Y) plots of well known stable $\frac{1}{2}^+$ baryons, 0^- pseudoscalar mesons and 1^- vector meson resonances. (Fig.4.4). These diagrams immediately suggest octet multiplets of SU(3) with an additional singlet vector meson. The meson octets are what we expect from the Sakata model. But an octet is suggested for $\frac{1}{2}^+$ baryons as well, in contrast to the p-n-λ triplet of the Sakata model. This classification of elementary particles is called the eightfold way or the octet model.

To formulate the problem mathematically we need eight operators satisfying the basic commutation relations. The three isospin operators we define in the usual manner. The baryon octet tells us that N should be identified with the hypercharge Y=B+S rather than with $\frac{1}{3}$ B+S as in the Sakata model. Now the operators $T_{\pm,0}$ and N are in hand, the rest are uniquely specified by the commutation relations. They are the ones changing T_0 by $\pm \frac{1}{2}$ and Y by $\pm$ 1. Eqn.4.9 gives the commutation relations amongst the step operators themselves. The multiplet structure has already been given by the Sakata model. Thus, the matrix elements of these operators can be calculated between any two states of a multiplet. This is analogous to the angular momentum case where given the commutation relations amongst $J_{\pm}, J_3$ and given the multiplet structure all matrix elements of these operators in a given multiplet are calculable.


n
Y
p
1/2
Σ⁻
Λ
Σ⁺
Σ⁰
-1
-1/2
1/2
1
T₀ = T₃
-1/2
Ξ⁻
-1
Ξ⁰
Fig 4.4a


K⁰
Y
K⁺
1
1/2
π⁻
π⁰
π⁺
-1
-1/2
1/2
1
T₀ = T₃
-1/2
K⁻
-1
K̄⁰
Fig 4.4b


Y
K*0
K*+
1
1/2
ρ⁻
ρ⁰
ω
φ
ρ⁺
-1
-1/2
1/2
1
T₀ = T₃
-1/2
K*-
-1
K̄*0

We have identified N with Y which we require to be integral. Hence, only those multiplets occur for which $\frac{1}{3}(\lambda-\mu)$ is an integer. The simplest of these are the singlet (0,0), octet (1,1), decuplets (3,0) and (0,3) and the 27-plet (2,2).

Now, when we talked about isospin we had a physical principle which we were trying to express in mathematical language. It was the charge-independence of nuclear forces and the mathematical expression was the invariance of the strong interaction Hamiltonian under the SU(2) transformations. Do we have such a physical basis for the eightfold way? In the Sakata model we did have, namely the equivalence of n,p and λ interactions. We do not have any obvious basis here. We have not even talked about the fundamental representation. And the properties of the fundamental triplet in this model are startling. Since we have put Y = N = B+S our baryon number for the triplet is one-third of that defined by eqn.4.8. Thus, the triplet would have $B = \frac{1}{3}$. The hypercharge is thus $\frac{1}{3}$, $\frac{1}{3}$ and $-\frac{2}{3}$ and the charges would be $\frac{2}{3}$, $\frac{1}{3}$ and $-\frac{1}{3}$, for $T_3 = \pm \frac{1}{2}$ and $T_3 = 0$ respectively. These peculiar objects are called quarks and so far none of these fractionally charged objects have been found experimentally. Perhaps they do not exist in free unbound states. If these are the fundamental objects the generalization of charge-independence would be the equivalence of the interaction between any two of these. Of course, one can believe that the quarks are more than just a mnemonic and instead hypothesize that they are the entities out of which the mesons and baryons are physically constructed. In the mid

1960's one felt that quarks were just a mnemonic devoid of physical meaning. Today we tend to believe that quarks are physical entities that are dynamically confined within mesons and baryons. The observed multiplet structure and approximate SU(N) symmetry of particle physics then has its origin in N flavours (types) of quarks filling the fundamental representation of SU(N) and the composite quark systems yield the hadron spectroscopy. Unfortunately, these exciting new developments are not the subject matter of this work and we better stop before we go astray.

Finally, for the sake of completeness let us review the successes of the hypothesis of approximate invariance of the strong interaction with respect to the group SU(3). These are numerous and impressive[2,7,10,11,12].

1. All stable hadrons and low lying resonances are distributed over SU(3) multiplets of low dimensions.

2. The quark model, which makes it possible to construct all hadrons from three fundamental particles and their antiparticles is very attractive.

3. Different mass-formulae have been obtained for isomultiplets within unitary multiplets based on educated guesses on the nature of SU(3) breaking strong interaction Hamiltonian. These are in good agreement with data.

4. A number of relations between the coupling constants of baryons and mesons have been established.

5. Various scattering and decay processes could be related. (The general procedure followed for all these items is exactly analogous to that described in the previous chapter on SU(2)).

6. Taking into account the electromagnetic interaction mass-formulae have been obtained for individual members of isomultiplets of a given SU(3) multiplet. Thus, one finds for example,

$$(m_n - m_p) - (m_{\Xi^0} - m_{\Xi^-}) = (m_{\Sigma^-} - m_{\Sigma^+}) \ ,$$

This is used to predict the sign of the mass-difference $(m_{\Xi^0} - m_{\Xi^-})$ and checks experimentally.

7. The relations between magnetic moments of the baryons of a given multiplet were derived. Thus, one finds,

$$\mu_p = \mu_{\Sigma^+} \ , \ \mu_{\Xi^-} = \mu_{\Sigma^-} \ , \ \mu_n = 2\mu_{\Lambda^0} = -2\mu_{\Sigma^0}$$

In appendix A the SU(2) subgroups of SU(3) are discussed. These will be exploited to derive the relations mentioned in the last two items.

We conclude this chapter with the observation that the group theoretic method contains no element of dynamics which must come from a theory of elementary particles. However, having agreed that there is an underlying symmetry the group theoretic method supplies a wealth of information.

CHAPTER 5

APPLICATIONS OF SU(3) TO OTHER BRANCHES OF PHYSICS

5.1 The Three-Dimensional Isotropic Oscillator

The harmonic oscillator problem finds many applications in physics. It is a subject that deserves a treatise by itself. In the harmonic oscillator shell model, a number of particles are assumed to move independently in a harmonic oscillator potential. This is the problem we shall discuss here. We first address ourselves to the problem of a three-dimensional isotropic oscillator.

The energy spectrum of such a quantum system was perplexing. The only manifest symmetry of the Hamiltonian is the rotational symmetry. Thus, the characteristic angular momentum degeneracy is to be expected. But, as is well known, the degeneracy is far greater, with sets of states of angular momenta differing by two units turning out to be degenerate. Since the cause of such a degeneracy was not understood, it was termed accidental degeneracy. However, the approach we are adopting here reveals at a glance that there is nothing accidental about this degeneracy, for the symmetry of the Hamiltonian is that of SU(3), so that degenerate levels correspond to multiplets of SU(3) and not just to those of angular momentum. This example (along with the other famous example of the ideal hydrogen atom-the Kepler problem, actually) emphasize in no uncertain terms the beauty and power of symmetry arguments.

Before giving a detailed treatment paralleling the Sakata model, let us see how the degeneracy is such an obvious result.

A three-dimensional oscillator is nothing but the sum of three independent equal mass, equal spring constant oscillators. Its hamiltonian is simply,

$$H = \omega(\underline{a}^{+}\cdot\underline{a} + \frac{3}{2}) \quad , \tag{5.1}$$

where $\underline{a}$, $\underline{a}^{+}$ are vectors with components,

$$a_\mu = \sqrt{\frac{m\omega}{2}}\, x_\mu + \frac{i}{(2m\omega)^{1/2}} p_\mu \quad , \quad a_\mu^{+} = \sqrt{\frac{m\omega}{2}}\, x_\mu - \frac{ip_\mu}{(2m\omega)^{1/2}} \quad , \tag{5.2}$$

where $\mu = 1,2,3$, correspond to the creation and annihilation operators in the three directions. Thus, H is essentially given by the length of a complex three-dimensional vector. Rotations in the associated $V^3(C)$ do not change the length of the vector. Thus, H is invariant under rotations in $V^3(C)$. The previous chapters tell us that the symmetry of H is that of U(3) and the multiplets correspond to SU(3). The generators of this symmetry group are the (1+8) bilinear products of the type $a_\mu^{+}a_\nu$. These are even parity operators connecting states of the same parity. Thus, a degenerate level contains states with angular momentum ℓ, $\ell+2$, $\ell+4$,... A little later, we shall come to the occupancy number of a given level. We note that the nine bilinear products break up into groups of one, three and five operators transforming irreducibly amongst each other. This has to be, for

given two vectors $\underline{a}$ and $\underline{a}^+$ we can form the scalar product $\underline{a}\cdot\underline{a}^+$, the vector product $\underline{a}\times\underline{a}^+$ and a symmetric second rank tensor. The vector product corresponds to the angular momentum operator and the tensor correspond to the quadrupole moment tensor. With all the expectations made explicit and plausible let us now approach the problem along the lines of the previous chapter.

The operators $\underline{a}$ and $\underline{a}^+$ satisfy the commutation relations,

$$[a_\mu, a_\nu^+] = \delta_{\mu\nu}, \quad [a_\mu, a_\nu] = [a_\mu^+, a_\nu^+] = 0 \ , \qquad (5.3)$$

The Hamiltonian is,

$$H = \omega\Big(\sum_{\mu=1}^{3} (a_\mu^+ a_\mu) + \frac{3}{2}\Big) \ . \qquad (5.4)$$

The angular momentum operators are given by,

$$\ell_{\mu\nu} = x_\mu p_\nu - p_\mu x_\nu = i(a_\mu a_\nu^+ - a_\nu a_\mu^+), \quad \mu \neq \nu \ . \qquad (5.5)$$

We have nine bilinear products of the type $a_\mu^+ a_\nu$ commuting with H. Except that they are boson operators, they look the same as the SU(3) generators of the previous chapter. Like the operator B of SU(3) we have the combination $\Sigma a_\mu^+ a_\mu$ commuting with the remaining eight.

Now we can draw the exact correspondence between the operators of eqn.5.2 and those of the Sakata model. There, we made use of the isospin subgroup by singling out the λ-direction in the n-p-λ Hilbert space. So, we now choose the

x_3-direction as a preferred direction and look at the SU(2) subgroup associated with the two-dimensional oscillator in the x_1-x_2 plane. Use of cylindrical coordinates is immediately suggested and we define,

$$a_{\pm} \equiv (a_x \mp i a_y)/\sqrt{2} \quad , \quad a_o \equiv a_3 \quad , \tag{5.6a}$$

so that,

$$a_{\pm}^{+} = (a_x^{+} \mp i a_y^{+})/\sqrt{2} \quad , \quad a_o^{+} = a_3 \quad . \tag{5.6b}$$

Interms of these operators,

$$H = \omega(a_{+}^{+}a_{+} + a_{-}^{+}a_{-} + a_o^{+}a_o + \tfrac{3}{2}) \, . \tag{5.7a}$$

We also define,

$$\lambda_{+} \equiv a_{+}^{+}a_{-} \equiv \lambda_{-}^{+} \, , \; \lambda_o \equiv \tfrac{1}{2}(a_{+}^{+}a_{+} - a_{-}^{+}a_{-}) = \tfrac{1}{2}\ell_3 \, , \tag{5.7b}$$

The operators a_+ and a_- are now analogous to the p and n operators and λ_+ , λ_- , λ_o are analogous to T_+ , T_- and T_o. Thus, λ_+ , λ_- , λ_o are the operators of the SU(2) group of the two-dimensional oscillator in the x_1 - x_2 plane. Incidentally, $\lambda_o = \tfrac{1}{2}\ell_3$, hence its eigenvalues are $0, \tfrac{1}{2}, 1, \tfrac{3}{2}$ ... For our quasi-spin operators we have the Casimir operator,

$$\lambda^2 = \tfrac{1}{2}\{\lambda_{+}\lambda_{-} - \lambda_{-}\lambda_{+}\} + \lambda_o^2 \, . \tag{5.8}$$

The remaining five operators of the algebra can be written down in direct analogy to the Sakata model as:

$$B_{+} \equiv a_{+}^{+}a_o, \; B_{-} \equiv a_{-}^{+}a_o, \; C_{+} \equiv a_o^{+}a_{-} \, , \; C_{-} = a_o^{+}a_{+} \, ,$$

$$N \equiv \frac{1}{3}(a_{+}^{+}a_{+} + a_{-}^{+}a_{-} - 2a_o^{+}a_o) = \frac{1}{3}(a_1^{+}a_1 - a_2^{+}a_2 - 2a_3^{+}a_3) \, , \tag{5.9}$$

We now have a set of eight operators in one-to-one correspondence with those of the Sakata model. Hence, they generate a SU(3) algebra and the Hamiltonian, which commutes with all these operators, has the invariance of SU(3). The energy levels corresponding to this Hamiltonian fall into multiplets arising out of the Sakata model. They can be plotted on a $N - \lambda_o$ plot. The physical meaning of N follows from its definition. It is one-third the difference between the sum of the quanta in the x_1 and x_2 directions and twice those in the x_3-direction. Thus, $N = 0$ if the average number of quanta in the x_1-x_2 direction equals that in x_3-direction, $N > 0$ if x_1-x_2 average exceeds the occupation number in the x_3-direction and $N < 0$ in the reverse situation. N is thus a measure of departure from spherical symmetry which makes its association with a quadrupole tensor clear.

The ground state of the oscillator in non-degenerate and is the SU(3) singlet (0,0). The first excited state is three-fold degenerate. We can put one oscillator quantum in any one of the three directions. This is the (1,0) multiplet of the Sakata model. The n^{th} oscillator state corresponds to a totally symmetric n-sakaton state. Its Young diagram is a row of n boxes so that it is the multiplet (n,0). The degeneracy follows from the general SU(3) formula. However, it can be written down directly by observing that the problem is one of distributing n oscillator quanta among the three oscillator directions, i.e., by finding all possible

combinations of n_1, n_2, n_3 such that $n = n_1 + n_2 + n_3$ and $n_1, n_2, n_3 = 0, 1, 2 \ldots, n$. This number is simply $\frac{1}{2}(n+1)(n+2)$. This is the known degeneracy of the oscillator in the n^{th} state. Thus, the (n,0) level exhausts the degeneracy of the n^{th} level. As in the case of the Sakata model the states of the (n,0) multiplet can be classified into quasi-spin multiplets having all possible values of the total quasi-spin, λ from 0 to $\frac{1}{2}$ n.

5.2 The Angular Momentum Classification

The quasi-spin classification of the previous section was useful to understand the degenercay problem. In actual use, we should be more interested in the angular momentum classification of the oscillator states. We solve the oscillator problem in either cartesian or spherical coordinates. In the latter case we use orbital angular eigenfunctions. The energy of the oscillator is, $E = (n+\frac{3}{2})\omega$ where the integer n and the orbital angular momentum ℓ are related as, $n = 2k + \ell$ where $k = 0,1,2,\ldots$ Thus, if n is even $\ell = 0,2,4..,n$, and if n is odd $\ell = 1,3,5 \ldots n$. The states in the quasi-spin classification are related to the states in the angular momentum classification (just as the isospin and U-spin classification are related). In both cases ℓ_3 is a good quantum number, if λ is integral then ℓ_3 is even, if λ is half integral, ℓ_3 is odd. Consider the n^{th} excited state of the oscillator. We have the (n,0) multiplet. The two largest quasi-spin multiplets ($\lambda = \frac{1}{2}n$ and $\lambda = \frac{n-1}{2}$) just contain all the eigenvalues of ℓ_3 appearing in the $\ell = n$ angular momentum multiplet.

Continuing in this manner, we get the one-to-one correspondence between the eigenvalues of ℓ_3 arising in the two classifications. Thus, for example, for the (3,0) multiplet the two largest quasi-spin multiplets are $\lambda = \frac{3}{2}$ and $\lambda = 1$ containing 7 states and corresponding to the seven angular momentum states for $\ell = 3$. The remaining quasi-spin multiplets with $\lambda = \frac{1}{2}$ and $\lambda = 0$ contain the three states with $\ell_3 = \pm 1, 0$ making up the angular momentum $\ell = 1$ states.

In the quasi-spin classification we had the operator N which commutes with all the quasi-spin operators. Thus, the states could be plotted on a $N - \lambda_o$ plot. In the angular momentum case there is no such operator available. Thus, there is no simple way of defining another quantum number to classify the states in conjuction with ℓ^2 and ℓ_3. For a single oscillator there is no problem for ℓ^2 and ℓ_3 like λ^2 and λ_o are enough to classify the states of (n,0) multiplet and N is redundant. However, when we deal with a system of several oscillators we get several states having the same quasi-spin with different values of N. But, since in the angular momentum case an N-like operator is not available we do not know how to display the several states with the same angular momentum. We notice here, how differences can arise between SU(2) and R_3 based classifications. The Lie algebra is the same in both cases but the geometrical and physical significance of the two groups is different. For SU(2) we work in $V^2(C)$ which is a subspace of $V^3(C)$ in which the SU(3) transformations are defined. R_3 is a group of real rotations in the whole of $V^3(C)$.

In using the angular momentum classification of harmonic oscillator states the following combinations are more suitable.

$$
\begin{aligned}
\ell_0 &\equiv (a_+^+ a_+ - a_-^+ a_-) = 2\lambda_0 \ , \\
\ell_\pm &\equiv \sqrt{2}\,(a_0^+ a_\mp - a_\pm^+ a_0) = \pm\sqrt{2}\,(C_\pm - B_\pm) \ , \\
q_{\pm 2} &\equiv -\sqrt{6} a_\pm^+ a_\mp = -\sqrt{6}\,\lambda_\pm \ , \\
q_{\pm 1} &\equiv \mp\sqrt{3}\,(a_0^+ a_\mp + a_\pm^+ a_0) = \mp\sqrt{3}\,(C_\pm + B_\pm) \ , \\
q_0 &\equiv 2a_0^+ a_0 - a_+^+ a_+ - a_-^+ a_- = 3N \ .
\end{aligned}
\tag{5.10}
$$

Their commutation rules are,

$$
\begin{aligned}
&[\ell_0 \ , \ \ell_\pm] = \pm \ell_\pm \ , \\
&[\ell_+ \ , \ \ell_-] = 2\ell_0 \ , \ [\ell_0 \ , \ q_m] = m q_m \ , \\
&[\ell_\pm \ , \ q_m] = \sqrt{6 - m(m\pm 1)}\ \ q_{m\pm 1} \ , \\
&[q_0 \ , \ q_{\pm 1}] = \pm \tfrac{3}{2}\sqrt{6}\ \ell_\pm \ , \\
&[q_1 \ , \ q_{-1}] = -3\ell_0 \ , \\
&[q_2 \ , \ q_{-2}] = 6\ell_0 \ , \\
&[q_\pm \ , \ q_{\pm 1}] = \pm 3\ell_\pm \ , \quad [q_0, q_{\pm 2}] = [q_{\pm 1}, q_{\pm 2}] = 0 \ .
\end{aligned}
\tag{5.11}
$$

We also want to write down one of the Casimir operators which commutes with all the eight operators here. Thus, the operator must be a scalar under rotation and we expect it to be quadratic in these operators (in analogy to the angular momentum Casimir operator ℓ^2). We have two such opera-

tors available, namely, ℓ^2 and the square of the quadrupole moment. Starting with a linear combination of these two and commuting them with the operators of the given set we find the combination to be,

$$C' = 3\ell^2 + \sum_{m=-2}^{2} (-)^m q_m q_{-m} \quad . \tag{5.12}$$

Conventionally, $C = \frac{1}{36} C'$ is employed.

In the nuclear shell model a number of particles are assumed to move independently in a harmonic oscillator potential. The above treatment can be trivially generalized to cover the situation. We only have to define the oscillator operators for each particle and define the operators of the Lie algebra as the sum over all independent particle operators. Let $x_{\mu i}$ and $p_{\mu i}$ be the canonical variables of the i^{th} particle and $a_{\mu i}$ and $a^{+}_{\mu i}$ be the corresponding annihilation and creation operators. Let λ_i , ℓ_i and q_i be the quasi-spin, angular momentum and quadrupole tensor operators respectively, for the i^{th} particle. The Lie algebra for the system is defined by the following operators,

$$L = \Sigma \ell_i \ , \ \lambda = \Sigma \lambda_i \ , \ Q_m = \Sigma q_{im} \ ,$$

and

$$C = \frac{1}{36} \left[3L^2 + \sum_m (-)^m Q_m Q_{-m} \right] \quad . \tag{5.13}$$

The operators $B_{\pm}$, $C_{\pm}$ and N are similarly defined appropriately. The particles being all independent, the commutation relations obviously remain unchanged.

Each individual particle belongs to some SU(3) multiplet. The state of the whole system is obtained by combining SU(3) multiplets just as one combines a number of angular momentum states to obtain the possible total angular momentum states. Suppose, for example, that we have the lowest s-shell filled and a number of particles are in the next p-shell. The three degenerate states of each particle correspond to the sakaton triplet (1,0). We have to combine n such triplets and find out $3 \otimes 3 \otimes 3 \ldots$ (n factors). Not all these would be allowed when the generalized Pauli principle is invoked. The second excited level of the oscillator has six states available. It is the (2,0) multiplet which corresponds to $L = 2$ and $L = 0$. We have the particles in the s-d shell. If we have four particles here with the two lower shells full then we have to combine four (2,0) multiplets. Pauli principle tells us that closed shells belong to the (0,0) multiplet just as the closed electronic shells carry zero angular momentum.

5.3 The Elliot Model[13]

A relevant question now arises. What has one gained by going through this SU(3) classification? For a single oscillator we did gain something concrete. Namely, we unearthed the precise symmetry of the oscillator Hamiltonian. We know the full quantum mechanical description of the oscillator. The principal quantum number and the angular momentum fully classify the states. However, in the case of many particles

in an oscillator potential several states with the same angular momentum appear. Additional quantum numbers are then required for a complete specification of the states and the SU(3) classification may become relevant.

An illustration of such a situation is the Elliot model in which some of the degeneracy is lifted by adding a two-body interaction of the quadrupole-type to the oscillator Hamiltonian. Thus,

$$H = H_{osc} - V \sum_{ij} (-)^m q_{mi}\, q_{-mj} \qquad (5.14a)$$

$$= H_{osc} - V \sum (-)^m Q_m \quad Q_{-m} \qquad (5.14b)$$

or,

$$H = H_{osc} - 36CV + 3VL^2 \quad . \qquad (5.14c)$$

Here, H_{osc} represents the Hamiltonian for the particles moving independently in the oscillator potential. The remaining term is a two-body interaction which might be considered as the quadrupole term in the expansion of a general two-body interaction in spherical harmonics.

At this point we are obliged to make a digression so as to appreciate the philosophy behind writing down a Hamiltonian like the above one. Let us discuss the symmetry group we have in hand, namely, SU(3)-generated by three angular momentum and five components of the quadrupole tensor operators. What has this group to do with a real nucleus? L and Q form a closed algebra but this group is not a true symmetry

of real nuclei. The quadrupole components are not really constants of motion, i.e.,

$$[H_{real}, Q_m] \neq 0 \ .$$

Still the group is useful. We have a non-conserved symmetry which gives useful results. If the observed properties of the nucleus are angular momentum and quadrupole moment this group may be relevant. This may not be true for other nuclei, e.g., those with spherical symmetry. So, one is trying to use a group that is at least important for certain nuclei. There are two possible scenarios.

Firstly, H_{real} may almost commute with the Q_m so that quadrupole moments are approximately conserved and SU(3) is an approximate symmetry. Secondly, one may say that one is not worried whether symmetry breaking is large or small but H_{real} must be a function of L and Q. So that H_{real} has matrix elements only within a given mutliplet. The multiplets are then split but not mixed by the Hamiltonian. The present model is a simple model where this kind of approach is followed. We see that the quadrupole interaction is just the sum of a term proportional to the Casimir operator C and a term proportional to L^2. Thus, the eigenfunctions of H are those linear combinations of the degenerate oscillator functions which are simultaneous eigenfunctions of C and L^2. The states of a given SU(3) multiplet are all eigenfunctions of C. The required eigenfunctions are obtained first by classifying the states into SU(3) multiplets

and then choosing the states within the multiplets to be eigenfunctions of L^2. What kind of splitting should one expect? Firstly, there is a splitting between the multiplets because C varies from multiplet to multiplet and then there is a splitting within a multiplet due to the L^2 term. Each SU(3) multiplet constitutes a rotational band of "moment of inertia" I given by $\frac{1}{2I} = 3V$. The Elliot Model has met with reasonable success in the range $16 < A < 40$. Finally, a word more about the term $\Sigma(-)^m Q_m Q_{-m}$. It is invariant under R_3 but not under the full SU(3) group. It represents a long range attractive interaction indicating that every pair of particles interacts mainly through the effect of nuclear shape as a whole. Thus, if particle one bulges in a given direction then particle two would try to do the same. Such a term would encourage cooperative quadrupole distortions and must be present in the real case. But this may not be the only term. The other way this term could be justified is to say that the force has a quadrupole-quadrupole component which dominates over the rest which can hence be ignored.

We have thus learnt that the SU(3) model can demonstrate how collective features such as "rotational bands" can be understood within an independent particle shell model. If the interaction not included above is somehow small it can be treated as a perturbation. The states to be used for this purpose are those classified and labelled by SU(3) quantum numbers. However, since the interaction does not

commute with SU(3) generators the choice of the basis functions has to be such that L and one other angular momentum operator have to be chosen diagonal. The more convenient quasi-spin classification can not be used. We have learnt here a very vital means of exploiting symmetry considerations in which the Hamiltonian is chosen to be a function of the group generators so that symmetry can be used to label the states and no multiplet mixing is allowed. Such considerations have given valuable results, both in nuclear and particle physics.

Finally, we would like to restress the fact that the collective behaviour resulting from an independent particle model obtained here has a deeper root. This problem will be discussed in greater details in Appendix B.

CHAPTER 6

THE GENERAL ALGEBRA OF BILINEAR PRODUCTS

6.1 The Case of Two Fermion Operators

So far, we have examined the algebra of bilinear products of number conserving boson and fermion operators. We would now like to include the remaining bilinear products which create or annihilate a particle - pair. Let us go back to the case of two fermion operators and look at all the left out operators, namely,

$$S_+ = a_p^+ a_n^+ \ , \quad S_- = a_p a_n \ , \tag{6.1}$$

These, along with the four introduced in the case of SU(2) are all the distinct bilinear products we have. In the bosonic case there are a total of four extra ones, since Pauli principle is not operational in that case.

A proton-neutron pair in the same quantum state has isospin zero. Hence, $S_\pm$ commute with the isospin operators. The full set of commutation relation is:

$$\begin{aligned} &[T_o, T_\pm] = \pm T_\pm \ , \quad [T_+, T_-] = 2T_o \\ &[S_o, S_\pm] = \pm S_\pm \ , \quad [S_+, S_-] = 2S_o \\ &[T_i, S_k] = 0, \ i, k = 1,2,3 \quad \text{and where,} \end{aligned} \tag{6.2}$$

$$S_o \equiv \tfrac{1}{2}(a_p^+ a_p - a_n a_n^+) = \tfrac{1}{2}(B - 1) .$$

We have the direct product of two angular momentum like algebras. The multiplet structures and the matrix elements of all the operators follow from angular momentum theory.

The addition or removal of a proton-neutron pair in the same quantum state has no physical relevance. But the algebra is valid for any two fermion quantum states. Such pairs are of great physical significance in studying pair-correlations in many-fermion systems. Consider two quantum states denoted by $+k$ and $-k$ which might, for example, be two states with equal and opposite momentum $\pm k$. We define, the quasi-spin operators,

$$S_+ \equiv a_k^+ a_{-k}^+ , \quad S_- \equiv a_k a_{-k} , \quad S_{ok} \equiv \tfrac{1}{2}(a_k^+ a_k + a_{-k} a_{-k}^+) \tag{6.3}$$

These quasi-spin operators correspond physically to the addition or removal from the system of a pair of particles having equal and opposite momentum. They can describe the creation or annihilation of a Cooper pair, for example. Without affecting the algebra, we can generalize the above definitions for any number of values of k. Thus, the generalized quasi-spin operators are,

$$S_{\pm} = \sum_k S_{\pm k} , \qquad S_o = \sum_k S_{ok} \tag{6.4}$$

To see their utility, consider a simple two-body pairing interaction,

$$V = - G \sum_{kk'} a_k^+ a_{-k}^+ a_{-k'} a_{k'} , \tag{6.5}$$

where the sum is over a particular set of states k. Such an interaction is used, for example, in the theory of superconductivity. Now,

$$V = -G\, S_+ S_- = -G\,(S^2 - S_o^2 + S_o)\ , \qquad (6.6)$$

where as usual,

$$S^2 = \tfrac{1}{2}(S_+S_- + S_-S_+) + S_o^2 \qquad (6.7)$$

is the Casimir operator. Thus, the states $|S,S_o\rangle$ are eigenstates of such an interaction. V is simply a diagonal operator with eigenvalues $-G\big(S(S+1) - S_o^2 + S_o\big)$ with S_o running from +S to -S in unit steps. If the pairing interaction is the dominant force, i.e., if the "strong coupling limit" applies we have a complete solution to the pairing problem. In such a problem the number conserving operators have no direct physical significance.

Let us now try to identify the Lie algebra - a direct product of two angular momentum algebras. Group theory tells us that this is the R_4 algebra. We see this as follows. (since the answer is known, we follow a procedure which is just the converse of what one needs for showing that R_4 has the same algebra as $R_3 \times R_3$).

Define,

$$\gamma_1 \equiv a_p^+ + a_p,\ \gamma_2 \equiv -i(a_p^+ - a_p),\ \gamma_3 \equiv a_n^+ + a_n,\ \gamma_4 = i(a_n^+ - a_n). \qquad (6.8)$$

One has,

$$\gamma_i^2 = I \quad \text{and} \quad [\gamma_i, \gamma_j]_+ = 2\gamma_{ij}\ , \qquad (6.9)$$

where,

$$i, j = 1,2,3,4\ .$$

The six independent bilinear products are simply,

$$L_{ij} \equiv \tfrac{1}{2} \gamma_i \gamma_j = - L_{ji} , \quad i \neq j \tag{6.10}$$

The L_{ij} satisfy,

$$[L_{ij}, L_{jk}] = L_{ik} = - L_{ki} , \quad i \neq k , \tag{6.11a}$$

$$[L_{ij}, L_{km}] = 0, \text{ if no two indices are equal.} \tag{6.11b}$$

These represent the natural extension of the angular momentum commutation relations to four dimensions.

Generalization of these results to the case of an arbitrary number of creation and annihilation operators is straight forward. First, consider the genralization to the case of the case of three states. As in the Sakata model we introduce the operators a_λ^+ and a_λ for the additional state λ. We then define,

$$\gamma_5 \equiv a_\lambda^+ + a_\lambda \text{ and } \gamma_6 \equiv i(a_\lambda^+ - a_\lambda),$$

The operators L_{ij} are similarly defined with i,j, running from 1 to 6. We now get the algebra of the rotation group in six dimensions whose generators are the 15 independent operators L_{ij}, $i \neq j$.

Now, the general result for the set of fermion operators a_k^+ and a_k for n values of k (these may either be n different kinds of fermions or n different states of the same fermion) is clear. The set of all possible bilinear products of these operators is a set of $n(2n - 1)$ operators

which constitute the Lie algebra of the rotation group in 2n dimensions-R_{2n}. The subset of these making up the number conserving bilinear products is a set of n^2 operators constituting the group U(n). One bilinear combination of these operators is the total number operator which commutes with the remaining $n^2 - 1$ operators. Thus, $U(n) = SU(n) \otimes U(1)$. The U(n), SU(n) and U(1) are subgroups of R_{2n}.

6.2 The Seniority Classification

In section 6.1, we could find two number changing isoscalar operator for the n = 2 case, namely, S_+ and S_-. This enabled us to separate the R_4 algebra into two commuting SU(2) algebras. For n = 3, this procedure will not work, for, combining two sakatons we get a triplet and a sextet of SU(3) but no singlet. We have no operators creating a pair of sakatons and commuting with all the SU(3) operators. But whenever n is even we can always define operators of the type S_{+k} of eqn.(64). Such quasi-spin operators are useful. Consider the classification of nuclear states in the jj coupling nuclear shell model. Consider the set of (2j+1) operators a^+_{jm} (hence forth written as a^+_m) creating a fermion in a state of angular momentum $|jm\rangle$. These operators transform amongst themselves as the components of an irreducible tensor of rank j under rotations. This is obvious, for, acting on the singlet vacuum state $|0\rangle$, they give us the 2j+1 states $|jm\rangle$ which form an irreducible

representation of $(2j+1)$ dimensions. The previous section tells us that the set of all bilinear products of a_m and a_m^+ generate the Lie algebra of the rotation group in $2(2j+1)$ dimensions. Consider the multiplets corresponding to this algebra, needed for the classification of states of any number of fermions distributed among these $(2j+1)$ single fermion states. All the many-fermion states should fall into two multiplets. One should contain the states having an odd number of particles and the other an even number of particles. This must be so, for the Lie Algebra involves bilinear products which can change particle numbers in multiples of two only. Thus, it does not mix states belonging to an odd and an even number of particles.

The number conserving bilinear products generate the algebr of SU_{2j+1} as per the previous section. Thus, all possible states of a given number of particles in a j-shell should belong to a single multiplet of the group SU_{2j+1}. Like before, let us define quasi-spin operators as,

$$S_+ \equiv \tfrac{1}{2} \sum (-)^{j-m} a_m^+ a_{-m}^+, \quad S_- \equiv S_+^+, \quad S_o \equiv \tfrac{1}{2}[S_+, S_-] \qquad (6.12)$$

The factor $\frac{1}{2}$ takes care of double counting in m and the factor $(-)^{j-m}$ is conventional. Our knowledge of spherical tensor tells us immediately that these quasi-spins are scalars under rotation. We can check this explicitly by verifying the $S_{+,o}$ commute with the angular momentum operators. Let us consider one such example. Let us show that,

$$[J_+ , S_-] = 0 \ . \qquad (6.13)$$

Using, $[J_+, a_m] = \sqrt{j(j+1)-m(m+1)}\, a_{m+1}$, we have,

$$[J_+, S_-] = \sum_m \Big[(-)^{j-m} \; j(j+1)-m(m+1) \; a_{m+1} a_{-m} - (-)^{j-m} \sqrt{j(j+1)-m(m+1)} \; a_m a_{-m+1} \Big]$$

$$= \sum_m \Big[(-)^{j-m} \sqrt{j(j+1)-m(m+1)} \; a_{m+1} a_{-m} + (-)^{j+m} \sqrt{j(j+1)-m(m+1)} \; a_{m+1} a_{-m} \Big]$$

In the second term we have changed the summation index from m to -m and used the anticommutation relation amongst the a's.

$$[J_+, S_-] = \sum_m (-)^{j-m} \sqrt{j(j+1)-m(m+1)} \; a_{m+1} a_{-m} (1+(-)^{2m})$$

$$= 0 \quad , \quad \text{for } (-)^{2m} = -1 \; .$$

S_- is therefore an operator which destroys a pair of particles in a state of total angular momentum zero, while S_+ similarly creates such a pair. This algebra defines two quantum numbers corresponding to the eigenvalues of S^2 and S_o. The states of this system can be classified into quasi-spin multiplets. We label the states as $|S, S_o, \alpha\rangle$ where α collectively represents all quantum numbers other than the quasi-spin ones, S and S_o. The operator S_o is given by,

$$S_o = \tfrac{1}{2}[S_+, S_-] = \frac{1}{4} \sum (a_m^+ a_m - a_{-m} a_{-m}^+) \qquad (6.14$$

$$= \frac{1}{4} \sum (a_m^+ a_m + a_{-m}^+ a_{-m} - 1)$$

From this the eigenvalue S_o is simply given by,

$$S_o = \frac{1}{4}(n+n - (2j+1)) = \frac{1}{2}(n-j-\tfrac{1}{2}) , \qquad (6.15)$$

so that, the total number of particles,

$$n = 2S_o + j + \tfrac{1}{2} . \qquad (6.16)$$

The particle number can vary between 0 and 2j + 1 so that S_o varies between the limits $\pm \frac{1}{4}(2j + 1)$. Since each value of S_o occurs only once in a quasi-spin multiplet, these multiplets consist of states each having a different number of particles. For a multiplet of a given total quasi-spin S, S_o varies from + S to -S and n varies from,

$$n_{min}(S_o = -S) = (j + \tfrac{1}{2} - 2S) \equiv v \qquad (6.17)$$

to

$$n_{max}(S_o = +S) = j+\tfrac{1}{2} + 2S = 2S + 1-v . \qquad (6.18)$$

Here, v is the minimum number of particles occuring in a given quasi-spin miltiplet and is called the seniority number. We can now label $|S, S_o, \alpha\rangle$ equivalently as $|n,v,\alpha\rangle$

Now we are in a position to build up these multiplets. We start with the state with the minimum number of particles i.e., the state $|n = v,v,\alpha\rangle$ for which $S_o = -S$. Hence,

$$S_- |v\ v\ \alpha\rangle = S_-|S, -S\ \alpha\rangle = 0 . \qquad (6.19)$$

The entire multiplet can now be built by repeated application of S_+ to this state. Each application of S_+ adds a

pair of particles coupled to total angular momentum zero. The state $|v\ v\ \alpha\rangle$ has no such pairs otherwise S_- would not give zero acting on this state. Thus, an arbitrary state contains v particles having no pairs coupled to angular momentum zero plus an arbitrary number $\frac{1}{2}(n - v)$ pairs coupled to angular momentum zero. The utility of this classification becomes obvious when we have to consider a pairing interaction. A simple example is the potential,

$$V = -\frac{1}{4} G \sum_{mm'} (-)^{2j-m-m'} a_m^+ a_{-m}^+ a_{-m'} a_{m'} \quad , \tag{6,20}$$

$$= -G\ S_+ S_- = -G(S^2 - S_0^2 + S_0) . \tag{6,21}$$

V is already diagonal in the quasi-spin or the seniority classification and the problem is trivially solved. In the seniority classification the eigenvalues of V are,

$$V = -\frac{1}{4} G\ (n - v)(2j + 3 - n - v) \quad . \tag{6,22}$$

If the pairing interaction represents the dominant term the seniority classification is clearly very useful. The situation is analogous to the utility of SU(3) classification when the quadrupole interaction is dominant.

We know the angular momentum behaviour of the operators a_m, a_m^+ . Let us now study their quasi-spin behaviour. These follow from the commutation relations for fermionic operators and the fact that m takes half odd integral values only. We find,

$$[S_+,\ a_m^+] = [S_-,\ a_m] = 0\ ,$$

$$[S_+,\ a_m] = (-)^{j-m} a_{-m}^+,\quad [S_-,\ a_m^+] = (-)^{j-m} a_{-m}, \tag{6.23}$$

$$[S_0,\ a_m^+] = \tfrac{1}{2} a_m^+\ ,\qquad [S_0,\ a_m] = -\tfrac{1}{2} a_m\ .$$

The content of these relations is clear. a_m^+ and $(-)^{j-m}a_{-m}$ behave like two-component spinors in quasi-spin space. Thus, bilinear products of these operators would yield a quasi-spin scalar (S) and a quasi-spin vector (V). These can be written down without any work by treating it like the problem of coupling two spin $\frac{1}{2}$ objects to yield a spin 1 and a spin 0 object. a_m^+ is like the spin-up state and $(-)^{j-m}a_{-m}$ is like the spin down state (applied to the vacuum state these are what they will generate in the quasi-spin space). We can thus write,

$$S(m,m') = \frac{1}{\sqrt{2}}\left(a_m^+(-)^{j-m'}a_{-m'} - (-)^{j-m}a_{-m}a_{m'}^+\right)\ ,$$

$$V_0(m,m') = \frac{1}{\sqrt{2}}\left(a_m^+(-)^{j-m'}a_{-m'} + (-)^{j-m}a_{-m}a_{m'}^+\right)\ , \tag{6.24a}$$

$$V_+(m,m') = a_m^+ a_{m'}^+\ ,$$

$$V_-(m,m') = (-)^{2j-m-m'}a_{-m}a_{-m'}\ .$$

That, this is a correct assertion can be verified by checking that,

$$\begin{aligned}
&[S_+, S(m,m')] = [S_-, S(m,m')] = [S_0, S(m,m')] = 0 \ , \\
&[S_+, V_+] = [S_-, V_-] = [S_0, V_0] = 0 \ , \\
&[S_0, V_\pm] = \pm V_\pm, \ [S_\pm, V_0] = \sqrt{2}\, V_\pm \ , \\
&[S_\pm, V_\mp] = \sqrt{2}\, V_0 \ .
\end{aligned} \tag{6.24b}$$

We shall dwell more on the implication of the commutation relations eqn.(6.24) in a later section. For the present note that any function of the scalar operators is diagonal in the quasi-spin classification. For $m = m'$, the vector vanishes.

We have already discussed the rotational behaviour of a_m and a_m^+. What about their bilinear products? The angular momentum theory prescribes how to construct irreducible tensors out of these bilinear products.

Thus,

$$\begin{aligned}
T_{kq} &= \langle jjmm' | kq \rangle \ a_m^+ a_{m'}^+ \\
T'_{kq} &= \langle jjmm' | kq \rangle \ (-)^{j-m'} a_m^+ a_{-m'} \ ,
\end{aligned} \tag{6.25}$$

are examples of two irreducible tensors of degree k we can so construct. These tensors are obviously combinations of quasi-spin scalars and vectors. These are simply two different ways of writing down the bilinear products of the single

particle fermionic operators, (eqns. (6.24a) and (6.25) are both invertible). The total number of independent tensor components would also be the same as the number of distinct bilinear products and this would be the same as the number of independent scalar and vector componets. It is like saying that if we have a system of two spin -½ objects we would need four basis states to describe it whatever basis we choose. Thus the bilinear products can be expanded in spherical tensors which can be expressed as combinations of quasi-spin scalars and vectors. We can be a little more specific about what to expect. The tensor T_{kq} contains only creation operators. Hence, it can be a component of a quasi-spin vector only. The Clebsch Gordon co-efficients obey the symmetry relation,

$$\langle jjmm'|kq\rangle = (-)^{2j-k} \langle jjm'm|kq\rangle \ . \qquad (6.26)$$

Hence,

$$\langle jjmm'|kq\rangle = \langle jjm'm|kq\rangle \ , \text{if } k \text{ is odd } (2j \text{ is odd})$$

and

$$\langle jjmm'|kq\rangle = -\langle jjm'm|kq\rangle \ , \text{ if } k \text{ is even.}$$

These, coupled with the relation $a^{+}_{m} a^{+}_{m'} + a^{+}_{m'} a^{+}_{m} = 0$, imply that T_{kq} for odd k vanishes identically. The same consideration tells us the T'_{kq} consists of paris of terms for given m and m' which are quasi-spin scalars if k is odd and vectors if k is even. Thus, all odd tensors are quasi-spin scalars. Hence, odd tensors are diagonal in the quasi-spin classification. An interaction consisting of odd tensors only is trivially solvable in the seniority classi-

fication. Thus, if we are dealing with pairing interactions or odd tensor interactions this classification is valuable. If we have to deal with multipole expansions i.e., expansions interms of $Y_{\ell m}$'s the contribution of the odd multipoles is transparent.

6.3 Symplectic Groups

The operators $S(m,m')$ constitute a Lie algebra. This is so, because they commute with the quasi-spin operators. Hence, their commutators also commute with the quasi-spin operators and so must be a linear combination of the members of this set. This algebra is included in SU_{2j+1} which includes all bilinear products of number conserving operators. The total number of independent operator. $S(m,m')$ is just $\frac{1}{2}(2j+1)(2j+2)$ since m,m' take $2j+1$ values each and $S(m,m')$ and $S(m',m)$ are not independent. It turns out that the associated group is the symplectic group in $2j+1$ dimensions whose basic properties can be easily analysed.

The operators $S(m,m')$ of eqn.(6.24) commute with the quasi-spin operators of eqn.(6.12). We can think of these as generating linear transformations on a_m, a_m^+ where a_m and a_m^+ can be regarded as vectors in a $(2j+1)$ dimensional space. These transformations leave some peculiar bilinear products of these vectors invariant. These are precisely the quasi-spin operators. We have thus strongly motivated the fact that one is discussing transformation in a space of $(2j+1)$

dimensions and that the group involved is not the rotation group in (2j+1) dimensions for that would have left the lengths of a and a^+ invariant. Let us now examine this peculiar invariant vector product. From eqn.(6.12) we see that the components of each vector are classified into pairs denoted by +m and -m and the product comprises terms in which the +m component of one vector is multiplied by the -m component of the second. This is also true for S_o since a_m annihilates angular momentum +m and thus transforms like a^+_{-m} . The product is antisymmetric under the interchange of two vectors or of the members of all the conjugate pair +m and -m. The general product of two vectors contains the combination $X_m Y_{-m} - X_{-m} Y_m$. The symplectic group is defined only in vector spaces of even dimensions and they leave invariant an antisymmetric product of two vectors requiring the classification of dimensions of the space into conjugate pairs.

Let us now go back to the example of jj coupling shell model. For half integral j, 2j+1 is even and the necessary requirement for defining a symplectic group is satisfied. How do the operators act on a given multi-particle state? They commute with $S_{\pm,0}$ and hence they affect neither n nor v. The seniority classification is preserved by our operators. They can only change the states of the v particles not coupled to angular momentum zero. They leave invariant a two particle state with angular momentum zero. This classification is thus complementary to the quasi-spin classification where the number of zero angular momentum pairs can be altered but not v.

Now the quasi-spin operators themselves have a Lie algebra for which the symplectic group S_{p_2} is the most natural assignment. We have noted that $(a_m^+ , (-)^{j-m} a_{-m})$ behave as quasi-spin spinors and the invariant object is S(m,m')-the typical antisymmetric invariant product we could associate with a S_{p_2} algebra.

The symplectic group in n-dimensions is denoted as S_{p_n}, S_{p_2} has the same algebra as that of SU(2). We again run into a Lie algebra of rank one which is the same as that of angular momentum. We would like to note that both neutrons and protons in the same j shell can be simultaneously treated by enlarging the group to S_{p_4}. We have 2(2j+1) single fermion states. The full algebra is that of the rotation group in 4(2j+1) dimensions with the number conserving bilinear products generating the U(2(2j+1)) algebra. Details can be obtained from reference 2.

6.4 The Full Lie Algebra of Boson Operators

Consider now the number changing bosonic bilinear products. First, consider a single oscillator with operators a and a^+ describing a single boson state. We define the convenient hermitian combinations as follows,

$$t_1 \equiv \frac{1}{4} (a^+a^+ + aa),$$

$$t_2 \equiv -\frac{1}{4}i (a^+a^+ - aa), \qquad (6.27)$$

$$t_3 \equiv \frac{1}{4} (a^+a + aa^+).$$

Experience tells us that a rank one algebra should result. However, the commutators show a non-trival sign difference compared to the angular momentum algebra. One finds,

$$[t_1,t_2] = -it_3, \quad [t_2,t_3] = it_1, \quad [t_3,t_1] = it_2 \qquad (6.28)$$

Introducing anti-hermitian operators $j_1 \equiv it$, and $j_2 \equiv it_2$, one gets the angular momentum algebra,

$$[j_1,j_2] = ij_3, [j_2,j_3] = ij_1, [j_3,j_1] = ij_2$$

where (6.29)

$$j_3 \equiv t_3 .$$

Thus, the results following from angular momentum commutation relations are still valid but those depending on the hermiticity of the operators would change. We get interesting differences. We define the "total angular momentum operator" i.e., the Casimir operator for this algebra as,

$$j^2 \equiv j_1^2 + j_2^2 + j_3^2 = t_3^2 - t_1^2 - t_2^2 . \qquad (6.30)$$

Our algebra provides two quantum numbers - the eigenvalues of j^2 and j_3, which we call j^2 , m. The eigenvalues of j^2 are less than those of j_3^2 in a given state. Like in the angular momentum case we define step operators,

$$j_{\pm} \equiv j_1 \pm ij_2 \text{ with } j_+^+ = - j_- . \qquad (6.31)$$

Clearly, the action of the step operators on jj_3 is going to be different. Let us work this out. Let,

$$j_{\pm}|jm\rangle = \Gamma_{\pm}|j,m\pm1\rangle,$$

Hence,

$$\langle jm|(j_{\pm})^{+} = \langle jm\pm1|\Gamma_{\pm}^{*} .$$

From this we find that,

$$\langle jm|(j_{\pm})^{+}(j_{\pm})|jm\rangle = |\Gamma_{\pm}|^2 ,$$

or

$$|\Gamma_{\pm}|^2 = - \langle jm | j_{\mp} j_{\pm}|j_m\rangle = - (j^2-m(m\pm1)).$$

Following, the angular momentum phase convention we have,

$$\Gamma_{\pm} = \sqrt{m(m\pm1)-j^2} , \text{ so that, } j_{\pm}|j^2 m\rangle = \sqrt{m(m\pm1)-j^2}\,|j^2,m\pm1\rangle .$$

Since j^2 is bounded by m^2 the sign reversal was to be expected. The j^2 under the square root stands for the eigenvalue of the operator j^2 . We follow the angular momentum argument with the roles of j^2 and j_3 reversed. Since $m^2 > j^2$ there must be a minimum value of m^2 for a given j^2 . The co-efficient under the square root must vanish at the end of the multiplet so as to prevent the creation of states beyond the end. We must have a m_{min} such that,

$$j^2 = |m_{min}|(|m_{min}|-1). \qquad (6.32)$$

It is convenient to define a number j such that the eigenvalues of j^2 are $j(j-1)$ with $|m_{min}| = j$. Hence, the allowed values of m run from $+j$ to $+\infty$ and $-j$ to $-\infty$ in unit steps.

No restriction on j is obtained (from the algebra) which can even continuous. However, the definition of t_3 pases restrictions. Its eigenvalue spectrum is simply $m = \frac{1}{4}(2n+1)$ with $n = 0,1,2.....$ So, the aloowed values of m are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, ...$ Incidently, this fixes the operator j^2 to be a mere complex number equal to $\frac{-3}{16}$. This can be checked directly from the definition of j^2 interms of a and a^+. Since, $j^2 = j(j-1) = \frac{-3}{16}$, we have $j = \frac{1}{4}$ or $\frac{3}{4}$. With $j = \frac{1}{4}$ one has $m = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}....$ and with $j = \frac{3}{4}$, $m = \frac{3}{4}, \frac{7}{4}, \frac{11}{4}$ .... These are the oscillator states with even and odd parity respectively, or even and odd numbers of quanta.

The generalization to the case of n oscillators is straightforward. This corresponds to a system of particles in an oscillator potential or to a multi-dimensional oscillator. We define the operators a_k and a_k^+ for the k^{th} oscillator, construct operators like t_1, t_2 and t_3 and sum over k. Introducing the symbols $T_i = \sum_k t_i^k$ $(i = 1,2,3....)$ we find,

$$[T_1,T_2] = iT_3, \quad [T_2,T_3] = iT_1 , \quad [T_3,T_1] = iT_2 \qquad (6.33)$$

and we define $J^2 \quad T_3^2 - T_1^2 - T_2^2$. Repitition of the procedure of the previous paragraph reveals that the eigenvalues of J^2 are of the form $J(J-1)$ where $J = \frac{n}{4}$ with $n = 0,1,2....$ For each J an infinite dimensional multiplet obtains as before starting at $J_3 = J$. A very interesting physical application is reviewed in Appendix B.

We now look into the group associated with the angular momentum algebra with one wrong sign. Experience with special relativity tells us that we are dealing with a Minkowisky type space which preserves the combination $x_3^2 - x_1^2 - x_2^2$.

No such physical space relevant to this problem (like the 2-space, 1-time dimension in relativity) suggests itself. So, we would try and look for a $V^2(c)$. We notice that (a_k^+, a_k) behave as the components of a vector in a two-dimensional space mixing with each other under the transformations generated by t_1, t_2, t_3. Morever, these operators preserve the antisymmetric product of two vectors $a_k^+ a_{k'}^+ - a_k a_{k'}^+$. Thus, the natural choice is S_{p2}. The transformations are symplectic transformations rather than rotations or Lorentz transformations. It is not the S_{p2} group of the seniority scheme. We need not forget the wrong sign in one of the commutators. Finally, the multiplets all contain an infinite number of states. We say, we have a non-compact group. The reason for infinite dimensional multiplets is transparent. With our two bason creation operators we can act on a given state as often as we please. For bosons there is no principle to terminate this process. For fermions the procedure of adding pair of particles must of neccessity come to an end, if there are only a finite number of states available. For the number conserving operators, we have only a finite number of states for the many

particle system and the multiplets must be finite whichever be the case.

The full algebra of all bosonic bilinear products can now be classified. The total number of bilinear products without the Pauli principle is $n(2n+1)$. Either the rotation group in $(2n+1)$ dimension or the symplectic group $S_{p_{2n}}$ is suggested. The $n = 1$ case indicates that $S_{p_{2n}}$ is the correct assignment. Finally, we can classify the Lie algebra of all bilinear products as shown in the tabe below.

Statistics	Number of particles	Number of bilinear products	Lie algebra	Compact
fermions	unchanged	n^2	U_n	yes
fermions	changed	$n(2n-1)$	R_{2n}	yes
bosons	unchanged	n^2	U_n	yes
bosons	changed	$n(2n+1)$	$S_{p_{2n}}$	no

This concludes the present programme of studies. However, we are deeply aware of one classic omission. This is the Wigner Supermultiplet Theory[14] of light nuclei based on the group SU(4). Limitation of space here has been the principal factor behind this. Even then, a few words are in order. This theory considered the equivalence of the four spin and isospin states of the nucleon which can be

denoted by $p\uparrow$, $p\downarrow$, $n\uparrow$, $n\downarrow$. An algebra of SU(4) will be obtained in the, by now, routine manner. The four states can be divided into pairs in a variety of ways, each with some physical significance. However, no combination of a triplet and singlet is suggested. So, no SU(3) subgroup would be physically relevant. The most natural SU(2)xSU(2) subgroup for this case is the direct product of spin and isospin. The four basis states are a spin and an isospin doublet and constitute a (2,2) multiplet of SU(2)xSU(2). If nuclear interactions are spin and isospin independent then this classification would be of physical importance. However, the SU(4) breaking part of the nuclear force is a good fraction of the SU(4) conserving part. That is why this theory never became popular. But, the successes of SU(3) have told us that even in the presence of large symmetry breaking pieces, a symmetry can give meaningful results. The success of the SU(3) mass formulae can hardly be ignored. We would not go into further details of the Wigner scheme. Incidently, SU(4) has acquired a new relevance, since the discovery of the charmed quark. There are other symmetries like SU(6) and other aspects of symmetries to which we could aslo do no justice at all. But, we have learnt so often (to our advantage) in this modest compilation that the operations have to terminate somewhere, their overaction only produces zeros!

REFERENCES

1. Schwinger, J., "On Angular Momentum", AEC Report NYO-3071 (1952).

2. Lipkin, H.J., "Lie Groups for Pedestrians", |North Holland Publishing Co.|, (1966). We have used this reference very extensively in our work.

3. Wigner, E.P., "Group Theory and its Application to the Quantum Mechanics of Atomic Spectra", |Academic Press N.Y.|, (1959).

4. Hammermesh, M., "Group Theory and its Application to Physical Problems", |Addison-Wesley, Reading, Mass|, (1962).

5. Shankar, R., "Principles of Quantum Mechanics", |Plenum Press, New York and London|, (1981).

6. Rose, M.E., "Elementary Theory of Angular Momentum", |John Wiley, New York|, (1957).

7. Gell-Mann, M., "The Eighfold Way: A Theory of Strong Interaction Symmetry", Cal-tech. Synchrotron Laboratory Report CTSL-20 (1961), (unpublished). See also, M. Gell-Mann and Y.Ne'eman, "The Eighfold Way", |W.A. Benjamin, Inc., New York and Amsterdam|, (1964).

8. Barnes et al, V.E., "Observation of a Hyperon with Strangeness minus three", PRL 12, 204, (1964).

9. Swart de, J.J., "The Octet Model and its Clebsch-Gordon Co-efficients", Rev. Mod. Phys. 35, 916, (1963).

10. Close, F.E., "An Introduction to Quarks and Partons", |Academic Press, London, N.Y., San Fransisco|, (1979). This is a very useful reference for SU(N) symmetries and quark models, specially for the parts which could not be incorporated in our work.

11. Okubo, S., "Some Consequences of Unitary Symmetry Model", Phys. Lett., 4, 14, (1963).

12. Carruthers, P., "Introduction to Unitary Symmetry", (Interscience Publishers), (1966). This a very useful reference for both SU(2) and SU(3) and their applications. Another useful reference, which we could not obtain is; S. Okubo, "Lectures on Unitary Symmetry", (Rochester University Preprint (unpublished).

13. Elliot, J.P., "Collective Motion in the Nuclear Shell Model, 1", Proc. Roy. Soc. (London) A245, 128, (1958). (See also Ref.2).

14. Wigner, E.P., "On the Consequences of the Symmetry of the Nuclear Hamiltonian on the Spectroscopy of Nuclei", Phys. Rev.51, 106, (1937). (See also Ref.2).

15. Goshen, S., (Goldstein) and Lipkin, H.J., "A Simple Independent-Particle System Having Collective Properties", Ann. Phys. (N.Y.), 6, 301, (1959). For additional examples, see Refs. (1-5) in this article.

Appendix A

The SU(2) Subgroups of SU(3) and the Physical Application of Unitary Symmetry

We do not wish to go into the C.G. machinery of SU(3). We now show that it is not necessary. For this we look at the isospin-like subgroups of SU(3) and first convince ourselves (in the Sakata model) that the simultaneous invariance under two of these implies full unitary symmetry.

In the Sakata model the equivalence of p, n and λ gives us SU(3) symmetry. The equivalence of p and n on the other hand implies SU(2) symmetry. Let us now examine the equivalence of n and λ ignoring the proton, just as we ignored λ in examining isospin. It is obvious that the algebra of the number conserving bilinear products again leads to an SU(2) algebra analogous to isospin. This algebra mixes n and λ states. Thus it can be characterized by an "equivalent angular momentum" similar to isospin. We call it U-spin. Clearly U-spin can be treated like angular momentum, e.g., we can classify states into U-spin multiplets and examine the consequences of U-spin conservation. Simultaneous holding of isospin and U-spin symmetries implies equivalence of p, n and λ in this model and hence full SU(3) symmetry. Thus all consequences of unitary symmetry follow from the simultaneous consideration of isospin and U-spin conservation which need just the angular momentum machinery.

Next let us look at the U-spin classification. The $n \leftrightarrow \lambda$ transformations change strangeness and conserve charge. The multiplet members have the same charge but variable strangeness. From eqn (4.8) the relevant operators are B_- and C_+ causing motion along a line at 120^0 to the horizontal isospin operators. Both in the Sakata and octet models this direction(120^0 to the horizontal) corresponds to lines of constant charge. We now write down the U-spin algebra without reference to any model. We want B_- and C_+ as generators. Since,

$$[B_-,\ C_+] = \tfrac{1}{2}(3N-2T_0), \tag{A.1}$$

we define U-spin operators as,

$$U_+ \equiv B_-,\ U_- \equiv C_+,\ U_0 \equiv \tfrac{1}{4}(3N-2T_0). \tag{A.2}$$

Thus
$$[U_+,U_-] = 2U_0,\ [U_0,U_+] = \pm U_\pm. \tag{A.3}$$

Also,
$$U^2 \equiv \tfrac{1}{2}(U_+U_- + U_-U_+) + U_0^2, \tag{A.4}$$

and
$$[U^2,\ U_{\pm,0}] = 0. \tag{A.5}$$

Since B_- and C_+ move one state into another along a line of constant $\frac{1}{2}N + T_0$ we define

$$Q = \tfrac{1}{2}N + T_0, \tag{A.6a}$$

and note that
$$[U_\pm,\ Q] = [U_0,Q] = 0. \tag{A.6b}$$

The operators $V_+ \equiv C_-$, $V_- \equiv B_+$ and $V_0 \equiv -\frac{1}{4}(3N+2T_0)$ define another SU(2) group called V-spin. This is physically not very interesting. Q for U-spin is the analogue of N for isospin. It causes motion in a direction perpendicular to that indicated by U-spin operators.

In the octet model $N = Y$ and Q is just the charge operator. In the Sakata model $N = Y - \frac{2}{3}B$, so that Q differs from the charge by a function of the baryon number which is fixed for a multiplet. Thus in both models same Q implies same charge and U-spin transformations are charge conserving transformations. Thus U-spin multiplets contain states of equal charge and variable strangeness. We can plot U-spin multiplets on a U_0-Q plot which is nothing but a 120^0 rotation of the previous T_0-N plots. We know that a 120^0 rotation leaves the shape unchanged. Thus the new diagrams have the same shape as before except that states along a horizontal line form U-spin multiplets and not isospin multiplets.

For SU(3) multiplets like (1,0), (0,1), (3,0) and (0,3). There is no difficulty in the transformation to U-spin for there is only one state to each point in the diagram. Just as each point has definite T and T_0 so it has definite U and U_0. But in multiplets like (1,1) or (2,2) there are certain values of N and T_0 (or Q and U_0) where more than one state occur. We are familiar with two states at the origin in the (1,1) case, One has $T = 1$ and the other has $T = 0$. Similarly in the U-spin classification one must belong to $U = 1$ triplet and the other to a U-spin singlet. However, the eigenstates of T^2 and U^2 are not the same, for these operators do not commute with one another. When there is only one state for given N and T_0, this state must be a simultaneous eigenfunction of T^2 and U^2 since both commute with N and T_0. If there is only one state with given T_0 and N

then the operators T^2 and U^2 cannot mix in other states. But if there is more than one state for given N and T_o then T^2 and U^2 do not have the same eigenfunctions.

Let us consider the two states at the centre of the baryon octet and determine the linear combinations of λ and Σ^o which belong to the singlet and triplet states.

Let $$|U = 1,\ U_o = 0\rangle = \alpha|\Sigma^o\rangle + \beta|\lambda\rangle . \qquad (A.7)$$

Now the neutron is a member of the same U-spin triplet with $U_o = +1$, Hence,

$$U_-|n\rangle = \sqrt{2}[\alpha|\Sigma^o\rangle + \beta|\lambda\rangle] \qquad (A.8)$$

and $$T_+U_-|n\rangle = 2\alpha|\Sigma^+\rangle . \qquad (A.9)$$

But $$\text{L.H.S.} = U_-T_+|n\rangle = U_-|p\rangle = |\Sigma^+\rangle \qquad (A.10)$$

so that $$\alpha = \tfrac{1}{2} . \qquad (A.11)$$

From normalization, $$\beta = \pm\frac{\sqrt{3}}{2} . \qquad (A.12)$$

We take β to be positive. Thus,

$$|U=1,\ U_o=0\rangle = \tfrac{1}{2}[|\Sigma^o\rangle + \sqrt{3}|\Lambda\rangle] . \qquad (A.13)$$

Apart from a phase factor the singlet is obtained by the orthogonality requirement to be,

$$|U=0,\ U_o=0\rangle = \tfrac{1}{2}[\sqrt{3}|\Sigma^o\rangle - |\lambda\rangle] . \qquad (A.14)$$

Incidently we have proved explicitly that two states are needed at the centre. If there were only one state it would be a triplet in both isospin and U-spin and eqns. (A.9) and (A.10) would be inconsistent.

We now present a few selected examples to illustrate the use of isospin and U-spin to get predictions of SU(3) symmetry. Consider the decay of a baryon resonance in the (3.0) decuplet $(N^*, Y^*, \Xi^*, \Omega^-)$ into a $\frac{1}{2}^+$ baryon and 0^- meson, the products being both in octets. Assume the interaction to be SU(3) symmetric. We will consider all decays conserving isospin and strangeness without regard to physical masses of the particles. We thus include decays such as $N^* \to \Sigma K$ disallowed by energy conservation (these may still be physically relevant). Since mass differences are not considered, corrections such as due to phase space should be separately included. To be definite consider the decays of $(N^{*-}, Y^{*-}, \Xi^{*-}, \Omega^-)$ into a negative baryon and a neutral meson $(K^0, \pi^0, \eta, \bar{K}^0)$. The above resonances form a $U = \frac{3}{2}$ multiplets, (Σ^-, Ξ^-) form a $U=\frac{1}{2}$ doublet and the 0^- neutral mesons form a $U=1$ triplet and a $U=0$ singlet. The U-spin singlet cannot contribute to the decays for we cannot get $U=\frac{3}{2}$ by coupling $U=0$ and $U=\frac{1}{2}$. Thus only the particular linear combination of π^0 and η which has $U = 1$ contributes. Wigner Eckart theorem tells us that all amplitudes can be expressed in terms of just one parameter A, the co-efficient being those of the coupling of spin $\frac{1}{2}$ to spin 1 and giving spin $\frac{3}{2}$. The results are:

$$
\begin{aligned}
&\langle N^{*-} | \Sigma^- K^0 \rangle = \langle \tfrac{1}{2} 1 \tfrac{1}{2} 1 | \tfrac{3}{2}\ \tfrac{3}{2} \rangle\ A, \\
&\langle Y^{*-} | \Xi^- K^0 \rangle = \langle \tfrac{1}{2} 1 \tfrac{1}{2} 1 | \tfrac{3}{2}\ \tfrac{1}{2} \rangle\ A, \\
&\langle Y^{*-} | \Sigma^- \pi^0 \rangle = \tfrac{1}{2} \langle \tfrac{1}{2} 1 \tfrac{1}{2} 0 | \tfrac{3}{2}\ \tfrac{1}{2} \rangle\ A, \\
&\langle Y^{*-} | \Sigma^- \eta \rangle = \sqrt{\tfrac{3}{2}} \langle \tfrac{1}{2} 1 \tfrac{1}{2} 0 | \tfrac{3}{2}\ \tfrac{1}{2} \rangle\ A, \\
&\langle \Xi^{*-} | \Sigma^- \bar{K}^0 \rangle = \langle \tfrac{1}{2} 1 \tfrac{1}{2} -1 | \tfrac{3}{2} -\tfrac{1}{2} \rangle\ A, \\
&\langle \Xi^{*-} | \Xi^- \pi^0 \rangle = \tfrac{1}{2} \langle \tfrac{1}{2} 1 -\tfrac{1}{2} 0 | 3/2\ -\tfrac{1}{2} \rangle A, \\
&\langle \Xi^{*-} | \Xi^- \eta \rangle = \sqrt{\tfrac{3}{2}} \langle \tfrac{1}{2} 1 -\tfrac{1}{2} 0 | \tfrac{3}{2} -\tfrac{1}{2} \rangle A, \\
&\langle \Omega^- | \Xi^- \bar{K} \rangle = \langle \tfrac{1}{2} 1 -\tfrac{1}{2} -1 | \tfrac{3}{2} -\tfrac{3}{2} \rangle A.
\end{aligned}
\qquad (A.15)
$$

We can similarly write down the eight amplitudes when the O^- is negative and the baryon neutral. Here again all processes involve one parameter B. Here, the neutral baryon has U=1 and the mesons have U=$\frac{1}{2}$. An illustrative example is,

$$< \Omega^- | \bar{K} \Xi^0 > = < \tfrac{1}{2} 1 -\tfrac{1}{2} -1 | \tfrac{3}{2} - \tfrac{3}{2} > B . \qquad (A.16)$$

Now A and B are not independent but are related by isospin considerations. Ω^- has T=0 and goes into opposite members of two isospin doublets. $|K^- \Xi^0>$ and $|K^0 \Xi^->$. The combination $|K^- \Xi^0> + |\bar{K}^0 \Xi^->$ has T=1. Hence $<\Omega^- | K^- \Xi^0 + \bar{K}^0 \Xi^- > = 0$. This implies

$$A = -B.$$

Thus the sixteen processes are expressed in terms of a single parameter. We next come to symmetry breaking interactions. SU(3) is broken by at least two interactions.

1. Electromagnetic interaction,
2. Strong interaction which produces mass splitting between members of a multiplet.

If we are given the transformation properties of these breaking terms, several predictions can be made. We can conveniently employ the isospin and U-spin behaviour of these terms so as to get away with simple angular momentum calculations only. Strong interactions conserve isospin. Thus the mass breaking SU(3) term must be an isoscalar. The charge operator commutes with all U-spin operators and hence the electromagnetic interaction which transforms like the charge operator must be a U-spin

scalar. Thus electromagnetic interaction conserves U-spin to all orders and the photon may be regarded as a U=0 object. The electromagnetic interaction is a sum of an isoscalar and an isovector as we have noted before. The mass splitting operator is an isoscalar and hence not a U-spin scalar (otherwise it will be a SU(3) scalar which cannot be. The mass splitting interaction is assumed to be a U=0 plus a U=1 object. The electromagnetic interaction being like the charge operator is a member of the octet. The mass splitting operator transforms like the hypercharge and hence is also a member of the octet. We can now use the octet co-efficients to write down the following isospin and U-spin behaviour of the electromagnetic interaction (E) and the mass splitting operator (M),

$$E \equiv U_S = \sqrt{\frac{3}{2}}\, T_V + \tfrac{1}{2}\, T_S \quad , \tag{A.17a}$$

$$(U_V = \tfrac{1}{2}\, T_V + \sqrt{\frac{3}{2}}\, T_S) \quad , \tag{A.17b}$$

$$M \equiv T_S' = \sqrt{\frac{3}{2}}\, U_V' - \tfrac{1}{2}\, U_S' \quad , \tag{A.18a}$$

$$(T_V' = \tfrac{1}{2}\, U_V' + \sqrt{\frac{3}{2}}\, U_S') \quad . \tag{A.18b}$$

We wrote down these relations directly from those for the particular linear combinations of isoscalar λ and isovector Σ^0 which are a U-scalar and a U-vector.

Consider now an example of a relation following from the U-spin property of E. Let us look at the electromagneic decays of η and π^0. Since U=0 for photons, only a U=0 state can decay into two photons. The linear combination $|\tfrac{1}{2}\pi^0 + \sqrt{\frac{3}{2}}\,\eta\rangle \equiv |U=1\rangle$.

$$\langle \lambda | E | \lambda \rangle = -\sqrt{\frac{3}{4}} \langle \lambda | U_V | \lambda \rangle + \frac{1}{4} \langle \lambda | U_S | \lambda \rangle \ , \qquad (A.25a)$$

$$\langle \Sigma^o | E | \Sigma^o \rangle = -\sqrt{\frac{3}{4}} \langle \Sigma^o | U_V | \Sigma^o \rangle + \frac{1}{4} \langle \Sigma^o | U_S | \Sigma^o \rangle . \qquad (A.25b)$$

Since $E \equiv U_S$ we find,

$$\langle \lambda | E | \lambda \rangle = -\sqrt{\frac{3}{3}} \langle \lambda | U_V | \lambda \rangle \qquad (A.26a)$$

and

$$\langle \Sigma^o | E | \Sigma^o \rangle = -\sqrt{\frac{3}{3}} \langle \Sigma^o | U_V | \Sigma^o \rangle . \qquad (A.26b)$$

But,

$$|\lambda\rangle = \sqrt{\frac{3}{2}} |U=1|\rangle - \frac{1}{2} |U=0\rangle \ , \qquad (A.27a)$$

$$|\Sigma^o\rangle = \frac{1}{2} |U=1\rangle + \sqrt{\frac{3}{2}} |U=0\rangle \ , \qquad (A.27b)$$

and $$\langle U=1 | U_o=0 | U_V | U=1, \ U_o=0 \rangle = \langle U=0 | U_V | U=0 \rangle = 0 . \qquad (A.28)$$

Thus, for example, we find

$$\mu_\lambda = -\mu_{\Sigma^o} . \qquad (A.30)$$

From eqns (A.27a,b) we also find that,

$$\langle \lambda | E | \lambda \rangle = \frac{3}{4} \langle U=1 | U_S | U=1 \rangle + \frac{1}{4} \langle U=0 | U_S | U=0 \rangle . \qquad (A.31a)$$

and $$\langle \Sigma^o | E | \Sigma^o \rangle = \frac{1}{4} \langle U=1 | U_S | U=1 \rangle + \frac{3}{4} \langle U=0 | U_S | U=0 \rangle . \qquad (A.31b)$$

Whence,

$$\langle U=1 | U_S | U=1 \rangle = -\langle U=0 | U_S | U=0 \rangle = 2 \langle \lambda | E | \lambda \rangle . \qquad (A.32)$$

Now, the neutron belongs to the U-spin triplet to which the linear combination of λ and Σ^o belongs. Thus we immediately have,

$$\langle n | E | n \rangle = 2 \langle \lambda | E | \lambda \rangle . \qquad (A.33)$$

and in particular, $$\mu_n = 2\mu_\lambda . \qquad (A.34)$$

Finally, we would like to end up with the remarkable success of SU(3) theory in predicting the mass splittings within SU(3) multiplets. First order degenerate perturbation theory is used giving

a large effect with no explanation why higher order effects should be small. Over the years this has remained a mystery. Further, for bosons the formula works if squares of the masses are used. No real explanation exists even for this. Anyhow, since the formulae work remarkably, let us see at least how they are derived.

i) The masses of $\frac{1}{2}^*$ baryon resonance:

Consider the U-spin quarlet Ω^-, Ξ^{*-}, Y^{*-}, N^{*-}, with $U_o = \frac{3}{2}$, $\frac{1}{2}$,$-\frac{1}{2}$ and $-\frac{3}{2}$ respectively. Let us calculate the expectation value of M in this U-spin multiplet. We immediately find the formula $m = a+b\,U_o$. We have four equispaced energy levels. It is this formula which predicted $m_{\Omega^-} \simeq 1680$ Mev.

ii) The baryon octet:

Consider the neutral members of the U-spin triplet, $(\Xi^o,\ \frac{1}{2}(\Sigma^o+\sqrt{3}\lambda),\ n)$, which are states with U=1 and U_o=-1, 0 and + 1, respectively. Again U_s gives a constant term S to the mass. The vector part contributes a term proportional to U_o. We thus find

$$\langle \Xi^o | M | \Xi^o \rangle = S-V \qquad (A.35a)$$

$$\langle n | M | n \rangle = S+V \qquad (A.35b)$$

$$\langle -\tfrac{1}{2}\Sigma^o+\sqrt{\tfrac{3}{2}}\lambda | M | -\tfrac{1}{2}\Sigma^o+\sqrt{\tfrac{3}{2}}\lambda \rangle = S \qquad (A.35c)$$

But M conserves isospin, hence it cannot connect a T= 0 to a T = 1 state. Thus eqn.(A.35b) becomes,

$$\frac{1}{4} < \Sigma^0 |M| \Sigma^0 > + \frac{3}{4} < \lambda |M| \lambda > = S \ , \qquad (A.36)$$

From these equations we easily find the relation ,

$$m_n + m_{\Xi^0} = \frac{m_{\Sigma^0} + 3m_\lambda}{2} \ . \qquad (A.37)$$

The SU(3) couplings are the same for any octet. The results for the 0^- octet can be read off directly from eqn (A.37). However, the formula works only if mass squares are used instead of the mass itself. The formula is,

$$m_k^2 = \frac{1}{4} \ (m_\pi^2 + 3m_\eta^2) \ . \qquad (A.38)$$

Appendix B

We report here, on the work of Goshen and Lipkin.[15] In this work symmetries are used to explain the collective behaviour of independent particle systems via a model calculation. The underlying group structure is responsible for this behaviour. In this work, a simple classical picture is presented showing the equivalence of the two descriptions. Then, collective coordinates are defined explicitly and the independent particle Hamiltonian is transformed into the standard format of the collective model. The collective states are classified using a naturally occuring group structure.

Consider a system of particles in a one-dimensional oscillator well, described by the hamiltonian,

$$H = \sum_i \left(\frac{p_i^2}{2m} + \frac{m\omega^2 x_i^2}{2}\right) . \qquad (B.1)$$

Classically it is easy to see how collective motion can arise in this independent particle system. At t = 0, let all particles be at the origin and be released with arbitrary initial velocities. Each particle excutes a simple harmonic motion of the same frequency. All particles move in phase. Hence, their motions are strongly correlated. An observer will see them leave the origin together and return to the origin together. He will see a "collective dilational oscillation" of the particles pulsating with twice the oscillator-frequency.

Thus the same motion is describable as independent particle motion and collective motion. The reason behind this is the

special symmetry of the oscillator potential in which all possible motions have the same frequency (energy levels are equally spaced). Any initial correlation persists for ever. This leads us to expect that the existence of symmetries in the independent particle Hamiltonian allows a correlated motion of the particles to be created by a proper choice of initial conditions. In quantum mechanics this implies choosing an appropriate linear combination of degenerate states. Let us try to be more quantitative now. We should be able to separate H into a collective part described by collective canonical variables (Q, P) and another part describing other degrees of freedom (i.e., in the collective model form),

$$H = H_{coll} + H', \tag{B.2}$$

with,

$$[H', P] = [H', Q] = 0 . \tag{B.3}$$

We look for an object describing the "degree of dilation" of the system. The deviation from the mean of the one-dimensional quadrupole moment $\Sigma\chi_i^2 - \langle\Sigma\chi_i^2\rangle$ is the only natural condidate. Since the mean kinetic energy equals the mean potential energy and their sum is always the total energy we can reduce the above form to a more useful form given by,

$$T_1 = \sum_i (p_i^2 - m^2\omega^2\chi_i^2)/4m\hbar\omega . \tag{B.4}$$

introducing also, the operators,

$$T_2 = \sum_i (x_i p_i + p_i x_i)/4\hbar \tag{B.5}$$

and

$$T_3 = \sum_i (p_i^2 + m^2\omega^2 x_i^2)/4m\hbar\omega = \frac{H}{2\hbar\omega} . \tag{B.6}$$

These are just the operators T_1, T_2, T_3 of Chapter 6. Hence, we find,

$$[T_1, T_2] = -iT_3, [T_2, T_3] = iT_1 \text{ and } [T_3, T_1] = iT_2. \tag{B.7}$$

Or, $$[J_1, J_2] = iJ_3, [J_2, J_3] = iJ_1 \text{ and } [J_3, J_1] = iJ_2 , \tag{B.8}$$

where, $$J_1 = iT_1, \; J_2 = iT_2 \text{ and } J_3 = T_3. \tag{B.9}$$

We have learnt all about this algebra. We have a Casimir operator J defined through,

$$J(J-1) \equiv J_1^2 + J_2^2 + J_3^2 . \tag{B.10}$$

The eigenvalues M of J_3 are integers, half integers and quarter integers and so are those of J. For a given J we have an infinite set of states having M = J, J+1, ... with energies,

$$E_{JM} = 2M\hbar\omega . \tag{B.11}$$

The states of the system divide into "bands". Each J value gives one band and containing an infinite number of states with the energy spectrum running from $2J\hbar\omega$ to ∞ in steps of $2\hbar\omega$. This is precisely the spectrum we need for our "double frequency collective vibration". Now $P \pm i2\omega Q$ create or destroy one quantum which is the same as the change in M value of one unit brought about by the step operators $J_\pm$. Thus $P \pm i2\omega Q$ can be expressed as function of $J_\pm$ respectively as shown in ref. (15) and are explicitly canonical variables. Finally, one finds that,

$$P^2 + 4\omega^2 Q^2 = (2J_3 - 2J + 1)\hbar\omega , \tag{B.12}$$

so that, $$\tfrac{1}{2}(P^2 + 4\omega^2 Q^2) + (2J-1)\hbar\omega = 2J_3\hbar\omega = H. \tag{B.13}$$

Thus, H has the desired form with,

$$H' = (2J-1)\hbar$$

and $$[H', P] = [H', Q] = 0 \qquad \text{(B.15)}$$

Eqn (B.15) follows from the fact that J commutes with $J_{\pm}$, J_3 and with any functions of these and hence with P and Q.

Finally, let us digest carefully what has been suggested by this simple example. An independent particle model with some symmetry properties was considered. One aspect of this was the existence of a group of operators which can be used to classify the states. The Hamiltonian transforms simply under this group (like J_3). Thus the group quantum numbers still classify the states but the states of an irreducible representation are no longer degenerate.

The band of states (irreducible representation of a given J) are classified by the quantum number M. the eigenvalue of one of the degree of freedom of the system separable from the rest. The states within a band differ only in the motion of this particular degrees of freedom. Since the group operators connect states only within one band and do not connect different bands they may be considered dynamical variables associated with this degree of freedom. Hence, this degree of freedom can be considered collective, for the group operators operate on all particles in a symmetric way.

Generalizing, we might expect to find an alternative collective description in any independent particle model possessing symmetries which allows the states to be classified according to the irreducible representations of a group. The states within a given representation would constitute a band, the group quantum numbers would characterize collective degrees of freedom and the group operators would be collective dynamical variables. Proper sets of canonical collective variables can then be constructed. The Elliot model in Chapter 5 is another illustration of the phenomenon discussed here.