

ADDIS ABABA UNIVERSITY



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DEPARTMENT OF MATHEMATICS
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ON

**COMPARISON BETWEEN NEWTON-COTES AND GAUSSIAN
QUADRATURE METHODES**

(SUBMITTED IN THE PARTIAL FULLFILMENT OF THE MSc DEGREE IN MATHEMATICS)

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Preface

The increasing desire for numerical solutions to mathematical problems are more difficult or impossible to solve explicitly has become the present day scientific research. This time it sounds more appropriate to find an approximate solution to a more complicated model of a physical problem than to find an exact solution of a simplified model. It is clear that numerical methods can give approximate solutions in an efficient manner, when ordinary analytical methods fail, for example in finding the roots of transcendental equations or in solving non-linear differential equations one of such methods is solving definite integrals of some functions. Numerical evaluation of integrals to find numerical solutions provides the basis for studying various problems which include problems of heat transfer, fluid flows and many others. In this report a discussion about the numerical solutions of definite integrals which can not be evaluated analytically is made. Various recent methods for the numerical solutions of the definite integral are sketched. The methods have been well illustrated by suitable examples with detailed solutions.

I would like to say thanks to God for his immeasurable help. Finally, I would like to thank my advisor Dr. Adinew Alemayehu for suggesting me the topic and for his constant and supportive guidance which made this report fruitful.

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1.1 Introduction to numerical integration

Definition 1: An integrable function $w(x)$ defined on an interval $[a, b]$ such that for all $x \in [a, b]$ $w(x) \geq 0$ and $w(x) > 0$ on any subinterval of $[a, b]$ is called a weight function.

The purpose of a weight function is to assign varying degrees of importance to approximations on a certain portions of the interval. For example the weight function

$w(x) = \frac{1}{\sqrt{1-x^2}}$ places less emphasis near the center of the interval $(-1, 1)$ and more

emphasis when $|x|$ near 1.

The general problem of numerical integration is to find an approximate value of the integral

$$I(f) = \int_a^b w(x)f(x)dx \quad (1.11)$$

since some times it may be difficult to find the anti- derivative of functions analytically.

The limits of integration in (1.11) may be finite, semi finite or infinite and also since the integral operator is linear the integral is approximated by a finite linear combinations of values of $f(x)$ in the form

$$I(f) = \int_a^b w(x)f(x)dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.12)$$

where x_k 's $k=0,1,\dots,n$ are called the nodes distributed in $[a, b]$ and λ_k 's $k=0,1,2,\dots,n$ are called the weights of the integration or the quadrature formula (1.12). The error approximation is given by

$$R_n = \int_a^b w(x)f(x)dx - \sum_{k=0}^n \lambda_k f_k \quad (1.13)$$

where $f_k = f(x_k)$ and $k=0,1,2,\dots,n$.

Definition 2: An integration method of the form (1.12) is said to be of order p if it produces exact results ($R_n = 0$) for all polynomials of degree less than or equal to p .

In (1.12) if we have $2n+2$ unknowns $(n+1)$ nodes and $(n+1)$ weights and the method can be exact for polynomials of degree less or equal to $2n+1$. Thus the method of the form (1.12) can have a maximum of order $2n+1$.

1.2 Numerical integration (quadrature) based on Interpolation

Given the $n+1$ nodes and the corresponding functional values $f_k = f(x_k)$ where $k=0,1,\dots,n$ then the Lagrange interpolating polynomial fitting the data (x_k, f_k) is given by

$$p_n(x) = \sum_{k=0}^n L_k(x) f_k$$

where $L_k(x)$ is the Lagrange fundamental polynomial given by

$$L_k(x) = \frac{\prod_{n+1}(x)}{(x-x_k) \prod_{n+1}^{(1)}(x_k)},$$

$$\prod_{n+1}(x) = (x-x_0)(x-x_1)\dots(x-x_n)$$

and

$\prod_{n+1}^{(1)}(x_k)$ is the first derivative of $\prod_{n+1}(x)$ at x_k then

$$f(x) = \sum_{k=0}^n L_k(x) f(x_k) + \frac{\prod_{n+1}(x) f^{(n+1)}(\xi)}{(n+1)!}, x_0 < \xi < x_n \quad (1.21)$$

now replace $f(x)$ in equation (1.11) and integrate in the given limits of integration then we have that

$$\begin{aligned} I(f) &= \int_a^b w(x) f(x) dx = \sum_{k=0}^n \left[\int_a^b w(x) L_k(x) dx \right] f_k + \frac{\int_a^b w(x) \prod_{n+1}(x) f^{(n+1)}(\xi) dx}{(n+1)!} \\ &= \sum_{k=0}^n \lambda_k f_k + R_n \end{aligned} \quad (1.22)$$

Where

$$\lambda_k = \int_a^b w(x)L_k(x)dx \quad (1.23)$$

and

$$R_n = \frac{\int_a^b \prod_{n+1}(x)w(x)f^{(n+1)}(\xi)dx}{(n+1)!} \quad (1.24)$$

1.2.1 Determination of the error term

Now $I = \int_a^b w(x)f(x)dx = \sum_{k=0}^n \lambda_k f_k + R_n$ from equation (1.22) and from equation (1.24)

the error term is

$$R_n = \frac{1}{(n+1)!} \int_a^b \prod_{n+1}(x)w(x)f^{(n+1)}(\xi)dx \quad (1.25)$$

If $\prod_{n+1}(x)$ does not change sign in $[a, b]$ and $f^{(n+1)}(x)$ is continuous in $[a, b]$ then by the mean value theorem of integral calculus we can write the error of the approximation in the form

$$R_n = \frac{f^{(n+1)}(\zeta)}{(n+1)!} \int_a^b \prod_{n+1}(x)w(x)dx \quad \zeta \in [a, b] \quad (1.26)$$

If $\prod_{n+1}(x)$ change sign in $[a, b]$ then we can obtain $|R_n|$ from equation (1.25) as

$$|R_n| \leq \frac{1}{(n+1)!} \int_a^b \left| \prod_{n+1}(x)w(x) \right| |f^{(n+1)}(\xi)| dx \leq \frac{M_{n+1}}{(n+1)!} \int_a^b \left| \prod_{n+1}(x)w(x) \right| dx$$

$$\text{Where } M_{n+1} = \max_{a \leq x \leq b} [f^{(n+1)}(x)] .$$

Alternatively the error term can also be obtained in the following way since we can make the method is exact for polynomials of degree less than or equal to n and we have

$$R_n = 0 \text{ when } f(x) = x^i \text{ where } i=0,1,\dots,n \text{ and } R_n \neq 0 \text{ when } f(x) = x^{n+1} .$$

Thus the error term is in the form

$$R_n = \frac{C}{(n+1)!} f^{(n+1)}(\zeta) \quad (1.27)$$

where $\zeta \in [a, b]$ and

$$C = \int_a^b w(x)x^{n+1} dx - \sum_{k=0}^n \lambda_k x_k^{n+1} \quad (1.28)$$

is called the error constant.

If C is zero for $f(x) = x^{n+1}$ then we take the next term $f(x) = x^{n+2}$.

1.2.2 Newton-Cotes quadrature methods

When $w(x) = 1$ and given $(n+1)$ nodes which are equi-spaced with

$x_0 = a, x_n = b$, and $h = \frac{(b-a)}{n}$ on $[a, b]$ then the methods

$$\int_a^b f(x) dx = \sum_{k=0}^n \lambda_k f_k$$

are called the Newton-Cotes integration (Quadrature) methods where the error is neglecting and the weights λ_k, s are called Cote's numbers.

Now setting $x = x_0 + sh$ in (1.21) above we obtain that

$$\prod_{n+1}(x) = h^{(n+1)} s(s-1)\dots(s-n), L_k(x) = \frac{(-1)^{(n-k)} s(s-1)\dots(s-k+1)(s-k-1)\dots(s-n)}{k!(n-k)!}$$

$$\lambda_k = \frac{(-1)^{(n-k)} h}{k!(n-k)!} \int_0^n s(s-1)\dots(s-k+1)(s-k-1)\dots(s-n) ds \quad (1.29)$$

1.2.2.1 Trapezoidal rule

For $n=1$ in (1.29) we have $x_0 = a, x_1 = b$ and $h = b - a$

$\lambda_0 = -h \int_0^1 (s-1) ds = \frac{h}{2}$ $\lambda_1 = h \int_0^1 s ds = \frac{h}{2}$ and we get the method

$$\int_a^b f(x) dx = \frac{h}{2} [f(a) + f(b)] = \frac{(b-a)[f(a) + f(b)]}{2} \quad (1.30)$$

which is called the trapezoidal rule. Geometrically it is the area of the trapezoid with width $(b - a)$ and ordinates $f(a), f(b)$ which is the area under the curve $y = f(x)$ above the x axis and the ordinates $x = a$ and $x = b$.

Error term:

Now the error constant is given by

$$C = \int_a^b x^2 dx - \sum_{k=0}^1 \lambda_k x_k^2 = \frac{b^3}{3} - \frac{a^3}{3} - \lambda_0 x_0^2 - \lambda_1 x_1^2 = \frac{1}{3}(b^3 - a^3) - \frac{1}{2}(b-a)(b^2 + a^2) \\ = -\frac{1}{6}(b-a)^3.$$

Then the error term is given by

$$R_1 = \frac{C}{2!} f^{(2)}(\eta) \Rightarrow R_1 = -\frac{(b-a)^3}{12} f^{(2)}(\eta) = -\frac{h^3}{12} f^{(2)}(\eta)$$

Where $\eta \in [a, b]$.

1.2.2.2 Simpson's-rule

SIMPSON'S $\frac{1}{3}$ RULE

For $n=2$ in (1.29) we have $h = \frac{(b-a)}{2}$, $x_0 = a$, $x_1 = (\frac{a+b}{2})$ & $x_2 = b$ then we get

$$\lambda_0 = \frac{h}{2} \int_0^2 (s-1)(s-2) ds = \frac{h}{3}, \lambda_1 = -h \int_0^2 s(s-2) ds = \frac{4h}{3}, \lambda_2 = \frac{h}{2} \int_0^2 s(s-1) ds = \frac{h}{3} \quad \text{and}$$

we obtain the method

$$I(f) = \int_a^b f(x) dx = \frac{b-a}{6} [f(a) + 4f(\frac{a+b}{2}) + f(b)] \quad (1.31)$$

which is called the Simpson's $\frac{1}{3}$ rule.

SIMPSON'S $\frac{3}{8}$ RULE

For $n=3$ we obtain $\lambda_0 = \frac{3}{8}h$, $\lambda_1 = \frac{9}{8}h$, $\lambda_2 = \frac{9}{8}h$, $\lambda_3 = \frac{3}{8}h$ and the method is given by

$$\int_a^b f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + 3f(x_3)] \quad (1.32)$$

which is called Simpson's $\frac{3}{8}$ -rule.

Error term:

Determine the error term in Simpson's $\frac{1}{3}$ rule.

Solution: Given $n=2$ then since the methods are exact for polynomials of degree ≤ 2 we have

$$C = \int_a^b x^3 dx - \sum_{k=0}^2 \lambda_k x_k^3 = \frac{1}{4}(b-a)^4 - \frac{(b-a)}{4}[2a^3 + (a+b)^3 + 2b^3] = \frac{1}{4}(b^4 - a^4) - \frac{1}{4}(b-a)^4 = 0$$

this shows that the method is exact for polynomial of degree three also hence the error term becomes

$$R_2 = \frac{Cf^{(4)}(\zeta)}{4!}, \zeta \in [a, b]$$

where

$$C = \int_a^b x^4 dx - \frac{(b-a)}{6}[a^4 + 4(\frac{a+b}{2})^4 + b^4] = -\frac{(b-a)^5}{120}$$

hence the error term becomes

$$R_2 = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta) = -\frac{h^5}{90} f^{(4)}(\zeta).$$

1.2.2.3 Newton-Cotes closed type methods

The method

$$\int_a^b f(x)dx \approx \sum_{k=0}^n \lambda_k f_k$$

which dose include the end points $x_0 = a, x_n = b$ are called the Newton-Cotes closed methods

Examples: The trapezoidal and Simpson's rule given in (1.30) and in (1.31) are Newton-Cotes closed type methods.

1.2.2.4 Newton-Cotes open type methods

The methods

$$\int_a^b f(x)dx \approx \sum_{k=0}^n \lambda_k f_k$$

which do not include the end points $x_0 = a$ and $x_n = b$ are called the Newton-Cotes open type methods.

Examples:

(1).Mid-point rule

When $n=2$ then we have $x_0 = a, x_0 + h, x_0 + 2h = b$ then the method is given by

$$\int_a^b f(x)dx = 2hf(x_0 + h) + \frac{h^3}{3} f^{(2)}(\xi) \quad (1.33)$$

Where $a < \xi < b$

(2).Two-point rule

When $n=3$ $x_0 = a, x_0 + h, x_0 + 2h, x_0 + 3h = b$ then the method given by

$$\int_a^b f(x)dx = \frac{3h}{2}[f(x_0 + h) + f(x_0 + 2h)] + \frac{3h^3}{4} f^{(2)}(\xi) \quad (1.34)$$

where $a < \xi < b$.

(3).Three-point rule

When $n=4$ we have $x_0 = a, x_0 + h, x_0 + 2h, x_0 + 3h, x_0 + 4h = b$ then the method is given by

$$\int_a^b f(x)dx = \frac{4h}{3}[2f(x_0 + h) - f(x_0 + 2h) + 2f(x_0 + 3h)] + \frac{14h^5}{45} f^{(4)}(\xi) \quad (1.35)$$

where $a < \xi < b$.

Example 1: Find the approximate value of $I = \int_0^1 \frac{dx}{x+1}$ using:

(a).TRAPEZOIDAL RULE.

(b). SIMPSON'S RULE ($\frac{1}{3}$).

Solutions:

(a). Using Trapezoidal rule given $I = \int_a^b \frac{dx}{1+x}$ then $f(x) = \frac{1}{1+x}$, $w(x) = 1$ for all

$x \in [a, b]$ and hence $\lambda_0 = \frac{h}{2}$ and $\lambda_1 = \frac{h}{2}$ hence

$$I = \frac{h}{2}[f(a) + f(b)] = \frac{(b-a)}{2}[f(a) + f(b)]$$

where $a=0$ and $b=1$ then $I = \frac{1}{2}[f(0) + f(1)] = \frac{1}{2}[1 + \frac{1}{2}] = 0.75$.

(b). Using Simpson's $\frac{1}{3}$ rule we have

$$I = \frac{(b-a)}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] = \frac{1}{6}[1 + \frac{8}{3} + \frac{1}{2}] = 0.694444$$

now the exact value of $I = \ln 2 = 0.693147$ correct to six decimal places hence comparing (a) and (b) Simpson's rule is the best approximation method.

Example2: Find the approximate value of $I = \int_0^1 \frac{\sin(x)}{x} dx$ using

(a). MID-POINT RULE

(b). TWO-POINT OPEN TYPE RULE

Solution: (a). Using mid-point rule we have $h = \frac{(b-a)}{2} = \frac{1}{2}$ hence

$$x_0 = 0, x_1 = \frac{1}{2}, x_2 = \frac{2}{3} \text{ and}$$

$$\begin{aligned} I &= \int_a^b f(x) dx = \frac{3h}{2}[f(x_0+h) + f(x_0+2h)] \\ &= \frac{1}{2}[f(\frac{1}{3}) + f(\frac{2}{3})] = \frac{1}{2}[3\sin(\frac{1}{3}) + \frac{3}{2}\sin(\frac{2}{3})] = 0.9546 \end{aligned}$$

1.3 Numerical integration (quadrature) based on Undetermined Coefficients

Now consider the quadrature formula given by

$$\int_a^b w(x)f(x)dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.36)$$

then we can make the method exact for polynomials of degree less than or equal to n if all the nodes are given or less than or equal to $2n+1$ if all the nodes are not given.

If all the nodes are given and the weights are to be determined by making the methods exact for polynomials of degree less than or equal to n then the methods are called methods of undetermined coefficients that means all the nodes x_k 's are given and all the weights λ_k 's for $k = 0, 1, \dots, n$ are to be determined.

If all the nodes are not given and all the weights are to be determined by making the methods exact for polynomials of degree less than or equal to $2n+1$ then the methods are called methods of undetermined coefficients.

1.3.1 Determination of the error term

The error term can be obtained in the following way since we can make the method exact for polynomials of degree less than or equal to n if all the nodes are given and we have

$$R_n = 0 \text{ when } f(x) = x^i \text{ where } i=0, 1, \dots, n \text{ and } R_n \neq 0 \text{ when } f(x) = x^{n+1}.$$

Thus the error term is in the form

$$R_n = \frac{C}{(n+1)!} f^{(n+1)}(\zeta) \text{ where } \zeta \in [a, b] \quad (1.37)$$

and

$$C = \int_a^b w(x)x^{n+1} dx - \sum_{k=0}^n \lambda_k x_k^{n+1} \quad (1.38)$$

is called the error constant. If C is zero for $f(x) = x^{n+1}$ then we take the next term

$$f(x) = x^{n+2}.$$

If all the nodes are not given then by making the methods exact for polynomials of degree less than or equal to $2n+1$ and we have $R_n = 0$ when $f(x) = x^i$ where $i = 0, 1, \dots, 2n+1$ and $R_n \neq 0$

when $f(x) = x^{2n+2}$. Thus the error term is given by

$$R_{2n+2} = \frac{C}{(2n+2)!} f^{(2n+2)}(\xi) \text{ where } \xi \in [a, b] \text{ and } C = \int_a^b w(x)x^{2n+2} dx - \sum_{k=0}^n \lambda_k x_k^{2n+2}.$$

1.3.2 Newton-Cotes quadrature methods

Given the $n+1$ nodes which are equally spaced then the Newton-Cotes methods derived in the interpolation method can also be obtained using the approach of methods of undetermined coefficients.

The Newton-Cotes methods are given by

$$\int_a^b f(x) dx \approx \sum_{k=0}^n \lambda_k f_k$$

where all the nodes x_k 's are given and all the weights λ_k 's for $k=0, 1, \dots, n$ are to be determined by making the method exact for polynomials of degree less than or equal to n .

1.3.2.1 Trapezoidal rule

When $n=1$ $x_0 = a, x_1 = b, h = x_1 - x_0$ we have that

$$\int_a^b f(x) dx = \int_{x_0}^{x_1} f(x) dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$$

then making the method exact for polynomials of degree less than or equal to 1.

$$\text{For } f(x) = 1, \int_{x_0}^{x_1} 1 dx = x_1 - x_0 = h = \lambda_0 + \lambda_1 \dots (1)$$

$$\text{For } f(x) = x, \int_{x_0}^{x_1} x dx = \frac{x^2 - x_0^2}{2} = \frac{1}{2}(x_1 - x_0)(x_1 + x_0) = \lambda_0 x_0 + \lambda_1 x_1$$

$$\Rightarrow \frac{1}{2}h(2x_0 + h) = \lambda_0 x_0 + \lambda_1(x_0 + h) = (\lambda_0 + \lambda_1)x_0 + \lambda_1 h \quad (2)$$

then solving (1) and (2) we get $\lambda_0 = \lambda_1 = \frac{h}{2}$. Hence the method becomes

$$\int_{x_0}^{x_1} f(x) dx = \frac{h}{2}[f(x_0) + f(x_1)] = \frac{(b-a)}{2}[f(a) + f(b)]$$

which is called the trapezoidal rule.

Error term: Now the error constant is given by

$$C = \int_{x_0}^{x_1} x^2 dx - \sum_{k=0}^1 \lambda_k x_k^2 = \frac{x^3}{3} \Big|_{x_0}^{x_1} - \lambda_0 x_0^2 - \lambda_1 x_1^2 = \frac{1}{3}(x_1^3 - x_0^3) - \frac{h}{2}(x_0^2 + x_1^2)$$

$$= \frac{1}{6}[2(x_0^2 + 3x_0^2 h + 3x_0 h^2 + h^3) - 2x_0^2 - 3x_0^2 h - 3h(x_0^2 + 2x_0 h + h^2)] = -\frac{h^3}{6}$$

and the truncation error becomes

$$R_1 = \frac{C}{2!} f^{(2)}(\xi) = -\frac{h^3}{12} f^{(2)}(\xi) \text{ for } x_0 < \xi < x_1.$$

1.3.2.2 Simpson's rule

For $n=2$ we have $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h = b, h = \frac{b-a}{2}$ then we have

$$\int_{x_0}^{x_1} f(x) dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

then the rule can be made exact for polynomials of degree up to two.

For $f(x) = 1, x, x^2$ then we get the following systems of equations.

$$\text{For } f(x) = 1, \int_{x_0}^{x_2} 2 dx = x_2 - x_0 = \lambda_0 + \lambda_1 + \lambda_2 \quad (1)$$

$$\text{For } f(x) = x, \int_{x_0}^{x_2} x dx = \frac{(x_2^2 - x_0^2)}{2} = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 \quad (2)$$

$$\text{For } f(x) = x^2, \int_{x_0}^{x_2} x^2 dx = \frac{(x_2^3 - x_0^3)}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 \quad (3)$$

Then from equation (1) we get $2h = \lambda_0 + \lambda_1 + \lambda_2 \dots (1^*)$.

$$\frac{1}{2}(x_2 - x_0)(x_2 + x_0) = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2$$

From equation (2) we get $h(2x_0 + 2h) = 2hx_0 + (\lambda_1 + 2\lambda_2)h$.

$$2h = \lambda_1 + 2\lambda_2 \dots (2^*)$$

From equation (3) above we get that

$$\frac{1}{3}(x_0^3 + 6x_0^2h + 12x_0h^2 + 8h^3 - x_0^3) = \lambda_0 x_0^2 + \lambda_1(x_0^2 + 2x_0h + h^2) + \lambda_2(x_0^2 + 4x_0h + 4h^2)$$

$$\text{then, } \frac{8}{3}h = \lambda_1 + 4\lambda_2 \dots (3^*)$$

Then solving (1*), (2*) and (3*) simultaneously we get that $\lambda_0 = \frac{h}{3}, \lambda_1 = \frac{4h}{3}, \lambda_2 = \frac{h}{3}$

hence we get the method

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

which is called Simpson's $\frac{1}{3}$ rule.

Error term: Now from (1.2.2.2) given above we have that $C = -\frac{(b-a)^5}{120} = -\frac{4h^5}{5}$ and the

truncation error is given by

$$R_2 = \frac{C}{4!} f^{(4)}(\eta) = -\frac{h^5}{90} f^{(4)}(\eta) \text{ where } \eta \in [a, b].$$

Now the method of undetermined coefficients can be used to derive quadrature formulas of a given type.

Example: Determine a, b and c such that the formula

$$\int_a^b f(x)dx = h[af(0) + bf(\frac{h}{3}) + cf(h)]$$

is exact for polynomials of as high degree as possible and determine the order of the truncation error.

Solution: Now making the method exact for polynomials of degree up to 2 then we obtain

For $f(x) = 1$, $h = h(a + b + c)$ or $a + b + c = 1$ (1)

For $f(x) = x$, $\int_0^h x dx = \frac{h^2}{2} = h(\frac{bh}{3} + ch)$ or $\frac{1}{3}b + c = \frac{1}{2}$ (2)

For $f(x) = x^2$, $\int_0^h x^2 dx = \frac{h^3}{3} = h(\frac{bh^2}{9} + ch^2)$ or $\frac{1}{9}b + c = \frac{1}{3}$ (3)

Then solving equations (1), (2) and (3) we get that $a = 0, b = \frac{3}{4}$ and $c = \frac{1}{4}$ hence the quadrature method is given by

$$\int_a^b f(x)dx = \frac{3}{4}f(\frac{h}{3}) + \frac{1}{4}f(h)$$

and the truncation error is given by

$$R_2 = \frac{C}{3!} f^{(3)}(\eta) \text{ where } 0 < \eta < h$$

and

$$C = \int_a^b x^3 dx - h[\frac{bh^3}{27} + ch^3] = -\frac{h^4}{36}$$

Hence the error term is given by

$$R_2 = -\frac{h^4}{216} f^{(3)}(\eta).$$

1.3.2.3 Comparison of Newton-Cotes Quadrature by Interpolation and by undetermined coefficients

A convenient method for determining the coefficients in the Newton-Cotes formulas is the method of undetermined coefficients.

In Newton-Cotes formula by interpolation the weights are independent of the interval limits a and b and the function f in fact the weights only depend on the relative distribution of the support points on the interval $[a, b]$

Each Newton-Cotes quadrature formula by interpolation and undetermined coefficients

$\sum_{k=0}^n \lambda_k f_k$ satisfies $\sum_{k=0}^n \lambda_k = b - a$ since its degree of exactness is at least $n \geq 0$ and hence

$$\sum_{k=0}^n \lambda_k = \int_a^b dx = b - a \text{ for } f(x) = 1.$$

Newton-cotes quadrature formulas by interpolation yield good approximations to the integral in question only over short interval $[a, b]$ that is $b - a \ll 1$. Hence for large n Newton-Cotes quadrature formulas by undetermined coefficients is better.

1.3.3 Gaussian quadrature (integration) methods

In the integration method

$$\int_a^b w(x) f(x) dx \approx \sum_{k=0}^n \lambda_k f_k$$

the nodes x_k 's and the weights λ_k 's for $k=0, 1, \dots, n$ can also be obtained by making the method exact for polynomials of degree up to m . When the nodes are known that is $m=n$ the corresponding methods are called Newton-cotes quadrature methods

with $w(x) = 1 \forall x \in [a, b]$.

When the nodes are also to be determined we have $m=2n+1$ and such a quadrature methods are called Gaussian quadrature (integration) methods

Now since any finite interval $[a, b]$ can always be transformed to $[-1, 1]$ using the transformation

$$x = \frac{(b-a)t}{2} + \frac{(b+a)}{2}$$

where $x \in [a, b], t \in [-1, 1]$ then we have

$$\int_{-1}^1 w(x)f(x)dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.39)$$

where $w(x) \geq 0 \forall x \in [-1, 1]$ is the weight function and $f_k = f(x_k)$.

1.3.3.1 Gauss-Legendre quadrature (integration) method

Let the weight function be $w(x) = 1$ then the integration method given by

$$\int_{-1}^1 f(x)dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.40)$$

is called Gauss-Legendre quadrature methods. In this case all nodes x_k 's and weights λ_k 's for $k=0, 1, \dots, n$ are unknown.

Examples:

(a). One-point rule $n=0$.

Then the formula is given by $\int_{-1}^1 f(x)dx = \lambda_0 f(x_0)$ and the method has two unknown's λ_0, x_0 . Then making the method exact for polynomials $f(x) = 1, x$ we get

$$\text{For } f(x) = 1 \quad \int_{-1}^1 dx = 2 = \lambda_0 \quad (1)$$

$$\text{For } f(x) = x \quad \int_{-1}^1 x dx = 0 = \lambda_0 x_0 \quad x_0 = 0 \text{ since } \lambda_0 \neq 0 \quad (2)$$

Hence by equation (1) and (2) the method is given by

$$\int_{-1}^1 f(x)dx = 2f(0)$$

which is called one point rule.

(b). Two-Point Rule (Formula) n=1.

The formula is given by

$$\int_{-1}^1 f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1).$$

The method has four unknowns $x_0, x_1, \lambda_0, \lambda_1$ then making the method exact for polynomials $f(x) = 1, x, x^2, x^3$.

For $f(x) = 1$ we have $\int_{-1}^1 dx = 2 = \lambda_0 + \lambda_1$ (1).

For $f(x) = x$ we have $\int_{-1}^1 x dx = 0 = \lambda_0 x_0 + \lambda_1 x_1$ (2).

For $f(x) = x^2$ we have $\int_{-1}^1 x^2 dx = \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2$ (3).

For $f(x) = x^3$ we have $\int_{-1}^1 x^3 dx = 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3$ (4)

Then eliminating λ_0 from equation (2) and (4) we get $\lambda_1 x_1^3 - \lambda_1 x_1 x_0^2 = 0$ or

$\lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0$ since $\lambda_1 \neq 0, x_0 \neq x_1$ we get or $x_1 = -x_0$. Note that if $x_1 = 0$

from equation (2) we get $x_0 = 0$ since $\lambda_0 \neq 0$ hence $x_1 \neq 0$ substituting in (2) we get that

$\lambda_0 = \lambda_1$ and substituting in (1) we get that $\lambda_0 = \lambda_1 = 1$ using (3) we get $x_0^2 = \frac{1}{3}$ or

$x_0 = \pm \frac{1}{\sqrt{3}}$ and $x_1 = \mp \frac{1}{\sqrt{3}}$. Hence the two point Gauss-Legendre formula is given by

$$\int_{-1}^1 f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

C. Three-Point Formula n=2. The formula is given by

$$\int_{-1}^1 f(x)dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

there are six unknowns in the method and it can be exact for polynomials of degree up to five.

For $f(x) = x^i, i = 0, 1, \dots, n$ we get the system of equations as follows:-

For $f(x) = 1$ we have $\int_{-1}^1 dx = 2 = \lambda_0 + \lambda_1 + \lambda_2 = 2$ (1).

For $f(x) = x$ we have $\int_{-1}^1 x dx = 0 = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2$ (2).

$$\text{For } f(x) = x^2 \text{ we have } \int_{-1}^1 x^2 dx = \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 \quad (3)$$

$$\text{For } f(x) = x^3 \cdot \int_{-1}^1 x^3 dx = 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 \quad (4)$$

$$\text{For } f(x) = x^4 \text{ we have } \int_{-1}^1 x^4 dx = \frac{2}{5} = \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 \quad (5)$$

$$\text{For } f(x) = x^5 \text{ we have } \int_{-1}^1 x^5 dx = 0 = \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 \quad (6)$$

Then solving the equations given above we get that $x_0 = \pm\sqrt{\frac{3}{5}}$

$$x_2 = \mp\sqrt{\frac{3}{5}} \quad x_1 = 0 \quad \lambda_0 = \frac{5}{9}, \lambda_1 = \frac{8}{9}, \lambda_2 = \frac{5}{9}.$$

Hence the three point Gauss-Legendre quadrature method is given by

$$\int_{-1}^1 f(x) dx = \frac{1}{9} [5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(\sqrt{\frac{3}{5}})]$$

Example: Evaluate the integral $I = \int_1^2 \frac{2x}{1+x^4} dx$ using:

- The Gauss-Legendre 1-Point method
- The Gauss-Legendre 2-Point method
- The Gauss-Legendre 3-Point Quadrature Method and compare with the exact

$$\text{Solution } I = \tan^{-1}(4) - \frac{\pi}{4} = 0.5406.$$

Solution: First to use the Gauss-Legendre methods the interval $[1,2]$ is to be reduced

to the interval $[-1,1]$ by $x = \frac{(b-a)}{2}t + \frac{(b+a)}{2}$ where $a=1, b=2$ then we get $x = \frac{t+3}{2}$

and hence the integral is changed in to the form

$$\int_{-1}^1 f(t) dt = \int_{-1}^1 \frac{8(t+3)}{16+(t+3)^4} dt.$$

$$(a). \text{ Using 1-point rule we have that } I = 2f(0) = 2 \frac{(24)}{16+81} = 0.4948.$$

$$(b). \text{ Using 2-point rule we have that } I = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) = 0.3842 + 0.1592 = 0.5434.$$

(c). Using 3-point rule we have that $I = \frac{1}{9}[5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(\sqrt{\frac{3}{5}})] = 0.5406$.

But the exact solution is $I = \tan^{-1}(4) - \frac{\pi}{4} = 0.5404$ now comparing the solutions in the above Gauss-Legendre rules with the exact solutions the 3-point rule is the best approximation.

1.3.3.2 Gauss-Chebyshev integration methods

Let the weight function be $w(x) = \frac{1}{\sqrt{1-x^2}}$ and $x \in (-1,1)$ then the methods given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx \approx \sum_{k=0}^n \lambda_k f_k$$

is called the Gauss-Chebyshev integration methods with nodes x_k 's and weights λ_k 's for $k = 0, 1, \dots, n$ are all unknowns.

Error analysis of the methods:

Now since we can make the method exact for polynomials of degree less than or equal to $2n+1$ then there is an error at the next polynomials of degree at least $2n+2$ then we can drive the error constant C by

$$C = \int_{-1}^1 \frac{x^{2n+2}}{\sqrt{1-x^2}} dx - \sum_{k=0}^n \lambda_k x_k^{2n+2}$$

and the error is given by

$$R_{2n+2} = \frac{C f^{(2n+2)}(\eta)}{(2n+2)!}, \eta \in (-1,1).$$

Examples:

(a). **One-point rule (method) for n=0:** The formula is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \lambda_0 f_0$$

now the method has two unknowns λ_0, x_0 .

Then making the method exact for polynomials $f(x) = 1, x$ we get:

$$\text{For } f(x) = 1 \text{ we have } \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \lambda_0 \text{ or } \sin^{-1}(x) \Big|_{-1}^1 = \lambda_0 = \pi$$

$$\text{For } f(x) = x \text{ we have } \int_{-1}^1 \frac{x}{\sqrt{1-x^2}} dx = \lambda_0 x_0 \text{ or } 0 = \lambda_0 x_0 \text{ or } x_0 = 0 \text{ since } \lambda_0 \neq 0$$

Hence the method is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \pi f(0).$$

Error: Now the error constant is given by

$$C = \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx = 2 \int_0^1 \frac{x^2}{\sqrt{1-x^2}} dx = 2 \int_0^{\frac{\pi}{2}} \sin^2(\theta) d\theta = \frac{\pi}{2}$$

hence the error term is given by

$$R_2 = \frac{C f^{(2)}(\xi)}{2!} = \frac{\pi}{4} f^{(2)}(\xi) \quad -1 < \xi < 1.$$

(b). **Two-point rule (method) for n=1:** The formula is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \lambda_0 f(x_0) + \lambda_1 f(x_1).$$

The method has four unknowns $x_0, x_1, \lambda_0, \lambda_1$ then making the method exact for

polynomials $f(x) = 1, x, x^2, x^3$ we obtain:

$$\text{For } f(x) = 1 \text{ we have } \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \lambda_0 + \lambda_1 \text{ or } \lambda_0 + \lambda_1 = \pi \quad (1)$$

$$\text{For } f(x) = x \text{ we have } \int_{-1}^1 \frac{x}{\sqrt{1-x^2}} dx = \lambda_0 x_0 + \lambda_1 x_1 \text{ or } \lambda_0 x_0 + \lambda_1 x_1 = 0 \quad (2)$$

$$\text{For } f(x) = x^2 \text{ we have } \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx = \lambda_0 x_0^2 + \lambda_1 x_1^2 \text{ or } \lambda_0 x_0^2 + \lambda_1 x_1^2 = \frac{\pi}{2} \quad (3)$$

For $f(x) = x^3$ we have $\int_{-1}^1 \frac{x^3}{\sqrt{1-x^2}} dx = \lambda_0 x_0^3 + \lambda_1 x_1^3$ or $\lambda_0 x_0^3 + \lambda_1 x_1^3 = 0$ (4)

Now eliminating λ_0 from equation (2) and (4) above we get $\lambda_1 x_1^3 - \lambda_1 x_1 x_0^2 = 0$ or

$$\lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0 \text{ since } \lambda_1 \neq 0 \text{ and } x_1 \neq x_0 \text{ we get } x_1 + x_0 = 0 \text{ or } x_1 = -x_0.$$

If $x_1 = 0$ then from equation (2) above we get $x_0 = 0$ since $\lambda_0 \neq 0$ we have

$x_1 \neq 0$. Substituting in equation (2) above we get $\lambda_0 = \lambda_1$ and substituting in equation (1)

above we get $\lambda_0 = \lambda_1 = \frac{\pi}{2}$ using (3) we get $2x_0^2 = 1$ or $x_0 = \pm \frac{1}{\sqrt{2}}$ and $x_1 = \mp \frac{1}{\sqrt{2}}$. Hence

the method is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right) \right].$$

Error: The error constant is given by

$$C = \int_{-1}^1 \frac{x^4}{\sqrt{1-x^2}} dx - \frac{\pi}{2} [x_0^2 + x_1^2] = \frac{\pi}{40}$$

hence the error is given by

$$R_4 = \frac{C}{4!} f^{(4)}(\eta) = \frac{\pi}{960} f^{(4)}(\eta), -1 < \eta < 1.$$

(C) . **Three-point rule (formula) for n=2.** The method is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

the method has six unknowns $x_0, x_1, x_2, \lambda_0, \lambda_1, \lambda_2$ then we can make the method exact

for following polynomials $f(x) = 1, x, x^2, x^3, x^4, x^5$. Then we get the following systems of

equations:-

For $f(x) = 1$ we have $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \lambda_0 + \lambda_1 + \lambda_2 = \pi$ (1).

For $f(x) = x$ we have $\int_{-1}^1 \frac{x}{\sqrt{1-x^2}} dx = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0$ (2).

For $f(x) = x^2$ we have $\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 = \frac{\pi}{2}$ (3).

For $f(x) = x^3$ we have $\int_{-1}^1 \frac{x^3}{\sqrt{1-x^2}} dx = \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0$ (4).

For $f(x) = x^4$ we have $\int_{-1}^1 \frac{x^4}{\sqrt{1-x^2}} dx = \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4 = \frac{3\pi}{2}$ (5).

For $f(x) = x^5$ we have $\int_{-1}^1 \frac{x^5}{\sqrt{1-x^2}} dx = \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5 = 0$ (6)

Then solving the above six equations we get

that $x_0 = -\frac{\sqrt{3}}{2}, x_1 = 0, x_2 = \frac{\sqrt{3}}{2}, \lambda_0 = \lambda_1 = \lambda_2 = \frac{\pi}{3}$.

Hence the three points Gauss-Chebyshev quadrature formula is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{3} [f(-\frac{\sqrt{3}}{2}) + f(0) + f(\frac{\sqrt{3}}{2})]$$

The error constant is given by

$$C = \int_{-1}^1 \frac{x^6}{\sqrt{1-x^2}} dx - \frac{\pi}{3} [\frac{27}{64} + 0 + \frac{27}{64}]$$

then using trigonometric substitution $x = \sin \theta$ we get

$$C = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^6(\theta) d\theta - \frac{9\pi}{32} = \frac{\pi}{32}$$

Hence the error is given by

$$R_6 = \frac{Cf^{(6)}(\eta)}{6!} = \frac{\pi f^{(6)}(\eta)}{23040}, -1 < \eta < 1.$$

Example: - Evaluate the integral $\int_{-1}^1 (1-x^2)^{\frac{3}{2}} \cos x dx$ using:

- (a).Gauss-Chebyshev one point rule:
- (b).Gauss-Chebyshev two-point rule:
- (c).Gauss-Chebyshev three-point rule:

Solutions: Given $(1-x^2)^{\frac{3}{2}} \cos x = \frac{(1-x^2)^2 \cos x}{\sqrt{1-x^2}}$ now put $f(x) = (1-x^2) \cos x$

Then

(a).using the Gauss-Chebyshev one point formula we have

$$\int_{-1}^1 (1-x^2)^{\frac{3}{2}} \cos x dx = \pi f(0) = \pi(1-0^2) \cos 0 = \pi$$

(b).using the Gauss-Chebyshev two-point formula we have

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{\frac{3}{2}} \cos x dx &= \int_{-1}^1 \frac{(1-x^2)^2 \cos x}{\sqrt{1-x^2}} dx = \\ \frac{\pi}{2} [f(-\frac{1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}})] &= \frac{\pi}{2} [2(\frac{1}{4}) \cos(\frac{1}{\sqrt{2}})] = 0.59709. \end{aligned}$$

(c).using Gauss-Chebyshev three-rule we have

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{\frac{3}{2}} \cos x dx &= \int_{-1}^1 \frac{(1-x^2)^2 \cos x}{\sqrt{1-x^2}} dx = \\ \frac{\pi}{3} [f(-\frac{\sqrt{3}}{2}) + f(0) + f(\frac{\sqrt{3}}{2})] &= \frac{\pi}{2} [2(\frac{1}{16}) \cos(\frac{\sqrt{3}}{2}) + 1] = 1.13200. \end{aligned}$$

1.3.3.3 Determination of nodes and weights through Orthogonal polynomials

Now consider the quadrature formula

$$I = \int_a^b w(x) f(x) dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.41)$$

Definition3: For a given weight function $w : [a, b] \rightarrow (0, \infty)$ let

$$\langle p, q \rangle = \int_a^b p(x)q(x)w(x)dx,$$

$\|p\| = \langle p, p \rangle^{\frac{1}{2}}$ for $p, q \in \Pi$ the mapping $\langle \cdot, \cdot \rangle : \Pi \times \Pi \rightarrow \mathbb{R}$ defines an inner product on the space of all real polynomials Π where in practice $\langle \cdot, \cdot \rangle$ is linear with respect to its arguments and $\langle p, p \rangle > 0$ for $0 \neq p \in \Pi$.

Definition4: Two polynomials $p, q \in \Pi$ are called orthogonal with respect to the weight function if $\langle p, q \rangle = 0$.

Theorem: If x_k 's are the zeros of an orthogonal polynomial orthogonal with respect to the weight function $w(x)$ over $[a, b]$ then the formula (1.39) has degree of exactness less than or equal to $2n+1$. Further $\lambda_k > 0$ for $k = 0, 1, \dots, n$.

Proof: Let $f(x)$ be a polynomial of degree less than or equal to $2n+1$.

Let $q_n(x)$ be the Lagrange interpolating polynomial of degree less than or equal to n interpolating the data (x_k, f_k) for $k = 0, 1, \dots, n$

$$q_n(x) = \sum_{k=0}^n L_k(x) f(x_k)$$

with

$$L_k(x) = \frac{\prod_{n+1}(x)}{(x-x_k) \prod_{n+1}^{(1)}(x_k)}$$

The polynomial $[f(x) - q_n(x)]$ has zeros at x_0, x_1, \dots, x_n .

Hence it can be written as

$$f(x) - q_n(x) = p_{n+1}(x) r_n(x) \tag{1.42}$$

Where $r_n(x)$ is a polynomial of degree at most n and $p_{n+1}(x_k) = 0$ for $k=0, 1, \dots, n$.

Integrating (1.42) we get

$$\int_a^b w(x)[f(x) - q_n(x)] dx = \int_a^b w(x) p_{n+1}(x) r_n(x) dx \Rightarrow \int_a^b w(x) f(x) dx = \int_a^b w(x) q_n(x) dx + \int_a^b w(x) p_{n+1}(x) r_n(x) dx$$

Now the integral $\int_a^b w(x) p_{n+1}(x) r_n(x) dx = 0$ if $p_{n+1}(x)$ is orthogonal polynomial with respect to the weight function to all polynomials of degree less than or equal to n . Then we have

$$\int_a^b w(x) f(x) dx = \int_a^b w(x) q_n(x) dx = \sum_{k=0}^n \lambda_k f_k \tag{1.43}$$

Where $\lambda_k = \int_a^b w(x) L_k(x) dx$ hence (1.41) has degree of accuracy $2n+1$

Observe that $L_j^2(x)$ is a polynomial of degree less than or equal to $2n$. Choosing

$f(x) = L_j^2(x)$ and substituting in (1.41) we have

$$\int_a^b w(x) L_j^2(x) dx = \sum_{k=0}^n \lambda_k f_k = \sum_{k=0}^n \lambda_k L_j^2(x_k)$$

and since $L_j(x_k) = \delta_{jk}$ we get

$$\lambda_j = \int_a^b w(x) L_j^2(x) dx > 0 \Rightarrow \lambda_k > 0.$$

1.3.3.3.1 Gauss-Legendre formulas

Now the Legendre polynomials can be obtained by using the Rodriguez's formula

$$p_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Examples: $p_0(x) = 1, p_1(x) = x, p_2(x) = \frac{(3x^2 - 1)}{2}, p_3(x) = \frac{(5x^3 - 3x)}{2}$ etc... are the

Gauss-Legendre polynomials. The Nodes are the zeros of the Legendre polynomials.

(a). **Two-point formula for n=2**

Then we have $p_2(x) = \frac{(3x^2 - 1)}{2}$ hence setting

$$p_2(x) = 0 \Rightarrow \frac{(3x^2 - 1)}{2} = 0 \Rightarrow x_0 = -\frac{1}{\sqrt{3}}, x_1 = \frac{1}{\sqrt{3}}.$$

The Lagrange fundamental polynomials are $L_0(x) = \frac{x - x_1}{x_0 - x_1} = -\frac{\sqrt{3}}{2} (x - \frac{1}{\sqrt{3}})$,

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{\sqrt{3}}{2} (x + \frac{1}{\sqrt{3}})$$

therefore

$$\lambda_0 = \int_{-1}^1 L_0(x) dx = -\frac{\sqrt{3}}{2} \int_{-1}^1 (x - \frac{1}{\sqrt{3}}) dx = 1, \lambda_1 = \int_{-1}^1 L_1(x) dx = \frac{\sqrt{3}}{2} \int_{-1}^1 (x + \frac{1}{\sqrt{3}}) dx = 1.$$

The two-point formula is given by

$$\int_{-1}^1 f(x) dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

(b). *Three-point formula for n=3*

In this case the nodes are the zeros of the Legendre polynomial $p_3(x) = \frac{(5x^3 - 3x)}{2}$

Setting $p_3(x) = 0$ we obtain $x_0 = -\sqrt{\frac{3}{5}}, x_1 = 0, x_2 = \sqrt{\frac{3}{5}}$. The Lagrange fundamental

Polynomials are

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{5}{6}x\left(x - \sqrt{\frac{3}{5}}\right).$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -\frac{5}{3}\left(x + \sqrt{\frac{3}{5}}\right)\left(x - \sqrt{\frac{3}{5}}\right) = -\frac{5}{3}\left(x^2 - \frac{3}{5}\right).$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{5}{6}x\left(x + \sqrt{\frac{3}{5}}\right). \text{Hence}$$

$$\lambda_0 = \int_{-1}^1 L_0(x) dx = \frac{5}{6} \int_{-1}^1 x\left(x - \sqrt{\frac{3}{5}}\right) dx = \frac{5}{6} * \frac{2}{3} = \frac{5}{9}.$$

$$\lambda_1 = \int_{-1}^1 L_1(x) dx = -\frac{5}{3} \int_{-1}^1 \left(x^2 - \frac{3}{5}\right) dx = -\frac{5}{3} \left(\frac{2}{3} - \frac{6}{5}\right) = \frac{8}{9}.$$

$$\lambda_2 = \int_{-1}^1 L_2(x) dx = \frac{5}{6} \int_{-1}^1 x\left(x + \sqrt{\frac{3}{5}}\right) dx = \frac{5}{9}.$$

Then the three-point Gauss-Legendre formula is given by

$$\int_{-1}^1 f(x) dx = \frac{1}{9} \left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right]$$

now in general using the Rodriguez's formula we can obtain the zeros of the Legendre polynomial $p_n(x)$ and also we can drive the corresponding Gauss-Legendre formula.

The weights of an (n+1) point-formula are given by

$$\lambda_k = \int_{-1}^1 \frac{p_{n+1}(x)}{(x-x_k)p_{n+1}'(x_k)} dx$$

where $k = 0, 1, \dots, n$.

1.3.3.3.2 Gauss-Chebyshev formulas

The nodes of an $(n+1)$ Gauss-Chebyshev formula are the roots of the Chebyshev Polynomial

$$T_{n+1}(x) = \cos[(n+1)\cos^{-1}x] = 0$$

$$\Rightarrow (n+1)\cos^{-1}x_k = (2k+1)\frac{\pi}{2} \Rightarrow \cos^{-1}x_k = \frac{(2k+1)\pi}{2(n+1)} \quad \text{hence}$$

$$x_k = \cos\left[\frac{(2k+1)\pi}{2(n+1)}\right], k = 0, 1, \dots, n$$

and the weights are given by

$$\lambda_k = \int_{-1}^1 \frac{T_{n+1}(x) dx}{\sqrt{1-x^2}(x-x_k)T_{n+1}'(x_k)} \quad \text{for } k = 0, 1, \dots, n.$$

Example:

(a). **Two-point Gauss-Chebyshev formula for $n=1$:** - Now we have that

$$T_2(x) = 0 = 2x^2 - 1 \Rightarrow x_0 = -\frac{1}{\sqrt{2}} \quad \text{and} \quad x_1 = \frac{1}{\sqrt{2}} \quad \text{and the weights are given by}$$

$$\lambda_0 = \int_{-1}^1 \frac{T_2(x)}{\sqrt{1-x^2}(x-x_0)T_2'(x_0)} dx = \int_{-1}^1 \frac{(2x^2-1)}{\sqrt{1-x^2}(x+\frac{1}{\sqrt{2}})(-2\sqrt{2})} dx$$

$$= -\frac{1}{\sqrt{2}} \int_{-1}^1 \frac{(x-\frac{1}{\sqrt{2}})}{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

$$\lambda_1 = \int_{-1}^1 \frac{T_2(x)}{\sqrt{1-x^2}(x-x_1)T_2'(x_1)} dx = \int_{-1}^1 \frac{(2x^2-1)}{\sqrt{1-x^2}(x-\frac{1}{\sqrt{2}})(2\sqrt{2})} dx$$

$$= \frac{1}{\sqrt{2}} \int_{-1}^1 \frac{(x+\frac{1}{\sqrt{2}})}{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

Hence the two-point Gauss-Chebyshev formula is given by

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right) \right].$$

1.3.3.3 Gauss-Laguerre Integration Methods

Let $w(x) = e^{-x}$ for $x \in [0, \infty)$ be the weight function then the integration methods of the form

$$I = \int_0^{\infty} e^{-x} f(x) dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.44)$$

are called the Gauss-Laguerre Integration Methods. The nodes x_k 's for $k = 0, 1, \dots, n$ are the roots of the Laguerre polynomials

$$L_{n+1}(x) = (-1)^{n+1} e^x \frac{d^{n+1}}{dx^{n+1}} (e^{-x} x^{n+1}).$$

We have $L_0(x) = 1, L_1(x) = x - 1, L_2(x) = x^2 - 4x + 2, L_3(x) = x^3 - 9x^2 + 18x - 6$ etc...

The Laguerre polynomials $L_n(x)$ are orthogonal with respect to the weight function

$$w(x) = e^{-x} \text{ on } [0, \infty) \text{ that is } \int_0^{\infty} e^{-x} L_m(x) L_n(x) dx = 0 \text{ for } m \neq n$$

and the weights are obtained from the relation

$$\lambda_k = \int_0^{\infty} \frac{e^{-x} L_{n+1}(x)}{(x - x_k) L_{n+1}'(x_k)} dx \quad (1.45)$$

Error term:-The method in (1.42) given above is exact for polynomials of degree up to $2n+1$ from (1.3.1) above the error term is given by

$$R_{2n+2} = \frac{C f^{(2n+2)}(\xi)}{(2n+2)!} \text{ where } 0 < \xi < \infty$$

and

$$C = \int_0^{\infty} e^{-x} x^{2n+2} dx - \sum_{k=0}^n \lambda_k x_k^{2n+2} \quad (1.46)$$

is the error constant.

Derivation of Some Gauss-Laguerre Integration Methods

(a). **One-point Gauss-Laguerre formula for n=0:** Then $L_1(x) = 0$ gives the node $x_0 = 1$ and the corresponding weight is

$$\lambda_0 = \int_0^{\infty} e^{-x} dx = 1$$

Hence the one-point Gauss-Laguerre formula is give by

$$\int_0^{\infty} e^{-x} f(x) dx = f(1).$$

Error term: The formula is exact for polynomials of degree less than or equal to 1 then by (1.44) above the error constant is given by

$$C = \int_0^{\infty} x^2 e^{-x} dx - \lambda_0 x_0^2 = \Gamma(3) - 1 = 2! - 1 = 2 - 1 = 1$$

hence the error term in the one-point formula is given by

$$R_2 = \frac{C f^{(2)}(\xi)}{2!} = \frac{1}{2} f^{(2)}(\xi)$$

where $0 < \xi < \infty$ by (1.45) above.

(b). **Two-point Gauss-Laguerre formula for n=1:-** The nodes are the roots of

$L_2(x) = x^2 - 4x + 2 = 0$ then solving this quadratic equation we get

$x_0 = 2 - \sqrt{2}, x_1 = 2 + \sqrt{2}, L_2(x) = (x - x_0)(x - x_1)$ and the corresponding weights are given by

$$\lambda_0 = \int_0^{\infty} \frac{e^{-x} L_2(x)}{(x - x_0) L_2^{(1)}(x_0)} dx = \int_0^{\infty} \frac{e^{-x} (x - x_1)}{(x_0 - x_1)} dx = -\frac{1}{2\sqrt{2}} [e^{-x} (x - x_1) - e^{-x}] \Big|_0^{\infty} = \frac{2 + \sqrt{2}}{4} \text{ and}$$

$$\lambda_1 = \int_0^{\infty} \frac{e^{-x} L_2(x)}{(x_0 - x_1) L_2^{(1)}(x)} dx = \int_0^{\infty} \frac{e^{-x} (x - x_0)}{(x_1 - x_0)} dx = \frac{2 - \sqrt{2}}{4}.$$

Hence the two-point Gauss-Laguerre formula is given by

$$\int_0^{\infty} e^{-x} f(x) dx = \frac{1}{4} [(2 + \sqrt{2}) f(2 - \sqrt{2}) + (2 - \sqrt{2}) f(2 + \sqrt{2})].$$

Error term: The formula is exact for polynomials of degree

less than or equal to 3 then by (1.44) above the error constant is given by

$$C = \int_0^{\infty} x^4 e^{-x} dx - (\lambda_0 x_0^2 + \lambda_1 x_1^2) = 24 - \frac{1}{4} [4 * 68 - 2\sqrt{2} * 48\sqrt{2}] = 4.$$

Hence the error term is given by

$$R_3 = \frac{Cf^{(4)}(\xi)}{4!} = \frac{1}{6} f^{(4)}(\xi) \text{ for } 0 < \xi < \infty.$$

Example: Evaluate $I = \int_0^{\infty} \frac{e^{-x}}{1+x^2} dx$ using the two-point Gauss-Laguerre formula.

Solution: Given $f(x) = \frac{1}{1+x^2}$ then using two-point formula given above we

$$\text{have } \int_0^{\infty} \frac{e^{-x}}{1+x^2} x \approx \frac{1}{4} [(2+\sqrt{2}) \left\{ \frac{1}{1+(2-\sqrt{2})^2} \right\} + (2-\sqrt{2}) \left\{ \frac{1}{1+(2+\sqrt{2})^2} \right\}] \approx 0.64706.$$

1.3.3.3.4 Gauss-Hermite Integration Methods.

Let $w(x) = e^{-x^2}$ for $x \in (-\infty, \infty)$ be the weight function then the integration methods of the form

$$I = \int_{-\infty}^{\infty} e^{-x^2} f(x) dx \approx \sum_{k=0}^n \lambda_k f_k \quad (1.47)$$

are called Gauss-Hermite integration methods. In this method the nodes x_k 's for $k = 0, 1, \dots, n$ are the roots of the Hermite polynomials

$$H_{n+1}(x) = (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}} (e^{-x^2})$$

Examples we have $H_1(x) = 2x, H_2(x) = 2(2x^2 - 1), H_3(x) = 4(2x^3 - 3x)$ etc...

Now the Hermite polynomials are orthogonal with respect to the weight function

$$w(x) = e^{-x^2} \text{ on } (-\infty, \infty) \text{ that is } \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 0 \text{ for } m \neq n.$$

The weights are determined by

$$\lambda_k = \int_{-\infty}^{\infty} \frac{e^{-x^2} H_{n+1}(x)}{(x-x_k)H_{n+1}^{(1)}(x_k)} dx \quad (1.48)$$

Error term:- The method is exact for polynomials of degree up to $2n+1$ hence the error term of the above method is given by

$$R_{2n+2} = \frac{Cf^{(2n+2)}(\xi)}{(2n+2)!}$$

where $-\infty < \xi < \infty$ and

$$C = \int_{-\infty}^{\infty} x^{2n+2} e^{-x^2} dx - \sum_{k=0}^n \lambda_k x_k^{2n+2} \text{ is the error constant.}$$

Derivation of Some Gauss-Hermite Formulas Explicitly

(a). **One-point Gauss-Hermite formula for $n=0$:-** We have $H_1(x) = 0$ gives the node as $x_0 = 0$ and the corresponding weight is given by

$$\lambda_0 = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Hence the one-point Gauss-Hermite formula is given by

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx = \sqrt{\pi} f(0).$$

Error term: The formula is exact for polynomials of degree less than or equal to 1 and hence the error constant is given by

$C = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx - \lambda_0 x_0^2 = 2 \int_0^{\infty} x^2 e^{-x^2} dx - 0 = 2 \int_0^{\infty} x^2 e^{-x^2} dx$ since the integrand $x^2 e^{-x^2}$ is even now using substitution $x^2 = t$ we get the error constant as

$$C = \int_0^{\infty} t^{1/2} e^{-t} dt = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \quad (*) \text{ then the error term is given by}$$

$$R_2 = \frac{Cf^{(2)}(\xi)}{2!} = \frac{\sqrt{\pi}}{4} f^{(2)}(\xi) \text{ for } -\infty < \xi < \infty.$$

(b). **Two-point Gauss-Hermite formula for $n=1$:-** The nodes are the roots of the Hermite polynomial and $H_2(x) = (x-x_0)(x-x_1)$. The corresponding weights are given by

$$\lambda_0 = \int_{-\infty}^{\infty} \frac{e^{-x^2} 4(x-x_0)(x-x_1)}{(x-x_0)[4(x_0-x_1)]} dx = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2} (x-x_1) dx = -\frac{1}{\sqrt{2}} \left[\int_{-\infty}^{\infty} e^{-x^2} x dx - x_1 \int_{-\infty}^{\infty} e^{-x^2} dx \right] = \frac{\sqrt{\pi}}{2}$$

and

$$\lambda_1 = \int_{-\infty}^{\infty} \frac{e^{-x^2} 4(x-x_0)(x-x_1)}{(x-x_0)[4(x_0-x_1)]} dx = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2} (x-x_0) dx = \frac{\sqrt{\pi}}{2}$$

Hence the two point Gauss-Hermite formula is given by

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx = \frac{\sqrt{\pi}}{2} \left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right) \right].$$

Example: Evaluate $I = \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x^2 + x + 1} dx$ using Gauss-Hermite two-point formula and determine the error term.

Solution: We have $f(x) = \frac{1}{x^2 + x + 1}$ then using two-point formula we get

$$I = \frac{\sqrt{\pi}}{2} \left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right) \right] = \frac{\sqrt{\pi}}{2} [1.26120 + 0.45308] = 1.51924 \text{ and the error constant}$$

is given by $C = \frac{\sqrt{\pi}}{2}$ by (*) above and $R_4 = \frac{C}{4!} f^{(4)}(\eta) = \frac{\sqrt{\pi}}{48} f^{(4)}(\eta)$ for $-\infty < \eta < \infty$.

1.3.3.3.5 Convergence of Gaussian Quadrature

Methods

In contrast to the Newton-cotes formula if $f(x)$ is continuous and $w(x)$ is any non negative integrable weight function on $[a, b]$ then any desired degree of accuracy is obtained by using a sufficiently high order Gaussian quadrature formula.

Proof: Let $\varepsilon > 0$ be given and let $p(x)$ be a polynomial of degree N such that

$$|f(x) - p(x)| < \varepsilon \text{ on } [a, b] \text{ where } p(x) \text{ exists by the Weierstrass approximation theorem.}$$

Now from $\int_a^b w(x)f(x)dx = \sum_{k=0}^n \lambda_k f(x_k) + R_n$ given above in (1.23) we have

$R_n = \int_a^b w(x)f(x)dx - \sum_{k=0}^n \lambda_k f(x_k)$ put $E = R_n$ then we have

$$|E| = \left| \int_a^b w(x)f(x)dx - \sum_{k=0}^n \lambda_k f_k \right|$$

$$\leq \left| \int_a^b w(x)[f(x) - p(x)]dx \right| + \left| \int_a^b w(x)p(x)dx - \sum_{k=0}^n \lambda_k f_k \right| + \left| \sum_{k=0}^n \lambda_k [p(x_k) - f_k] \right| \dots (*)$$

If $2n+1 \geq N$ then the second term on the right side of (*) is zero since it is exact for polynomials of degree less than or equal to $2n+1$. Then since the sum of the weights in any formula is the integral of the weight function over the given interval by taking $f(x)=1$ and also since all the Gaussian weights are positive from (*) we have

$|E| \leq \varepsilon \int_a^b w(x)dx + \varepsilon \int_a^b w(x)dx = 2\varepsilon \int_a^b w(x)dx \dots (**)$ which proves the convergence of the method since ε is arbitrarily small. Therefore using sequence of Gaussian formulas with increasing n will lead to a convergent sequence of approximations.

1.4 Composite Newton-Cotes Integrations Methods

As the order of integration method $I = \int_a^b f(x)dx \approx \sum_{k=0}^n \lambda_k f_k$ is increased the order of the

derivative in the error term in the methods also increases.

For any numerical quadrature method to produce a meaningful results these higher order derivative remain continuous in the integral of integration.

An alternative to obtain accurate results while using lower order methods is the use of composite integration methods. -By subdividing the given interval $[a, b]$ in to a number of subintervals and evaluate the integral in each subinterval by a particular method.

1.4.1 The Composite Trapezoidal Rule

Divide the interval $[a, b]$ in to N subintervals each of length $h = \frac{b-a}{N}$ then denote the subintervals as $(x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$

Where $x_0 = a, x_n = b$ and $x_i = x_0 + ih$ for $i = 1, \dots, n-1$ hence

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \dots + \int_{x_{n-1}}^{x_n} f(x)dx \quad (1.49)$$

then evaluating each of the integrals in (14.1) by trapezoidal rule then we get

$$\begin{aligned} I &= \int_a^b f(x)dx = \frac{h}{2}[(f_0 + f_1) + (f_1 + f_2) + \dots + (f_{n-1} + f_n)] \\ &= \frac{h}{2}[f_0 + 2(f_1 + f_2 + \dots + f_{n-1}) + f_n] \end{aligned} \quad (1.50)$$

Where $f_k = f(x_k)$ for $k = 0, 1, \dots, n$. The error in the integration method (14.2) becomes

$$R_1 = -\frac{h^3}{12}[f^{(2)}(\xi_1) + f^{(2)}(\xi_2) + \dots + f^{(2)}(\xi_n)] \quad (1.51)$$

where $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1, 2, \dots, n$. Now if $f^{(2)}(x)$ is constant for all $x \in [a, b]$ or if

$$f^{(2)}(\eta) = \max_{a \leq x \leq b} |f^{(2)}(x)|, a < \eta < b \text{ we can write (1.49) as } |R_1| \leq \frac{nh^3}{12} f^{(2)}(\eta) \text{ since}$$

$$h = \frac{b-a}{n} \text{ we have } |R_1| \leq \frac{(b-a)^3}{12n^2} f^{(2)}(\eta) = \frac{(b-a)h^2}{12} f^{(2)}(\eta), \eta \in [a, b].$$

1.4.2 Composite Simpson's Rule

In using the Simpson's rule of integration we need three nodes. Divide the interval $[a, b]$ in to an even number of subintervals of equal length giving an odd number of nodes. If

we divide the interval $[a, b]$ in to $2n$ sub intervals each of length $h = \frac{b-a}{2n}$ then we get

$2n+1$ nodes $x_0, x_1, \dots, x_{2n}, x_0 = a, x_{2n} = b, x_i = x_0 + ih, i = 1, 2, \dots, 2n-1$. Then

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx \quad (1.52)$$

Then evaluating each of the integrals on the right hand side of (1.44) by Simpson's rule we get

$$\begin{aligned} I &= \int_a^b f(x)dx = \frac{h}{3}[(f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4) + \dots + (f_{2n-2} + 4f_{2n-1} + f_{2n})] \\ &= \frac{h}{3}[f_0 + 4(f_1 + f_3 + f_5 + \dots + f_{2n-1}) + 2(f_2 + f_4 + f_6 + \dots + f_{2n-2}) + f_{2n}] \end{aligned} \quad (1.53)$$

Hence the method in (1.53) is called Simpson's composite method (rule) and the error in the integration method in (1.51) becomes

$$R_2 = -\frac{h^5}{90}[f^{(4)}(\xi_1) + \dots + f^{(4)}(\xi_n)] \dots \quad (1.54)$$

where $x_{2i-2} < \xi_i < x_{2i}$, for $i = 1, 2, \dots, n$ using $f^{(4)}(\eta) = \max_{a \leq x \leq b} |f^{(4)}(x)|$, $a < \eta < b$ we can write (1.46) in the

$$\text{form } |R_2| \leq \frac{nh^5}{90} f^{(4)}(\eta) = \frac{(b-a)^5 f^{(4)}(\eta)}{2880n^4} = \frac{(b-a)^5}{180} h^5 f^{(4)}(\eta), \eta \in [a, b].$$

Now since the number of required function calls for the composite Simpson's rule is twice as high as for the composite Trapezoidal rule. However, for sufficiently smooth functions f the composite Simpson's rule is to be preferred since compared to the composite Trapezoidal rule, the accuracy is squared.

1.5 Comparison of Newton-Cotes and Gaussian Quadrature Methods

In Newton-Cotes integration methods: The negative weights occur when $n \geq 8$ this contradicts the idea that the integral is a limit of a sum of functional values whose weights are only positive in value and the sum of the absolute values of the weights exceeds the value $b - a$ and this leads to an amplification of round off errors hence the Newton-Cotes formulas are used only for small values of n .

The nodes should be equally spaced in the given interval (limit of integration).

The weight function is 1.

In Gauss Quadrature Methods: Gauss derived a formula which uses the same number of function values but with different spacing and gives better accuracy.

The weights and the nodes of the Gauss-quadrature formula are symmetrical with respect to the middle point of the given interval.

Gauss-Quadrature formulas were seldom used in practice before the advent of digital computers.

In Gauss-Quadrature formulas the nodes may not be equally spaced hence we relax this restriction and try to choose the nodes as well as the weights so as to maximize the order of integration method that is to maximize the degree for which all polynomials are

exactly integrated by
$$I(f) = \int_a^b w(x)f(x)dx \approx \sum_{k=0}^n \lambda_k f(x_k).$$

All the weights are positive.

If a single quadrature formula of high order is to be used the case for using some Gaussian formula is very good the reason for this is that the Gaussian formulas achieve higher accuracy with the use of fewer nodes than Newton-Cotes methods.

Example: Table of comparison of accuracy of Gaussian Quadrature and Newton-Cotes Quadrature (Closed) Formulas.

Order of accuracy n	Gauss-Legendre Formula(Quadrature)		Newton-Cotes Formula(Quadrature)	
	Number of nodes	Error coefficients	Number of nodes	Error coefficients
3	2	$\frac{1}{135}$	3	$-\frac{1}{90}$
5	3	$\frac{1}{15,701}$	5	$-\frac{1}{15,120}$
7	4	$\frac{1}{3,472,875}$	7	$-\frac{1}{3,061,800}$
9	5	$8.08 * 10^{-10}$	9	$-1.21 * 10^{-9}$

Finally as Conclusion we have:-By comparing the integration rules of Newton-Cotes and Gaussian methods we find that computational efforts being equal but Gaussian Quadrature Methods yields the most accurate results.

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