ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENTOF MATHEMATICS

HERMITE DIFFERENTIAL EQUATION AND ITS APPLICATIONS

By Kaleay Hagos

A project submitted to Department of Mathematics in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Advisor: Dr. Tadesse Bekeshie

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Approval

This project has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

Examining board members

Name	Signature	Date
1. Dr. Tadesse Bekeshie	(Advisor)__________	06/09/2018
2. Dr.Addisalem Abathun	(Examiner)__________	06/09/2018
3. Dr.Tesfa Biset	(Examiner) __________	06/09/2018
4. Dr. Tesfa Biset	(Chairperson)__________	06/09/2018

Permission

"This is to certify that this project is prepared by Kaleay Hagos in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that I have read all part of the project, so it can be submitted for the evaluation by examiners and eventual defense.

______________________________"

Advisor's signature

Declaration

"I declare that this project has been composed by me and that no part of the project has formed the basis for the award for any degree, associate ship, fellowship or any other similar title. All relevant sources of materials have been duly acknowledged.

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Acknowledgement

First and foremost I would like to give my thanks to God who gave me chance to live and have all such experiences in learning up to this level of which I did not think of.

Secondly; my heartfelt gratitude goes to my advisor Dr.Tadesse Bekeshie for his constructive comments and friendly approaches all through the work.

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Abstract

In this project we study various solution techniques of Hermite differential equations, properties of Hermite polynomials and physical application of Hermite equations and Hermite polynomials to quantum harmonic oscillator.

Introduction

This project consists of two chapters, each one divided into several sections .In the chapter one we present an introduction to second order homogeneous differential equations. In this chapter we consider basic definition and concepts, series solution of ordinary differential equations and some examples are also considered.

In the second chapter we give the definition and solution method of Hermite differential equation includes power series solution and Hermite polynomial. It also includes some properties of Hermite polynomial and their proof. Such as generating function, recurrence relation, ortho-normality and Rodrigues formulae.

Finally an application of Hermite differential equation to quantum harmonic oscillator is included.

CHAPTER 1

PRELIMINARIES

1.1 Basic definition and concepts

Definition 1.1: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1.1)$$

Examples: 1. $y'' - 2y' + y = 0$

2. $\frac{\partial^2 u}{\partial x^2} = -2\frac{\partial u}{\partial t}$

Classification of differential equations

(i) Ordinary differential equation: A differential equation which involves derivatives with respect to a single independent variables.
(ii) Partial differential equations: A differential equation which contains two or more independent variables and partial derivatives with respect to them.(PDE)

1.2 Series solutions of ordinary differential equations

1.2.1 Second –order linear ordinary Differential equations

Any homogeneous second order linear ODE can be written in the form

$$y'' + p(x)y' + q(x)y = 0, \qquad (1.2)$$

Where, p(x) and q(x) are given functions of x.

The most general

solution to Eq. (1.2) *is*

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where $y_1(x)$and $y_2(x)$ are linearly independet Solutions of Eq.(1.2), and c_1 *and* c_2 are constants. Their linear independence may be verified by the evaluation of the Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1 \qquad (1.3)$$

If W $(x) \neq 0$ *in a* given interval, then y_1 and y_2 are linearly independent in that interval.

1.2.2 Ordinary and singular points of an ODE

Definition 1.2: If x= x_0 is singularity of equation (1.2) and if (x- x_0)P(x) and $(x-x_0)^2$Q(x) result in functions, each of which is analytic at x=x_0, then the point x=x_0 is called a regular singular point of (1.2).

Definition 1.3: If x= x_0 is singularity of equation (1.2) and if (x- x_0) P(x) and $(x-x_0)^2$Q(x) result in functions, one or both of which are not analytic at x=x_0 , then the point x=x_0 is called an irregular point of (1.2)

Theorem 1.2.2 If x=0 is an ordinary point of the equation (1.2), then the general solution in an interval containing this point has the form

$y=\sum_{n=0}^{\infty} a_n\ x^n$

$Where\ a_0\ and\ \ a_1$ are arbitrary constants
and $y_1(x)$ and $y_2(x)$ are linearly independent functions analytic at x = 0

Consider the second-order linear homogeneous ODE

$$y''+p(x)y'+q(z)y=0, \tag{1.4}$$

where the functions are complex functions of a complex variable z. If at some point $z=z_0$ the functions $p(z)$ and $q(z)$ are finite, and can be expressed as complex power series

$$p(z)=\sum_{n=0}^{\infty} p_n(z-z_0)^n \quad , \quad q(z)=\sum_{n=0}^{\infty} q_n(z-z_0)^n,$$

then $p(z)$ and $p(z)$ are said to be analytic at $z=z_{0,}$ and this point is called an ordinary point of the ODE.

Even if an ODE is singular at a given point $z=z_0$, it may still possess a non- singular (finite) solution at that point. The necessary and sufficient condition for such a solution to exist is that $(z-z_0)p(z)\ and\ (z-z_0)^2q(z)$ are both analytic at $z=z_0$. Singular points that have this property are regular singular points, where as any singular point not satisfying both of these criteria is termed an irregular or essential singularity.

Series solutions about an ordinary point

If $z = z_0$. is an ordinary point of Eq. (1.4), then every solution y(z) of the equation is also analytic at $z = z_0$. We shall take z_0 as the origin. If this is not the case, then a substitution $Z = z - z_0$ will make it so. Then $y(z)$ can be written as

$$y(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (1.5)$$

Such a power series converges for $|z| < R, where\ R$ is the radius of convergence.

Since every solution of Eq. (1.4) is analytic at an ordinary point, it is always possible to obtain two independent solutions of the form, Eq. (1.5)

The derivatives of y with respect to z are given by

$$y' = \sum_{n=0}^{\infty} na_n\, z^{n-1} = \sum_{n=1}^{\infty}(n+1)a_n\, z^n \qquad (1.6)$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}z^n \qquad (1.7)$$

By substituting Eqs.$(1.5) - (1.6)$ Into the ODE Eq.(1.4), And requiring that the coefficients of each power of z sum to zero, we obtain a recurrence relation expressing a_n as a function of the previous $a_r(0 \le r \le n-1)$.

Example

Find the series solutions, about $z = 0$, of

$$y'' + y = 0.$$

Solution

Z=0 is an ordinary point of the equation, and so we may obtain two independent solutions by making the substitution

$y = \sum_{n=0}^{\infty} a_n z^n$

using Eq.(1.2) and (1.4), We find

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+a_n]z^n = 0$$

For this equation to be satisfied, we require that the coefficient of each power of z vanishes separately, and so we obtain the two –term recurrence relation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}\ for\ n \ge 0$$

Two independent solutions of the ODE may be obtained by setting either $a_0 = 0$ *or* a_1= 0. If we first set $a_1 = 0$ and choose $a_0 = 1$, then we obtain the solution

$$y_1(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

However, if we set $a_0 = 0$ and chose $a_1 = 1$, we obtain a second

independent solution

$$y_2(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

Recognizing these two series as $\cos z$ and $\sin z$, we can write the general solution as $y(z) = c_1 \cos z + c_2 sinz$

Where c_1 *and* c_2 are arbitrary constants

Example 2

$y' = 2xy$

Assume the series solutions as

$$y = a_0 + a_1 x + a_2\, x^2 + a_3 x^3 + \cdots + a_n\, x^n + \cdots = \textstyle\sum_{n=0}^{\infty} a_n\, x^n \qquad (1.5)$$

We would like to find a value for constant a.

We differentiate the series once

$$y' = a_1 + 2a_2 x + 3a_3\, x^2 + \cdots + na_n\, x^{n-1} + \cdots = \textstyle\sum_{n=1}^{\infty} na_n\, x^{n-1} \qquad (1.6)$$

Substitute Equ.(1.5) and Equ.(1.6) in to the given equations. We have

$$a_1 + 2a_2 x + 3a_3\, x^2 + \cdots = 2a_0 + 2a_1 x + 2a_2\, x^2 + \cdots$$

By collecting x in term, we get $a_1 = 0$ $\quad a_2 = a_0$

$a_3 = \frac{2}{3} a_1 \quad a_4 = \frac{1}{2} a_2$

Or in general $\quad na_n = 2a_{n-2}, \quad a_n = \begin{cases} 0, & odd\ n \\ \frac{2}{n} a_{n-2}, & even\ n \end{cases}$

Putting n= 2m, since only even term exists, we get

$$a_{2m} = \frac{2}{2m} a_{2m-2} = \frac{1}{m} a_{2m-2} \qquad \frac{1}{m} a_{2m-2} = \frac{1}{m} a_{2-4} = \ldots = .\frac{1}{m!} a_0 \ldots \qquad (1.7)$$

Substituting Equ.(1.7) in to Equ.(1.5), we get the assumed solutions.

$$y = a_0 + a_0 x^2 + \frac{1}{2!} a_0 x^4 + \ldots \frac{1}{m!} a_0 x^{2m} + \ldots = a_0\, \textstyle\sum_{m=0}^{\infty} \frac{x^{2m}}{m!} \ldots$$

CHAPTER 2

HERMITE DIFFERENTIAL EQUATION AND ITS APPLICATIONS

2.1 HERMITE DIFFERENTIAL EQUATIONS

The Hermite equation is the second order differential equation of the form

$$y'' - 2xy' + 2py = 0 \qquad (2\ .8)$$

where, the constant p can be any real number. This differential equation can be compared with the homogeneous second order linear differential equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2.9)$$

Where $p(x) = -2x \; and \; Q(x) = 2p$

Both $p(x)$ and $Q(x)$ are analytic at $x \in \mathbb{R}$, there are no singular points. To find an ordinary point, we check for same at x= 0

$\lim_{x\to 0}(x-0)\, p(x) = \lim_{x\to 0} x\,(-2x) = 0$ = Finite

And $\lim_{x\to 0}(x-0)^2 Q(x) = \lim_{x\to 0} x^2 2p = 0 =$ Finite (2.10)

Therefore x=0 is an ordinary point of the differential equation. We can find the power series solutions centered at x = 0 that converges for all $|x| < \infty$

2.2 Power series solution

We assume the solution is in the form of

$y = \sum_{m=0}^{\infty} a_m x^m = a_{0+} a_1 x + a_2 x^2 + a_3 x^3 \ldots + a_m x^m + \ldots$

And differentiate it to find y' and y'' , then substituting the series in to (2.8) leads

$$y' = \sum_{m=1}^{\infty} m a_m\, x^{m-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + m a_{m x^{m-1}}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} x^m$$

$$= 2a_2 + 6a_3 x \ldots \qquad + \; m(m-1) a_m x^{m-2}$$

We get the following expression for the individual terms on the left sides of the equation.

$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

$-2xy' = -2x \sum_{m=1}^{\infty} m a_m\, x^{m-1}$

$2py = 2p \sum_{m=0}^{\infty} a_m x^m$

Which implies that $y'' - 2xy' + 2py$ =0 leads to

$$\sum_{m=2}^{\infty}(m\ \ )(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty}(m)\, a_m x^{m-1} + 2p\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty}(m+2\ \ )(m+1)a_{m+2} x^{m} - 2\sum_{m=1}^{\infty}(m)\, a_m x^{m} + 2p\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\textstyle\sum_{m=0}^{\infty}(m+2\ \ )(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} 2p\, a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m = 0 \qquad (2.11)$$

$$\left[\sum_{m=0}^{\infty}(m+2\ \ )(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} 2p\, a_m x^m\right] - \sum_{m=1}^{\infty} 2m a_m x^m = 0$$

We separate out the x^0 terms and collect the coefficients of higher x^m to get

$$(0+2)(0+1)a_2 + 2pa_0 + \sum_{m=1}^{\infty}[(m+2)(m+1)a_{m+2}x^m + 2pa_m x^m - 2ma_m x^m] = 0$$

$$2(a_2 + pa_0) + \sum_{m=1}^{\infty}[(m+2)(m+1)a_{m+2} + 2(p-m)a_m]x^m = 0$$

Thus we have, $a_2 = -pa_0$&

$$a_{m+2} = \frac{2(m-p)}{(m+2)(m+1)}\mathrm{a_m} \qquad \forall \mathrm{m}=1,2,\ldots \qquad (2.12)$$

This recurrence formula also written as in the form of (using m-2 in place of m) to get

$$a_m = \frac{2(m-p-2)}{m(m-1)}a_{m-2}\ \ for\ m \geq 2 \qquad (2.13)$$

$$a_2 = \frac{-2p}{2.1}a_0$$

$$a_3 = \frac{-2(p-1)}{3.2}a_1$$

$$a_4 = \frac{-2(p-2)}{4.3}a_2 = \frac{2^2p(p-2)}{4!}a_0$$

And also $a_5 = \frac{-2(p-3)}{5.4}a_3 \ = \frac{2^2(p-1)(p-3)}{5!}a_1$

$a_6 = \dfrac{-2(p-4)}{6.5}a_4 = \dfrac{-2^3p(p-2)(p-4)}{6!}a_0$ and so on

Inserting these coefficient in to the power series

$$y = \sum_{m=0}^{\infty} a_m x^m, \quad \text{we obtain}$$

y= $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_{5x^5} + a_6x^6 + \ldots$

Hence, y=$a_0 + a_1x - \frac{2p}{2.1}a_0x^2 - \frac{2(p-1)}{3.2}a_1x^3 + \frac{2^2(p-2)}{4!}a_0x^4$

$$+\frac{2^2(p-1)(p-3)}{5!}a_1x^5 - \frac{2^3p(p-2)(p-4)}{6!}a_0x^6 + \cdots$$

$$y = a_0\left[1 - px^2 + \frac{p(p-2)}{6}x^4 - \frac{p(p-2)(p-4)}{90}x^6 + \cdots\right]$$

$$+ a_1\left[x - \frac{(p-1)}{3}x^3 + \frac{(p-1)(p-3)}{30}x^5 + \cdots\right]$$

Suppose $y_1(\text{even}) = 1 - px^2 + \frac{p(p-2)}{6}x^4 - \frac{p(p-2)(p-4)}{90}x^6 + \cdots$

And $y_2(\text{odd}) = x - \frac{p-1}{3}x^3 + \frac{(p-1)(p-3)}{30}x^5 + \cdots$

We have, y = $a_0Y_1 + a_1Y_2$ is the general solution of the Hermite differential equations.

Hence y_1 *and* y_2 are our solutions of the Hermite equation. When p is not an integer, each series in bracket has a radius of converges R=1

This is most easily seen by using the recursion formula

For the first series this formula with m replaced by 2m.

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}x^{2m+2}}{a_{2m}x^{2m}}\right| = \lim_{m\to\infty}\left|\frac{-2(p-2m)}{(2m+2)(2m+1)}\right||x|^2$$

$$= \lim_{x\to 0} 2|x|^2 = 0$$

Thus both series converges for all x.

Note: For p= n, where n is an integer, we find that Hermite differential equation becomes

$$y'' - 2xy' + 2ny = 0 \tag{2.18}$$

2.3 HERMITE POLYNOMIAL

The Hermite polynomial of order **n** is denoted and defined by

$$H_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}$$

Where $[\frac{n}{2}] = \begin{cases} n/2, if\ n\ is\ even \\ \frac{n-1}{2}\ , if\ n\ is\ odd \end{cases}$

The solutions of Hermites differential equations are often referred to as Hermite polynomials $H_n(x)$.

From equation.(1.6),the recurrence relation can be written for p=n as

$$a_{m+2} = \frac{-2(n-m)}{(m+2)(m+1)} a_m \qquad (2.9)$$

$for\ m = 0,1,2.3$ ...This then implies

Since $m = 0,$ $\quad a_2 = \frac{-2n_{a_0}}{2} = -n_{a_0}$

$m = 1,$ $\quad a_3 = \frac{-2(n-1)}{3.2} a_1 = \frac{-n+1}{3.2.1} a_1$

$m = 2$ $\quad a_4 = \frac{-2(\text{n}-2)}{(2+2)(2+1)} a_2$

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m= n-1, $\quad a_{n+1} = \frac{-2}{(\text{n}+1)\text{n}} a_{\text{n}-1}$

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For m= n, $\quad a_{n+2} = \frac{-2(0)}{(n+2)(n+1)} a_n = 0$

As m increases taking the integral value up to m=n we find the coefficients $a_{n+2} = 0$. Due to the recurrence relation it follows that all $a_{n+4} = a_{n+6} = a_{n+8} = \cdots = 0$.This means that the series terminates after a_n. An odd value of the parameter n, will lead to the termination of the odd series while if n is even then series will terminate i.e., depending on n the terminating solution will be either one of the linearly independent solutions y_{even} ory_{odd}.

This means that the terminating series will be called the Hermite polynomial $H_n(x)$.

We recall that the two linearly independents solution had the form.

$y_{even} = a_0\left[1+\frac{a_2}{a_0}x^2+\frac{a_4}{a_0}x^4+\frac{a_6}{a_0}x^6+\cdots+\frac{a_n}{a_0}x^n\right]$ And

$$y_{odd} = a_1 x\left[1+\frac{a_3}{a_1}x^3+\frac{a_5}{a_1}x^5+\frac{a_7}{a_1}x^7+\cdots+\frac{a_n}{a_1}x^n\right]$$

Therefore y=$a_0 H_n(x) + a_1 y_n(x)$ is the general solution. We must remember that out of the two lineally in depended solution if one solution is terminating, then the other solution we will be non-terminating.

The non-terminating solution is the Hermite polynomial of second kind which we represent as $y_n(x)$

Now, let's have a look at the Hermite polynomial

$$H_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} \ldots + a_{n-2r}x^{n-2r}$$

From equation (2.31), we have

$$a_m = \frac{(m+2)(m+1)}{-2(n-m)}a_{m+2} \qquad (2.20)$$

If n is the last non vanishing term putting m=n-2, n-4,..+ n-2r in Equ.(2.20)

We get $\quad a_{n-2} = \frac{-(n-1)n}{2(n-n+2)}a_n = \frac{-n(n-1)a_n}{2.2}$

$$a_{n-4} = \frac{-(n-2)(n-3)}{2(n-n+4)}a_{n-2} = \frac{(-1)^2 n(n-1)(n-2)(n-3)}{2^2 \qquad (4.2)}a_n$$

$$a_{n-6} = \frac{-(n-4)(n-5)}{2(n-n+6)}a_{n-4} = \frac{(-1)^3 n(n-1)(n-2)(n-3)(n-4)(n-5)}{2^3 \qquad (6.4.2)}a_n$$

And so on thus in general we can write

$$a_{n-2r} = \frac{(-1)^r n(n-1)(n-2)\ldots(n-2r+1)}{2^r(2r)\ldots 4.2}a_n$$

$$a_{n-2r} = \frac{(-1)^r n(n-1)(n-2)\ldots(n-2r+1)}{2^{2r} \qquad r\ldots 4..1}a_n$$

Multiply by $(n-2r)!$ in the numerator and denominator

We get $\quad a_{n-2r} = \frac{(-1)^r n(n-1)(n-2)\ldots(n-2r+1)(n-2r)!}{2^{2r} \qquad r!(n-2r)!}a_n$

It can be

$$a_{n-2r} = \frac{(-1)^r n!}{2^{2r}(n-2r)!r!}a_n$$

Assuming

$$a_n = 2^n$$

$a_{n-2r} = (-1)^r 2^{n-2r}\frac{n!}{(n-2r)!r!}$

We can now reach the form of the Hermite polynomials as

$H_n(x) = a_n x^n + a_{n-2}x^{n-2} + \ldots + a_{n-2r}x^{n-2r} + \cdots$

$$H_n(x) = \sum_{r=0}^{n/2 \, or \frac{n-1}{2}} a_{n-2r}x^{n-2r} = \sum_{r=0}^{n/2 \, or \frac{n-1}{2}} (-1)^r 2^{n-2r} \frac{n!}{(n-2r)!r!} x^{n-2r}$$

$H_n(x) = \sum_{\mathrm{r=0}}^{\mathrm{n}/2 \text{ or } \frac{\mathrm{n-1}}{2}} (-1)^r \frac{\mathrm{n!}}{\mathrm{(n-2r)!r!}} (2\mathrm{x})^{\mathrm{n-2r}}$ (2.21)

The summation from r=0 to r $=\frac{n}{2}$ for n even and from $r = 0$ to $\frac{\mathrm{n-1}}{2}$ for n odd

Now depending on n being even or odd we find the Hermite polynomial using equation (2.21)

$$H_0(x) = 1$$

$$H_1(x) = (2x)^1 = 2x$$

$$H_2(x) = (2x)^2 + (-1)\frac{2!}{(2-2)!1!}(4x)^{2-2} = 4x^2 - 2$$

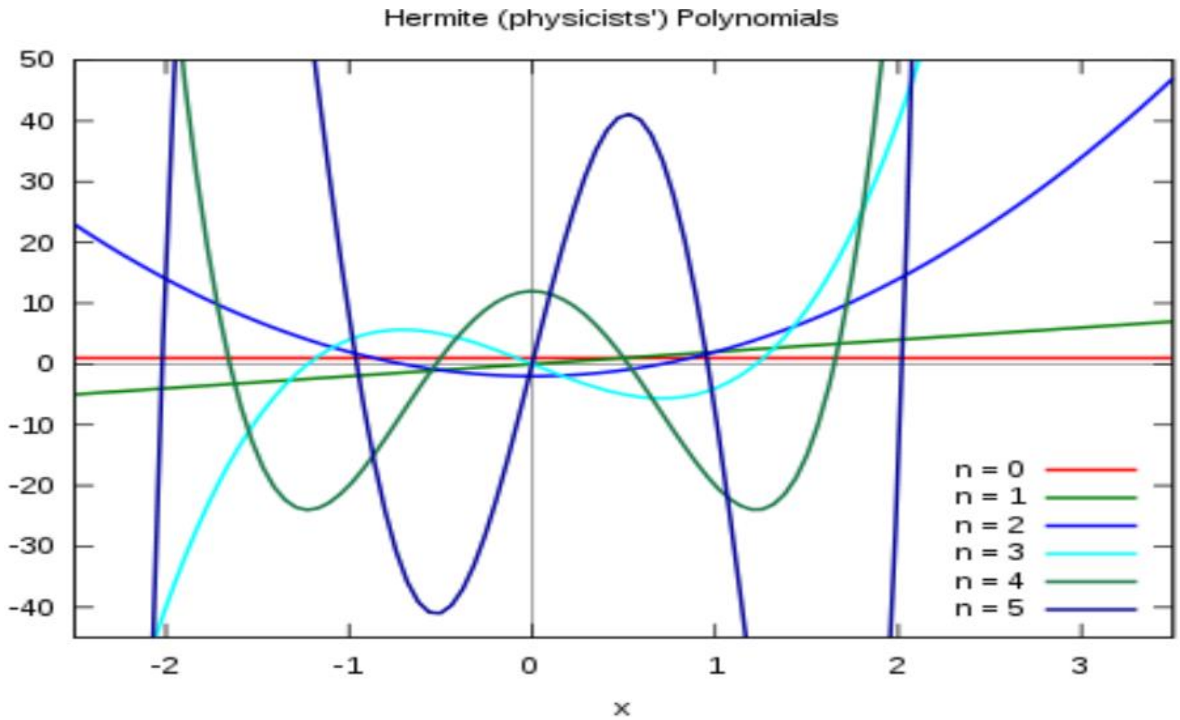
$$H_3(x) = (2x)^3 + (-1)\frac{3!}{(3-2)!1!}(2x)^{3-2} = 8x^3 - 12x$$

$$H_4(x) = (2x)^4 + (-1)\frac{4!}{(4-2)!1!}(2x)^{4-2} + (-1)^2\frac{4!}{(4-4)!2!}(2x)^{4-4}$$

$$=16x^4 - 48x^2 + 12$$

$$H_5(x) = (2x)^5 + (-1)\frac{5!}{(5-2)!\,1!}(2\mathrm{x})^{5-2}\frac{2_5!}{(5-4)!\,2!}(2x)^{5-4}$$

$$= 32x^5 - 160x^3 + 60x$$



Example: Solve the differential equation

$$y'' - 2xy' + 6y(x) = 0 \quad (2.22)$$

Step1: We compose the given equation with the Hermite –DE

$y''(x) - 2xy'(x) + 2ny(x) = 0$. In this example the parameter n= 3 and we find that x=0 is an ordinary point. The DE thus admits a Frobeninus series solution .

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \quad (2.23)$$

Step 2; since the parameter n is odd, we expect the Hermite polynomial solution to be a polynomial of odd power. The recurrence relation equation.(2.7)yields

$$a_m = \frac{2(3+2-m)}{m(m-1)} a_{m-2} = \frac{-2(5-m)}{m(m-1)} a_{m-2} \quad (2.24)$$

We get an in finite series for m= even as

$$a_2 = -3a_0 \qquad a_4 = -\frac{1}{6} a_2 \qquad a_6 = \frac{2}{6.5} a_4$$

While a finite polynomial for m= odd as

$$a_3 = \frac{-2}{3} a_1 \qquad a_5 = \frac{-2.0}{5.4} a_3 = 0 \qquad a_7 = \frac{2.2}{7.6} a_5 = 0 \quad (2.25)$$

Since the finite solution is that of the Hermite polynomial we write it as

$$H_3(x) = a_1 x \left(1 + \frac{a_3}{a_1} x^2 + 0\right) = a_1 x \left(1 - \frac{2}{3} x^2\right) \quad (2.26)$$

Step 3; we can determine directly in the polynomial form of

$H_n(x)$ by putting n=3

$H_3(x) = \sum_{r=0}^{\frac{3-1}{2}}(-1)^r \frac{3!}{(3-2r)!r!}(2x)^{3-2r} = \sum_{r=0}^{1}(-1)^r \frac{3!}{(3-2r)!r!}(2x)^{3-2r}$

$$\Rightarrow H_3(x) = (-1)^0 \frac{3!}{(3)!\,0!}(2x)^3 + (-1)^1 \frac{3!}{1!\,1!}(2x)$$

$H_3(x) = 8x^3 - 12x = -12x\left(1 - {}^2/_3\, x^2\right)$ (2.27)

on comparing equation (2.38) and equation (3.39),we find that the arbitrary coefficient $a_1 = -12$

2.4 PROPERTIES OF HERMITE POLYNOMIAL

2.4.1 **The generating function**

The function $G_H(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}$ is known as the generating function of all the Hermite polynomials.

Proof

We prove this starting from Theorem 2.4.1and expand the expontial function as

$$e^{2tx-t^2} = e^{2tx}e^{-t^2} = \left[\sum_{r=0}^{\infty} \frac{(2tx)^r}{r!}\right] x \left[\sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!}\right]$$

$$e^{2tx-t^2} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left[\frac{(2tx)^r}{r!}\frac{(-t^2)^s}{s!}\right]$$

$\mathrm{e}^{2\mathrm{tx}-\mathrm{t}^2} = \sum_{\mathrm{r}=0}^{\infty}\sum_{\mathrm{s}=0}^{\infty}\left[\frac{(-1)^{\mathrm{s}}}{\mathrm{r!s!}}(2\mathrm{x})^{\mathrm{r}\cdot}\mathrm{t}^{2\mathrm{s}+\mathrm{r}}\right]$

The above relation suggests that t can take powers from 0 to ∞, so if we introduce another variable n= 2s+r for powers of t and on replacing r by n-2s in equation (2.4.1), we get

$e^{2tx-t^2} = \sum_{n=0}^{\infty}\left\{\left[\frac{(-1)^s}{(n-2s)!s!}(2x)^{n-2s}t^n\right]\right\}$

Note that this would restrict s to a maximum value of ${}^n/_2$ for even value of n and$\frac{(n-1)}{2}$for odd value of. this is required as thevariable x would assume negativ$e$ powers and a minimum of 0 for the coefficient $\frac{(-1)^s}{(n-2s)!s!}$ to be well defined. Thus

$$e^{2tx-t^2} = \sum_{n=0}^{\infty}\sum_{s=0}^{{}^n/_2 or \frac{n-1}{2}}\left[\frac{(-1)^s}{(n-2s)!\,s!}(2x)^{n-2s}t^n\right]$$

$$= \sum_{n=0}^{\infty} t^n \left\{\sum_{s=0}^{{}^n/_2 or \frac{n=1}{2}}\left[\frac{(-1)^s}{(n-2s)!s!}(2x)^{n-2s}\right]\right\}$$

Companying with the form of the Hermite polynomial which we have read as

$$H_n(x) = \sum_{r=0}^{n/2 or \frac{n-1}{2}} (-1)^r \frac{n!}{(n-2r)!\, r!} (2x)^{n-2r}$$

We note that, we can write

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} t^n \left\{\frac{H_n(x)}{n!}\right\} \quad (2.28)$$

Thus the function e^{2tx-t^2} generates all the Hermite polynomials and hence is called the generating functions of Hermite polynomial.

2.4.2 The Recurrence Relation of the Hermite polynomials

The generating function for the Hermite polynomials yields some interesting relations among the polynomials and their derivatives. They are known as the recurrence relation of the polynomials.

Theorem 2.4.2

(i) $H'_n(x) = 2nH_{n-1}(x)(n \geq 1)$
(ii) $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)(n \geq 1);$

Proof

(i) Let's see how to get this from the generating function $G_H(x,t)$ of Hermite polynomials

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad (2.29)$$

Differentiating with respect x, we get

$$e^{-t^2} e^{2tx}. 2t = \sum_{n=0}^{\infty} \frac{t^n}{n!} H'_n(x)$$

$$2t\, e^{2tx} e^{-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, H'_n(x) \quad (2.30)$$

Since e^{2tx-t^2} is generating function we use (2.29) in (2.30) to get

$$2xt \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (x) H'_n(x)$$

$$= \sum_{n=0}^{\infty} 2 \frac{t^{n+1}}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, H'_n(x) \quad (2.31)$$

Displacing the dummy count n by n-1 on the left hand sides

$$LHS = \sum_{n-1=0}^{\infty} 2\frac{t^n}{(n-1)!}H_{n-1}(X) = \sum_{n=1}^{\infty} 2\frac{t^n}{(n-1)}H_{n-1}(x)$$

But since the contribution of the above summation for n=0 is zero we can rewrite out of brevity that

$LHS = \sum_{n=0}^{\infty} 2\frac{t^n}{(n-1)!}H_{n-1}(X)$ (2.32)

Thus equation (2.4.23) can be rewritten as

$$\sum_{n-1=0}^{\infty} 2\frac{t^n}{(n-1)!}H_{n-1}(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}H'_n(x)$$

Comparing the coefficient of t^n we can written in general

$$\frac{2H_{n-1}(x)}{(n-1)!} = \frac{H'_n(x)}{n!}$$

Which when simplified yields the recurrence relation

$H'_n(x) = 2nH_{n-1}(x)$ (2.33)

- between the polynomial them selves
  $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$, Again from the generating function $G_H(x,t)$ of the Hermite polynomials we have
  $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}H_n$ (2.34)

Differentiating with respect to t

$$e^{2tx-t^2}x(2x-2t) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!}H_n(x)$$

$2(x-t)e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!}H_n(x)$ (2.35)

Since $e^{2tx-}t^2$ is the generating function we use (2.4.26) in (2.4.27) to get

$$2(x-t)x\sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!}H_n(x)$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}2xH_n(x) - \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!}2H_n(x) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!}H_n(x)$$

$\sum_{n=0}^{\infty}\left[\frac{2x}{n!}H_n(x)\right]t^{n+1} - \sum_{n=0}^{\infty}\left[\frac{2}{n!}H_n(x)\right]t^{n-1} - \sum_{n=0}^{\infty}\left[\frac{n}{n!}H_n(x)\right]t^{n-1} = 0$

Displacing the dummy count n by n-1 in the 2[nd] term and dummy count n by n+1 in the 3[rd] term, we rewrite

$$\sum_{n=0}^{\infty}\left[\frac{2x}{n!}H_n(x)\right]t^n - \sum_{n=0}^{\infty}\left[\frac{2}{(n-1)!}H_{n-1}(x)\right]t^n - \sum_{n=0}^{\infty}\left[\frac{n+1}{(n+1)!}H_{n+1}(x)\right]t^n = 0$$

there fore, $$\sum_{n=0}^{\infty}\left[\frac{2x}{n!}H_n(x) - \frac{2}{(n-1)!}H_{n-1}(x) - \frac{(n+1)}{(n+1)!}H_{n+1}(x)\right]t^n - \sum_{n=0}^{\infty}\left[\frac{2}{n!}H_n(x)\right]t^{n=0}$$

Equation the coefficient of t^n to zero we get

$$\frac{2x}{n!}H_n(x) - \frac{2}{(n-1)!}H_{n-1}(x) - \frac{(n+1)}{(n+1)!}H_{n+1}(x) = 0$$

Multiply by n! Throughout, we get

$$2xH_n(x) - \frac{2n!}{(n-1)!}H_{n-1}(x) - \frac{(n+1)}{(n+1)!}H_{n+1}(x) = 0$$

$$2xH_n(x) - 2nH_{n-1}(x) - H_{n+1}(x) = 0$$

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \qquad (2.36)$$

Using the first two recurrence relations to get a relation between the polynomials and its derivative;

$$H'_n(x) = 2xH_n(x) - H_{n+1}(x)$$

We rewrite the previous two recurrence relations as

$$H'_n(x) = 2nH_{n-1}(x)$$

And $2xH_n(x) = 2nH_{n-1}(x) - H_{n+1}(x) + H_{n+1}(x)$

Replacing $2nH_{n-1}(x)$ in(2.4.28) by $H'_n(x)$in(2.4.25)

We get a new recurrence relation

$$H'_n(x) = 2xH_n(x) - H_{n+1}(x) \qquad (2.37)$$

Example: the Hermite equation $y'' - 2xy' + 2ny = 0$ has Hermite polynomials as its solution. They themselves satisfy a recurrence relation. Establish it as

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

Step1: the third recurrence relation is

$H'_n(x) = 2xH_n(x) - H_{n+1}(x)$ From (2.37)

Step 2: differentiating w.r.t. x, we get

$$H''_n(x) = \{2xH'_n(x) + 2H_n(x) - H_{n+1}(x)$$

Step3: using the first two recurrence relation after replacing n by n+1, we get

$$H''_n(x) = \{2xH'_n(x) + 2H_n(x)\} - 2(n+1)H_n(x)\}$$

$$H''_n(x) = \{2xH'_n(x)\} - \{2nH_n(x)\}$$

Step 4: on rearrangement. We get

$$H''(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

2.4.3 The Rodrigues formula for Hermite polynomials

Theorem 2.4.3

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Let's see how to get this from the generating function again which we re-express

$$GH(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

And expanding the summation we get

$$GH(x,t) = \frac{H_0(x)}{0!}t^0 + \frac{H_1(x)}{1!}t^1 + \frac{H_2(x)}{2!}t^2 + \frac{H_3(x)}{3!}t^3 + \cdots . + \frac{H_n(x)}{n!}t^n + \cdots$$

Differentiating partially $GH(x,t)$with respect to t once

$$\frac{\partial}{\partial t} GH(x,t) = \frac{H_1(x)}{1!} + \frac{H_2(x)}{2!}2t + \frac{H_3(x)}{3!}3t^2 + \frac{H_4(x)}{4!}4t^3 + \cdots .. + \frac{H_n(x)nt^{n-1}}{n!} + \cdots$$

At t=0 its values is

$$\frac{\partial}{\partial t} GH(x,t)|_{t=0} = H_1(x)$$

Differentiating partially $GH(x,t)$ with respect to t twice

$$\frac{\partial^2}{\partial t^2} GH(x,t) = \frac{H_2(x)}{1!} + \frac{H_3(x)}{2!}2t$$

$$\frac{\mathrm{H_4(x)}}{3!}3\mathrm{t}^2+\cdots+\frac{\mathrm{H_n(x)}}{(\mathrm{n}-1)!}(\mathrm{n}-1)\mathrm{t}^{\mathrm{n}-2}+\cdots \text{ at } \mathrm{t}=0 \quad \text{its value is}$$

$$\frac{\partial^2}{\partial t^2}GH(x,t)|_{t=0.}=H_2(x)$$

Likewise if we differentiate for some n times and then substitute t=0 we get

$$\frac{\partial^n}{\partial t^n}GH(x)|_{t=0}=H_n(x)$$

Now, we get

$$H_n(x)=\frac{\partial^n}{\partial t^n}\ GH\ (x\ ,t)\Big|_{t=0}=\frac{\partial^n}{\partial t^n}\{e^{2tx-t^2}|_{t=0}=\frac{\partial^n}{\partial t^n}\{e^{2tx-t^2-x^2+x^2}\}$$

$$=\frac{\partial^n}{\partial t^n}\{e^{-(t-x)^2+x^2}|_{t=0}H_n(x)=\left[e^{x2}\frac{d^n}{dt^n}\{e^{-(t-x)^2}\}\right]\Big|_{t=0}$$

Therefore by substituting y = x-t, we find

$$H_n(y+t)=\left[e^{(y+t)^2}\frac{d^n}{d(-y)^n}\{e^{-y^2}\}\right]\Big|_{t=0}$$

$$H_n(y)=(-1)^n e^{y2}\frac{d^n}{d_y n}e^{-y2}$$

This differential form of the Hermite polynomials is called the Rodrigues formula

Example: determine the Hermite polynomials from the Rodrigues formula

$$H_n(y)=(-1)^n e^{y2}\frac{d^n}{d_y n}e^{-y2}$$

Step1: we verify the correctness of this formula by taking different values of n

$$H_0(y)=e^{y2}\frac{d^0}{dy^0}e^{-y2}=e^{-y2}xe^{-y2}=1$$

$$H_1(y)=-e^{y2}\frac{d}{dy}e^{-y2}=-e^{y2}e^{-y2}(-2y)=2y$$

$$H_2(y)=e^{y2}\frac{d^2}{dy^2}e^{-y2}=e^{y2}\frac{d}{dy}\left(-2ye^{-y^2}\right)$$

$$=-2e^{y2}\left(e^{-y2}-2y^2e^{-y2}\right)$$

$$H_2(y) = 4y^2 - 2$$

2.4.4 Ortho-normality of the Hermite polynomials

The Hermite polynomials are orthonormal to each other making a complete orthonormal set.

Theorem 2.4.4

$\int_{-\infty}^{\infty} H_n(x)H_m(x)\, e^{-x^2}dx = 2^n n!\sqrt{\pi}\sigma_{nm}$.

Where σ_{nm} is the Kronecker delta function σ_{nm}= 1 $\forall_n$=m,$\sigma_{nm} = 0\forall_{n \neq m}$

Proof

To prove this we start with the fact that all the Hermite polynomials satisfy the equation

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

If we multiply throughout by e^{-x^2} and rearranging subsequently we get

$$e^{-x^2}H''_n(x) - 2xe^{-x^2}H'_n(x) + 2ne^{-x^2}H_n(x) = 0$$

$$e^{-x^2}H''_n(x) + H'_n(x)\frac{d}{dx}e^{-x^2} + 2ne^{-x^2}2nH_n(x) = 0$$

$$\frac{d}{dx}\left\{e^{-x^2}H'_n(x)\right\} + 2ne^{-x^2}H_n(x) = 0$$

If we take two Hermite polynomials of order n & m, then for both

$$\frac{d}{dx}\left\{e^{-x^2}H'_n(x)\right\} + 2ne^{-x^2}H_n(x) = 0$$

$$\frac{d}{dx}\left\{e^{-x^2}H'_m(x)\right\} + 2me^{-x^2}H_m(x) = 0$$

And up on multiplication by $H_m(x)\ and\ H_n(x)$ respectively we get

$$H_m(x)\frac{d}{dx}\left\{e^{-x^2}H'_n(x)\right\} + 2ne^{-x^2}H'_m(x)H_n(x) = 0$$

$$H_n(x)\frac{d}{dx}\left\{e^{-x^2}H'_m(x)\right\} + 2me^{-x^2}H_n(x)H_m(x) = 0$$

Subtracting one from the other

$$H_m(x)\frac{d}{dx}\left\{e^{-x^2}H'_n(x)\right\} - H_n(x)\frac{d}{dx}\left\{e^{-x^2}H'_m(x) + 2(n-m)e^{-x^2}H_n(x)H_m(x) = 0\right.$$

Which we rewrite as

$$\left[\frac{d}{dx}\left\{H_m(x)e^{-x2}H'_n(x)\right\} - \left\{e^{-x2}H'_n(x)\right\}\left\{H'_n(x)\right\}\right] -$$

$$\left[\frac{d}{dx}\left\{H_n(x)e^{-x2}H'_m(x)\right\} - \left\{e^{-x2}H'_m(x)\right\}\left\{H'_n(x)\right\}\right]$$

$$+2(n-m)e^{-x2}H_n(x)H_m(x) = 0$$

$$\frac{d}{dx}\left[H_m(x)e^{-x2}H'_n(x) - H_n(x)e^{-x2}H'_m(x)\right] + 2(n-m)e^{-x2}H_n(x)H_m(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left\{e^{-x2}\left[H_m(x) - H'_n(x)\right]\right\} +$$

$$2(n-m)e^{-x2}H_n(x)H_n(x) = 0$$

Now integrating the above over the limits $-\infty\ \ to\ \infty$ we find

$$\int_{-\infty}^{\infty} dx\ \frac{d}{dx}\left\{e^{-x2}\left[H_m(x)H'_n(x) - H_n(x)H'_m(x)\right]\right\}$$

$$+\int_{-\infty}^{\infty} 2(n-m)e^{-x2}\,H_n(x)H_m(x)dx = 0$$

$$\left\{e^{-x2}\left[H_m(x)H'_n(x) - H_n(x)H'_m\right]\right\}\Big|{-\infty}^{\infty}\ +\ \int_{-\infty}^{\infty} 2(n-m)e^{-x2}\,H_n(x)H_m(x)dx = 0$$

There fore

$$0 + \int_{-\infty}^{\infty} 2(n-m)e^{-x2}\,H_n(x)H_m(x)dx = 0$$

$$(n-m)\int_{-\infty}^{\infty} e^{-x2}\,H_n(x)H_m(x)dx = 0$$

If $n \neq m, then\ we\ must\ have$

$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0$ (2.38)

Which is the **orthogonally conditions**

Now from the generating function of the Hermite polynomials we can white

$$G_H(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}$$

$$G_H(x,s) = e^{2tx-s^2} = \sum_{m=0}^{\infty}\frac{H_n(x)s^m}{n!}$$

Multiplying the above two together along with the function e^{-x^2} $we\ get$

$$e^{-x^2}e^{2tx-t^2}\ e^{2tx-s^2} = e^{-x^2}\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n\sum_{m=0}^{\infty}\frac{H_m(x)}{m!}s^m$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-x^2}\frac{H_n(x)}{n!}\frac{H_m(x)}{m!}t^n s^m$$

Integrating it for the variable x over the limits $-\infty\ to\ \infty\ we\ get$

$$\int_{-\infty}^{\infty}e^{-x^2}\,e^{2tx-t^2}e^{2sx-s^2}dx = \int_{-\infty}^{\infty}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-x^2}\frac{H_n(x)}{n!}\frac{H_n(x)}{m!}t^n s^m\right]dx$$

Taking the summations out we write

$$\int_{-\infty}^{\infty}e^{-x^2}\,e^{2tx-t^2}e^{2sx-s^2}dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[\int_{-\infty}^{\infty}e^{-x^2}\frac{H_n(x)}{n!}\frac{H_m(x)}{m!}dx]t^n s^m$$

Separating the RHS for n=m and n$\neq m\ we\ get$

$$\int_{-\infty}^{\infty}e^{-x^2}\,e^{2tx-t^2}e^{2sx-s^2}dx$$

$$=\sum_{n=0}^{\infty}\left[\int_{-\infty}^{\infty}e^{-x^2}\left(\frac{H_n(x)}{n!}\right)^2dx\right]t^n s^n$$

$$+\sum_{\substack{n=0,m=0\\ n\neq m}}^{\infty}\frac{1}{n!\,m!}\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\}dxt^n s^m$$

The second summation on the RHS vanishes by virtue of orthogonally and therefore we rewrite.

$$\sum_{n=0}^{\infty}\left[\int_{-\infty}^{\infty}e^{-x^2}\left(\frac{H_n(x)}{n!}\right)^2dx\right]t^n t^n$$

$$=\int_{\infty}^{\infty}e^{-x^2}e^{2tx-t^2}e^{2sx-s^2}dx = \int_{-\infty}^{\infty}e^{2tx-x^2+2sx-s^2}dx$$

$$\sum_{n=0}^{\infty}\left[\int_{-\infty}^{\infty}e^{-x^2}\left(\frac{H_n(x)}{n!}\right)^2dx\right]t^n s^n = \sum_{-\infty}^{\infty}e^{-t^2}e^{2tx-(x-s)^2}dx$$

$$= \sum_{-\infty}^{\infty} e^{-t^2} e^{2t(y+s)-y^2}$$

Where we have put x= y +s

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2} \left(\frac{H_n(x)}{n!} \right)^2 dx \right] t^n = e^{2ts} \int_{-\infty}^{\infty} e^{-t^2+2ty+-y^2} dy = e^{2ts} \int_{-\infty}^{\infty} e^{-(y-t)^2} dy = e^{2ts} \sqrt{n}$$

Now expanding the exponential term in the RHS we find

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2} \left(\frac{H_n(x)}{n!} \right)^2 dx \right] t^n s^n = \sqrt{\pi} \left[\sum_{n=0}^{\infty} \frac{(2ts)^2}{n!} \right]$$

$$= \sum_{n=0}^{\infty} \left[\sqrt{\pi} \frac{2^n}{n!} \right] t^n s^n$$

Thus equating the coefficients $of\ t^n\ s^n\ on\ both\ sides$

$$\int_{-\infty}^{\infty} e^{-x^2} \left(\left(\frac{H_n(x)}{n!} \right)^2 dx = \sqrt{\pi} \frac{2^n}{n!} \right)$$

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n n! \sqrt{\pi} \qquad (2.39)$$

Combining the Equ.(2.4.41) and the Equ.(2.4.42) we can rewrite them as.

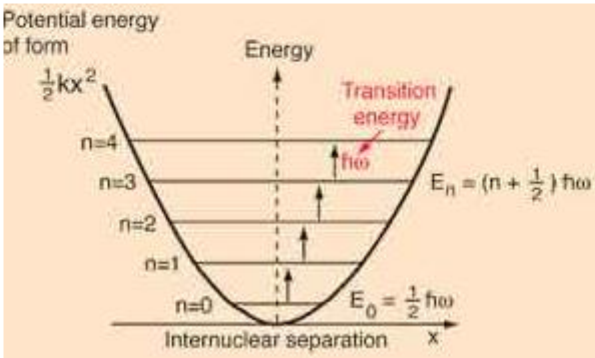
$$\int_{-\infty}^{\infty} e^{-x^2} H_{n(x)}\ H_{m(x)} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

2.5 APPLICATION

2.5.1 Quantum Harmonic Oscillator

The quantum harmonic oscillator is quantum mechanical analog of the classic harmonic.

A diatomic molecule vibrates somewhat like too masses on a spring with potential energy that depends up on the square of the displacement from equilibrium. But the energy levels are quantized at equally spaced values.



X= 0 represents the equilibrium separation between the nuclei.

The energy level of the quantum molecule the natural frequency of the form

Angular frequency $(\omega) = \sqrt{\frac{k}{m_r}}$ k= bond of force constant and m_r =reduced mass

Where the reduced mass is given by $m_r = \frac{m_1 \, m_2}{m_1 + m_2}$

This form of the frequency is the same as that for the classical simple harmonic oscillator.

The most surprising difference for the quantum case is so called "zero- point vibration" of the n=0 ground state. This implies that molecules are not completely at rest. Even at absolute zero temperature.

The quantum harmonic oscillator has implications far beyond the simple diatomic molecule.

The connection of Hermite polynomial with quantum Harmonic oscillator. First of all, the analogue of the classical Harmonic oscillator in quantum mechanics is described by the Schrodinger equation

$$\psi'' + \frac{2M}{h^2}\big(E - v(y)\big)\psi = 0 \tag{2.40}$$

Where ψ is the state of a particles of mass M in the potential v(y), with energy E .we will suppose that the potential has the form $v(y) = y^2$, and therefore .we consider the following equation;

$$\psi'' + \frac{2M}{h^2}(E - y^2)\psi = 0 \tag{2.41}$$

In order to simplify this equation, we make a change of variable y= k x equation is transformed to; $\psi'' - \frac{2Mk^2}{h^2} x^2\psi = -\frac{2M}{h^2} E\psi$

Where the differentiation is now with respect to the new variable the constant k appropriately our equation becomes;

$$\psi'' - x^2\psi = \beta\psi \quad (2.42)$$

Where $\beta = -\sqrt{\frac{2M}{h^2}} E$

Equation (2.5.12) is a second order differential equation with variable coefficients. To solve this equation, we first notice the $\psi_{*(x)} = e^{\frac{-x^2}{2}}$ is a solution of the differential equation $\psi'' - x^2\psi = -\psi$. Using the method of variation of parameters,

$$\psi_{n(x)} = \psi_*(x) H_{(x)} \quad (2.43)$$

Here $\psi_{*(x)}$is the solution defined above, and H(x) is a function to be determined.

To find the form of $H(x)$, We substitute $\psi_{n(x)}$ given by (2.43) in to equation (2.42);

$$H'' - 2xH' + (\beta - 1)H = 0 \quad (2.44)$$

Setting $\beta - 1 = 2n \Leftrightarrow \beta = 2n + 1$in (2.44), we obtain none other than the Hermite differential equation (2.11) whose solution are H(x) =$H_n(x)$ the polynomials.

SUMMARY

A homogenous second order linear differential equation of the type

$$y'' - 2xy' + 2py = 0$$

Where, the constant p can be any real number is known as Hermite differential equation. This differential equation has x= 0 as ordinary point and the series solution for it is

$$y(x) = \sum_{n=0}^{\infty} a_m x^m$$

The recursive relation is $a_m = \frac{2(m-p-2)}{m(m-1)} a_{m-2}$ for m= 2, 3, 4, 5 ,..

Thus solution to the Hermite DE is

$y = a_0 \mathrm{y_{even}} + a_1 y_{odd}$,With linearly independent solutions

$$y_{even} = 1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \frac{a_6}{a_0}x^6 + \cdots$$

$$y_{odd} = x(1 + \frac{a_3}{a_1}x^{,3} + \frac{a_5}{a_1}x^5 + \frac{a_7}{a_1}x^7 + \cdots$$

The form of the Hermite polynomial (p=n)

$$H_{n(x)} = \sum_{r=0}^{n/2 \; or \; \frac{n-1}{2}} (-1)^r \frac{n!}{(n-2r)!r!}(2x)^{n-2r}$$

The recurrence Relations of the Hermite polynomials are

- Between the polynomial and its derivative ;$H'_{n(x)} = 2nH_{n-1(x)}$
- Between polynomial themselves; $2xH_{n(x)} = 2nH_{n-1(x)} + H_{n+1(x)}$
- Using the first two recurrence relation to get a relation between the polynomial and its derivative ; $H'_{n(x)} = 2xH_{n(x)} - H_{n+1(x)}$

The Rodrigues Formula for Hermite polynomials has the form

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

The Hermite polynomials are orthonormal to each other making a complete orthonormal set

$$\int_{-\infty}^{\infty} H_n(x) H_m(x)\, e^{-x^2} dx = 2^n n! \sqrt{\pi} \sigma_{nm}$$

Where σ_{nm} is the Kronecker delta function σ_{nm}= 1 $\forall_n$=m &$\sigma_{nm} = 0 \; \forall_{n \neq m}$

The Hermite differential equations have applications in solving quantum harmonic oscillator problems.

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