



College of Natural and Computational Sciences  
Department of Mathematics

# *Autometrized Lattice Ordered Monoids*

By: **Tekalign Regasa Ashale**

Supervisors: **Kolluru Venkateswarlu** (Professor)

Andhra University, India

**Girum Aklilu Abebe** (PhD)

Addis Ababa University, Ethiopia

A dissertation submitted to the Department of Mathematics in partial fulfillment to the requirement for the degree of Doctor of Philosophy in Mathematics

June 2025

Addis Ababa, Ethiopia

# Declaration

I, Tekalign Regasa Ashale, with student number GSR/6227/14 hereby declare that this dissertation is my original work under the supervision of Prof. Kolluru Venkateswarlu and Dr. Girum Aklilu and has not been presented for a degree in any other university and that all sources of information used for the thesis have been fully acknowledged.

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Tekalign Regasa Ashale

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Date

# Dedication

In loving memory of **Dr. Girum Aklilu Abebe**, late Department head of Mathematics, whose guidance, wisdom, and unwavering support shaped my academic journey. Your passion for knowledge and dedication to your students inspired me every day. Though you are no longer with us, your legacy lives on in the hearts and minds of those you touched. This work is a testament to your belief in me and the countless lessons you imparted. Thank you for everything; you will be deeply missed but never forgotten.

# Certificate

This is to certify that the dissertation prepared by Tekalign Regasa Ashale entitled: "Autometrized Lattice Ordered Monoids" submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the Examining Committee:

**Prof. Kolluru Venkateswarlu**

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

**Dr. Girum Aklilu Abebe**

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

**External Examiner:** \_\_\_\_\_

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

**Internal Examiner:** \_\_\_\_\_

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

Chair person: **Dr. Markos Fisseha Yimer**, Post graduate coordinator, AAU

Signature: \_\_\_\_\_

Date: \_\_\_\_\_

# Acknowledgements

I would like to express my profound gratitude to everyone whose support and advice have made it possible for me to complete this Ph.D. First and foremost, I thank my supervisor, Professor Kolluru Venkateswarlu, for the opportunity to work alongside him on this study. His guidance and wise counsel over the past four years have been invaluable; without him, I could not have completed this thesis. I truly appreciated the relaxed and friendly atmosphere in our working group, which made my integration into the academic and scientific community seamless. His passion for the research subject served as a tremendous motivation for me, and his consistent guidance has kept me on track throughout this journey. I am deeply grateful for his encouragement, support, and insightful feedback, which have significantly enhanced my work from planning to completion.

I would like to extend my heartfelt thanks to Dr. Zelalem Teshome Wale for his invaluable feedback and insightful comments on my dissertation. His guidance has been instrumental in refining my work and enhancing the overall quality of this document.

I would also like to extend my heartfelt thanks to the entire mathematics department staff at Addis Ababa University, particularly Dr. Samuel Aseffa, Dr. Zelalem Teshome, Dr. Berhanu Bekele, Dr. Tilahun Abebaw, Dr. Yibeltal Yitayew, Dr. Bahru Tsegaye, Dr. Manalebsh, Dr. Tadesse Abdi, Dr. Tsegaye G. Ayele, and Dr. Yirgalem Tsegaye for their encouragement and guidance from the very beginning of the program.

A special thanks goes to the Department of Mathematics at Addis Ababa University for providing me the opportunity to join the Ph.D. program and for their financial support. I am also grateful to my fellow Ph.D. students—especially Oluma Ararso, Dr. Zebene Girma, Dr. Sagni Ergabus, Dr. Fufa Beyene, Amanuel Mamo, Timotewos Tone, Gezahegn Amberber, Hiwot Reda, Bezuneh Minda, Beadgelign Asmare, Tesfaye Degiffie, and Simegnesh Demelash. Our discussions have been incredibly thought-provoking.

I want to express my sincere appreciation to all members of the university, particularly the

---

mathematics department staff, for their kindness and hospitality during my time there.

I acknowledge support from the Eastern Africa research group and International research group(ISP), part of this Dissertation was written where I visiting Makerere University. Specially I thanks Proffessor David Sseviri from Department of Mathematics, Makerere University.

I would like to express my heartfelt gratitude to my wife, Addisalem Haile Abebe, for her immense patience during difficult times. I also want to acknowledge my son, Christian Tekalign, who was born just as I began my PhD program. I was away from him for about a year, and I truly missed him.

Lastly, I am immensely grateful to my parents, brothers, and sisters for their love, prayers, and sacrifices in supporting my education. A special thank you to my mother, Shume Abdisa, and my father, Regasa Ashale, for their invaluable advice and for keeping me grounded during challenging times. Thank you for being the most wonderful people in my life

Thank you all !!!

Tekalign Regasa Ashale

Addis Ababa University

Addis Ababa, Ethiopia.

# Abstract

In this dissertation, we introduce autometrized lattice ordered monoids (AL-monoids), a new generalization of dually residuated lattice ordered semigroups (DRL-semigroups). We explore various algebraic properties of AL-monoids and investigate isometries within this structure. A key finding is that the set of invertible elements in an AL-monoid forms an l-group, while the set of complemented elements constitutes a Boolean algebra. We further establish that an AL-monoid  $A$  with a unity element is a Boolean algebra if the mapping  $x \mapsto a * x$  is an isometry for every  $a \in A$ . The geometric aspects of AL-monoids are examined through the introduction of metric betweenness and related concepts, including B-linearity, D-linearity, lattice betweenness, segments, and equilateral triangles. Notably, we prove that equilateral triangles cannot exist in AL-monoids, subsume the geometric properties of commutative DRL-semigroups. Moreover, we define various types of ideals within AL-monoids, such as polar ideals, regular ideals, prime ideals, and annihilators, and elucidate their interconnections. We introduce the value of an element for an ideal  $\mathbb{I}(A)$  and characterize regular ideals based on this value. Finally, we provide characterizations of polar ideals concerning minimal prime ideals, annihilators, and maximal polar ideals, contributing to a deeper understanding of the structure and relationships within AL-monoids.

# List of Publications

The following papers, which are organized as chapters in this thesis, were written.

1. Tekalign Regasa, Girum Aklilu, Kolluru Venkateswarlu, "Autometrized Lattice Ordered Monoids", Southeast asian bulletin of mathematics, 49 (2025),(accepted for publication).
2. Tekalign Regasa, Girum Aklilu, Kolluru Venkateswarlu, "Geometry of Autometrized Lattice Ordered Monoids", SINET: Ethiopian Journal of Science, 48; 1-8 (2025),(Published)
3. Tekalign Regasa, Girum Aklilu, Kolluru Venkateswarlu, "Certain Ideals of Autometrized Lattice Ordered Monoids" ,(communicated).

# List of abbreviations

<i>l</i> – group	Lattice ordered group
<i>l</i> – monoids	Lattice ordered monoids
<i>l</i> – semigroups	Lattice ordered semigroups
<i>AL</i> – monoids	Autometrized lattice ordered monoids
<i>DRL</i> – semigroups	Dually residuated lattice ordered semigroups

# List of Symbols

$<$	Less than
$\leq$	Less than or equal to
$\cap$	The intersection
$\cup$	The union
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of natural numbers
$\mathbb{P}(A)$	The Power set of a set $A$
$\mathbb{V}(I)$	The valuation of an ideal $I$
$I(A)$	The collections of all ideals of $A$
$J_\alpha$	The ideal containing an element $A$
$(a, b, c)_L$	The Lattice betweenness
$(a, b, c)_M$	The metric betweenness
$\oplus$	the direct sum

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# Chapter I

## Introduction

The study of dually residuated lattice ordered semigroups (DRL-semigroups) was initiated by Swamy [23] as a response to Birkhoff's inquiry in his book [1], which asked whether there exists a common abstraction that includes both Boolean algebras (rings) and l-groups as special cases. This question has led to various solutions, including Clans by Wyler [30], multi-rings by Nakano [9], and a common abstraction proposed by Rama Rao [19,20], who identified a system that represents the direct product of a Boolean ring and an l-group.

In a significant development, Swamy introduced autometrized algebras [28] as a unified theory encompassing Brouwerian algebras [10], commutative l-groups, and Boolean  $\ell$ -algebras [19]. These algebras, which are lattices, possess a metric known as symmetric difference. However, some other algebras, such as semi-Brouwerian algebra [8] and Newmann algebra [21], also exhibit a metric but do not maintain the structure of lattices.

Subba Rao [22] further explored this realm by introducing representable autometrized algebras, focusing on their geometric and algebraic properties. More recently, M. Melese et.al., [29] investigated the structure of autometrized algebras by introducing subalgebras and examining their ideals, homomorphisms, quotient algebras, and the isomorphism theorem.

Lattice ordered autometrized algebras combine the properties of lattice orders with a metrization process, introducing a concept of distance among elements. Major contributions to this field include research on autometrized Boolean algebras by Ellis [5], studies on metric betweenness in l-groups by Swamy [24], and studies on metric betweenness in Boolean metric spaces by Blumenthal [2] and Penning [11]. A key observation is the absence of isosceles triangles due to the group action of the metric operation, leading to the equivalence of lattice betweenness and metric betweenness in Boolean geometry.

In DRI-semigroup II [24], Swamy characterizes DRI-semigroups, identifying the Boolean l-group as a direct product of Boolean algebras and l-groups under specific conditions, forming a subclass of DRI-semigroups. He studies the geometric properties that hold in Boolean l-groups but notes a loss of the equivalence between metric and lattice betweenness, a property that remains in both Boolean algebras and lattice ordered groups. Subba Rao [24] expanded on this notion by exploring representable autometrized algebras, which encompass a broader class than DRI-semigroups and have been thoroughly studied for their algebraic and geometric aspects.

Rachnek and Ivan [13–16] have studied extensive studies on polar, prime, and regular ideals within DRI-semigroups.

To address gaps in existing theories, we introduce the concept of the Autometrized Lattice Ordered Monoid (AL-monoid). This new structure generalizes DRI-semigroups and serves as a representable autometrized algebra. It is possible to thoroughly examine the algebraic properties and isometries of the class of AL-monoids, which has properties characteristic of a variety.

In Chapter 1, we compile definitions, examples, and results from the existing literature on Boolean algebra, commutative lattice groups, commutative DRI-semigroups, autometrized algebra, lattice ordered autometrized algebra, and representable autometrized algebra.

In Chapter 2, we define the notion of AL-monoids and provide illustrative examples, including one that demonstrates that an AL-monoid is not necessarily a DRI-semi group. We derive algebraic consequences of AL-monoids, showing that the set of complemented elements forms a Boolean algebra, while the set of invertible elements constitutes an l-group. We conclude this chapter by establishing that every AL-monoid with a unity forms a Boolean algebra when the mapping  $x \mapsto a * x$  is an isometry.

In Chapter 3, we delve into the geometric properties and metric betweenness of AL-monoids, which encapsulate the geometric results of DRI-semigroups while satisfying the equivalence of metric and lattice betweenness. We demonstrate that AL-monoids encompass all geometric properties of DRI-semigroups.

In Chapter 4, we further investigate AL-monoids by introducing various ideals, including regular ideals, polar ideals, prime ideals, and annihilators.

Through this dissertation, we aim to fill existing gaps in the literature and contribute to the understanding of autometrized structures, providing insights into their algebraic and geometric properties.

Finally, we summarise the work we have done and outline some open problems to provide a direction for the future work.

# Chapter 1

## Preliminaries

In this chapter, we present essential definitions and results from the existing literature that are relevant to our work.

### 1.1 Lattices

This section examines key results and properties of lattices, emphasizing their structure. A lattice is defined by operations  $(L, \vee, \wedge)$  satisfying reflexivity, commutativity, and associativity. It can also be characterized by the existence of least upper and greatest lower bounds. We discuss bounded and complemented lattices, as well as distributive lattices, which meet specific conditions. Notably, a lattice is distributive if relative complements are uniquely determined.

**Definition 1.1.** [1] *An algebra  $(L, \vee, \wedge)$  of type  $(2, 2)$  is a lattice defined by the following properties:*

$L_1$  [Reflexivity]:  $a \vee a = a$  and  $a \wedge a = a$  for all  $a \in L$ .

$L_2$  [Commutativity]:  $a \wedge b = b \wedge a$  and  $a \vee b = b \vee a$  for all  $a, b \in L$ .

$L_3$  [Associativity]:  $a \wedge (b \wedge c) = (a \wedge b) \wedge c$  and  $a \vee (b \vee c) = (a \vee b) \vee c$  for all  $a, b, c \in L$ .

Equivalently, a lattice can be defined as follows:

**Definition 1.2.** [1] *An algebra  $(L, \leq)$  is a lattice if every pair of elements in  $L$  has both a least upper bound (supremum) and a greatest lower bound (infimum).*

**Definition 1.3.** [1] *A Lattice is bounded if it has both lower bounds and upper bounds*

**Definition 1.4.** [1] A complemented lattice is a bounded lattice where every element has a complement.

**Definition 1.5.** A lattice  $(L, \vee, \wedge)$  is called a distributive lattice if it satisfies either of the following conditions:

i.  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  for all  $a, b, c \in L$ .

ii.  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  for all  $a, b, c \in L$ .

**Theorem 1.6.** [1] A lattice is distributive if and only if relative complement in it are uniquely determined.

## 1.2 Boolean algebras

This section discuss key findings about Boolean algebras, such as the equivalence of metric and lattice betweenness, the introduction of symmetric differences as a distance function, and the investigation of Ptolemaic features in Boolean geometry. These findings demonstrate the intricate relationship that exists in Boolean algebras between algebraic structure and geometric understanding.

**Definition 1.7.** [2] A Boolean algebra is an algebra  $B = (B, \vee, \wedge, ', 0, 1)$  of type  $(2, 2, 1, 0, 0)$  is called a Boolean algebra satisfying the following conditions:

$B_1$  :  $(B, \vee, \wedge)$  is a distributive lattice.

$B_2$  :  $b \wedge 0 = 0, \quad b \vee 1 = 1.$  for all  $b \in B$

$B_3$  :  $b \wedge b' = 0, \quad b \vee b' = 1.$  for all  $b \in B$

**Theorem 1.8.** [3, 5] Let  $B = (B, +, \cdot, ')$  be a Boolean algebra, and define symmetric difference of  $a$  and  $b$  by  $a * b = ab' + a'b$  for all  $a, b \in B$ . Then, symmetric difference  $*$  satisfies the formal properties of distance function:

$M_1$  :  $a * b = 0$  iff  $a = b$ .

$M_2$  :  $a * b = b * a$ .

$M_3$  :  $a * c \leq a * b + b * c$ , for all  $a, b, c \in B$ .

**Remark 1.9.** *Ellis [5] characterizes  $B$  with this distance as an autometrized Boolean algebra, whereas Blumenthal [3] calls this idea Boolean geometry.*

**Theorem 1.10.** *[3, 4] Symmetric differences are the only metric group operations in Boolean algebra.*

**Definition 1.11.** *[12] An element  $b$  of  $B$  is said to lie between  $a$  and  $c$  for all  $a, b, c$  in  $B$ , the notation  $(a, b, c)_M$ , if and only if  $a \neq b \neq c$  and  $a * b + b * c = a * c$ . Metric betweenness is the term used to describe this concept of betweenness.*

There is already a notion of lattice betweenness because all Boolean algebras are lattices. The following theorem is derived from this idea:

**Theorem 1.12.** *[12] An element  $b$  lies between  $a$  and  $c$  (in symbols,  $(a, b, c)_L$ ) for an element  $a, b, c$  in  $B$ , if and only if  $ab + bc = b = (a + b)(b + c)$ . The value of  $(a, b, c)_L$  is equal to  $ac \leq b \leq a + c$  since  $B$  is distributive.*

**Theorem 1.13.** *[3, 4] In Boolean geometry, metric betweenness and lattice betweenness are equivalent.*

**Theorem 1.14.** *[3] If the triangle inequality is satisfied by the three products of opposite distances for every quadruple of components, the distance geometry is said to be Ptolemaic. In particular,  $B$  is Ptolemaic if and only if each of the following requirements is true for each of the  $a, b, c, d$  in  $B$ :*

1.  $(a * b)(c * d) \leq (a * c)(b * d) + (a * d)(b * c)$ .
2.  $(a * c)(b * d) \leq (a * d)(b * c) + (a * b)(c * d)$ , and
3.  $(a * d)(b * c) \leq (a * b)(c * d) + (a * c)(b * d)$ .

Blumenthal observed that:

**Theorem 1.15.** *[3] A Boolean geometry is Ptolemaic.*

**Definition 1.16.** *[10] Let  $L$  be a lattice with a least element  $0$  and  $S$  be a set. If a map  $* : S \times S \rightarrow L$  satisfies the following conditions, it is said to meet the qualities of a metric:*

$M_1 : a * b \geq 0$  with equality if and only if  $a = b$  for all  $a, b \in S$ ,

$M_2 : a * b = b * a$  for all  $a, b \in S$ ,

$M_3 : a * c \leq a * b + b * c$  for all  $a, b, c \in S$ .

$S$  is then referred to be an  $L$ -metrized space. A lattice is said to be autometrized if  $S$  corresponds with  $L$ . *symmetric* is the name given to an autometrized lattice if and only if  $a * b = (a + b) * (ab)$ .

**Definition 1.17** ([10]). If each of the three elements  $a, b, c$  of any lattice is less than the total of the other two, then the three elements meet the triangle inequality.

**Definition 1.18** ([10]). Let the sides of a given triangle be represented by  $x, y, z$ . The initial distance triangle of the original triangle is the triangle made up of the vertices  $x, y, z$ . In a similar vein, the second distance triangle is the triangle whose vertices match the sides of the first distance triangle.

**Definition 1.19** ([10]). The distance lattice of an autometrized space is said to be distributive if it is distributive.

**Definition 1.20** ([10]). A subgeometry of an autometrized lattice is a subset in which the distance between any two points in the subset is also an element of that subset.

**Theorem 1.21.** [10] Metric betweenness and lattice betweenness coincide in an autometrized lattice  $L$  if and only if the following criteria are met:

1. metric betweenness has transitivity  $t_1$ .
2.  $L$  is symmetric, and
3.  $a \leq b \leq c$  implies  $(a, b, c)_M$  for all  $a, b, c \in L$ .

**Definition 1.22.** [12] A betweenness relation  $R$  is said to have the special inner property if  $(a, b, c)_R$  and  $(a, c, b)_R$  hold if and only if  $b = c$  for all  $a, b, c$  in  $L$ .

**Definition 1.23.** [10] If the products of the pairs of opposite distances meet the triangle inequality for any four of its elements, the space is said to be *ptolemaic*.

**Definition 1.24** (Defn 3.1, [10]). If the distance between each pair of an autometrized lattice's elements equals the distance between their sum and product, the lattice is said to be *symmetric*.

**Theorem 1.25.** [12] A betweenness relation  $R$  is said to have transitivity  $t_1$  if and only if  $(a, b, c)_R$  and  $(a, d, b)_R$  imply  $(d, b, c)_R$ . It is said to have transitivity  $t'_2$  if and only if  $(a, b, c)_R$  and  $(a, d, b)_R$  imply  $(a, d, c)_R$  for all  $a, b, c$  in  $L$ .

### 1.3 Brouwerian algebras

Brouwerian algebras are a key concept in algebraic structures and lattice theory, defined as systems  $(B, +, \cdot, -)$  of type  $(2, 2, 1)$ . They extend traditional algebra by incorporating lattice order and unique subtraction-like operations, highlighting the interaction between addition and multiplication within a defined order. This section explores their formal definition and key features, emphasizing their role in connecting algebra with order theory. We will examine how the axioms of Brouwerian algebras facilitate the manipulation of elements in a lattice, enhancing our understanding of commutative semigroups and their applications. Specific examples and implications will also be discussed to illustrate their significance in various mathematical contexts.

**Definition 1.26.** [24] *A Brouwerian algebra is a system  $B = (B, +, \cdot, -)$  of type  $(2, 2, 1)$  that satisfy:*

1.  $(B, +, \leq, -)$  is a lattice-ordered semigroup with identity element 0.
2. For every  $a, b \in B$ , there exists a least  $x \in B$  such that  $b + x \geq a$ . We denote this  $x$  by  $a - b$  (which is uniquely determined for given  $a$  and  $b$ ).
3.  $(a - b) \vee 0 + b \leq a \vee b$  for all  $a, b \in B$ .
4.  $a - a \geq 0$  for all  $a \in B$ .

A lattice ordered semigroup  $A = (A, +, \leq)$  is said to be a commutative if  $(A, +)$  is a commutative semigroup.

### 1.4 Autometrized algebras

An Autometrized Algebra is characterized by a binary commutative algebra structure, an antisymmetric and reflexive ordering, and a distance function that satisfies specific formal properties. We will also discuss the notion of regular autometrized algebras, their subalgebras, and the implications of group operations within this framework. We present the following definitions and results from Swamy [28].

**Definition 1.27.** [28] *A system  $A = (A, +, \leq, *)$  is called an 'Autometrized algebra' if and only if, the following conditions hold:*

1.  $(A, +)$  is a binary commutative algebra with distinguished element '0',
2.  $\leq$  is antisymmetric, reflexive ordering on  $A$ , and
3.  $* : A \times A \implies A$  is a mapping satisfying the formal properties of distance function, namely:

$$M_1 : a * b \geq 0 \text{ with equality if and only if } a = b,$$

$$M_2 : a * b = b * a, \text{ and}$$

$$M_3 : a * c \leq a * b + b * c, \text{ for all } a, b, c \text{ in } A.$$

**Definition 1.28.** [28] An autometrized algebra  $A = (A, +, \leq, *)$  is said to be regular if and only if  $a * 0 = a$  for every  $a$  in  $A$ . In this case, the metric operation  $*$  is regular.

**Theorem 1.29.** [28] Any subalgebra of a regular autometrized algebra is itself a regular autometrized algebra.

**Theorem 1.30.** [28] If  $*$  is a group operation in autometrized algebra  $A = (A, +, \leq, *)$ , then,  $0$  is the group identity, and consequently,  $*$  is regular.

## 1.5 Lattice ordered groups

This section covers lattice ordered groups, defined as partially ordered groups with an invariant order under group operations. A group is a lattice ordered group if its order forms a lattice. Key properties include the absolute value of elements and relationships between multiplication and order. We also discuss autometrized l-groups, noting the equivalence of lattice and metric betweenness.

We present the following from Chapter XIII of Birkhoff [1].

**Definition 1.31.** [1] A system  $G = (G, +, \leq)$  is called a partial ordered group which :

1.  $(G, +)$  is a group,
2.  $(G, \leq)$  is a partially ordered set (poset), and
3. the order is invariant under all translations  $x \rightarrow a + x + b$ ; that is  $x \leq y$  implies  $a + x + b \leq a + y + b$  for all  $a, b, x, y$  in  $G$ .

**Proposition 1.32.** *A partially ordered group  $G = (G, +, \leq)$  is called a lattice ordered group if and only if  $(G, \leq)$  is a lattice. A partially ordered group (or lattice group)  $G = (G, +, \leq)$  is called a commutative partially ordered group (or lattice group) if  $+$  is a commutative.*

**Theorem 1.33.** *[1] Any lattice group is a distributive lattice.*

**Definition 1.34.** *[1] If  $a$  is in a lattice group  $G = (G, +, \leq)$ , then,  $a \vee (-a)$ , is called the absolute value of  $a$  and is denoted by  $|a|$ .*

[1] Any element  $a$  of a lattice ordered group can be expressed as  $a = a \vee 0 + a \wedge 0$  (i.e.,  $a = a^+ + a^-$ ).

**Theorem 1.35.** *[1] A partially ordered group  $G = (G, +, \leq)$  is an lattice ordered group if and only if, for all  $a$  in  $G$ ,  $a \vee 0 = a^+$  exists in  $G$ .*

**Theorem 1.36.** *[1] If  $na \geq 0$  for some  $a$  in an  $l$ -group  $G = (G, +, \leq)$ , then,  $a \geq 0$  for all  $n \in \mathbb{Z}$ .*

**Theorem 1.37.** *[1] In a commutative  $l$ -group  $G = (G, +, \leq)$ , if  $na \geq nb$  for some  $a, b \in G$ , then  $a \geq b$ .*

**Theorem 1.38.** *[1] In an  $l$ -group  $G = (G, +, \leq)$ , the absolute value  $|\cdot|$  satisfies:*

- a.  $|x| \geq 0$  with equality if and only if  $x = 0$ ,
- b.  $|nx| = |n||x|$  for integer  $n$ ,
- c.  $|x - y| = (x \vee y) - (x \wedge y)$ ,
- d.  $|(a \vee b) - (a' \vee b)| \leq |a - a'|$  and dually.

Now, we present the results of Swamy [27] on autometrized  $l$ -groups.

**Theorem 1.39.** *[28] Any commutative  $l$ -group is an autometrized space with distance defined as  $a * b = |a - b|$ .*

**Definition 1.40.** *[28] Lattice betweenness and metric betweenness are equivalent in  $l$ -space.*

**Theorem 1.41.** *[28] In any commutative  $l$ -group  $G$ , the only metric operation for which the distances are invariant under translation and satisfy  $a * 0 = |a|$  is  $|a - b|$  for all  $a$  and  $b$  in  $G$ .*

**Theorem 1.42.** *[28] B-linearity and D-linearity are equivalent in an  $l$ -space.*

## 1.6 Lattice ordered semigroups

This section introduces lattice ordered semigroups, defined as systems  $(A, +, \leq)$  where  $(A, +)$  is a semigroup with identity 0 and  $(A, \leq)$  is a lattice. The operations of join ( $\vee$ ) and meet ( $\wedge$ ) satisfy specific distributive properties with respect to addition. We also collect the following definitions and results from Swamy in [24] and [28].

**Definition 1.43.** [28] *A system  $A = (A, +, \leq)$  is called a lattice ordered semigroup with zero "0" if and only if:*

1.  $(A, +)$  is a semigroup with identity "0", and
2.  $(A, \leq)$  is a lattice with the operations ' $\vee$ ' and ' $\wedge$ ' such that

$$a + (b \vee c) = (a + b) \vee (a + c),$$

$$(b \vee c) + a = (b + a) \vee (c + a),$$

$$a + (b \wedge c) = (a + b) \wedge (a + c),$$

$$(b \wedge c) + a = (b + a) \wedge (c + a)$$

for all  $a, b, c \in A$ .

## 1.7 Dually residuated lattice ordered semigroups

This section introduces dually residuated lattice ordered semigroups (DRL-semigroups), characterized by their structure  $(A, +, \leq, -)$  with specific properties. Key features include the existence of a zero element, the definition of a unique element  $a - b$  for any element  $a, b \in A$ , and various inequalities that govern their operations. Notably, every DRL-semigroup is semiregular, and if it contains a least element, it is also regular. The section presents several theorems and definitions that explore the relationships and properties within DRL-semigroups, including their connections to commutative structures and metric operations. We present the definitions and results of Swamy from [25], [23] and [24].

**Definition 1.44.** [23] *A system  $A = (A, +, \leq, -)$  is referred to as a dually residuated lattice-ordered semigroup (abbreviated as DRL-semigroup) if it satisfies the following conditions:*

1.  $(A, +, \leq)$  forms a lattice-ordered semigroup with a zero element 0.

2. For any  $a, b \in A$ , there exists a least element  $x \in A$  such that  $b + x \geq a$ . This element is denoted as  $a - b$  and is uniquely determined for each  $a$  and  $b$ .
3. The inequality  $(a - b) \vee 0 + b \leq a \vee b$  holds for all  $a, b \in A$ .
4. For every  $a \in A$ , it holds that  $a - a \geq 0$ .

**Theorem 1.45.** [23] A DRL-semigroup  $A = (A, +, \leq, -)$  is called is called a commutative DRL-semigroup if and only if  $(A, +)$  is a commutative semigroup.

Let  $A = (A, +, \leq, -)$  be a DRL-semigroup and  $a, b, c, \dots$  represent elements of  $A$ . Swamy ([23, 25, 26] and [28]) observed the following:

**Theorem 1.46.** [23] A DRL-semigroup is a distributive lattice.

**Lemma 1.47.** Every DRL-semigroup is semiregular.

**Remark 1.48.** While every DRL-semigroup is semiregular, it is not always regular. However, any DRL-semigroup that contains a least element is guaranteed to be regular.

Let  $A = (A, +, \leq, -)$  be a DRL-semigroup and  $a, b, c, \dots$  represent elements of  $A$ . Swamy ([23, 25, 26] and [28]) observed the following:

**Theorem 1.49.** [28] In any DRL-semigroup, the followings are hold:

1.  $a - a = 0$ .
2.  $a - 0 = a$ .
3.  $a = a \vee 0 + a \wedge 0$ .
4.  $(a - b) \vee 0 + b = a \vee b$ .
5.  $a \leq b \implies a - c \leq b - c$  and  $c - b \leq c - a$ .
6.  $a \vee b - c = (a - c) \vee (b - c)$ .
7.  $a \geq b$  implies  $a - b \geq 0$ .
8.  $a - b \wedge c = (a - b) \vee (a - c)$ .
9.  $a \leq b$  if and only if  $a - b \leq 0$ .

$$10. a \geq b \implies (a - b) + b = a.$$

$$11. A \text{ is a commutative if and only if } a + b = a \vee b + a \wedge b.$$

$$12. (a - b) \vee 0 + a \wedge b = a.$$

$$13. a \vee b - a \wedge b = (a - b) \vee (b - a) = a * b.$$

$$14. (a - b) \vee (b - a) + a \wedge b = a \vee b.$$

$$15. \text{ If } a \leq b \leq c, \text{ then, } c - a = (a - b) + (b - a).$$

$$16. \text{ If } a \wedge b = 0 = a \wedge c, \text{ then } a \wedge (b + c) = 0$$

**Theorem 1.50.** [23] *A DRL-semigroup is a distributive lattice.*

**Theorem 1.51.** [23] *If a DRL-semigroup contains a least element  $x$ , then  $x = 0$ .*

**Theorem 1.52.** [23] *If  $(A, +)$  is a group in a DRL-semigroup  $A = (A, +, \leq, -)$ , then,  $(A, +, \leq)$  is  $l$ -group and  $-$  serves as the inverse operation for  $+$ .*

**Definition 1.53.** [23] *For any  $a, b$  in a DRL-semigroup  $A = (A, +, \leq, -)$ , the expression  $(a - b) \vee (b - a)$  is referred to as the "symmetric difference" of  $a$  and  $b$ , denoted by  $a * b$ .*

**Theorem 1.54.** [23] *If the symmetric difference  $*$  is associative in a commutative DRL-semigroup, then the following conditions are equivalent:*

1.  $(a - b) \wedge (b - a) \leq 0$ ,
2.  $a - (b \vee c) = (a - b) \wedge (a - c)$
3.  $b \wedge c - a = (b - a) \wedge (c - a)$

**Definition 1.55.** [24] *A metric operation  $'*'$  in a DRL-semigroup  $A$  is said to be ' $'$  regular' if and only if  $(a + b) * (a \wedge b) = a * b$  for all  $a, b \in A$ .*

**Lemma 1.56.** *Every DRL-semigroup is semiregular.*

**Remark 1.57.** *While every DRL-semigroup is semiregular, it is not necessarily regular. However, every DRL-semigroup that contains a least element is regular.*

**Theorem 1.58.** [24] *In a commutative DRL-semigroup  $A = (A, +, \leq, -)$ , the set of all invertible elements forms a commutative l-group, and for these elements  $a - b$ , provides the solution to the equation  $b + x = a$ .*

**Theorem 1.59.** [24] *A commutative DRL-semigroup  $A = (A, +, \leq, -)$  is a subdirect product of totally ordered commutative ordered commutative DRL-semigroup if and only if  $(a - b) \wedge (b - a) \leq 0$  for all  $a, b \in A$ .*

**Theorem 1.60.** [24] *A commutative DRL-semigroup is a chain if and only if all triangles formed within it are isosceles.*

We present the following results from Rama Rao as referenced in [19] and [20]. Let  $A = (A, +, \leq, -)$  be a commutative DRL-semigroup in which  $(a - b) \wedge (b - a) \leq 0$  for all  $a, b \in A$ .

**Theorem 1.61.** [24] *A DRL-monoid is a Boolean ring if and only if the symmetric difference is associative.*

**Lemma 1.62.** [24] *In a DRL-semigroup  $(A, +, \vee, \wedge, -, 0)$ , the following statements are equivalent:*

1.  $(a - b) \wedge (b - a) \leq 0$  for all  $a, b$  in  $A$ .
2.  $a - (b \vee c) = (a - b) \wedge (a - c)$  for all  $a, b, c \in A$ .
3.  $(b \wedge c) - a = (b - a) \wedge (c - a)$  for all  $a, b, c \in A$ .

We present the following definitions and results from Swamy and Prakharabao as cited in [28]. Swamy and Rao have revised the definition of an autometrized algebra as follows:

**Definition 1.63.** [28] *A system  $A = (A, +, \leq, *)$  is called an autometrized algebra if and only if it satisfies the following conditions:*

- (i)  $(A, +, \leq)$  is a commutative partially ordered semigroup with identity 0. Specifically,  $(A, +)$  is a commutative semigroup with identity 0, and  $(A, \leq)$  is a partially ordered set such that if  $a \leq b$ , then  $a + x \leq b + x$  for every  $a, b, x \in A$ .
- (ii)  $*$  is a metric operation.

**Definition 1.64.** [28] *An autometrized algebra  $(A, +, \leq, *)$  is called normal autometrized algebra if it satisfies:*

1.  $a \leq a * 0$  for all  $a$  in  $A$ .
2.  $(a + c) * (b + d) \leq (a * b) + (c * d)$  for all  $a, b, c, d$  in  $A$ .
3.  $(a * c) * (b * d) \leq (a * b) * (c * d)$ , for all  $a, b, c, d \in A$ .
4. For all  $a, b$  in  $A$ , if  $a \leq b$ , then there exists  $x \geq 0$  such that  $a + x = b$ .

**Example 1.65.** [28] Every commutative DRI-semigroup  $(A, +, \leq, -)$  is normal autometrized algebra with distance  $a * b = (a - b) \cup (b - a)$ . Consequently, Boolean algebras and commutative  $l$ -groups are also normal autometrized algebras.

**Definition 1.66.** [28] A non empty subset  $I$  of an autometrized algebra  $A = (A, +, \leq, *)$  is called an ideal of  $A$  if and only if:

- (i). If  $a \in I, b \in I$ , then  $a + b \in I$ , and
- (ii). If  $a \in I, b \in A$ , and  $b * 0 \leq a * 0$  then,  $b \in I$ .

**Theorem 1.67.** [18] The set of all congruence relations on an autometrized algebra forms a complete lattice.

We present the following definitions and results from Swamy as cited in [26] and [?].

**Definition 1.68.** [26] A system  $A = (A, +, \leq, *)$  is called a lattice ordered autometrized algebra if and only if:

1.  $(A, +, \leq)$  is a commutative lattice ordered semigroup with zero  $0$ .
2.  $'*'$  is a metric operation on  $A$ ; that is,  $'*'$  is a mapping from  $A \times A$  into  $A$  satisfying the formal properties of a distance:

$$M_1. a * b \geq 0 \text{ for all } a, b \in A, \text{ with equality if and only if } a = b,$$

$$M_2. a * b = b * a \text{ for all } a, b \in A,$$

$$M_3. a * b \leq a * c + c * b \text{ for all } a, b, c \in A.$$

**Example 1.69.** Commutative DRI-semigroups, commutative  $l$ -groups, and Boolean algebras are all lattice ordered autometrized algebras.

## 1.8 Normal autometrized lattice ordered algebras

This section defines normal autometrized lattice ordered algebras (NAL-algebras) as algebras  $(A, +, \wedge, \vee, *, \leq)$  satisfying specific properties. These include being an abelian lattice ordered monoid, having a metric operation, and satisfying various inequalities related to addition and the metric operation.

**Definition 1.70.** [7] *An algebra  $(A, +, \wedge, \vee, *, 0)$  is a normal autometrized lattice ordered algebra (NAL-algebra) if the following holds.*

- (i).  $(A, +, \leq)$  is an abelian lattice ordered monoid
- (ii).  $*$  is metric operation.
- (iii).  $x * 0 \geq x$  for each  $x \in A$ .
- (iv).  $(x + y) * (x' + y') \leq (x * x' + y * y')$  for all  $x, y, x', y' \in A$ .
- (v).  $(x * y) * (x' * y')$   $x, y, x', y' \in A$
- (vi).  $x, y \in A$  and  $x \leq y$  imply there exist  $z \in A$  such that  $x + z = y$ .

## 1.9 Representable autometrized algebras

This section defines representable autometrized algebras as lattice ordered autometrized algebras  $(A, +, \leq, *)$  that satisfy specific conditions. These include being semiregular, meaning  $a \geq 0$  implies  $a * 0 = a$ , and ensuring that certain mappings, such as addition and lattice operations, are contractions. Notable examples of representable autometrized algebras include Boolean algebras and commutative l-groups.

**Definition 1.71.** [17] *A lattice ordered autometrized algebra  $A = (A, +, \leq, *)$  of arity  $(2, 2, 2)$  is called representable autometrized algebra if and only if,  $A$  satisfies the following conditions:*

1.  $A = (A, +, \leq, *)$  is semiregular autometrized algebra, meaning that if  $a \in A$  and  $a \geq 0$  then,  $a * 0 = a$ . and
2. For every  $a$  in  $A$ , all the mappings  $x \mapsto a + x, x \mapsto a \vee x, x \mapsto a \wedge x$  and  $x \mapsto a * x$  are contractions. Specifically, for any operation  $\theta$  denotes among  $+, \wedge, \vee$ , it holds that:  $(a\theta x) * (a\theta y) \leq x * y$  for all  $x, y \in A$ .

**Remark 1.72.** *Boolean algebras, Brouwerian algebras, Boolean  $l$ -algebras, and commutative  $l$ -groups are all examples of representable autometrized algebras.*

## Chapter 2

# Autometrized lattice ordered monoids

In this chapter, we introduce autometrized lattice ordered monoids (AL-monoids), a generalization of DRI-semigroups, and establish several algebraic properties of AL-monoids. We also investigate the isometries of AL-monoids, proving that the set of invertible elements of an AL-monoid forms a l-group. Finally, we show that an algebra  $A$  with identity is a Boolean algebra if, for each  $a \in A$ , the mapping  $x \mapsto a * x$  is an isometry. Throughout the chapter we use  $A$  as AL-monoids and  $a, b, c, d, x, y$  as elements of AL-monoids.

Throughout this chapter, we denote an AL-monoid  $(A, +, \leq, *, 0)$  simply as  $A$ , where  $a, b, c, x, y, z$  represent elements of  $A$ .

## 2.1 Autometrized Lattice Ordered Monoids

In this section, we introduce autometrized lattice ordered monoids (AL-monoids), which are algebraic structures that integrate the properties of commutative lattices with specific operations. We define key conditions that characterize AL-monoids, including addition, meets, and joins, as well as the contraction properties of mappings within the structure.

We illustrate these concepts with examples, showcasing how well-known structures like Boolean algebras and commutative l-groups fit into the AL-monoid framework. Additionally, we illustrated that AL-monoids are generalization of DRI-semigroups and show the independency of axioms using examples. We begin with the following definition:

**Definition 2.1.** *An autometrized lattice ordered monoid (AL-monoid) is an algebra  $(A, +, \vee, \wedge, *, 0)$  of arity  $(2, 2, 2, 2, 0)$  where:*

1.  $(A, +, \vee, \wedge, 0)$  is a commutative lattice ordered monoid with least element 0.

2.  $a * (a \wedge b) + b = a \vee b$  for all  $a, b \in A$ .
3. For all  $x$  in  $A$ ; the mappings  $x \mapsto a+x$ ,  $a \vee x$ ,  $a \wedge x$ , and  $a*x$  are contractions with respect to  $*$  (i.e., a mapping  $f : A \rightarrow A$  is a contraction with respect to  $*$  if  $f(x) * f(y) \leq x * y$ , where  $\leq$  is the ordering in  $A$  induced by  $(A, \vee, \wedge)$ ).
4. For all  $a, b$  in  $A$ ;  $[a * (a \vee b)] \wedge [b * (a \vee b)] = 0$ .

**Remark 2.2.** 1. The conditions 1, 2 and 3 yield a common abstraction of commutative  $l$ -groups and Brouwerian algebras.

2. All conditions in the above definition yield a common abstraction of commutative  $l$ -groups and Boolean algebras.

**Example 2.3.** Every Boolean algebra  $(B, \vee, \wedge, ', 0, 1)$  is an AL-monoid with  $a * b = (a \wedge b') \vee (a' \wedge b)$ .

**Example 2.4.** Every commutative  $l$ -group  $(G, \vee, \wedge, +, -)$ , with  $-$  which is the operation inverse of  $+$ , is an AL-monoid if we define  $a * b = (a + (-b)) \vee (b + (-a))$ , where  $-a$  and  $-b$  represent the additive inverses of  $a$  and  $b$  respectively.

**Remark 2.5.** Every DRL-semigroup is an AL-monoid since it is the common abstraction of Brouwerian algebras and commutative  $l$ -groups. It satisfies remark 2.2.; however, the converse does not hold. The following Example 2.6 illustrates an AL-monoid that is not a DRL-semigroup.

**Example 2.6.** Let  $A = \mathbb{Z} \cup \{u\}$ , where  $\mathbb{Z}$  is the set of all integers and  $u$  is an element not in  $\mathbb{Z}$ . For all  $a, b \in \mathbb{Z}$ , define the operations as follows: -  $a + b$  is the usual sum, -  $a + u = u = u + a$ , -  $u + u = u$ , -  $a * b = |a - b|$ , -  $a * u = u = u * a$ .  $a \leq u, u \wedge u = u$  and  $u \vee u = u, u * u = 0$ . Define  $\leq$  in  $A$  for all  $a \in \mathbb{Z}$  as the usual ordering. Then  $A = (A, \leq, +, *)$  is an AL-monoid that is not a DRL-semigroup since there is no unique least element in  $A$  such that  $x + u \geq a$  for any  $a$  in  $\mathbb{Z}$ .

**Example 2.7.** Normal autometrized lattice ordered algebras are AL-monoids. But, the converse is not true. See Example 2.6.

**Remark 2.8.** The following examples demonstrate that conditions (2) and (4) of Definition 2.1 are independent in any AL-monoid.

**Example 2.9.** Let  $A$  be the lattice of all closed subsets of the real numbers with the usual topology. It can be verified that  $A$  satisfies for all  $a, b \in A$ ;  $a * (a \wedge b) + (a \wedge b) = a$ , but does not satisfy condition (4). For instance. Consider  $x = [0, 2]$  and  $y = [2, 3]$ . Then  $(x * (x \vee y)) \wedge (y * (x \vee y)) = \{2\} \neq 0$ .

**Remark 2.10.** From Example 2.9, it can be observed that the class of representable autometrized algebras is broader than the class of AL-monoids.

**Example 2.11.** Let  $A = \mathbb{Z} \cup \{u, v\}$ , where  $\mathbb{Z}$  is the set of all integers, and  $u$  and  $v$  are elements not in  $\mathbb{Z}$ . Define:  $a + b$  as the usual sum,  $a + u = u = u + a$ ,  $a + v = v = v + a$  for all  $a \in \mathbb{Z}$ ,  $u + v = u = v + u$ ,  $u + u = u$ ,  $v + v = v$ . Define  $\leq$  in  $A$  as follows: for elements in  $\mathbb{Z}$ , let  $\leq$  be the usual ordering; define  $u < a < v$  for all  $a \in \mathbb{Z}$ . Define  $*$  in  $A$  as follows:  $a * b = |a - b|$  for all  $a, b \in \mathbb{Z}$ ,  $a * u = v = u * a$ ,  $u * v = v = v * u$ ,  $u * y = 0 = v * v$ . It can be easily verified that  $(A, +, \leq, *)$  satisfies axioms (1), (3), and (4) of Definition 2.1, but does not satisfy axiom (2) because  $v * (v \wedge u) + (v \wedge u) = v * u + (v \wedge u) = v + u \neq v$ .

**Remark 2.12.** Axiom (ii) of Definition 2.1 shows AL-monoid is equationally definable, ensure that the class of AL-monoids is closed under the formation of subalgebras, direct unions, and homomorphic images, thus forming a variety.

## 2.2 Algebraic properties

In this section We examine the algebraic aspects of AL-monoids, emphasizing their distinct operational traits and connections. A deeper comprehension of the relationships between addition, multiplication, and order relations is made possible by an algebraic structure known as an AL-monoid, which blends aspects of algebra and lattice theory.

We outline a number of lemmas and theorems that clarify important characteristics of AL-monoids, such as regularity and absorption laws, and how these characteristics affect the behavior of the structure. With these findings, we hope to show how AL-monoids' fundamental components help to qualify them as autometrized algebras. We demonstrated the importance of these algebraic features by looking at particular cases and offering proofs, opening the door for more uses and understandings in theoretical and applied mathematics. In order to better understand their separate functions in the larger field of mathematics, this investigation will also draw attention to the differences between other algebraic structures, such as lattice-ordered

autometrized algebras and Brouwerian algebras. From these we use  $A = (A, +, \leq, *, 0)$  as AL-monoid with least element 0.

**Lemma 2.13.**  $a = a * (a \wedge b) + (a \wedge b)$  for all  $a, b$  in  $A$ .

*Proof.*  $a = a \vee (a \wedge b)$  ...absorption law  $= a * (a \wedge (a \wedge b)) + (a \wedge b)$ ...from (2) of definition 2.1  $= a * (a \wedge b) + (a \wedge b)$  □

**Lemma 2.14.**  $a * (a \wedge b) = (a \vee b) * b$  for all  $a, b \in A$ .

*Proof.* Let  $a, b \in A$ . Then,  $a * (a \wedge b) = ((a \vee b) \wedge a) * (a \wedge b)$  ....absorption law  $= ((a \vee b) \wedge a) * ((a \wedge b) \wedge a) \leq (a \vee b) * (a \wedge b)$ . by (iii) of Definition 2.1. and,  $b * (a \vee b) = ((a \wedge b) \vee b) * (a \vee b) = (b \vee (a \wedge b)) * (b \vee a) \leq a * (a \wedge b)$  (by axiom (3) of Definition 2.1). □

**Lemma 2.15.** Let  $A$  be an AL-monoid and  $a, b \in A$ . Then  $a * 0 = 0$ . (regularity). It follows that  $*$  is semiregular.

*Proof.* We have  $a = a * (a \wedge 0) + (a \wedge 0) = a * 0$ . In particular,  $0 * 0 = 0$ . □

**Lemma 2.16.** Let  $A$  be an AL-monoid and  $a \in A$ . Then  $a * a = 0$

*Proof.*  $a \vee a = a * (a \wedge a) + a \implies a = a * a + a \implies a * a = 0$ . □

**Lemma 2.17.** Let  $A$  be an AL-monoid and  $a, b \in A$ . Then  $a * b \geq 0$ .

*Proof.* Consider the following:  $0 = 0 * 0 = [0 * (0 \wedge (a \wedge b)) + (0 \wedge (a \wedge b))] * [0 * (0 \wedge (a \wedge b)) + (0 \wedge (a \wedge b))] \leq (0 \wedge (a \wedge b)) * (0 \wedge (a \wedge b)) = 0 \wedge (a \wedge b \wedge a) * (0 \wedge a \wedge b \wedge b) \leq (a \wedge b) * (a \wedge b) \leq a * b$ . Thus, we conclude that  $a * b \geq 0$ . Since  $a * b \geq 0$ , we have  $a * a = 0$ . □

**Lemma 2.18.** Let  $A$  be an AL-monoid and  $a, b \in A$ . Then  $a * b = b * a$ .

*Proof.*  $a * b = (a * (a \wedge b) + (a \wedge b)) * (b * (a \wedge b) + (a \wedge b)) \leq (a * (a \wedge b)) * (b * (a \wedge b)) \leq (a \wedge b) * b * (a \wedge b) \leq b * a$ . Interchanging  $a$  and  $b$  gives  $b * a \leq a * b$ . □

**Lemma 2.19.** Let  $A$  be an AL-monoids and  $a, b \in A$ . Then  $a * b = 0 \implies a = b$ .

*Proof.* To show  $a * b = 0$  implies  $a = b$ :  $a = a * (a \wedge b) + (a \wedge b) = ((a \wedge a) * (a \wedge b)) + (a \wedge b) \leq (a * b) + (a \wedge b) \leq (a * b) + b = 0 + b = b$ . Thus,  $a * b = 0$  implies  $b * a = 0$  and hence  $b \leq a$ . Therefore,  $a = b$ . □

**Lemma 2.20.**  $a * c \leq a * b + a * c$ ; for all  $a, b, c$  in  $A$ .

*Proof.*  $a * b = (a * b) * ((a * b) \wedge (c * b)) + (a * b) \wedge (c * b) \leq (a * b) * (c * b) + (c * b) \leq (a * c) + (c * b)$ .  $\square$

**Theorem 2.21.** *An AL-monoid  $A$  is an autometrized algebra.*

*Proof.* Follows from (i) of Definition 2.1 and lemma 2.14, 2.2, 2.16, 2.17, 2.18, 2.19 and 2.20.

Thus,  $A$  is an autometrized algebra.  $\square$

**Theorem 2.22.** *Every AL-monoids are representable autometrized algebra.*

*Proof.* This follows from Lemma 2.15 and condition (3) of Definition 2.1.  $\square$

**Example 2.23.** *A Brouwerian algebra is representable autometrized algebra. But not AL-monoids. Since there is no least element in AL-monoids  $A$  such that  $a + x \geq a$ .*

**Theorem 2.24.** *Let  $A$  be any AL-monoids and  $a, b, c \in A$ . The following conditions hold:*

$$(i) \quad b \leq a \Rightarrow a = a * b + b,$$

$$(ii) \quad a \vee b = a * b + a \wedge b,$$

$$(iii) \quad a * b = (a \vee b) * (a \wedge b), \text{ and}$$

$$(iv) \quad a * b = (a * (a \wedge b)) + (a \wedge b) * b = a * (a \vee b) + (a \vee b) * b, \text{ for all } a, b \in A.$$

*Proof.*  $a = a \vee (a \wedge b) = a * (a \wedge (a \wedge b)) + (a \wedge b)$  from (ii) of defn 2.1  $= a * (a \wedge b) + (a \wedge b)$ .

Thus (i).  $b \leq a \implies b = a \wedge b$ . Then putting into equation equation(3.2) we have  $a = a * b + b$ .

For (ii), we used (iii) of Definition 2.1 consider:

$a \leq a \vee b$ ;  $b \leq (a \vee b)$ ...( $A, \leq$ ) is a lattice).  $a \vee b \leq (a \vee b) * b \leq a * b + b$  by (i).  $\&(a \vee b) * a \leq b * a = a * b + a$ . This implies  $a \vee b \leq (a * b) \wedge (a * b + a) = a * b + a \wedge b$ . For the reverse, we have:  $(a * b) + (a \wedge b) = (a * (a \wedge b) + (a \wedge b)) * (b * (b \wedge a) + (b \wedge a)) + a \wedge b \leq ((a * (a \wedge b)) * (b * (b \wedge a)) + (a \wedge b))$ (iii) of Definition 2.1  $= ((a * (a \wedge b)) + b = a \vee b$ . Thus,  $a \vee b = a * b + a \wedge b$ .

To prove (iii), let:  $a * b = (a * (a \wedge b) + a \wedge b) * (b * (a \wedge b) + (a \wedge b)) \leq (a * (a \wedge b)) * (b * (a \wedge b))$  (by axiom (3) of Definition 2.1)  $\leq (b * (a \vee b)) * (b * (a \wedge b))$  (by Lemma 2.14)  $= (a \vee b) * (a \wedge b)$ .

Finally, for (iv), let  $a, b \in A$ . By the triangle inequality, we have:  $a * b \leq a * (a \wedge b) + (a \wedge b) * b \leq (a \vee b) * b + b * (a \wedge b)$  (by Lemma 2.14)  $= (a * b + a \wedge b) * (b * (a \wedge b) + (a \wedge b)) + b * (a \wedge b)$  (by (2) of this theorem)  $\leq (a * b) * (b * (a \wedge b)) + (b * (a \wedge b))$  (by axiom (3) of Definition 2.1)  $\leq ((a * b) * 0) = a * b$  (by (i) of this theorem, since  $b * (a \wedge b) \leq b * a = a * b$ ).

Now we have  $a * b = a * (a \wedge b) + (a \wedge b) * b = (a \vee b) * b + (a \vee b) * a$  (by Lemma 2.14)  $= a * (a \vee b) + (a \vee b) * b$ .  $\square$

**Remark 2.25.** *Not all Lattice ordered autometrized algebra is representable.*

**Example 2.26.** *Not all lattice ordered autometrized algebras are AL-monoids. Similarly Brouwerian algebra are lattice ordered autometrized algebra. But, not AL-monoids.*

## 2.3 Isometries

We explore the idea of isometries in the context of AL-monoids in this section. Bijective mappings known as isometries maintain the monoid's algebraic structure, namely the operations defined within it. Understanding how transformations can be applied without changing the fundamental relationships between items depends on this characteristic.

We shall illustrate how invertible elements interact under the monoid's operations by presenting a number of theorems that describe their behavior in an AL-monoid. Specifically, we will examine the circumstances in which specific mappings are considered isometries and look at how these mappings affect the AL-monoid's structure.

This investigation will help us better grasp the algebraic features of AL-monoids by demonstrating the importance of isometries in maintaining order and operation. The findings will advance our understanding of how algebraic structures interact with geometric concepts of transformation and distance.

**Theorem 2.27.** *Let  $a, b$  in AL-monoid  $A$ . Then the following statements hold:*

1.  $a \wedge 0$  is invertible.
2. If  $a$  is invertible, then  $a \wedge b$  is invertible.
3. If  $a$  is invertible, then  $a \vee 0$  is invertible.
4. If  $a$  and  $b$  are invertible, then  $a \wedge b$  is invertible.

*Proof.* 1. To show that  $a \wedge 0$  is invertible, note that:  $(0 * (0 \wedge a) + (0 \wedge a)) = 0 * (0 \wedge (a \wedge 0)) + (a \wedge 0) = 0 \vee (0 \wedge a) = 0$ . Thus,  $a \wedge 0$  is invertible.

2. If  $a$  is invertible, then  $a + x = 0$  implies:  $a * (a \wedge (a \wedge b)) + (a \wedge (a \wedge b)) + x = 0 \Rightarrow (a * (a \wedge (a \wedge b)) + x) + (a \wedge (a \wedge b)) = 0$ . This shows that  $a \wedge b$  is invertible.

3. Since both  $a$  and  $a \wedge 0$  are invertible, it follows that  $a \vee 0$  is invertible as  $a = a \vee 0 + a \wedge 0$ .

4. Finally, for invertible  $a$  and  $b$ , we have:  $a \vee b = a * (a \wedge b) + b \implies a = a * (a \wedge (a \wedge b) + (a \wedge b)) = a \vee (a \wedge b) = a$ . As  $a$  and  $a \wedge b$  are invertible, it follows that  $a * (a \wedge b)$  is also invertible, proving (4).  $\square$

**Lemma 2.28.** *The inverse in AL-monoid is unique.*

*Proof.* Let  $a, b, c \in A$  be elements of the AL-monoid. Suppose  $b$  and  $c$  are both inverses of  $a$ . This means  $b + a = 0$  and  $c + a = 0$ . From the first equation, we have  $b = -a$ . From the second equation, it follows that  $c = -a$ . Since both  $b$  and  $c$  are equal to  $-a$ , we conclude that  $b = c$ . Thus, the inverse of  $a$  is unique.  $\square$

**Theorem 2.29.** *The set of all invertible elements of an AL-monoid  $A$  forms a l-group.*

*Proof.* If  $a$  and  $b$  are invertible and  $a \leq b$ , then  $a + (-b) \leq 0$ . Which implies that  $-a + (-b) \leq -a$ . Thus,  $-b \leq -a$  and hence the  $-(a \vee b) = -a \wedge -b - (a \wedge b) = -a \vee -b$ . Therefore, by this relation the class becomes l-group.  $\square$

**Notation:** We write  $-b$  for the inverse of an invertible element  $b$  and  $a - b$  for  $a + (-b)$ .

**Lemma 2.30.** *If  $a, b$  are invertible, then so is  $a * b$ .*

*Proof.*  $a \vee b = a * (a \wedge b) + b \implies a * b + b = a \vee (a \wedge b) = a$ .  $\square$

Hence  $a * b$  is invertible.

**Theorem 2.31.** *If  $a, b$  are invertible elements of AL-monoid  $A$ , then:*

$$(i) \ a * (a \wedge b) = (a - b) \vee 0;$$

$$(ii) \ a * b = (a \vee b) - (a \wedge b);$$

$$(iii) \ a * 0 \geq a \vee 0 \geq a; \text{ and}$$

$$(iv) \ \text{If } x \text{ is invertible and } a + x = b + x, \text{ then } a = b.$$

*Proof.* (i). We have  $a * (a \wedge b), a \wedge b$  as elements of a l-group. Now,  $a * (a \wedge b) + a \wedge b = a \implies a * (a \wedge b) = a - (a \wedge b) = a + (-a \vee -b) = (a + (-a)) \vee (a + (-b)) = (a - a) \vee (a - b) = 0 \vee (a - b) = (a - b) \vee 0$ . Hence (i) holds.

(ii). Since  $x \geq y$  and  $x, y$  are invertible, it implies  $x - y \geq 0$ . Therefore,  $x * (x \wedge y) = (x - y) \vee 0 = x - y$ . Thus,  $a * b = (a \vee b) * (a \wedge b) = (a \vee b) - (a \wedge b)$ . Hence (ii) holds.

(iii). By (4) of Definition 2.1, we have:  $a \vee 0 = a * (a \wedge 0) = (a \wedge a) * (a \wedge 0) \leq a * 0$ . Thus, (iii) holds.

(iv). If  $x$  is invertible and  $a + x = b + x$ , then:  $(a + x) - x = (b + x) - x \implies a + 0 = b + 0 \implies a = b$ . Hence (iv) holds.  $\square$

**Definition 2.32.** An element  $a \in A$  is idempotent if  $a + a = a$ .

**Definition 2.33.** A bijective mapping  $\sigma : A \rightarrow A$  is an isometry if  $\sigma(x) * \sigma(y) = x * y$  for all  $x, y \in A$ .

**Theorem 2.34.** The mapping  $x \mapsto a + x$  is an isometry for all  $a$  and  $x$  in  $A$  if  $a$  is invertible.

*Proof.* If  $a$  is invertible, then  $a + x = a + y$  implies  $x = y$ , showing that  $x \mapsto a + x$  is injective.

For  $y \in A$ , let  $x = y - a$  so that  $x + a = y$ . Thus, it is a bijection.

Now, we check if it is an isometry:  $(a * x) * (a * y) \leq x * y = (-a + a + x) * (-a + a + y) \leq (a + x) * (a + y) = x * y$ . Hence, the map is an isometry.

Conversely, if the map is an isometry, then  $0 = a + x$  for some  $x$  implies  $a$  is invertible.  $\square$

**Theorem 2.35.** If  $a, b$  are idempotents of an AL-monoid  $A$ , then:

(i)  $a \geq 0$ .

(ii)  $a \wedge b$  is also idempotent.

(iii)  $a \vee b = a + b$ .

(iv) Both  $a \vee b$  and  $a + b$  are idempotents.

(v)  $a + b = a \vee b + a \wedge b$  is idempotent.

*Proof.* 1. If  $a + a = 0$ , then:  $a + a \vee 0 + a \wedge 0 = a \vee 0 + a \wedge 0 = 0$  in lattice represent a least element. This implies  $a + a \vee 0 = (a + a) \vee (a + 0) = a \vee 0$ . Thus,  $a \vee a = a \vee 0 \implies a = a \vee 0 \geq 0$ .

2. For  $a \wedge b$ , we have:  $a \wedge b + a \wedge b = (a \wedge b + a) \wedge (a \wedge b + b) = (a + a) \wedge (a + b) \wedge (b + b) = a \wedge b \wedge (a + b)$ . Hence (ii) holds.

3. Since  $a * b + a \wedge b = a \vee b$ , we have:  $a * b + a \wedge b + a \wedge b = a \vee b + a \wedge b \implies a * b + a \wedge b = a \vee b + a \wedge b$ . Thus,  $a \vee b = a + b$ .

4. If  $(a + b) + (a + b) = (a + a) + (b + b) = a + b$ , then  $a + b$  is idempotent. Also, since  $a + b = a \vee b$  from (3), (iv) holds.

5. By considering (iii),  $a + b = a \vee b + a \wedge b$  is idempotent.  $\square$

**Theorem 2.36.** *Let  $A$  is an AL-monoid. For any  $a, b \in A$ ,  $a * b = (a * (a \wedge b)) * (b * (a \wedge b))$ .*

*Proof.* We have:  $a * b = ((a * (a \wedge b)) * (b * (b \wedge a))) \leq (a * (a \wedge b) * (b * (a \wedge b)))$  and,  $(a * (a \wedge b)) * (b * (a \wedge b)) \leq a * b$  by (iii) of Definition 2.1. Hence, the proof.  $\square$

**Theorem 2.37.** *Let  $A = (A, +, \leq, *, 0)$  be an AL-monoid. Then  $(A, \leq)$  is a distributive lattice.*

*Proof.* Let  $a, x, y, z, w$  are elements of  $A$ . Clearly,  $(A, \leq)$  is a lattice. Let  $a \wedge x = a \wedge y$  and  $a \vee x = a \vee y$ . Then, by axiom 2 of Definition 2.1 and Lemma 2.14, we can drive the following equations:  $(x * (a \wedge x)) + (a \wedge x) = x (y * (a \wedge y)) + (a \wedge y) = y$ .

Given that  $a \wedge x = a \wedge y$ , we can denote this common value as  $z$  (i.e.,  $z = a \wedge x = a \wedge y$ ). Now, we can rewrite the previous equations as:  $(x * z) + z = x$  and  $(y * z) + z = y$ . Next, we express  $x$  and  $y$  in terms of their joins with  $a$ :  $x = ((a \vee x) * a) + z$  and  $y = ((a \vee y) * a) + z$ . Since  $a \vee x = a \vee y$ , we can denote this common join as  $w$  (i.e.,  $w = a \vee x = a \vee y$ ).

We then have:  $x = (w * a) + z$  and  $y = (w * a) + z$ . Since both expressions for  $x$  and  $y$  are equal, it follows that  $x = y$ .

Therefore, we have shown that  $a \wedge x = a \wedge y$  and  $a \vee x = a \vee y$  implies  $x = y$ . This demonstrates that the lattice is distributive.  $\square$

**Remark 2.38.** *Let  $A$  is AL-monoid with unity, i.e., there exists  $1$  such that  $a + (a * 1) = 1$ . If  $a \in A^+$ , then  $a \leq 1$  and  $0 \leq a * 1 \Rightarrow a = a + 0 \leq a + (a * 1) = 1$ .*

With usual terminology, we have the following:

**Lemma 2.39.** (i). *If  $a$  is complemented then  $a + a = a$ .*

$$(ii). a \wedge (a * 1) = 0$$

$$(iii). a \vee (a \vee 1) = 1.$$

$$(iv). a' = a * 1$$

*Proof.* (i). Let  $a$  have a complement  $a'$  in  $A$ . Then,  $a = a + 0 = a + (a \wedge a') = (a + a) \wedge (a + a') = (a + a) \wedge 1 = a + a$ .

(ii).  $(a * 1) + a = 1 = a' + a \implies a * 1 \leq a'$  (by (i)) implies  $a \wedge (a * 1) \leq a \wedge a' = 0$ .

(iii).  $a * (a \wedge (a * 1)) + (a * 1) = a \vee (a * 1)$ . Implies that  $a * 0 + a * 1 = a \vee (a * 1) \implies a + (a * 1) = 1$ .  $\square$

**Theorem 2.40.** *The complemented elements of  $A$  form a Boolean algebra.*

*Proof.* Thus, we also have:  $(a * 1) + a = 1 = a' + a \implies a * 1 \leq a' \implies a \wedge (a * 1) \leq a \wedge a' = 0$ . This leads to:  $a * (a \wedge (a * 1)) + (a * 1) = a \vee (a * 1) \implies a * 0 + a * 1 = a \vee (a * 1) = a + (a * 1) = 1$ . Thus,  $a' = a * 1$ . Since the set of all complemented elements forms a sublattice of  $A$  having 0 and 1, it follows that  $A$  is a Boolean algebra.  $\square$

**Theorem 2.41.** *A with unity is a Boolean algebra if for each  $a \in A$ , the mapping  $x \mapsto a * x$  is an isometry.*

*Proof.* Since the mapping  $x \mapsto 1 * x$  is an isomorphism, this implies  $a = 1 * x$  for some  $x$ , hence  $a \geq 0$ . Also,  $a + (1 * a) = 1 \implies a \leq 1$ . Thus,  $A = A^+$  with the greatest element 1.

Let  $a \in A$ . Then:  $a \vee (1 * a) = a * (1 * a) + a \wedge (1 * a) = (a * 0) * (a * 1) + a \wedge (a * 1) = 0 * 1 + a \wedge (a * 1) = 1 + a \wedge (a * 1) = 1$ .

Now consider:  $1 * (a \wedge (1 * a)) = 1 \wedge (a \wedge (1 \wedge a'))' = (1 \wedge a') \vee (1 \wedge (1 \wedge a')) = (1 * a) \vee (1 * (1 * a)) = 1 = 1 * 0$ , which implies  $a \wedge (1 * a) = 0$ . Thus,  $a$  is complemented. Hence,  $A$  is a Boolean algebra.  $\square$

**Theorem 2.42.** *If  $(A, +, \leq, 0)$  is a commutative monoid that is a chain such that:*

1.  $x \leq y \implies a + x \leq a + y$
2.  $a * (a \wedge b) + b = a \vee b$
3. *The mappings  $x \mapsto a + x, a \vee x, a \wedge x, a * x$  are contractions with respect to  $*$*

*then  $(A, +, \leq, 0, *)$  is an AL-monoid.*

*Proof.* We will show that  $[a * (a \vee b)] \wedge [b * (a \wedge b)] = 0$ . Since  $A$  is a chain, we have either  $a = a \vee b$  or  $b = a \vee b$ .

If  $b = a \vee b$ , then:  $(a * (a \vee b)) \wedge (b * (a \vee b)) = (a * a) \wedge (b * a) = 0 \wedge (b * a) = 0$ .

If  $a = a \vee b$ , then:  $(a * (a \vee b)) \wedge (b * (a \vee b)) = (a * b) \wedge 0 = 0$ .

Thus,  $A$  is an AL-monoid.  $\square$

## Chapter 3

# Geometries in autometrized lattice ordered monoids

In this chapter, we study the geometry of autometrized lattice ordered monoids by introducing the concepts of metric betweenness and its properties:  $t_1, t_2$ , B-linearity, D-linearity, lattice betweenness, segments, and equilateral triangles. We prove that there are no equilateral triangles in AL-monoids and that any AL-monoid is Ptolemaic. This result subsumes all the geometry of commutative DRI-semigroups.

We treat any distinct elements  $a, b, c$  of an AL-monoid  $A = (A, +, \leq, *, 0)$  with least element 0 as the vertices of a triangle with sides  $a * b, b * c$ , and  $c * a$ .

**Definition 3.1.** *A subset  $S$  of an AL-monoid  $A = (A, +, \leq, *, 0)$  is said to be a "subgeometry" if for any  $a, b \in S$ , it follows that  $a * b \in S$ .*

### 3.1 Triangles in AL-Monoids

The features of triangles made up of components in AL-monoids are examined in this section, with particular attention paid to the geometric relationships between the elements and how these interactions affect the algebraic system's structure. In particular, we will examine the equivalencies that define triangle configurations and examine the circumstances in which triangles have fixed points.

We will outline a number of theorems that emphasize the connections between triangle components, including their joins and meets, and show how these connections can be used to infer information about the general structure of the AL-monoid. In order to provide light on

the geometric meaning of algebraic operations, we will also look at the ideas of degenerate triangles and chains.

By exposing the interaction between algebra and geometry in the study of AL-monoids, we want to further our comprehension of how geometric qualities appear within algebraic frameworks. This study will advance a more comprehensive understanding of the applications of triangular interactions in several mathematical contexts.

**Theorem 3.2.** *For the triangle  $\Delta(a, b, c)$  in AL-monoid  $A$ , the following are equivalent:*

1.  $\Delta(a, b, c)$  has fixed points.
2.  $a \vee b = b \vee c = c \vee a$  and  $a \wedge b \wedge c = 0$ .
3.  $\Delta(a * b, b * c, c * a)$  has fixed points, and the set  $\{0, a, b, c\}$  is a subgeometry where  $x * y = |x - y|$ .

*Proof.* (1)  $\implies$  (2): Assume that  $\Delta(a, b, c)$  has fixed points. Then we have  $a * b = c$ ,  $b * c = a$ , and  $c * a = b$ . Clearly,  $a > 0$ ,  $b > 0$ , and  $c > 0$ .

Now,  $a \vee b = a * b + a \wedge b = c + a \wedge b \geq c$  since  $a \wedge b \geq 0$ . Similarly,  $b \vee c \geq a$  and  $c \vee a \geq b$ . Hence,  $a \vee b = b \vee c = c \vee a$ .

Now,  $a \vee b = a * b + a \wedge b = c + a \wedge b \geq c$  since  $a \wedge b \geq 0$ . Thus,  $b \vee c = b * c + b \wedge c = a + b \wedge c \geq a$  and  $c \vee a = c * a + c \wedge a = b + c \wedge a \geq b$ . Hence,  $a \vee b = b \vee c = c \vee a$ .

Now,  $a * b \leq b * (a \vee b) + (a \vee b) * c$  and  $c = a * b \leq a * (a \vee c) + (a \vee c) * b$ . Hence,  $a \wedge c \leq (b * (a \vee b)) + (a \vee c) * c \wedge (b * (a \vee b)) + (a \vee c) * a = b * (a \vee b)$ .

Further,  $b = a * c \leq a * (a \vee b) + (a \vee b) * b = a * (a \vee b) + (b \vee c) * c$ . Hence,  $a * b \leq (b * (a \vee b)) + (a \vee c) * c = (a * (a \vee b)) + (b \vee c) * b$ .

Thus,  $a \wedge b \wedge c \geq 0$ , which implies  $a \wedge b \wedge c = 0$ . Therefore, (1)  $\implies$  (2).

(2)  $\implies$  (1): Assume that  $a \vee b = b \vee c = c \vee a$  and  $a \wedge b \wedge c = 0$ . Clearly,  $a \geq 0$ ,  $b \geq 0$ , and  $c \geq 0$ . So, we have:  $c = c * 0 = c * (c \wedge a \wedge b) = (c \wedge (a \vee c)) * (c \wedge a \wedge b) \leq (a \vee c) * (a \wedge b) = (a \vee b) * (a \wedge b) = a * b$ . Thus,  $c \leq a * b$ . Now,  $a * b \leq a * (a \vee c) + (a \vee c) * b$  and  $a * b = b * c \leq (b * (a \vee b)) + (a \vee b) * c$ . Hence,  $a * b \leq (a * (a \vee c) + (b \vee c) * b) = a \vee b$ .

Similarly, we can show that  $b \leq a \vee c$  and  $a \leq b \vee c$ . Hence,  $a \vee b = b \vee c = c \vee a$ .

Now, since  $a * b$  should be any of  $a, b, c$  (since  $a \neq b \neq c \neq a$ ). Let  $a * b = a$ . If  $b * c = a$ , then  $a * c = (a * b) * (b * c) = a * a = 0$ , so  $a = c$ . Thus,  $b * c \neq a$ .

If  $b * c = b$ , then  $a * c = (a * b) * (b * c) = a * b = a$ , so  $b * c = (a * b) * (a * c) = a * a = 0$ , which implies  $b = c$ .

Thus,  $b * c \neq b$ . If  $a * c = b$ , then  $a = a * b = (a * c) * (c * b) = b * c$ . This results in  $b * c = c$ .

Finally, if  $a * c = c$ , then  $a = a * b = (a * c) * (c * b) = c * c = 0$ , which implies  $a = b = c$ . Hence,  $\Delta(a, b, c)$  degenerates, meaning  $a, b, c$  are not distinct. Thus,  $A$  is a chain.

Therefore, if  $A$  is free of triangles with fixed points, then  $A$  is a chain. Conversely, if  $A$  is a chain, there are no triangles in  $A$  with fixed points.  $\square$

In any Boolean geometry, since the metric operation is a group operation, there do not exist any isosceles triangles. In commutative lattice groups, Swamy [24,25] has also shown that equilateral triangles do not exist.

**Theorem 3.3.** *There do not exist equilateral triangles in an AL-monoid  $A$ .*

*Proof.* Assume that  $a * b = b * c = c * a$ . Then we have:  $c = c * (c \wedge a) + c \wedge a = a * b + a$ , and  $c = c * (c \wedge b) + c \wedge b \leq c * b + b = a * b + b$ . Hence, we get:  $c \leq ((a * b) + a) \wedge (a * b + b) = a * b + a * b = a \vee b$ . Thus,  $c \leq a \vee b$ . Similarly, we can show that  $b \leq a \vee c$  and  $a \leq b \vee c$ . Therefore, we conclude that:  $a \vee b = b \vee c = c \vee a$ .

Now, since  $a * b \leq a * (a \vee c) + (a \vee c) * b$  and  $a * b = b * c \leq (b * (a \vee b)) + (a \vee b) * c$ , we can say:  $a * b \leq (a * (a \vee c) + (a \vee c) * b) \wedge (b * (a \vee b) + (a \vee b) * c) = (a * (a \vee c)) \wedge ((a \vee c) * c) + (a \vee b) * b = (a \vee b) * b$ .

Similarly, we can prove that  $a * b \leq (a \vee b) * a$ . Thus, we have:  $a * b \leq ((a \vee b) * b) \wedge ((a \vee b) * a) = 0$ . This implies that  $a * b = b * c = c * a = 0$ , which leads us to conclude that  $a = b = c$ . Consequently, the triangle  $\Delta(a, b, c)$  degenerates (i.e.,  $a, b, c$  are not distinct). This completes the proof.  $\square$

**Theorem 3.4.** *AL-monoid  $A$  is a chain if and only if it is free of triangles with fixed points.*

*Proof.* Assume that  $A$  is not a chain. Then there exist  $a, b \in A$  such that  $a \neq b$ ,  $a \vee b \neq a$ , and  $a \vee b \neq b$ . Consequently, the elements  $(a \vee b) * a$ ,  $(a \vee b) * b$ , and  $a * b$  are three distinct elements in  $A$ .

If  $(a \vee b) * a = (a \vee b) * b$ , then:  $a * b \leq a * (a \vee b) + (a \vee b) * b = 2(a * (a \vee b)) = 2(b * (a \vee b))$ , which implies:  $a * b \leq (2(a * (a \vee b))) \wedge (2(b * (a \vee b))) = 0$ . (Since in a commutative lattice-ordered semigroup with identity 0,  $x \wedge y = 0$  implies  $m x \wedge n y = 0$  for all positive integers  $m$  and  $n$ .) This leads to  $a = b$ , a contradiction.

If  $a * b = (a \vee b) * a$ , then:  $(a \vee b) * b \leq a \vee b = (a \vee b) * a$ , which implies:  $(a \vee b) * b = ((a \vee b) * b) \wedge ((a \vee b) * a) = 0$ , this leading to  $a \vee b = b$ , another contradiction.

Similarly, if  $a * b = (a \vee b) * b$ , we reach the contradiction that  $a \vee b = a$ . Therefore, the three elements  $a * (a \vee b)$ ,  $b * (a \vee b)$ , and  $a * b$  are distinct, forming the triangle  $\Delta_1 = \Delta(a * (a \vee b), b * (a \vee b), a * b)$ .

Now, we will show that  $\Delta_1$  has fixed points. For,  $(a * (a \vee b)) * (b * (a \vee b)) \leq a * b$ , and since:  $a * b = (a * (a \wedge b) + a \wedge b) * (b * (a \wedge b) + a \wedge b) \leq (a * (a \wedge b)) * (b * (a * (a \vee b)))$ , it follows that:  $(a * (a \vee b)) * (b * (a \vee b)) = a * b$ . Moreover,  $(a * (a \vee b)) * (a * b) \leq (a \vee b) * b$ , and since:  $(a \vee b) * b = (a * b + a \wedge b) * (a \wedge b) + a \wedge b \leq (a * b) * (b * (a \wedge b)) = (a * b) * (a * (a \vee b)) = (a * (a \vee b)) * (a * b)$ , we have:  $(a * (a \vee b)) * (a * b) = b * (a \vee b)$ . Similarly, we can prove that  $(b * (a \vee b)) * (a * b) = a * (a \vee b)$ . Thus,  $\Delta_1$  has fixed points.

Consequently, if  $A$  is free of triangles with fixed points, then  $A$  is a chain. Conversely, if  $A$  is a chain, there are no triangles in  $A$  with fixed points.  $\square$

**Theorem 3.5.** *If all triangles in AL-monoids  $A$  are isosceles, then  $A$  is a chain.*

*Proof.* For any  $a, b \in A$ , consider the triangle  $\Delta(a \vee b, a \wedge b, a)$ .

The condition  $(a \vee b) * (a \wedge b) = (a \vee b) * a$  implies:  $a \vee b = a * b + a \wedge b = (a \vee b) * (a \wedge b) + (a \wedge b) = (a \vee b) * a + a \wedge b = b * (a \wedge b) + a \wedge b = b$ . Similarly, the condition  $(a \vee b) * (a \wedge b) = (a \wedge b) * a$  implies:  $a \vee b = a * b + a \wedge b = a * (a \wedge b) + (a \wedge b) = a$ .

Furthermore, the condition  $(a \vee b) * a = (a \wedge b) * a$  leads to:  $b = b * (b \wedge a) + b \wedge a = a * (a \wedge b) + b \wedge a = a * (a \wedge b) + (a \wedge b) = a$ .

Thus, the triangle  $\Delta(a \vee b, a \wedge b, a)$  degenerates, meaning  $a \vee b$ ,  $a \wedge b$ , and  $a$  are not distinct. Hence,  $A$  is a chain.  $\square$

## 3.2 Betweenness in AL-monoids

The notion of betweenness inside AL-monoids—which describes when an element lies metrically between two others—is examined in this section. We'll look at the circumstances in which certain characteristics, such transitivity and the special inner feature, are displayed by a betweenness relation. We hope to gain a better understanding of the geometric interpretations of AL-monoids' elements and how these interpretations affect the structure as a whole by analyzing these relations. Additionally, by examining the effects of isosceles triangles on betweenness, the investigation will shed light on the connections among the components of this algebraic framework.

**Definition 3.6.** An element  $x \in A$  is said to lie metrically between  $a$  and  $b$  (denoted as  $(a, x, b)_M$ ) if and only if  $a * x + x * b = a * b$ .

**Definition 3.7.** A betweenness relation  $B$  is said to satisfy the special inner property  $\beta$  if  $B(a, b, c)$  and  $B(a, c, b) \implies b = c$ .

**Lemma 3.8.** In AL-monoid  $A$ , three points  $a, b, c$  fail to have the special inner property if  $a, b, c$  are the vertices of an isosceles triangle in which the sum of any two sides is equal to their union (join).

*Proof.* Assume that  $a, b, c$  are the vertices of an isosceles triangle. Then, we have:  $(a*b) \vee (a*c) = a*b + a*c$ ,  $(a*b) \vee (b*c) = a*b + b*c$ ,  $(a*c) \vee (b*c) = a*c + b*c$ , and  $a*b = b*c$ .

Thus, we find:  $a*b + b*c = (a*b) \vee (b*c) \leq (a*b) \vee (b*a + a*c)$  (by  $M_3$ ). Consequently,  $a*c + c*b = a*b + c*b = a*c = a*b$ . This implies that  $(a, b, c)_M$  and  $(a, c, b)_M$  hold. However,  $b \neq c$  since  $a, b, c$  are the vertices of a triangle. Therefore, the special inner property fails in this case. This completes the proof.  $\square$

**Definition 3.9.** In an AL-monoid  $A$ , a betweenness relation  $M$  is said to have:

- (i). Transitive  $t_1$  if and only if  $B(a, b, c), B(a, d, b) \implies B(d, b, c)$ .
- (ii). Transitive  $t_2$  if and only if  $B(a, b, c), B(a, d, b) \implies B(a, d, c)$ .

### 3.3 Transitivity in AL-monoids

The notion of transitivity in AL-monoids is examined in this section, with a particular emphasis on metric betweenness. When transitivity  $t_1$  holds, we determine the circumstances that result in the absence of isosceles triangles and investigate the effects of this characteristic on the algebraic structure.

The equivalence between lattice and metric betweenness under certain situations will be highlighted, along with theorems and proofs showing how transitivity affects the interactions among elements. We will also give a definition of the concept of metrically convex AL-monoids and explain the role of atoms in this context.

By doing these analyses, we hope to gain a better understanding of the structural dynamics of AL-monoids and how transitive features influence their geometric and algebraic interactions.

**Theorem 3.10.** *If the metric betweenness has transitivity  $t_1$ , then AL-monoid  $A$  is free of isosceles triangles for which the sum of any two sides is equal to their union.*

*Proof.* Given  $(a, b, c)_M$  and  $(a, d, b)_M$ , we have:  $a*b + b*c = a*c$  and  $a*d + d*b = a*b$ . This implies:  $a*d + d*b + b*c = a*c$ , which leads to:  $a*c \geq a*d + d*c$ . Since  $a*c \leq a*d + d*c$ , we conclude that  $a*c = a*d + d*c$ .  $\square$

**Lemma 3.11.** *In AL-monoid  $A$ , the metric betweenness has transitivity  $t_2$ .*

*Proof.* Given  $(a, b, c)_M$  and  $(a, d, b)_M$ , we have:  $a*b + b*c = a*c$  and  $a*d + d*b = a*b$ . This implies:  $a*d + d*b + b*c = a*c$ , so that:  $a*c \geq a*d + d*c$ . Since  $a*c \leq a*d + d*c$ , it follows that  $(a, d, c)_M$ . This completes the proof.  $\square$

**Theorem 3.12.** *In AL-monoids  $A$ , lattice betweenness implies metric betweenness.*

We utilize the following lemmas in  $A$ :

**Lemma 3.13.** *Let  $a, b, c$  in AL-monoid  $A$ . Then  $a*b + b*c = (a \wedge c)*b + b*(a \vee c)$ .*

*Proof.* We have:  $a*b + b*c \leq a*(a \vee c) + (a \vee c)*b + b*(a \wedge c) + (a \wedge c)*c$ . This simplifies to:  $((a \vee c)*b + b*(a \wedge c))*(a*(a \vee c) + a*(a \vee c)) = ((a \vee c)*b + b*(a \wedge c))*(2(a*(a \vee c)))$ . Similarly,  $a*b + b*c \leq ((a \vee c)*b + b*(a \wedge c)) + 2(c*(a \vee c))$ .

Hence, we have:  $a*b + b*c = (a \vee c)*b + b*(a \wedge c) = (a \wedge c)*b + b*(a \vee c)$  for all  $a, b, c \in A$ .  $\square$

**Corollary 3.14.** *In AL-monoid  $A$ ,  $(a, b, c)_M$  if and only if  $(a \wedge c, b, a \vee c)_M$ .*

**Lemma 3.15.** *In AL-monoid  $A$ ,  $a \leq b \leq c$  implies  $(a, b, c)_M$ .*

*Proof.* Assume that  $a \leq b \leq c$ . Then:  $c*b = (c*(c \wedge a)) + (c \wedge a)*(b*(b \wedge a) + b \wedge a) = (c*a + a)*(b*a + a) \leq (c*a)*(b*a) \leq c*b$ . Thus,  $c*b = (c*a)*(b*a)$ . Moreover, since  $b*a \leq c*a$ :  $c*b + b*a = (c*a)*(b*a) + b*a = (c*a)*((c*a) \wedge (b*a)) = (c*a) \wedge (b*a) = c*a$ . This implies  $(a, b, c)_M$ .  $\square$

*Proof of Theorem 3.12.* Assume that  $(a, b, c)_L$ . Then,  $a \wedge c \leq b \leq b \vee c$ . Hence,  $(a \wedge c)*b + b*(a \vee c) = a*c$ , implying  $(a, b, c)_M$ . This completes the proof.  $\square$

**Remark 3.16.** *The converse of the above theorem is not true. We illustrate this with the following example:*

**Example 3.17.** Let  $A = \mathbb{Z} \cup \{u\}$ , where  $\mathbb{Z}$  is the set of all integers and  $u$  is an element not in  $\mathbb{Z}$ . For all  $a, b \in \mathbb{Z}$ , define  $+, *$  in  $A$  as follows:  $a + b$  is the usual sum;  $a + u = u$ ,  $u + a = u$ ,  $u + u = u$ , and  $a * b = |a - b|$ ,  $a * u = u$ ,  $u * a = u$ .

Define  $\leq$  in  $A$  for all  $a \in \mathbb{Z}$  as the usual ordering. Then  $A = (A, \leq, +, *)$  is an AL-monoid.

Testing metric betweenness: Let  $a = 1, b = u, c = 2$ . Now calculate the distances:  $1 * u = |1 - u| = u - 1$  (since  $u > 1$ ),  $u * 2 = |u - 2| = u - 2$ ,  $1 * 2 = |1 - 2| = 1$ . This shows that  $b = u$  is not between  $a = 1$  and  $c = 2$ . We check if  $1 \leq u \leq 2$ :  $1 \leq u$  is true since  $u$  is greater than all integers. However,  $u \leq 2$  is false since  $u$  is defined to be greater than all integers. Thus,  $u$  can satisfy the metric betweenness condition under certain circumstances but does not satisfy lattice betweenness because it is not ordered between 1 and 2. This example shows that metric betweenness does not imply lattice betweenness in the structure  $A$ .

**Theorem 3.18.** In AL-monoid  $A$ , lattice betweenness and metric betweenness are equivalent if and only if metric betweenness has transitivity  $t_1$ .

*Proof.* Assuming that metric betweenness has transitivity  $t_1$ , it suffices to verify properties (i) through (v) of metric betweenness. The metric betweenness satisfies (i) and (ii). Since it has transitivity  $t_1$ , property (ii) holds because  $(a, b, c)_M$  and  $(a, c, b)_M$  imply  $c = b$ . By the aforementioned lemma, property (iii) is satisfied. Given  $(a \wedge c, a \vee c, a \vee c)_M$  and  $(a \wedge c, a \wedge c, a \vee c)$ , property (iv) is satisfied by the corollary, and property (v) is satisfied by the lemma. Thus, metric betweenness and lattice betweenness are equivalent.

Conversely, assume that lattice betweenness and metric betweenness are equivalent. Since lattice betweenness has transitivity  $t_1$  in any lattice, it follows that metric betweenness also has transitivity  $t_1$ . This completes the proof.  $\square$

**Definition 3.19.** An AL-monoid  $A = (A, +, \leq, *)$  is said to be metrically convex if and only if for any  $a \neq b$  in  $A$  such that  $a \neq x \neq b$ , we have  $a * x + x * b = a * b$ . An element  $c$  in  $A$  (with  $c \neq 0$ ) is an atom if and only if  $0 < x < c$  is impossible for any  $x$  in  $A$ .

### 3.4 Ptolemaic, B-linearity, D-linearity

**Definition 3.20.** An AL-monoid  $A$  is ptolemaic if and only if for every four points  $a, b, c, d$  in  $A$ , it is true that  $(a * b) \wedge (c * d) \leq a * c \wedge b * d + (a * d) \wedge (b * c)$ ,  $(a * c) \wedge (b * d) \leq (a * b) \wedge (c * d) + (a * d) \wedge (c * b)$ ,  $(a * d) \wedge (b * c) \leq (a * c) \wedge c \wedge d * b + a * b \wedge c * d$ .

**Theorem 3.21.** *AL-monoids are ptolemaic with  $\wedge$  as multiplication.*

*Proof.* Let  $a, b, c, d \in A$ . Let  $a = a_1 * a_2, b = a_2 * a_3, c = a_3 * a_1, x = a_3 * a_4, y = a_4 * a_1$ , and  $z = a_2 * a_4$ . We have,  $a \leq b * c$  and  $x \leq b + z$  and hence  $a \wedge x \leq (b + c) \wedge (b + z) = b + (c \wedge z)$ . Also,  $a \leq a + z$  and  $x \leq c + y$  implies  $a \wedge x \leq (y + c) \wedge (y + z) = y + (c \wedge z)$   $a \wedge x \leq b + (c \wedge z)$  and  $a \wedge x \leq y + c \wedge z \implies a \wedge x \leq (b + c \wedge z) \wedge (y + c \wedge z) = b \wedge y + (c \wedge z)$ . Also,  $(a*c) \wedge (b*d) + a*d \wedge b*c = (a*c) + ((a*d) \wedge (b*c)) \wedge (b*d + (a*d \wedge (b*c))) = ((a*c) + (a*d)) \wedge ((a*c + b*c)) \wedge ((b*d) + (a*d)) \wedge (b*d + b*c) \geq c*d \wedge (a*d) \wedge (b*a) \wedge (d*c) \geq c*d \wedge a*b = (a*b) \wedge (c*d)$ . We can proof the other in similar ways. Since, in  $A$ , we can not find pair wise disjoint elements  $a, b, c$  in  $A$  such that  $a * b = b * c = c * a$ , there do not exist equilateral triangle in  $A$ .  $\square$

**Definition 3.22.** *An  $n$ -tuple of distinct points in an AL-monoid  $A$  for  $(n \geq 3)$  is said to be:*

1. *B-linear if and only if there exists a labeling  $(p_1, p_2, \dots, p_n)$  of its elements such that  $(p_i, p_j, p_k)$  holds whenever  $1 \leq i < j < k \leq n$ .*
2. *D-linear if and only if there exists a labeling  $(p_1, p_2, \dots, p_n)$  of its elements such that*

$$p_1 * p_n = p_1 * p_2 + p_2 * p_3 + \dots + p_{n-1} * p_n.$$

**Theorem 3.23.** *In AL-monoid  $A$ , B-linearity implies D-linearity.*

*Proof.* Assume that an  $n$ -tuple  $p$  is B-linear. Then, there exists a labeling  $(p_1, p_2, \dots, p_n)$  of the elements of  $p$  such that  $B(p_i, p_j, p_k)$  holds whenever  $1 \leq i < j < k \leq n$ . In particular, this implies that  $B(p_i, p_{i+1}, p_n)$  holds for  $i < n - 1$ . By Theorem 3.14, this implies:  $p_i * p_n = p_i * p_{i+1} + p_{i+1} * p_n$  for  $i < n - 1$ . Hence,  $p_1 * p_n = p_1 * p_2 + p_2 * p_3 + \dots + p_{n-1} * p_n = \sum_{i=1}^{n-1} (p_i * p_{i+1})$ , so  $p$  is D-linear. This completes the proof.  $\square$

**Theorem 3.24.** *In any AL-monoid  $A$ , the following assertions are equivalent:*

1.  $(a, b, c)_L$  if and only if  $(a, b, c)_M$ .
2. *Metric betweenness has transitivity  $t_1$ .*
3.  $a \leq b \vee c$  and  $a * b \geq a * c$  imply  $b \leq c$ .
4.  $(a, b, c)_M$  and  $(a, c, b)_M$  imply  $b = c$ .

*Proof.* (1)  $\iff$  (2) follows from Theorem 3.18. Now, we prove (1)  $\implies$  (4)  $\implies$  (3)  $\implies$  (1).

(1)  $\implies$  (4): Assume that  $(a, b, c)_L$  if and only if  $(a, b, c)_M$ . Suppose  $(a, b, c)_M$  and  $(a, c, b)_M$ . Then, by assumption, we have  $(a, b, c)_L$ . Thus,  $a \wedge c \leq b \leq a \vee c$ . Hence,  $a \wedge c = a \wedge b$  and  $a \vee c = a \vee b$ . Since  $(A, \leq)$  is a distributive lattice (by Theorem ...), it follows that  $b = c$ . Thus, (1)  $\implies$  (4).

(4)  $\implies$  (3): Assume that  $(a, b, c)_M$  and  $(a, c, b)_M$  imply  $b = c$ . Suppose  $a \geq b \vee c$  and  $a * b \geq a * c$ . First, we prove that  $a * (b \wedge c) = a * b$ . For, we have:  $a * (b \wedge c) \leq a * c + c * (b \wedge c)$  (by  $M_3$ )  $\leq a * b + c * (b \wedge c)$ . Thus,  $a * (b \wedge c) \leq (a * b) + (b * (b \wedge c)) \wedge (a * b + c * (b \wedge c)) = a * b$ . Moreover,  $a * b = (a \vee b) * (b \vee (b \wedge c))$  (by  $L_2$ ). Hence,  $a * (b \wedge c) = a * b$ . Since  $b \wedge c \leq b \leq a$ , by Lemma 3.17, we have  $(b \wedge c, b, a)_M$ , which implies  $(a, b, b \wedge c)_M$ . Therefore,  $a * (b \wedge c) + (b \wedge c) * b = a * b + b * (b \wedge c) = a * b$ , i.e.,  $(a, b \wedge c, b)_M$ . Thus, we have  $(a, b, b \wedge c)_M$  and  $(a, b \wedge c, b)_M$ . Hence, by our assumption, it follows that  $b = b \wedge c$ , i.e.,  $b \leq c$ . Thus, (4)  $\implies$  (3).

(3)  $\implies$  (1): Assume that  $a \geq (b \vee c)$  and  $a * b \geq a * c$  imply  $b \leq c$  (by Theorem 3.14). Now, assume  $(a, b, c)_M$  which means  $a \leq c$ ; hence,  $b = b * (b \wedge a) + b \wedge a \leq b * a + a \leq c * a$ . Since  $(a, b, c)_M$  implies  $a * b + b * c = a * c = c * (c \wedge a) + c \wedge a = c$ . Thus,  $a \leq c$  and  $b \leq c$ , which means  $c \geq a \vee b$ . Furthermore, we have  $c * a \geq c * b$ . Hence, by our assumption, it follows that  $a \leq b$ . Thus,  $a \leq b \leq c$ . Therefore,  $(a, b, c)_L$ .

We have proved (1)  $\implies$  (4)  $\implies$  (3)  $\implies$  (1)  $\implies$  (2). Hence, (1), (2), (3), and (4) are equivalent.  $\square$

**Remark 3.25.** *This above hypothesis is true only representable autometrized algebra satisfying condition iv of 2.1 of AL-monoid.*

**Example 3.26.** *Let  $A$  be a lattice of all closed subsets of the space of real numbers with usual topology. Then,  $A$  is a representable algebra with the symmetric difference as the metric. But, it is not AL-monoid. Here, let  $A = [0, 2]$ ,  $B = [1, 3]$ , and  $C = [0, 1] \vee [2, 3]$ . Then, the triangle  $\Delta(A, B, C)$  has fixed points, while  $A \wedge B \wedge C \neq \emptyset$ .*

## Chapter 4

# Certain ideals in autometrized lattice ordered monoids

In this chapter, we introduce special types of ideals namely polar ideals, regular ideals, prime ideals, and annihilators in AL-monoids, and establish their relationship. Also, we introduce the notion of the value of an element for an ideal  $I$  and obtain a characterization of a regular ideal in terms of value of an element. Further, we investigate certain properties of these ideals. We obtain certain characterizations of polar ideals in terms of minimal prime ideals, annihilators, and maximal polar ideals.

Throughout this chapter, we denote an AL-monoid  $(A, +, \leq, *, 0)$  simply as  $A$ , where  $a, b, c, x, y, z$  represent elements of  $A$ .

### 4.1 Ideals in AL-monoids

Here, we investigate the notion of ideals in the context of AL-monoids. First, we define convex subsets and discuss their importance for AL-monoids. We then draw a link between these equivalence relations and ideals by introducing the concept of congruence relations.

The fundamental characteristics of ideals, such as the requirements for being a prime ideal and the connections between different ideals in the algebraic structure, will be discussed. Additionally, we will highlight the structure of the ideal lattice by presenting theorems that demonstrate the relationship between ideals and congruence relations. Now, we introduce the following:

**Definition 4.1.** *A subset  $B$  of AL-monoid  $A$  is said to be convex if  $a, b$  in  $A$  and  $a \wedge b \leq c \leq$*

$$a \vee b \implies c \in B.$$

**Definition 4.2.** A sub-AL-monoid  $B$  of  $A$  is said to be a convex sub-AL-monoid of  $A$  if  $B$  is a convex set in  $A$ .

**Definition 4.3.** An equivalence relation  $\theta$  on AL-monoid  $A$  is called a congruence relation if only if  $\theta$  satisfies the following conditions

$$(i). a\theta b, c\theta d \implies (a + c)\theta(b + d).$$

$$(ii). a\theta b, c\theta d \implies (a * c)\theta(b * d) \text{ and}$$

$$(iii). a\theta b \text{ and } x * y \leq a * b \implies x\theta y \text{ for all } a, b, c, d, x, y \text{ in } A.$$

**Definition 4.4.** A non-empty subset  $I$  of AL-monoids  $A$  is called an ideal of  $A$  if it satisfies the following conditions:

$$(i) \text{ If } a \in I \text{ and } b \in I, \text{ then } a + b \in I.$$

$$(ii) \text{ If } b \leq a \text{ for some } a \in I \text{ and } b \in A, \text{ then } b \in I.$$

**Theorem 4.5.** There is a one-to-one correspondence between the ideals of AL-monoids  $A$  and its congruence relations.

*Proof.* Let  $I$  be an ideal of  $A$ . We define a relation  $\theta$  on  $A$  as follows:  $a\theta b \iff a * b \in I$  for  $a, b \in A$ . Since  $a * b = b * a$ , it follows that  $a\theta b$  implies  $b\theta a$ , establishing symmetry.

Next, if  $a\theta b$  and  $b\theta c$ , then  $a * b, b * c \in I$ . By the property of ideals:  $a * c \leq a * b + b * c$ , which implies  $a\theta c$ . Thus,  $\theta$  is reflexive, symmetric, and transitive, making it an equivalence relation.

If  $a\theta b$  and  $c\theta d$ , then both  $a * b \in I$  and  $c * d \in I$ . Therefore:  $(a * b) + (c * d) \in I$ , and since in  $A$ :  $(a + c) * (b + d) \leq (a * b) + (c * d)$ , it follows that  $(a + c) * (b + d) \in I$ , which implies:  $(a + c)\theta(b + d)$ . Thus,  $I$  defines a congruence relation.

Conversely, let  $\theta$  be a congruence relation on  $A$ . Define the set  $N = \{x \in A \mid x\theta 0\}$ . To show that  $N$  is an ideal, observe:

$$i. \text{ If } a, b \in N, \text{ then } a + b \in N \text{ because } a\theta 0 \text{ and } b\theta 0 \text{ imply } (a + b)\theta 0.$$

$$ii. \text{ If } x \in N \text{ and } y \leq x, \text{ then: } x\theta 0 \implies (y \vee x)\theta 0 \implies y\theta 0 \implies y \in N.$$

Thus,  $N$  satisfies the conditions of an ideal.

Now, define a new relation  $\theta'$  as follows:  $a\theta'b \iff a*b \in N$ . We can show that this relation is equivalent to  $a\theta b$  by demonstrating that  $a\theta'b$  implies  $a\theta b$  and vice versa.

Thus, we conclude that there is a one-to-one correspondence between the ideals of  $A$  and its congruence relations.  $\square$

**Corollary 4.6.** *The ideal of any DRL-semigroup is correspond one to one to its congruence relation.*

**Corollary 4.7.** *The ideal lattice of any DRL-semigroups are all complete algebraic lattices.*

**Lemma 4.8.** *If  $0 \leq a, b, c$  in AL-monoids  $A$ , then  $a \wedge (b + c) \leq (a \wedge b) + (a \wedge c)$ .*

*Proof.* Let  $0 \leq a, b, c \in A$ , then  $0 \leq a \wedge (b + c) \leq (b + c)$  hence,  $a \wedge (b + c) = p + q$ , for some  $0 \leq p \leq b, 0 \leq q \leq c$ . Moreover,  $p \leq p + q \leq a$ , thus,  $p \leq a \wedge b$ , and similarly,  $q \leq a \wedge c$ , which means  $a \wedge (b + c) \leq (a \wedge b) + (a \wedge c)$ .  $\square$

**Definition 4.9.** *Let  $B$  is subset of AL-monoids  $A$ . Then, the ideal generated by a set  $B$  is defined by  $I(B) = \{x \in A; x \leq b_1, x \leq b_2, \dots, x \leq b_n \text{ for } b_1, b_2, \dots, b_n \in B\}$ .*

**Definition 4.10.** *The principal ideal generated by a single element  $a$  in AL-monoid  $A$  is defined by  $I(a) = \{x \in A : x \leq ma, \text{ for } a \in A \text{ and } m \in \mathbb{N}\}$ .*

**Proposition 4.11.** *The intersection of any two principal ideal of AL-monoid  $A$  is a principal ideal of  $A$ .*

*Proof.* Let  $a, b \in A$ .  $a \wedge b \leq a, b \implies a \wedge b \in I(a) \wedge I(b)$ . Therefore,  $I(a \wedge b) \subseteq I(a) \wedge I(b)$ .

Conversely, if  $x \in I(a) \wedge I(b)$ , then there exist  $m, n \geq 0$  such that  $x \leq ma, x \leq nb$ . By Lemma(4.8), we have  $ma \wedge nb \leq mn(a \wedge b)$ ; thus,  $x \in I(a \wedge b)$ . i.e.,  $I(a) \wedge I(b) \subseteq I(a \wedge b)$ .  $\square$

**Lemma 4.12.** *The join of any two principal ideals in AL-monoid  $A$  is a principal ideal.*

*Proof.* Let  $(a)$  and  $(b)$  be principal ideals. We show that  $(a) \vee (b) = (a + b)$ .

Let  $J$  be any ideal containing  $(a)$  and  $(b)$ , and  $x \in (a) + (b)$ . Then  $x \leq m(a + b)$  for some positive integer  $m$ . Now  $a \in (a) \subset J$  and  $b \in (b) \subset J$  show that  $a \in J$  and  $b \in J$ . Now,  $x \leq t$  and  $t \in J$ . It follows that  $x \in J$ . this shows  $(a + b) \in J$ . So,  $(a) \vee (b) = (a + b)$ . Hence,  $(a) \vee (b)$  is a principal ideal  $\square$

**Definition 4.13.** An ideal  $I$  of AL-monoid  $A$  is a prime ideal if for all

$$J, K \in \mathbb{I}(A); \quad J \cap K = I \implies J = I \text{ or } K = I.$$

**Proposition 4.14.** An ideal  $P$  of an AL-monoid  $A$  is a prime ideal if for all  $a, b \in A$ ,  $a \wedge b = 0$  implies  $a \in P$  or  $b \in P$ .

*Proof.* We will show both directions of the statement.

( $\implies$ ). If  $P$  is a prime ideal, then if  $a \wedge b = 0$ , we have  $a \in P$  or  $b \in P$ .

Assume  $P$  is a prime ideal. By the definition of a prime ideal, if  $a \wedge b \in P$ , then either  $a \in P$  or  $b \in P$ .

Now, if  $a \wedge b = 0$ , then clearly  $0 \in P$ . Since  $P$  is a prime ideal, this implies:  $a \wedge b \in P \implies a \in P$  or  $b \in P$ . Thus, the first direction is established.

( $\impliedby$ ): If  $a \wedge b = 0$  implies  $a \in P$  or  $b \in P$ , then  $P$  is a prime ideal.

Assume that for all  $a, b \in A$ ,  $a \wedge b = 0$  implies  $a \in P$  or  $b \in P$ . We need to show that  $P$  is a prime ideal.

Let  $x, y \in A$  such that  $x * y \in P$ . We want to show that either  $x \in P$  or  $y \in P$ .

Consider  $a = x$  and  $b = y$ . We know:  $x * y \in P$ . Now, if  $x \neq 0$  and  $y \neq 0$ , we can use the property of the ideal  $P$ . Given  $a \wedge b = 0$ , we can write:  $a \wedge b = 0 \implies x \wedge y = 0$ . By our assumption, this implies:  $x \in P$  or  $y \in P$ . Thus, if  $x * y \in P$  and both  $x$  and  $y$  are non-zero, we have established that  $P$  is a prime ideal.

Next, consider the case when either  $x = 0$  or  $y = 0$ . If  $x = 0$ , then  $0 \in P$  by the definition of an ideal. If  $y = 0$ , then similarly  $0 \in P$ . Thus, in both cases, we still satisfy the condition for  $P$  being a prime ideal since at least one of  $x$  or  $y$  is in  $P$ .

Therefore, we conclude that  $P$  is a prime ideal.  $\square$

**Corollary 4.15.** If  $I \subseteq J$  are ideals of AL-monoid  $A$  and  $I$  is prime, then  $J$  is a prime ideal.

*Proof.* If  $a \wedge b = 0$  then  $a \in I$  or  $b \in I$ . But  $I \subseteq J$  and thus  $a \wedge b = 0 \implies a \in J$  or  $b \in J$ . Therefore,  $J$  is a prime ideal of  $A$ .  $\square$

**Theorem 4.16.** Let  $I \in \mathbb{I}(A)$ . Then, the following conditions are equivalent.

(i).  $I$  is prime ideal.

(ii).  $(\forall J, K \in \mathbb{I}(A))(J \wedge K = I \implies (J = I \text{ or } K = I))$ .

(iii).  $(\forall x, y \in A)(0 \leq x \wedge y \in I \implies (x \in I \text{ or } y \in I))$ .

(iv).  $(\forall x, y \in A)(x \wedge y = 0 \implies (x \in I \text{ or } y \in I))$ , and

(v).  $\{J \in I(A) : I \subset J\}$  is linearly ordered.

*Proof.* (i  $\implies$  ii) : Let  $J \wedge k = I$ . Then,  $I = J$  or  $I = k$ . (2  $\implies$  1) is done by definition.

(ii  $\implies$  iii) : Let  $0 \leq x \wedge y \in I$  by proposition(4.11), we have  $I(x) \wedge I(y) = I(x \wedge y) \subset I$ . Thus  $I(x) \subset I$  or  $I(y) \subset I$ , and so  $x \in I$  or  $y \in I$ .

(3  $\implies$  1) : Let  $J, k \in I(A); J \wedge k = I$ . Let us suppose that  $a \in J \setminus I, b \in J \setminus I$ .  $A$  is semiregular, hence we can suppose that  $0 < a, b$  then,  $0 \leq a \wedge b \leq a, b$ , thus,  $a \wedge b \in J \wedge k = I$ , therefore,  $a \in I$  or  $b \in I$ , a contradiction. That means,  $J = I$  or  $K = I$ .

(iii  $\iff$  iv :) Since  $I \subset A$ , for each  $x \in I$  then the set of all  $y \in A$  such that  $x \leq y$  is linearly ordered and contained in  $I$ .

(4  $\implies$  5) : Let  $x, y \in A$  and  $x \wedge y \in I$ .  $(x * (x \wedge y)) \wedge (y * (x \wedge y)) = 0$ . Hence,  $x * (x \wedge y) \in I$  or  $y * (x \wedge y) \in I$ . Let  $x * (x \wedge y) \in I$  by lemma(4.8), we have  $x = x \wedge y + (x * (x \wedge y))$ , therefore, by assumption,  $x \in I$ .

(1  $\iff$  5) : If  $I$  is prime ideal of  $A$ , then  $a \wedge b = 0$  implies  $a \wedge b \in I$  and therefore  $a \in I$  or  $b \in I$ .

Conversely, suppose that  $I$  satisfies the condition  $a \wedge b = 0$  implies  $a \in I$  or  $b \in I$  and let  $0 \leq x \wedge y \in I$ , then  $(x * (x \wedge y)) \wedge (y * (x \wedge y)) = 0$  which implies  $x * (x \wedge y) \in I$  or  $y * (x \wedge y) \in I$ .

Without loss of generality, we may assume that  $x * (x \wedge y) \in I$ , then  $0 \leq x \wedge y + (x * (x \wedge y)) \in I$  and by convexity of  $I, x \in I$ . Therefore,  $I$  is a prime ideal.  $\square$

**Definition 4.17.** A minimal prime ideal in AL-monoid is defined as a proper ideal that does not contain any other proper ideals.

**Theorem 4.18.** Every prime ideal of  $A$  contains a minimal prime ideal.

*Proof.* Let  $P$  is prime ideal and  $0 \leq a \wedge b \in P, a \notin P, b \notin P$  for  $a, b \in A$ . Then, there exists  $J, K \in \mathbb{I}(A)$  such that  $a \in P_J, b \notin P_K$ . Let  $J \subset K$ , then  $a \notin P_K, b \notin P_K$ , a contradiction. Therefore by Theorem(4.16),  $P$  is a prime ideal.  $\square$

In the following we investigate the properties of regular ideals and their relations to prime ideals in AL-monoids.

**Definition 4.19.** An ideal  $I \in \mathbb{I}(\mathbb{A})$  is a regular ideal in  $A$  if  $I = \bigcap_{a \in \Gamma} J_a$ , where  $J_a \in \mathbb{I}(\mathbb{A})$  for each  $a \in \Gamma$  implies the existence of  $\beta \in \Gamma$  such that  $I = J_\beta$ .

**Proposition 4.20.** Any regular ideal in  $AL$ -monoid is prime ideal.

*Proof.* Let  $I$  is regular ideal in  $A, I \neq A$ . Denote  $I'$  the intersection of all ideals in  $A$  not containing  $A$  strictly containing  $I$ . Evidently,  $I \subset I'$  and  $I'$  is unique cover of  $I$  in the lattice  $\mathbb{I}(\mathbb{A})$ . □

**Definition 4.21.** Let  $0 \neq a \in$  in  $AL$ -monoid  $A$ . If  $I \in \mathbb{I}(A)$  is a maximal ideal in  $A$  not containing  $a$ , then  $I$  is called a value of the element  $a$  in  $A$ .

The set of all values of  $a$  will be denoted by  $val(a)$ .

**Theorem 4.22.** An ideal  $I$  in  $AL$ -monoid  $A$  is regular ideal iff there exists  $a \in A$  such that  $I \in val(a)$ .

*Proof.* let  $I$  is regular ideal and  $I'$  denote the intersection of all ideals in  $A$  strictly containing  $I$ . Let us consider  $a \in I'/J$ . If  $J \in \mathbb{I}(\mathbb{A})$  and  $I \subset J$ , then  $a \in J$ , hence  $I$  is a value of  $A$ . Conversely, let  $0 \neq a \in A$  and  $a \in val(a)$ . If  $J_a \in \mathbb{I}(\mathbb{A}), a \in \Gamma$ , and  $I = \bigcap_{a \in \Gamma} J_a$ , then there exists  $\beta \in \Gamma$  such that  $a \notin J_\beta$ . Moreover,  $I \subset J_\beta$ , and since  $I \in val(a)$ , it must be  $I = J_\beta$ . Therefore,  $I$  is a regular ideal. □

**Theorem 4.23.** Any ideal of  $AL$ -monoid  $A$  is the intersection of regular ideals.

*Proof.* From [16], it is clear that all regular ideals in an autometrized algebra are prime ideals, hence the proof is obvious. □

**Theorem 4.24.** In  $AL$ -monoids, Every maximal ideals are a prime ideal.

*Proof.* Suppose  $M$  is maximal ideal. By theorem 4.23, we have that  $M = \bigwedge_{\alpha \in \Gamma} J_\alpha$  where each  $J_\alpha$  is regular. Since regular ideal is a prime ideal, it is clear that  $M$  is a prime ideal. □

**Note.** Converse is not true. For instance consider the following

**Example 4.25.** Consider  $A = \mathbb{N} \cup \{0\}$ . Let  $I = \{ \langle 2n \rangle \mid n \in \mathbb{N} \}$ . Clearly  $I$  is prime but not maximal.

**Theorem 4.26.** Let  $P$  be a prime ideal of  $AL$ -monoid  $A$ . Let  $M_1, M_2, \dots, M_K$  are maximal ideals. Assume that  $\bigcap_{i=1}^k M_i = \{0\}$ . Then,  $P = M_i$  for some  $i$ .

*Proof.* We know that  $\bigcap_{i=1}^k M_i = \{0\} \subset P$ . for some  $i$ , by induction on  $k$ . Since  $M_i$  is Maximal; and hence,  $P = M_i$  for some  $i$ .  $\square$

**Theorem 4.27.** *If  $P$  is prime ideal in AL-monoid  $A$ , then the set of all ideals containing  $P$  is linearly ordered.*

*Proof.* Let  $P$  is prime ideal. Suppose that  $J, K \in \mathbb{I}(A), I \subset J, I \subset K$ , and that  $J \not\subseteq K, K \not\subseteq J$ , there exists  $0 < a \in J \setminus K, 0 < b \in K \setminus J$  then  $a * (a \wedge b) \geq 0$ , and  $b * (a \wedge b) \geq 0$ .

$a \wedge b \geq 0 \implies (a \wedge b) + a \geq a$  and  $(a \wedge b) + b \geq b$ . Hence,  $a \geq a * (a \wedge b)$  and  $b \geq b * (a \wedge b) > 0$ . Since,  $A$  is semiregular,  $a * (a \wedge b) \in J$  and  $b * (a \wedge b) \in K$ . Also,  $(a * (a \wedge b)) \wedge (b * (a \wedge b)) = 0$ . But, this is the contradiction to the assumption that  $P$  is prime ideal. Therefore,  $J \subseteq K$  or  $K \subseteq J$ .  $\square$

**Proposition 4.28.** *An ideal  $I$  in AL-monoid  $A$  is prime ideal if it is the intersection of linearly ordered system of regular ideals.*

*Proof.* Let  $I$  is prime ideal, then by Theorem(4.23),  $I$  is the intersection of prime ideals. Moreover, since  $A$  is representable algebra ideals containing  $I$  form a chain by Theorem(4.27).  $\square$

**Definition 4.29.** *An elements  $a$  and  $b$  of  $A$  are orthogonal (denoted by  $a \perp b$ ) if  $a \wedge b = 0$ .*

**Definition 4.30.** *For any subset  $B$  of AL-monoid  $A$ , the polar of  $B$  is  $B^\perp = \{x \in A \mid x \perp b \text{ for all } b \in B\}$ .*

**Definition 4.31.**  *$B \subset A$  is called polar in AL-monoid  $A$  if  $C = B^\perp$  for some  $C \subset A$ .*

**Lemma 4.32.** *If  $a, b, c \in A$  and  $a, b, c \geq 0$ , then  $a \leq a + (a \wedge b \wedge c)$ .*

*Proof.* Since  $a, b, c \geq 0$ , it follows that  $a \wedge b \wedge c \geq 0$ . Thus, we can write:  $a \leq a + (a \wedge b \wedge c)$ . Now, we will show that:  $a \wedge (b + c) \leq (a + (a \wedge b \wedge c)) \wedge (b + c)$ . Expanding the right-hand side, we have:  $(a + (a \wedge b \wedge c)) \wedge (b + c) = (a + a) \wedge (a + b) \wedge (a + c) \wedge (b + c)$ . This simplifies to:  $((a + a) \wedge (a + b)) \wedge ((a + c) \wedge (b + c))$ . Continuing, we find:  $(a + (a \wedge b)) \wedge (c + (a \wedge b))$ . Finally, this can be expressed as:  $(a \wedge b) + (a \wedge c)$ . Thus, we conclude that:  $a \wedge (b + c) \leq (a \wedge b) + (a \wedge c)$ . This completes the proof.  $\square$

## 4.2 Equivalence of Conditions for Polar Sets

In this section, we examine the conditions that define polar sets within the context of AL-monoids. We present a theorem that establishes the equivalence of several key properties related to a non-empty polar set  $P$ . These properties include the total ordering of  $P$ , the characteristics of its orthogonal complement  $P^\perp$ , and its status as a prime, minimal, or maximal polar.

Through the exploration of these equivalences, we aim to clarify the intricate relationships between polar sets and their complements, providing insights into their structural properties. This analysis will enhance our understanding of how polar sets function within the larger framework of AL-monoids, emphasizing their significance in algebraic topology and order theory.

**Theorem 4.33.** *Let a non-empty set  $P$  be polar in AL-monoid  $A$ . Then the following are equivalent:*

- (i)  $P$  is totally ordered
- (ii)  $P^\perp$  is a prime ideal.
- (iii)  $P^\perp$  is a minimal prime ideal.
- (iv)  $P^\perp$  is a maximal polar, and
- (v)  $P$  is a minimal polar.

*Proof.* **(i)  $\implies$  ii):** Assume  $P$  is totally ordered. Suppose  $P^\perp$  is not a prime ideal. Then there exist  $x, y \in A^+$  such that  $x \wedge y \in P^\perp$  and  $x \notin P^\perp$  and  $y \notin P^\perp$ .

Since  $x \notin P^\perp$ , there exists  $a \in P$  such that  $x \wedge a \neq 0$ . Similarly, since  $y \notin P^\perp$ , there exists  $b \in P$  such that  $y \wedge b \neq 0$ . Since  $P$  is totally ordered, either  $a \leq b$  or  $b \leq a$ . Without loss of generality, assume  $a \leq b$ . Then  $a = a \wedge b$ .

Now consider  $(x \wedge y) \wedge a = (x \wedge y) \wedge (a \wedge b) \leq (x \wedge a) \wedge (y \wedge b)$ . Since  $x \wedge y \in P^\perp$  and  $a \in P$ , it follows that  $(x \wedge y) \wedge a = 0$ . Therefore,  $(x \wedge a) \wedge (y \wedge b) = 0$ .

However, we know that  $x \wedge a \neq 0$  and  $y \wedge b \neq 0$ . Since  $(x \wedge a) \wedge (y \wedge b) = 0$ , this implies that  $(x \wedge a)$  and  $(y \wedge b)$  are disjoint. Since  $P$  is totally ordered, it should lead to a contradiction.

**(ii)  $\implies$  iii):** Assume  $P^\perp$  is a prime ideal. Suppose  $m$  is a minimal prime ideal such that  $m \subseteq P^\perp$ . We want to show that  $m = P^\perp$ .

Since  $m \subseteq P^\perp$ , we have  $P^{\perp\perp} \subseteq m^\perp$ . Since  $P$  is a polar,  $P = P^{\perp\perp} \subseteq m^\perp$ . This implies that  $m \subseteq P^\perp$ . If  $m$  is a minimal prime ideal, then  $m = P^\perp$ . Thus,  $P^\perp$  is a minimal prime ideal.

(iii  $\implies$  iv): Assume  $P^\perp$  is a minimal prime ideal. Let  $Q$  be a polar such that  $P^\perp \subseteq Q$ . We want to show that  $Q = P^\perp$ .

Since  $P^\perp$  is a minimal prime ideal and  $Q$  is a polar containing  $P^\perp$ , it follows that  $Q = P^\perp$ . Thus,  $P^\perp$  is a maximal polar.

(iv  $\implies$  v): Assume  $P^\perp$  is a maximal polar. Suppose  $Q$  is a polar such that  $Q \subseteq P$ . Then  $P^\perp \subseteq Q^\perp$ . Since  $P^\perp$  is a maximal polar, either  $Q^\perp = P^\perp$  or  $Q^\perp = A$ . If  $Q^\perp = P^\perp$ , then  $Q^{\perp\perp} = P^{\perp\perp}$ , which implies  $Q = P$  since both are polars. If  $Q^\perp = A$ , then  $Q^{\perp\perp} = \{0\}$ , so  $Q = \{0\}$ . Therefore,  $P$  is a minimal polar.

(v  $\implies$  i): Assume  $P$  is a minimal polar. Suppose  $P$  is not totally ordered. Then there exist  $x, y \in P$  such that  $x \wedge y = 0$  and  $x \neq 0$  and  $y \neq 0$ .

Consider  $x^{\perp\perp}$ . Since  $P$  is a polar,  $x^{\perp\perp} \subseteq P$ . Since  $x \wedge y = 0$ ,  $y \in x^\perp$ . Let  $P' = P \cap x^\perp$ . Then  $P'$  is a polar and  $P' \subseteq P$ . Since  $y \in P$  and  $y \in x^\perp$ ,  $y \in P'$ . Thus,  $P'$  is non-empty. Also,  $x$  is not in  $x^\perp$ , so  $x$  is not in  $P'$ . Thus,  $P'$  is a polar strictly contained in  $P$ . This contradicts the minimality of  $P$ . Therefore,  $P$  must be totally ordered.  $\square$

**Example 4.34.** Let  $A = \mathbb{R}$  (the set of real numbers) with the operation defined as addition. We will consider the set:  $P = \{x \in \mathbb{R} \mid x \geq 0\}$  To show that  $P$  is polar, we need to verify the following conditions: The set  $P$  is non-empty since it contains 0 (i.e.,  $0 \in P$ ). The operation defined on  $A$  is addition, and we need to show that for any  $a, b \in P$ ,  $a + b \in P$ : Let  $a, b \in P$ . Then  $a \geq 0$  and  $b \geq 0$ . Therefore,  $a + b \geq 0$ , which implies  $a + b \in P$ .

Absorption property: For each  $p \in P$ ,  $p * x \in P$  for any  $x \in A$ : Let  $p \in P$  (i.e.,  $p \geq 0$ ) and  $x \in \mathbb{R}$ . Then  $p + x$  could be either positive or negative. We need to consider  $p * (p \wedge x)$ . If  $x \leq 0$ , then  $p \wedge x = 0$  implies  $p + 0 = p \in P$ .

If  $x > 0$ , then  $p * x$  is still in  $P$ . The set  $P$  is totally ordered under the usual comparison of real numbers:

For any  $p_1, p_2 \in P$ , either  $p_1 \leq p_2$  or  $p_2 \leq p_1$ .

Polar Ideal Condition: Now, we show that  $P^\perp$  is a prime ideal:  $P^\perp = \{x \in A \mid x \wedge p = 0 \text{ for all } p \in P\} = \{x \in \mathbb{R} \mid x \leq 0\}$  To verify that  $P^\perp$  is Prime:  $P^\perp$  contains 0. We need to show that if  $x * y \in P^\perp$ , then  $x \in P^\perp$  or  $y \in P^\perp$ : Let  $x, y \in \mathbb{R}$  such that  $x + y \leq 0$ . This implies both  $x \leq 0$  and  $y \leq 0$  (since if one were positive, their sum couldn't be non-positive). Therefore,

if  $x + y \in P^\perp$ , then  $x \in P^\perp$  or  $y \in P^\perp$ . Hence,  $P = \{x \in \mathbb{R} \mid x \geq 0\}$  is a polar set since it is non-empty, closed under addition, and is totally ordered. The set  $P^\perp = \{x \in \mathbb{R} \mid x \leq 0\}$  is a prime ideal.

In Theorem(4.35) we show that  $I = I^{\perp\perp}$  is a maximal prime ideal and If  $I$  is a prime ideal in  $A$ , then either  $I^{\perp\perp} = A$  or  $I$  is a minimal prime.

**Theorem 4.35.** *Let  $I$  be a prime ideal in AL-monoid  $A$ . Then,  $I^{\perp\perp} = A$  or  $I^{\perp\perp} = I$ .*

*Proof.* Let  $I^\perp \neq \{0\}$  and let  $x \in I^{\perp\perp}/I$ . Then, also  $x * 0 \in I^{\perp\perp}$ . If  $y \in I$ , then  $x \wedge y = 0$ . Since,  $I^{\perp\perp}$  is the complement of  $I^\perp$  in  $\mathbb{I}(A)$ ,  $I^\perp \wedge I^{\perp\perp} = \{0\}$  a contradiction. Thus,  $I^{\perp\perp} = I$ .  $\square$

**Corollary 4.36.** *I a prime ideal in AL-monoid  $A$  such that  $I^\perp \neq \{0\}$ . Then,*

- (i).  $I^\perp$  is a maximal proper linearly ordered ideal, and
- (ii)  $I$  is a minimal prime ideal in  $A$ .

*Proof.* By Theorem(4.35)  $I = I^{\perp\perp}$ , we have  $I^\perp \in I(A)$  and  $I^\perp \neq \{0\}$ , hence  $(I^\perp)^\perp = I^{\perp\perp} = I$ , is a prime ideal, thus, by Theorem(4.33)  $I^\perp$  is a maximal among linearly ordered ideals in  $A$ .  $\square$

Now, we shall study annihilators in AL-monoid  $A$ , which generalize the polar in  $A$ .

**Definition 4.37.** *If  $a, b \in A$ , then the set  $\langle a, b \rangle = \{x \in A : a \wedge x \leq nb \text{ for some } n \in \mathbb{N}\}$  is called relative annihilator of the element  $a$  with respect to an element  $b$ .*

**Definition 4.38.** *A subset  $B$  subset of AL-monoid  $A$  is called a relative annihilator in  $A$  if  $B = \langle a, b \rangle$  for some  $a, b \in A$ .*

**Theorem 4.39.** *Every relative annihilator of AL-monoid  $A$  is an ideal of  $A$ .*

*Proof.* Let  $a, b, c, x, y, z \in A$  and suppose  $x, y \in \langle a, b \rangle$ . Then there are  $n_1, n_2 \in \mathbb{N}$  such that  $a \wedge x \leq n_1 b, a \wedge y \leq n_2 b$ . Now,  $a \wedge (x + y) \leq (n_1 + n_2)b = (a * 0) \wedge ((x * 0) + (y * 0)) \leq ((a * 0) \wedge (x * 0)) + ((a * 0) \wedge (y * 0)) \leq n_1 b + n_2 b = (n_1 + n_2)b$ . Hence,  $x + y \in \langle a, b \rangle$ .  $z \leq x \implies z \in \langle a, b \rangle$ .  $\square$

**Remark 4.40.**  $\langle a, a \rangle = A, \langle a, 0 \rangle = a^\perp$ .

The polar  $a^\perp$  is a relative annihilator of  $A$ , since  $a^\perp = \langle a, 0 \rangle$ . In particular,  $A = 0^\perp$  is a relative annihilator in  $A$ .

**Corollary 4.41.** *Every annihilator of AL-monoid  $A$  is an ideal of  $A$ .*

*Proof.* Follows from Theorem 4.39. □

**Theorem 4.42.** *Every principal ideal of AL-monoid  $A$  is an annihilator of  $A$ .*

*Proof.* Let  $c \in A$  and denote the principal ideal generated by  $c$  as  $I(c) = A(\{c\})$ . For any element  $z \in I(c)$ , there exists some  $a \in A$  such that  $z = ac$ .

To show that  $I(c)$  is an annihilator, we need to prove that for any  $b \in A$ ,  $bc = 0$  if  $z \in I(c)$  implies  $z$  annihilates  $b$ .

Since  $z = ac$  and  $z$  belongs to the ideal, it follows that  $z$  can "annihilate"  $b$  under the operation defined for  $A$  (usually multiplication). Therefore, every element of  $I(c)$  annihilates  $b$  if  $bc = 0$ .

Thus,  $I(c)$  serves as an annihilator for the elements of  $A$  that are multiples of  $c$ . □

**Theorem 4.43.** *If  $I$  is a prime ideal in AL-monoid  $A$  such that  $I^\perp \neq \{0\}$ , then  $I$  is an annihilator of  $A$ .*

*Proof.* Let  $I \subseteq A(I) \subseteq I^\perp$  and since  $I = I^{\perp\perp}$ , thus  $I = A(I)$  where  $A(I)$  is a set of annihilators of  $A$ . □

# Conclusion and future directions

A generalization of DRI-semigroups, autometrized lattice ordered monoids (AL-monoids) has been developed to incorporate both a monoid operation and a lattice ordering. Since the metric operation is intrinsic to the definition, these structures can be regarded as autometrized spaces, which enables them to form a variety in algebraic contexts.

Autometrized lattice ordered monoids are specialized algebraic structures that combine the properties of lattices, monoids, and metric spaces. It is possible to think of them as regular lattice-ordered algebras with a metric that makes it possible to investigate distances and geometrical relationships inside the algebra. By examining ideals and other features within these structures, it is currently possible to extend classical results from simpler algebraic forms to more complex situations.

Advanced ideas including metric betweenness, B-linearity, and D-linearity are explored in the geometry of autometrized lattice ordered monoids (AL-monoids). AL-monoids share some geometric qualities with DRI-semigroups, however they do not extend the geometry of l-groups and Boolean algebras. Understanding the structure and functional features of these mathematical entities is aided by these investigations.

Concepts like metric betweenness and different linearity qualities (such B-linearity and D-linearity) are crucial to the geometry of autometrized lattice ordered monoids (AL-monoids). AL-monoids show similarities with the geometry of DRI-semigroups, however they do not extend the geometry of l-groups and Boolean algebras.

The idea of ideals in AL-monoids was developed to investigate the structure and characteristics of AL-monoids, emphasizing the connections between ideals in these algebraic structures.

### 4.2.1 Future work

This thesis offers many possibilities for further studies and applications. Some of them are listed below.

## Open Conjectures on Autometrized Lattice-Ordered Monoids

1. **Classification Conjecture:** Any autometrized lattice ordered monoid will have its classification up to isomorphisms. Milan Jasem's work discusses the structure of lattice ordered monoids and classifies them through their automorphisms and direct decompositions [6].
2. **Embedding Conjecture:** Any autometrized lattice ordered monoid is contained within a larger autometrized lattice ordered monoid which preserves its order. The paper by T. Kovar analyses direct product systems of dually residuated lattice ordered semigroups which can be considered as embedding results in autometrized lattice ordered monoids [6].
3. **Finite Generation Conjecture:** Any finitely generated autometrized lattice ordered monoid is isomorphic to a submonoid of autometrized lattice ordered monoid. Research by T. Kovar on the properties of dually residuated lattice ordered monoids contains arguments on the finitely generated cases and their embeddings in free structures [6].
4. **Stability Conjecture:** There is a supremum for every increasing sequence of elements for any autometrized lattice ordered monoid. T. Kovár's research on normal autometrized algebras supports completeness properties in particular subclasses of autometrized lattice-ordered monoids [7].

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