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GRADUATE SEMINAR REPORT ON

COMPLETENESS AND COMPACTNESS

OF

FUNCTION SPACES

By

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May 1998.

A C K N O L E D G E M E N T

I would like to thank my advisor Dr. Seid Mohamed for his helps to complete this seminar report.

C O N T E N T S

PREFACE

This seminar report has two parts. In the first part we prove that C(X,Y), the set of all continuous functions from a topology X to a complete metric space Y is complete in the uniform topology. As a consequence we prove the famous peano space - Filling curve.

In the second part we shall prove a classical revsion of Ascolis Theorem. We also define three topologies on C(X,Y) namely, the point-open topology, the topology of compact convergence and the compact-open topology and we study their inclusion relation. We complete this report defining the notion of compactly generated spaces and showing that the evaluation map is continuous.

Definition 1.1: Let (X,d) be a metric space A sequeuce (x_n) of points of X is said to be a Cauchy-sequence, if for each $\epsilon > 0$ there is a natural number N such that

$$d(x_n, x_m) < \epsilon \quad , \forall n, \ m \geq N$$

The metric space (X,d) is said to be complete if every Cauchy sequence in X converges.

Lemma 1.2: Let (X,d) be a metric space and Let (x_n) be a sequence of points of X.

Then, (i) If (x_n) converges then (x_n) is a cauchy sequence.

(ii) If (x_n) is a cauchy-sequence and it has a convergent subsequence (x_{n_k}), that converges to a point $x \in X$, then the sequence (x_n) it self converges to x.

(iii) If (x_n) is Cauchy then the set $A = \{x_n : n \in \mathbb{N}\}$ is bounded.

Proof: (i) Let $x \in X : d(x_n, x) \longrightarrow 0$ and let $\epsilon > 0$

Then, there is $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon/2, \ \forall n \geq N$ and hence

$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon, \ \forall n, m \geq N$

(ii) Let $\epsilon > 0$. Choose $N_1 \in \mathbb{N} : d(x_n, x_m) < \epsilon/2 \ , \ \forall n, m \geq N_1$ (1)

(using the fact that (x_n) is a Cauchy-sequence)

Then using the fact that $d(xn_k, x) \underset{k \longrightarrow \infty}{\longrightarrow} 0$ choose $N_2 \in \mathbb{N}$ such that

$d(x_{n_k}, x) < \epsilon/2 \ , \ \forall \ n_k \geq N_2$ put $N = \max \{N_1, N_2\}$

then from (1) we get: $d(x_n, x_N) < \epsilon/2 \ , \ \forall n \geq N$ and from (2)

we get $d(x_N, x) < \epsilon/2$.

$d(x_n, x) \leq d(x_n, x_N) + d(x_N, x) < \epsilon \quad , \ \forall n \geq N$

Therefore, the Cauchy-sequence (x_n) converges to x.

(iii) Let (x_n) be a Cauchy-sequence in X.

choose $N \in \mathbb{N}$: $d(x_n, x_m) < 1$, $\forall n, m \geq N$. (We need to show that $\{x_n : n \in \mathbb{N}\} \subseteq B_d(x, M)$ for some $x \in X$ and $M > 0$). Let $x \in X$ be fixed.

Since $|d(x_n, x) - d(x_m, x)| \leq d(x_n, x_m)$ we have:

$d(x_n, x) < 1 + d(x_m, x)$, $\forall n, m \geq N$ and hence $d(x_n, x) < 1 + d(x_N, x)$, $\forall n \geq N$

Set $M := \max \{d(x_1, x), d(x_2, x), \ldots, d(x_{N-1}, x), d(x_N, x) + 1\}$

then, $d(x_n, x) \leq M$, $\forall n \in \mathbb{N}$

and hence $x_n \in B_d(x, M)$, $\forall n \in \mathbb{N}$

Remarks 1.3 By the above lemma a metric space X is complete iff every Cauchy-sequence in X has a convergent subsepuence. This criterion we are going to use it several times in the next proofs.

(2) It is easy to prove but important to observe that any closed subset A of a complete metric space (X,d) is necessarily complete.

To see this, Let (x_n) be a Cauchy-sequence in A. Then (x_n) is a Cauchy-sequence in X whence it converges to a point $x \in X$. Because A is a closed subset of X, the point x must lie in A.

Theorem 1.4: Consider the Euclidean space $\mathbb{R}^k, k \in \mathbb{N}$ and let d be the Euclidean metric on $\mathbb{R}^k$. Let ρ be the square metric on R^k. Then the spaces $(\mathbb{R}^k, d)$ and $(\mathbb{R}^k, \rho)$ are both complete.

Proof: To show first that $(\mathbb{R}^k, \rho)$ is complete. Let (x_n) be a Cauchy-sequence in $(\mathbb{R}^k, \rho)$. By Lemma 1.2 (iii) the set $\{x_n : n \in \mathbb{N}\}$ is a bounded subset of $(\mathbb{R}^k, \rho)$. Hence, $\exists\, M > 0 : \rho(x_n, 0) \leq M$, $\forall n \in \mathbb{N}$.

Let $x_n = \left[x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}\right]$. Then $\rho(x_n, 0) = \max \left\{|x_i^{(n)}| : i=1, \ldots, k\right\}$

Therefore, $|x_i^{(n)}| \leq M$, $\forall n \in \mathbb{N}$ and $\forall i = 1, 2, \ldots, k$

So $-M \le x_i^{(n)} \le M$ or $x_i^{(n)} \in [-M,M]$ $\forall n\in\mathbb{N}$ and $\forall i = 1,\dots,k$.

Thus $\quad x_n \in [-M,M]^k$, $\forall n\in\mathbb{N}$.

So all the points of the sequence (x_n) lie in the cube $[-M,M]^k$ in other words , (x_n) is a sequence of points of the set $[-M,M]^k$ which is compact (and hence sequentially compact). Therefore, (x_n) has a convergent subsequence and hence $(\mathbb{R}^k,\rho)$ is complete, by Lemma 1.2

To show that $(\mathbb{R}^k,d)$ is complete , note that

$$\rho(x,y) \le d(x,y) \le \sqrt{k}\ \rho(x,y) \ , \ \forall x\in\mathbb{R}^k.$$

Hence d and p give the same topology on $\mathbb{R}^k$ (the product topology) and so, they are equivalent metrics.

Since $(\mathbb{R}^k,\rho)$ is complete, the space $(\mathbb{R}^k,d)$ is also complete.

Remarks 1.5: Let (X.d) be a metric space

Define, $\bar{d}(x,y) := \min \{d(x,y),1\}$ (standard bounded metric). Then the metrics d and $\bar{d}$ are equivalent since d and $\bar{d}$ induce the same topology on X. To see this let $\epsilon > 0$ and put $\delta = \min \{\epsilon,1\}$

then, $B_d(x,\epsilon) \subseteq B_{\bar{d}}(x,\epsilon)$ and

$$B_{\bar{d}}(x,\delta) \subseteq B_d(x,\epsilon) \qquad (*)$$

Hence if X is complete with regard to metric d then X is complete with regard to the standard bounded metric $\bar{d}$ corresponding to d, and conversely.

EXAMPLE 1: Let X be any set and d be the discrete metric i.e

$$d(x,y) = 1 \quad \text{if } x \ne y$$
$$d(x,y) = 0 \quad \text{if } x = y$$

Then the metric space (X,d) is complete since any Cauchy-sequence in X is a constant sequence except for finitely many terms.

Example 2: Let Q be the set of rational numbers in the usual metric $d(x,y) = |x-y|$.

Consider the sequence $x_n = \left(1 + \frac{1}{n}\right)^n, n\in\mathbb{N}$

Clearly, (x_n) is a Cauchy sequence of rationals and $x_n \longrightarrow e \notin Q$. Hence Q is noncomplete.

Example 3: Another noncomplete space is the open interval $(-1,1)$ of $\mathbb{R}$, in the usual metric $d(x,y) = |x - y|$.

Consider the sequence (x_n) with $x_n := 1 - \frac{1}{n}$ it is a Cauchy sequence in $(-1,1)$ but $x_n \longrightarrow 1 \notin (-1,1)$

In this example we can see that completeness is not a topological property (it is not preserved by homeomorphisms) since the interval $(-1,1)$ is homeomorphic to $\mathbb{R}$ and $\mathbb{R}$ is complete in the above metric.

Definition 1.6 Let (Y,d) be a mertic space and let $\bar{d}$ be the standard bounded metric on Y corresponding to d. Given an arbitrary index set J, consider the set Y^J of all functions $f: J \longrightarrow Y$. Define $\bar{\rho} : Y^J \times Y^J \longrightarrow \mathbb{R}$ by

$$\bar{\rho}(f,g) = \sup \left\{\bar{d}(f(a),g(a)) : a \in J\right\}.$$

Then $\bar{\rho}$ is a metric ; it is called the uniform metric on Y^J , corresponding to the metric d on Y. It is easy to check that $\bar{\rho}$ is a metric:

1. Clearly, $\bar{\rho}(f,g) \geq 0, \ \forall f, g \in Y^J$.
2. $\bar{\rho}(f,g) = 0 \Rightarrow \bar{d}(f(a),g(a)) \leq 0 , \ \forall a \in J$

and since $\bar{d}(f(a),g(a)) \geq 0 \ \ \forall a \in J$ we have

$\bar{d}(f(a),g(a)) = 0 , \ \forall a \in J \Leftrightarrow f(a) = g(a), \ \forall a \in J \Leftrightarrow f = g.$

3. $\bar{d}(f(a),g(a)) \leq \bar{d}(f(a),h(a)) + \bar{d}(h(a),g(a)) \leq \bar{\rho}(f,h) + \bar{\rho}(h,g)$

   Hence , $\sup_{a \in J} \{\bar{d}(f(a),g(a)\} \leq \bar{\rho}(f,h) + \bar{\rho}(h,g)$

   $\Rightarrow \ \bar{\rho}(f,g) \leq \bar{\rho}(f,h) + \bar{\rho}(h,f)$

The metric $\bar{\rho}$ induces the uniform topology on Y^X

Theorem 1.7: Let (Y,d) be a metric space. If Y is complete then $(Y^J,\bar{\rho})$ is complete.

Proof: Let $(f_n)n \in \mathbb{N}$ be a Cauchy-sequence in $(Y^J,\bar{\rho})$

Then for each $a \in J$.

$$\bar{d}(f_n(a),f_m(a)) \leq \bar{\rho}(f_n,f_m) \quad \text{holds}$$

Hence, the sequence $\{f_n(a)\}_{n\in\mathbb{N}}$ is a cauchy sequence in $(Y,\bar{d})$. Therefore $\{f_n(a)\}$ is also a cauchy sequence in (Y,d)

Since (Y,d) is complete $\{f_n(a)\}$ converges in Y, say to a point $y_a \in Y$.

Define,

$f.J \longrightarrow Y$ by $f(a) = y_a$

We claim that $f_n \longrightarrow f$ under $\bar{\rho}$

Given $\epsilon > 0$ choose $N : \bar{\rho}(f_n, f_m) < \epsilon/2, \forall n,m \geq N$

then $\quad \bar{d}(f_n(a), f_m(a)) < \epsilon/2 , \forall n,m \geq N , \forall a\in J. \qquad (1)$

For fixed $a\in J$ and $n\in\mathbb{N}$ taking the limit of (1), for $m \longrightarrow \infty$. we get:

$$\bar{d}(f_n(a), f(a)) \leq \epsilon/2 \qquad (2)$$

The above inequality holds for all $a\in J$ and for all $n \geq N$.

Therefore, $\bar{\rho}(f_n,f) = \sup\left\{\bar{d}(f_n(a),f(a)):\ a\in J\right\} \leq \epsilon/2 < \epsilon \ , \ \forall n \geq m.$

Remarks 1.8(1) For $Y = \mathbb{R}$ and $J = \mathbb{N}$ we get that the space $\mathbb{R}^\omega$ is complete in the uniform topology (induced by $\bar{\rho}$). $\mathbb{R}^\omega$ in the box topology can not be complete since it is not metrizable space. In the product topology $\mathbb{R}^\omega$ is metrizable. Consider the metric: $D(x,y) = \sup\left\{\dfrac{\bar{d}(x_i,y_i)}{i}\right\}$

The metric D induces the product topology on $\mathbb{R}^\omega$

where $\bar{d}(a,b) = \min\left\{|a-b|,1\right\}$. $\mathbb{R}^\omega$ is complete under D.

To show this , let (x_n) be a Cauchy-sequence of R^ω under D, where

$x_n = \left[\lambda_1^{(n)},\lambda_2^{(n)},\ldots,\lambda_i^{(n)},\ldots\right]\in \mathbb{R}^\omega$, Let i be fixed for the moment.

Since, $\bar{d}\left[\lambda_i^{(n)},\lambda_i^{(m)}\right] \leq i\ D(x_n,x_m)$ holds, the sequence $\left\{\lambda_i^{(n)}\right\}_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(\mathbb{R},\bar{d})$. Therefore it converges to a real number say α_i.

Let $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, $\alpha \in \mathbb{R}^\omega$. We claim that $x_n \longrightarrow \alpha$.

Let U be a basis element of the product topology on $\mathbb{R}^{\omega}$: such that a∈U. Then, $U = \Pi U_i$ where $U_i = \mathbb{R}$ except for finitely manyt ihdices i. Therefore, $U_i = R$ for all $i > m$, where m is the maximum value of i such that $U_i = \neq \mathbb{R}$

if $i \le m$, choose $N_i \in \mathbb{N}$ (using the fact that $\lambda_i^{(n)} \longrightarrow a_i$) such that $\lambda_i^{(n)} \in U_i, \forall n \ge N_i$ Put $N = \max \{N_i, i = 1,2,\dots,m\}$ then $\lambda_i^{(n)} \in U_i$, $\forall n \ge N$, $\forall i = 1,2,\dots m$. Since $\lambda_i^{(n)} \in U_i = \mathbb{R}$, $\forall n \in \mathbb{N}$ and $\forall i > m$ we get

$x_n = \left[\lambda_1^{(n)} , \lambda_2^{(n)}, \dots\right] \in \Pi U_i = U$, $\forall n \ge N$

Since U was an arbitrary n.b.d of α, $x_n \longrightarrow \alpha$

Remark 2: Clearly Theorem 1.7. holds for J = X where X is a topological space. In this case consider the subset C(X,Y) of Y^X , consisting of all continuous functions f: X⟶Y. We shall show that C(X,Y) is closed in Y^X and hence C(X,Y) is complete, provided of course that Y is complete.

Theorem 1.9.: Let X be a topological space and Let (Y,d) be a metric space. The set C(X,Y) is closed in Y^X under the uniform metric $\bar{\rho}$. Therefore if Y is complete, C(X,Y) is complete under $\bar{\rho}$.

Proof: Let $f \in Y^X$ be a limit point of C(X,Y). Then there is a sequence (f_n) of continuous functions f_n: X ⟶Y such that $f_n \longrightarrow f$ under $\bar{\rho}$. Hence, given $\epsilon > 0 \ \exists \ N$: $\bar{\rho}(f_n,f) < \epsilon$, $\forall n \ge N$

since, $\bar{d}(f_n(x),f(x)) \le \bar{\rho}(f_n,f)$, $\forall \ x \in X$

we get that $\bar{d}(f_n(x),f(x)) < \epsilon$, $\forall n \ge N$ and for all $x \in X$. Therefore, $f_n \longrightarrow f$ uniformly (under $\bar{d}$). By the uniform limit theorem (p.130-Munkres), $f \in C(X,Y)$. By Remark 1.3 C(X,Y) is complete under $\bar{\rho}$.//

Lemma 1.11: Let X be a topological space. The set $B(X,\mathbb{R})$ is complete under ρ (sup-metric).

Proof: It suffices to show that $B(X,\mathbb{R})$ is a closed subset of $\mathbb{R}^X$ Let f be a limit point of $B(X,\mathbb{R})$. Then there is a sequence of bounded function $f_n: X \longrightarrow \mathbb{R}$, such that $f_n \longrightarrow f$ under $\bar{\rho}$, or $\bar{\rho}(f_n, f) \longrightarrow 0$.

Choose N such that $\bar{\rho}(f_n, f) < 1$, $\forall n \geq N$. In particular we have $\bar{\rho}(f_N, f) < 1$. Hence, $\bar{d}(f_N(x), f(x)) < 1$, $\forall x \in X$, where $\bar{d}(f_N(x), f(x)) =$

$\min \{|f_N(x) - f(x)|, 1\}$

Then, $|f_N(x) - f(x)| < 1$ $\forall x \in X$ and so, $|f(x)| < 1 + |f_N(x)|$, $\forall x$

But $f_N \in B(X,\mathbb{R})$.

This implies that $\exists M > 0$: $|f_N(x)| \leq M$, $\forall x \in X$

So, $|f(x)| \leq 1 + M$, $\forall x \in X$. i.e $f \in B(X,\mathbb{R})$. Since $f: X \longrightarrow \mathbb{R}$ was an arbitrary limit point of $B(X,\mathbb{R})$, $B(X,\mathbb{R})$ contains all its limit points. Therefore $B(X,\mathbb{R})$ is closed under $\bar{\rho}$.

Theorem 1.12: Let (X,d) be a metric space. There is an isometric imbedding of X into a complete metric space.

Proof: Let $B(X,\mathbb{R})$ be the set of all bounded functions mapping X into $\mathbb{R}$. It is a complete metric space under the sup metric ρ. Let $x_0 \in X$ be a fixed point For each $a \in X$,define $\varphi_a: X \longrightarrow \mathbb{R}$ by: $\varphi_a(x) = d(x,a) - d(x,x_0)$.

Then $|\varphi_a(x)| = |d(x,a) - d(x,x_0)| \leq d(a,x_0) := M_a > 0$

Hence $\varphi_a \in B(X,\mathbb{R})$, $\forall$ $a \in X$.

Define , $\Phi: X \longrightarrow B(x,\mathbb{R})$ by $\phi(a) = \varphi_a$

we claim that Φ is an isometric imbedding of (X,d) into the complete space $(B(X,\mathbb{R}),\rho)$. We need to show $d(a,b) = \rho(\varphi_a, \varphi_b)$.

Observe that $|\varphi_a(x) - \varphi_b(x)| = |d(x,a) - d(x,x_0) - d(x,b) + d(x,x_0)|$

$= |d(x,a) - d(x,b)|$ $\forall x$. Using the quadrangle inequality we get that $|d(x,a) - d(x,b)| \leq d(x,x) + d(a,b) = d(a,b)$. So $|\varphi_a(x) - \varphi_b(x)| \leq d(a,b)$, $\forall x \in X$.

Hence, $\rho(\varphi_a,\varphi_b) = \sup_{x\in X}\left\{|\varphi_a(x) - \varphi_b(x)|\right\} \le d(a,b)$. But since $a\in X$ we have $d(a,b) = |\varphi_a(a) - \varphi_b(a)| \le p(\varphi_a,\varphi_b)$

Therefore, $d(a,b) = p(q_a,q_b)$ and hence the proof of the theorem is completed.

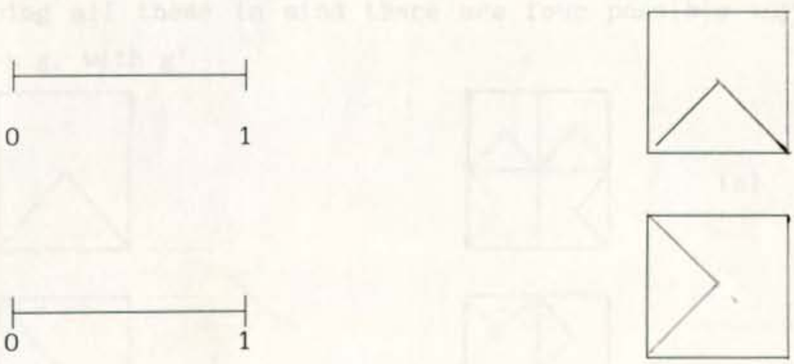
2. A space-Filling Curve

As an application of the completeness of the space C(X,Y) in the uniform metric when Y is complete, we shall construct the famous "Peano space-filling curve".

Theorem: Let $I = [0,1]$. There exists a continnous map $f\colon I \longrightarrow I^2$ which is surjective.

Our aim is to construct a sequence of continuous functions (f_n), $f_n\colon I \longrightarrow I^2$, n 0,1,... which is a Cauchy sequence of $C(I,I^2)$ If $f_n \longrightarrow f$ then $f\colon I \longrightarrow I^2$ is continuous.

Step 1: Let [a,b] be a closed interval in the real line and let H,be any square in the plane with sides parallel to the coordinate axes. Then always we can define a triagular path g (continuous) , $g\colon[a,b] \xrightarrow{\text{into}} H$. in particular for $I = [0,1]$ consider a triagular path $g\colon I \longrightarrow I^2$. Since g always connect two adjacent corners of the square I^2 there are four possible ways to define a such that g



(a)

(b)

(c)

(d)

Fig. 1

Step 2: Consider the map $g':[0,1]\longrightarrow I^2$ pictured in Figure 2.

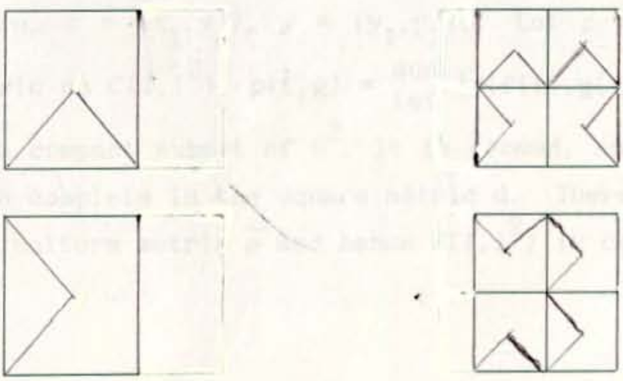
Fig.2

It is made up of forur triagular paths such that

1. Each one is half of the size of g
2. Each one lies in a sub-square of I^2 (The lenght of the side of each sub-square is half the lengtht of the side of I^2)
3. The end points of each triagular path) are adjacent cornens of the corresponding sub-square.

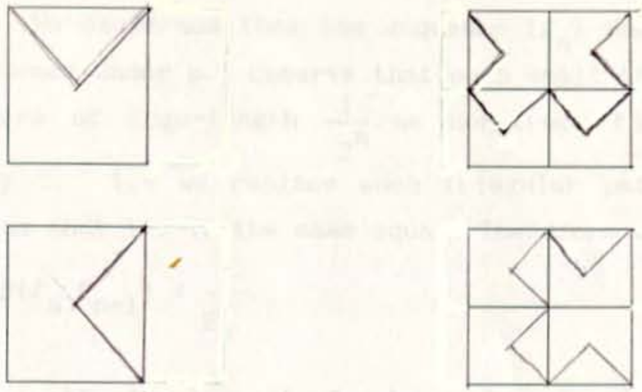
Clearly, each one of them has been defined on **Step 1**.

g' is a continuous map of $I = [0,1]$ into $I^2 = [0,1] \times [0,1]$ Note that, if a path g is given (from step. 1) we can always define a path g' with the above properties and such that g and g' have the same end points. Keeping all these in mind there are four possible ways to "replace" a given g. with g'



(a)

(b)



(c)

(d)

Figure 3

Step. 3 Now we define a sequence of functions $f_n : I \longrightarrow I^2$ for $n = 0, 1, \ldots$

put $f_0 = g$, (see Fig.1,a)

$f_1 = g$, (see Fig.3,a) where f_1 consists from $4 = 4^1$ triagular paths. The next function f_2 is the function obtained by "replacing" each of the four triagular paths of f_1 with four triagular paths using step 2. Then f_2 consists of $4^2 = 16$ triagular paths, each one lies in a square of edge Legth $1/2^2$. And so on! At the general step, f_n is a continuous map, $f_n : I \longrightarrow I^2$ made up of 4^n triagular paths of the type considered in step 1, each one lying in a square of edge-length $1/2^n$. We obtain the function f_{n+1} by applying the operation of step 2 to each of the 4^n triagular paths of f_n, replacing each one by four smaller triagular paths.

Step 4: For the needs of the proof Let d denote the square metric on $\mathbb{R}^2$.

$$d(X, Y) = \max \{|x_1 - y_1|, |x_2 - y_2|\}$$

Where, $x = (x_1, x_2)$, $y = (y_1, y_2)$. Let ρ denote the corresponding sup metric on $C(I, I^2)$, $\rho(f,g) = \sup_{t \in I} \{d(f(t), g(t))\}$ since $I^2 = [0,1] \times [0,1]$ is a compact subset of $\mathbb{R}^2$, it is closed, and since $\mathbb{R}^2$ is complete I^2 is also complete in the square metric d. Therefore, $C(I, I^2)$ is complete in the uniform metric $\bar{\rho}$ and hence $C(I, I^2)$ is complete under ρ.

We claim now that the sequence (f_n) defined in step 3 is a Cauchy sequence under ρ. Observe that each small triagular path in f_n lies in a square of edgc-length $\frac{1}{2^n}$ we construct the function f_{n+1} using the step 2. i.e we replace each triagular path of f_n by four triagular paths that lie in the same square Therefore $d(f_n(t), f_{n+1}(t)) \le \frac{1}{2^n}$, $\forall t \in I$. So $\rho(f_n, f_{n+1}) \le \frac{1}{2^n}$.

Since $\rho(f_n, f_{n+m}) \le \rho(f_n, f_{n+1}) + \rho(f_{n+1}, f_{n+2}) + \ldots + \rho(f_{n+m-1}, f_{n+m})$

we have: $\rho(f_n, f_{n+m}) \le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+m-1}} = \frac{1}{2^{n+m-1}}$

$= \frac{1}{2^n}\left[1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-1}}\right] \le \frac{1}{2^n}\sum_{i=0}^{\infty}\frac{1}{2^i} = \frac{2}{2^n}$, for all n, m

Hence (f_n) is a Cauchy sequence in $C(I, I^2)$. Since $I^2 = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ is compact (finite product of compact spaces), $C(I, I^2)$ is complete and so (f_n) converges to a continuous function $f: I \longrightarrow I^2$.

Now we need to show that f is surjective, it suffices to show that $I^2 \subseteq f(I)$ observe that $f(I)$ is a compact subset of I^2 (continuous image of a compact set) and hence $f(I)$ is closed. So if we show that $I^2 \subseteq \overline{f(I)}$ then we are done.

Let $x \in I^2$ and $n \in \mathbb{N}$. By the prerious considerations f_n is made up of 4^n triagular paths each lying in a square, say I_n^2 of edge-length $\frac{1}{2^n}$. Since $x \in I^2$, x lies in one of those I_n^2s. At the same square there is (also) a triagular path. The distance (in the square metric) between x and any point f(t) of this triagular path is at most $\frac{1}{2^n}$. Therefore there exist $t_0 \in I: d(x, f_n(t_0)) \le \frac{1}{2^n}$, for all $n \in \mathbb{N}$. (1)

Hence, given $\epsilon > 0$ choose N such that $\frac{1}{2^N} < \epsilon/2$ and $\rho(f_N, f) < \epsilon/2$. Then by (1) $d(x, f_N(t_0)) \le \frac{1}{2^N} < \epsilon/2$ and $d(f_N(t_0), f(t_0)) \le \rho(f_N, f) < \epsilon/2$

So, $d(x, f(t_0)) \le d(x, f_N(t_0)) + d(f_N(t_0), f(t_0)) < \epsilon$. Hence, $f(t_0) \in B_d(x,\epsilon) \cap f(I) \Rightarrow B_d(x,\epsilon) \cap f(I) \ne \emptyset$. Since $\epsilon > 0$ was arbitrary $x \in \overline{f(I)}$. Since $x \in I^2$ was arbitrary $I^2 \subseteq \overline{f(I)}$. The proof is completed.//

3. Compactness - Completeness in Metr. Spaces.

Definition 3.1: A metric space (X,d) is said to be totally bounded if for every $\epsilon > 0$ there is a finite covering of X by ϵ-balls.

Note: Since in the metric topology the bases elements are ϵ-balls, any covering of any space X is a collection of ϵ-balls. Observe that total boundedness implies boundedness. To show this let $\epsilon > 0$. Since X is totally bounded there are $x_1, x_2, \ldots, x_n \in X : X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$. Then for any two points α_1, α_2 of X each one of them lies in some of the above ϵ-balls. If $M := \max \{d(x_i, x_j) : i \neq j, \ i,j = 1,2,\ldots,n\}$ then $d(\alpha_1, \alpha_2) \leq M + 2\epsilon$.

But the converge does not hold. Under the metric $\bar{d}(x,y) = \min\{|x-y|, 1\}$ the real line is bounded but not totally bounded (for $\epsilon < 1$).

Theorem 3.2: Let (X,d) be a metric space. Then the following statements are equivalent.

(a) X is compact

(b) X is complete and totally bounded.

(c) X is sequentially compact.

Proof: By Theorem 7.4 (p.181- Munkres) (a) and (c) are equivalent.

(a) $\Rightarrow$ (b) Let X be compact. Then X is a sequentially compact space. Hence every Cauchy sequence converges in X. Therefore X is complete.

On the other hand given $\epsilon > 0$ a covering of X by ϵ-balls has a finite sub-covering that means X is totally bounded.

(b) $\Rightarrow$ (c) Let (x_n) be a sequence in X. We shall construct a Cauchy subsequence of (x_n). Using the fact that X is totally bounded cover X by finitely many balls of radius 1. These 1-balls cover (also) the infinite set: $\{x_n : n \in \mathbb{N}\}$. Necessarily, at least one of then constains infinitely many terms of the sequence (x_n) Let B_1 be this 1-ball, and

Let $J_1 = \{n\in\mathbb{N}: x_n\in B_1\}$ choose an arbitrary point in J_1, say $n_1 \in J_1$ Next cover X by finitely many balls of radius $\frac{1}{2}$ and consider the set $\{x_n : n\in J_1\}$. Since it is an infinite set and it is covered by the above $\frac{1}{2}$- balls, at least one of them, say B_2, contains x_n for infinitely many values of $n(n\in J_1)$ Let $J_2 = \{n\in J_1 : x_n\in B_2\}$. Choose $n_2\in J_2$: $n_2 > n_1$ In general given an infinite set J_k, $(J_k \subseteq \mathbb{N})$ and $n_k\in J_k$ cover X by finitely many balls of radius $\frac{1}{k+1}$ and consider the (infinite) set $\{x_n: n\in J_k\}$; some of the above $\frac{1}{k+1}$ - balls, say B_{k+1}, contains infinitely many elements of the set $\{x_n : n\in J_k\}$. Let J_{k+1} be the infinite subset of J_k such that $J_{k+1} = \{n\in J_k: x_n\in B_{k+1}\}$. Since J_{k+1} is infinite we can choose $n_{k+1}\in J_{k+1} : n_{k+1} > n_k$.

Consider the subsequence (x_{n_k}) of (x_n). For $i,j \geq k$ we have $n_i\in J_i$ and $n_j\in J_j$. Since $J_k \supseteq J_{k+1}$, $\forall k\in\mathbb{N}$ (by construction). Then $J_i \subseteq J_k$ and $J_j \subseteq J_k$ hence, $n_i, n_j\in J_k$ Hence x_{n_i}, x_{n_j} are both contained in B_k That means $d(x_{n_i}, x_{n_j}) < \frac{1}{k}$ So, let $\epsilon > 0$. Choose $k\in\mathbb{N}$: $\frac{1}{k} < \epsilon$; then $d(xn_i, xn_j) < \epsilon$, $\forall\ i,j \geq k$. Since (x_{n_k}) is Cauchy and X is complete it converges. Therefore, since (x_n) was an arbitrary sequence in X the space X is sequentially compact.

Corollary 3.4: Let (X,d) be a metric space and Let $Y \subseteq X$ Then Y is compact if and only if it is complete and totally bounded.

Proposition 3.5: Let (X,d) be a metric space Let $Y \subseteq X$ be a compact subset of X. Then Y is closed and bounded in X.

Proof: Since X is a metric space and hence a Hausdorff space, every compact set is closed (Th.5 5.3, p. 166. Mankres). To show now that Y is bounded. Observe that the collection $\{B_d(x,n): n\in\mathbb{N}\}$ is an open coreving of Y (because for $y\in Y$ there is $n\in\mathbb{N}$ such that $d(x,y) < n$, i.e $y\in B_d(x,n)$).

Since Y is compact, there is a finite sub-collection which covers Y. Therefore there is $M > 0$: $Y \subseteq B_d(x,M)$ and so Y is bounded.

Remarks: Let $X = \mathbb{R}^n$. A subset Y of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (in the euclidean metric or in the square metric). Consider now the function space $\mathbb{C}(X,\mathbb{R}^n)$ in the uniform topology. Let X be a compact space and let d denote either the euchidean or the square metric on $\mathbb{R}^n$. Since the set $\{d(f(x),g(x)): x\in X\}$ is bounded for all $f,g\in \mathbb{C}(X,\mathbb{R}^n)$ we can define the sup metric $\rho(f,g) = \sup \{d(f(x),g(x)):x\in X\}$ on $\mathbb{C}(X,\mathbb{R}^n)$.

The space $\mathbb{C}(X,\mathbb{R}^n)$ is complete under $\bar{\rho}$ and ρ. Let $\mathbb{F}$ be a subset of $\mathbb{C}(X,\mathbb{R}^n)$. We say that F is uniformly bounded if F is bounded under ρ. We say that $\mathbb{F}$ is pointwise bounded if for each $x\in X$ the set $\mathbb{F}_x=\{f(x):f\in F\}$ is a bounded subset of $\mathbb{R}^n$. Now we show that if $\mathbb{F}$ is uniformly bounded, there is a compact subset Y of $\mathbb{R}^n$ such that $f(x)\in Y$ for all $f\in\mathbb{F}$ and all $x\in X$. Let $f_o \in\mathbb{F}$ be a fixed point. Since $\mathbb{F}$ is bounded under ρ $\exists$ $M > 0$: $\rho(f_o,f) < M$, $\forall$ $f\in\mathbb{F}$. Then, $d(f_o(x),f(x)) < M$ $\forall f\in\mathbb{F}$ and $\forall x\in X$ (1). On the other hand since f_o is continuous and X is compact, $f_o(X)$ is also compact in $\mathbb{R}^n$. Hence $f_o(X)$ is bounded. This implies that there exists $N\in\mathbb{N}$: $f_o(X) \subseteq B_d(0,N) \subseteq \mathbb{R}^n$. Then $f_o(x) \in B_d(0,N)$, $\forall x\in X \Rightarrow d(0,f_o(x)) < N$ $\forall x$ (2). (1), (2) $\Rightarrow d(0,f(x) \le d(0,f_o(x)) + d(f_o(x),f(x)) \le M + N \Rightarrow f(x) \in B_d(0,M + N)$, $\forall f\in\mathbb{F}$ and $\forall x\in X$. So $f(X) \subseteq B_d(0,M+N)$ $\forall f\in\mathbb{F}$.

Set $Y = \overline{B_d(0, M + N)}$

Then Y is closed and bounded subset of $\mathbb{R}^n$ Hence Y is a compact set such that $f(x) \in Y$, $\forall f\in\mathbb{F}$ and $\forall x\in X$. Note that since $Y \subseteq \mathbb{R}^n$, $\mathbb{C}(X,Y) \subseteq \mathbb{C}(X,\mathbb{R}^n)$ and F is contained in $\mathbb{C}(X,Y)$.

Definition 3.6: Let (Y,d) be a metric space Let $\mathbb{F} \subseteq \mathbb{C}(X,Y)$. The set $\mathbb{F}$ is said to be equicouinuous at a point $x_o \in X$, if given $\epsilon > 0$ there is a neighborhood U of x_o such that for all $x\in U$ and for all $f\in\mathbb{F}$ we have $d(f(x),f(x_o)) < \epsilon$. If the set $\mathbb{F}$ is equicoutinuous at each point $x\in X$ then $\mathbb{F}$ is equicoutinuous on X.

Theorem 3.7: Let $\mathbb{F}$ be a subset of $\mathbb{C}(X,Y)$ where X is compact space and (Y,d) is a compact metric space. Then $\mathbb{F}$ is equcontinuous if and only if $\mathbb{F}$ is totally bounded (under ρ).

Proof: Let $\mathbb{F}$ be totally bounded under ρ. Let $x_o \in X$ be fixed, and let $\epsilon > 0$. We shall show that $\mathbb{F}$ is equicoutinuous at x_o. First choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$: $2\epsilon_1 + \epsilon_2 \le \epsilon$. Since $\mathbb{F}$ is totally bounded there is a finite cover of $\mathbb{F}$ by ϵ_1 - balls. Therefore, there are $f_1, f_2, \ldots, f_n \in \mathbb{C}(X,Y)$ such that $\mathbb{F} \subseteq \bigcup_{i=1}^{n} B_\rho(f_i, \epsilon_1)$. Each function f_i, $i = 1,\ldots,n$ is continuous; hence it is continuous at x_o. Therefore $\exists$ U_i(n.b.d of x_o) such that $d(f_i(x), f_i(x_o)) < \epsilon_2$, for all $x \in U_i$. Let $U = \bigcap_{i=1}^{n} U_i$. Then U is open and $x_o \in U$. Moreover $d(f_i(x), f_i(x_o)) < \epsilon_2$, $\forall x \in U$, $i = 1,\ldots,n$ (1)

We assert that if $x \in U$ and $f \in \mathbb{F}$ then $d(f(x), f(x_o)) < \epsilon$ holds. Let $f \in \mathbb{F}$ be arbitrary. then $f \in B_\rho(f_i, \epsilon_1)$ for some i and so , $\rho(f_i, f) < \epsilon_1$. Then $d(f_i(x), f(x)) < \epsilon_1$ $\forall x \in X$ ... (2)

In particular $d(f_i(x), f(x)) < \epsilon_1$ $\forall x \in U$ (3)

By (1),(2) and (3) we get: $d(f(x), f(x_o)) \le d(f(x), f_i(x)) + d(f_i(x), f_i(x_o)) + d(f_i(x_o), f(x_o)) \le \epsilon_1 + \epsilon_2 + \epsilon_1 = 2\epsilon_1 + \epsilon_2 \le \epsilon$ Conversely, let $\epsilon > 0$. Since $\mathbb{F}$ is equicontinuous on X, for each $x \in X$ $\exists$ U_x (a neighborhood of x) such that $d(f(x), f(y)) < \epsilon/3$ $\forall y \in U_x$ and $\forall f \in \mathbb{F}$. We have $X = \bigcup_{x \in X} U_x$. That means the collection $\{U_x : x \in X\}$ is an open covering of X, and since X is compact there are $x_1, x_2, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} U_{x_i}$ and $d(f(x), f(x_i)) < \epsilon/3$, $\forall x \in U_{x_i}$ and for all $f \in \mathbb{F}$.

Now, cover Y by finitely many open sets $V_1, V_2, \ldots, V_m$, with $\text{diam}V_j < \epsilon/3$, $\forall_i = 1,2,\ldots,m$ For $f \in \mathbb{F}$ and $i \in \{1,2,\ldots,k\}$, $f(x_i)$ is contained in at least one of V_j's for $j = 1,2,\ldots,m$.

J is the collection of all functions $\alpha:\{1,2,\ldots,k\}\longrightarrow\{1,2.\ldots,m\}$. Let $\alpha\in J$; Then $\alpha(i) = j\in\{1,2,\ldots,m\}$, $\forall i\in\{1,2,\ldots,k\}$. If there is $f\in\mathbb{F}$ such that $f(x_i) \in V_j = V_{\alpha(i)}$, $\forall i$ take this function and denote it with $f_\alpha, f_\alpha\in\mathbb{F}$. Consider the collection $\{f_\alpha\} \subseteq \mathbb{F}$. Then $\alpha\in J'\subseteq J$. Clearly J is a finite set and hence $\{f_\alpha\}$ is a finite subset of $\mathbb{F}$.

Claim: $\mathbb{F}\subseteq \bigcup_{\alpha\in J'} B_\rho(f_\alpha,\epsilon)$. Let $f\in\mathbb{F}$. Then $f(x_i)\in V_j, \forall i = 1,2,\ldots,k$ where $j = 1,2,\ldots m$. Choose $\alpha^*\in J$: $\alpha^*(i) = j$

Then $f(x_i) \in V_{\alpha^*(i)}$, $\forall i\in\{1,2,\ldots,k\}$ and hence $\alpha^*\in J'$. We assert that $f\in B_\rho(f_{\alpha^*},\epsilon)$. Let $x\in X$; then

there exists $i\in\{1,2,\ldots,k\}$ such that $x\in U_{x_i}$ Therefore

$$d(f(x),f(x_i)) < \epsilon/3 \qquad \forall f\in\mathbb{F} \text{ and } x\in U_i \tag{1}$$

hence $$d(f_{\alpha^*}(x),f_{\alpha^*}(x_i)) < \epsilon/3 \qquad \forall x\in U_i \tag{2}$$

Since $f(x_i) \in V_{\alpha^*}(i)$ and $f_{\alpha^*}(x_i)\in V_{\alpha^*}(i)$ we have

$$d(f_{\alpha^*}(x_i),f(x_i)) < \epsilon/3 \tag{3}$$

(1),(2),(3) $\Rightarrow d(f(x),f_{\alpha^*}(x)) < \epsilon$. Since $x\in X$ was arbitrary, $d(f(x),f_{\alpha^*}(x)) < \epsilon\ \forall x\in X$. Hence $\rho(f,f_{\alpha^*}) = \max\{d(f(x),f_{\alpha^*}(x))\} < \epsilon$. Therefore, $f \in B_\rho(f_{\alpha^*},\epsilon) \subseteq \bigcup_{\alpha\in J'} B_\rho(f_\alpha,\epsilon)$. Since $f\in\mathbb{F}$ was arbitrary $\mathbb{F} \subseteq \bigcup_{\alpha\in J'} B_\rho(f_\alpha,\epsilon)$

Theorem 3.8: (Ascoli's theorem, classical version). Let X be compact space; A subset F of $C(X,\mathbb{R}^n)$ is compact if and only if it is closed, bounded and equicontinuous.

Proof: Let F be a compact subset of the metric space $(C(X,\mathbb{R}^n)$. Then F is closed and bounded under ρ by pr 3.5. Hence there is a compact subset Y of $\mathbb{R}^n$ such that $F\subseteq C(X,Y)$. Since F is compact, F is totally bounded under ρ(Th.3.2,p.20). By Th.3.7, we get that F is equicoutiuuous. Conversely, suppose F is closed, bounded and equicoutinuous, where $F\subseteq C(X,\mathbb{R}^n)$. Since $C(X,\mathbb{R}^n)$ is complete and F closed in $C(X,\mathbb{R}^n)$, we have that F is complete. The fact that F is bounded (under ρ) implies that F is coutained in some subspace $C(X,Y)$ of $C(X,\mathbb{R}^n)$, where Y is a compact subset of $\mathbb{R}^n$. Since, F is equicontinuous in $C(X,Y)$, it is also totally bounded. Therefore, F is compact.■

4. Pointwise and Compact Convergence

Consider the function space Y^X in the product topology. Let $x\in X$ and let U be an open subset of Y. Then $\pi_x^{-1}(U)$ is a typical subbasis element. Using functional notation, $\pi_x^{-1}(U) = \left\{f:X\longrightarrow Y. \mid f(x)\in U\right\}$. Set $S(x,U) = \pi_x^{-1}(U)$.

According the text book (ξ2.8,p.112-Munkres) the typical basis element for this topology is a finite intersection of subbasis elements $S(X,U)$. Let $f\in Y^X$ and let B_f to be a basis element about f. Then $B_f = \bigcap_{l=1}^{n} S_i(x_i,U_i)$, where $x_i\in X$ and U_i are open in Y. Thus B_f contains all functions $g:X\longrightarrow Y$ that are "closed" to f at finitely many points $x_1 x_2,\ldots,x_n$; i.e $g(x_i)\in U_i$ $\forall i=1,\ldots,n$. (see Fig.1)

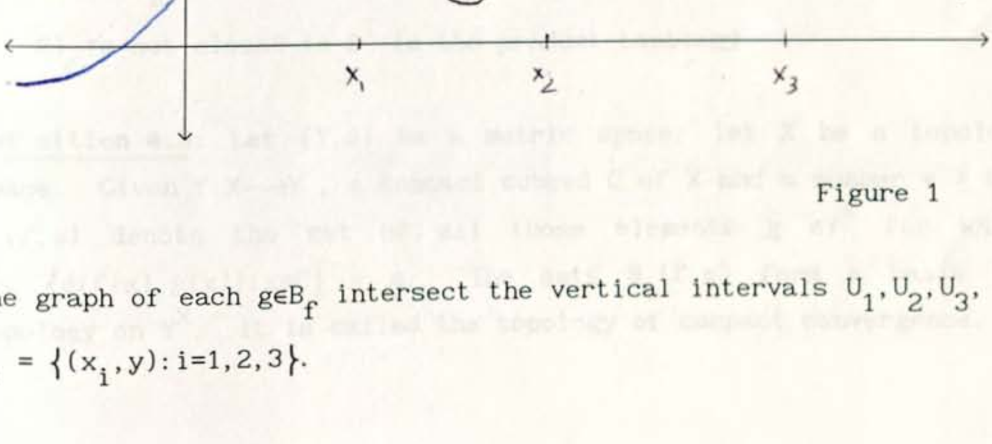


Figure 1

The graph of each $g\in B_f$ intersect the vertical intervals U_1, U_2, U_3, where $U_i = \{(x_i,y): i=1,2,3\}$.

Theorem 4.1: A sequence (f_n) of functions $f_n: X \longrightarrow Y$ converges to a function $f: X \longrightarrow Y$ in the product topology if and only if for each $x \in X$ the sequence $\{f_n(x)\}$ of points of Y converges to the point $f(x) \in Y$.

Proof: Let $f_n: X \longrightarrow Y$ such that $f_n \longrightarrow f$, in the product topology on Y^X. Hence, given $x \in X$ and a neighborhood U of $f(x)$ the set $S(x,U)$ is a neighborhood of f. Therefore $\exists N$: $f_n \in S(x,U)$, $\forall n \geq N$. Then, $f_n(x) \in U, \forall n \geq N$. Since U is an arbitarary neighborhood of $f(x)$, $f_n(x) \longrightarrow f(x)$. Conversely, Let $S(x,U)$ be an arbitrary subbasis element neighborhood of f. Since $f_n(x) \longrightarrow f(x)$ there is an $N: f_n(x) \in U$ for all $n \geq N$. This implies $f_n \in S(x,U)$, $\forall n \geq N$ Therefore, $f_n \longrightarrow f$.

Remarks 4.2 By theorem 4.1 the convergence in the product topology on Y^X is equivalent with the pointwise convergence. That is the reason we call the product topology, topology of poiutwise convergence. (or point-open topology).

Example 1: In the space $\mathbb{R}^I$, where $I = [0,1]$ the sequence (f_n) of continuous functions given by, $f_n(x) = x^n$ converges in the topology of pointwise convergence to the function $f: I \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$ clearly f is not a continuous function. In the uniform topology the sequence (f_n) does not converge. This example shows that the subspace $C(I,\mathbb{R})$ is not closed in $\mathbb{R}^I$ in the product topology.

Definition 4.3: Let (Y,d) be a metric space; let X be a topological space. Given $f: X \longrightarrow Y$, a compact subsed C of X and a number $\epsilon > 0$, let $B_C(f,\epsilon)$ denote the set of all those elements $g \in Y^X$ for which : $\sup\{d(f(x),g(x)): x \in C\} < \epsilon$. The sets $B_C(f,\epsilon)$ form a basis for a topology on Y^X. It is called the topology of compact convergence.

Remarks 4.4: 1) Compering the pointwise convergence topology and the compact convergence topology the typical basis element $B_C(f,\epsilon)$ contains all function that are "closed" to f not juct at finitely many points, but at all points of a compact set C.

2) Note also that if $g\in B_C(f,\epsilon)$ then there is $\delta > 0 : B_C(g,\delta)\subset B_C(f,\epsilon)$ where $\delta = \epsilon - \sup\{d(f(x),g(x)) : X\in C\}$. The sets $B_C(g,\delta)$ form a basis for this topology.

Theorem 4.5: A sequence $f_n : X \longrightarrow Y$ of functions converges to the function f in the compact convergence topology, if and only if for each compact subset C of X the sequence $f_n|C$ converges uniformuly to $f|C$.

Proof: Suppose that (f_n) converges to f in the topology of compact convergence. Let C be an arbitrary compact subset of X and let $\epsilon > 0$. Then $B_C(f,\epsilon)$ is a n.b.d of f. Therefore there is an N:such that $f_n\in B_C(f,\epsilon)$, $\forall n\geq N$. Then $d(f_n(x),f(x)) < \epsilon$, $\forall n\geq N$ and $\forall x\in C$. So f_n converges uniformly to f on C ; i.e $f_n|C$ converges uniformly to $f|C$.

Conversely, let C be any compact set in X and let $\epsilon > 0$. Since $\{f_n|C\}$ converges uniformly to $f|C$ there is an N such that $d(f_n|C,f|C) < \epsilon/2$ $\forall n \geq N$ or $d(f_n(x),f(x)) < \epsilon/2$, $\forall n \geq N$ and $\forall x\in C$.

Hence $\sup\left\{d\left[f_N(x),f(x)\right] : x\in C\right\} < \epsilon, \forall n \geq N$. This implies $f_n\in B_C(f,\epsilon)$ $\forall n \geq N$ Since $B_C(f,\epsilon)$ is an arbitrary n.b.d of f in the topology of compact convergence, f_n converges to f.

Definition 6.6: A space X is said to be compactly generated if it satisfies the following condition: A set $A \subseteq X$ is open in X if and only if $A\cap C$ is open in C, for each compact set C in X. The above condition is equivalent to say that a set B is closed in X if and only if $B\cap C$ is closed in C for any compact set C.

Lemma 4.7: If a topological space X is locally compact or if X satisfies the first countability axiom, then X is compactly qenerated.

Proof: Let X be a locally compact space. Let $A \subseteq X$ such that $A \cap C$ is open in C for every compact set C. We show that A is open in X. Let $x \in A$; since X is locally compact some neighborhood U of x is contained in some compact set C. Hence $x \in U \subseteq C$. Since $A \cap C$ is open in C and $U \subseteq C$ $A \cap U = (A \cap C) \cap U$ is open in U (consider U as a subspace of C).

Since $A \cap U$ is open in U and U is open in X we get that $A \cap U$ is open in X. Clearly $x \in A \cap U \subseteq A$ so $A \cap U$ is a neighborhood of x contained in A. Therefore , A is open in X.

Let X be a space which satisfies the first countability axiom. If $B \cap C$ is closed in C for all compact subsets C of X we need to show that B is closed in X. Let $x \in \bar{B}$; since X has a countable basis at the point x there is a sequence (x_n) of elements of B converging to x. Let V be a neighborhood of x. Then there is a positive integer N: $x_n \in V$, $\forall n \geq \mathbb{N}$. Since infinitely many points of the sequence (x_n) lie in V ($x \in V$) the set $C := \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact , so $B \cap C$ is closed in C. Clearly $x_n \in B \cap C$, , $\forall n \in \mathbb{N}$. So we have a sequence in $B \cap C$ which converges to $x \in C$. Then $x \in B \cap C$. Therefore $x \in B$. Since x was an arbitrary limit point of B, the set B contains all its limit points and hence it is closed in X. //

Theorem 4.8: If X is a compactly generated space and (Y,d) is a metric space, then C(X,Y) is closed in Y^X in the topology of compact convergence.

Proof: Let $f \in Y^X$ be a limit point of C(X,Y); we need to show that f is continuous. Let C be an arbitrary compact subset of X. For each $n \in \mathbb{N}$, the ball $B_c(f, 1/n)$ is a neighborhood of f. Since f is a limit point of C(X,Y) we get that $B_c(f, 1/n) \cap C(X,Y) \neq \emptyset$, $\forall n \in \mathbb{N}$. Let $f_n \in C(X,Y)$ such that $f_n \in B_c(f, 1/n)$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. Then $1/n < \epsilon$, $\forall n \geq N$. Since $f_n \in B_c(f, 1/n)$ we get that $\sup_{x \in C} \{d(f(x), f_n(x))\}$ <

1/n. Therefore $\sup_{x\in C} \{d(f(x),f_n(x))\} < \epsilon$, $\forall n \geq N$. This implies that $f_n \in B_C(f,\epsilon)$, $\forall n \geq N$ that means the sequence (f_n) of points of C(X,Y) convergences to f in the topology of compact convergence. By Theorem 4.5, $f_n|C$ converges uniformly to $f|C$. By the uniform limit theorem $f|C$ iscontinuous. Let now V be an open set in Y. We want to show that $f^{-1}(V)$ is open in X.

Obverve that $f^{-1}(V) \cap C = (f|C)^{-1}(V)$, where $f|C:C\longrightarrow Y$ and C is any subset of X. Since $f|C$ is continuous $(f|C))^{-1}(V)$ is open in C ; i.e $f^{-1}(V) \cap C$ is open in C. Considering C as a compact subspace of X and using the fact that X is compactly generated we get that $f^{-1}(V)$ is open in X. //

Corolary 4.9: Let X be a compactly generated space and let Y be a metric space with metric d. If a sequence of continuous functions $f_n:X\longrightarrow Y$ converges to a function $f:X\longrightarrow Y$ in the topology of compact convergence, then f is continuous.

Proof: Let (f_n) be a sequence of continuous functions, $f_n:X\longrightarrow Y$: i.e $f_n \in C(X,Y) \subseteq Y^X$. Assume that $f_n \longrightarrow f$, $f\in Y^X$ in the topology of compact convergence. By Theorem 4.8, C(X,Y) is closed in Y^X in the topology of compact convergence. Hence $f\in C(X,Y)$.

Theorem 4.10: Let X be a topologyical space and let Y be a metric space with metric d. On the function space Y^X we have define three topologies. The uniform topology, the topology of compact convergence and the topology of pointwise convergence. Then, (pointwise convergence) $\subset$(compact convergence)$\subset$ (uniform convergence) If X is compact , the last two coinside, and if X is discrete the first two coinside.

Proof: Let J_1, J_2 and J_3 denote the pointwise convergence topology, the compact convergence topology and uniform topology respectively

1. we need to show $J_1 \subset J_2$. Let $f = (f(x))_{x\in X}$ be any point of Y^X.

Recall that $Y^X = \prod_{x\in X} Y_x$, where $Y_x = Y$, $\forall x\in X$. Let B be a basis element about f in the topology of pointwise convergence J_1 (product topology) on Y^X. Let $x_1, x_2, \ldots, x_n \in X$ be the indices for which $U_x \neq Y$. Then $B = \prod U_x$. Since Y is a metric space each U_{x_i} is a ball $B_d(f(x_i), \epsilon_i)$. We need to show that there is $\delta > 0$ and a compact subset C of X such that $B_C(f,\delta) \subset B = \prod_{x\in X} U_x$, where $B_C(f,\delta)$ is a basis element about f in the topology of compact convergence. Let $C := \{x_1, x_2, \ldots, x_n\}$ and $\delta := \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then C is a compact set Let $g \in B_C(f,\delta)$; then we have that $\sup\{d(f(x),g(x)) : x\in C\} = \max\{d(f(x),g(x)) : x\in C\} < \delta$. So, $d(f(x_i),g(x_i)) < \delta \leq \epsilon_i$ $\forall i = 1,\ldots,n$. Therefore $g(x_i) \in B_d(f(x_i),\epsilon_i) = U_{x_i}$; i.e $g \in \prod U_{x_i} = B$

2. $J_2 \subset J_3$. Let $f \in Y^X$, $\epsilon > 0$ and C compact in X. Set $\delta := \min\{1,\epsilon\}$; we claim that $B_{\bar{\rho}}(f,\delta/2) \subset B_C(f,\epsilon)$. Let $g \in B_{\bar{\rho}}(f,\delta/2)$, then $\sup\{\bar{d}(f(x),g(x)) : x\in X\} < \delta/2$. So we get $\bar{d}(f(x),g(x)) < \delta/2$, $\forall x\in X$. Clearly this implies $d(f(x), g(x)) < \epsilon/2$, $\forall x\in X$, therefore $\sup\{d(f(x), g(x)) : x\in C\} \leq \sup\{d(f(x), g(x) : x\in X\} \leq \epsilon/2 < \epsilon$. Then $g \in B_C(f,\epsilon)$.

Note that if $g \in B_C(f,\epsilon)$ then $B_C(g,\delta) \subset B_C(f,\epsilon)$....(1) where $\delta := \epsilon - \sup\{d(f(x),g(x)) : x\in C\}$.

Let X be compact; it remains to show that $J_3 \subset J_2$. Let $f \in Y^X$ and $\epsilon > 0$ it suffices to show: $B_X(f,\epsilon) \subset B_{\bar{\rho}}(f,\epsilon)$. Let $g \in B_X(f,\epsilon)$; then $\sup\{d(f(x),g(x)) : x\in X\} < \epsilon$. Since $\bar{d}(f(x),g(x)) \leq d(f(x),g(x))$, we get that $\sup\{\bar{d}(f(x),g(x)) : x\in X\} < \epsilon$ which means that $\bar{\rho}(f,g) < \epsilon \Leftrightarrow g \in B_{\bar{\rho}}(f,\epsilon)$.

Now let X be discrete. Then a subset C of X is compact if and only if C is finite. We show $J_2 \subset J_1$. Let $B_C(f,\epsilon)$ be an arbitrary basis element about f in the topology of compact convergence J_2, where C is a compact subset of X and $\epsilon > 0$.

Then $C = \{x_1, x_2, \ldots, x_n\} \subseteq X$. Consider the following basis element B_f(about f) in the topology of pointwise convergence: $B_f := \Pi V_x$, where $V_x = Y$ for all $x \neq x_i$, $i = 1,2,\ldots,n$ and $V_{x_i} = B_d(f(x_i),\epsilon)$. We claim that $B_f \subset B_C(f,\epsilon)$. Let $g \in B_f$; then $g(x_i) \in B_d(f(x_i),\epsilon)$, $\forall i = 1,2,\ldots,n$ Hence $d(f(x_i),g(x_i)) < \epsilon$ for all $i = 1,2,\ldots,n$. sup $\{d(f(x_i),g(x_i): x_i \in C\} = \max\{d(f(x_i),g(x_i)): x_i \in C\} < \epsilon \Leftrightarrow g \in B_C(f,\epsilon)$. //

5. The Compact Open Topology

Defining the topology of compact convergence on the function space Y^X we considered the space Y as a metric space, with metric d. But a topology on a space can be induced by different metrics. So, the question is whether the above topology depends only on the topology of Y rather that on the particular metric d. We can answer if our space is $C(X,Y) \subseteq Y^X$; for this space the topology of compact convergence is independent of the metrics which generate the topology of Y.

Definition 5.1: Let X and Y be topological spaces. Let C be a compact subset of X and U be an open set in Y. Define, $S(C,U) := \{f \in C(X,Y): f(C) \subset U\}$. The sets $S(C,U)$, form a subbasis for a topology on $C(X,Y)$. We call this topology, the compact-open topology. Clearly, if C is one-point set it is compact and $S(C,U) = S(x,U)$, where $S(x,U)$ is a subbasis element for the point-open topology.

Note that the compact-open topology is defined without assuming a metric on Y. Let now d be a metric on Y which generates the topology of Y. Then we can define the topology of compact convergence of Y^X and hence on $C(X,Y)$. If we show that this topology concides with the compact-open topology then the topology of compact convergence is independent of the particular metric chosen for Y.

Theorem 5.2: Let X be a space and let (Y,d) be a metric space consider the function space $C(X,Y)$ and let J_1 and J_2 denote the compact-open topology and the topology of compact convergence (on $C(X,Y)$). Then $J_1 = J_2$.

Proof: Step 1. For each subset A of Y and for each $\epsilon > 0$ we can define $U(A,\epsilon) := \bigcup_{a\in A} B_d(a,\epsilon)$. The set $U(A,\epsilon)$ is called "ϵ-neighborhood" of A. If A is compact and V is an open set in Y containing A then we can show that there is $\epsilon > 0$ such that $U(A,\epsilon) \subset V$: Since $A \subset V$, for each $a\in A$ choose $\delta(a) > o$ such that $B_d(a,\delta(a)) \subset V$. Note that since A is compact the collection $\{B_d(a, 1/2\ \delta(a)) : a\in A\}$ contain a finite covering of A, say $B_d(a_1, 1/2\delta(a_1)),\dots,B_d(a_n, 1/2\delta(a_n))$. Set $\epsilon := \min\{1/2\delta(a_i)\}$ and let $a\in A$. Then $b\in B_d(a,\epsilon)$ implies $d(a,b) < \epsilon$.

Note that since $a\in A$ and $A \subset \bigcup_{i=1}^{n} B_d(a_i, 1/2\ d(a_i))$, $a \in B_d(a_k, 1/2d(a_k))$, for some $k\le n$. Hence $d(a_k,a) < 1/2\ d(a_k)$. Then $d(a_k,b) \le d(a_k,a) + d(a,b)$. So $d(a_k,b) < 1/2\ \delta(a_k) + \epsilon \le 1/2\ \delta(a_k) + 1/2\ \delta(a_k) = \delta(a_k)$ $\Rightarrow b\in B_d(a_k,\delta(a_k))$. Therefore, $B_d(a,\epsilon) \subset B_d(a_k,\delta(a_k)) \subset V$. Since $a\in A$ was arbitrary $B_d(a,\epsilon) \subset V$, $\forall a\in A$; that means $U(A,\epsilon) = \bigcup_{a\in A} B_d(a,\epsilon) \subset V$. So, we proved that for each compact subset A of Y with $A\subset V$ (where V is an open set) there is $\epsilon > 0$ such that $U(A,\epsilon) \subset V$.

Step 2. Claim $J_1 \subset J_2$. Let $S(C,U)$ be any subbasis element for J_1(compact-open topology) on $C(X,Y)$ and let $f\in S(C,U)$. Then $f(C) \subset U$. Since C is compact and f is continuous $f(C)$ is a compact subset of Y. By step 1 there is $\epsilon > 0$ such that the ϵ-neighborhood of $f(C)$ lies in U. Then $B_c(f,\epsilon)$ is a basis element for the topology of compact convergence J_2. We want to show: $B_c(f,\epsilon) \subset S(C,U)$. Let $g\in B_c(f,\epsilon)$; then $d(f(x),g(x)) < \epsilon$, $\forall x\in C \Rightarrow g(x)\in B_d(f(x),\epsilon)$, $\forall x\in C \Rightarrow g(C) \subset \bigcup_{c\in C} B_d(f(x),\epsilon) \subset U$. Since $g(C) \subset U$, $g\in S(C,U)$.

Step 3. Claim $J_2 \subset J_1$. Let $f\in C(X,Y)$, let C be a compact set in X and let $\epsilon > 0$. Then $B_c(f,\epsilon)$ is a basis element for J_2 on $C(X,Y)$, neighborhood of f. We shall find a basis element of J_1 that contains f and lies in $B_c(f,\epsilon)$.

Using the fact that f is continuous at each point $x\in C$ we can choose a neighborhood V_x such that $f(V_x) \subset B_d(f(x),\epsilon/4)$. We claim that $f(\overline{V_x}) \subset B_d(f(x),\epsilon/3)$; it suffices to show $f(\overline{V_x}) \subset \overline{B_d(f(x),\epsilon/4)}$. Let $y\in f(\overline{V}_x)$; then $y = f(x_o)$, $x_o \in \overline{V}_x$. Note that $y\in \overline{B_d(f(x),\epsilon/4)}$ if and only if every neighborhood of $y = f(x_o)$ intersects $B_d(f(x),\epsilon/4)$. Let $B_d(f(x_o),\delta)$ be a neighborhood of y ; set $A:= f^{-1}[B_d(f(x_o),\delta)] \cap V_x$. Since $x_o \in \overline{V}_x$, $A \neq \emptyset$ and $f(A) \subset B_d(f(x_o),\delta) \cap f(V_x) \subset B_d(f(x_o),\delta) \cap B_d(f(x),\epsilon/4) \neq \emptyset$. Hence $y\in \overline{B_d(f(x),\epsilon/4)}$.

Since the collection $\{V_x : x\in C\}$ is an open covering of C and C is compact, C has a finite subcovering, say $V_{x_1},\dots,V_{x_n}$. Set $C_{x_i} = \overline{V}_{x_i} \cap C$. Then each C_{x_i} is compact in X. If $U_x := B_d(f(x),\epsilon/3)$ we claim that $S(C_{x_1},U_{x_1}) \cap \dots \cap S(C_{x_n},U_{x_n}) \subset B_c(f,\epsilon)$. Let $g\in S(C_{x_1},U_{x_1}) \cap \cdots \cap S(C_{x_n},U_{x_n})$ and $x\in C$. Then $x\in V_{x_i}$ and hence $x \in C_{x_i}$ for some i. Then $f(x)\in U_{x_i}$ therefore $d(f(x),f(x_i)) < \epsilon/3$. On the other hand $g \in S(C_{x_i},U_{x_i})$, hence $g(x) \in U_{x_i} \Rightarrow d(f(x_i),g(x)) < \epsilon/3$. Then $d(f(x),g(x)) \le d(f(x),f(x_i)) + d(f(x_i),g(x)) < 2\epsilon/3$ So , $\sup_{x\in C} \{d(f(x),g(x))\} < \epsilon$ and so $g\in B_c(f,\epsilon)$. //

Theorem 5.3 Let X be a locally compact Hausdorff space and consider the set C(X,Y) in the compact open topology. Then the map $e: X\times C(X,Y) \longrightarrow Y$ given by $e(x,f) := f(x)$ is a continuous map. It is called the evaluation map.

Proof: Given a point $(x,f)\in X\times C(X,Y)$ and an arbitrary open set V in Y, neighborhood of the image point $e(x,f) = f(x)$ we need to find a neighborhood of (x,f) that e maps into V. Since f is continuous $f^{-1}(V)$ is an open set in X, neighborhood of x. Since X is locally compact at x we can find a neighborhood U of x with compact closure such that $\overline{U} \subset f^{-1}(V)$ or $f(\overline{U}) \subset V$. Then $S(\overline{U},V)$ is a basis element for the compact-open topology on $C(X,Y)$ and since $f(\bar{U}) \subset V$ we get that $f \in S(\bar{U},V)$. Observe that the set $U \times S(\bar{U},V)$ is open in $X \times C(X,Y)$ and it is a neighborhood of (x,f). It remains to show that $f'(x') \in V$, for every $(x',f') \in U \times S(\bar{U},V)$. Let $(x',f') \in U \times S(\bar{U},V)$; then $x' \in U \subset \bar{U}$ and $f(\bar{U}) \subset V$. Hence $f'(x') \in V$.

R E F E R E N C E S

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