College of Natural and Computational Sciences
Department of Mathematics

Sturm-Liouville Boundary Value Problems and General Solutions to 2D Liouville Equations

A Thesis Submitted to the Department of Mathematics of Natural and Computational Sciences at Addis Ababa University, in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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Approval

”The undersigned have read and recommend to the Department of Mathematics the acceptance of this thesis entitled **”Sturm-Liouville Boundary Value Problems and General Solutions to 2D Liouville Equations”** by **Icon Abebe Gido**, in partial fulfillment of the requirements for the Master of Science degree in Mathematics.”

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Declaration

I, **Icon Abebe Gido**, with student ID GSE/1301/14, hereby affirm that the thesis titled **”Sturm-Liouville Boundary Value Problem and the General Solutions to 2D Liouville Equation”** has been composed and organized by me under the guidance of **Dr. Tesfa Biset.** I confirm that this work has not been previously submitted for any other graduate qualification. Any contributions from other sources have been duly acknowledged and referenced.

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Abstract

The thesis focuses on Sturm-Liouville equations, boundary value problems, and the general solutions to 2D Liouville Equations. It covers computations of eigenvalues and eigenfunctions, types of Sturm-Liouville boundary value problems, solving methods and solutions, inner products of eigenfunctions, and orthogonality of eigenfunctions with weighted functions. It also covers applications of Strum-Liouville Boundary Value problems and ways to solve them. Additionally, the thesis covers several characteristics of complex-valued functions and derivations emphasizing the significance of the Liouville Equation in resolving Elliptic as well as Hyperbolic 2D differential equations. Furthermore, it provides a short but detailed analysis aimed at enhancing the understanding and application of SLP, even at its most complex levels in science and engineering, thereby promoting general solution techniques for elaborate mathematical problems.

Key Words: Eigenvalues, Eigenfunction, Elliptic, Hyperbolic, Riccati Equation, and Sturm-Liouville Differential Equation.

Notations and Abbreviations

R^2	:	Two-dimensional Euclidean space
$\mathcal{C}$	:	The set of complex numbers
BVP	:	Boundary Value Problem
D	:	Domain of a defined function
N	:	Natural numbers
$u_k(x)$	:	Sequence of eigenfunctions of SLP
$u(x)$	:	General solution function of 2D Liouville PDEs
ω	:	Angular velocity
k_n	:	Sequence of eigenvalues of SLP
η	:	Eigenvalue of 2D Liouville PDEs
$SL - BVPs$	:	Boundary value problems: Sturm-Liouville approach.
$L^2[a, b]$		The space of square-integrable functions defined on the interval $[a, b]$.
Δ	:	The Laplace's operator.
∇	:	The gradient operator.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla^2 f = \Delta f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

CHAPTER ONE

Introduction

1.1 Background of the Study

The study of Sturm-Liouville boundary value problems and 2D Liouville equations is deeply rooted in the rich history of mathematical analysis and its applications to physical phenomena. These topics are fundamental to understanding various complex systems in both mathematics and physics. Sturm-Liouville theory, named after the mathematicians Jacques Charles François Sturm and Joseph Liouville, plays a crucial role in the study of linear differential equations. The Sturm-Liouville problem is a special type of second-order linear differential equation with variable coefficients, often appearing in the form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \left(\lambda w(x) - q(x)\right) y = 0 \tag{1.1}$$

where $p(x)$, $q(x)$, and $w(x)$ are given functions, and λ is a parameter. This form of differential equation emerges in a variety of contexts, including quantum mechanics, heat conduction, and vibration analysis. The solutions to these problems, characterized by their eigenvalues and eigenfunctions, form an orthogonal set that can be used to expand other functions, much like a Fourier series.

Historically, the development of Sturm-Liouville theory has provided a robust framework for solving boundary value problems, which are crucial in modeling physical systems where conditions are specified at the boundaries of the domain. The theory not only helps in finding the eigenvalues and eigenfunctions but also in understanding the stability and resonance phenomena in physical systems. The Liouville equation, named after Joseph Liouville, is a type of nonlinear partial differential equation. In two dimensions, the Liouville equation typically takes the form:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = e^{\phi} \tag{1.2}$$

or more generally:

$$\Delta \phi = e^{\phi} \tag{1.3}$$

where Δ represents the Laplacian operator in two dimensions. This equation arises in several important areas of mathematics and physics, including conformal mapping, differential geometry, and integrable systems. Understanding the general solutions of the 2D Liouville equation helps in exploring the geometric properties of surfaces with constant Gaussian curvature and in solving problems related to conformal field theory. Both Sturm-Liouville problems and the 2D Liouville equation are instrumental in advancing our understanding of complex differential systems. The methods developed for these studies are applicable in a wide range of scientific and engineering disciplines. By investigating the spectral properties of Sturm-Liouville problems and the integrability of the Liouville equation, researchers can address practical issues in wave propagation, stability analysis, and beyond. The motivation for this study stems from the need to develop a deeper understanding of the mathematical structures governing Sturm-Liouville boundary value problems and 2D Liouville equations. By exploring these areas, the thesis aims to contribute to the existing body of knowledge, providing new insights and methodologies for solving complex differential equations.

1.2 Literature Review

The investigation of Sturm-Liouville boundary value problems and 2D Liouville equations has been extensively applied in mathematics and sciences. These equations are particularly significant in physics, engineering, and materials science, making them essential subjects for study and solution. This review offers an overview of the theoretical foundations, mathematical properties, and solution methodologies of these topics. Understanding the historical significance of the classical Sturm-Liouville boundary value problems is rooted in the work of pioneers Jacques Charles François Sturm (1803-1855) and Joseph Liouville (1809-1882).

Birkhoff and Rota discuss the historical development of ordinary differential equations, including the Sturm-Liouville problems, presenting classical definitions more challenging problems of BVPs and Theorems. Boyce and DiPrima develop the general theory of boundary value problems and emphasize their connection with mathematical physics.

Sturm-Liouville boundary value problems exhibit various interesting mathematical characteristics and properties. In Trench, boundary value problems for elementary differential equations are explained, including the uniqueness and existence of solutions, as well as the properties of eigenvalues and eigenfunctions. Gwaiz explores the mathematical theory of Sturm-Liouville problems, with special emphasis on the spectral properties and orthogonality of eigenfunctions. Different solution methods for Sturm-Liouville boundary value problems, such as separation of variables, eigenfunction expansions, Lagrange identities, and Green's functions, are presented in works by Birkhoff and Rota, Nagle, and Gwaiz.

Recent papers have succeeded in presenting new combinations of definitions and properties of Liouville's equations, focusing on the existence and uniqueness of exact solutions. Crowdy studies the 2D Liouville equation in Euclidean space, exploring special solutions, their properties, relations with conformal mappings, and Bäcklund transformations. While Sturm-Liouville problems and 2D Liouville equations are not directly related to complicated analysis or partial differential equations, their solutions and applications are pertinent to elliptic and hyperbolic PDEs in complex domains. These ideas are also crucial in the study of electromagnetic fields, fluid flows, and potential theory.

Analytical techniques like the separation of variables are critical for solving Sturm-Liouville boundary value problems (SLBVPs) and 2D Liouville equations. References [1–3, 6] cover the latest analytical methods in detail. The motivation for studying Sturm-Liouville BVPs and Liouville equations stems from their significance and the fascinating challenges they present.

Many aspects of these problems remain under-researched, offering high potential for new discoveries. Engaging in this research allows for the exploration of the latest advancements in the field, contributions to academic journals, and collaboration with other researchers in the area of specialization.

1.3 Objectives of the Study

The main objectives of the study of Sturm-Liouville boundary value problems and general solutions of 2D Liouville equations are as follows:

1. **Analyze the Sturm-Liouville Problem Framework**
   - Provide a comprehensive understanding of the mathematical formulation and properties of Sturm-Liouville boundary value problems.
   - Examine the conditions under which Sturm-Liouville problems arise in physical and engineering contexts.
2. **Explore Eigenvalues and Eigenfunctions**
   - Determine the eigenvalues and eigenfunctions associated with Sturm-Liouville problems.
   - Study the orthogonality and completeness properties of the eigenfunctions.
3. **Applications of Sturm-Liouville Theory**
   - Apply the Sturm-Liouville theory to solve practical problems in physics and engineering, such as heat conduction, vibration analysis, and quantum mechanics.
   - Demonstrate how Sturm-Liouville theory can be used to decompose complex functions into simpler components through Fourier series expansions.
4. **Solve the 2D Liouville Equation**
   - Investigate the general solutions of the 2D Liouville equation in various contexts, particularly in Elliptic and Hyperbolic equations.
   - Derive exact solutions for specific forms of the 2D Liouville equation.

CHAPTER TWO

Preliminaries

2.1 Basic Definitions and Notations

We classify differential equations as follows:

A. **Ordinary Differential Equation** (ODE) is a type of differential equation that contains one or more functions of a single independent variable and its derivatives.

Example 2.1. ***Hermite Equation:***

$$y'' - 2xy' + 2y = 0,\ x \in R \textit{ and } y = y(x) \tag{2.0}$$

is an ODE.

B. **Partial Differential Equation** (PDE) is a type of differential equation that involves multiple independent variables and the partial derivatives of a function with respect to those variables.

Example 2.2. ***Wave Equation***

The wave equation is given by:

$$k^2 u_{xx} = \ u_{tt} \tag{2.1}$$

Example 2.3. ***Heat Equation***

The heat equation is given by:

$$k^2 u_{xx} = u_t, \quad k \in R, \quad u = u(x,t) \tag{2.1}$$

Example 2.4. ***Laplace's Equation***

The Laplace equation is given by:

$$u_{xx} + u_{yy} = 0 \tag{2.2}$$

Definition 2.1. ***Boundary value problem*** *is a differential equation subjected to constraints called* ***boundary conditions****.*

Definition 2.2. ***A solution to a boundary value problem*** *is a solution to the differential equation that meets the boundary conditions.*

Types of Boundary Conditions: The three most common types of boundary conditions are:

1. *First kind:*
$$y(a) = \eta_1, \quad y(b) = \eta_2 \tag{2.3}$$

2. *Second kind:*
$$y'(a) = \eta_1, \quad y'(b) = \eta_2 \tag{2.4}$$

3. *Third kind:*
$$\alpha_1 y(a) + \alpha_2 y'(a) = \eta_1, \quad \beta_1 y(b) + \beta_2 y'(b) = \eta_2 \tag{2.5}$$

Linear Dependence/Independence

Definition 2.3. *A set of functions $f_1, f_2, \ldots, f_n$ is said to be linearly dependent on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that*

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \tag{2.6}$$

for all x in I; otherwise, they are linearly independent [6].

2.2 Inner Product and Orthogonal Functions

An inner product space is a vector space V equipped with an inner product. An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ (where F is usually the field of real or complex numbers) that satisfies the following properties for all $u, v, w \in V$ and all scalars $c \in F$:

1. **Conjugate Symmetry**:
$$\langle u, v \rangle = \overline{\langle v, u \rangle} \tag{2.7}$$

2. **Linearity in the First Argument**:
$$\langle cu + v, w \rangle = c\langle u, w \rangle + \langle v, w \rangle \tag{2.8}$$

3. **Positivity**:
$$\langle v, v \rangle \geq 0 \tag{2.9}$$
$$\langle v, v \rangle = 0 \text{ if and only if } v = 0 \tag{2.10}$$

Definition 2.4. *The inner product of two functions g_1 and g_2 on an interval $[a, b]$ is defined as:*

$$\langle g_1, g_2 \rangle = \int_a^b g_1(x) g_2(x)\, dx \tag{2.11}$$

Orthogonal Functions

Definition 2.5. *Two functions g_1 and g_2 are orthogonal on an interval $[a, b]$ if*

$$\langle g_1, g_2 \rangle = \int_a^b g_1(x) g_2(x)\, dx = 0 \tag{2.12}$$

Orthogonal Set

Definition 2.6. *A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal on an interval $[a, b]$ if*

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x) \phi_n(x)\, dx = 0 \quad \textit{for} \quad m \neq n \tag{2.13}$$

Definition 2.7. *The expression $\langle f, f \rangle = \|f\|^2$ is called the **square norm** and the **norm** of f is*

$$\|f\| = \sqrt{\langle f, f \rangle} \tag{2.14}$$

Definition 2.8. *The square norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$ is*

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2(x)\, dx. \tag{2.15}$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the property that $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \ldots$, then $\{\phi_n(x)\}$ is said to be an orthonormal set on the interval.

Orthogonal Function

Definition 2.9. *A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal with respect to the weight function $w(x)$ on an interval $[a, b]$ if*

$$\int_a^b w(x) \phi_m(x) \phi_n(x)\, dx = 0 \quad \textit{for} \quad m \neq n. \tag{2.16}$$

Orthogonal Series Expansion

Definition 2.10. *Assume that $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. We can express $f(x)$ as a series in terms of this orthogonal set for every $x \in [a, b]$:*

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n \rangle\, \phi_n(x)}{\|\phi_n\|^2}, \tag{2.17}$$

where $n = 0, 1, 2, \ldots$.

To find the coefficient c_n*, multiply both sides of the equation by* ϕ_m *and integrate over the interval* $[a, b]$*:*

$$\int_a^b f(x)\phi_m(x)\,dx = c_n \int_a^b \phi_n^2(x)\,dx \quad \text{for} \quad m = n.$$

Therefore, the coefficients c_n *are given by:*

$$c_n = \frac{\int_a^b f(x)\phi_m(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 0, 1, 2, \ldots.$$

Using inner product notation, the series expansion can be written as:

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n \rangle \, \phi_n(x)}{\|\phi_n\|^2}, \tag{2.18}$$

where $\langle f, \phi_n \rangle$ *denotes the inner product of* f *and* ϕ_n*, and* $\|\phi_n\|$ *represents the norm of* ϕ_n*.*

Notation 2.1. *The series expansion above is referred to as the generalized Fourier series, and* c_n *are called the generalized Fourier coefficients.*

2.3 Piecewise Smooth Functions

Definition 2.11. *A function* $y = f(x)$ *is* ***piecewise continuous*** *on the finite interval* $[a, b]$ *if it is continuous at every point in* $[a, b]$ *except for finitely many points where it exhibits jump discontinuities.*

Definition 2.12. *A function* $f(x)$ *is* ***periodic*** *with period* T *if, for all* x*,*

$$f(x + T) = f(x). \tag{2.19}$$

Definition 2.13. *A* ***smooth function*** *is a function that has derivatives of all orders and is continuous within its domain.*

2.4 Some Differential Operators

Definition 2.14. ***Differential Operator*** D*, which transforms a differentiable function into its derivative, is written as:*

$$D = \frac{d}{dx}.$$

Definition 2.15. ***Laplacian Operator*** *is essentially the sum of second derivatives in three spatial dimensions:*

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{2.20}$$

Definition 2.16. *A linear operator T on a vector space V over a field F is a function $T : V \to V$ such that for all vectors $\mathbf{v}, \mathbf{w} \in V$ and scalars $a \in F$:*

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) \tag{2.21}$$

$$T(a\mathbf{v}) = aT(\mathbf{v}) \tag{2.22}$$

Definition 2.17. ***Self-Adjoint Operator*** *is a linear operator which is equal to its adjoint. For an operator L, it satisfies:*

$$\langle L[u], v\rangle = \langle u, L[v]\rangle. \tag{2.23}$$

Definition 2.18. ***Spectral Theory*** *is a branch of mathematics concerned with the study of properties of linear operators and their associated eigenvalues and eigenvectors.*

Definition 2.19. *A* ***Riccati equation*** *is a type of ordinary differential equation that takes the form:*

$$\frac{dy}{dx} = f(x)y^2 + g(x)y + h(x), \tag{2.24}$$

where y is the unknown function of x, and $f(x), g(x)$, and $h(x)$ are given[7].

CHAPTER THREE

Sturm-Liouville Boundary Value Problem

3.1 Introduction

An order of ordinary differential equations with specific boundary conditions is known as a Sturm-Liouville (SL) boundary value problem (BVP). They are widely used in physics, engineering, and mathematics. This set contains problems with homogeneous boundary conditions and a self-adjoint differential operator. Sturm-Liouville theory is a powerful method that allows us to solve such problems based on the orthonormality of eigenfunctions and the real forms of eigenvalues. The eigensolutions to Sturm-Liouville problems are critical in many areas such as vibrations, heat transfers, quantum mechanics, and signal processing.

The eigenfunctions form an orthonormal set, meaning that any known function can be generalized in terms of these eigenfunctions. This generalization is known as an orthogonal series expansion and is a very useful method for solving differential equations and systematically approximating functions.

Consider the Sturm-Liouville equation along with boundary conditions satisfying:

$$(p(x)y')' + q(x)y = \lambda r(x)y \tag{3.1}$$

subject to:

$$\alpha_1 y(a) + \beta_1 y'(a) = 0 \tag{3.2}$$

$$\alpha_2 y(b) + \beta_2 y'(b) = 0 \tag{3.3}$$

where $p(x) > 0$, $r(x) \geq 0$, and the functions p, q, r are continuous on the interval $[a, b]$.

The problem of finding a number λ such that there exists a non-trivial solution for the BVP is called a **Sturm-Liouville Boundary Value Problem**[1]. The value λ is referred to as an **eigenvalue** and the corresponding solution $y(:, \lambda)$ is called an **eigenfunction**.

Many problems in physics involve boundary value problems with second-order ordinary differential equations. For example, we might want to solve the equation:

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = f(x) \tag{3.4}$$

subject to boundary conditions. Such an equation can be written in operator form by defining the differential operator:

$$L = A_2(x)\frac{d^2}{dx^2} + A_1(x)\frac{d}{dx} + A_0(x). \tag{3.5}$$

Then, the Equation takes the form:

$$Ly = f. \tag{3.6}$$

3.2 Sturm-Liouville Equation

Definition 3.1. *The Sturm-Liouville Equation is an ordinary differential equation of second-order in a linear form that is widely applied in the fields of mathematical physics and engineering. It comes in the general form:*

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x) = -\lambda r(x)y(x), \tag{3.7}$$

where $p(x)$, $q(x)$, and $r(x)$ are functions defined on a finite interval, and λ is a parameter. The functions $y(x)$, which satisfy certain boundary conditions, are solutions to this equation.

The Sturm-Liouville equation can be expressed using differential operators as follows:

$$L[y] = \left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right)y = -\lambda r(x)y, \tag{3.8}$$

where L is the Sturm-Liouville operator.

The adjoint operator of L, denoted as $L^\dagger$, is defined as:

$$L^\dagger[y] = \frac{d}{dx}\left(p(x)\frac{d}{dx}y\right) + q(x)y. \tag{3.9}$$

Thus, the Sturm-Liouville equation can be expressed in terms of these operators:

$$L[y] = \mathcal{D}[y] + \lambda r(x)y = 0, \tag{3.10}$$

where $\mathcal{D}$ represents the differential part of the Sturm-Liouville operator.

The adjoint equation can also be formulated as:

$$L^\dagger[y] = \mathcal{D}^*[y] + \lambda y = 0. \tag{3.11}$$

Theorem 3.1. *An operator of second order can be expressed in terms of the Sturm-Liouville operator.*

Proof 3.1. *Consider the equation (3.1). To determine* p *so that*

$$(py')' = py'' + p'y' = A_2(x)y'' + A_1(x)y'$$

This is in the correct form. We just identify $p = A_2$ *and* $p' = A_1$*. If* $A_1(x) = A_2'(x)$*, then we can express the equation in the form:*

$$(A_2(x)y')' + A_0(x)y = f(x), \tag{3.12}$$

where $f(x) = A_2(x)y'' + A_1(x)y' + A_0(x)y$*.*

The resulting equation is now in Sturm-Liouville form. We simply identify $p(x) = A_2(x)$ *and* $q(x) = A_0(x)$*.*

Not all second-order differential equations are equally straightforward to transform. For instance, let's examine the differential equation:

$$x^2y'' + xy' + 2y = 0.$$

Here, the coefficient function $A_2(x) = x^2$*, and its derivative* $A_2'(x) = 2x$*, which notably does not match* $A_1(x)$*.*

This equation does not initially appear in Sturm-Liouville form. However, we can transform it into such a form by multiplying the equation by $\frac{1}{x}e^{\int \frac{dx}{x}} = \frac{1}{x}$*:*

$$xy'' + xy' + \frac{2}{x}y = 0.$$

This simplifies to:

$$(xy')' + \frac{2}{x}y = 0.$$

Hence, straightforward conversion is sometimes not possible because of the necessity of continuous coefficients and $A_2(x) \neq 0$*. To achieve a Sturm-Liouville operator, we require an integrating factor* $\mu(x)$ *such that when multiplied, it satisfies* $(\mu A_2)' = \mu A_1$*. Setting* $p = \mu A_2$*, the equation transforms to* $p' = \mu A_1 = \frac{pA_1}{A_2}$*. Solving this separable equation for* p*, we find:*

$$p(x) = ce^{\int \frac{A_1(x)\,dx}{A_2(x)}},$$

where $A_1(x)$ *and* $A_2(x)$ *are continuous functions, and* $A_2(x) \neq 0$ *for all* x*, and where the constant* c *is arbitrary.*

Example 3.1. *Convert the following equation into the form of a Sturm-Liouville equation:*

$$3x^2y'' + 4xy' + 6y + \lambda y = 0$$

Solution 3.1. *Here, the coefficients are* $A_2(x) = 3x^2$, $A_1(x) = 4x$, $A_0(x) = 6$. *Using the transformation formula, we find the integrating factor* $\mu(x)$*:*

$$\mu(x) = \frac{1}{A_2(x)} e^{\int \frac{A_1(x)}{A_2(x)}\,dx} = \frac{1}{3x^2} e^{\int \frac{4x}{3x^2}\,dx} = \frac{1}{3x^2} e^{\frac{4}{3}\ln x} = \frac{1}{3x^2} x^{\frac{4}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

Multiplying the original equation by $\mu(x)$*, we obtain:*

$$\frac{1}{3x^{\frac{2}{3}}} \left(3x^2y'' + 4xy' + 6y + \lambda y\right) = 0$$

Simplifying this gives us:

$$x^{\frac{4}{3}}y'' + \frac{4}{3}x^{\frac{1}{3}}y' + 2y + \frac{\lambda}{3x^{\frac{2}{3}}}y = 0$$

As a result, the equation is now represented by a Sturm-Liouville equation.

Example 3.2. *Convert the following equation into a Sturm-Liouville equation:*

$$x^2y'' + xy' + 2y + \lambda xy = 0$$

Solution 3.2. *Here, the coefficients are* $A_2(x) = x^2$, $A_1(x) = x$, $A_0(x) = 2$. *Using the transformation formula, we find the integrating factor* $\mu(x)$*:*

$$\mu(x) = \frac{1}{A_2(x)} e^{\int \frac{A_1(x)}{A_2(x)}\,dx} = \frac{1}{x^2} e^{\int \frac{1}{x}\,dx} = \frac{1}{x^2} e^{\ln x} = \frac{1}{x}$$

Multiplying the original equation by $\mu(x)$*, we obtain:*

$$\frac{1}{x} \left(x^2y'' + xy' + 2y + \lambda xy\right) = 0$$

Simplifying this gives us:

$$xy'' + y' + \frac{2}{x}y + \lambda y = 0$$

Thus, the equation is now in the form of a Sturm-Liouville equation.

3.3 Lagrange's and Green's Identities

1. **Lagrange's Identity:** Lagrange's identity states:

$$uLv - vLu = [p(uv' - vu')]'$$

Proof 3.2. *To prove this identity for second-order linear differential operators, consider* $L[u] = p(x)u'' + q(x)u' + r(x)u$*, where* $p(x), q(x), r(x)$ *are continuous on* $[a, b]$.

Start with:

$$uLv = u(pv'' + qv' + rv)$$

$$vLu = v(pu'' + qu' + ru)$$

Subtracting vLu *from* uLv*:*

$$u(pv'' + qv' + rv) - v(pu'' + qu' + ru)$$

$$= upv'' + uqv' + urv - (vp'' + vq' + vr)$$

$$= upv'' - vpu'' + uqv' - vqu' + urv - vru$$

$$= [p(uv' - vu')']$$

Therefore, Lagrange's identity is proved.

2. **Green's Identity:** Green's identity is obtained by integrating Lagrange's identity. over $[a, b]$:

$$\int_a^b (uLv - vLu)\, dx = \left. [p(uv' - vu')] \right|_a^b$$

Proof 3.3. *Applying integration by part:*

$$\int u\, dv = uv - \int v\, du$$

Step 1: Express Lv and Lu

$$Lv = \frac{d}{dx}\left(p(x)v'\right) + q(x)v$$

$$Lu = \frac{d}{dx}\left(p(x)u'\right) + q(x)u$$

Step 2: Substitute Lv and Lu into the Integral

$$\int_a^b \left(u \left[\frac{d}{dx} (p(x)v') + q(x)v \right] - v \left[\frac{d}{dx} (p(x)u') + q(x)u \right] \right) dx$$

Step 3: Simplify the Integral

Separate the integrals:

$$\int_a^b \left(u \frac{d}{dx} (p(x)v') - v \frac{d}{dx} (p(x)u') \right) dx + \int_a^b (uq(x)v - vq(x)u) \, dx$$

The second integral simplifies to zero because $uq(x)v - vq(x)u = 0$:

$$\int_a^b (uq(x)v - vq(x)u) \, dx = 0$$

Thus, we have:

$$\int_a^b \left(u \frac{d}{dx} (p(x)v') - v \frac{d}{dx} (p(x)u') \right) dx$$

Step 4: Apply Integration by Parts

First, consider $\int_a^b u \frac{d}{dx} (p(x)v') \, dx$:

$$\int_a^b u \frac{d}{dx} (p(x)v') \, dx = [up(x)v']_a^b - \int_a^b p(x)u'v' \, dx$$

Next, consider $\int_a^b -v \frac{d}{dx} (p(x)u') \, dx$:

$$\int_a^b -v \frac{d}{dx} (p(x)u') \, dx = - [vp(x)u']_a^b + \int_a^b p(x)v'u' \, dx$$

Step 5: Combine the Results

Combining the results from both integration by parts:

$$\int_a^b \left(u \frac{d}{dx} (p(x)v') - v \frac{d}{dx} (p(x)u') \right) dx = [up(x)v' - vp(x)u']_a^b$$

Notice that the integrals involving u' and v' cancel each other out:

$$\int_a^b p(x)u'v'\,dx - \int_a^b p(x)v'u'\,dx = 0$$

Thus, we obtain:

$$\int_a^b (uLv - vLu)\,dx = [p(x)(uv' - vu')]_a^b$$

Thus, Green's identity is proved[6].

Theorem 3.2. *The Sturm-Liouville operator is self-adjoint.*

Proof 3.4. *Let us consider the Sturm-Liouville equation:*

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda r(x)y = 0$$

where λ is an eigenvalue associated with the Sturm-Liouville problem.

Let $u(x)$ and $v(x)$ be two functions that satisfy the Sturm-Liouville problem. We define the inner product:

$$\langle u, L(v)\rangle = \int_a^b u(x)L[v(x)]r(x)\,dx$$

Integrating the second-order derivative term by parts:

$$\langle u, L(v)\rangle = \int_a^b \left[\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - \frac{d^2p}{dx^2}uv\right] r(x)\,dx$$

$$= \int_a^b \left[-p(x)\frac{du}{dx}\frac{dv}{dx} + \frac{d^2p}{dx^2}uv\right] r(x)\,dx$$

$$= \int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \frac{d^2p}{dx^2}uv\right] r(x)\,dx$$

$$= \int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)uv\right] r(x)\,dx$$

Now, consider the adjoint operator $L^\dagger$:

$$L^\dagger[u] = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u$$

To prove self-adjointness, we need to show:

$$\langle u, L(v)\rangle = \langle L^\dagger[u], v\rangle$$

Substituting the expressions, we obtain:

$$\int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)uv\right] r(x)\,dx = \int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)uv\right] r(x)\,dx$$

Simplifying gives:

$$\int_a^b \frac{d}{dx}\left[p(x)\frac{du}{dx} - p(x)\frac{dv}{dx}\right] r(x)\,dx = 0$$

Assuming the boundary terms vanish (or considering appropriate boundary conditions), we conclude:

$$p(x)\frac{du}{dx} - p(x)\frac{dv}{dx} = 0$$

Therefore, $u'(x) = v'(x)$, which implies $u(x)$ and $v(x)$ are related by their derivatives being equal.

Hence, $L^\dagger[u] = L[v]$, illustrating the self-adjoint relationship of the Sturm-Liouville operator L and its adjoint $L^\dagger$.

3.4 Different Types of Sturm-Liouville Boundary Value Problems

Consider the SLP along with boundary conditions satisfying Equation (3.1) for

$$\begin{aligned} \alpha_1^2 + \alpha_2^2 \neq 0 \\ \beta_1^2 + \beta_2^2 \neq 0 \end{aligned} \tag{3.13}$$

we categorize into three classes of SLPs:

1. The functions $p(x), p'(x), q(x)$, and $r(x)$ are continuous on (a, b), and $p(x) > 0, r(x) > 0$ on $[a, b]$. If these conditions on the coefficients hold over the finite interval $[a, b]$, then the problem is termed a **regular** Sturm-Liouville problem.

2. A Sturm-Liouville problem (SLP) is called **singular** if:

   (a) $p(x) > 0$ on (a, b), and $p(a) = p(b) = 0$.

   (b) $r(x) \geq 0$ on $[a, b]$.

3. A Sturm-Liouville problem (SLP) is called **periodic** if $p(a) = p(b)$, $p(x) > 0$,$r(x) > 0$, and $p(x), q(x), r(x)$ are continuous functions on $[a, b]$, along with the following boundary conditions:

$$y(a) = y(b), \quad y'(a) = y'(b)$$

[4].

3.4.1 Regular Sturm-Liouville Problem

A Sturm-Liouville problem along with the boundary condition is considered regular if $p(x)$>0, $r(x)$ >0 and $p(x)$, $p'(x)$,$q(x)$ and $r(x)$ are all continuous functions over the finite interval $[a, b]$.

Example 3.3. *For $\lambda \in R$, solve the Sturm-Liouville problem:*

$$y'' + \lambda y = 0, \quad y(0) - y(\pi) = 0, \quad y'(0) - y'(\pi) = 0.$$

Solution 3.3. *p(x) =1>0, r(x) =1>0, Thus , it is regular SLP.*
We consider three cases corresponding to values of λ.
Case 1: $\lambda = -\mu^2 < 0$

The general solution of the ODE is given by:

$$y = Ae^{-\mu x} + Be^{\mu x}$$

By substituting the boundary condition we obtain the following system:

$$\begin{cases} A(1 - e^{-\mu\pi}) + B(1 - e^{\mu\pi}) = 0 \\ A(-1 + e^{-\mu\pi}) + B(1 - e^{\mu\pi}) = 0 \end{cases}$$

This system has only a trivial solution $A = B = 0$. (The determinant of its matrix of coefficients is different from 0). That is, $y = 0$ is the only solution.
Case 2: $\lambda = 0$
In this case, the problem has the general solution $y = Ax + B$ and by substitution of the boundary condition one can check that $A = 0$ and B is an arbitrary constant.This corresponds to the eigenvalue $\lambda_0 = 0$ and the eigenfunction $f_0 = 1$ after we set $B = 1$. Note that this eigenvalue is called simple, if it is eigenspace of dimension one, otherwise an eigenvalue is multiple.
Case 3: $\lambda = \mu^2 > \mathbf{0}$

The general solution of the ODE is given as $y = Acos(\mu x) + Bsin(\mu x)$ *By substituting the boundary conditions we obtain the following system:*

$$\begin{cases} A(1 - cos\mu\pi) - Bsin(\mu\pi) = 0 \\ A(sin\mu\pi) + B(1 - cos\mu\pi) = 0 \end{cases}$$

This problem has a non-trivial solution when the determinant of the matrix of coefficients $D(\mu) = 1 - \cos(\mu\pi) = 0$*. This corresponds to* $\mu = 2n$*, where* $n = \pm 1, \pm 2, \ldots$*. Hence* $\lambda_n = 4n^2$*. Eigenfunctions corresponding to* λ_n *are given by* $A = B = 1$*:*

$$y_{1n} = \cos(\sqrt{\lambda_n}\, x) \quad \textit{and} \quad y_{2n} = \sin(\sqrt{\lambda_n}\, x), \quad n = \pm 1, \pm 2, \ldots$$

Notation 3.1. *All the eigenvalues* λ_n *are positive, and there exist two linearly independent eigenfunctions corresponding to each eigenvalue, hence they are not unique.*

3.4.2 Properties of regular SLP

Theorem 3.3. *All the eigenvalues of the regular Sturm-Liouville problem are real.*

Proof 3.5. *Given a self-adjoint Sturm-Liouville operator* L*, which is defined on sufficiently smooth functions satisfying the regular boundary conditions,* L *is an operator such that* $Lu = \lambda u$ *for some scalar* λ *and function* $u \neq 0$*.*

Since L *is self-adjoint, we have:*

$$\langle Lu, u\rangle = \langle u, Lu\rangle$$

This implies:

$$\langle \lambda u, u\rangle = \langle u, \lambda u\rangle$$

Therefore:

$$\lambda\langle u, u\rangle = \overline{\lambda}\langle u, u\rangle$$

Given that $\langle u, u\rangle > 0$*, it follows that* $\lambda = \overline{\lambda}$*, implying* λ *is real[6].*

Theorem 3.4. *An eigenfunctions of a regular Sturm-Liouville problem (SLP) corresponding to distinct eigenvalues are orthogonal with respect to the weight function* r *on* $[a, b]$*. That is, if* u *and* v *are eigenfunctions corresponding to distinct eigenvalues* λ *and* μ *respectively, then*

$$\int_a^b r(x)u(x)v(x)\, dx = 0. \tag{3.14}$$

Proof 3.6. *Let us consider the regular Sturm-Liouville operator* $L[y] = -(p(x)y')' + q(x)y$*, where* $p(x)$ *and* $q(x)$ *are continuous functions on the interval* $[a, b]$*. For distinct eigenvalues* λ_1 *and* λ_2*, the following eigenvalue equations hold for the corresponding eigenfunctions* $u(x)$ *and* $v(x)$*:*

$$L[u] = \lambda_1 w(x)u(x)$$

$$L[v] = \lambda_2 w(x)v(x)$$

Here, $w(x)$ *denotes the weight function.*

To demonstrate orthogonality, we compute the inner product of these eigenfunctions with respect to the weight function $w(x)$*:*

$$\langle u, v\rangle = \int_a^b u(x)v(x)w(x)\,dx$$

Applying integration by parts:

$$\langle u, v\rangle = -\int_a^b (u''(x)v(x) + u(x)v''(x))\,dx + [u(x)v(x)w(x)]_a^b$$

Since $u(x)$ *and* $v(x)$ *satisfy the boundary conditions of the Sturm-Liouville problem, and substituting the eigenvalue equations:*

$$\langle u, v\rangle = -(\lambda_1 + \lambda_2)\int_a^b u(x)v(x)w(x)\,dx + [u(x)v(x)w(x)]_a^b$$

Given $\lambda_1 \neq \lambda_2$*, it follows that* $\langle u, v\rangle = 0$*, showing that the eigenfunctions* $u(x)$ *and* $v(x)$ *are orthogonal with respect to the weight function* $w(x)$ *on the interval* $[a, b]$ *when corresponding to distinct eigenvalues [7].*

Theorem 3.5. *Eigenvalues of a regular Sturm-Liouville problem (SLP) are simple. This means the eigenfunction corresponding to an eigenvalue is unique to a constant multiple.*

Proof 3.7. *Consider a typical Sturm-Liouville operator* $L[y] = -(p(x)y')' + q(x)y$*, defined on the interval* $[a, b]$*, where* $p(x)$ *and* $q(x)$ *are continuous functions. Assume* λ *is an eigenvalue of the problem, and* $u(x)$ *is an eigenfunction satisfying the associated boundary conditions.*

Suppose there exists another linearly independent eigenfunction $v(x)$*, also corresponding to* λ*.*

The Wronskian of two solutions $u(x)$ *and* $v(x)$ *of a linear homogeneous differential equation* $y'' + p(x)y' + q(x)y = 0$ *is given by:*

$$W[u, v] = u(x)v'(x) - u'(x)v(x) \tag{3.15}$$

Since $u(x)$ and $v(x)$ are eigenfunctions corresponding to the same eigenvalue λ:

$$L[u] = \lambda u(x)$$

$$L[v] = \lambda v(x)$$

Substituting into the Wronskian equation:

$$W[u, v] = u(x)v'(x) - u'(x)v(x) = \lambda(u(x)v(x)) - \lambda(u(x)v(x)) = 0$$

However, this contradicts the assumption that the Wronskian is non-zero. Therefore, there cannot exist another linearly independent eigenfunction corresponding to the same eigenvalue λ.

Thus, each eigenvalue of a regular Sturm-Liouville problem has multiplicity one, and they are simple.

Theorem 3.6. *A self-adjoint Sturm-Liouville operator has an infinite sequence of real eigenvalues λ_n, each of which is simple and satisfies $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, with $\lim_{n\to\infty} \lambda_n = \infty$.*

Theorem 3.7. *The set of eigenfunctions forms a complete orthogonal system. This means any function can be represented by a generalized Fourier series expansion involving these eigenfunctions [3].*

Theorem 3.8. *The eigenvalues satisfy the Rayleigh quotient*

$$\lambda_n = \frac{-p(y_n)\frac{dy_n}{dx}\Big|_b^a + \int_a^b \left[p\left(\frac{dy_n}{dx}\right)^2 - qy_n^2\right] dx}{\langle y_n, y_n\rangle} \tag{3.16}$$

Proof 3.8. *To establish the Rayleigh quotient, we start with the eigenvalue problem for a self-adjoint operator. Consider the Sturm-Liouville problem:*

$$-\frac{d}{dx}\left(p(x)\frac{dy_n}{dx}\right) + q(x)y_n = \lambda_n r(x) y_n$$

where $p(x)$, $q(x)$, and $r(x)$ are real-valued functions defined on $[a, b]$, and y_n is an eigenfunction corresponding to λ_n.

Multiplying both sides of the eigenvalue equation by y_n and integrating over $[a, b]$ using the weight function $r(x)$, we obtain:

$$-\int_a^b \frac{d}{dx}\left(p(x)\frac{dy_n}{dx}\right) y_n\, dx + \int_a^b q(x)y_n^2\, dx = \lambda_n \int_a^b r(x)y_n^2\, dx$$

Integrating the first term by parts gives:

$$\int_a^b p(x)\left(\frac{dy_n}{dx}\right)^2 dx - p(x)\frac{dy_n}{dx}y_n\bigg|_a^b + \int_a^b q(x)y_n^2\,dx = \lambda_n \int_a^b r(x)y_n^2\,dx$$

Dividing through by $\int_a^b r(x)y_n^2\,dx$ *(assuming it's non-zero) gives the Rayleigh quotient:*

$$\lambda_n = \frac{-p(y_n)\frac{dy_n}{dx}\Big|_b^a + \int_a^b \left[p\left(\frac{dy_n}{dx}\right)^2 - qy_n^2\right] dx}{\langle y_n, y_n\rangle}$$

Thus, the Rayleigh quotient for the Sturm-Liouville problem is established.

3.4.3 The Periodic Sturm-Liouville Problem

The primary distinction between regular and periodic Sturm-Liouville problems lies in their eigenvalues, where periodic eigenvalues are not necessarily simple. Here, we outline the properties of the periodic Sturm-Liouville problem:

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0 \tag{3.17}$$

$$y(0) = y(2T); \quad y'(0) = y'(2T) \tag{3.18}$$

where $p(x), p'(x), q(x)$, and $r(x)$ are continuous functions, and $2T$-periodic functions with p and r being positive.

Properties of Periodic Sturm-Liouville Problem

For the periodic Sturm-Liouville boundary value problem :

1. Eigenvalues λ_n are real and form a countable, increasing sequence $\{\lambda_n\}$ such that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ with $\lim_{n\to\infty} \lambda_n = \infty$.

2. Eigenfunctions can be chosen to be real-valued.

3. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to r on $[0, 2T]$.

4. Let $\{y_n\}$ be an orthonormal sequence of all the eigenfunctions, and let f be a $2T$-periodic function such that f is continuous and f is piecewise continuous. Then,

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \tag{3.19}$$

where $a_n = \int_0^{2T} f(x)y_n(x)r(x)\,dx$. Moreover, the series converges uniformly on $[0, 2T]$ and hence on $(-\infty, \infty)$ [3].

Example 3.4. *Periodic SL-BVP:*

$$y'' + \lambda y = 0; \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$$

Solution 3.4. *Let's solve the SL-BVP $y'' + \lambda y = 0$ with boundary conditions $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$ using the theory of eigenvalues and eigenfunctions.*

Step 1: *Calculate the eigenvalues and eigenfunctions of the given SL-BVP. The general solution to the differential equation can be written as $y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$, where A and B are constants. Applying the boundary conditions gives us the characteristic equation:*

$$\cos(\sqrt{\lambda} \cdot 0) = \cos(0) = 1$$

$$\cos(\sqrt{\lambda} \cdot 2\pi) = \cos(2\pi\sqrt{\lambda}) = 1$$

This implies $2\pi\sqrt{\lambda} = 2n\pi$ for some integer n. Solving for λ, we find $\lambda = n^2$ for $n = 0, 1, 2, \ldots$.

Therefore, the eigenvalues are $\lambda_n = n^2$ and the corresponding eigenfunctions are $y_n(x) = A_n\cos(nx) + B_n\sin(nx)$.

Step 2:*Verification via Fourier Series: A complete solution of the SL-BVP can be expressed as a Fourier series:*

$$y(x) = \sum_{n=0}^{\infty}(A_n\cos(nx) + B_n\sin(nx))$$

where the coefficients A_n and B_n are determined by the orthogonality of the eigenfunctions:

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y(x)\cos(nx)\,dx$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y(x)\sin(nx)\,dx$$

By substituting the eigenfunctions into the Fourier series and utilizing the orthogonal property of sine and cosine functions, we extend the Fourier series notation to the solution of the SL-BVP [1].

3.4.4 Singular SL

A Sturm-Liouville differential equation on an interval $[a, b]$ with any of the following conditions is called a singular SL-BVP:

$$(p(x)y')' + q(x)y = \lambda r(x)y, \quad a < x < b \tag{3.20}$$

1. $p(a) = 0$: Boundary condition at a is dropped, boundary condition at b is homogeneous mixed.

2. $p(b) = 0$: Boundary condition at b is dropped, boundary condition at a is homogeneous mixed.

3. $p(a) = p(b) = 0$ and no boundary condition.

4. Interval $[a, b]$ is infinite.

Notation 3.2. *1. If $p(a) = 0$ and there is no boundary condition at a, then y is considered a solution if $y(a) < \infty$.*

2. If the interval is infinite, y must be square integrable to qualify as a solution [3].

Example 3.5. ***Legendre equation*** *is given by*

$$\left(1 - x^2\right) y'' - 2xy' + l(l+1)y = 0, \quad x \in (-1, 1), \quad l \in R.$$

Solution 3.5. *This can be rewritten as*

$$-\frac{d\left(\left(1 - x^2\right) y'\right)}{dx} = l(l+1)y,$$

where $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, and $\lambda = l(l+1)$. However, since $p(1) = p(-1) = 0$, this constitutes a singular SL-BVP.

Example 3.6. ***Hermite equation*** *is given by:*

$$y'' - 2xy' + 2\alpha y = 0, \quad x \in (-\infty, \infty), \quad \alpha \in R.$$

Solution 3.6. *This can be transformed into Sturm-Liouville form by multiplying by e^{-x^2}:*

$$-\frac{d\left(e^{-x^2} y'\right)}{dx} = 2\alpha e^{-x^2} y.$$

Here, $p(x) = e^{-x^2}$, $q(x) = 0$, $r(x) = e^{-x^2}$, and $\lambda = 2\alpha$. This is a singular SL-BVP due to the infinite interval.

Example 3.7. *Verify completeness of the set of eigenfunctions for a Sturm-Liouville boundary value problem.*

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda r(x)y = 0$$

subject to boundary conditions $y(0) = y(\pi) = 0$.

Solution 3.7. *Let $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$.*

1. The Eigenvalue Problem: *The differential equation in Sturm-Liouville form becomes:*

$$L[y] = -\frac{d}{dx}\left(\frac{dy}{dx}\right) + \lambda y$$

Applying the boundary conditions, the problem is:

$$-\frac{d^2y}{dx^2} + \lambda y = 0$$

with boundary conditions $y(0) = y(\pi) = 0$.

2. Solving the Eigenvalue Problem: *Solving this differential equation for eigenvalues λ and corresponding eigenfunctions $y(x)$, we get the general solution:*

$$y(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

Applying the boundary conditions:

- $y(0) = 0$ *implies* $B = 0$.
- $y(\pi) = 0$ *implies* $\sqrt{\lambda}\pi = n\pi$, *where n is an integer.*

Thus, $\lambda_n = n^2$, and the corresponding eigenfunctions are $y_n(x) = \sin(nx)$ for $n = 1, 2, 3, \ldots$.

3. Completeness of Set of Eigenfunctions: *To demonstrate that the set of eigenfunctions $\{\sin(nx)\}$ is complete, we consider the Fourier series expansion of any piecewise smooth function $f(x)$ that satisfies the boundary conditions:*

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

where c_n are the Fourier coefficients. By the theory of Fourier series, the set of sine functions is complete, meaning the series converges to $f(x)$ in the mean square sense. Hence, within the interval $(0, \pi)$, the set of eigenfunctions forms a complete basis. This implies that any periodic function (with period 2π) can be represented as an infinite series of sine functions. Therefore, the eigenfunctions $\{\sin(nx)\}$ form a complete set on the interval $[0, \pi]$.

3.5 Applications of Sturm-Liouville Boundary Value Problems

Sturm-Liouville boundary value problems have a wide range of applications in various scientific and engineering disciplines, especially in the analysis of differential equations and eigenvalue problems. Below are some examples of their applications:

1. **Quantum Mechanics:** SL-BVPs play a crucial role in solving the Schrödinger equation, which governs the behavior of quantum systems. In this context, the eigenfunctions represent the allowed states of the system, and the eigenvalues correspond to the energy levels of these states.

2. **Heat Conduction:** These problems are essential in modeling the distribution of temperature within solid objects undergoing heat conduction. The eigenfunctions describe the spatial temperature distribution, while the eigenvalues indicate the rate of heat transfer.

3. **Fluid Dynamics:** SL-BVPs are applied in the analysis of fluid flow stability and the behavior of acoustic waves in fluid-filled cavities. The eigenfunctions and eigenvalues provide important information about the oscillation modes and stability characteristics of the fluid system.

4. **Structural Mechanics:** In structural mechanics, SL-BVPs are used to investigate the stability of structures, such as the buckling of beams, plates, and shells under various loads. Here, the eigenvalues determine the critical buckling loads, and the eigenfunctions describe the deformation shapes of the structures [7].

CHAPTER FOUR

GENERAL SOLUTIONS OF 2D LIOUVILLE EQUATIONS

4.1 Liouville Equation

Consider the Liouville equations of the form

$$\psi_{xx} + \psi_{yy} + ke^{\psi} = 0 \tag{4.1}$$

and

$$\psi_{xx} - \psi_{yy} + ke^{\psi} = 0 \tag{4.2}$$

where k is a real constant and $\psi = \psi(x, y)$. Equations (4.1) and (4.2) are commonly recognized as forms of the two-dimensional **Elliptic** and **Hyperbolic** Liouville equations, respectively [5].

4.2 The Elliptic Case

To illustrate the method for solving the elliptic case, we consider the elliptic Liouville equation in the (x, y) plane:

$$\psi_{xx} + \psi_{yy} = \bar{c}e^{d\psi} \tag{4.3}$$

where $\bar{c}$ and d are non-zero real constants. We start with a complex function $\psi(z)$ where $z = x + iy$. Viewing it as a function of the two real variables x and y, the partial derivatives $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ can be defined. In this framework, the differential $d\psi$ is related to these partial derivatives as follows:

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \tag{4.4}$$

However, in practice, different rules such as the product rule, quotient rule, and chain rule are often used to compute $d\psi$. These rules are given by:

$$d(uv) = u\,dv + v\,du, \quad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}, \quad df = \frac{df}{du}\,du \text{ for } f = f(u).$$

Now consider changing to characteristic coordinates, defined as $z = x + iy$ and $\bar{z} = x - iy$. This introduces a new set of variables $\{z, \bar{z}\}$, related to the original variables $\{x, y\}$ by:

$$z = x + iy, \quad \bar{z} = x - iy, \quad x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

These relationships extend to the differentials as follows:

$$dz = dx + i\,dy, \quad d\bar{z} = dx - i\,dy, \quad dx = \frac{dz + d\bar{z}}{2}, \quad dy = \frac{dz - d\bar{z}}{2i}$$

As we know, the real variables x and y are mutually independent, and therefore, so are their differentials. However, the complex variables z and $\bar{z}$ are related and cannot be treated as independent variables. In equation (4.4), the differential $d\psi$ of ψ is expressed in terms of dx and dy. Using equation (4.5), we can rewrite $d\psi$ in terms of dz and $d\bar{z}$. Specifically,

$$\begin{aligned} d\psi &= \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \\ &= \frac{\partial\psi}{\partial x}\cdot\frac{1}{2}(dz + d\bar{z}) + \frac{\partial\psi}{\partial y}\cdot\left(-\frac{i}{2}\right)(dz - d\bar{z}) \\ &= \frac{1}{2}\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial\psi}{\partial x} + i\frac{\partial\psi}{\partial y}\right)d\bar{z} \end{aligned} \tag{4.5}$$

Now we define $\frac{\partial\psi}{\partial z}$ and $\frac{\partial\psi}{\partial\bar{z}}$ such that the identity

$$d\psi = \frac{\partial\psi}{\partial z}dz + \frac{\partial\psi}{\partial\bar{z}}d\bar{z} \tag{4.6}$$

holds. By comparing this with the previous identity, we naturally define:

$$\frac{\partial\psi}{\partial z} = \frac{1}{2}\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right), \quad \frac{\partial\psi}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial\psi}{\partial x} + i\frac{\partial\psi}{\partial y}\right)$$

The operator $\Delta \equiv 4\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\bar{z}}$ represents the **Laplacian**. Indeed, a straightforward computation shows that

$$\Delta = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

This final expression is the familiar definition of the Laplacian. Therefore,

$$4\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\bar{z}} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \tag{4.7}$$

This process leads to solving the given elliptic case, though the specific form of the solutions will depend on the constants c, d, as well as the boundary conditions. For instance,

$$\psi_{z\bar{z}} = \bar{c}e^{d\psi} \tag{4.8}$$

indicates that solutions ψ satisfy $c = \bar{c}/4$. By a linear change of the dependent variable,

$$\psi = d\psi + \log(|cd|), \tag{4.9}$$

Equation (4.9) can be rewritten in the canonical form:

$$\psi_{z\bar{z}} = \operatorname{sgn}(cd)e^{\psi} \tag{4.10}$$

Equations (4.9) and (4.10) are equivalent, and we can derive one from the other using the relationships we have established. Starting from the original equation and noting that $c = \bar{c}/4$, and with the linear change of the dependent variable $\psi = d\psi + \log(|cd|)$, we substitute these relations into the original equation to convert it into the canonical form as shown: To begin, let us apply these substitutions to our initial equation:

$$\psi_{z\bar{z}} = \bar{c}e^{d\psi}$$

First, substitute $c = \bar{c}/4$:

$$\psi_{z\bar{z}} = \left(\frac{\bar{c}}{4}\right) e^{d\psi}$$

Next, use the linear change of the dependent variable $\psi = d\psi + \log(|cd|)$:

$$\psi_{z\bar{z}} = \left(\frac{\bar{c}}{4}\right) e^{d(d\psi+\log(|cd|))}$$

Simplify the exponential term:

$$\psi_{z\bar{z}} = \left(\frac{\bar{c}}{4}\right) e^{d^2\psi+d\log(|cd|)}$$

Rewrite the exponential term using the properties of logarithms:

$$\psi_{z\bar{z}} = \left(\frac{\bar{c}}{4}\right) |cd|e^{d^2\psi}$$

Here, $|cd|$ is a positive constant, so we can absorb it into $\bar{c}/4$, and we get:

$$\psi_{z\bar{z}} = \operatorname{sgn}[cd]e^{\psi}$$

Therefore, the original equation can be written in the canonical form as provided.

Theorem 4.1. *Any function $\psi(z, \bar{z})$ that is twice differentiable with respect to z and $\bar{z}$ and is a real solution to*

$$\psi_{z\bar{z}} = \bar{c}e^{d\psi} \tag{4.11}$$

also satisfies

$$\psi_{z\bar{z}} - \frac{d}{2}\psi_{\bar{z}z}^2 = \bar{E}(\bar{z}) \tag{4.12}$$

where $\bar{E}$ is some analytic function of $\bar{z}$.

Proof 4.1. *Integrating (4.12) with respect to* z *gives* $\psi_{\bar{z}} = \bar{c} \int_{z_0}^{z} e^{d\psi}\, dz + \bar{F}(\bar{z})$ *for some arbitrary analytic function* $\bar{F}(\bar{z})$*. Differentiating with respect to* $\bar{z}$ *and using (4.12), we obtain*

$$\psi_{\bar{z}\bar{z}} = \frac{d}{2}\psi_{\bar{z}z}^2 + \bar{E}(\bar{z}) \tag{4.13}$$

where $\bar{E}(\bar{z}) = \bar{F}'(\bar{z})$*.*

Theorem 4.2. *Any real-valued solution* $\psi(z, \bar{z})$ *to (4.11) that is sufficiently differentiable with respect to* z *and* $\bar{z}$ *also satisfies (4.12) for some constant* C*.*

Proof 4.2. *Taking the second-order derivative of (4.11) with respect to* z *gives*

$$\psi_{\bar{z}zzz} - d\psi_{\bar{z}}\psi_{\bar{z}zz} - d\psi_{\bar{z}z}^2 = 0 \tag{4.14}$$

Taking the complex conjugate of (4.13) gives

$$\psi_{z\bar{z}z} - d\psi_{z}\psi_{z\bar{z}z} - d\psi_{z\bar{z}}^2 = 0 \tag{4.15}$$

Define $\beta(z, \bar{z}) = \psi_{\bar{z}z}$*. Subtracting (4.13) from (4.14) yields*

$$\psi_{\bar{z}}\beta_z - \beta_{\bar{z}}\psi_z = 0 \tag{4.16}$$

It follows from (4.15) that $\beta = f(\psi)$ *for some real-valued function* f*. Differentiating (4.11) once with respect to* z *yields*

$$\psi_{\bar{z}zz} - d\psi_{\bar{z}z}\psi_{\bar{z}} = 0 \tag{4.17}$$

Using $\beta(z, \bar{z}) = f(\psi)$ *then implies*

$$\psi_{\bar{z}}\left(f' - df\right) = 0 \tag{4.18}$$

From this, we conclude that any non-trivial real-valued function $\psi(z, \bar{z})$ *that satisfies (4.11) also satisfies (4.12) for some constant* c*.*

Theorem 4.3. *Every real-valued solution to (4.11) is of the form*

$$\psi = \frac{-2}{d} \log\left[c_1 y_1(z)\bar{y}_1(\bar{z}) + c_4 y_2(z)\bar{y}_2(\bar{z}) + c_2 y_1(z)\bar{y}_2(\bar{z}) + \bar{c}_2\bar{y}_1(\bar{z})y_2(z)\right] \tag{4.19}$$

where $\bar{y}_1(\bar{z})$ *and* $\bar{y}_2(\bar{z})$ *are two independent solutions to*

$$y_{\bar{z}\bar{z}} + \frac{d}{2}\bar{E}(\bar{z})y = 0 \tag{4.20}$$

for some analytic $\bar{E}(\bar{z})$*, while* c_1 *and* c_4 *are real constants and* c_2 *is a complex constant.*

Lemma 4.1. *The complex conjugate of $f(\bar{z})$ is $\overline{f(\bar{z})}$.*

Let's observe a characteristic $f(z)$ where $z = x + iy$ and $\bar{z} = x - iy$. Now let us write $f(z)$ in terms of its real and imaginary parts:

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions. Then it follows that:

$$f(\bar{z}) = u(x, -y) + iv(x, -y)$$

From the definition above we can specify the conjugate as:

$$\overline{f(\bar{z})} = u(x, y) - iv(x, y)$$

Comparing $f(\bar{z})$ and $\overline{f(\bar{z})}$, we can observe that they are indeed equal:

$$f(x, -y) = \overline{f(x, y)}$$

Thus we have proven that:

$$f(\bar{z}) = \overline{f(\bar{z})}$$

Therefore, we have proven that $f(\bar{z}) = \overline{f(\bar{z})}$.

Proof 4.3. *Note that (4.11) is in the form of a Riccati equation and can be transformed into the linear second-order differential equation (4.19) for $y = e^{-\frac{d\psi}{2}}$. Therefore, it follows that*

$$y(z, \bar{z}) = E_1(z)\bar{y}_1(z) + E_2(z)\bar{y}_2(\bar{z}) \tag{4.21}$$

for some functions $E_1(z)$ and $E_2(z)$. Since y is real, by taking the complex conjugate of (4.20), it follows that

$$y(z, \bar{z}) = \bar{E}_1(\bar{z})y_1(z) + \bar{E}_2(\bar{z})y_2(z) \tag{4.22}$$

Now, since (4.19) is a solution to (4.19), it follows that

$$\begin{aligned} \bar{E}_1(\bar{z}) &= \bar{c}_1\bar{y}_1(\bar{z}) + \bar{c}_2\bar{y}_2(\bar{z}) \\ \bar{E}_2(\bar{z}) &= \bar{c}_3\bar{y}_1(\bar{z}) + \bar{c}_4\bar{y}_2(\bar{z}) \end{aligned} \tag{4.23}$$

for some constants $\bar{c}_1, \bar{c}_2, \bar{c}_3$, and $\bar{c}_4$. Substituting (4.22) back into (4.20) and (4.21) and equating the two different expressions for y, we obtain the conditions that c_1 and c_4 are real, and $c_2 = \bar{c}_3$. Thus, we can write

$$y(z, \bar{z}) = c_1\bar{y}_1(\bar{z})y_1(z) + c_4y_2(z)\bar{y}_2(\bar{z}) + c_2y_1(z)\bar{y}_2(\bar{z}) + \bar{c}_2\bar{y}_1(\bar{z})y_2(z) \tag{4.24}$$

Theorem 4.4. *Let $Y_1(z)$ and $Y_2(z)$ be two arbitrary but independent analytic functions. Denote their Wronskian by $W(z) = Y_1(z)Y_2'(z) - Y_1'(z)Y_2(z)$. Then, $y_1(z) = \frac{Y_1(z)}{\sqrt{W(z)}}$ and $y_2(z) = \frac{Y_2(z)}{\sqrt{W(z)}}$ are two independent analytic functions with unit Wronskian.*

Proof 4.4. *Since $Y_1(z)$ and $Y_2(z)$ are independent, $W(z) \neq 0$. The relation $c_1y_1 + c_2y_2 = 0$ implies $c_1y_1 + c_2y_2 = 0$ in some open set. From the independence of Y_1 and Y_2, this implies $c_1 = 0$ and $c_2 = 0$, i.e., y_1 and y_2 are independent. Substituting for y_1 and y_2 in terms of Y_1 and Y_2, the Wronskian $W(z)$ of y_1 and y_2 is*

$$W(z) = \frac{Y_1(z)Y_2'(z) - Y_1'(z)Y_2(z)}{W(z)} = 1 \tag{4.25}$$

Theorem 4.5. *Any real solution to (4.11) is of the form*

$$\psi = \frac{-2}{d}\log\left[c_1y_1(z)\overline{y}_1(\overline{z}) + c_4y_2(z)\overline{y}_2(\overline{z}) + c_2y_1(z)\overline{y}_2(\overline{z}) + \overline{c}_2\overline{y}_1(\overline{z})y_2(z)\right] + \frac{1}{d}\log[W(z)\overline{W}(\overline{z})] \tag{4.26}$$

for some independent analytic functions $Y_1(z)$ and $Y_2(z)$, where c_1 and c_4 are real constants, c_2 is a complex constant, and $W(z)$ is the Wronskian of $Y_1(z)$ and $Y_2(z)$.

Proof 4.5. *It follows by substitution of $y_1(z)$ and $y_2(z)$ in terms of $Y_1(z)$ and $Y_2(z)$ in Equation (4.18).*

Theorem 4.6. *The most general real solution to (4.11) is given by Equation (4.17), where $Y_1(z)$ and $Y_2(z)$ are any independent analytic functions of z, $W(z)$ is the Wronskian, with c_1 and c_4 real constants, and c_2 a complex constant satisfying the constraint*

$$cd = -2\left(c_1c_4 - |c_2|^2\right), \tag{4.27}$$

which are otherwise arbitrary.

Proof 4.6. *Since any solution to (4.11) is of the form (4.25), by directly substituting (4.25) into (4.11), it is followed that (4.26) is satisfied if and only if the constraint is satisfied.*

4.3 The Hyperbolic Case

The two-dimensional hyperbolic Liouville equation is given by

$$u_{\xi\xi} - u_{\eta\eta} = \overline{c}e^{du}, \tag{4.28}$$

where $\overline{c}$ and d are real constants different from zero. By shifting to characteristic coordinates (x, t) where $x = \xi + \eta$ and $t = \xi - \eta$, we can equivalently reduce it to

$$u_{xt} = ce^{du}, \tag{4.29}$$

where c and d are real constants different from zero, and $c = \frac{\overline{c}}{4}$.

$$\phi = d\phi + \log(|cd|) \tag{4.30}$$

Using Equation (4.29), Equation (4.30) can be written in the canonical form

$$\phi_{xt} = \text{sgn}[cd]e^{\phi} \tag{4.31}$$

Theorem 4.7. *Any function $u(x, t)$ that is twice differentiable with respect to x and t and is a real solution to (4.29) also satisfies*

$$u_{xt} - \frac{d}{2}u_{xt}^2 = \overline{E}(t) \tag{4.32}$$

where $\overline{E}$ is some analytic function of t.

Proof 4.7. *Integrating Equation (4.29) with respect to x gives $u_t = c\int_{x_0}^{x} e^{du}\,dx + \overline{F}(t)$ for some arbitrary analytic function $\overline{F}(t)$. On differentiating with respect to t and using (4.29), we obtain*

$$u_{tt} = \frac{d}{2}u_t^2 + \overline{E}(t) \tag{4.33}$$

where $\overline{E}(t) = \overline{F}'(t)$.

Theorem 4.8. *Any function $u(x, t)$ that is twice differentiable with respect to x and t and is a real solution to (4.29) also simultaneously satisfies the two equations*

$$u_{xx} - \frac{d}{2}u_x^2 = E(x) \tag{4.34}$$

$$u_{tt} - \frac{d}{2}u_t^2 = F(t) \tag{4.35}$$

for some choice of functions $E(x)$ and $F(t)$.

Proof 4.8. *Integrating equation (4.29) with respect to* t *gives* $u_x = c\int_{t_0}^{t} e^{du}\,dt + G(x)$ *for some arbitrary analytic function* $G(x)$. *On differentiating with respect to* x *and using (4.29), we obtain*

$$u_{xx} - \frac{d}{2}u_x^2 = E(x) \tag{4.36}$$

where $E(x) = G'(x)$. *Hence (4.34) is satisfied.*

Next, integrating equation (4.29) with respect to x *gives* $u_t = c\int_{x_0}^{x} e^{du}\,dx + K(t)$ *for some arbitrary analytic function* $K(t)$. *On differentiating with respect to* t *using (4.29), we have*

$$u_{tt} - \frac{d}{2}u_t^2 = F(t) \tag{4.37}$$

where $F(t) = K'(t)$. *Hence (4.35) is satisfied.*

Theorem 4.9. *Any sufficiently differentiable solution of both (4.34) and (4.36) satisfies equation (4.29) for some choice of the constant* c.

Proof 4.9. *Differentiating (4.34) twice with respect to* t *gives*

$$u_{xxtt} - du_{xtt}u_x - du_{xt}^2 = 0 \tag{4.38}$$

Similarly, differentiating (4.35) twice with respect to x *gives*

$$u_{ttxx} - du_{txx}u_t - du_{tx}^2 = 0 \tag{4.39}$$

Now, subtracting (4.38) from (4.39) implies

$$u_{xtt}u_x - u_{txx}u_t = 0 \tag{4.40}$$

From differential equations

$$\begin{cases} u_{xtt} = f'(u)u_t \\ u_{xtx} = f'(u)u_x \end{cases}$$

which implies

$$u_{xt} = f(u) \tag{4.41}$$

for some real function $f(u)$. *Differentiating (4.35) once with respect to* t *and using (4.41) gives*

$$u_x(f\prime - df) = 0 \tag{4.42}$$

Hence, any non-trivial simultaneous solution of (4.35) and (4.36) satisfies (4.29) for some choice of constant c.

Theorem 4.10. *Every solution of (4.29) is of the form*

$$u = \frac{2}{d} \log \left[c_1 y_1(t) w_1(x) + c_2 y_1(t) w_2(x) + c_3 y_2(t) w_1(x) + c_4 y_2(t) w_2(x) \right] \tag{4.43}$$

where $y_1(t)$ and $y_2(t)$ are two independent solutions to

$$y_{tt} + \frac{d}{2} F(t) y = 0 \tag{4.44}$$

and where $w_1(x)$ and $w_2(x)$ are two independent solutions to

$$w_{xx} + \frac{d}{2} E(x) w = 0 \tag{4.45}$$

for some analytic functions $F(t)$ and $E(x)$, while c_1, c_2, c_3, c_4 are real constants.

Proof 4.10. *Note that (4.35) is of the Riccati form and can be made into a linear second-order equation for $M(x,t) = e^{-\frac{du}{2}}$. Using this transformation, the resulting differential equation for $M(x,t)$ is*

$$M_{xx} + \frac{d}{2} E(x) M = 0 \tag{4.46}$$

Therefore,

$$M(x,t) = E_1(t) w_1(x) + E_2(t) w_2(x) \tag{4.47}$$

for some functions $E_1(t)$ and $E_2(t)$. Now since u is also a solution of (4.36), then $M(x,t)$ is also a solution to (4.44) and we deduce that

$$E_1(t) = c_1 y_1(t) + c_3 y_2(t) \tag{4.48}$$

and

$$E_2(t) = c_2 y_1(t) + c_4 y_2(t) \tag{4.49}$$

for some real constants c_1, c_2, c_3, and c_4. Now substituting (4.48) and (4.49) into (4.47) gives the result (4.43).

Theorem 4.11. *Any real solution to (4.29) is of the form*

$$u = \frac{-2}{d} \log \left[c_1 Y_1(t) W_1(x) + c_2 Y_1(t) W_2(x) + c_3 Y_2(t) W_1(x) + c_4 Y_2(t) W_2(x) \right] + \frac{1}{d} \log \left[Y(t) W(x) \right] \tag{4.50}$$

where $Y_1(t)$ and $Y_2(t)$ are independent sufficiently differentiable functions with Wronskian $Y(t)$, $W_1(x)$ and $W_2(x)$ are independent sufficiently differentiable functions with Wronskian $W(x)$, and c_1, c_2, c_3, c_4 are real constants.

Theorem 4.12. *The most general real solution to (4.29) is given by (4.49), where $Y_1(t)$, $Y_2(t)$ are any independent functions of t with Wronskian $\mathbf{Y}(\mathbf{t})$, $W_1(x)$, $W_2(x)$ are any independent functions of x with Wronskian $W(x)$, and c_1, c_2, c_3, and c_4 are real constants satisfying the constraint*

$$cd = -2\left(c_1c_4 - c_2c_3\right). \tag{4.51}$$

Proof 4.11. *Since we know that any solution of (4.29) is of the form (4.50), by substituting (4.51) into equation (4.30), we find that (4.29) is satisfied if and only if the constraint (4.51) is satisfied.*

Example 4.1. *Let*

$$Y_1(t) = \delta(t) + z_0, \quad Y_2(t) = 1, \quad W_1(x) = \theta(x), \quad W_2(x) = 1 \tag{4.52}$$

with $c_1 = c_4 = 0$, $c_2 = c_3 = \frac{\sqrt{2}}{2}$, and with $\delta(t)$ and $\theta(x)$ arbitrary real functions, and z_0 a real constant, represents a well-known general solution to the hyperbolic Liouville (4.29) (with $c = d = 1$) [1]:

$$u(x,t) = \log\left[\frac{2\theta'(x)\delta'(t)}{\left(\theta(x) + \delta(t) + z_0\right)^2}\right] \tag{4.53}$$

CHAPTER FIVE

CONCLUSION

In this master's thesis, we have explored different basic concepts such as Sturm-Liouville boundary value problems and general solutions to specific types of 2D Liouville equations, focusing on elliptic and hyperbolic equations.

Firstly, we focused on some fundamental properties of Sturm-Liouville boundary value problems. Based on these properties we solved second-order differential equations subject to boundary points on a given set. It detail explained about types of Sturm-Liouville boundary value problems and their solutions, how to solve eigenvalue problems, inner products of eigenfunctions and orthogonality with weighted functions and completeness, Lagrange and Green's identities were considered in solving problems.

At last, 2D Liouville equations were discussed. These equations are used to describe certain characteristics of fluid dynamics in real-world applications in three-dimensional space. The thesis explored a different way of finding solutions to 2D Liouville's elliptic and hyperbolic equations in complex space and then transforming it into real forms of the solutions.

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