The Output Light from a Three-Level Laser and a Coherently Driven Cavity Mode

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The undersigned hereby certify that they have read and recommended to the Faculty of Science School of Graduate Studies for acceptance a thesis entitled "**The Output Light from a Three-Level Laser and a Coherently Driven Cavity Mode**" by Assegid Mengistu in partial fulfillment of the requirement for the degree of **Master of Science in Physics**.

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Abstract

In this thesis we study the statistical and squeezing properties of the output light from a degenerate three-level laser and a coherently driven cavity mode coupled to a vacuum reservoir. Applying the propagator method [1], we first obtain the Q function for the cavity coherent light as well as the light generated by the three-level laser. We then determine the Q function for the superposition of the two light beams. Applying the input-output relation, we determine the Q function for the output light.

Employing the Q function for the output light, we have calculated the mean photon number, the variance of the photon number, the quadrature variance, and the squeezing spectrum. It is found that the mean photon number we have obtained is greater than the mean photon number obtained using the Q function derived from a single master equation for the cavity mode (i.e when the cavity coherent photons are interacting with the three-level atoms). However, this interaction doesn't affect the quadrature variance and the squeezing spectrum of the output light.

Contents

1 Introduction

There has been a considerable interest in the analysis of the squeezing and statistical properties of the light generated by a three-level laser and a coherently driven cavity mode [2, 5, 8]. A three-level laser may be defined as a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels, are injected into a cavity coupled to a vacuum reservoir via a single port-mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have same frequency, then the three-level atom is called degenerate three-level atom otherwise it is called non degenerate.

The squeezing and the statistical properties of the light produced by a degenerate three-level laser has been studied by different authors [2, 5, 8]. These studies show that the degenerate three-level laser can generate squeezed light under certain conditions. Using c-number Langevin equation Fesseha K. [5] has shown that the light produced by a degenerate three-level laser is in a squeezed state when the probability for the injected atom to be in the bottom level is greater than that of the upper level and the degree of squeezing increases with the linear gain coefficient. Using the master equation for the cavity mode of a coherently driven degenerate three-level laser, Misrak G. [8] has shown that the coherent light has no effect on the quadrature variance.

In this thesis we study the statistical and squeezing properties of the output light from a three-level laser and a coherently driven cavity mode coupled to a vacuum reservoir. In our system, we let the driving coherent light into the cavity in such a way that it has no direct interaction with the three-level atoms, this can be possible if the driving coherent light strike the inside part of the port-mirror along perpendicular direction while the three-level atoms enter and leave the cavity, at a constant rate, along one part of the cavity box, as it is shown in the Fig. 1.1. Using the propagator method of solving a Fokker-Planck equation [1], we first obtain the Q function for the cavity coherent light as well as the light produced by the degenerate three-level laser. We then determine the Q function for the superposition of the two light beams. Applying the resulting Q function and using the input-output relation, we

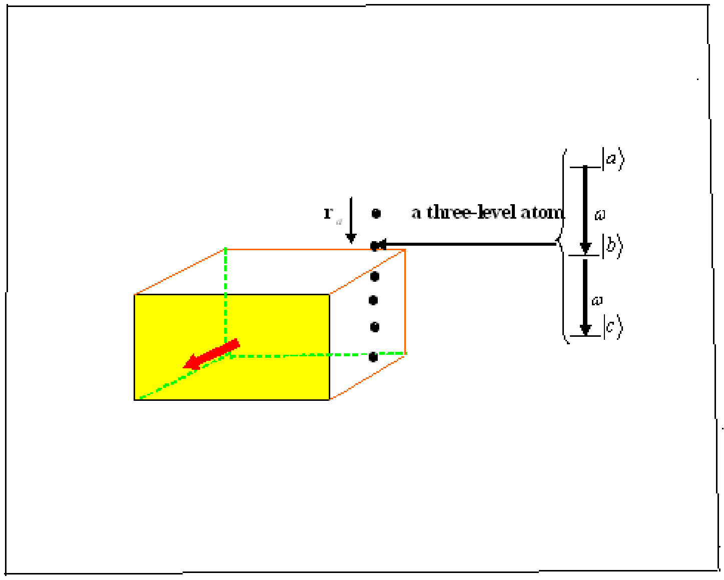

Fig. 1.1: Schematic representation of a degenerate three-level laser (black) and a driving coherent light (red).

find the Q function representing the output light from a three-level laser and a coherently driven cavity mode.

Employing the Q function for the output light, we calculate the mean photon number, the variance of the photon number, the quadrature variance, and the squeezing spectrum.

2 The Q function

2.1 The Q function for a coherently driven cavity mode

The quantum dynamics of a cavity mode driven by coherent light and coupled to a vacuum reservoir via a single-port mirror can be described by Fokker-Planck equation for the Q function. The interaction between the cavity mode and the driving light, treated classically, can be described by

$$\hat{H}_s = i\varepsilon(\hat{a}^\dagger - \hat{a}), \tag{2.1}$$

where $\hat{a}$ is the annihilation operator for the cavity mode and ε is proportional to the amplitude of the driving light. Hence the Master equation for a coherently driven cavity mode coupled to a vacuum reservoir is given by [5]

$$\frac{d}{dt}\hat{\rho} = -\varepsilon(\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger) + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}). \tag{2.2}$$

The Fokker-Planck equation for the Q function corresponding to the master equation (2.2) can be constructed by putting all operators in the normal-order. Thus using the relations

$$[\hat{a}, f(\hat{a}^\dagger, \hat{a})] = \frac{\partial}{\partial \hat{a}^\dagger} f(\hat{a}^\dagger, \hat{a}), \tag{2.3}$$

$$[\hat{a}^\dagger, f(\hat{a}^\dagger, \hat{a})] = -\frac{\partial}{\partial \hat{a}} f(\hat{a}^\dagger, \hat{a}), \tag{2.4}$$

One easily gets

$$\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger = \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger} + \frac{\partial\hat{\rho}}{\partial\hat{a}}, \tag{2.5}$$

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} = 2\frac{\partial^2\hat{\rho}}{\partial\hat{a}^\dagger\partial\hat{a}} + \frac{\partial(\hat{a}^\dagger\hat{\rho})}{\partial\hat{a}^\dagger} + \frac{\partial(\hat{\rho}\hat{a})}{\partial\hat{a}}. \tag{2.6}$$

Now applying Eq.(2.5) and (2.6), we can put Eq.(2.2) in the form

$$\frac{d}{dt}\rho = -\varepsilon\left(\frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger} + \frac{\partial\hat{\rho}}{\partial\hat{a}}\right) + \frac{\kappa}{2}\left(2\frac{\partial^2\hat{\rho}}{\partial\hat{a}^\dagger\partial\hat{a}} + \frac{\partial(\hat{a}^\dagger\hat{\rho})}{\partial\hat{a}^\dagger} + \frac{\partial(\hat{\rho}\hat{a})}{\partial\hat{a}}\right). \tag{2.7}$$

in which $\hat{\rho} = \hat{\rho}(\hat{a}^\dagger, \hat{a})$ is assumed to be in the normal-order. Upon replacing $\hat{a}^\dagger, \hat{a}$ and $\hat{\rho}(\hat{a}^\dagger, \hat{a})$ by α^*, α and $Q(\alpha^*, \alpha, t)$, there emerges the Fokker-Planck equation for the Q function

$$\frac{\partial}{\partial t}Q(\alpha^*, \alpha, t) = \left[-\varepsilon\left(\frac{\partial}{\partial\alpha^*} + \frac{\partial}{\partial\alpha}\right) + \frac{\kappa}{2}\left(2\frac{\partial^2}{\partial\alpha^*\partial\alpha} + \frac{\partial}{\partial\alpha^*}\alpha^* + \frac{\partial}{\partial\alpha}\alpha\right)\right]Q(\alpha^*, \alpha, t). \tag{2.8}$$

Introducing Cartesian coordinate defined by

$$\alpha = x + iy, \tag{2.9}$$

$$\alpha^* = x - iy, \tag{2.10}$$

we have

$$\frac{\partial}{\partial \alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{2.11}$$

$$\frac{\partial}{\partial \alpha}\alpha = \frac{1}{2}\left(\frac{\partial}{\partial x}x + i\frac{\partial}{\partial x}y - i\frac{\partial}{\partial y}x + \frac{\partial}{\partial y}y\right), \tag{2.12}$$

$$\frac{\partial^2}{\partial \alpha^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - 2i\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}\right), \tag{2.13}$$

$$\frac{\partial^2}{\partial \alpha^* \partial \alpha} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{2.14}$$

$$\frac{\partial}{\partial \alpha^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.15}$$

$$\frac{\partial}{\partial \alpha^*}\alpha^* = \frac{1}{2}\left(\frac{\partial}{\partial x}x - i\frac{\partial}{\partial x}y + i\frac{\partial}{\partial y}x + \frac{\partial}{\partial y}y\right). \tag{2.16}$$

Therefore on substituting Eqs. (2.11), (2.12), (2.14), (2.15) and (2.16) into (2.8), we obtain

$$\frac{\partial}{\partial t}Q(x,y,t) = \left[\frac{\kappa}{4}\frac{\partial^2}{\partial x^2} + \frac{\kappa}{4}\frac{\partial^2}{\partial y^2} + \frac{\kappa}{2}\frac{\partial}{\partial x}\left(x - \frac{2\varepsilon}{\kappa}\right) + \frac{\kappa}{2}\frac{\partial}{\partial y}y\right]Q(x,y,t) \tag{2.17}$$

Upon setting

$$x' = x - \frac{2\varepsilon}{\kappa}, \tag{2.18}$$

and

$$y' = y, \tag{2.19}$$

Eq.(2.17) becomes

$$\frac{\partial}{\partial t}Q(x',y',t) = \left[\frac{\kappa}{4}\frac{\partial^2}{\partial x'^2} + \frac{\kappa}{4}\frac{\partial^2}{\partial y'^2} + \frac{\kappa}{2}\frac{\partial}{\partial x'}x' + \frac{\kappa}{2}\frac{\partial}{\partial y'}y'\right]Q(x',y',t). \tag{2.20}$$

Upon replacing $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, x', y', Q(x',y',t)$ by $i\hat{p}_{x'}, i\hat{p}_{y'}, \hat{x'}, \hat{y'}, |Q(t)\rangle$. We can transform the differential equation (2.20) into a Schrödinger type equation of the form

$$i\frac{d}{dt}|Q(t)\rangle = \hat{H}|Q(t)\rangle, \tag{2.21}$$

where

$$\hat{H} = -i\frac{\kappa}{4}\hat{p}_{x'}^2 - i\frac{\kappa}{4}\hat{p}_{y'}^2 - \frac{\kappa}{2}\hat{p}_{x'}\hat{x'} - \frac{\kappa}{2}\hat{p}_{y'}\hat{y'}. \tag{2.22}$$

The formal solution of Eq. (2.21) is

$$|Q(t)\rangle = e^{-i\hat{H}t}|Q(0)\rangle. \tag{2.23}$$

Now multiplying Eq. (2.23) by $\langle x', y'|$ on both sides, we get

$$\langle x', y'|Q(t)\rangle = \langle x', y'|e^{-i\hat{H}t}|Q(0)\rangle, \tag{2.24}$$

and introducing the completeness relation for the two-dimensional position eigenstates, one can write

$$\langle x', y'|Q(t)\rangle = \int dx''dy''\langle x', y'|e^{-i\hat{H}t}|y'', x''\rangle\langle x'', y''|Q(0)\rangle, \tag{2.25}$$

or

$$Q(x', y', t) = \int dx''y''Q(x', y', t|x'', y'', 0)Q_0(x'', y''). \tag{2.26}$$

in which

$$Q(x', y', t) = \langle x', y'|Q(t)\rangle, \tag{2.27}$$

and

$$Q(x', y', t|x'', y'', 0) = \langle x', y'|e^{-i\hat{H}t}|y'', x''\rangle, \tag{2.28}$$

is the Q function propagator and

$$Q_0(x'', y'') = \langle x'', y''|Q(0)\rangle, \tag{2.29}$$

is the initial Q function.

According to Fesseha [1], the propagator associated with a quadratic Hamiltonian in one-dimension

$$\hat{H}(\hat{x}, \hat{p}_x) = a\hat{p}_x^2 + b(t)\hat{p}_x\hat{x} + c(t)\hat{x}^2, \tag{2.30}$$

is expressible in the form

$$K(x, t|x', 0) = \left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\right)^{1/2} \exp\left[-\frac{1}{2}\int_0^t b(t) + iS_c\right]. \tag{2.31}$$

where S_c is the classical action.

By writing Eq. (2.22) as

$$\hat{H} = a_x \hat{p}_{x'}^2 + a_y \hat{p}_{y'}^2 + b_x \hat{p}_{x'} \hat{x'} + b_y \hat{p}_{y'} \hat{y'}, \tag{2.32}$$

where

$$a_x = a_y = -i\frac{\kappa}{4}, \tag{2.33}$$

$$b_x = b_y = -\frac{\kappa}{2}, \tag{2.34}$$

and employing the two-dimensional forms of Eqs. (2.30) and (2.31), the Q function propagator associated with the Hamiltonian in Eq. (2.32) is expressible as

$$\begin{aligned} Q(x', y', t|x'', y'', 0) \quad &= \left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'' \partial x'}\right)^{1/2} \left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial y'' \partial y'}\right)^{1/2} \\ &\exp\left[-\frac{1}{2}\int_0^t b_x(t')dt' - \frac{1}{2}\int_0^t b_y(t')dt' + iS_c\right]. \end{aligned} \tag{2.35}$$

The classical Lagrangian corresponding to the Hamiltonian in Eq. (2.32) is

$$L = \dot{x}' p_{x'} + \dot{y}' p_{y'} - H, \tag{2.36}$$

or

$$L = \dot{x}' p_{x'} + \dot{y}' p_{y'} - a_{x'} p_{x'}^2 - a_{y'} p_{y'}^2 - b_x p_{x'} x' - b_{y'} p_{y'} y'. \tag{2.37}$$

Applying Hamilton's equation

$$\frac{\partial H}{\partial p_{x'}} = \dot{x}', \tag{2.38}$$

and

$$\frac{\partial H}{\partial x'} = -\dot{p_{x'}}, \tag{2.39}$$

one can find that

$$p_{x'} = \frac{\dot{x} - b_x x'}{2a_x}, \tag{2.40}$$

and

$$p_{y'} = \frac{\dot{y}' - b_y y'}{2a_y}, \tag{2.41}$$

Upon substituting Eqs. (2.40) and (2.41) into Eq. (2.37), we get

$$L = \frac{1}{4a_x}\left(\dot{x}'^2 - 2b_x \dot{x}'^2 x' + b_x^2 x'^2\right) + \frac{1}{4a_y}\left(\dot{y}'^2 - 2b_x \dot{y}'^2 y' + b_x^2 y'^2\right), \tag{2.42}$$

or

$$L = \frac{i}{\kappa}\left(\dot{x}' + \frac{\kappa}{2}x'\right)^2 + \frac{i}{\kappa}\left(\dot{y}' + \frac{\kappa}{2}y'\right)^2. \tag{2.43}$$

Applying Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}'}L\right) - \frac{\partial}{\partial x'}L = 0, \tag{2.44}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}'}L\right) - \frac{\partial}{\partial y'}L = 0, \tag{2.45}$$

one easily obtains

$$\ddot{x}' - \frac{\kappa^2}{4}x' = 0, \tag{2.46}$$

$$\ddot{y}' - \frac{\kappa^2}{4}y' = 0, \tag{2.47}$$

On accounting of the solution of these equations

$$x'(t) = A_1 e^{\frac{\kappa}{2}t} + B_1 e^{-\frac{\kappa}{2}t}, \tag{2.48}$$

and

$$y'(t) = C_1 e^{\frac{\kappa}{2}t} + D_1 e^{-\frac{\kappa}{2}t}. \tag{2.49}$$

the classical lagrangian (2.43) takes the form

$$L = i\kappa e^{\kappa t}(A_1^2 + C_1^2). \tag{2.50}$$

Using the boundary condition $x'(T) = x''', y'(T) = y''', x'(0) = x''$ and $y'(0) = y''$, we obtain

$$x''' = A_1 e^{\frac{\kappa}{2}T} + B_1 e^{-\frac{\kappa}{2}T}, \tag{2.51}$$

$$x'' = A_1 + B_1. \tag{2.52}$$

It then follows that

$$A_1 = -\frac{(x'' - e^{\frac{\kappa}{2}T}x''')}{(e^{\kappa T} - 1)}. \tag{2.53}$$

Moreover

$$y''' = C_1 e^{\frac{\kappa}{2}T} + D_1 e^{-\frac{\kappa}{2}T}, \tag{2.54}$$

$$y'' = C_1 + D_1, \tag{2.55}$$

leads to

$$C_1 = -\frac{(y'' - e^{-\frac{\kappa}{2}T} y''')}{(e^{\kappa T} - 1)}. \tag{2.56}$$

Therefore using Eq. (2.50) the classical action S_c becomes

$$S_c = \int_0^T L(t)dt = i\kappa(A_1^2 + C_1^2) \int_0^T e^{\kappa t} dt \tag{2.57}$$

After carrying out the integration we found that

$$S_c = i(A_1^2 + C_1^2)(e^{\kappa T} - 1), \tag{2.58}$$

where A_1 and C_1 are given by (2.53) and (2.56).

Upon replacing (x''', y''', T) by (x', y', t), the classical action can be put in the form

$$S_c = i \left[\frac{(x'' - e^{-\frac{\kappa}{2}t} x')^2 + (y'' - e^{-\frac{\kappa}{2}t} y')^2}{(e^{\kappa t} - 1)} \right], \tag{2.59}$$

or

$$S_c = i \left[\frac{(e^{-\frac{\kappa}{2}t} x'' - x')^2 + (e^{-\frac{\kappa}{2}t} y'' - y')^2}{(1 - e^{-\kappa t})} \right]. \tag{2.60}$$

We see from the above expression that

$$\frac{\partial^2 S_c}{\partial x'' \partial x'} = \frac{-2i e^{-\frac{\kappa}{2}t}}{(1 - e^{-\kappa t})}, \tag{2.61}$$

$$\frac{\partial^2 S_c}{\partial y'' \partial y'} = \frac{-2i e^{-\frac{\kappa}{2}t}}{(1 - e^{-\kappa t})}. \tag{2.62}$$

Therefore on account of Eqs. (2.60), (2.61), and (2.62), the Q function propagator (2.35) becomes

$$Q(x', y', t | x'', y'', 0) = \frac{1}{\pi(1 - e^{-\kappa t})} \exp \left[-\frac{(e^{-\frac{\kappa}{2}t} x'' - x')^2}{(1 - e^{-\kappa t})} - \frac{(e^{-\frac{\kappa}{2}t} y'' - y')^2}{(1 - e^{-\kappa t})} \right]. \tag{2.63}$$

Taking the Q function satisfying the initial condition

$$Q_0(x'', y'') = \frac{1}{\pi} \exp[-(x'' - x_0')^2 - (y'' - y_0')^2], \tag{2.64}$$

along with the help of Eq. (2.63), we see that Eq. (2.26) takes the form

$$\begin{aligned} Q(x', y', t) = \quad & \frac{1}{\pi^2(1 - e^{-\kappa t})} \left(\int dx'' \exp \left[-(x'' - x_0')^2 - \frac{(e^{-\frac{\kappa}{2}t} x'' - x')^2}{(1 - e^{-\kappa t})} \right] \right) \\ & \left(\int dy'' \exp \left[-(y'' - y_0')^2 - \frac{(e^{-\frac{\kappa}{2}t} y'' - y')^2}{(1 - e^{-\kappa t})} \right] \right). \end{aligned} \tag{2.65}$$

Upon setting

$$A = \frac{1}{1 - e^{-\kappa t}}, \tag{2.66}$$

$$B = e^{-\frac{\kappa}{2}t}, \tag{2.67}$$

the first integral of Eq. (2.65) can be written as

$$I_1 = \int dx'' \exp[-(x'' - x_0')^2 - A(Bx'' - x')^2], \tag{2.68}$$

or

$$I_1 = \int dx'' \exp[-x''^2 + 2x_0' x'' - x_0'^{\,2} - AB^2 x''^2 + 2ABx' x'' - Ax'^2], \tag{2.69}$$

One can see from the above expression that

$$I_1 = \exp[-x_0'^{\,2} - Ax'^2] \int dx'' \exp[-(1 + AB^2)x''^2 + 2(x_0' + ABx')x''. \tag{2.70}$$

Using the relation

$$\int d\eta e^{-a\eta^2 + b\eta} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}, \tag{2.71}$$

for $a > 0$, we find that

$$I_1 = \sqrt{\left(\frac{\pi}{1 + AB^2}\right)} \exp\left[-x_0'^{\,2} - Ax'^2 + \frac{(x_0' + ABx')^2}{(1 + AB^2)}\right], \tag{2.72}$$

or

$$I_1 = \sqrt{\left(\frac{\pi}{1 + AB^2}\right)} \exp\left[\frac{-A}{(1 + AB^2)}(x'^2 - 2Bx_0' x' + B^2 x_0'^{\,2})\right], \tag{2.73}$$

or simply we can put the above expression as

$$I_1 = \sqrt{\left(\frac{\pi}{1 + AB^2}\right)} \exp\left[\frac{-A}{(1 + AB^2)}(x' - Bx_0')^2\right]. \tag{2.74}$$

In a similar manner applying Eq. (2.71) to the second integral in Eq. (2.65), there follows

$$I_2 = \sqrt{\left(\frac{\pi}{1 + AB^2}\right)} \exp\left[\frac{-A}{(1 + AB^2)}(y' - By_0')^2\right]. \tag{2.75}$$

Using Eqs. (2.66), (2.67), (2.74), and (2.75), we write Eq.(2.65) as

$$Q(x', y', t) = \frac{1}{\pi} \frac{A}{(1 + AB^2)} \exp\left[-\frac{A}{(1 + AB^2)}(x' - Bx_0')^2 - \frac{A}{(1 + AB^2)}(y' - By_0')^2\right]. \tag{2.76}$$

Taking into account Eqs. (2.18) and (2.19), we have

$$Q(x,y,t) = \frac{1}{\pi}\frac{A}{(1+AB^2)}\exp\left(-\frac{A}{(1+AB^2)}\left[\left(x-\frac{2\epsilon}{\kappa}\right)-B\left(x_0-\frac{2\epsilon}{\kappa}\right)\right]^2-(y-By_0)^2\right) \tag{2.77}$$

$$\begin{aligned} = \quad & \frac{1}{\pi}\frac{A}{(1+AB^2)}\exp\left[-\frac{A}{(1+AB^2)}\left((x^2+y^2)-\frac{4\epsilon}{\kappa}(1-B)x-2B(xx_0+yy_0)\right)\right] \\ & \exp\left[-\frac{A}{(1+AB^2)}\left(B^2(x_0^2+y_0^2)+\frac{4\epsilon}{\kappa}B(1-B)x_0+\left(\frac{4\epsilon^2}{\kappa^2}(1-B)^2\right)\right],\right. \end{aligned} \tag{2.78}$$

or

$$\begin{aligned} Q(x,y,t) = \quad & \frac{1}{\pi}\frac{A}{(1+AB^2)}\exp\left[-\frac{A}{(1+AB^2)}\left(\frac{4\epsilon^2}{\kappa^2}(1-B)^2\right)\right] \\ & \exp\left[-\frac{A}{(1+AB^2)}[(x^2+y^2)-\frac{4\epsilon}{\kappa}(1-B)x-2B(xx_0+yy_0)\right] \\ & \exp\left[-\frac{A}{(1+AB^2)}\left(B^2(x_0^2+y_0^2)+\frac{4\epsilon}{\kappa}B(1-B)x_0\right)\right]. \end{aligned} \tag{2.79}$$

We note that

$$x = \frac{1}{2}(\alpha+\alpha^*), \tag{2.80}$$

$$y = \frac{-i}{2}(\alpha-\alpha^*), \tag{2.81}$$

$$x^2 = \frac{1}{4}(\alpha^2+\alpha^{*2}+2\alpha\alpha^*), \tag{2.82}$$

$$y^2 = \frac{-1}{4}(\alpha^2+\alpha^{*2}-2\alpha\alpha^*), \tag{2.83}$$

$$x_0.x = \frac{1}{4}(\alpha_0\alpha+\alpha_0\alpha^*+\alpha_0^*\alpha+\alpha_0^*\alpha^*), \tag{2.84}$$

$$y_0.y = \frac{-1}{4}(\alpha_0\alpha-\alpha_0\alpha^*-\alpha_0^*\alpha+\alpha_0^*\alpha^*). \tag{2.85}$$

Now employing these relations, the Q function in Eq. (2.79) can be expressible in terms of complex variables as

$$\begin{aligned} Q(\alpha^*,\alpha,t) \quad &= \frac{1}{\pi}\frac{A}{(1+AB^2)}\exp\left[-\frac{A}{(1+AB^2)}\left(\frac{2\epsilon}{\kappa}(1-B)\right)^2\right] \\ &\times\exp\left[-\frac{A}{(1+AB^2)}\left(\alpha^*\alpha-(\frac{2\epsilon}{\kappa}(1-B)+B\alpha_0^*)\alpha-(\frac{2\epsilon}{\kappa}(1-B)+B\alpha_0)\alpha^*\right)\right] \\ &\times\exp\left[-\frac{A}{(1+AB^2)}\left(B^2\alpha_0^*\alpha_0+\frac{2\epsilon}{\kappa}B(1-B)\alpha_0+\frac{2\epsilon}{\kappa}B(1-B)\alpha_0^*\right)\right]. \end{aligned} \tag{2.86}$$

Using Eqs. (2.66) and (2.67), one can easily see that

$$1+AB^2=1+\frac{e^{-\kappa t}}{1-e^{-\kappa t}}, \tag{2.87}$$

which leads to

$$1+AB^2=\frac{1}{1-e^{-\kappa t}}=A. \tag{2.88}$$

Using Eqs. (2.66), (2.67), and (2.88), Eq.(2.86) takes the form

$$\begin{aligned} Q(\alpha^*,\alpha,t) \quad &= \frac{1}{\pi}\exp\left[-\left(\frac{2\epsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})\right)^2-e^{-\kappa t}\alpha_0^*\alpha_0-\frac{2\epsilon}{\kappa}e^{-\frac{\kappa}{2}t}\left(1-e^{-\frac{\kappa}{2}t}\right)(\alpha_0+\alpha_0^*)\right] \\ &\times\exp\left[-\alpha^*\alpha+\left(\frac{2\epsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})+e^{-\frac{\kappa}{2}t}\alpha_0^*\right)\alpha+\left(\frac{2\epsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})+e^{-\kappa/2t}\alpha_0\right)\alpha^*\right], \end{aligned} \tag{2.89}$$

or

$$Q(\alpha^*,\alpha,t)=\frac{A(t)}{\pi}\exp\left[-\alpha^*\alpha+C(t)\alpha^*+C^*(t)\alpha\right], \tag{2.90}$$

where

$$A(t)=\exp\left[-\left(\frac{2\epsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})\right)^2-e^{-\kappa t}\alpha_0^*\alpha_0-\frac{2\epsilon}{\kappa}e^{-\frac{\kappa}{2}t}(1-e^{-\frac{\kappa}{2}t})(\alpha_0+\alpha_0^*)\right], \tag{2.91}$$

and

$$C(t)=\frac{2\epsilon}{\kappa}\left(1-e^{-\frac{\kappa}{2}t}\right)+e^{-\frac{\kappa}{2}t}\alpha_0. \tag{2.92}$$

2.2 The Q function for a three-level laser

In this section we find the Q function for a three-level laser using the pertinent Fokker-Planck equation. The three-level laser in which degenerate three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate r_a into a cavity coupled to a vacuum reservoir via a single-port mirror and removed from the cavity after a certain time τ. We denote the top, middle and bottom levels by $|a\rangle$, $|b\rangle$ and $|c\rangle$ respectively. In addition, we assume the cavity mode to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be electric-dipole forbidden. The interaction of a three-level atom with the cavity mode can be described by the Hamiltonian

$$\hat{H} = ig\left[(|a\rangle\langle b| + |b\rangle\langle c|)\,\hat{a} - \hat{a}^\dagger\,(|b\rangle\langle a| + |c\rangle\langle b|)\right], \tag{2.93}$$

where $\hat{a}$ is the annihilation operator for the cavity mode and g is the coupling constant. Thus applying the adiabatic approximation scheme and in the linear analysis [5], we obtain the master equation for the three-level laser to take the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho} = \quad & -\frac{1}{2}A\rho_{aa}^{(0)}\left[\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\right] + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right] \\ & + \frac{1}{2}A\rho_{ac}^{(0)}\left[\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\right] + \frac{1}{2}A\rho_{ca}^{(0)}\left[\hat{\rho}\hat{a}^2 + \hat{a}^2\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}\right]\end{aligned} \tag{2.94}$$

where

$$A = \frac{2r_a g^2}{\gamma^2} \tag{2.95}$$

is the linear gain coefficient and γ and κ are the atomic decay and the cavity damping constants respectively.

The Fokker-Planck equation for the Q function corresponding to the master equation (2.94) can be constructed by putting all operators in the normal-order. Thus using the relations (2.3) and (2.4), one easily gets

$$\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho} = -\frac{\partial(\hat{\rho}\hat{a})}{\partial\hat{a}} - \frac{\partial(\hat{a}^\dagger\hat{\rho})}{\partial\hat{a}^\dagger}, \tag{2.96}$$

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho} = 2\frac{\partial^2\hat{\rho}}{\partial\hat{a}^\dagger\partial\hat{a}} + \frac{\partial(\hat{a}^\dagger\hat{\rho})}{\partial\hat{a}^\dagger} + \frac{\partial(\hat{\rho}\hat{a})}{\partial\hat{a}}, \tag{2.97}$$

$$\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger} = \frac{\partial^2 \hat{\rho}}{\partial \hat{a}^2}, \tag{2.98}$$

$$\hat{\rho}\hat{a}^{2} + \hat{a}^{2}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a} = \frac{\partial^2 \hat{\rho}}{\partial \hat{a}^{\dagger 2}}. \tag{2.99}$$

Now applying Eqs. (2.96), (2.97), (2.98) and (2.99), we can put Eq. (2.94) in the form

$$\begin{aligned} \frac{d}{dt}\hat{\rho} = \quad & -\frac{1}{2}A\rho_{aa}^{(0)}\left[-\frac{\partial(\hat{\rho}\hat{a})}{\partial \hat{a}} - \frac{\partial(\hat{a}^{\dagger}\hat{\rho})}{\partial \hat{a}^{\dagger}}\right] + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\left[2\frac{\partial^2 \hat{\rho}}{\partial \hat{a}^{\dagger}\partial \hat{a}} + \frac{\partial(\hat{a}^{\dagger}\hat{\rho})}{\partial \hat{a}^{\dagger}} + \frac{\partial(\hat{\rho}\hat{a})}{\partial \hat{a}}\right] \\ & +\frac{1}{2}A\rho_{ac}^{(0)}\frac{\partial^2 \hat{\rho}}{\partial \hat{a}^2} + \frac{1}{2}A\rho_{ca}^{(0)}\frac{\partial^2 \hat{\rho}}{\partial \hat{a}^{\dagger 2}} \end{aligned} \tag{2.100}$$

in which $\hat{\rho} = \hat{\rho}(\hat{a}^{\dagger}, \hat{a})$ is assumed to be in the normal order. Upon replacing $\hat{a}^{\dagger}, \hat{a}$ and $\hat{\rho}(\hat{a}^{\dagger}, \hat{a})$ by α^*, α and $Q(\alpha^*, \alpha, t)$, there emerges the Fokker-Planck equation for the Q function

$$\begin{aligned} \frac{\partial}{\partial t}Q(\alpha^*, \alpha, t) = \quad & -\frac{1}{2}A\rho_{aa}^{(0)}\left[-\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right]Q(\alpha^*, \alpha, t) \\ & +\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\left[2\frac{\partial^2}{\partial \alpha^*\partial \alpha} + \frac{\partial}{\partial \alpha^*}\alpha^* + \frac{\partial}{\partial \alpha}\alpha\right]Q(\alpha^*, \alpha, t) \\ & +\frac{1}{2}A\rho_{ac}^{(0)}\frac{\partial^2}{\partial \alpha^2}Q(\alpha^*, \alpha, t) + \frac{1}{2}A\rho_{ca}^{(0)}\frac{\partial^2}{\partial \alpha^{*2}}Q(\alpha^*, \alpha, t). \end{aligned} \tag{2.101}$$

Using the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{2.102}$$

and

$$\rho_{ac}^{(0)} = \rho_{ca}^{(0)}, \tag{2.103}$$

we can write Eq.(2.101) as

$$\begin{aligned} \frac{\partial}{\partial t}Q(\alpha^*, \alpha, t) = \quad & A\rho_{aa}^{(0)}\left[-\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^* - \frac{\partial^2}{\partial \alpha^*\partial \alpha}\right]Q(\alpha^*, \alpha, t) \\ & +\frac{1}{2}(A + \kappa)\left[2\frac{\partial^2}{\partial \alpha^*\partial \alpha} + \frac{\partial}{\partial \alpha^*}\alpha^* + \frac{\partial}{\partial \alpha}\alpha\right]Q(\alpha^*, \alpha, t) \\ & +\frac{1}{2}A\rho_{ac}^{(0)}\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}}\right)Q(\alpha^*, \alpha, t). \end{aligned} \tag{2.104}$$

Applying the relations (2.12), (2.13), (2.14) along with their conjugates, one can write the Fokker-Planck equation (2.104) in Cartesian system as

$$\begin{aligned}\frac{\partial}{\partial t}Q(x,y,t) &= \frac{1}{4}\left[\left(A(\rho_{ac}^{(0)}-\rho_{aa}^{(0)}+1)+\kappa\right)\frac{\partial^2}{\partial x^2}+\left(A(1-\rho_{ac}^{(0)}-\rho_{aa}^{(0)})+\kappa\right)\frac{\partial^2}{\partial y^2}\right]Q(x,y,t)\\ &+\frac{1}{2}\left[\left(A(1-2\rho_{aa}^{(0)})+\kappa\right)\frac{\partial}{\partial x}x+\left(A(1-2\rho_{aa}^{(0)})+\kappa\right)\frac{\partial}{\partial y}y\right]Q(x,y,t),\end{aligned} \tag{2.105}$$

One can also write the above equation as

$$\frac{\partial}{\partial t}Q(x,y,t) = \left[\frac{a'}{4}\frac{\partial^2}{\partial x^2}+\frac{b'}{4}\frac{\partial^2}{\partial y^2}+\frac{\mu}{2}\frac{\partial}{\partial x}x+\frac{\mu}{2}\frac{\partial}{\partial y}y\right]Q(x,y,t), \tag{2.106}$$

where

$$a' = A(1-\rho_{aa}^{(0)}+\rho_{ac}^{(0)})+\kappa, \tag{2.107}$$

$$b' = A(1-\rho_{aa}^{(0)}-\rho_{ac}^{(0)})+\kappa, \tag{2.108}$$

$$\mu = A(1-2\rho_{aa}^{(0)})+\kappa. \tag{2.109}$$

Upon replacing $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, x, y, Q(x,y,t) by $i\hat{p}_x$, $i\hat{p}_y$, $\hat{x}$, $\hat{y}$, $|Q(t)\rangle$, we can transform this differential equation into a Schrödinger type equation of the form

$$i\frac{d}{dt}|Q(t)\rangle = \hat{H}|Q(t)\rangle, \tag{2.110}$$

where

$$\hat{H} = -i\frac{a'}{4}\hat{p}_x^2 - i\frac{b'}{4}\hat{p}_{y'}^2 - \frac{\mu}{2}\hat{p}_x\hat{x} - \frac{\mu}{2}\hat{p}_y\hat{y}. \tag{2.111}$$

The formal solution of Eq. (2.110) is

$$|Q(t)\rangle = e^{-i\hat{H}t}|Q(0)\rangle. \tag{2.112}$$

Now multiplying Eq. (2.112) by $\langle x,y|$ on the both side, we get

$$\langle x,y|Q(t)\rangle = \langle x,y|e^{-i\hat{H}t}|Q(0)\rangle, \tag{2.113}$$

and introducing the completeness relation for the two-dimensional position eigenstates as we do on Eq. (2.25), we arrive at

$$\langle x,y|Q(t)\rangle = \int dx'dy'\langle x,y|e^{-i\hat{H}t}|y',x'\rangle\langle x',y'|Q(0)\rangle, \tag{2.114}$$

or

$$Q(x,y,t)=\int dx'y'Q(x,y,t|x',y',0)Q_0(x',y'), \tag{2.115}$$

in which

$$Q(x,y,t)=\langle x,y|Q(t)\rangle, \tag{2.116}$$

and

$$Q(x,y,t|x',y',0)=\langle x,y|e^{-i\hat{H}t}|y',x'\rangle, \tag{2.117}$$

is the Q function propagator and

$$Q_0(x',y')=\langle x',y'|Q(0)\rangle, \tag{2.118}$$

is the initial Q function.

According to Eqs. (2.30) and (2.31), The Q function propagator associated with Eq. (2.111) is expressible as

$$Q(x,y,t|x',y',0)=\left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x}\right)^{1/2}\left(\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial y'\partial y}\right)^{1/2}\exp\left[-\frac{\mu}{2}t+iS_c\right], \tag{2.119}$$

where S_c is the classical action.

The classical Lagrangian corresponding to the hamiltonian (2.111) is

$$L=\dot{x}p_x+\dot{y}p_y-H. \tag{2.120}$$

Applying Hamilton equation's (2.38) and (2.39), one can see that

$$L=\frac{i}{a'}\left(\dot{x}+\frac{\mu}{2}x\right)^2+\frac{i}{b'}\left(\dot{y}+\frac{\mu}{2}y\right)^2, \tag{2.121}$$

and with the aid of Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}L\right)-\frac{\partial}{\partial x}L=0, \tag{2.122}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{y}}L\right)-\frac{\partial}{\partial y}L=0, \tag{2.123}$$

one easily obtains

$$\ddot{x}-\frac{\mu^2}{4}x=0 \tag{2.124}$$

$$\ddot{y}-\frac{\mu^2}{4}y=0 \tag{2.125}$$

On accounting of the solution of these equations

$$x(t) = A_1 e^{\frac{\mu}{2}t} + B_1 e^{-\frac{\mu}{2}t}, \tag{2.126}$$

and

$$y(t) = C_1 e^{\frac{\mu}{2}t} + D_1 e^{-\frac{\mu}{2}t}, \tag{2.127}$$

the classical lagrangian Eq. (2.121) takes the form

$$L = i\mu^2 \left(\frac{A_1^2}{a'} + \frac{C_1^2}{b'} \right) e^{\mu t}. \tag{2.128}$$

Using the boundary condition $x(T) = x''$, $y(T) = y''$, $x(0) = x'$ and $y(0) = y'$, we obtain

$$x'' = A_1 e^{\frac{\mu}{2}T} + B_1 e^{-\frac{\mu}{2}T}, \tag{2.129}$$

$$x' = A_1 + B_1, \tag{2.130}$$

It then follows that

$$A_1 = -\frac{(x' - e^{\frac{\mu}{2}T} x'')}{(e^{\mu T} - 1)}. \tag{2.131}$$

Moreover

$$y'' = C_1 e^{\frac{\mu}{2}T} + D_1 e^{-\frac{\mu}{2}T}, \tag{2.132}$$

$$y' = C_1 + D_1, \tag{2.133}$$

leads to

$$C_1 = -\frac{(y' - e^{-\frac{\mu}{2}T} y'')}{(e^{\mu T} - 1)}. \tag{2.134}$$

Therefore using Eq. (2.128), the classical action S_c becomes

$$S_c = \int_0^T L(t)dt = i\mu^2 \left(\frac{A_1^2}{a'} + \frac{C_1^2}{b'} \right) \int_0^T e^{\mu t} dt. \tag{2.135}$$

After carrying out the integration, we arrive at

$$S_c = i\mu^2 \left(\frac{A_1^2}{a'} + \frac{C_1^2}{b'} \right) (e^{\mu T} - 1) \tag{2.136}$$

where A_1 and C_1 are given by Eqs. (2.131) and (2.134) respectively.

Upon replacing (x'', y'', T) by (x, y, t), the classical action can be put in the form

$$\begin{aligned} S_c \quad &= i \left[\frac{b}{a'} \frac{(x' - e^{\frac{\mu}{2}t} x)^2}{(e^{\mu t} - 1)} + \frac{\mu}{b'} \frac{(y' - e^{\frac{\mu}{2}t} y)^2}{(e^{\mu t} - 1)} \right] \\ &= i \left[\frac{\mu}{a'} \frac{(e^{-\frac{\mu}{2}t} x' - x)^2}{(1 - e^{-\mu t})} + \frac{\mu}{b'} \frac{(e^{-\frac{\mu}{2}t} y' - y)^2}{(1 - e^{-\mu t})} \right]. \end{aligned} \tag{2.137}$$

We see that

$$\frac{\partial^2 S_c}{\partial x' \partial x} = \frac{-2i\mu e^{-\frac{\mu}{2}t}}{a'(1-e^{-\kappa t})}, \tag{2.138}$$

$$\frac{\partial^2 S_c}{\partial y' \partial y} = \frac{-2i\mu e^{-\frac{\mu}{2}t}}{b'(1-e^{-\kappa t})}. \tag{2.139}$$

Using Eqs. (2.137), (2.138), (2.139), the Q function propagator (2.119) becomes

$$Q(x,y,t|x',y',0) = \frac{\mu}{\pi(1-e^{-\mu t})} \frac{1}{(a'b')^{1/2}} \exp\left[-\frac{\mu(e^{-\frac{\mu}{2}t}x'-x)^2}{a'(1-e^{-\mu t})} - \frac{\mu(e^{-\frac{\mu}{2}t}y'-y)^2}{b'(1-e^{-\mu t})}\right], \tag{2.140}$$

Taking the Q function satisfying the initial condition

$$Q_0(x',y') = \frac{1}{\pi} \exp\left[-(x'-x_0)^2 - (y'-y_0)^2\right], \tag{2.141}$$

and making use of Eqs. (2.140) and (2.141), Eq. (2.115) takes the form

$$\begin{aligned} Q(x,y,t) = \quad & \frac{1}{\pi^2(1-e^{-\mu t})} \frac{\mu}{(a'b')^{1/2}} \left(\int dx' \exp\left[-(x'-x_0)^2 - \frac{\mu}{a'} \frac{(e^{-\frac{\mu}{2}t}x'-x)^2}{(1-e^{-\mu t})}\right]\right) \\ & \left(\int dy' \exp\left[-(y'-y_0)^2 - \frac{\mu}{b'} \frac{(e^{-\frac{\mu}{2}t}y'-y)^2}{(1-e^{-\mu t})}\right]\right). \end{aligned} \tag{2.142}$$

Upon setting

$$A = \frac{\mu}{a'(1-e^{-\mu t})}, \tag{2.143}$$

$$B = e^{-\frac{\kappa}{2}t}, \tag{2.144}$$

$$C = \frac{\mu}{b'(1-e^{-\mu t})}, \tag{2.145}$$

then one can see that

$$\left(\frac{\mu}{(a'b')^{1/2}}\right) \frac{1}{1-e^{-\mu t}} = \sqrt{AC}. \tag{2.146}$$

Using Eqs. (2.143), (2.144), (2.145), and (2.146), the two integral inside the bracket of Eq. (2.142) can be written as

$$I = I_1 I_2, \tag{2.147}$$

where

$$I_1 = \int dx' \exp[-(x'-x_0)^2 - A(Bx'-x)^2], \tag{2.148}$$

and

$$I_2 = \int dy' \exp[-(y'-y_0)^2 - A(By'-y)^2]. \tag{2.149}$$

But one can see that

$$I_1 = \exp[-x_0^2 - Ax^2] \int dx' \exp[-(1+AB^2)x'^2 + 2(x_0 + ABx)x']. \tag{2.150}$$

Using the relation (2.71), we find that

$$\begin{aligned} I_1 &= \sqrt{\left(\frac{\pi}{1+AB^2}\right)} \exp\left[-x_0^2 - Ax^2 + \frac{(x_0+ABx)^2}{(1+AB^2)}\right] \\ &= \sqrt{\left(\frac{\pi}{1+AB^2}\right)} \exp\left[\frac{-A}{(1+AB^2)}(x^2 - 2Bx_0x + B^2x_0^2)\right], \end{aligned} \tag{2.151}$$

or simply

$$I_1 = \sqrt{\left(\frac{\pi}{1+AB^2}\right)} \exp\left[\frac{-A}{(1+AB^2)}(x - Bx_0)^2\right]. \tag{2.152}$$

In a similar manner applying Eq. (2.71) to the second integral in Eq. (2.149), there follows

$$I_2 = \sqrt{\left(\frac{\pi}{1+CB^2}\right)} \exp\left[\frac{-C}{(1+CB^2)}(y - By_0)^2\right]. \tag{2.153}$$

Using Eqs. (2.142), (2.147), (2.152) and (2.153) along with Eqs. (2.80), (2.81), (2.82), (2.83), (2.84) and (2.85), we can express the Q function in terms of complex variables as

$$Q(\alpha^*, \alpha, t) = \frac{A_1(t)}{\pi} \exp\left[-B_1(t)\alpha^*\alpha + D_1(t)\alpha^* + D_1^*(t)\alpha + \frac{C_1}{2}(\alpha^2 + \alpha^{*2})\right], \tag{2.154}$$

where

$$A_1(t) = \sqrt{(B_1^2 + C_1^2)} \exp E(t), \tag{2.155}$$

$$B_1(t) = \frac{1}{2}\left[\frac{\mu}{a' + (\mu - a')e^{-\mu t}} + \frac{\mu}{b' + (\mu - b')e^{-\mu t}}\right], \tag{2.156}$$

$$C_1(t) = \frac{1}{2}\left[\frac{\mu}{b' + (\mu - b')e^{-\mu t}} - \frac{\mu}{a' + (\mu - a')e^{-\mu t}}\right], \tag{2.157}$$

$$D_1(t) = e^{-\frac{\mu}{2}t}\left(B_1\alpha_0 - C_1\alpha_0^*\right), \tag{2.158}$$

and

$$E(t) = e^{-\mu t}\left(-B_1\alpha_0\alpha_0^* + \frac{1}{2}C_1(\alpha_0^2 + \alpha_0^{*2})\right). \tag{2.159}$$

Moreover, a', b', μ is given by Eqs. (2.107), (2.108) and (2.109) respectively.

2.3 The Q function for the superposition of Laser and Cavity coherent light

We know from sections (2.1) and (2.2) that the Q function for a coherently driven cavity mode is given by

$$Q(\beta^*,\beta,t)=\frac{A_1(t)}{\pi}\exp[-\beta^*\beta+b(t)\beta^*+b^*(t)\beta], \tag{2.160}$$

where

$$A_1(t)=\exp\left(-\left[\frac{2\epsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})\right]^2-e^{-\kappa t}\beta_0^*\beta_0-\frac{2\epsilon}{\kappa}e^{-\frac{\kappa}{2}t}(1-e^{-\frac{\kappa}{2}t})(\beta_0+\beta_0^*)\right), \tag{2.161}$$

and

$$b(t)=\frac{2\epsilon}{\kappa}(1-e^{-\frac{\kappa}{2}t})+e^{-\frac{\kappa}{2}t}\beta_0, \tag{2.162}$$

and also the Q function for the three level laser is

$$Q(\gamma^*,\gamma,t)=\frac{A_2(t)}{\pi}\exp\left[-a_1(t)\gamma^*\gamma+c_1(t)\gamma^*+c_1^*(t)\gamma+\frac{b_1}{2}(\gamma^2+\gamma^{*2})\right], \tag{2.163}$$

where

$$A_2(t)=\sqrt{a_1^2+b_1^2}\exp E(t), \tag{2.164}$$

$$a_1(t)=\frac{1}{2}\left[\frac{\mu}{a'+(\mu-a')e^{-\mu t}}+\frac{\mu}{b'+(\mu-b')e^{-\mu t}}\right], \tag{2.165}$$

$$b_1(t)=\frac{1}{2}\left[\frac{\mu}{b'+(\mu-b')e^{-\mu t}}-\frac{\mu}{a'+(\mu-a')e^{-\mu t}}\right], \tag{2.166}$$

$$c_1(t)=e^{-\frac{\mu}{2}t}\left(a_1\gamma_0-b_1\gamma_0^*\right), \tag{2.167}$$

and

$$E(t)=e^{-\mu t}\left(-a_1\gamma_0\gamma_0^*+\frac{1}{2}b_1(\gamma_0^2+\gamma_0^{*2})\right). \tag{2.168}$$

It is clear to see that a', b', μ in the above expression refers to Eqs. (2.107), (2.108), and (2.109) respectively. One can write the Q function representing the superposition of the two light beams in the form [5]

$$\begin{aligned}Q(\alpha^*,\alpha,t)\quad&=\frac{1}{\pi}\int d^2\beta d^2\gamma Q\left(\beta^*,\beta+\frac{\partial}{\partial\beta*},t\right)Q\left(\gamma^*,\gamma+\frac{\partial}{\partial\gamma*},t\right)\\&\times\exp\left[-\alpha^*\alpha-\beta^*\beta-\gamma^*\gamma+\alpha^*\beta+\alpha^*\gamma+\alpha\gamma^*-\beta^*\gamma-\beta\gamma^*\right].\end{aligned} \tag{2.169}$$

Using Eqs. (2.160) and (2.163), it is clear to see that

$$Q(\beta^*,\beta+\frac{\partial}{\partial\beta*},t)=\frac{A_1(t)}{\pi}\exp\left[-\beta^*\beta+b(t)\beta^*+b^*(t)\beta+(-\beta^*+b^*)\frac{\partial}{\partial\beta*}\right], \quad (2.170)$$

and

$$Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma*},t)=\frac{A_2(t)}{\pi}\times$$
$$\exp\left[-a_1\gamma^*\gamma+c_1\gamma^*+c_1^*\gamma+\frac{b_1}{2}(\gamma^2+\gamma^{*2})+(-a_1\gamma^*+b_1\gamma+c_1^*)\frac{\partial}{\partial\gamma^*}+\frac{b_1}{2}\frac{\partial^2}{\partial\gamma^{*2}}\right]. \quad (2.171)$$

Using Eqs. (2.170) and (2.171), Eq.(2.169) becomes

$$\begin{aligned}Q(\alpha^*,\alpha,t) &= \frac{A_1}{\pi}\int d^2\gamma Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma*},t)\\ &\times\int\frac{d^2\beta}{\pi}\exp\left[-\beta^*\beta+b(t)\beta^*+b^*(t)\beta+(-\beta^*+b^*)\frac{\partial}{\partial\beta*}\right]\\ &\times\exp[-\alpha^*\alpha-\beta^*\beta-\gamma^*\gamma+\alpha^*\beta+\alpha^*\gamma+\alpha\gamma^*-\beta^*\gamma-\beta\gamma^*],\end{aligned} \quad (2.172)$$

One can also rewrite Eq. (2.172) as

$$\begin{aligned}Q(\alpha^*,\alpha,t)= \quad & A_1\int\frac{d^2\gamma}{\pi}Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma*},t)\exp[-\alpha^*\alpha-\gamma^*\gamma+\alpha^*\gamma+\alpha\gamma^*]\\ &\times\int\frac{d^2\beta}{\pi}\exp\left[-\beta^*\beta+b\beta^*+b^*\beta+(-\beta^*+b^*)\frac{\partial}{\partial\beta*}\right]\\ &\times\exp[-\beta^*\beta+(\alpha^*-\gamma^*)\beta+(\alpha-\gamma)\beta^*].\end{aligned} \quad (2.173)$$

Let the second integral be

$$\begin{aligned}I_1 &= \int\frac{d^2\beta}{\pi}\exp\left[-\beta^*\beta+b\beta^*+b^*\beta+(-\beta^*+b^*)\frac{\partial}{\partial\beta*}\right]\\ &\times\exp[-\beta^*\beta+(\alpha^*-\gamma^*)\beta+(\alpha-\gamma)\beta^*].\end{aligned} \quad (2.174)$$

Applying the relation

$$\exp\left[a\frac{\partial}{\partial\eta}\right]f(\eta)=f(a+\eta), \quad (2.175)$$

we easily see that

$$I_1 = \exp[\beta^*(\alpha-\gamma)] \int \frac{d^2\beta}{\pi} \exp[-\beta^*\beta + b(t)\beta^* + (\alpha^* - \gamma^*)\beta]. \tag{2.176}$$

In view of the relation

$$\int \frac{d^2\eta}{\pi} \exp\left[-a\eta^*\eta + b\eta + +c\eta^*\right] = \frac{1}{a} e^{bc/a}, \tag{2.177}$$

for $a > 0$, one easily gets

$$I_1 = \exp[\beta^*(\alpha-\gamma) + b(\alpha^* - \gamma^*)]. \tag{2.178}$$

Therefore, making use of Eq. (2.178), Eq.(2.173) turns out to be

$$Q(\alpha^*, \alpha, t) = \frac{A_1 A_2}{\pi} \exp[-\alpha^*\alpha + b\alpha^* + b^*\alpha] \times I_2, \tag{2.179}$$

where

$$\begin{aligned} I_2 \quad &= \int \frac{d^2\gamma}{\pi} \exp\left[-a_1\gamma^*\gamma + c_1\gamma^* + c_1^*\gamma + \frac{b_1}{2}(\gamma^2 + \gamma^{*2}) + (-a_1\gamma^* + b_{1\gamma + c_1^*})\frac{\partial}{\partial\gamma^*} + \frac{b_1}{2}\frac{\partial^2}{\partial\gamma^{*2}}\right]. \\ &\times \exp[-\gamma^*\gamma + (\alpha^* - b^*)\gamma + (\alpha - b)\gamma^*], \end{aligned} \tag{2.180}$$

and the integrand can be written as

$$\exp[\gamma(\alpha^* - b^*)] \exp\left[a' + b'\frac{\partial}{\partial\gamma^*} + c'\frac{\partial^2}{\partial\gamma^{*2}}\right] \exp[d'\gamma^*], \tag{2.181}$$

where

$$\begin{aligned} a' &= -a_1\gamma^*\gamma + c_1\gamma^* + c_1^*\gamma + \frac{b_1}{2}(\gamma^2 + \gamma^{*2}), \\ b' &= -a_1\gamma^* + b_1\gamma + c_1^*, \\ c' &= \frac{b_1}{2}, \\ d' &= \alpha - b - \gamma. \end{aligned} \tag{2.182}$$

Considering a differential operator $\hat{A}$ satisfying an eigenvalue equation

$$\hat{A} f(\eta) = a f(\eta),$$

and using power series one can easily see that

$$e^{\hat{A}}f(\eta) = e^{a}f(\eta).$$

Using this result, we see that

$$\begin{aligned}&\exp[\gamma(\alpha^* - b^*)]\exp\left[a' + b'\frac{\partial}{\partial\gamma^*} + c'\frac{\partial^2}{\partial\gamma^{*2}}\right]\exp[d'\gamma^*]\\ &= \exp[a' + b'd' + c'd'^2]\exp[d'\gamma^* + \gamma(\alpha^* - b^*)].\end{aligned} \tag{2.183}$$

Now using Eqs. (2.182) and (2.183), we can put Eq. (2.180) in the form

$$\begin{aligned}I_2 \quad &= \exp\left[c_1^*\alpha - c_1^*b + \frac{b_1}{2}(\alpha - b)^2\right]\\ &\int\frac{d^2\gamma}{\pi}\exp\left[-\gamma^*\gamma + (\alpha^* - b^*)\gamma + (c_1 + (1 - a_1)(\alpha - b))\,\gamma^* + \frac{b_1}{2}\gamma^{*2}\right].\end{aligned} \tag{2.184}$$

Applying the relation

$$\int\frac{d^2\eta}{\pi}\exp\left[-a\eta^*\eta + b\eta + +c\eta^* + d\eta^2 + e\eta^{*2}\right] = \frac{1}{[a^2 - 4de]^{1/2}}e^{\frac{abc + dc^2 + eb^2}{a^2 - 4de}}, \tag{2.185}$$

for $a > 0$, we readily obtain

$$I_2 = \exp[c_1^*\alpha - c_1^*b + \frac{b_1}{2}(\alpha - b)^2]\exp\left[(\alpha^* - b^*)\left(c_1 + (1 - a_1)(\alpha - b)\right) + \frac{b_1}{2}(\alpha^* - b^*)^2\right].$$

In view of this results and Eq. (2.179), the Q function for the superposition of a coherently driven cavity mode and the laser turns out to be

$$Q(\alpha^*, \alpha, t) = \frac{A_3(t)}{\pi}\exp\left[-p_1(t)\alpha^*\alpha + q_1^*(t)\alpha + q_1(t)\alpha^* + \frac{r_1(t)}{2}(\alpha^2 + \alpha^{*2})\right], \tag{2.186}$$

where

$$A_3(t) = A_1(t)A_2(t)\exp\left[-(bc_1^* + b^*c_1) + \frac{b_1}{2}(b^2 + b^{*2}) + (1 - a_1)b^*b\right], \tag{2.187}$$

$$p_1(t) = a_1(t), \tag{2.188}$$

$$q_1(t) = c_1(t) + a_1(t)b(t) - b_1(t)b^*(t), \tag{2.189}$$

$$r_1(t) = b_1(t), \tag{2.190}$$

in which A_1, A_2, a_1, b_1, c_1 and b are given by Eqs. (2.161), (2.164), (2.165), (2.166), (2.167) and (2.162) respectively.

Applying the well known input-output relation

$$\alpha_{out} = \sqrt{\kappa}\alpha, \tag{2.191}$$

and associating the initial condition with the Q function for the three-level laser, the normalized Q function for the output light from a three-level laser and a coherently driven cavity mode take the form

$$Q(\alpha^*, \alpha, t) = \frac{A(t)}{\pi} \exp\left[-p(t)\alpha^*\alpha + q^*(t)\alpha + q(t)\alpha^* + \frac{r(t)}{2}(\alpha^2 + \alpha^{*2})\right], \tag{2.192}$$

where

$$\begin{aligned} A(t) \quad &= \frac{\sqrt{p^2 - r^2}}{\kappa} \exp\left[-p\alpha_0^*\alpha_0 + \frac{r}{2}(\alpha_0^{*2} + \alpha_0^2) - \left[\frac{2\epsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t})\right]^2\right] \\ &\times \exp\left[-(b(t)c_1^*(t) + b^*(t)c_1(t)) + \frac{r(t)}{2}(b^2(t) + b^{*2}(t)) + (1 - p(t))b^*(t)b(t)\right], \end{aligned} \tag{2.193}$$

$$p(t) = \frac{\mu}{2\kappa}\left[\frac{1}{a' + (\mu - a')e^{-\mu t}} + \frac{1}{b' + (\mu - b')e^{-\mu t}}\right], \tag{2.194}$$

$$r(t) = \frac{\mu}{2\kappa}\left[\frac{1}{b' + (\mu - b')e^{-\mu t}} - \frac{1}{a' + (\mu - a')e^{-\mu t}}\right], \tag{2.195}$$

$$q(t) = c_1(t) + p(t)b(t) - r(t)b^*(t), \tag{2.196}$$

$$c_1(t) = e^{-\frac{\mu}{2}t}\left[p(t)\alpha_0 - r(t)\alpha_0^*\right], \tag{2.197}$$

$$b(t) = \frac{2\epsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}t}), \tag{2.198}$$

$$a' = A(1 - \rho_{aa}^{(0)} + \rho_{ac}^{(0)}) + \kappa, \tag{2.199}$$

$$b' = A(1 - \rho_{aa}^{(0)} - \rho_{ac}^{(0)}) + \kappa, \tag{2.200}$$

$$\mu = A(1 - 2\rho_{aa}^{(0)}) + \kappa, \tag{2.201}$$

where κ is the damping constant of the cavity and also ε and A are the amplitude of the coherent driving light and the linear gain coefficient of the three-level laser respectively.

3 Photon Statistics

The statistical property of the light generated from a three-level laser and a coherently driven cavity mode is described by the mean and the normally-ordered variance of the photon number. In this section we apply the Q function for the output light of a three-level laser and a coherently driven cavity mode to calculate the mean and the variance of the photon number.

3.1 The mean photon number

The mean photon number of the output light, resulted from a three-level laser and a coherently driven cavity mode, can be calculated using the Q function. In view of the well known input-output relation (2.191) and making use of the relation

$$\langle\hat{A}\rangle = \frac{1}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta)|\langle\alpha|\beta\rangle|^2 A_n(\beta^*, \alpha), \tag{3.202}$$

to find the expectation value of the operator $\hat{A}$, in which $A_n(\beta^*, \alpha)$ is a c-number corresponding to the operator $\hat{A}$ in normal-ordering, we observe that the mean photon number of the output light can be expressed as

$$\bar{n} = \frac{a}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta)\exp[-a\alpha^*\alpha - a\beta^*\beta + a\alpha\beta^* + a\beta\alpha^*]A_n(\beta^*, \alpha). \tag{3.203}$$

where $a = \frac{1}{\kappa}$ and

$$A_n(\beta^*, \alpha) = \beta^*\alpha, \tag{3.204}$$

is c-number corresponding to the number operator in normal-ordering, and

$$Q(\alpha^*, \beta, t) = \frac{A(t)}{\pi}\exp\left[-p(t)\alpha^*\beta + q^*(t)\beta + q(t)\alpha^* + \frac{r(t)}{2}(\beta^2 + \alpha^{*2})\right]. \tag{3.205}$$

Therefore, one can see that

$$\begin{aligned}\bar{n} \quad &= Aa\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp[-p\alpha^*\beta + q^*\beta + q\alpha^* + \frac{r}{2}(\beta^2 + \alpha^{*2})] \\ &\times\exp[-a\alpha^*\alpha - a\beta^*\beta + a\alpha^*\beta + a\alpha\beta^*]\beta^*\alpha,\end{aligned} \tag{3.206}$$

or

$$\begin{aligned}\bar{n} \quad &= Aa\int \frac{d^2\alpha}{\pi}\exp[-a\alpha^*\alpha + q\alpha^* + \alpha^{*2}]\alpha \\ &\times \left(\int \frac{d^2\beta}{\pi}\exp\left[-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right]\beta^*\right).\end{aligned} \tag{3.207}$$

One can see that the second integral in the bracket can be written as

$$\begin{aligned}&\int \frac{d^2\beta}{\pi}\exp\left(-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right)\beta^* \\ &= \frac{\partial}{\partial(a\alpha)}\int \frac{d^2\beta}{\pi}\exp\left(-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right),\end{aligned} \tag{3.208}$$

applying Eq. (2.185) to integrate the above relation and carrying out the differentiation, one readily obtains

$$\begin{aligned}&\int \frac{d^2\beta}{\pi}\exp\left(-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right)\beta^* \\ &= \left(\frac{q^* + r\alpha + (a-p)\alpha^*}{a^2}\right)\exp\left[(a-p)\alpha^*\alpha + q^*\alpha + \frac{r}{2}\alpha^2\right],\end{aligned} \tag{3.209}$$

Therefore Eq. (3.207) take the form

$$\bar{n} = \frac{A}{a}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2 + \alpha^{*2})\right] \times \left[(a-p)\alpha^*\alpha + q^*\alpha + r\alpha^2\right], \tag{3.210}$$

or

$$\bar{n} = I_1 + I_2 + I_3, \tag{3.211}$$

where

$$I_1 = \frac{A}{a}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2 + \alpha^{*2})\right](a-p)\alpha^*\alpha, \tag{3.212}$$

$$I_2 = \frac{A}{a}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2 + \alpha^{*2})\right]q^*\alpha, \tag{3.213}$$

$$I_3 = \frac{A}{a}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2 + \alpha^{*2})\right]r\alpha^2. \tag{3.214}$$

But we know that

$$\begin{aligned} I_1 &= \frac{A}{a}(a-p)\frac{\partial^2}{\partial q\partial q^*}\int\frac{d^2\alpha}{\pi}\exp[-p\alpha^*\alpha+q\alpha^*+q^*\alpha+\frac{r}{2}(\alpha^2+\alpha^{*2})] \\ &= \frac{A}{a}(a-p)\frac{\partial^2}{\partial q\partial q^*}\left(\frac{1}{\sqrt{p^2-r^2}}\exp\left[\frac{pqq^*+\frac{r}{2}(q^2+q^{*2})}{p^2-r^2}\right]\right) \\ &= \frac{a-p}{a}\left[\frac{p}{p^2-r^2}+\left(\frac{pq+rq^*}{p^2-r^2}\right)\left(\frac{pq^*+rq}{p^2-r^2}\right)\right], \end{aligned} \tag{3.215}$$

$$\begin{aligned} I_2 &= \frac{A}{a}\frac{\partial}{\partial\epsilon}\int\frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha+q\alpha^*+\epsilon q^*\alpha+\frac{r}{2}(\alpha^2+\alpha^{*2})\right]\Big|_{\epsilon=1} \\ &= \frac{A}{a}\frac{\partial}{\partial\epsilon}\left(\frac{1}{\sqrt{p^2-r^2}}\exp\left[\frac{\epsilon pqq^*+\frac{r}{2}(q^2+\epsilon^2q^{*2})}{p^2-r^2}\right]\right)\Big|_{\epsilon=1} \\ &= \frac{q^*}{a}\left[\frac{pq+rq^*}{p^2-r^2}\right], \end{aligned} \tag{3.216}$$

and

$$\begin{aligned} I_3 &= \frac{Ar}{a}\frac{\partial^2}{\partial q^{*2}}\int\frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha+q\alpha^*+q^*\alpha+\frac{r}{2}(\alpha^2+\alpha^{*2})\right] \\ &= \frac{Ar}{a}\frac{\partial^2}{\partial q^{*2}}\left(\frac{1}{\sqrt{p^2-r^2}}\exp\left[\frac{pqq^*+\frac{r}{2}(q^2+q^{*2})}{p^2-r^2}\right]\right) \\ &= \frac{r}{a}\left[\frac{r}{p^2-r^2}+\left(\frac{pq+rq^*}{p^2-r^2}\right)^2\right]. \end{aligned} \tag{3.217}$$

On account of Eqs. (3.215), (3.216), (3.217) and (3.211), and taking into consideration that initially the cavity is in a vacuum state, one easily see that

$$\bar{n} = \frac{1}{a}\left[\frac{ap}{p^2-r^2}-1+a\left(\frac{pq+rq}{p^2-r^2}\right)^2\right]. \tag{3.218}$$

With the help of Eqs. (2.194), (2.195), (2.196), (2.197), (2.198), (2.199), (2.200), (2.201), the mean photon number of the output light from a three-level laser and a coherently driven cavity mode take the form

$$\bar{n} = \frac{\kappa A\rho_{aa}^{(0)}}{\mu}(1-e^{-\mu t})+\frac{4\varepsilon^2}{\kappa}\left(1-e^{-\kappa t/2}\right)^2. \tag{3.219}$$

The above expression tells that the mean photon number of the output light is just the sum of the mean photon number of transmitted cavity coherent light and three-level laser.
At steady state Eq. (3.219) turns out to be

$$\bar{n} = \frac{\kappa A\rho_{aa}^{(0)}}{\mu} + \frac{4\varepsilon^2}{\kappa}. \tag{3.220}$$

Introducing a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{3.221}$$

and in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{3.222}$$

and

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{3.223}$$

one easily finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{3.224}$$

and

$$\rho_{ac}^{(0)} = \frac{1}{2}(1-\eta^2)^{1/2}. \tag{3.225}$$

One can use the above result to see that the mean photon number in Eq. (3.220) also be expressible as a function of the initial condition η as

$$\bar{n} = \frac{\kappa A(1-\eta)}{2(A\eta+\kappa)} + \frac{4\varepsilon^2}{\kappa}, \tag{3.226}$$

where $\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}$.

As it is seen from the above expression and Fig. 3.1, the mean photon number of the output light increases with the linear gain coefficient and with the amplitude of the driving light. Applying the Q function obtained using a single master equation, Misrak G. [8], has found the mean photon number of a coherently driven degenerate three-level laser (in which the cavity coherent photons interact with the three-level atoms) is given by

$$\bar{n} = \frac{\kappa A(1-\eta)}{2(A\eta+\kappa)} + \frac{4\varepsilon^2\kappa}{(A\eta+\kappa)^2}. \tag{3.227}$$

Comparison of this result with Eq. (3.226) shows that the mean photon number is decreased when the cavity coherent photons interact with the three-level atoms. This must be due to

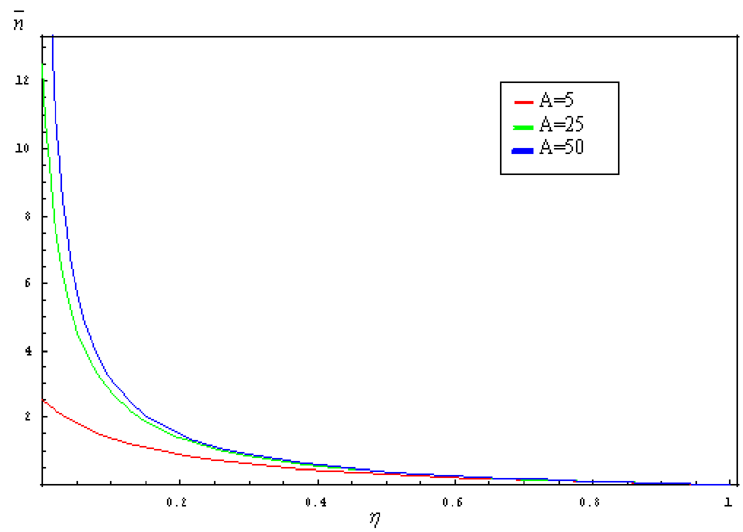

Fig 3.1 Plots of the mean photon number $\bar{n}$ at steady state versus η for $\kappa = 0.8$, $\varepsilon = 0.2$, and for different values of the linear gain coefficient.

the absorption of cavity coherent photons followed by spontaneous emission by the three-level atoms.

We now proceed to consider some special cases, first we consider the case in which the cavity mode is not driven by coherent light (i.e $\varepsilon = 0$), then Eq. (3.226) turns to be

$$\bar{n} = \frac{\kappa A(1-\eta)}{2(A\eta + \kappa)}, \tag{3.228}$$

which is the mean photon number of the output light from the three-level laser, this result is the same as that obtained in [5, 8].

We next consider the case in which there is no three-level laser in the cavity, thus upon setting $A = 0$ in Eq. (3.226), we get

$$\bar{n} = \frac{4\varepsilon^2}{\kappa}, \tag{3.229}$$

which is the mean photon number of the output light from a coherently driven cavity mode, this result agrees with that obtained in [5].

3.2 The normally-ordered variance of the photon number

The normally-ordered variance of the photon number is defined by

$$\begin{aligned} :(\triangle n)^2: \quad &= \langle :\hat{n}^2:\rangle - \bar{n}^2 \\ &= \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle - \bar{n}^2. \end{aligned} \tag{3.230}$$

where :: represents normal-ordering of operators under consideration.

One can also use Eq. (3.202) along with the input-output relation and express the first term in terms of c-number variables associated with the normal-ordering as

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \frac{a}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*,\beta)\exp[-a\alpha^*\alpha - a\beta^*\beta + a\alpha^*\beta + a\alpha\beta^*]\beta^{*2}\alpha^2, \tag{3.231}$$

where $a = \frac{1}{\kappa}$. Applying Eq. (3.205) into Eq. (3.231), we obtain

$$\begin{aligned} \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle \quad &= Aa\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp\left[-p\alpha^*\beta + q^*\beta + q\alpha^* + \frac{r}{2}(\beta^2 + \alpha^{*2})\right] \\ &\times \exp\left[-a\alpha^*\alpha - a\beta^*\beta + a\alpha^*\beta + a\alpha\beta^*\right]\beta^{*2}\alpha^2 \\ &= Aa\int \frac{d^2\alpha}{\pi}\exp[-a\alpha^*\alpha + q\alpha^* + \alpha^{*2}]\alpha^2 \\ &\times \left(\int \frac{d^2\beta}{\pi}\exp\left[-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right]\beta^{*2}\right). \end{aligned} \tag{3.232}$$

The second integral can be written as

$$\begin{aligned} I \quad &= \int \frac{d^2\beta}{\pi}\exp\left[-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + (a\alpha + \epsilon)\beta^* + \frac{r}{2}\beta^2\right]\beta^{*2}\Big|_{\epsilon=0} \\ &= \frac{\partial^2}{\partial\epsilon^2}\int \frac{d^2\beta}{\pi}\exp\left[-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + (a\alpha + \epsilon)\beta^* + \frac{r}{2}\beta^2\right]\Big|_{\epsilon=0}, \end{aligned} \tag{3.233}$$

By applying Eq. (2.185) and carrying out the differentiation and finally putting $\epsilon = 0$, one gets

$$I = \frac{1}{a^3}[r + (a-p)\alpha^* + q^* + r\alpha]^2\exp\left[(a-p)\alpha^*\alpha + q^*\alpha + \frac{r}{2}\alpha^2\right]. \tag{3.234}$$

Therefore, Eq.(3.232) takes the form

$$\begin{aligned} \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle &= \frac{A}{a^2}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2 + \alpha^{*2})\right] \\ &\times [r\alpha^2 + (a-p)^2\alpha^{*2}\alpha^2 + r^2\alpha^4 + 2r(a-p)\alpha^*\alpha^3 + q^{*2}\alpha^2 + 2q^*(a-p)\alpha^*\alpha^2 + 2q^{*r}\alpha^3], \end{aligned} \tag{3.235}$$

or

$$\langle \hat{a}^{\dagger 2}\hat{a}^{2}\rangle = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \tag{3.236}$$

where

$$I_1 = \frac{Ar}{a^2}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^2, \tag{3.237}$$

$$I_2 = \frac{A}{a^2}(a-p)^2\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^2\alpha^{*2}, \tag{3.238}$$

$$I_3 = \frac{Ar^2}{a^2}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^4, \tag{3.239}$$

$$I_4 = \frac{2Ar}{a^2}(a-p)\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^*\alpha^3, \tag{3.240}$$

$$I_5 = \frac{Aq^{*2}}{a^2}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^2, \tag{3.241}$$

$$I_6 = \frac{2Aq^*}{a^2}(a-p)\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^*\alpha^2, \tag{3.242}$$

$$I_7 = \frac{2Aq^*r}{a^2}\int \frac{d^2\alpha}{\pi}\exp\left[-p\alpha^*\alpha + q\alpha^* + q^*\alpha + \frac{r}{2}(\alpha^2+\alpha^{*2})\right]\alpha^3. \tag{3.243}$$

Applying the technique to integrate the above seven integrals along with the relation in Eq. (2.185), one readily obtains

$$I_1 = \frac{r}{a^2}\left[\frac{r}{p^2-r^2} + \left(\frac{pq+rq^*}{p^2-r^2}\right)^2\right], \tag{3.244}$$

$$\begin{aligned} I_2 \quad &= \frac{(a-p)^2}{a^2}\left[\frac{4p}{p^2-r^2}\left(\frac{pq+rq^*}{p^2-r^2}\right)^2 + \frac{2r}{p^2-r^2}\left(\frac{pq^*+rq}{p^2-r^2}\right)^2 + \frac{2p^2}{p^2-r^2}\right] \\ &+ \frac{(a-p)^2}{a^2}\left[\frac{r^2}{p^2-r^2} + \left(\frac{pq^*+rq}{p^2-r^2}\right)^4\right], \end{aligned} \tag{3.245}$$

$$I_3 = \frac{r^2}{a^2}\left[3\left(\frac{r}{p^2-r^2}\right)^2 + \frac{6r}{p^2-r^2}\left(\frac{pq+rq^*}{p^2-r^2}\right)^2 + \left(\frac{pq^*+rq}{p^2-r^2}\right)^4\right], \tag{3.246}$$

$$\begin{aligned} I_4 \quad &= \frac{2r(a-p)}{a^2}\left[\frac{3p}{p^2-r^2}\frac{r}{p^2-r^2} + \frac{3p}{p^2-r^2}\left(\frac{pq+rq^*}{p^2-r^2}\right)^2 + \frac{3r}{p^2-r^2}\left(\frac{pq+rq^*}{p^2-r^2}\right)^2\right] \\ &+ \frac{2r(a-p)}{a^2}\left[\left(\frac{pq^*+rq}{p^2-r^2}\right)^4\right], \end{aligned} \tag{3.247}$$

$$I_5 = \frac{q^{*2}}{a^2}\left[\frac{r}{p^2-r^2} + \left(\frac{pq+rq^*}{p^2-r^2}\right)^2\right], \tag{3.248}$$

$$I_6 = \frac{2q^*(a-p)}{a^2}\left[\frac{2p}{p^2-r^2}\left(\frac{pq+rq^*}{p^2-r^2}\right) + \frac{r}{p^2-r^2}\left(\frac{pq^*+rq}{p^2-r^2}\right) + \left(\frac{pq+rq^*}{p^2-r^2}\right)^3\right], \tag{3.249}$$

$$I_7 = \frac{2q^*r}{a^2}\left[\frac{3r}{p^2-r^2}\left(\frac{pq+rq^*}{p^2-r^2}\right) + \left(\frac{pq+rq^*}{p^2-r^2}\right)^3\right]. \tag{3.250}$$

Using Eq. (3.218) one can write the square of the mean photon number as

$$\bar{n}^2 = \frac{1}{a^2}\left[\frac{ap}{p^2-r^2} - 1 + a\left(\frac{pq+rq}{p^2-r^2}\right)\right]^2, \tag{3.251}$$

With the aid of Eqs. (3.236) and (3.251), we can put the normally-ordered variance of the photon number as

$$: (\triangle n)^2 : \quad = (I_1+I_2+I_3+I_4+I_5+I_6+I_7) - \frac{1}{a^2}\left[\frac{ap}{p^2-r^2} - 1 + a\left(\frac{pq+rq}{p^2-r^2}\right)\right]^2, \tag{3.252}$$

In view of Eqs. (4.244), (4.245), (4.246), (4.247), (4.248), (4.249), (4.250), (4.252), and (2.194), (2.195), (2.196), (2.197), (2.198), the normally-ordered variance of the photon number is expressible as

$$: (\triangle n)^2 : \quad = \left[\frac{8A\varepsilon^2\kappa^2}{\mu}(\rho_{aa}^{(0)} + \rho_{ac}^{(0)})(1-e^{-\kappa t/2}) - \frac{2A\kappa^2}{\mu}\rho_{aa}^{(0)}\right](1-e^{-\mu t}) + \frac{\kappa^2}{2\mu^2}\left([a' + (\mu-a')e^{-\mu t}]^2 + [b' + (\mu-b')e^{-\mu t}]^2\right) - \kappa^2. \tag{3.253}$$

where a', b' and μ are given by Eqs. (2.199), (2.200) and (2.201) respectively.
As it is introduced in Eqs. (3.221), (3.224), and (3.225), one can also describe the normally-ordered variance of the photon number in terms of the initial condition $\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}$ as

$$: (\triangle n)^2 : \quad = \left[\frac{4A\varepsilon^2\kappa^2[(1-\eta)+(1-\eta^2)^{1/2}](1-e^{-\kappa t/2}) - A\kappa^2(1-\eta)}{A\eta+\kappa}\right](1-e^{-(A\eta+\kappa t)}) + \frac{\kappa^2}{2}\left[\left(\frac{a' - c'e^{-(A\eta+\kappa)t}}{A\eta+\kappa}\right)^2 + \left(\frac{b' - d'e^{-(A\eta+\kappa)t}}{A\eta+\kappa}\right)^2\right] - \kappa^2, \tag{3.254}$$

where

$$a' = \frac{A}{2}\left[(1+\eta) + (1-\eta^2)^{1/2}\right] + \kappa, \tag{3.255}$$

$$b' = \frac{A}{2}\left[(1+\eta) - (1-\eta^2)^{1/2}\right] + \kappa, \tag{3.256}$$

$$c' = \frac{A}{2}\left[(1-\eta) + (1-\eta^2)^{1/2}\right], \tag{3.257}$$

$$d' = \frac{A}{2}\left[(1-\eta) - (1-\eta^2)^{1/2}\right], \tag{3.258}$$

and at steady state Eq. (3.254) turns out to be

$$\begin{aligned} :(\triangle n)^2: \quad &= \left[\frac{4A\varepsilon^2\kappa^2[(1-\eta)+(1-\eta^2)^{1/2}] - A\kappa^2(1-\eta)}{A\eta+\kappa}\right] \\ &+\frac{\kappa^2}{2}\left[\frac{A(1+\eta)(A+2\kappa)+2\kappa^2}{(A\eta+\kappa)^2}\right] - \kappa^2. \end{aligned} \tag{3.259}$$

When $A = 0$, $:(\triangle n)^2 := 0$, this is the normally-ordered variance of the photon number of the cavity coherent light.

We observe from Fig. 3.2 that the increase in the normally-ordered variance of the photon number with the driving coherent light shows the fluctuations in the number of output photons increase as the result of the superposition of the two different light beams. The increase in the fluctuations of the photon number must be due to the increase in the mean photon number of the output light with the driving coherent light.

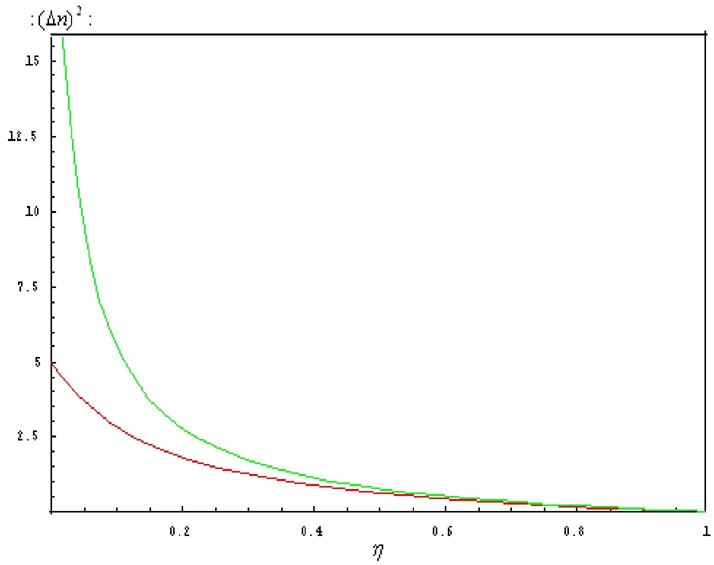

Fig 3.2 Plots of the normally-ordered variance of the photon number $:(\Delta n)^2:$ at steady state versus η when $\varepsilon = 10$ (green) and when $\varepsilon = 0$ (red)

4 Quadrature squeezing

In the previous chapter with the aid of the Q function for the output light generated from a three-level laser and a coherently driven cavity mode we study the statistical property of the output light. In this chapter we apply the Q function to study the squeezing property of the output light. The squeezing property of the light is described by calculating the quadrature variance and the squeezing spectrum of the light mode under consideration.

4.1 Quadrature variance

The normally-ordered variance of the output quadratures represented by the operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{4.260}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{4.261}$$

can be expressed as

$$: (\triangle a_\pm)^2 := \pm\langle : (\hat{a}^\dagger \pm \hat{a})^2 : \rangle \mp \langle(\hat{a}^\dagger \pm \hat{a})\rangle^2, \tag{4.262}$$

where : : represents normal-ordering of operators under consideration. And the two Hermitian quadrature operators satisfy the commutation relation $[\hat{a}_+, \hat{a}_-] = 2i$.
One can also express the above quadrature variance in normal-ordering as

$$: (\triangle a_\pm)^2 := \pm\langle(\hat{a}^{\dagger 2} + \hat{a}^2 \pm 2\hat{a}^\dagger\hat{a})\rangle \mp \langle(\hat{a}^\dagger \pm \hat{a})^2\rangle^2. \tag{4.263}$$

In view of the well known input-output relation (2.191) and making use of the relation

$$\langle\hat{A}\rangle = \frac{1}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta)|\langle\alpha|\beta\rangle|^2 A_n(\beta^*, \alpha), \tag{4.264}$$

to find the expectation value of the operator $\hat{A}$, in which $A_n(\beta^*, \alpha)$ is a c-number corresponding to the operator $\hat{A}$ in normal order, we easily put Eq. (4.263) as

$$\begin{aligned} : (\triangle a_+)^2 : \quad &= \left[\frac{a}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta)|\langle\alpha|\beta\rangle|^2_{out} A_{n1}(\beta^*, \alpha)\right] \\ &- \left[\frac{a}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta)|\langle\alpha|\beta\rangle|^2_{out} A_{n2}(\beta^*, \alpha)\right]^2, \end{aligned} \tag{4.265}$$

where

$$\begin{aligned}
Q(\alpha^*,\beta) &= \frac{A}{\pi}\exp[-p\alpha^*\beta + q^*\beta + \frac{r}{2}(\beta^2+\alpha^{*2})],\\
|\langle\alpha|\beta\rangle|^2_{out} &= \exp\left[-a\alpha^*\alpha - a\beta^*\beta + a\alpha^*\beta + a\alpha\beta^*\right],\\
A_{n1}(\beta^*,\alpha) &= \beta^{*2}+\alpha^2+2\beta^*\alpha,\\
A_{n2}(\beta^*,\alpha) &= \beta^*+\alpha,
\end{aligned} \tag{4.266}$$

and $a = \frac{1}{k}$.

Hence one can use the above expressions and put Eq. (4.265) as

$$:(\triangle a_+)^2 := I_1 - I_2^2, \tag{4.267}$$

in which

$$\begin{aligned}
I_1 \quad &= Aa\int \frac{d^2\alpha}{\pi}\exp\left[-a\alpha^*\alpha + q\alpha^* + \frac{r}{2}\alpha^{*2}\right]\\
&\times \int \frac{d^2\beta}{\pi}\exp\left(-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right)(\beta^{*2}+\alpha^2+2\beta^*\alpha),
\end{aligned} \tag{4.268}$$

where $a = \frac{1}{\kappa}$. After carrying out the integration, we arrive at

$$I_1 = \left[\kappa(4b^2-2) + \frac{2}{p-r}\right] \tag{4.269}$$

and

$$\begin{aligned}
I_2 \quad &= Aa\int \frac{d^2\alpha}{\pi}\exp[-a\alpha^*\alpha + q\alpha^* + \alpha^{*2}]\\
&\times \int \frac{d^2\beta}{\pi}\exp\left(-a\beta^*\beta + [(a-p)\alpha^* + q^*]\beta + a\alpha\beta^* + \frac{r}{2}\beta^2\right)(\beta^*+\alpha).
\end{aligned} \tag{4.270}$$

Similarly after carrying out the integration, we obtain

$$I_2 = \frac{2\kappa}{\sqrt{\kappa}}b. \tag{4.271}$$

Therefore applying Eqs. (4.269) and (4.271) into Eq. (4.267), one easily gets

$$:(\triangle a_+)^2 := \left[\frac{2}{p-r} - 2\kappa\right], \tag{4.272}$$

Using Eqs. (2.194), (2.195), we also see that

$$:(\triangle a_+)^2 := 2\kappa\left[\frac{a'-\mu}{\mu}\right](1-e^{-\mu t}), \tag{4.273}$$

where a' and μ are given by Eqs. (2.199) and (2.201). And similarly we see from Eqs. (4.262) and (4.264), the minus quadrature variance takes the form

$$\begin{aligned}:(\triangle a_-)^2: \quad &= -Aa\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\exp\left[-p\alpha^*\beta+q^*\beta+q\alpha^*+\frac{r}{2}(\beta^2+\alpha^{*2})\right]\\ &\times\exp\left[-a\alpha^*\alpha-a\beta^*\beta+a\alpha^*\beta+a\alpha\beta^*\right]\left(\beta^{*2}+\alpha^2-2\beta^*\alpha\right).\end{aligned}$$

After carrying out the integration, we obtain

$$:(\triangle a_-)^2 := \left[2\kappa-\left(\frac{2}{p-r}\right)\right], \tag{4.274}$$

Using Eqs. (2.194) and (2.195), we see that

$$:(\triangle a_-)^2 := 2\kappa\left[\frac{\mu-b'}{\mu}\right](1-e^{-\mu t}), \tag{4.275}$$

where b' and μ are given by Eqs. (2.200) and (2.201) respectively.
In general, making use of Eqs. (4.273) and (4.275) along with (2.199), (2.200) and (2.201), the normally-ordered quadrature variance of the output light from a three-level laser and a coherently driven cavity mode take the form

$$:(\triangle a_\pm)^2 := \pm\frac{2A\kappa}{\mu}(\rho_{ac}^{(0)}\pm\rho_{aa}^{(0)})(1-e^{-\mu t}). \tag{4.276}$$

At steady state, we see that

$$:(\triangle a_\pm)^2 := \pm\frac{2A\kappa(\rho_{ac}^{(0)}\pm\rho_{aa}^{(0)})}{A(1-2\rho_{aa}^{(0)})+\kappa}. \tag{4.277}$$

As it is seen from the above expression, the squeezing occurs on the minus quadrature and the probability for the injected atom to be in the bottom level is larger than the probability for the atom to be in the upper level [i.e ($\rho_{cc}^{(0)} > \rho_{aa}^{(0)}$) and $\rho_{aa}^{(0)} < 0.5$]. As it is seen from Fig. 4.1, the squeezing of the output light increases exponentially as time progresses and reach at steady state (maximum squeezing) after some time.

Using the parameter, describing the initial condition of the three-level atoms, defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{4.278}$$

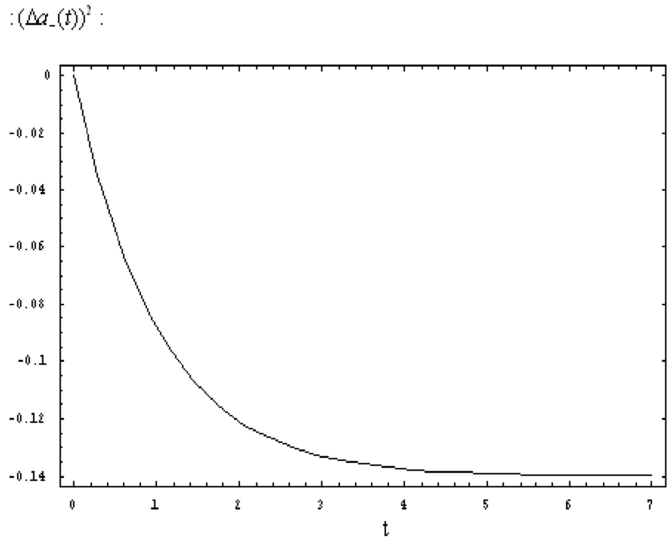

Fig 4.1 Plots of the normally-ordered quadrature variance $:(\Delta a)^2:$ **at steady state versus time** t **for** $\kappa = 0.8,\ A = 1$**.**

and in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{4.279}$$

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{4.280}$$

one easily finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{4.281}$$

and

$$\rho_{ac}^{(0)} = \frac{1}{2}(1-\eta^2)^{1/2}. \tag{4.282}$$

Hence

$$\rho_{ac}^{(0)} \pm \rho_{aa}^{(0)} = \frac{1}{2}\left[(1-\eta^2)^{1/2} \pm (1-\eta)\right]. \tag{4.283}$$

Using Eqs. (4.278), (4.283), along with (4.276), the normally-ordered quadrature variance can be expressed in terms of the initial condition $\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}$ in the form

$$:(\triangle a_{\pm})^2 := \pm\frac{A\kappa}{A\eta+\kappa}\left[(1-\eta^2)^{1/2} \pm (1-\eta)\right](1-e^{-(A\eta+\kappa)t}). \tag{4.284}$$

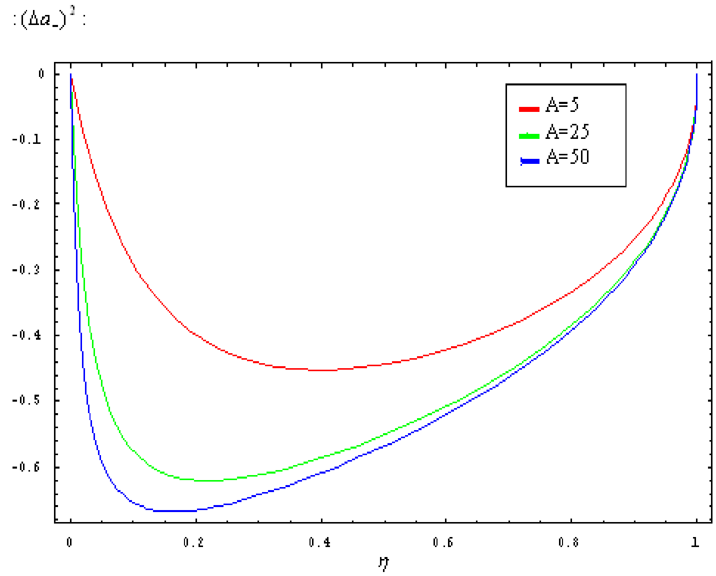

Fig. 4.2: Plots of the normally-ordered quadrature variance $:(\Delta a_{-})^2:$ at steady state versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient.

This result is exactly the same as the one obtained using the Q function derived from a single master equation for the cavity mode [8], therefore we assert that the interaction between the cavity coherent photons and the three-level atoms have no effect on the quadrature variance.

As it is seen from Fig. 4.2 the light mode is in the squeezed state (the squeezing occurs in the minus quadrature) for all values of η between zero and one, and the degree of squeezing increases with the linear gain coefficient. One can also see from the same graph, almost perfect squeezing can be obtained for large values of A and for small values of η close to zero (but different from zero). In other word, the more the squeezing occurs when the probability for the injected atoms to be in the bottom level is slightly larger than the probability for the atoms to be in the upper level for an increasing value of the linear gain coefficient.

We also noticed from Eq. (4.284), the cavity coherent light has no effect on the degree of squeezing.

We note that the noise in the minus quadrature is below the coherent-state level at the expense of enhanced noise in the plus quadrature, hence, our system satisfies the uncertainty principle $\Delta a_+ \Delta a_- \geq 1$.

4.2 Squeezing spectrum

We next seek to obtain the spectrum of quadrature fluctuations (squeezing spectrum) of the output light from a three-level laser and a coherently driven cavity mode. To this end, we define the squeezing spectrum of the output light with central frequency ω_o by [5]

$$S_{\pm}^{out}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_o)\tau} \langle : \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau) : \rangle_{ss}, \tag{4.285}$$

where $\hat{a}_+$ and $\hat{a}_-$ are the output quadrature operators defined by Eqs. (4.260) and (4.261) respectively, and the subscript "ss" stands for steady state and also

$$\langle \alpha, \beta \rangle = \langle \alpha\beta \rangle - \langle \alpha \rangle \langle \beta \rangle. \tag{4.286}$$

Upon integrating both sides of the expression (4.285) over ω, we readily get

$$\int_{-\infty}^{\infty} S_{\pm}^{out}(\omega) d\omega =: (\triangle a_{\pm})^2 : . \tag{4.287}$$

On the basis of this relation, we note that $S_{\pm}^{out}(\omega)d\omega$ is the normally-ordered quadrature variance of the output light in the interval between ω and $\omega + d\omega$.

One can also express the normally-ordered squeezing spectrum in terms of c-number variables associated with the normal ordering as

$$S_{\pm}^{out}(\omega) = \pm \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_o)\tau} \langle \alpha_{\pm}(t), \alpha_{\pm}(t+\tau) \rangle_{ss}. \tag{4.288}$$

Using Eq. (4.286), we see that

$$\begin{aligned} \langle \alpha_{\pm}(t), \alpha_{\pm}(t+\tau) \rangle_{ss} \quad &= \langle \alpha^*(t)\alpha^*(t+\tau) \rangle + \langle \alpha(t)\alpha(t+\tau) \rangle \pm \langle \alpha^*(t)\alpha(t+\tau) \rangle \\ &\pm \langle \alpha(t)\alpha^*(t+\tau) \rangle - \left(\langle \alpha^*(t) \rangle \pm \langle \alpha(t) \rangle \right) \left(\langle \alpha^*(t+\tau) \rangle \pm \langle \alpha(t+\tau) \rangle \right), \end{aligned} \tag{4.289}$$

but one can apply Schrödinger picture and write as

$$\langle \alpha(t+\tau) \rangle = Tr \left(\hat{a}(\tau) \hat{\rho}(t) \right), \tag{4.290}$$

Upon expanding the density operator in normal order and introducing the identity relation

$$I = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|, \tag{4.291}$$

the above expression can be put in the form

$$\langle\alpha(t+\tau)\rangle = \int \frac{d^2\alpha}{\pi} \sum_{i,m}^{\infty} C_{lm}(t) Tr\left(\hat{a}(\tau)|\alpha\rangle\langle\alpha|\hat{a}^{\dagger l}(0)\hat{a}^m(0)\right). \tag{4.292}$$

Now using the relation

$$\hat{a}(0)|\alpha\rangle = \alpha|\alpha\rangle, \tag{4.293}$$

and applying the identity

$$|\alpha\rangle\langle\alpha|\hat{a}^n = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)^n |\alpha\rangle\langle\alpha|, \tag{4.294}$$

we have

$$\langle\alpha(t+\tau)\rangle = \int \frac{d^2\alpha}{\pi} \sum_{i,m}^{\infty} C_{lm}(t)\alpha^{*l} \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)^m Tr(\hat{a}(\tau)|\alpha\rangle\langle\alpha|), \tag{4.295}$$

or

$$\langle\alpha(t+\tau)\rangle = \int d^2\alpha Q\left(\alpha, \alpha + \frac{\partial}{\partial\alpha^*}, t\right) Tr\left(\hat{a}(\tau)|\alpha\rangle\langle\alpha|\right). \tag{4.296}$$

We note that

$$Tr(\hat{a}(\tau)|\alpha\rangle\langle\alpha|) = Tr\left(\hat{a}(0)\hat{\rho}(\tau)\right), \tag{4.297}$$

in which

$$\hat{\rho}(0) = |\alpha\rangle\langle\alpha|. \tag{4.298}$$

Therefore one can write

$$Tr(\hat{a}(\tau)|\alpha\rangle\langle\alpha|) = \int d^2\xi Q(\xi^*, \xi, \tau)\xi, \tag{4.299}$$

where at $\tau = 0$, $Q(\xi^*, \xi, \tau)$ reduce to

$$Q(\xi^*, \xi, 0) = \frac{1}{\pi} \exp[-|\xi - \alpha|^2]. \tag{4.300}$$

Thus substitution of Eq. (4.299) into Eq. (4.296) leads to

$$\langle\alpha(t+\tau)\rangle = \int d^2\alpha d^2\xi Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t) Q(\xi^*, \xi, \tau)\xi. \tag{4.301}$$

Now replacing $(\alpha^*, \alpha, \alpha_0^*, \alpha_0, t)$ by $(\xi^*, \xi, \alpha^*, \alpha, \tau)$ in the Q function (2.192), we have

$$Q(\xi^*, \xi, \tau) = \frac{A'(\tau)}{\kappa\pi} \exp\left[-p'(\tau)\xi^*\xi + q'^*(\tau)\xi + q'(\tau)\xi^* + \frac{r'(\tau)}{2}(\xi^2 + \xi^{*2})\right], \tag{4.302}$$

where A', p', q', q'^* and r' are described by expressions (2.193), (2.194), (2.195), (2.196), (2.197), (2.198) with $(\alpha_0^*, \alpha_0, t)$ replaced by (α^*, α, τ). It then follows that

$$\int d^2\xi Q(\xi^*, \xi, \tau)\xi = A'(\tau) \int \frac{d^2\xi}{\pi} \exp\left[-p'(\tau)\xi^*\xi + q'^*(\tau)\xi + q'(\tau)\xi^* + \frac{r'(\tau)}{2}(\xi^2 + \xi^{*2})\right] \xi, \tag{4.303}$$

or

$$\int d^2\xi Q(\xi^*, \xi, \tau)\xi = \frac{\partial}{\partial q'^*} A'(\tau) \int \frac{d^2\xi}{\pi} \exp\left[-p'(\tau)\xi^*\xi + q'^*(\tau)\xi + q'(\tau)\xi^* + \frac{r'(\tau)}{2}(\xi^2 + \xi^{*2})\right]. \tag{4.304}$$

In view of the relation (2.185), we find that

$$\int d^2\xi Q(\xi^*, \xi, \tau)\xi = \frac{\partial}{\partial q'^*} \frac{A'}{\sqrt{p'^2 - r'^2}} \exp\left[\frac{p'q'q'^* + (q'^2 + q'^{*2})r'/2}{p'^2 - r'^2}\right] = \frac{p'q' + r'q'^*}{p'^2 - r'^2}. \tag{4.305}$$

with the help of Eqs. (4.299) and (2.194), (2.195), (2.196), (2.197), (2.198), we readily obtains

$$Tr(\hat{a}(\tau)|\alpha\rangle\langle\alpha|) = \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}\tau}) + e^{\frac{-\mu}{2}\tau}\alpha. \tag{4.306}$$

Putting Eq. (4.306) into Eq. (4.296), one easily see that

$$\langle\alpha(t+\tau)\rangle = \int d^2\alpha Q(\alpha, \alpha + \frac{\partial}{\partial\alpha^*}, t) \left[\frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}\tau}) + e^{\frac{-\mu}{2}\tau}\alpha\right]. \tag{4.307}$$

on account of the identity (4.294), we readily see that

$$\langle\hat{A}\rangle = \int \frac{d^2\alpha}{\pi} Q(\alpha, \alpha + \frac{\partial}{\partial\alpha^*}, t) A_n(\alpha^*\alpha), \tag{4.308}$$

where $A_n(\alpha^*\alpha)$ is the c-number equivalent to $\hat{A}(\hat{a}^\dagger, \hat{a})$ for the normal-ordering.
With the help of Eq. (4.308), Eq. (4.307) can be put as

$$\langle\alpha(t+\tau)\rangle = \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}\tau}) + e^{\frac{-\mu}{2}\tau}\langle\alpha(t)\rangle. \tag{4.309}$$

Similarly one can see that

$$\langle\alpha^*(t+\tau)\rangle = \frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa}{2}\tau}) + e^{\frac{-\mu}{2}\tau}\langle\alpha^*(t)\rangle. \tag{4.310}$$

Applying the quantum regression theorem to the above expressions, we readily find that

$$\langle\alpha^*(t)\alpha^*(t+\tau)\rangle = \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}\tau})\langle\alpha^*(t)\rangle + e^{\frac{-\mu}{2}\tau}\langle\alpha^{*2}(t)\rangle, \tag{4.311}$$

$$\langle\alpha(t)\alpha^*(t+\tau)\rangle = \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}\tau})\langle\alpha(t)\rangle + e^{\frac{-\mu}{2}\tau}\langle\alpha^{2}(t)\rangle, \tag{4.312}$$

$$\langle\alpha^*(t)\alpha(t+\tau)\rangle = \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}\tau})\langle\alpha^*(t)\rangle + e^{\frac{-\mu}{2}\tau}\langle\alpha^*(t)\alpha(t)\rangle, \tag{4.313}$$

$$\langle\alpha(t)\alpha^*(t+\tau)\rangle = \frac{2\varepsilon}{\kappa}(1-e^{-\frac{\kappa}{2}\tau})\langle\alpha(t)\rangle + e^{\frac{-\mu}{2}\tau}\langle\alpha^*(t)\alpha(t)\rangle. \tag{4.314}$$

Using the above expressions, one can write Eq. (4.289) as

$$\langle\alpha_\pm(t), \alpha_\pm(t+\tau)\rangle_{ss} = \pm e^{\frac{-\mu}{2}\tau}\left(\pm\langle(\alpha^* \pm \alpha)^2\rangle \mp \langle(\alpha^* \pm \alpha)\rangle^2\right). \tag{4.315}$$

Using the fact that the normally-ordered quadrature variance of the output mode is defined by the c-number variable as

$$:(\triangle a_\pm)^2 := \pm\langle:(\alpha^* \pm \alpha):\rangle \mp \langle(\alpha^* \pm \alpha)\rangle^2, \tag{4.316}$$

one easily put Eq. (4.315) as

$$\langle\alpha_\pm(t), \alpha_\pm(t+\tau)\rangle_{ss} = \pm e^{\frac{-\mu}{2}\tau}\left(:(\triangle a_\pm)^2:\right). \tag{4.317}$$

Therefore the normally-ordered squeezing spectrum in Eq. (4.288) becomes

$$S_\pm^{out}(\omega) = \pm\frac{1}{\pi}\left(:(\triangle a_\pm)^2:\right) Re\int_0^\infty d\tau e^{-[\frac{\mu}{2}-i(\omega-\omega_o)]\tau}. \tag{4.318}$$

Upon carrying out the integration, we finally arrive at the squeezing spectrum to take the form

$$S_\pm^{out}(\omega) =: (\triangle a_\pm)^2 : \left[\frac{\mu/2\pi}{(\mu/2)^2 + (\omega-\omega_0)^2}\right]. \tag{4.319}$$

We can see from the above result that the cavity coherent light has no effect on the squeezing spectrum, this result agrees with the one obtained using the Q function derived from a single master equation for the cavity mode [5, 8]. Therefore, we can assert that the interaction between the cavity coherent photons and the three-level atoms do not change the squeezing spectrum.

As it is seen from Fig. 4.3 like the quadrature variance, the spectrum of quadrature fluctuations of the output light increased as the linear gain coefficient of the three-level laser increased.

We can see from Eqs. (4.319) and (4.284), at steady state and for $\eta = 0.2$, $A = 100$, and $\kappa = 0.8$ there is a 50% squeezing of the output light in the interval between $(\omega - \omega_0) = -10$ and $(\omega - \omega_0) = 10$, there is a 64% squeezing of the output light in the interval between $(\omega - \omega_0) = -100$ and $(\omega - \omega_0) = 100$, but it is clearly seen from Eq. (4.284) the maximum squeezing obtained is 69% below the coherent-state level, Hence we observe that most of the squeezing is concentrated in the vicinity of the central frequency. This assertion agrees with the observation from Fig. 4.4 .

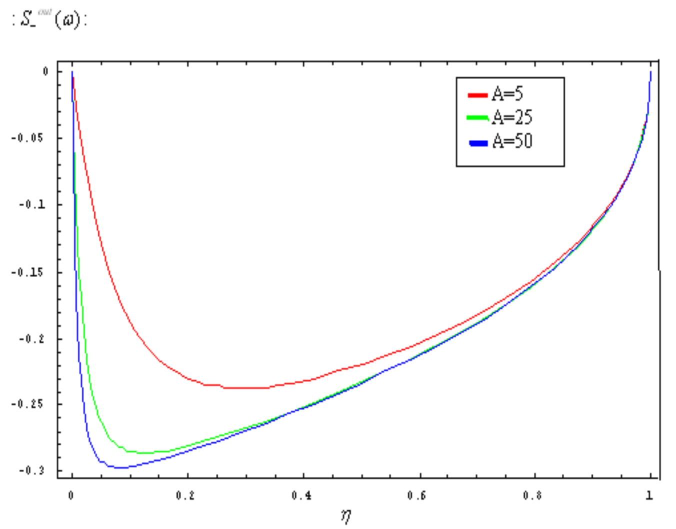

Fig. 4.3: Plots of the normally-ordered squeezing spectrum $: S_-^{out}(\omega_o) :$ **at steady state and at resonance versus** η **for** $\kappa = 0.8$ **and for different values of the linear gain coefficient.**

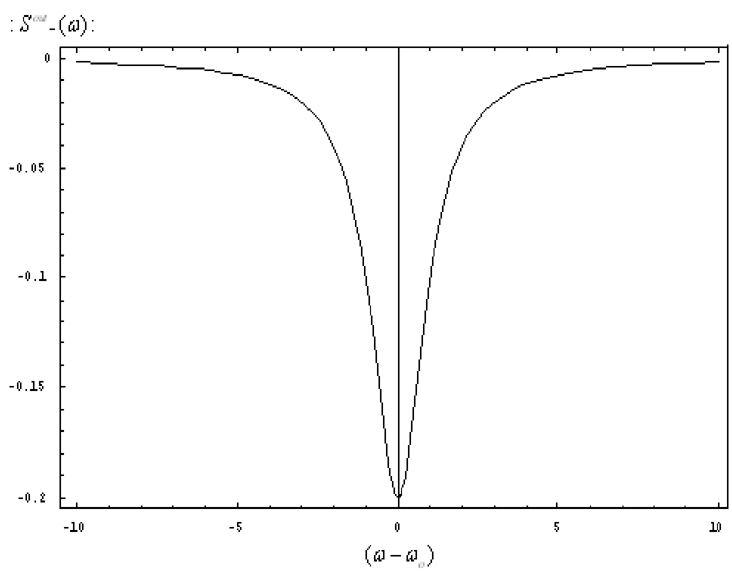

Fig. 4.4: Plots of the normally-ordered squeezing spectrum $: S_-^{out}(\omega) :$ **versus** ω **for** $\eta = 0.2, A = 100$**, and** $\kappa = 0.8$**.**

5 Conclusion

In this thesis we have studied the statistical and squeezing properties of the output light generated from a three-level laser and a coherently driven cavity mode coupled to a vacuum reservoir. Using the propagator method [1], we have obtained the Q function for the cavity coherent light as well as the light produced by a degenerate three level laser, we have then obtained the Q function for the superposition. Applying the Q function for the output light, we have calculated the mean photon number and the variance of the photon number.

It is found that the mean photon number of the output light increased when the three-level atoms do not interact with the cavity coherent photons. The absorption of the coherent cavity photons followed by spontaneous emission by the three-level atoms must be the reason for the decrease in the mean photon number when the coherent cavity photons and the three-level atoms are interacting. We have also seen that the linear gain coefficient and the driving coherent light enhance the mean photon number. We also observe that the fluctuations in the mean number of the output photons (the normally-ordered variance of the photon number) increased as the result of the driving coherent light, this is because of the increase in the mean photon number as the result the cavity coherent light.

Employing the Q function for the output light, We also have determined the quadrature variance and the squeezing spectrum of the light. The output light generated from a three-level laser and a coherently driven cavity mode is in a squeezed state and the squeezing property is unaffected by the interaction of the cavity coherent photons with the three-level atoms. And the squeezing occurs when the probability for the injected three-level atom to be in the bottom level is larger than the probability for the atom to be in the upper level ($\rho_{cc}^{(0)} > \rho_{(aa)}^{(0)}$), and better squeezed light is obtained for an increasing value of the linear gain coefficient and when $\rho_{cc}^{(0)}$ is slightly larger than $\rho_{aa}^{(0)}$. We have also seen that most of the squeezing (i.e quadrature variance) is concentrated in the vicinity of the central frequency.

Although the cavity coherent light enhance the brightness of the output light, it has no effect on the degree of squeezing.

In general, using the quantum optical system under consideration we can obtain a bright and highly squeezed light.

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university and that all sources of material used for the thesis have been duly acknowledged.

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