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GRADUATE SEMINAR REPORT ON
THE HARTMAN-GROBMAN THEOREM AND PLANAR SYSTEMS

Submitted to the Department of Mathematics in Partial Fulfillment of the
Requirements for Masters Degree of Science in Mathematics

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I. Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Solomon Balcha

Signature_____ Date_____

II. Permission

This is to certify that this project is compiled by Mr. Solomon Balcha in the Department of Mathematics, Addis Ababa University, under my supervision.

I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. Tadesse Abdi

Signature_____ Date_____

III. Abstract

The purpose of this study was to investigate the overall qualitative behavior of two dimensional linear systems such as classification of the systems from the dynamical point of view and in particular.

- ✓ To obtain insight in the classification of fixed point using trace and determinant of the coefficient matrix of planar systems;
- ✓ To develop classification criteria using trace and determinant of the coefficient matrix of the system, and also the way how to draw the trace determinant plane is discussed.
- ✓ To discuss the more subtle issue of topological equivalence (conjugacy) of these systems. Starting from simple planar linear systems and then we develop insight in investigating the relationship between two system that are topological conjugate and equivalent.
- ✓ Moreover, there is very important theorem mentioned in this project, the Hartman-Grobman theorem, which results in the local behavior theory of differential equations. The theorem shows near the hyperbolic fixed point, the non-linear system has the same qualitative structure as its linearization.

IV. Acknowledgment

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V. Definition of Key Terms

☞ A flow $\phi(t, x_0)$ is the solution of initial value problem

$$\begin{cases} X' = AX \\ X(0) = X_0 \end{cases} \quad (1)$$

☞ Equilibrium point for a system $X' = F(X)$ is a vector X_0 for which $F(X_0) = 0$.

☞ Orbit or trajectory of a point X is

$$\Gamma_X = \{\phi(t, x_0) : t \in R\} \text{ where } \phi(t, x_0) \text{ is flow of the initial value problem (1)}$$

☞ The generalized eigenspace E_k is defined as $E_k = \ker(A - \lambda_k)^{n_k}$ where λ_k is an eigenvalue a matrix A with algebraic multiplicity $n_k \geq 1$

☞ Surjective or onto function is a function $h: A \rightarrow B$ if for all $b \in B$ there exist at least one $a \in A$ such that $h(a) = b$.

☞ Injective or one to one function is a function $h: A \rightarrow B$ if $h(a) = h(a')$ implies $a = a'$ for all $a, a' \in A$

☞ Bijective function $h: A \rightarrow B$ is a function that is both surjective and injective.

☞ Homeomorphic function is a function that is continuous, bijective and has continuous inverse.

☞ Diffeomorphic function is a function that is bijective, its first derivative is continuous, and its first derivative has continuous inverse.

☞ Linearly independent a set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ in a vector space V over a field K if there exist constants $c_1, c_2, \dots, c_n \in K$ such that $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$, then $c_i = 0$ for all $i = 1, 2, \dots, n$

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VII. Introduction

This project is all about the overall behaviors of planar systems and the Hartman-Grobman theorem, which deals with the relationship between the non-linear systems and its linearization at a fixed point. The project also gives unified presentation of trace-determinant plane and topological conjugacy between two planar systems.

The project has three chapters. The first chapter is directed toward the canonical form that says the phase portrait of the system is qualitatively the same to the system with its canonical form, trace determinant theorem which gives the classification of fixed points rather than the classification as in phase portrait and the trace-determinant plane which generalizes the information in the trace-determinant theorem plotting the *TD*-plane.

However, the qualitative relationship, that of conjugacy and equivalence, between two systems is discussed in the second chapter.

Finally, the last chapter of the project deals with the Hartman-Grobman theorem classifying it into deffiomorphic version and homeomorphic version. This chapter is all about local behavior of non-linear planar systems starting from the linearization to the qualitative behavior of non-linear planar system. And then it generally says the non-linear planar systems have the same qualitative behavior with the system of its linearization.

Chapter 1

1. Planar Linear Systems and Canonical Forms

In this chapter we begin the study of systems of differential equations. A system of differential equations is a collection of n interrelated differential equations of the form

$$\begin{aligned}x_1' &= f_1(t, x_1, x_2, \dots, x_n) \\x_2' &= f_2(t, x_1, x_2, \dots, x_n) \\&\vdots \\x_n' &= f_n(t, x_1, x_2, \dots, x_n)\end{aligned}$$

Here the functions f_j , $j=1, 2, \dots, n$ are real-valued functions of the $n+1$ variables x_1, x_2, \dots, x_n and t . Unless otherwise specified, we will always assume that the f_j are C^∞ functions. This means that the partial derivatives of all orders of the f_j exist and are continuous. To simplify the notation, let

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{then our system is written as } X' = F(t, X) \text{ where}$$

$$F(t, X) = \begin{pmatrix} f_1(t, x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \dots, x_n) \end{pmatrix}$$

Definition 1: The system of equations is called autonomous if none of the f_j depends on t , so the system becomes $X' = F(X)$.

For most of the rest of this paper we will be concerned with autonomous systems.

Definition 2: A vector X_0 for which $F(X_0) = 0$ is called an equilibrium point. Sometimes it may be called critical point, fixed point, and rest point and so on.

For the remainder of this project we will deal with autonomous systems in R^2 that is planar autonomous systems, which we will write in the form

$$\begin{aligned}x' &= f(x, y) \\y' &= g(x, y)\end{aligned}$$

thus eliminating the annoying subscripts on the functions and variables. As above, we often use the abbreviated notation $X' = F(X)$ where $X = (x, y)$ and $F(X) = F(x, y) = (f(x, y), g(x, y))$

We now further restrict our attention to the most important class of planar systems of differential equations, namely, linear systems. In the autonomous case, these systems assume the simple form

$$\begin{aligned}x' &= ax + by \\y' &= cx + dy\end{aligned}$$

Where a, b, c and d are constants. We can rewrite this system as

$$X' = AX \tag{1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

And $X = (x, y)^T$ from this we also has $X' = (x', y')^T$.

Note that the origin is always an equilibrium point for a linear system. This system has a nonzero solution if and only if $\det A \neq 0$.

Now we turn to the question of finding non-equilibrium solutions of the linear system (1). The key observation here is this: Suppose V_0 is a nonzero vector for which we have $AV_0 = \lambda V_0$ where $\lambda \in \mathbb{R}$. Then the function $X(t) = e^{\lambda t} V_0$ is a solution of the system (1). To see this we can compute as

$$X'(t) = \lambda e^{\lambda t} V_0 = e^{\lambda t} (\lambda V_0) = e^{\lambda t} (AV_0) = A(e^{\lambda t} V_0) = AX(t).$$

So, $X(t)$ does indeed solve the system of equations. Such a vector V_0 and its associated scalar λ have names:

Definition 3: A nonzero vector V_0 is called an eigenvector of A if $AV_0 = \lambda V_0$ for some λ . The constant λ is called an eigenvalue of A .

We can also mention the solution of the linear system (1) together with the initial condition $X(0) = X_0$ is given by $X(t) = e^{At} X_0$ where e^{At} is a 2×2 matrix function which is defined by its Taylor series. That means for $t \in R$,

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

This is done using the definition of derivative. Since A is square matrix it commutes with itself that

$$\begin{aligned} \frac{d}{dt} e^{At} &= \lim_{h \rightarrow 0} \frac{e^{A(t+h)} - e^{At}}{h} \\ &= \lim_{h \rightarrow 0} e^{At} \frac{(e^{Ah} - I)}{h} \\ &= e^{At} \lim_{h \rightarrow 0} \lim_{k \rightarrow \infty} \left(A + \frac{A^2 h}{2!} + \dots + \frac{A^k h^{k-1}}{k!} \right) \\ &= A e^{At} \end{aligned}$$

This shows that $X'(t) = \frac{d}{dt} e^{At} X_0 = A e^{At} X_0 = AX(t)$. This solution can also be coinciding with the above mentioned solution. A good portion of this chapter is concerned with the computation of the matrix e^{At} in terms of the eigenvalues and eigenvectors of the square matrix A .

1.1 Canonical Forms

To compute the matrix e^{At} for any 2×2 matrix A , first, we consider a special case. We say that a 2×2 matrix is a diagonal matrix if it is in the form of

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

For any 2×2 matrix there is an invertible 2×2 matrix P (whose columns consist of generalized eigenvectors of A) such that the matrix $B = P^{-1}AP$ has one of the following forms

$$\text{i. } B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \text{ ii. } B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ or iii. } B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Where λ may equal to μ in the first case.

Any 2 x 2 matrix that is in one of these three forms is said to be in *canonical form*. Given any linear system $X' = AX$, we can always change coordinates to $Y' = BY$ where $B = P^{-1}AP$ using the linear transformation of coordinates $X = PY$, So that the new system's coefficient matrix B is in one of the above canonical form and hence easily solved. Clearly this property holds if a matrix A is diagonal matrix. But the question is what if matrix A is non diagonal matrix. Let us prove this as follows from eigenvalue point of view.

Case 1: Suppose the matrix A has two real, distinct eigenvalues λ_1 and λ_2 with associated eigenvectors V_1 and V_2 . Let P be the matrix whose columns are V_1 and V_2 . Thus $PE_j = V_j$ for $j = 1, 2$ where the E_j form the standard basis of R^2 i.e. ($E_1 = (1,0)^T$ and $E_2 = (0,1)^T$) also, $P^{-1}V_j = E_j$. Therefore we have

$$\begin{aligned} BE_j &= (P^{-1}AP)E_j = P^{-1}AV_j = P^{-1}(\lambda_j V_j) \\ &= \lambda_j P^{-1}V_j \\ &= \lambda_j E_j \end{aligned}$$

Thus the matrix B assumes the canonical form

$$B = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and the corresponding system is easy to solve

Case 2: Suppose A has a single real eigenvalue λ that means A has repeated eigenvalues. To proof this case let us assume that V is an eigenvector and that every other eigenvector is a multiple of V . Let W be any vector for which V and W are linearly independent. Then we have $AW = \mu V + \nu W$ for some constants $\mu, \nu \in R$. Note that $\mu \neq 0$, for otherwise we would have a second linearly independent eigenvector W with eigenvalue ν . We claim that $\nu = \lambda$. If $\nu - \lambda \neq 0$, a computation shows that

$$A\left(W + \left(\frac{\mu}{\nu - \lambda}\right)V\right) = \nu\left(W + \left(\frac{\mu}{\nu - \lambda}\right)V\right).$$

This says that ν is a second eigenvalue different from λ . Hence we must have $\nu = \lambda$.

Finally, let $U = (1/\mu)W$. Then

$$AU = V + (\lambda/\mu)W = V + \lambda U.$$

Thus if we define $PE_1 = V$, $PE_2 = U$, we get

$$B = P^{-1}AP = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Case 3: Now suppose that the matrix A has complex eigenvalues $\alpha \pm i\beta$ with $\beta \neq 0$. Then we may find a complex eigenvector $V_1 + iV_2$ corresponding to $\alpha + i\beta$, where both V_1 and V_2 are real vectors. We claim that V_1 and V_2 are linearly independent vectors in R^2 . If this were not the case, then we would have $V_1 = cV_2$ for some $c \in R$. But then we have

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2) = (\alpha + i\beta)(c + i)V_2.$$

But we also have

$$A(V_1 + iV_2) = (c + i)AV_2.$$

So we conclude that $AV_2 = (\alpha + i\beta)V_2$. This is a contradiction since the left-hand side is a real vector while the right is complex.

Since $V_1 + iV_2$ is an eigenvector associated to a $\alpha + i\beta$, we have

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2)$$

Equating the real and imaginary components of this vector equation, we find

$$AV_1 = \alpha V_1 - \beta V_2$$

$$AV_2 = \beta V_1 + \alpha V_2.$$

Let P be the matrix whose columns are V_1 and V_2 . Hence $PE_j = V_j$ for $j = 1, 2$. Now consider $P^{-1}AP$. We have

$$\begin{aligned} (P^{-1}AP)E_1 &= P^{-1}(\alpha V_1 - \beta V_2) \\ &= \alpha E_1 - \beta E_2 \end{aligned}$$

and similarly

$$(P^{-1}AP)E_2 = \beta E_1 + \alpha E_2$$

Thus the matrix $P^{-1}AP$ is in the canonical form

$$B = P^{-1}AP = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

This complete the proof and hence any 2 x 2 matrix A has one of the above given canonical forms.

Example: Suppose

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

Then find the general solution of the system $X' = AX$.

Solution: the characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$ which yields eigenvalues $\lambda = -1$ and $\lambda = -2$. an eigenvector corresponding to $\lambda = -1$ is given by solving

$$(A - \lambda I)V = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields an eigenvector $V_1 = (1,1)$. Similarly an eigenvector associated to $\lambda = -2$ is given by $V_2 = (0,1)$. To put this system in canonical form, we choose P to be the matrix whose columns are these eigenvectors:

$$P = [V_1, V_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ so that } P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Finally, we compute

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

So $P^{-1}AP$ is in canonical form. The general solution of the system $Y' = BY$ where $B = P^{-1}AP$ is

$$Y(t) = \alpha e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So the general solution of $X' = AX$ is

$$\begin{aligned} PY(t) &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\alpha e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \alpha e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Note that we really do not have to go through the step of converting a specific system to one in canonical form; once we have the eigenvalues and eigenvectors, we can simply write down the general solution. We take this extra step because, when we attempt to classify all possible linear systems, the canonical form of the system will greatly simplify this process.

Proposition: if a matrix A is similar to a matrix B (there exist an invertible matrix P such that $B = P^{-1}AP$), then $e^B = P^{-1}e^A P$.

Proof: this follows from the definition of e^A that

$$e^B = e^{P^{-1}AP} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(P^{-1}AP)^k}{k!} = P^{-1} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{A^k}{k!} P = P^{-1} e^A P$$

This proposition shows that a matrix e^A is also belong to one of the canonical forms discussed earlier. Now we are going to show this as follows.

Proof :

Case 1: when

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

Then by definition

$$e^A = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{bmatrix}}{k!} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{\lambda^k}{k!} & 0 \\ 0 & \frac{\mu^k}{k!} \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \end{bmatrix} = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{bmatrix}$$

Case 2: when

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

We can write

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda I + B \text{ Letting } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then λI commutes with B and $e^A = e^{\lambda I + B} = e^{\lambda I} e^B = e^\lambda e^B$ and from the definition of exponential matrix we have $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$, $e^B = I + B + \frac{B^2}{2!} + \dots = I + B$ thus

$$e^A = e^\lambda (I + B) = e^\lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ since by direct computation } B^2 = B^3 = \dots = 0$$

Case 3: when

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \operatorname{Re}(\lambda^k) & -\operatorname{Im}(\lambda^k) \\ \operatorname{Im}(\lambda^k) & \operatorname{Re}(\lambda^k) \end{bmatrix}$$

Where Re and Im denote the real and imaginary parts of the complex number λ respectively. Thus

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \operatorname{Re}(\frac{\lambda^k}{k!}) & -\operatorname{Im}(\frac{\lambda^k}{k!}) \\ \operatorname{Im}(\frac{\lambda^k}{k!}) & \operatorname{Re}(\frac{\lambda^k}{k!}) \end{bmatrix}}{k!} \\ &= \begin{bmatrix} \operatorname{Re}(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}) & -\operatorname{Im}(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}) \\ \operatorname{Im}(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}) & \operatorname{Re}(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}) \end{bmatrix} \\ &= \begin{bmatrix} \operatorname{Re}(e^\lambda) & -\operatorname{Im}(e^\lambda) \\ \operatorname{Im}(e^\lambda) & \operatorname{Re}(e^\lambda) \end{bmatrix} \\ &= \begin{bmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{bmatrix} \end{aligned}$$

1.2 Topological Type of Phase Portraits

Let us start this portion by defining phase portrait which is a picture of a collection of representative solution curves of the system in R^2 , which we call the phase plane.

In this section we discuss the various phase portraits that are possible for the linear planar system.

$$X' = AX \quad (1)$$

Where $X \in R^2$ and A is a 2×2 matrix. We begin by describing the phase portraits for the linear system

$$Y' = BY \quad (2)$$

where the matrix $B = P^{-1}AP$ has one of the canonical forms given in the earlier topic. The phase portrait for the linear system (1) above is then obtained from the phase portrait for (2) under the linear transformation of coordinates $X = PY$.

First of all, if

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ or } B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

Then the solution of the initial value problem (2) with $X(0) = X_0$ is given by

$$X(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} X_0, \quad X(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} X_0, \quad \text{or } X(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} X_0$$

respectively. We now list the various phase portraits that result from these solutions, grouped according to their topological type:

Case I: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda < 0 < \mu$.

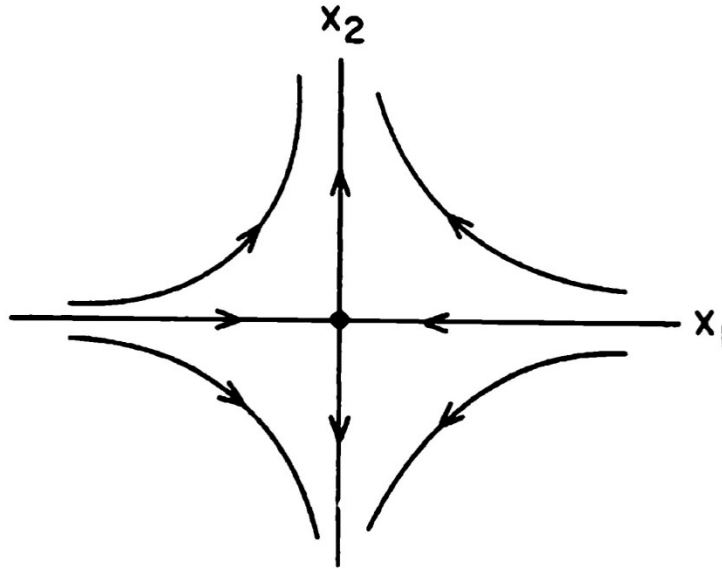


Figure 1.1: A saddle at the origin.

The equilibrium point of a system (2) of this type (eigenvalues satisfying $\lambda < 0 < \mu$) is called a saddle. If $\mu < 0 < \lambda$, the arrows in Figures 1.1 are reversed. Whenever A has two real eigenvalues of opposite sign, $\lambda < 0 < \mu$, the phase portrait for the linear system (1) is linearly equivalent to the phase portrait shown in Figure 1.1; i.e., it is obtained from Figure 1.1 by a linear transformation of coordinates.

Case II: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ and

The phase portraits for the linear system (2) in these cases are given in Figure 1.2 below.

I. $\lambda < \mu < 0$

The phase portrait for this system is displayed in Figure 1.2 (a). In this case the equilibrium point is called a sink. Whenever A has two negative eigenvalues $\lambda < \mu < 0$, the phase portrait of the linear system (1) is linearly equivalent to phase portraits shown in figure 1.2 (a) below.

II. $0 < \lambda < \mu$

In this case the equilibrium point is called a source. Our vector field may be regarded as the negative of the previous case $\lambda < \mu < 0$. The general solution and phase portrait remain the same, except that all solutions now tend away from the origin along the same paths as displayed on figure 1.2 (b) below.

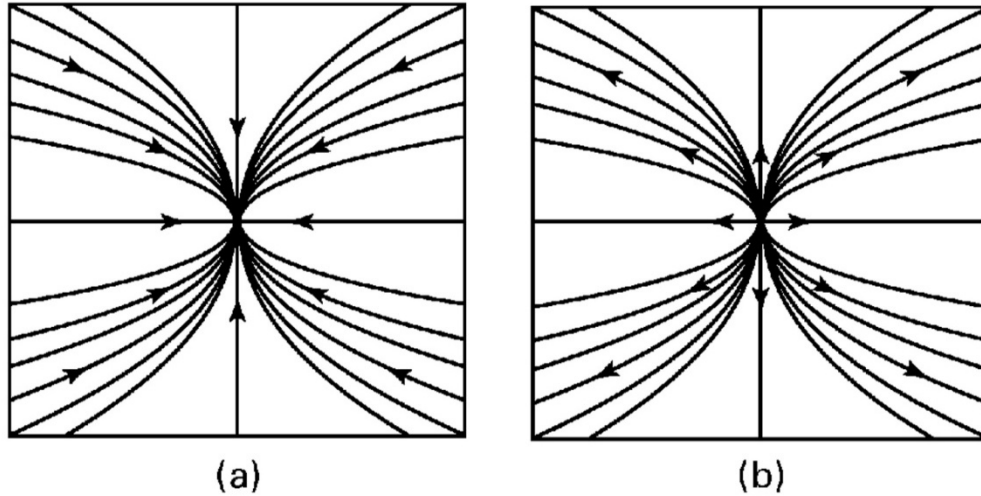


Figure 1.2: Phase portraits for (a) a sink and (b) a source.

Case III: $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a, b \neq 0$.

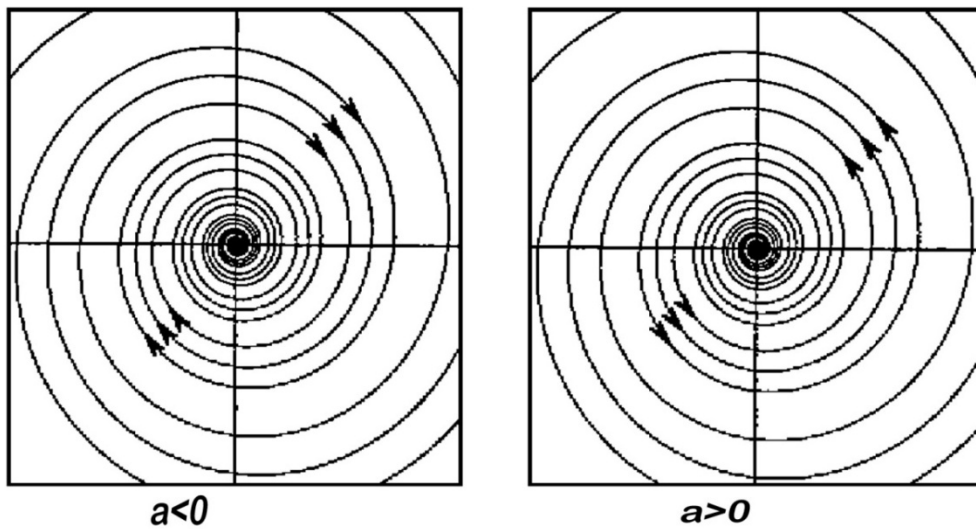


Figure 1.3: A Phase portraits for a spiral sink and a spiral source at origin.

The phase portrait for the linear system (2) case is given in Figure 1.3. If $a < 0$, the trajectories are spiral into the origin with increasing t and spiral away from the origin if $a > 0$. In these cases the equilibrium point is called a spiral sink or spiral source, respectively. Whenever A has a pair of complex conjugate eigenvalues with nonzero real

part, $a \pm ib$, with $a, b \neq 0$, the phase portraits for the system (1) is linearly equivalent to one of the phase portraits shown in Figure 1.3 above.

Case IV: $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ and $b \neq 0$.

The phase portrait for the linear system (2) in this case is given in figure 1.4. The equilibrium point of this system is called center. Whenever A has a pair of pure imaginary complex conjugate eigenvalues, $\pm ib$, the phase portrait of the linear system (1) is linearly equivalent to one of the phase portraits shown in figure 1.4 below.

Note that the trajectories or solution curves in Figure 1.4 lie on circles $|X(t)| = \text{constant}$ centered at origin. These circles are traversed in the clockwise direction if $b > 0$, counterclockwise if $b < 0$

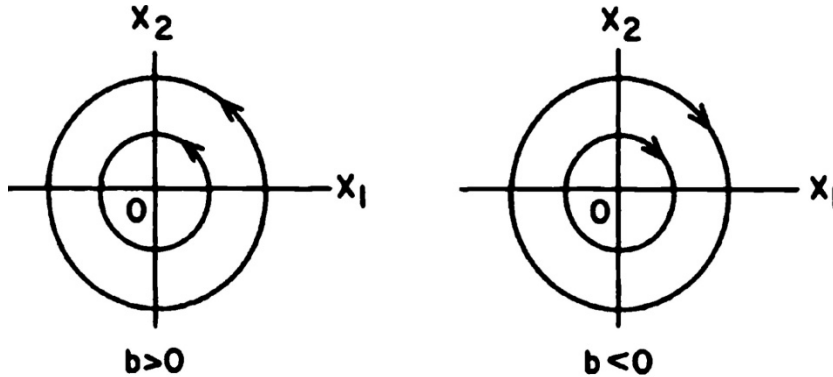


Figure 1.4: A center at origin

1.3 Trace-Determinant Theorem

The matrix A is similar to the matrix B , i.e., if there is a nonsingular matrix P such that $B = P^{-1}AP$, then the system (1) is transformed into the system (2) by the linear transformation of coordinates $X = PY$. If B has the form III, then the phase portrait for the system (2) consists of either a counterclockwise motion (if $b > 0$) or a clockwise motion (if $b < 0$) on either circles (if $a = 0$) or spirals (if $a \neq 0$). Furthermore, the direction of rotation of trajectories in the phase portraits for the systems (1) and (2) will be qualitatively the same if $\det P > 0$ (i.e., if P is orientation preserving) and it will be opposite if $\det P < 0$ (i.e., if P is orientation reversing). In either case, the two systems (1) and (2) are qualitatively the same.

For $\det A \neq 0$ there is an easy method for determining if the linear system has a saddle, sink, source or center at the origin. This is given in the next theorem. Before starting the classification of autonomous planar linear system using trace determinant let us see some preliminaries from linear algebra.

Taking the coefficient matrix of the system (1)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. The quantity $a + d$ is called Trace of A and is denoted by $\text{tr}A$ and
2. The quantity $ad - bc$ is the determinant of A and denoted by $\det A$

The characteristic equation for the coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\det(A - \lambda I) = 0$ where λ is the eigenvalue, the roots of the characteristic equation $\det(A - \lambda I) = 0$, of A and I is 2×2 identity matrix. Thus the characteristic equation of the coefficient matrix A will be

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

Here we can rewrite the last equation as $\lambda^2 - (\text{tr}A)\lambda + (\det A) = 0$ and, letting by $\text{tr}A = T$

and $\det A = D$ the roots can be given by $\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$ that means

$$\lambda_+ = \frac{T + \sqrt{T^2 - 4D}}{2} \text{ and } \lambda_- = \frac{T - \sqrt{T^2 - 4D}}{2}. \text{ From this one can simply find the following}$$

facts:

1. $\lambda_+ + \lambda_- = T$ which means the sum of eigenvalues of A is equal to $\text{tr}A$ and
2. $\lambda_+ \cdot \lambda_- = D$ Implies the multiplication of the eigenvalues of A is equal to $\det A$.

Theorem (Trace-Determinant): let $\text{tr}A = T$ and $\det A = D$ and consider the linear system

$$X' = AX \tag{1}$$

1. If $T^2 - 4D > 0$ and
 - a. If $D < 0$, then (1) has a saddle equilibrium point.
 - b. If $D > 0$ and $T < 0$, then (1) has a sink equilibrium point.
 - c. If $D > 0$ and $T > 0$ then (1) has source equilibrium point.
2. If $T^2 - 4D < 0$ and
 - a. If $T < 0$, then the equilibrium point is spiral sink.
 - b. If $T > 0$, then the equilibrium point of (1) is spiral source and.
 - c. If $T = 0$, then (1) has the equilibrium of center type.

Proof: the eigenvalues of the matrix A are given by

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

1. let $T^2 - 4D > 0$ this shows there are two distinct real eigenvalues.
 - a. If $D < 0$, for whatever value of T , then $\lambda_+ \cdot \lambda_- < 0$ this shows that the sign of λ_+ and λ_- should not be the same this is followed by $\lambda_{\pm} < 0 < \lambda_{\mp}$. Thus the equilibrium point is saddle.
 - b. If $D > 0$ and $T < 0$, then $\lambda_+ \cdot \lambda_- > 0$ and $\lambda_+ + \lambda_- < 0$ we see that λ_+ and λ_- must have the same sign and from $\lambda_+ + \lambda_- < 0$ we get that λ_+ and λ_- are less than zero because they must have the same sign and they are distinct. Therefore $\lambda_{\pm} < \lambda_{\mp} < 0$ and hence the given equilibrium point is sink.

- c. If $D > 0$ and $T > 0$, then $\lambda_+ \cdot \lambda_- > 0$ and $\lambda_+ + \lambda_- > 0$ this means λ_+ and λ_- has the same sign and $\lambda_+ + \lambda_- > 0$ from this we can also deduce that λ_+ and λ_- are greater than zero since they must has the same sign and they are also distinct. Hence the equilibrium point is source.
2. Let $T^2 - 4D < 0$ which shows the eigenvalues λ_{\pm} are complex whose real part is $T/2$.
- If the real part $T/2 < 0$, then $T < 0$ and the equilibrium point is spiral sink.
 - If $T > 0$, then the equilibrium point is of type spiral source and
 - If $T = 0$, then the equilibrium point is of the center type.

This completes the proof of the theorem

1.4 Trace-Determinant Plane

Plotting all of the above verbal information in the TD -plane gives us a visual summary of all of the different types of linear systems. The equations above partition the TD -plane into various regions in which systems of a particular type reside. See Figure 1.4 below this yields a geometric classification of 2×2 linear systems.

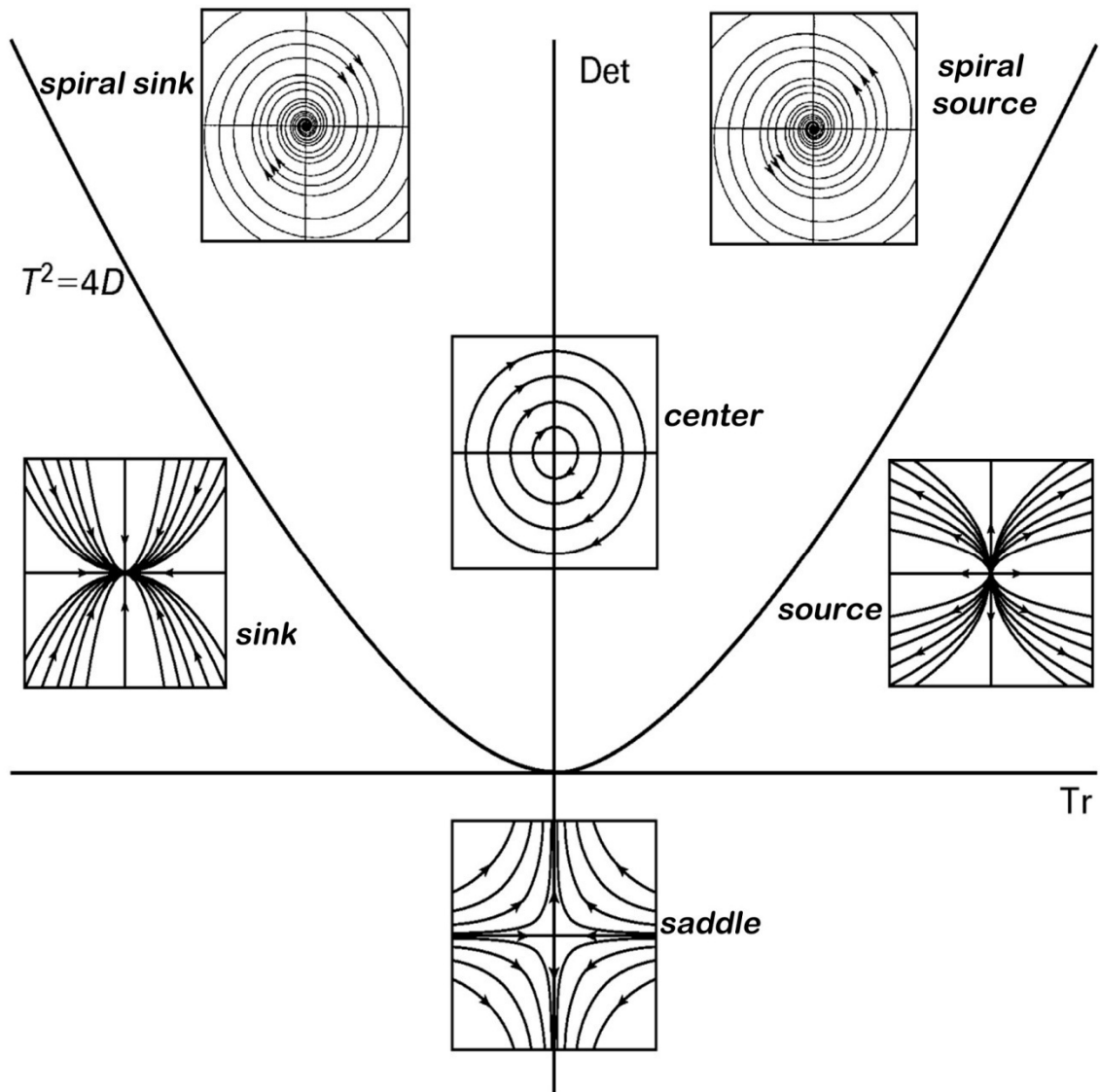


Figure 1.5 The trace-determinant plane.

Remarks

1. TD -plane is a two dimensional representation of what is really a four dimensional space, since 2×2 matrices are determined by four parameters, the entries of the matrix. Thus there are infinitely many different matrices to each point in the TD -plane. These all matrices share the same Eigenvalue configuration.
2. We also think of the trace determinant plane as the analog of the bifurcation diagram for planar linear system. A one parameter family of linear systems corresponds to a curve in the TD-plane. When this curve crosses the T-axis, the positive D-axis, or the parabola $T^2 - 4D = 0$, the phase portrait of the linear system undergoes a bifurcation: A major change occurs in the geometry of the phase portrait.
3. Finally, note that we may obtain quite a bit of information about the system from D and T without ever computing the eigenvalues. For example, if $D < 0$, we know that we have a saddle at the origin. Similarly, if both D and T are positive, then we have a source at the origin.

Example 1:

Consider the one parameter family of linear systems given by

$$X' = \begin{bmatrix} a & \sqrt{2} + (a/2) \\ \sqrt{2} - (a/2) & 0 \end{bmatrix} X$$

Sketch the path traced out by this family of linear systems in the trace determinant plane as a varies.

Solution: let

$$A = \begin{bmatrix} a & \sqrt{2} + (a/2) \\ \sqrt{2} - (a/2) & 0 \end{bmatrix}$$

First let us find the eigenvalues of A . That is computing the characteristic equation $\lambda^2 - a\lambda + (-2 + a^2/4) = 0$ then we get the roots $\lambda_+ = \sqrt{2} + a/2$ and $\lambda_- = -\sqrt{2} + a/2$

That is $\text{tr}(A) = a$ and $\det(A) = -2 + a^2/4$. Thus

$T^2 - 4D = a^2 - 4(-2 + a^2/4) = 8 > 0$. This shows that the eigenvalues of A are distinct and real. Now let us consider the following cases:

Case 1: Let $D < 0$ that is $-2 + a^2/4 < 0 \Rightarrow -2\sqrt{2} < a < 2\sqrt{2}$

Hence the equilibrium point is a saddle if $-2\sqrt{2} < a < 2\sqrt{2}$.

Case 2: Let $D > 0$ and $T < 0$ that is $|a| > 2\sqrt{2}$ and $T = a < 0$ thus combining

these we have $a < -2\sqrt{2}$. Hence the equilibrium point is a sink if $a < -2\sqrt{2}$.

Case 3: $D > 0$ and $T > 0 \Rightarrow |a| > 2\sqrt{2}$ and $a > 0 \Rightarrow a > 2\sqrt{2}$. Therefore if

$a > 2\sqrt{2}$ the equilibrium point will be a source.

We can generalize the above classification in the following trace-determinant plane.

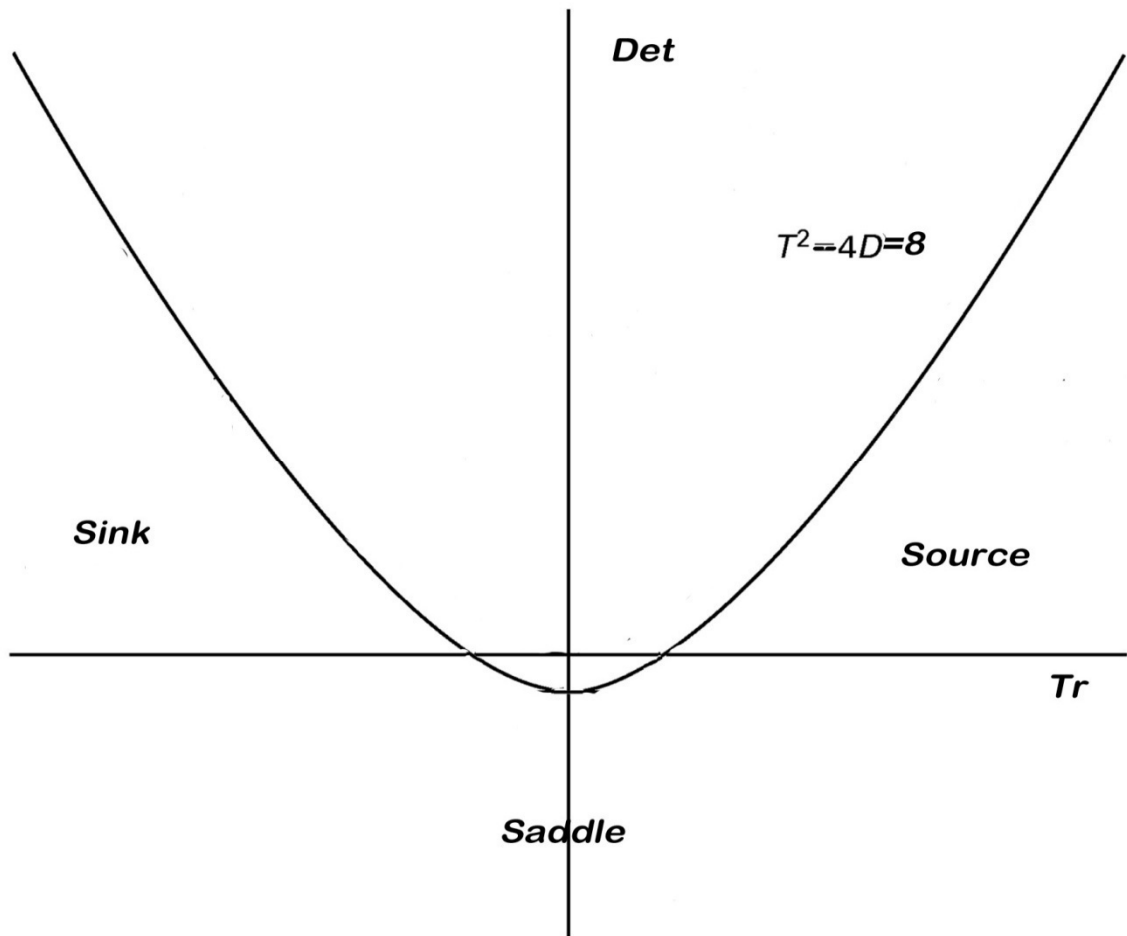


Figure 1.6 Trace determinant plane.

Example 2:

Sketch the analog of the trace determinant plane for the two parameters family of systems

$X' = \begin{bmatrix} a & b \\ b & a \end{bmatrix} X$ in the ab -plane. That is, identify the regions in the ab - plane where

this system has similar phase portraits.

Solution: let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ here the characteristic equation is $\det(A - \lambda I) = 0$

This implies $\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = 0$ computing this determinant and equating it

zero we have $\lambda^2 - 2a\lambda + a^2 - b^2 = 0$ Which means that $T = 2a$ and

$D = a^2 - b^2$. Thus $T^2 - 4D = (2a)^2 - 4(a^2 - b^2) = 4b^2$. Now we are going to consider the following cases.

Case 1: let $b = 0$. This means $T^2 - 4D = 0 \Rightarrow \lambda_{\pm} = a$. This shows that we have a repeated Eigenvalue.

Case 2: let $b \neq 0$ from this we have $T^2 - 4D = 4b^2 > 0$ which means that our eigenvalues are real and distinct.

Case 2.1 let $D = a^2 - b^2 < 0$ this implies $a^2 < b^2$. Here we have saddle equilibrium points.

Case 2.2 let $a^2 - b^2 > 0$ and $a < 0$. This implies that $a^2 > b^2$ and $a < 0$. And now the equilibrium point is sink.

Case 2.3 let $a^2 - b^2 > 0$ and $a > 0$ that is $a^2 > b^2$ and $a > 0$. And this shows we have source type equilibrium point. Now this time using the above mentioned information we can simply plot the (a, b) graph as follows.

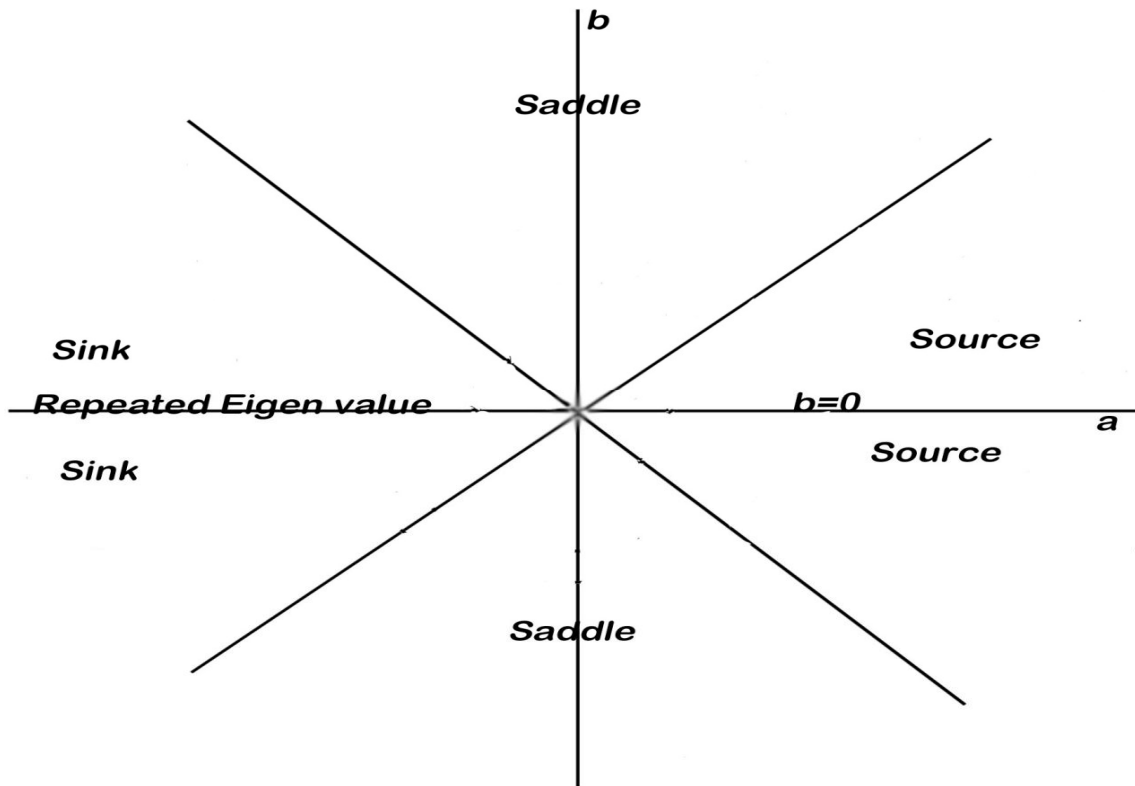


Figure 1.7 trace determinant plane for the two parameters family of systems.

Chapter 2

2 Classification of Planar Linear System

This section is concerned with dynamical classification of planar linear systems from dynamical point of view. A dynamical system is a phenomenon that changes with time, or a physical setting together with rules for how the settings changes or evolves from one moment of time to the next.

One basic goal of mathematical theory of dynamical system is to determine or to characterize the long term behavior of the system. Most commonly, dynamical systems are described mathematically in terms of linear systems of differential equation. A realistic model however often leads to none linear systems of differential equations. But in this chapter we will focus only on planar linear systems.

We are usually interested primarily in the long term behavior of solutions of differential equations. Thus two systems are equivalent if their solutions share the same fate or events.

2.1 Initial Value Problems and Flows

Consider the following pair of equation

$$\begin{cases} X' = AX \\ X(0) = X_0 \end{cases} \quad (1)$$

where the second equation is called an initial condition. If the differential equation is defined as equation $X' = AX$ with initial condition, then the pair of equations is called an *initial value problem*.

In chapter one we saw that the solution of any linear system (1) is given by $X(t) = e^{At} X_0$ is defined for all $t \in R$ through each point X_0 in the phase space R^2 . A solution of the system that depends on both time t and initial value x_0 denoted by $\phi(t, x_0)$ or simply $\phi_t(x_0)$ is called flow of the differential equation. Here the function $\phi : RXR \rightarrow R$ is called the time t map of the flow. That is flow $\phi_t(x_0) = e^{At} x_0$ is the solution of initial value problem.

The mapping $\phi_t = e^{At}$ satisfies the following basic properties for all $X \in R^2$:

- i. $\phi_0(x) = x$
- ii. $\phi_s(\phi_t(x)) = \phi_{s+t}(x)$ for all $s, t \in R$
- iii. $\phi_{-t}(\phi_t(x)) = \phi_t(\phi_{-t}(x)) = x$ for all $t \in R$

Proof: let $\phi_t = e^{At}$ be flows of the system (1)

- i. From definition of exponential matrix we have $\phi_0(x) = e^0 x = x$
- ii. $\phi_s(\phi_t(x)) = \phi_s(e^{At} x) = e^{As} (e^{At} x) = e^{As} e^{At} x = e^{A(s+t)} x = \phi_{s+t}(x)$ for all $s, t \in R$
- iii. $\phi_{-t}(\phi_t(x)) = \phi_{-t}(e^{At} x) = e^{A(-t)} (e^{At} x) = e^{A(0)} x = x = \phi_0(x)$ for all $t \in R$

2.2 Hyperbolic Systems and Conjugacy

Before we will go to classify the system we have to be familiar with the following terminologies:

- ✓ A function h is a homeomorphism if it is continuous, bijective and has a continuous inverse.
- ✓ A function h is a diffeomorphism if its derivative h' is continuous, bijective and has a continuous inverse.

Example 1: let $h: R \rightarrow R$ defined by $h(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases}$ whose inverse is given by

$h^{-1}(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{|x|}, & x \leq 0 \end{cases}$ is a homeomorphism but not a diffeomorphism. Note that

$h^{-1}(x)$ is not differentiable at 0 .

We will consider two systems to be dynamically equivalent if there is a function h that takes one flow to other where this function is homeomorphism.

Definition 2.1 Two flows ϕ_t and φ_t are topologically equivalent if there exist a homeomorphism function h that takes flow ϕ_t to flow φ_t while preserving the direction of time.

Here such homeomorphism function h is called equivalence.

Definition 2.2: Suppose $X' = AX$ and $X' = BX$ have flows ϕ_t and φ_t . These two systems are (Topologically) *conjugate* if there exists a homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that satisfies

$$\varphi_t(h(x_0)) = h(\phi_t(x_0))$$

The homeomorphism h is called *conjugacy*. Thus conjugacy takes the solution curves of $X' = AX$ to those of $X' = BX$.

We can also rewrite the above definition as $\varphi_t(h(x_0)) = h(\phi_t(x_0))$ as $\varphi_t \circ h = h \circ \phi_t$ or $\phi_t = h^{-1} \circ \varphi_t \circ h$.

Example 2: for one dimensional linear differential equations $x' = \lambda_1 x$ and $x' = \lambda_2 x$ we have the flows $\phi_t(x_0) = x_0 e^{\lambda_1 t}$ and $\varphi_t(x_0) = x_0 e^{\lambda_2 t}$ respectively. Suppose that λ_1 and λ_2 are

nonzero and have the same sign. Then let $h(x) = \begin{cases} x^{\lambda_2/\lambda_1}, & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1}, & \text{if } x < 0 \end{cases}$ where we recall that

$x^{\lambda_2/\lambda_1} = \exp\left(\frac{\lambda_2}{\lambda_1} \log(x)\right)$. Note that h is a homeomorphism of the real line. We claim that h

is a conjugacy between $x' = \lambda_1 x$ and $x' = \lambda_2 x$. To see this, we consider the following two cases.

Case 1: let $x_0 > 0$ thus

$$h(\phi_t(x_0)) = h(x_0 e^{\lambda_1 t}) = (x_0 e^{\lambda_1 t})^{\lambda_2/\lambda_1} = x_0^{\lambda_2/\lambda_1} e^{\lambda_2 t} = \varphi_t\left(x_0^{\lambda_2/\lambda_1}\right) = \varphi_t(h(x_0))$$

Case 2: let $x_0 < 0$ thus

$$h(\phi_t(x_0)) = h(x_0 e^{\lambda_1 t}) = -|x_0 e^{\lambda_1 t}|^{\lambda_2/\lambda_1} = -|x_0^{\lambda_2/\lambda_1}| e^{\lambda_2 t} = \varphi_t\left(-|x_0^{\lambda_2/\lambda_1}|\right) = \varphi_t(h(x_0))$$

Therefore from the above two cases we see that h is conjugacy and hence the systems $x' = \lambda_1 x$ and $x' = \lambda_2 x$ are topologically conjugate. There are several things to note here.

1. λ_1 and λ_2 must have the same sign because if their sign is different $\lambda_2/\lambda_1 < 0$ which results in $h(0)$ which is undefined or infinite. This means h is not a homeomorphism.
2. If λ_1 and λ_2 have the same sign, then their solutions behave similarly. That is both tend to the origin and both tend away from the origin which agrees with our notion of dynamical equivalence.
3. If $\lambda_2 < \lambda_1$ h is not differentiable at origin, if $\lambda_2 > \lambda_1$, then h^{-1} is not differentiable at origin and if $\lambda_2 = \lambda_1$, then h is differentiable. This note has no problem here because we are not requiring that h is being a diffeomorphism.

Example 3: Consider the linear systems $X' = AX$ and $Y' = BY$ with

$$A = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}.$$

Let $H(X) = RX$ where the matrix

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } R^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then $B = RAR^{-1}$ and letting $Y = H(X) = RX$ or $X = R^{-1}Y$ gives us

$$Y' = RAR^{-1}Y = BY.$$

Thus, if $X(t) = e^{At}X_0$ is the solution of the first system through X_0 , then

$$Y(t) = H(X(t)) = RX(t) = Re^{At}X_0 = e^{Bt}RX_0$$

is the solution of the second system through RX_0 ; i.e., H maps flows of the first system onto flows of the second system and it preserves the parameterization since.

$$He^{At} = e^{Bt}H$$

The mapping $H(X) = RX$ is simply a rotation through 45° and it is clearly a homeomorphism. The phase portraits of these two systems are shown in Figure 2.1 below

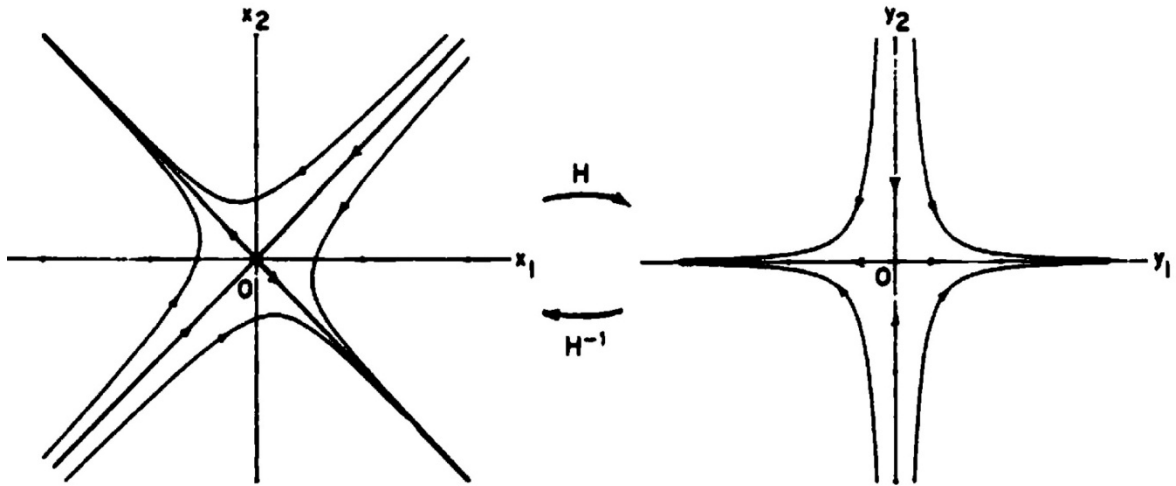


Figure 2.1 the rotation of phase portrait

Given two equilibrium points, x_{01} and x_{02} associated with diffeomorphisms h_1 and h_2 respectively, the equilibrium points are of the same topological type if the associated diffeomorphisms are topologically conjugate. Similarly, two equilibrium points of flows are of the same topological type if the flows are topologically equivalent. This implies that if h is a conjugacy then it is automatically an equivalence, but the converse is not necessary true.

Theorem 1.1: (one dimensional equivalence)

Two flows ϕ_t and φ_t in R are topologically equivalent if and only if their equilibrium points can be put into one to one correspondence and have the same topological type (sink, source or saddle).

This gives a classification of (autonomous) linear first-order differential equations. There are three conjugacy "classes" the sinks, the sources, and the special "in-between" case, $x' = 0$, where all solutions are constants.

Definition 2.3: A matrix A is hyperbolic if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is hyperbolic.

Theorem 1.2: Suppose that the 2×2 matrices A_1 and A_2 are hyperbolic. Then the linear systems $X' = A_i X, i = 1, 2$ are conjugate if and only if each matrix has the same number of eigenvalues with negative real part.

Remark: Actually, the inverse result is also valid, i.e. one can write “if and only if” in the above Theorem. However, this fact is only of theoretical interest, since it is rarely known a priori that two linear ODE systems are topologically conjugate.

Thus two hyperbolic matrices yield conjugate linear systems if both sets of eigenvalues fall into the same category below:

1. One eigenvalue is positive and the other is negative;
2. Both eigenvalues have negative real parts;
3. Both eigenvalues have positive real parts.

Before proving this theorem, note that this theorem implies that a system with a spiral sink is conjugate to a system with a (real) sink. Of course! Even though their phase portraits look very different, it is nevertheless the case that all solutions of both systems share the same fate: They tend to the origin as $t \rightarrow \infty$.

Proof: Assume all systems are in canonical form. Then the proof divides into three distinct cases.

Case I: Suppose we have two linear systems $X' = A_i X$ for $i = 1, 2$ such that each A_i has eigenvalues $\lambda_i < 0 < \mu_i$. Thus each system has a saddle at the origin. This is the easy case.

As we saw previously, the real differential equations $x' = \lambda_i x$ have conjugate flows via the homeomorphism.

$$h_1(x) = \begin{cases} x^{\lambda_2/\lambda_1}, & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1}, & \text{if } x < 0 \end{cases}$$

Similarly, the equations $y' = \mu_i y$ also have conjugate flows via an analogous function h_2

Now define $H(x, y) = (h_1(x), h_2(y))$. Then one can check immediately that H provides a conjugacy between these two systems.

Case II: Consider the system $X' = AX$ where A is in canonical form with eigenvalues that have negative real parts. We further assume that the matrix A is not in the form

$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda < 0$. Thus, in canonical form, A assumes one of the following two forms

$$\text{a) } \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{b) } \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $\alpha, \lambda, \mu < 0$ We will show that, in either case, the system is conjugate to $X' = BX$ where

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

It then follows that any two systems of this form are conjugate. Consider the unit circle in the plane parameterized by the curve $0 \leq \theta \leq 2\pi$. We denote this circle by S' . We first claim that the

vector field determined by a matrix in the above form must point inside S' . In case (a), we have that the vector field on S' is given by

$$AX(\theta) = \begin{pmatrix} \alpha \cos \theta + \beta \sin \theta \\ -\beta \cos \theta + \alpha \sin \theta \end{pmatrix}$$

The outward pointing normal vector to S' at $X(\theta)$ is

$$N(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

The dot product of these two vectors satisfies

$$AX(\theta) \cdot N(\theta) = \alpha(\cos^2 \theta + \sin^2 \theta) < 0$$

Since $\alpha < 0$. This shows that $AX(\theta)$ does indeed point inside S' . Case (b) is even easier.

As a consequence, each nonzero solution of $X' = AX$ crosses S' exactly once. Let ϕ_t denote the time t map for this system, and let $\tau = \tau(x, y)$ denote t the time at which $\phi_t(x, y)$ meets S' . Thus $|\phi_\tau(x, y)| = 1$. Let φ_t denote the time t map for the system $X' = BX$. Clearly, $\varphi_t(x, y) = (e^{-t}x, e^{-t}y)$.

We now define a conjugacy H between these two systems. If $(x, y) \neq (0, 0)$

let $H(x, y) = \varphi_{-\tau(x, y)} \phi_{\tau(x, y)}$ and set $H(0, 0) = (0, 0)$ Geometrically, the value of $H(x, y)$ is given by the following solution curve of $X' = AX$ exactly $\tau = \tau(x, y)$ time units (forward or backward) until the solution reaches S' , and then following the solution of $X' = BX$ starting at that point on S' and proceeding in the opposite time direction exactly τ time units. See the Figure below.

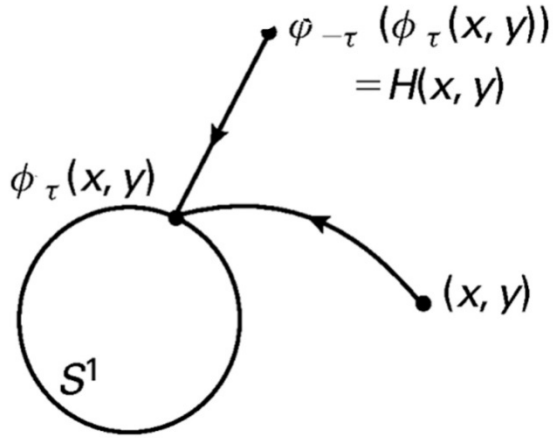


Figure 2.2 Definition of $\tau(x, y)$

To see that H gives a conjugacy, note first that $\tau(\phi_s(x, y)H(x, y)) = \tau(x, y) - s$ since

$$\phi_{\tau-s}\phi_s(x, y) = \phi_\tau(x, y) \in S^1$$

Therefore we have

$$\begin{aligned} H(\phi_s(x, y)) &= \phi_{-\tau+s}\phi_{\tau-s}(\phi_s(x, y)) \\ &= \phi_s\phi_{-\tau}\phi_\tau(x, y) \\ &= \phi_s(H(x, y)) \end{aligned}$$

So H is a conjugacy.

Now we show that H is a homeomorphism. We can construct an inverse for H by simply reversing the process defining H . That is, let

$$G(x, y) = \phi_{-\tau_1(x, y)}\phi_{\tau_1(x, y)}(x, y)$$

and set $G(0, 0) = (0, 0)$. Here $\tau_1(x, y)$ is the time for the solution of through (x, y) to reach

S^1 . An easy computation shows that $\tau_1(x, y) = \log r$ where $r^2 = x^2 + y^2$. Clearly,

$G = H^{-1}$ so H is one to one and onto. Also, G is continuous at $(x, y) \neq (0, 0)$ since G

may be written as $G(x, y) = \phi_{-\log r}\left(\frac{x}{r}, \frac{y}{r}\right)$ which is a composition of continuous

functions. For continuity of G at the origin, suppose that (x, y) is close to the origin, so that r is small. Observe that

as $r \rightarrow 0, \log r \rightarrow \infty$. Now $\left(\frac{x}{r}, \frac{y}{r}\right)$ is a point on S^1 and for r sufficiently small, $\phi_{-\log r}$

maps the unit circle very close to $(0, 0)$. This shows that G is continuous at $(0, 0)$. We

thus need only show continuity of H . For this, we need to show that $\tau(x, y)$ is continuous. But τ is determined by the equation $|\phi_t(x, y)| = 1$. We write $\phi_t(x, y) = (x(t), y(t))$. Taking the partial derivative of $\phi_t(x, y)$ with respect to t , we find

$$\begin{aligned} \frac{\partial}{\partial t} |\phi_t(x, y)| &= \frac{\partial}{\partial t} \sqrt{(x(t))^2 + (y(t))^2} \\ &= \frac{1}{\sqrt{(x(t))^2 + (y(t))^2}} (x(t)x'(t) + y(t)y'(t)) \\ &= \frac{1}{|\phi_t(x, y)|} \left(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \right). \end{aligned}$$

But the latter dot product is nonzero when $t = \tau(x, y)$ since the vector field given by $(x'(t), y'(t))$ points inside S' . Hence $\frac{\partial}{\partial t} |\phi_t(x, y)| \neq 0$ at $(\tau(x, y), x, y)$. Thus we may apply the implicit function theorem to show that τ is differentiable at (x, y) and hence continuous. Continuity of H at the origin follows as in the case of $G = H^{-1}$. Thus H is a homeomorphism and we have a conjugacy between $X' = AX$ and $X' = BX$. Note that this proof works equally well if the eigenvalues have positive real parts.

Case III: Finally, suppose that $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda < 0$. The associated vector field need not point inside the unit circle in this case. However, if we let $T = \begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix}$ then the vector field given by $Y' = (T^{-1}AT)Y$. Now does have this property, provided $\varepsilon > 0$ is sufficiently small. Indeed

$$T^{-1}AT = \begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix} \text{ so that } \left(T^{-1}AT \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right) \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \lambda + \varepsilon \sin \theta \cos \theta.$$

Thus if we choose $\varepsilon < -\lambda$ this dot product is negative. Therefore the change of variables T puts us into the situation where the same proof as in Case 2 applies. This completes the proof of the theorem.

Definition 2.4: Two flows ϕ_t and φ_t in R are said to be diffeomorphic when there is diffeomorphism h satisfying

$$\varphi_t(h(x_0)) = h(\phi_t(x_0))$$

Theorem 1.3 (Linear conjugacy):

The flows ϕ_t and φ_t of the linear systems $x' = Ax$ and $y' = By$ are diffeomorphic if and only if the matrix A is similar to the matrix B .

Proof: Note that $\phi_t(x) = e^{tA}x$ and $\varphi_t(y) = e^{tB}y$ are flows of $x' = Ax$ and $y' = By$ respectively.

(backward proof) Assume that A is similar to B . That means there a nonsingular matrix H such that $B = H^{-1}AH$. Then clearly $y = h(x) = Hx$ is a diffeomorphism and $h(\phi_t(x)) = H^{-1}e^{tA}HHx = e^{tH^{-1}AH}Hx = e^{tB}Hy = \varphi_t(h(x))$, which implies that both flows ϕ_t and φ_t are diffeomorphism

(forward proof) assume that ϕ_t and φ_t are diffeomorphism. Then there is $h \in C^1$ such that $h(0) = 0$ and $h(\phi_t(x)) = \varphi_t(h(x))$ for all $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Differentiating this we obtain $Dh(\phi_t(x))D\phi_t = D\varphi_t Dh(x)$ when we evaluate this at $x = 0$ and set $H = Dh(0)$ to find $He^{tA} = e^{tB}H$. Differentiate with respect to t and set $t = 0$ to finally obtain $HA = BH$. Thus A is similar to B .

Example 4: The matrices $A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ are not similar.

The matrix A is similar to no other matrix than itself. To see this, suppose that H is any invertible matrix. Then $H^{-1}AH = A$ since H commutes with A . It follows that the flows ϕ_t and φ_t of $x' = Ax$ and $y' = By$ cannot be diffeomorphic.

More generally, recall the classification of linear systems from chapter one by trace determinant plane. Let $A \in L(\mathbb{R}^2)$, $\text{tr}A = T$ and $\det A = D$. Similarity transformations preserve both the algebraic and geometric multiplicities of eigenvalues. Therefore, the flows of linear systems on the parabola $T^2 - 4D = 0$ that has only a single Eigen space cannot be diffeomorphic to flows that have two Eigen spaces. In particular, the only Eigen space of B is $\{(c, 0)^T : c \in \mathbb{R}\}$.

Here one can simply show that the flows associated with the matrices A and B in this example are topologically conjugate.

2.3 Classification of Fixed Points of Flows

A classification of (autonomous) linear first order differential equations, which agrees with our qualitative observations in Chapter 1. There are three conjugacy "classes" the sinks, the sources, and the special "in-between" case, $x' = 0$, where all solutions are constants. Now we move to the planar version of this scenario. We first note that we only need to decide on conjugacies among systems whose matrices are in canonical form. For, as we saw in Chapter 1, if the linear map $P: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ puts A in canonical form, then P takes the time t map of the flow of $Y' = (P^{-1}AP)Y$ to the time t map for $X' = AX$. Our classification of planar linear systems now proceeds just as in the one-dimensional case. But here we will stay away from the case where the system has eigenvalues with real part equal to zero.

Chapter 3

3. Local Behavior of Non-Linear Planar Systems

In Chapter 1 we saw that any linear system

$$X' = AX \quad (1)$$

has a solution through each point X_0 in the phase space R^n ; the solution is given by

$X(t) = e^{At} X_0$ and it is defined for all $t \in R^n$. In this chapter we begin our study of nonlinear systems of differential equations.

$$x' = f(x) \quad (2)$$

Where $f : E \rightarrow R^n$ and E is an open subset of R^n . In general, it is not possible to solve the nonlinear system (2); however, a great deal of qualitative information about the local behavior of the solution is determined in this chapter. In particular, we establish the Hartman-Grobman Theorem which show that topologically the local behavior of the nonlinear system (2) near an equilibrium point x_0 where $f(x_0) = 0$ is typically determined by the behavior of the linear system (1) near the origin when the matrix $A = Df(x_0)$ the derivative of f at x_0 .

Now let us define the partial derivatives of the system (2). If f is a differentiable function, the derivative Df is given by the $n \times n$ Jacobian matrix

$$Df = \begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}$$

Example 1: Find the derivative of the function

$$f(x) = \begin{bmatrix} x_1 - x_2^2 \\ -x_2 + x_1 x_2 \end{bmatrix}$$

and evaluate it at the point $x_0 = (1, -1)^T$.

We first compute the Jacobian matrix of partial derivatives,

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & -2x_2 \\ x_2 & -1 + x_1 \end{bmatrix}$$

And then

$$Df(1, -1) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

3.1 Linearization

A good place to start analyzing the nonlinear system

$$x' = f(x) \tag{1}$$

is to determine the equilibrium points of (1) and to describe the behavior of (1) near its equilibrium points. In the next sections it is shown that the local behavior of the nonlinear system (1) near a hyperbolic equilibrium point x_0 is qualitatively determined by the behavior of the linear system

$$X' = AX \tag{2}$$

with the matrix $A = Df(x_0)$ near the origin. The linear function $AX = Df(x_0)X$ is called the linear part of f at x_0 .

The linear system (2) with the matrix $A = Df(x_0)$ is called the linearization of (1) at x_0 .

If $x_0 = 0$ is an equilibrium point of (1), then $f(0) = 0$ and, by Taylor's Theorem,

$$f(x) = Df(0)x + \frac{1}{2}D^2f(0)(x, x) + \dots$$

It follows that the linear function $Df(x_0)x$ is a good first approximation to the nonlinear function $f(x)$ near $x = 0$ and it is reasonable to expect that the behavior of the nonlinear system (1) near the point $x = 0$ will be approximated by the behavior of its linearization at $x = 0$.

Example: consider the non linear planar system

$$x' = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix}$$

Clearly we can find the equilibrium point $x_0 = 0$, thus

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2x_1 & 1 \end{bmatrix}$$

And then

$$A = Df(x_0) = Df(0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The linearization of the system

$$x' = \begin{bmatrix} -x_1 \\ x_2 + x_1^2 \end{bmatrix} \text{ is } X' = AX \text{ where } A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Note: 1. The classification of fixed point of the nonlinear system (1) follows from the linear system (2) that, an equilibrium point x_0 of (1) is called a sink if all of the eigenvalues of the matrix $Df(x_0)$ have negative real part; it is called a source if all of the eigenvalues of $Df(x_0)$ have positive real part; and it is called a saddle if it is a hyperbolic equilibrium point and $Df(x_0)$ has at least one eigenvalue with a positive real part and at least one with a negative real part. If the real part of the eigenvalues of the matrix $Df(x_0)$ are all zero, then the matrix $Df(x_0)$ is inhyperbolic in that we cannot classify the nonlinear system from its linearization.
2. if x_0 is a hyperbolic equilibrium point of (1) then the local behavior of the nonlinear system (1) is topologically equivalent to the local behavior of the linear system (2).

3.2 The Hartman-Grobman Theorem

The Hartman-Grobman Theorem is another very important result in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0 , the nonlinear system

$$x' = f(x)$$

has the same qualitative structure as the linear system

$$X' = AX$$

With $A = Df(x_0)$.

3.2.1 Hartman-Grobman Theorem for Homeomorphisms

In this section we will prove the following version of the Hartman–Grobman theorem for a hyperbolic fixed point of an autonomous differential equation.

Theorem 3.1 (Hartman–Grobman): Suppose X_0 be a hyperbolic equilibrium point of $x' = f(x)$ where $f(x) \in C^1$ with flow $\phi_t(x)$. Then there is a neighborhood N of X_0 such that ϕ is topologically conjugate to its linearization on N .

Example: consider the system

$$\begin{aligned} x_1' &= \mu x_1 - x_2 + x_1(x_1^2 + x_2^2), \\ x_2' &= x_1 + \mu x_2 + x_2(x_1^2 + x_2^2) \end{aligned} \quad (*)$$

Study this system as the parameter μ varies. The system linearized about the origin is

$Y' = AY$, where $A = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$. The eigenvalues of A are $\lambda_{\pm} = \mu \pm i$. The linearized

system is sink for $\mu < 0$, a source for $\mu > 0$, and a center for $\mu = 0$. The Hartman-Grobman theorem guarantees that the flow of the nonlinear system is topologically conjugate to that of the linearized system in some neighborhood of the origin so long as the equilibrium point is hyperbolic, that is $\mu \neq 0$.

At $\mu = 0$, there is a bifurcation, a qualitative change in the character of the solution. The Hartman-Grobman theorem does not apply because the equilibrium point is not hyperbolic.

Proof of theorem 3.1: let $x' = f(x)$ writing this equation in the form of $x' = Ax + g(x)$ where $g(x) \in C^1$ represents the nonlinear terms, so that $g(x) = o(x)$.

Let $\phi_t(x)$ be the flow of the nonlinear ODE and the $\Phi_t(x) = e^{At}x$ flow for the linear part.

We wish to find a homeomorphism h satisfying

$$\begin{aligned} \Phi_t(h(x)) &= h(\phi_t(x)) \\ h(x) &= \Phi_t^{-1}(h(\phi_t(x))) \\ h(x) &= \Phi_t^{-1} \circ h \circ \phi_t(x) \\ h(x) &= e^{-At} \circ h \circ \phi_t(x) \end{aligned} \quad (1)$$

Suppose H_1 is a homeomorphism that satisfies this equation for one value of time, say $t = 1$, Then

$$H_1(x) = e^{-A} \circ H_1 \circ \phi_1(x) \quad (2)$$

Now let

$$H_t(x) = e^{-At} \circ H_1 \circ \phi_t(x) \quad (3)$$

Then it follows from the group property of composition of flow and (2) that

$$\begin{aligned} e^{-A} \circ H_t \circ \phi_1(x) &= e^{-A} \circ e^{-At} \circ H_1 \circ \phi_t \circ \phi_1(x) \\ &= e^{-At} \circ e^{-A} \circ H_1 \circ \phi_1 \circ \phi_t(x) \\ &= e^{-At} \circ H_1 \circ \phi_t(x) \\ &= H_t(x) \end{aligned}$$

Thus H_t also satisfies (2)

From our let $\phi_t(x)$ is the flow of the system $x' = Ax + \bar{g}(x)$ where $\bar{g} = g$ in some neighborhood N of the hyperbolic equilibrium point X_0 and $\bar{g} = 0$ outside N . Then, under the hypotheses of the theorem, it is possible to show that the solution to [2] is unique. It follows that $H_t = H_1$. Therefore, by (3)

$$H_1 = e^{-At} \circ H_1 \circ \phi_t(x),$$

Satisfying (1) and $H_1 = h$ is the homeomorphism that we originally sought!

The time 1 homeomorphism can be found iteratively, by starting with the $H_1^{(0)}() = x$, and then successively solving

$$H_1^{i+1} = e^{-A} \circ H_1^i \circ \phi_1(x), \quad i = 0, 1, 2, \dots$$

In general, convergence of the iteration scheme requires $\bar{g} = 0$ outside of a neighborhood of X_0 . this completes the proof.

3.2.2 Hartman-Grobman Theorem for Diffeomorphisms

The next theorem is a version of the Hartman–Grobman theorem for diffeomorphisms. Informally, it states that the phase portrait near a hyperbolic fixed point is the same, up to a continuous change of coordinates, as the phase portrait of the dynamical system induced by the derivative of the diffeomorphism evaluated at the fixed point.

Theorem 3.2 Suppose X_0 be a hyperbolic equilibrium point of $x' = f(x)$ where $f(x) \in C^2$ with flow $\phi_t(x)$. Then there is a neighborhood N of X_0 such that ϕ is diffeomorphic to its linearization on N .

Proof: the proof of this theorem is similar to the proof of the theorem 3.1, there are some subtle differences that we have to do further on the function h that its derivative is continuous and the derivative of its inverse is continuous.

4. Conclusion

In this project we investigate some sorts of behaviors of the planar linear systems. Moreover, we have dealt with the topological conjugacy and equivalence of two planar linear systems and the Hartman-Grobman theorem which is the core of the paper. The project also tries to investigate the classification of planar linear system from the dynamical point of view and, how to draw trace determinant plane.

As we observed, this topic is so wide and it needs further investigations to address more behaviors of the planar linear system. For example, for an inhyperbolic case of the coefficient matrix of the planar linear system, the conjugacy and equivalence is not addressed in this project.

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