REVIEW OF COHERENTLY DRIVEN RESONANCE LIGHT OF DEGENERATE THREE LEVEL LASER

By

Gebresenbet Entehabu Berhe

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS (QUANTUM OPTICS)
AT
ADISS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
ADISS ABABA, ETHIOPIA
DECEMBER 04, 2018

ADISS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommend to the College of Natural and Computational Sciences for acceptance a project entitled "**REVIEW OF COHERENTLY DRIVEN RESONANCE LIGHT OF DEGENERATE THREE LEVEL LASER**" by **Gebresenbet Entehabu Berhe** in partial fulfillment of the requirements for the degree of **Master of Science in Physics (Quantum Optics)**.

Dated: December 04, 2018

Advisor: ______________________
Dr. Deribe Hirpo

Examiners: ______________________
Dr. Fesseha Kassahun

____
Dr.Teshome Senbeta

Chairman: ______________________
Dr. Belayneh Mesfin

ADISS ABABA UNIVERSITY

Date: **December 04, 2018**

Author: **Gebresenbet Entehabu Berhe**

Title: **REVIEW OF COHERENTLY DRIVEN RESONANCE LIGHT OF DEGENERATE THREE LEVEL LASER**

Department: **College of Natural and Computational Sciences**
Department of Physics

Degree: **MSc.**
Convocation: **December 04**
Year: **2018**

____
Signature of Author

Table of Contents

List of Figures

Abstract

We analyze a degenerate three-level laser driven by coherent light and coupled to a vacuum reservoir via a port-mirror. Employing the solutions of the c-number Langevin equations we have determined the mean photon number, variance of the photon number and quadrature variance of the cavity radiation.The mean photon number would be zero when there is no driving light and all atoms are initially in the bottom level, and the most intense light is produced when all atoms are initially in the upper level. We observed that the more atoms are injected into the cavity at a time, the more the degree of the squeezing of the cavity radiation would be.

Acknowledgements

First and foremost I would like to express my deepest thanks to my advisor Dr. Deribe Hirpo, for his invaluable guidance. It is also my pleasure to acknowledge the great support, invaluable guidance, and constructive criticism, I received from him during the work of this project.

My strong appreciation and respect goes to physics department of Addis Ababa university, all my instructors, ministry of education and my families for facilitating conditions for my education and financial support.

Chapter 1

Introduction

In recent years the interaction of three-level atoms with a radiation has attracted a great deal of interest [1-12]. According to [1-7] the atomic coherence can be induced in a three-level atom by coupling the levels between which direct transition is dipole forbidden by an external radiation or by preparing the atom initially in a coherent superposition of these two levels [9, 10, 11]. It is found that the cavity radiation exhibits squeezing under certain conditions for both cases [6, 8, 11, 12].

In a three-level atom the top, intermediate, and bottom levels are denoted by $|a\rangle, |b\rangle$, and $|c\rangle$ in which a direct transition between levels $|a\rangle$ and $|c\rangle$ is dipole forbidden. When the three-level atom decays from $|a\rangle$ to $|c\rangle$ via the level $|b\rangle$ two photons are generated. If the two photons have identical frequency, then the three-level atom is referred to as a degenerate. Hence we define a degenerate three-level laser as a quantum optical system in which degenerate three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate into a cavity. These atoms are removed from the cavity after some time. We hence realize that, a degenerate three-level laser is a two photon device in which squeezing properties are expected to occur due to the correlation between these two photons [4, 6].

We consider a degenerate three-level laser coupled to a vacuum reservoir via a single-port mirror and the bottom level of the atom on the other hand is coupled to the top level by an external resonant coherent light as shown in Fig. 1.1. Some authors have already studied such a scheme in which the atomic coherence is induced by an external radiation and when initially the atoms are prepared in a coherent superposition of the top level and bottom level [6, 7, 12]. They found that the three-level laser in these cases resemble the parametric oscillator for a strong radiation. Moreover, recently Saavedra [3] studied the λ three-level laser when the atoms are initially prepared in a coherent superposition and the forbidden transition is induced by driving with strong external radiation.

In this project, we calculate the mean photon number, the variance of the photon number, and we also study the squeezing properties of the cavity radiation produced by degenerate three level laser using the c-number Langevin equations associated with the normal ordering. In particular, we calculate the mean photon number and the quadrature variance for cases when the atoms are initially prepared to be in the top and bottom levels.

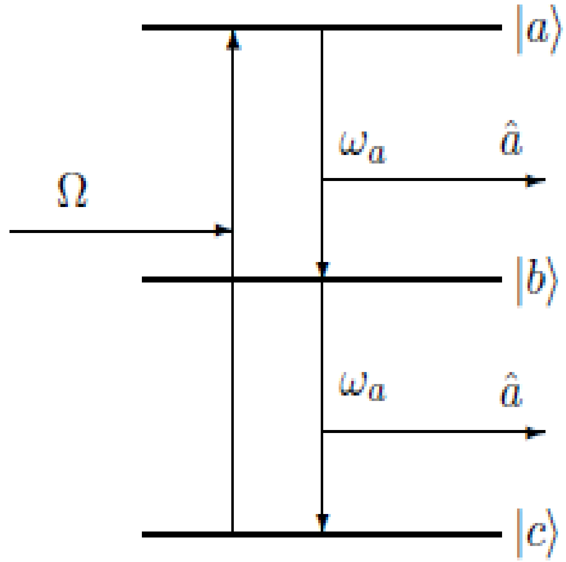

Figure 1.1: Schematic representation of a coherently driven degenerate three-level atom. The transitions from $|a\rangle$ to $|b\rangle$ and $|b\rangle$ to $|c\rangle$ at frequency ω_a each are taken to be resonant with the cavity. The transition from $|a\rangle$ to $|c\rangle$ is dipole forbidden and can be induced by driving the atom externally with resonant radiation of frequency $2\omega_a$.

Chapter 2

Master Equation

In this chapter we obtain the master equation describing the cavity radiation of the driven degenerate three-level laser coupled to a vacuum reservoir.

The interaction of a degenerate three-level atom with a single-mode light can be described in the rotating- wave and electric dipole approximations by the Hamiltonian [1]

$$\hat{H}_{AR} = \imath g[\hat{a}(|a\rangle\langle b| + |b\rangle\langle c|) - (|b\rangle\langle a| + |c\rangle\langle b|)\hat{a}^{\dagger}], \tag{2.1}$$

where AR stands for atom-radiation interaction, g is the coupling constant, which is taken to be the same for both transitions, and $\hat{a}$ is the annihilation operator for the cavity mode, $|a\rangle\langle b|$ and $|b\rangle\langle c|$ are atomic operators. On the other hand, the three-level atom for which its bottom level is coupled to the top level by a resonant coherent light can be expressed in the rotating-wave approximation by the Hamiltonian [1]

$$\hat{H}_C = \imath\frac{\Omega}{2}[|c\rangle\langle a| - |a\rangle\langle c|], \tag{2.2}$$

where C stands for coupling, and Ω is a real-positive constant proportional to the amplitude of the coherent driving radiation. Hence on the basis of Eqs. (2.1) and (2.2) the interaction of a coherently driven three-level atom with the cavity radiation can be represented in the rotating-wave approximation by the Hamiltonian,

$$\hat{H} = \imath g[\hat{a}(|a\rangle\langle b| + |b\rangle\langle c|) - (|b\rangle\langle a| + |c\rangle\langle b|)\hat{a}^{\dagger}] + \imath\frac{\Omega}{2}[|c\rangle\langle a| - |a\rangle\langle c|]. \tag{2.3}$$

We can write the initial state of a three-level atom whose top and bottom levels are coupled by resonant driving coherent light as [1]

$$|\Phi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle, \tag{2.4}$$

where $C_a(0) = \langle a|\Phi_A(0)\rangle$ and $C_c(0) = \langle c|\Phi_A(0)\rangle$ are probability amplitudes for the atom to be in the top and bottom levels, respectively. This corresponds to the fact that the three-level atom is initially prepared to be in a coherent superposition of the top and bottom levels. Hence the initial density operator for the atom described by the quantum state Eq. (2.4) would be

$$\begin{aligned}
\hat{\rho}_A(0) &= |\Phi_A(0)\rangle\langle\Phi_A(0)|, \\
&= (C_a(0)|a\rangle + C_c(0)|c\rangle)(C_a^*(0)\langle a| + C_c^*(0)\langle c|), \\
&= |C_a(0)|^2|a\rangle\langle a| + C_a(0)C_c^*(0)|a\rangle\langle c| + C_c(0)C_a^*(0)|c\rangle\langle a| + |C_c(0)|^2|c\rangle\langle c|, \\
&= \rho_{aa}(0)|a\rangle\langle a| + \rho_{ac}(0)|a\rangle\langle c| + \rho_{ca}(0)|c\rangle\langle a| + \rho_{cc}(0)|c\rangle\langle c|,
\end{aligned} \tag{2.5}$$

where

$$\rho_{\alpha\beta}(0) = C_\alpha^*(0)C_\beta(0), \tag{2.6}$$

with $\alpha, \beta = a, b, c$.

It proves to be convenient to introduce a new parameter η defined by [2]

$$\rho_{aa}(0) = \frac{1-\eta}{2}. \tag{2.7}$$

Using the fact that

$$\rho_{aa}(0) + \rho_{cc}(0) = 1, \tag{2.8}$$

along with

$$|\rho_{ac}(0)|^2 = \rho_{aa}(0)\rho_{cc}(0), \tag{2.9}$$

one easily finds

$$\rho_{cc}(0) = \frac{1+\eta}{2}, \tag{2.10}$$

and

$$|\rho_{ac}(0)| = \frac{1}{2}\sqrt{1-\eta^2}, \tag{2.11}$$

with $-1 \leq \eta \leq 1$.

We note that the parameter η describes the initial preparation of a three-level atom. Upon setting

$$\rho_{ac}(0) = |\rho_{ac}(0)|e^{i\theta}, \tag{2.12}$$

the expression given by Eq. (2.5) can be put in the form

$$\hat{\rho}_A(0) = \frac{1-\eta}{2}|a\rangle\langle a| + \frac{1}{2}\sqrt{1-\eta^2}e^{i\theta}|a\rangle\langle c| + \frac{1}{2}\sqrt{1-\eta^2}e^{-i\theta}|c\rangle\langle a| + \frac{1+\eta}{2}|c\rangle\langle c|. \tag{2.13}$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity mode at time t, with the atom injected at time t_j, such that $(t - \tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as [2]

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t_j, \tag{2.14}$$

where $r_a \Delta t_j$ represents the number of atoms injected into the cavity in a time Δt_j. Assuming that the atoms are continuously injected into the cavity and taking the limit that $\Delta t_j \to 0$, the summation over j can be converted into integration with respect to t', we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt'. \tag{2.15}$$

From the Leinbnitz rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x, x')dx' = f(x, v)\frac{dv(x)}{dx} - f(x, u)\frac{du(x)}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x, x')dx', \tag{2.16}$$

there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_{AR}(t, t) - \hat{\rho}_{AR}(t, t-\tau)) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt'. \tag{2.17}$$

We notice that $\hat{\rho}_{AR}(t, t)$ represents the density operator for an atom plus the cavity radiation at a time when the atom is injected into the cavity, whereas $\hat{\rho}_{AR}(t, t - \tau)$ represents the density operator when the atom is removed from the cavity. Since the atomic and radiation variables are not correlated at the instant the atoms are injected into or removed from the cavity[2],

$$\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(0)\hat{\rho}(t) \tag{2.18}$$

and

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t), \tag{2.19}$$

where, $\hat{\rho}_A(0) = \hat{\rho}_A(t)$.

With $\hat{\rho}(t)$ being the density operator for the cavity mode alone can be obtained by tracing the atom-radiation density operator over atomic variables.

Now in view of Eq. (2.18) and (2.19), one can write Eq. (2.17) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau))\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.20}$$

In the absence of the damping of the cavity mode by vacuum reservoir, the density operator $\hat{\rho}_{AR}(t,t')$ evolves in time according to[1,2]

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -\imath[\hat{H}, \hat{\rho}_{AR}(t,t')], \tag{2.21}$$

so that using Eq. (2.21) and taking in to account Eq. (2.15), one can put Eq. (2.20) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a\big(\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau)\big)\hat{\rho}(t) - \imath[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{2.22}$$

Furthermore, tracing over the atomic variables, we have

$$Tr_A\frac{d}{dt}(\hat{\rho}_A(t)\hat{\rho}(t)) = Tr_A\Bigg(r_a\big(\hat{\rho_A}(0) - \hat{\rho_A}(t-\tau)\big)\hat{\rho}(t) - \imath[\hat{H}, \hat{\rho}_{AR}(t)]\Bigg),$$

$$\hat{\rho}(t)(\frac{d}{dt}(Tr_A\hat{\rho}_A(t))) + Tr_A\hat{\rho}_A(t)(\frac{d}{dt}(\hat{\rho}(t))) = r_aTr_A(\hat{\rho_A}(0) - \hat{\rho_A}(t-\tau))\hat{\rho}(t) - Tr_A(\imath[\hat{H}, \hat{\rho}_{AR}(t)]),$$

$$0 + \frac{d}{dt}(\hat{\rho}(t)) = r_a(1-1)\hat{\rho}(t) - Tr_A(\imath[\hat{H}, \hat{\rho}_{AR}(t)]),$$

$$\frac{d}{dt}(\hat{\rho}(t)) = 0 - Tr_A(\imath[\hat{H}, \hat{\rho}_{AR}(t)]),$$

$$\frac{d}{dt}\hat{\rho}(t) = -\imath Tr_A[\hat{H}, \hat{\rho}_{AR}(t)], \tag{2.23}$$

where Tr_A denotes trace over atomic variables and we have used the fact

$$Tr\hat{\rho}_A(t) = Tr\hat{\rho}_A(t-\tau) = 1. \tag{2.24}$$

The master equation associated with the interactions described by the Hamiltonian Eq. (2.3) can be put in the form

$$\frac{d}{dt}\hat{\rho}(t) = -\imath Tr_A\big[\imath g\big(\hat{a}(|a\rangle\langle b| + |b\rangle\langle c|) - (|b\rangle\langle a| + |c\rangle\langle b|)\hat{a}^\dagger\big) + \imath\frac{\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|), \hat{\rho}_{AR}(t)\big], \tag{2.25}$$

$$\frac{d}{dt}\hat{\rho}(t) = Tr_A\bigg(\big(g\big(\hat{a}(|a\rangle\langle b| + |b\rangle\langle c|) - \hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|)\big) + \frac{\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|)\big)\hat{\rho}_{AR}(t),$$
$$- \hat{\rho}_{AR}(t)\big(g\big((|a\rangle\langle b| + |b\rangle\langle c|)\hat{a} - (|b\rangle\langle a| + |c\rangle\langle b|)\hat{a}^\dagger\big) + \frac{\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|)\big)\bigg),$$
$$= g\big(\hat{a}(\rho_{ba} + \rho_{cb}) - \hat{a}^\dagger(\rho_{ab} + \rho_{bc}) - (\rho_{ba} + \rho_{cb})\hat{a} + (\rho_{ab} + \rho_{bc})\hat{a}^\dagger\big) + \frac{\Omega}{2}(\rho_{ac} - \rho_{ca}) - \frac{\Omega}{2}(\rho_{ac} - \rho_{ca}$$
$$= g(\hat{a}\rho_{ba} + \hat{a}\rho_{cb} - \rho_{ba}\hat{a} - \rho_{cb}\hat{a} - \hat{a}^\dagger\rho_{ab} - \hat{a}^\dagger\rho_{bc} + \rho_{ab}\hat{a}^\dagger + \rho_{bc}\hat{a}^\dagger), \tag{2.26}$$

in which the matrix element $\rho_{\alpha\beta}$ is defined by

$$\rho_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.27}$$

with $\alpha, \beta = a, b, c$.

We next proceed to determine the matrix elements $\rho_{\alpha\beta}$, by using Eq. (2.22). In the relation

$$\frac{d}{dt}\rho_{\alpha\beta} = \langle\alpha|\frac{d}{dt}\hat{\rho}_{AR}(t)|\beta\rangle. \tag{2.28}$$

It then follows that

$$\frac{d}{dt}\rho_{\alpha\beta} = r_a(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle)\hat{\rho}(t) - \imath(\langle\alpha|[\hat{H}, \hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}, \tag{2.29}$$

where the last term is included to account for the decay of the atoms due to spontaneous emission. Here γ, considered to be the same for both transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$ is the atomic decay constant. We assume that the atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0, \tag{2.30}$$

and Eq.(2.29) becomes

$$\frac{d}{dt}\rho_{\alpha\beta} = r_a(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle)\hat{\rho}(t) - \imath(\langle\alpha|[\hat{H}, \hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\rho_{\alpha\beta}. \tag{2.31}$$

Applying this equation and taking into account (2.3) and (2.5), we obtain

$$\frac{d}{dt}\rho_{ab} = g(\hat{a}\rho_{bb} - \rho_{aa}\hat{a} + \rho_{ac}\hat{a}^\dagger) - \frac{\Omega}{2}\rho_{cb} - \gamma\rho_{ab}, \tag{2.32}$$

$$\frac{d}{dt}\rho_{bc} = g(\hat{a}\rho_{cc} - \hat{a}\rho_{ac} - \rho_{bb}\hat{a}) + \frac{\Omega}{2}\rho_{ba} - \gamma\rho_{bc}, \tag{2.33}$$

$$\frac{d}{dt}\rho_{bb} = g(\hat{a}\rho_{cb} - \hat{a}^\dagger\rho_{ab} - \rho_{ba}\hat{a} + \rho_{ac}\hat{a}^\dagger) - \gamma\rho_{bb}, \tag{2.34}$$

$$\frac{d}{dt}\rho_{aa} = r_a\rho_{aa}(0)\hat{\rho}(t) + g(\hat{a}\rho_{ba} + \rho_{ab}\hat{a}^\dagger) - \frac{\Omega}{2}(\rho_{ca} + \rho_{ac}) - \gamma\rho_{aa}, \tag{2.35}$$

$$\frac{d}{dt}\rho_{cc} = r_a\rho_{cc}(0)\hat{\rho}(t) - g(\hat{a}\rho_{bc} - \rho_{cb}\hat{a}) + \frac{\Omega}{2}(\rho_{ac} + \rho_{ca}) - \gamma\rho_{cc}, \tag{2.36}$$

$$\frac{d}{dt}\rho_{ac} = r_a\rho_{ac}(0)\hat{\rho}(t) + g(\hat{a}\rho_{bc} - \hat{\rho}_{ab}\hat{a}) - \frac{\Omega}{2}(\rho_{cc} - \rho_{aa}) - \gamma\rho_{ac}. \tag{2.37}$$

We confine ourselves to linear analysis and this can be achieved by dropping the g terms in Eqs. (2.34), (2.35), (2.36), and (2.37). We then see that

$$\frac{d}{dt}\rho_{bb} = -\gamma\rho_{bb}, \tag{2.38}$$

$$\frac{d}{dt}\rho_{aa} = r_a\rho_{aa}(0)\hat{\rho}(t) - \frac{\Omega}{2}(\rho_{ca} + \rho_{ac}) - \gamma\rho_{aa}, \tag{2.39}$$

$$\frac{d}{dt}\rho_{cc} = r_a\rho_{cc}(0)\hat{\rho}(t) + \frac{\Omega}{2}(\rho_{ac} + \rho_{ca}) - \gamma\rho_{cc}, \tag{2.40}$$

$$\frac{d}{dt}\rho_{ac} = r_a\rho_{ac}(0)\hat{\rho}(t) - \frac{\Omega}{2}(\rho_{cc} - \rho_{aa}) - \gamma\rho_{ac}. \tag{2.41}$$

In addition, imposing the good cavity limit ($\kappa \ll \gamma$) in which the atomic variables reach steady state in a relatively short period of γ^{-1}, we can take the time derivatives of such variables to be zero, while keeping the zero order atomic and cavity mode variables at time t. This procedure may be referred as adiabatic approximation scheme. Thus applying the adiabatic approximation scheme, we get from Eqs. (2.38), (2.39), (2.40), and (2.41) that

$$\rho_{bb} = 0, \tag{2.42}$$

$$\rho_{aa} = \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) - \frac{\Omega}{2\gamma}(\rho_{ca} + \rho_{ac}), \tag{2.43}$$

$$\rho_{cc} = \frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) + \frac{\Omega}{2\gamma}(\rho_{ac} + \rho_{ca}), \tag{2.44}$$

$$\rho_{ac} = \frac{r_a}{\gamma}\rho_{ac}(0)\hat{\rho}(t) - \frac{\Omega}{2\gamma}(\rho_{cc} - \rho_{aa}). \tag{2.45}$$

Upon setting $\rho_{ac}(0) = \rho_{ca}(0)$, we see from Eq. (2.45) that

$$\rho_{ac}(t) = \rho_{ca}(t). \tag{2.46}$$

By subtracting Eq.(2.43) from Eq.(2.44) and by taking in to account Eq. (2.46)

$$\begin{aligned}\rho_{cc} - \rho_{aa} &= \frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) - \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) + 2\frac{\Omega}{2\gamma}(\rho_{ac} + \rho_{ca}),\\ &= \frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) - \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) + 2\frac{\Omega}{\gamma}(\rho_{ac}).\end{aligned} \tag{2.47}$$

By substituting Eq. (2.47) in to Eq. (2.45), we get

$$\begin{aligned}\rho_{ac} &= \frac{r_a}{\gamma}\rho_{ac}(0)\hat{\rho}(t) - \frac{\Omega}{2\gamma}\big(\frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) - \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) + \frac{2\Omega}{\gamma}(\rho_{ac})\big)\\ \rho_{ac}(1 + \frac{\Omega^2}{\gamma^2}) &= \frac{r_a}{\gamma}\rho_{ac}(0)\hat{\rho}(t) - \frac{\Omega r_a}{2\gamma^2}\rho_{cc}(0)\hat{\rho}(t) + \frac{\Omega r_a}{2\gamma^2}\rho_{aa}(0)\hat{\rho}(t)\\ \rho_{ac} &= \frac{r_a}{2(\gamma^2 + \Omega^2)}\hat{\rho}(t)(\Omega(\rho_{aa}(0) - \rho_{cc}(0)) + 2\gamma\rho_{ac}(0)).\end{aligned} \tag{2.48}$$

By substituting Eq.(2.48) in to Eqs.(2.43) and (2.44), and by taking in to account Eq.(2.46), we see that

$$\begin{aligned}\rho_{aa} &= \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) - \frac{\Omega}{2\gamma}(\rho_{ca} + \rho_{ac}),\\ &= \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) - \frac{\Omega}{\gamma}(\rho_{ac}),\\ &= \frac{r_a}{\gamma}\rho_{aa}(0)\hat{\rho}(t) - \frac{\Omega}{\gamma}\left(\frac{r_a}{2(\gamma^2 + \Omega^2)}\hat{\rho}(t)\big(\Omega(\rho_{aa}(0) - \rho_{cc}(0)) + 2\gamma\rho_{ac}(0)\big)\right),\\ &= \frac{r_a\hat{\rho}(t)}{2\gamma(\gamma^2 + \Omega^2)}(2(\gamma^2 + \Omega^2)\rho_{aa}(0) - \Omega^2\rho_{aa}(0) + \Omega^2\rho_{cc}(0) - 2\Omega\gamma\rho_{ac}(0)),\\ &= \frac{r_a\hat{\rho}(t)}{2\gamma(\gamma^2 + \Omega^2)}((2\gamma^2 + \Omega^2)\rho_{aa}(0) - 2\Omega\gamma\rho_{ac}(0) + \Omega^2\rho_{cc}(0)),\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\rho_{cc} &= \frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) + \frac{\Omega}{2\gamma}(\rho_{ca}+\rho_{ac}),\\
&= \frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) + \frac{\Omega}{\gamma}(\rho_{ac}),\\
&= \frac{r_a}{\gamma}\rho_{cc}(0)\hat{\rho}(t) + \frac{\Omega}{\gamma}\Big(\frac{r_a}{2(\gamma^2+\Omega^2)}\hat{\rho}(t)\big(\Omega(\rho_{aa}(0)-\rho_{cc}(0))+2\gamma\rho_{ac}(0)\big)\Big),\\
&= \frac{r_a\hat{\rho}(t)}{2\gamma(\gamma^2+\Omega^2)}(2(\gamma^2+\Omega^2)\rho_{cc}(0)+\Omega^2\rho_{aa}(0)-\Omega^2\rho_{cc}(0)+2\Omega\gamma\rho_{ac}(0)),\\
&= \frac{r_a\hat{\rho}(t)}{2\gamma(\gamma^2+\Omega^2)}(\Omega^2\rho_{aa}(0)+2\Omega\gamma\rho_{ac}(0)+(2\gamma^2+\Omega^2)\rho_{cc}(0)). \qquad (2.50)
\end{aligned}$$

Now making use of Eqs.(2.32), (2.33), (2.42), (2.49) and (2.50), we have

$$\begin{aligned}
\rho_{ab} = {}& \frac{gr_a\hat{\rho}(t)}{(4\gamma^2+\Omega^2)(\gamma^2+\Omega^2)}[\hat{a}[(4\gamma^2+\Omega^2)\rho_{aa}(0)-6\gamma\Omega\rho_{ac}(0)+3\Omega^2\rho_{cc}(0)]\\
&+\hat{a}^\dagger[-\frac{\Omega(\gamma^2-\Omega^2)}{\gamma}\rho_{aa}(0)+2(\Omega^2-2\gamma^2)\rho_{ac}(0)+\frac{\Omega(\Omega^2+4\gamma^2)}{\gamma}\rho_{cc}(0)]], \qquad (2.51)
\end{aligned}$$

$$\begin{aligned}
\rho_{cb} = {}& \frac{gr_a\hat{\rho}(t)}{(4\gamma^2+\Omega^2)(\gamma^2+\Omega^2)}[\hat{a}[\frac{-\Omega(4\gamma^2+\Omega^2)}{\gamma}\rho_{aa}(0)-2(2\gamma^2-\Omega^2)\rho_{ac}(0)\\
&-\frac{\Omega(\Omega^2-2\gamma^2)}{\gamma}\rho_{cc}(0)]+\hat{a}^\dagger[3\Omega^2\rho_{aa}(0)+6\Omega\gamma\rho_{ac}(0)+(\Omega^2+4\gamma^2)\rho_{cc}(0)]], \qquad (2.52)
\end{aligned}$$

By substituting Eqs. (2.51) and (2.52) in to Eq. (2.26)

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) = {}& \frac{AC}{2B}[2\hat{a}^\dagger\hat{\rho}\hat{a}-\hat{a}\hat{a}^\dagger\hat{\rho}-\hat{\rho}\hat{a}\hat{a}^\dagger]\\
&+\frac{AD}{2B}[2\hat{a}\hat{\rho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^\dagger\hat{a}]\\
&+\frac{AE}{2B}[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger-\hat{a}^2\hat{\rho}-\hat{\rho}\hat{a}^{\dagger 2}+\hat{a}\hat{\rho}\hat{a}]\\
&+\frac{AF}{2B}[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger-\hat{a}^{\dagger 2}\hat{\rho}-\hat{\rho}\hat{a}^2+\hat{a}\hat{\rho}\hat{a}], \qquad (2.53)
\end{aligned}$$

where

$$A = \frac{2r_ag^2}{\gamma^2}, \qquad (2.54)$$

is the linear gain coefficient,

$$B = (1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}), \qquad (2.55)$$

$$C = \rho_{aa}(0)(1+\frac{\Omega^2}{4\gamma^2})-\rho_{ac}(0)\frac{3\Omega}{2\gamma}+\rho_{cc}(0)\frac{3\Omega^2}{4\gamma^2}, \qquad (2.56)$$

$$D = \rho_{aa}(0)\frac{3\Omega^2}{4\gamma^2} + \rho_{ac}(0)\frac{3\Omega}{2\gamma} + \rho_{cc}(0)(1 + \frac{\Omega^2}{4\gamma^2}), \tag{2.57}$$

$$E = -\rho_{aa}(0)\frac{\Omega}{2\gamma}(1 - \frac{\Omega^2}{2\gamma^2}) - \rho_{ac}(0)(1 - \frac{\Omega^2}{2\gamma^2}) + \rho_{cc}(0)\frac{\Omega}{\gamma}(1 + \frac{\Omega^2}{4\gamma^2}), \tag{2.58}$$

$$F = -\rho_{aa}(0)\frac{\Omega}{\gamma}(1 + \frac{\Omega^2}{4\gamma^2}) - \rho_{ac}(0)(1 - \frac{\Omega^2}{2\gamma^2}) + \rho_{cc}(0)\frac{\Omega}{2\gamma}(1 - \frac{\Omega^2}{2\gamma^2}). \tag{2.59}$$

The general master equation for a system coupled to a reservoir is given by

$$\frac{d\hat{\rho}}{dt} = -\imath[\hat{H}_S, \hat{\rho}(t)] - \imath[\langle\hat{H}_{SR}\rangle_R, \hat{\rho}(t-h)] - h[\langle\hat{H}_{SR}\rangle_R, [\hat{H}_s, \hat{\rho}]] - hTr_R(H_{SR}^2\hat{R})\hat{\rho} - h\hat{\rho}Tr_R(\hat{R}H_{SR}^2) + 2hTr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}), \tag{2.60}$$

where Tr_R is tracing over the reservoir, $\hat{H}_{SR}$ is the Hamiltonian describes the interaction of the system with the reservoir, and $\hat{H}$ is the Hamiltonian describes the interaction of the system.

If we assume a thermal reservoir the hamiltonian describing the interaction of cavity mode and reservoir mode is

$$\hat{H}_{SR} = \imath\lambda(\hat{a}^\dagger\hat{a}_{in} - \hat{a}_{in}^\dagger\hat{a}), \tag{2.61}$$

where λ is coupling constant, $\hat{a}_{in}$ is the annihilation operator for the reservoir mode, and $\hat{a}$ is the annihilation operator for the cavity mode.

$$\begin{aligned}\langle\hat{H}_{SR}\rangle &= Tr_R(\hat{R}\hat{H}_{SR}),\\ &= Tr_R(\hat{R}\imath\lambda(\hat{a}^\dagger\hat{a}_{in} - \hat{a}_{in}^\dagger\hat{a}),\\ &= \imath\lambda(\hat{a}^\dagger Tr_R(\hat{R}\hat{a}_{in}) - \hat{a}Tr_R(\hat{R}\hat{a}_{in}^\dagger)),\end{aligned} \tag{2.62}$$

$$\begin{aligned}\langle\hat{a}_{in}\rangle_R &= Tr_R(\hat{R}\hat{a}_{in}),\\ &= Tr_R(\sum\frac{\bar{n}^n}{(1+n)^{n+1}}|n\rangle\langle n|\hat{a}_{in}),\\ &= \sum\frac{\bar{n}^n}{(1+n)^{n+1}}\langle n|\hat{a}_{in}|n\rangle,\\ &= 0.\end{aligned} \tag{2.63}$$

Similarly, one can verify that $\langle \hat{a}_{in}^{\dagger} \rangle_R = 0$, $\langle \hat{a}_{in}^2 \rangle_R = 0, \langle \hat{a}_{in}^{\dagger 2} \rangle_R = 0$.

Therefore Eq. (2.62) becomes

$$\langle \hat{H}_{SR} \rangle_R = 0. \tag{2.64}$$

Using the Hamiltonian Eq. (2.61), we see that

$$\begin{aligned}
hTr_R(H_{SR}^2\hat{R})\hat{\rho} &= hTr_R[\imath\lambda(\hat{a}^{\dagger}\hat{a}_{in} - \hat{a}_{in}^{\dagger}\hat{a})\imath\lambda(\hat{a}^{\dagger}\hat{a}_{in} - \hat{a}_{in}^{\dagger}\hat{a})\hat{R}\hat{\rho}], \\
&= -\lambda^2 hTr_R[\hat{a}^{\dagger}\hat{a}_{in}\hat{a}^{\dagger}\hat{a}_{in}\hat{R}\hat{\rho} - \hat{a}^{\dagger}\hat{a}_{in}\hat{a}_{in}^{\dagger}\hat{a}\hat{R}\hat{\rho} - \hat{a}_{in}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}_{in}\hat{R}\hat{\rho} + \hat{a}_{in}^{\dagger}\hat{a}\hat{a}_{in}^{\dagger}\hat{a}\hat{R}\hat{\rho}], \\
&= -\lambda^2 h[\hat{a}^{\dagger 2}\hat{\rho}Tr_R(\hat{a}_{in}^2 R) - Tr_R(\hat{a}_{in}\hat{a}_{in}^{\dagger}R)\hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{a}\hat{a}^{\dagger}\hat{\rho}Tr_R(\hat{a}_{in}^{\dagger}\hat{a}_{in}R) + \hat{a}^2\hat{\rho}Tr_R(\hat{a}_{in}^{\dagger 2}R)], \\
&= -\lambda^2 h[0 - Tr_R(\hat{a}_{in}\hat{a}_{in}^{\dagger}R)\hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{a}\hat{a}^{\dagger}\hat{\rho}Tr_R(\hat{a}_{in}^{\dagger}\hat{a}_{in}R) + 0], \\
&= \lambda^2 h((\bar{n}+1)\hat{a}^{\dagger}\hat{a}\hat{\rho} + \bar{n}\hat{a}\hat{a}^{\dagger}\hat{\rho}), \\
&= \frac{\kappa}{2}((\bar{n}+1)\hat{a}^{\dagger}\hat{a}\hat{\rho} + \bar{n}\hat{a}\hat{a}^{\dagger}\hat{\rho}).
\end{aligned} \tag{2.65}$$

Similarly one can verify that

$$h\hat{\rho}Tr_R(\hat{R}H_{SR}^2) = \frac{\kappa}{2}((\bar{n}+1)\hat{\rho}\hat{a}^{\dagger}\hat{a} + \bar{n}\hat{\rho}\hat{a}\hat{a}^{\dagger}), \tag{2.66}$$

$$hTr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = \frac{\kappa}{2}(\bar{n}\hat{a}^{\dagger}\hat{\rho}\hat{a} + (\bar{n}+1)\hat{a}\hat{\rho}\hat{a}^{\dagger}), \tag{2.67}$$

where $\langle \hat{a}_{in}^{\dagger}\hat{a}_{in} \rangle = \bar{n}$, $\langle \hat{a}_{in}\hat{a}_{in}^{\dagger} \rangle = \bar{n}+1$, and $\kappa = 2h\lambda^2$ is cavity dumping constant. By substituting Eq. (2.64), (2.65), (2.66) and (2.67) in to Eq. (2.60), we have

$$\frac{d}{dt}\hat{\rho}(t) = -\imath[\hat{H}_s(t), \hat{\rho}(t)] + \frac{\kappa(\bar{n}+1)}{2}[2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}] + \frac{k\bar{n}}{2}[2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}]. \tag{2.68}$$

On the other hand, the time evolution of the density operator for a single-mode cavity radiation coupled to a vacuum reservoir via a single-port mirror is found

$$\frac{d}{dt}\hat{\rho}(t) = -\imath[\hat{H}_s(t), \hat{\rho}(t)] + \frac{\kappa}{2}[2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}]. \tag{2.69}$$

In over case the deriving coherent light couples the top and bottom levels of the atom.That is, the cavity mode is not driven(pumped) by the coherent light. consequently, we set $\hat{H}_s = 0$ in Eq. (2.69). With the aid of Eqs. (2.53) and (2.69), the master equation describing the cavity radiation of the driven degenerate three-level laser coupled to a vacuum reservoir turns out to be

$$\frac{d}{dt}\hat{\rho}(t) = \frac{AC}{2B}[2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}]$$
$$+\frac{1}{2}(\frac{AD}{B} + k)[2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}]$$
$$+\frac{AE}{2B}[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2} + \hat{a}\hat{\rho}\hat{a}]$$
$$+\frac{AF}{2B}[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{2} + \hat{a}\hat{\rho}\hat{a}]. \tag{2.70}$$

Chapter 3

c-number Langevin Equations

In this Chapter we obtain the c-number Langevin equations, their solutions associated with normal ordering and correlation properties of the noise force. Employing the master equation along with the relation $\frac{d\langle\hat{A}(t)\rangle}{dt} = Tr(\frac{d\hat{\rho}\hat{A}}{dt})$, we see that

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}(t)\rangle &= Tr(\frac{d\hat{\rho}}{dt}\hat{a}), \\
&= \frac{AC}{2B}Tr[2\hat{a}^{\dagger}\hat{\rho}\hat{a}^{2} - \hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}] \\
&+ \frac{1}{2}(\frac{AD}{B} + \kappa)Tr[2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}] \\
&+ \frac{AE}{2B}Tr[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} + \hat{a}\hat{\rho}\hat{a}^{2}] \\
&+ \frac{AF}{2B}Tr[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{3} + \hat{a}\hat{\rho}\hat{a}^{2}], \qquad (3.1)
\end{aligned}$$

or

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = T_1 + T_2 + T_3 + T_4, \qquad (3.2)$$

in which

$$\begin{aligned}
T_1 &= \frac{AC}{2B}Tr[2\hat{a}^{\dagger}\hat{\rho}\hat{a}^{2} - \hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}], \\
T_2 &= \frac{1}{2}(\frac{AD}{B} + \kappa)Tr[2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}], \\
T_3 &= \frac{AE}{2B}Tr[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} + \hat{a}\hat{\rho}\hat{a}^{2}], \\
T_4 &= \frac{AF}{2B}Tr[\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{3} + \hat{a}\hat{\rho}\hat{a}^{2}]. \qquad (3.3)
\end{aligned}$$

The above traces can be simplified taking in to account the cyclic property and the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. It then follows that

$$\begin{aligned} T_1 &= \frac{AC}{2B} Tr[2\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}], \\ &= \frac{AC}{2B} Tr[\hat{\rho}\hat{a}(\hat{a}^\dagger\hat{a} + 1) - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}], \\ &= \frac{AC}{2B} Tr[\hat{\rho}\hat{a}], \\ &= \frac{AC}{2B} \langle\hat{a}\rangle, \end{aligned} \tag{3.4}$$

$$\begin{aligned} T_2 &= \frac{1}{2}(\frac{AD}{B} + \kappa) Tr[2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2], \\ &= \frac{1}{2}(\frac{AD}{B} + \kappa) Tr[\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}(1 + \hat{a}^\dagger\hat{a})\hat{a}], \\ &= \frac{1}{2}(\frac{AD}{B} + \kappa) Tr[-\hat{\rho}\hat{a}], \\ &= -\frac{1}{2}(\frac{AD}{B} + \kappa)\langle\hat{a}\rangle, \end{aligned} \tag{3.5}$$

$$\begin{aligned} T_3 &= \frac{AE}{2B} Tr[\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} + \hat{\rho}\hat{a}^3], \\ &= \frac{AE}{2B} Tr[\hat{\rho}\hat{a}^\dagger(1 + \hat{a}^\dagger\hat{a}) - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}), \\ &= \frac{AE}{2B} Tr[\hat{\rho}\hat{a}^\dagger], \\ &= \frac{AE}{2B}\langle\hat{a}^\dagger\rangle, \end{aligned} \tag{3.6}$$

$$\begin{aligned} T_4 &= \frac{AF}{2B} Tr[\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^{\dagger 2} - \hat{\rho}\hat{a}^3 + \hat{\rho}\hat{a}^3], \\ &= \frac{AF}{2B} Tr[\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}(1 + \hat{a}^\dagger\hat{a})\hat{a}^\dagger], \\ &= \frac{AF}{2B} Tr[-\hat{\rho}\hat{a}^\dagger], \\ &= -\frac{AF}{2B}\langle\hat{a}^\dagger\rangle. \end{aligned} \tag{3.7}$$

By substituting Eqs. (3.4), (3.5), (3.6) and (3.7) in to Eq.(3.2) we have

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}(t)\rangle &= \frac{AC}{2B}\langle\hat{a}\rangle - \frac{1}{2}(\frac{AD}{B} + k)\langle\hat{a}\rangle + \frac{AE}{2B}\langle\hat{a}^\dagger\rangle - \frac{AF}{2B}\langle\hat{a}^\dagger\rangle, \\ &= -\frac{1}{2}(\frac{A}{B}(D - C) + \kappa)\langle\hat{a}\rangle + \frac{A}{2B}(E - F)\langle\hat{a}^\dagger\rangle, \\ &= -\frac{\mu}{2}\langle\hat{a}(t)\rangle + \beta\langle\hat{a}^\dagger(t)\rangle, \end{aligned} \tag{3.8}$$

where

$$\mu = \frac{A}{B}(D - C) + \kappa, \tag{3.9}$$

$$\beta = \frac{A}{2B}(E - F). \tag{3.10}$$

Furthermore, we note that

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a}^2(t) \rangle &= Tr(\frac{d\hat{\rho}}{dt}\hat{a}^2), \\
&= \frac{AC}{2B}Tr[2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2], \\
&+ \frac{1}{2}(\frac{AD}{B} + \kappa)Tr[2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3], \\
&+ \frac{AE}{2B}Tr[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^2\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}\hat{\rho}\hat{a}^3], \\
&+ \frac{AF}{2B}Tr[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^{\dagger 2}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^4 + \hat{a}\hat{\rho}\hat{a}^3],
\end{aligned} \tag{3.11}$$

or

$$\frac{d}{dt}\langle \hat{a}^2(t) \rangle = S_1 + S_2 + S_3 + S_4 \tag{3.12}$$

where

$$\begin{aligned}
S_1 &= \frac{AC}{2B}Tr[2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2], \\
S_2 &= \frac{1}{2}(\frac{AD}{B} + \kappa)Tr[2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3], \\
S_3 &= \frac{AE}{2B}Tr[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^2\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}\hat{\rho}\hat{a}^3], \\
S_4 &= \frac{AF}{2B}Tr[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^{\dagger 2}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^4 + \hat{a}\hat{\rho}\hat{a}^3].
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
S_1 &= \frac{AC}{2B}Tr[2\hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2], \\
&= \frac{AC}{2B}Tr[\hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}(\hat{a}^\dagger\hat{a} + 1)\hat{a}^2], \\
&= \frac{AC}{2B}Tr[\hat{\rho}\hat{a}^2], \\
&= \frac{AC}{2B}\langle \hat{a}^2 \rangle,
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
S_2 &= \frac{1}{2}(\frac{AD}{B}+\kappa)Tr[2\hat{\rho}\hat{a}^\dagger\hat{a}^3-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^\dagger\hat{a}^3],\\
&= \frac{1}{2}(\frac{AD}{B}+\kappa)Tr[\hat{\rho}\hat{a}^\dagger\hat{a}^3-\hat{\rho}\hat{a}(1+\hat{a}^\dagger\hat{a})\hat{a}],\\
&= \frac{1}{2}(\frac{AD}{B}+\kappa)Tr[\hat{\rho}\hat{a}^\dagger\hat{a}^3-\hat{\rho}\hat{a}^2-\hat{\rho}(1+\hat{a}^\dagger\hat{a})\hat{a}^2,\\
&= \frac{1}{2}(\frac{AD}{B}+\kappa)Tr[-2\hat{\rho}\hat{a}^2],\\
&= -(\frac{AD}{B}+\kappa)\langle\hat{a}^2\rangle,
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
S_3 &= \frac{AE}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger-\hat{\rho}\hat{a}^4-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2+\hat{\rho}\hat{a}^4],\\
&= \frac{AE}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}(1+\hat{a}^\dagger\hat{a})-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2],\\
&= \frac{AE}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger(1+\hat{a}^\dagger\hat{a})-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2],\\
&= \frac{AE}{2B}Tr[2\hat{\rho}\hat{a}^\dagger\hat{a}],\\
&= \frac{AE}{B}\langle\hat{a}^\dagger\hat{a}\rangle,
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
S_4 &= \frac{AF}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger-\hat{\rho}\hat{a}^2\hat{a}^{\dagger 2}-\hat{\rho}\hat{a}^4+\hat{\rho}\hat{a}^4],\\
&= \frac{AF}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}(1+\hat{a}^\dagger\hat{a})-\hat{\rho}\hat{a}(1+\hat{a}^\dagger\hat{a})\hat{a}^\dagger,\\
&= \frac{AF}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{a}\dagger\hat{a}-\hat{\rho}\hat{a}\hat{a}^\dagger-\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger],\\
&= \frac{AF}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger(1+\hat{a}^\dagger\hat{a})\hat{a}-\hat{\rho}(1+\hat{a}^\dagger\hat{a})-\hat{\rho}(1+\hat{a}^\dagger\hat{a})(1+\hat{a}\dagger\hat{a})],\\
&= \frac{AF}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2-\hat{\rho}-\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}(1+2\hat{a}\hat{a}\dagger+\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a})],\\
&= \frac{AF}{2B}Tr[\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2-\hat{\rho}-\hat{\rho}-2\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^\dagger(1+\hat{a}^\dagger\hat{a})\hat{a}],\\
&= \frac{AF}{2B}Tr[-2\hat{\rho}-2\hat{\rho}\hat{a}^\dagger\hat{a}],\\
&= -2(\frac{AF}{2B})[1+\langle\hat{a}^\dagger\hat{a}\rangle].
\end{aligned} \tag{3.17}
$$

By substituting Eqs. (3.14), (3.15), (3.16) and (3.17) in to (3.12) we have

$$
\begin{aligned}
\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= \frac{AC}{2B}\langle\hat{a}^2\rangle-(\frac{AD}{B}+\kappa)\langle\hat{a}^2\rangle+\frac{AE}{B}\langle\hat{a}^\dagger\hat{a}\rangle-2(\frac{AF}{2B})Tr[1+\langle\hat{a}^\dagger\hat{a}\rangle],\\
&= (\frac{AC}{2B}-(\frac{AD}{B}+\kappa))\langle\hat{a}^2\rangle+2(\frac{AE}{2B}-\frac{AF}{2B})\langle\hat{a}^\dagger\hat{a}\rangle-2\frac{AF}{2B},\\
&= -\mu\langle\hat{a}^2(t)\rangle+2\beta\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle-\frac{AF}{B}.
\end{aligned} \tag{3.18}
$$

Following similar procedure, it can be verified that

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\mu\langle\hat{a}^\dagger\hat{a}\rangle + \beta[\langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^2\rangle] + \frac{AC}{B}, \tag{3.19}$$

where μ and β are given in Eqs. (3.9) and (3.10).

The operators in Eqs. (3.8), (3.18), and (3.19) are in the normal order. Hence we can express these equations in terms of the c-number variables associated with the normal ordering as

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\mu}{2}\langle\alpha(t)\rangle + \beta\langle\alpha^*(t)\rangle, \tag{3.20}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + 2\beta\langle\alpha^*(t)\alpha(t)\rangle - \frac{AF}{B}, \tag{3.21}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \beta[\langle\alpha^{*2}(t)\rangle + \langle\alpha^2(t)\rangle] + \frac{AC}{B}. \tag{3.22}$$

On the basis of Eqs. (3.20), it is possible to write

$$\frac{d}{dt}\alpha(t) = -\frac{\mu}{2}\alpha(t) + \beta\alpha^*(t) + F(t), \tag{3.23}$$

where F(t) is the corresponding noise force the properties of which remain to be determined. Correlation properties of noise force can be written as the expectation value of Eq. (3.23)

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\mu}{2}\langle\alpha(t)\rangle + \beta\langle\alpha^*(t)\rangle + \langle F(t)\rangle. \tag{3.24}$$

By comparing Eqs. (3.20) and (3.24) we get

$$\langle F(t)\rangle = 0. \tag{3.25}$$

By using Eq. (3.23) together with the relation $\frac{d}{dt}\langle\alpha^2(t)\rangle = \langle\alpha(t)\frac{d\alpha(t)}{dt}\rangle + \langle\frac{d\alpha(t)}{dt}\alpha(t)\rangle$, we see that

$$\begin{aligned}\frac{d}{dt}\langle\alpha^2(t)\rangle &= \langle\alpha(t)(-\frac{\mu}{2}\alpha(t) + \beta\alpha^*(t) + F(t))\rangle + \langle(-\frac{\mu}{2}\alpha(t) + \beta\alpha^*(t) + F(t))\alpha(t)\rangle \\ &= -\mu\langle\alpha^2(t)\rangle + 2\beta\langle\alpha^*(t)\alpha(t)\rangle + \langle\alpha(t)F(t)\rangle + \langle F(t)\alpha(t)\rangle \end{aligned} \tag{3.26}$$

Comparison of this with Eq. (3.21) indicates that

$$\langle \alpha(t)F(t)\rangle + \langle F(t)\alpha(t)\rangle = -\frac{AF}{B}. \tag{3.27}$$

The solution of Eq. (3.23) is

$$\alpha(t) = \alpha(0)e^{\frac{-\mu}{2}t} + \beta e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\alpha^*(t')dt' + e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}F(t')dt'. \tag{3.28}$$

Multiplying this by F(t) from right and taking the expectation value of the resulting expression, we obtain

$$\langle \alpha(t)F(t)\rangle = \langle \alpha(0)F(t)\rangle e^{\frac{-\mu}{2}t} + \beta e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle \alpha^*(t')F(t)\rangle dt' + e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F(t')F(t)\rangle dt'. \tag{3.29}$$

Multiplying Eq(3.28) by F(t) from left and taking the expectation value of the resulting expression, we get

$$\langle F(t)\alpha(t)\rangle = \langle F(t)\alpha(0)\rangle e^{\frac{-\mu}{2}t} + \beta e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F(t)\alpha^*(t')\rangle dt' + e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F(t)F(t')\rangle dt' \tag{3.30}$$

On account of Eq. (3.25) along with the fact that the noise operator F(t) at some time t should not affect system variables at an earlier times, we can write

$$\begin{aligned}\langle \alpha(0)F(t)\rangle &= \langle \alpha(0)\rangle\langle F(t)\rangle = 0,\\ \langle F(t)\alpha(0)\rangle &= \langle F(t)\rangle\langle \alpha(0)\rangle = 0,\\ \langle \alpha^*(t')F(t)\rangle &= \langle \alpha^*(t')\rangle\langle F(t)\rangle = 0,\\ \langle F(t)\alpha^*(t')\rangle &= \langle F(t)\rangle\langle \alpha^*(t')\rangle = 0.\end{aligned} \tag{3.31}$$

In view of these results, Eqs. (3.29) and (3.30) reduce to

$$\langle \alpha(t)F(t)\rangle = e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F(t')F(t)\rangle dt', \tag{3.32}$$

and

$$\langle F(t)\alpha(t)\rangle = e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F(t)F(t')\rangle dt'. \tag{3.33}$$

By substituting Eqs. (3.32) and (3.33) in to Eq.(3.27) and assuming $\langle F(t')F(t)\rangle = \langle F(t)F(t')\rangle$, we have

$$\int_0^t e^{\frac{-\mu(t-t')}{2}}\langle F(t')F(t)\rangle dt' = -\frac{1}{2}\frac{AF}{B}. \tag{3.34}$$

Now on the basis of the relation

$$\int_0^t e^{-a(t-t')}\langle F(t')G(t)\rangle dt' = C, \tag{3.35}$$

it can be expressed as

$$\langle F(t')G(t)\rangle = 2C\delta(t-t'), \tag{3.36}$$

with a being a constant. Therefore, in view of Eqs. (3.34), (3.35) and (3.36), we get

$$\langle F(t')F(t)\rangle = -\frac{AF}{B}\delta(t-t'). \tag{3.37}$$

The complex conjugate of Eq. (3.23) is

$$\frac{d}{dt}\alpha^*(t) = -\frac{\mu}{2}\alpha^*(t) + \beta\alpha(t) + F^*(t). \tag{3.38}$$

Moreover, applying Eqs. (3.23) and (3.38) along with the relation $\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\frac{d\alpha^*(t)}{dt}\alpha(t)\rangle + \langle\alpha^*(t)\frac{d\alpha(t)}{dt}\rangle$, we see that

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -\frac{\mu}{2}\langle\alpha^*(t)\alpha(t)\rangle + \beta\langle\alpha^2(t)\rangle + \langle F^*(t)\alpha(t)\rangle - \frac{\mu}{2}\langle\alpha^*(t)\alpha(t)\rangle + \beta\langle\alpha^{*2}(t)\rangle + \langle\alpha^*(t)F(t)\rangle\\ &= -\mu\langle\alpha^*(t)\alpha(t)\rangle + \beta(\langle\alpha^2(t)\rangle + \langle\alpha^{*2}(t)\rangle) + \langle F^*(t)\alpha(t)\rangle + \langle\alpha^*(t)F(t)\rangle.\end{aligned} \tag{3.39}$$

By comparing Eq. (3.39) with Eq. (3.22) we get

$$\langle F^*(t)\alpha(t)\rangle + \langle\alpha^*(t)F(t)\rangle = \frac{AC}{B}. \tag{3.40}$$

Multiplying Eq. (3.28) from left by $F^*(t)$ and its complex conjugate from right by F(t) and taking the expectation value in both cases, we have

$$\langle F^*(t)\alpha(t)\rangle = \langle F^*(t)\alpha(0)\rangle e^{\frac{-\mu}{2}t} + \beta e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F^*(t)\alpha^*(t')\rangle dt' + e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F^*(t)F(t')\rangle dt' \tag{3.41}$$

and

$$\langle\alpha^*(t)F(t)\rangle = \langle\alpha^*(0)F(t)\rangle e^{\frac{-\mu}{2}t} + \beta e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle\alpha(t')F(t)\rangle dt' + e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F^*(t')F(t)\rangle dt'. \tag{3.42}$$

On account of Eq. (3.25) along with the fact that the noise operators F(t) and $F^*(t)$ at time t should not affect system variables at an earlier times, we have

$$\langle F^*(t)\alpha(0)\rangle = \langle F^*(t)\rangle\langle\alpha(0)\rangle = 0,$$
$$\langle F^*(t)\alpha^*(t'))\rangle = \langle F^*(t)\rangle\langle\alpha^*(t')\rangle = 0,$$
$$\langle\alpha^*(0)F(t)\rangle = \langle\alpha^*(0)\rangle\langle F(t)\rangle = 0,$$
$$\langle\alpha(t')F(t)\rangle = \langle\alpha(t')\rangle\langle F(t)\rangle = 0. \tag{3.43}$$

According to Eq. (3.43), we can put Eqs. (3.41) and (3.42) in the form

$$\langle F^*(t)\alpha(t)\rangle = e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F^*(t)F(t')\rangle dt' \tag{3.44}$$

and

$$\langle\alpha^*(t)F(t)\rangle = e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F^*(t')F(t)\rangle dt'. \tag{3.45}$$

By substituting Eqs. (3.44) and (3.45) in to Eq. (3.40) and assuming $\langle F^*(t)F(t')\rangle = \langle F^*(t')F(t)\rangle$, we have

$$e^{\frac{-\mu}{2}t}\int_0^t e^{\frac{\mu}{2}t'}\langle F^*(t)F(t')\rangle dt' = \frac{1}{2}\frac{AC}{B} \tag{3.46}$$

and taking Eq. (3.35) and (3.36) into account, we obtain

$$\langle F^*(t)F(t')\rangle = \frac{AC}{B}\delta(t-t'). \tag{3.47}$$

We notice that Eqs. (3.25), (3.37), and (3.47) represent the correlation properties of the noise force.

We now proceed to obtain the solution of the c-number Langevin equations. Adding Eqs. (3.23) and (3.38) results in

$$\frac{d}{dt}(\alpha^*(t)+\alpha(t)) = -\frac{\mu}{2}(\alpha^*(t)+\alpha(t)) + \beta(\alpha^*(t)+\alpha(t)) + F^*(t) + F(t)$$
$$\frac{d}{dt}\alpha_+(t) = -\frac{1}{2}(\mu-2\beta)\alpha_+(t) + F^*(t) + F(t), \tag{3.48}$$

and subtracting Eq. (3.23) from Eq. (3.38) we get

$$\frac{d}{dt}(\alpha^*(t)-\alpha(t)) = -\frac{\mu}{2}(\alpha^*(t)-\alpha(t)) - \beta(\alpha^*(t)-\alpha(t)) + F^*(t) - F(t)$$
$$\frac{d}{dt}\alpha_-(t) = -\frac{1}{2}(\mu+2\beta)\alpha_-(t) + F^*(t) - F(t) \tag{3.49}$$

On the basis of Eq. (3.48) and (3.49), we see that

$$\frac{d}{dt}\alpha_\pm(t) = -\frac{\lambda_\mp}{2}\alpha_\pm(t) + F^*(t) \pm F(t), \tag{3.50}$$

where

$$\alpha_\pm(t) = \alpha^*(t) \pm \alpha(t), \tag{3.51}$$

and

$$\lambda_\mp = \mu \mp 2\beta. \tag{3.52}$$

The solution of Eq. (3.50) is expressible as

$$\alpha_\pm(t) = \alpha_\pm(0)e^{-\frac{\lambda_\mp}{2}t} + \int_0^t e^{-\frac{\lambda_\mp}{2}(t-t')}F^*(t')dt' \pm \int_0^t e^{-\frac{\lambda_\mp}{2}(t-t')}F(t')dt'. \tag{3.53}$$

It then follows that

$$\alpha_+(t) = \alpha_+(0)e^{-\frac{\lambda_-}{2}t} + \int_0^t e^{-\frac{\lambda_-}{2}(t-t')}F^*(t')dt' + \int_0^t e^{-\frac{\lambda_-}{2}(t-t')}F(t')dt', \tag{3.54}$$

and

$$\alpha_-(t) = \alpha_-(0)e^{-\frac{\lambda_+}{2}t} + \int_0^t e^{-\frac{\lambda_+}{2}(t-t')}F^*(t')dt' - \int_0^t e^{-\frac{\lambda_+}{2}(t-t')}F(t')dt'. \tag{3.55}$$

Now using Eq. (3.51), we have

$$\alpha^*(t) + \alpha(t) = \alpha^*(0)e^{-\frac{\lambda_-}{2}t} + \alpha(0)e^{-\frac{\lambda_-}{2}t} + e^{-\frac{\lambda_-}{2}t}\int_0^t e^{\frac{\lambda_-}{2}t'}(F^*(t') + F(t'))dt', \tag{3.56}$$

and

$$\alpha^*(t) - \alpha(t) = \alpha^*(0)e^{-\frac{\lambda_+}{2}t} - \alpha(0)e^{-\frac{\lambda_+}{2}t} + e^{-\frac{\lambda_+}{2}t}\int_0^t e^{\frac{\lambda_+}{2}t'}(F^*(t') - F(t'))dt' \tag{3.57}$$

By subtracting Eq. (3.57) from Eq. (3.56), we have

$$\begin{aligned}
2\alpha(t) &= \alpha^*(0)(e^{-\frac{\lambda_-}{2}t} - e^{-\frac{\lambda_+}{2}t}) + \alpha(0)(e^{-\frac{\lambda_-}{2}t} + e^{-\frac{\lambda_+}{2}t}) \\
&\quad + e^{-\frac{\lambda_-}{2}t}\int_0^t e^{\frac{\lambda_-}{2}t'}(F^*(t') + F(t'))dt' - (e^{-\frac{\lambda_+}{2}t}\int_0^t e^{\frac{\lambda_+}{2}t'}(F^*(t') - F(t'))dt') \\
\alpha(t) &= \frac{1}{2}\alpha^*(0)(e^{-\frac{\lambda_-}{2}t} - e^{-\frac{\lambda_+}{2}t}) + \frac{1}{2}\alpha(0)(e^{-\frac{\lambda_-}{2}t} + e^{-\frac{\lambda_+}{2}t}) \\
&\quad + \frac{1}{2}e^{-\frac{\lambda_-}{2}t}\int_0^t e^{\frac{\lambda_-}{2}t'}(F(t') + F^*(t'))dt' + \frac{1}{2}(e^{-\frac{\lambda_+}{2}t}\int_0^t e^{\frac{\lambda_+}{2}t'}(F(t') - F^*(t'))dt').
\end{aligned} \tag{3.58}$$

or

$$\alpha(t) = a_+(t)\alpha(0) + a_-(t)\alpha^*(0) + E_-(t) + E_+(t), \tag{3.59}$$

in which

$$a_\pm(t) = \frac{1}{2}(e^{-\frac{\lambda_- t}{2}} \pm e^{-\frac{\lambda_+ t}{2}}), \tag{3.60}$$

$$E_\pm(t) = \frac{1}{2}\int_0^t e^{-\frac{\lambda_\mp (t-t')}{2}}[F(t') \pm F^*(t')]dt'. \tag{3.61}$$

We observe that a well-behaved solution of Eq. (3.54) exists at steady state for $\lambda_- > 0$. As a result, $\mu = 2\beta$ is designated as a threshold condition. It may worth mentioning that photon statistics as well as the quadrature variance of the cavity radiation can be studied making use of Eq. (3.59).

Chapter 4

Photon Statistics

4.1 Mean photon number

In this chapter we calculate the mean and variance of the photon number for the cavity light produced by the degenerate three-level laser coupled to a vacuum reservoir.
The mean photon number of the cavity radiation is

$$\overline{n} = \langle \alpha^*(t)\alpha(t)\rangle. \tag{4.1}$$

By using the solution of the c-number Langevin equation written in Eq. (3.59), the mean photon number can be written as

$$\begin{aligned}
\overline{n} &= \langle \alpha^*(t)\alpha(t)\rangle \\
&= \langle (a_+(t)\alpha^*(0) + a_-(t)\alpha(0) + E_-^*(t) + E_+^*(t))(a_+(t)\alpha(0) + a_-(t)\alpha^*(0) + E_-(t) + E_+(t))\rangle, \\
&= a_+^2(t)\langle \alpha^*(0)\alpha(0)\rangle + a_+(t)a_-(t)\langle \alpha^{*2}(0)\rangle + a_+(t)\langle \alpha^*(0)E_-(t)\rangle + a_+(t)\langle \alpha^*(0)E_+(t)\rangle \\
&+ a_-(t)a_+(t)\langle \alpha^2(0)\rangle + a_-^2(t)\langle \alpha(0)\alpha^*(0)\rangle + a_-(t)\langle \alpha(0)E_-(t)\rangle + a_-(t)\langle \alpha(0)E_+(t)\rangle \\
&+ a_+(t)\langle E_-^*(t)\alpha(0)\rangle + a_-(t)\langle E_-^*(t)\alpha^*(0)\rangle + \langle E_-^*(t)E_-(t)\rangle + \langle E_-^*(t)E_+(t)\rangle \\
&+ a_+(t)\langle E_+^*(t)\alpha(0)\rangle + a_-(t)\langle E_+^*(t)\alpha^*(0)\rangle + \langle E_+^*(t)E_-(t)\rangle + \langle E_+^*(t)E_+(t)\rangle.
\end{aligned} \tag{4.2}$$

At steady state, Eq. (3.60) becomes

$$a_\pm(t) = \frac{1}{2}(e^{-\frac{\lambda_- t}{2}} \pm e^{-\frac{\lambda_+ t}{2}}) = 0. \tag{4.3}$$

On account of Eq. (3.25) along with the fact that the noise forces F(t) and $F^*(t)$ at time t should not affect system variables at earlier times and for the cavity mode assumed initially to be in a vacuum state, Eq. (4.2) can be expressed as

$$\overline{n} = \langle \alpha^*(t)\alpha(t)\rangle = \langle E_-^*(t)E_-(t)\rangle + \langle E_-^*(t)E_+(t)\rangle + \langle E_+^*(t)E_-(t)\rangle + \langle E_+^*(t)E_+(t)\rangle. \quad (4.4)$$

Employing Eq. (3.61) and considering the correlation of properties of the noise forces,

$$\begin{aligned}
\langle E_-^*(t)E_-(t)\rangle &= \langle(\frac{1}{2}\int_0^t e^{-\frac{\lambda_+(t-t')}{2}}[F^*(t') - F(t')]dt')(\frac{1}{2}\int_0^t e^{-\frac{\lambda_+(t-t'')}{2}}[F(t'') - F^*(t'')]dt'')\rangle, \\
&= \frac{1}{4}e^{-\lambda_+ t}\int_0^t e^{\frac{\lambda_+ t'}{2}}[\int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F^*(t')F(t'')\rangle dt'' - \int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F^*(t')F^*(t'')\rangle dt'' \\
&- \int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F(t')F(t'')\rangle dt'' + \int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F(t')F^*(t'')\rangle dt'']dt', \\
&= \frac{1}{4}e^{-\lambda_+ t}\int_0^t e^{\frac{\lambda_+ t'}{2}}[\frac{AC}{B}\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'' - (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'' \\
&- (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'' + \frac{AC}{B}\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'']dt', \\
&= \frac{1}{4}e^{-\lambda_+ t}[\frac{2AF}{B} + \frac{2AC}{B}]\int_0^t e^{\frac{\lambda_+ t'}{2}}\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt''dt', \\
&= \frac{1}{2}[\frac{AF}{B} + \frac{AC}{B}]e^{-\lambda_+ t}\int_0^t e^{\lambda_+ t'}dt', \\
&= \frac{1}{2}[\frac{AF}{B} + \frac{AC}{B}]e^{-\lambda_+ t}\frac{1}{\lambda_+}(e^{\lambda_+ t} - 1), \\
&= \frac{A}{2B\lambda_+}[F + C](1 - e^{-\lambda_+ t}), \quad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
\langle E_-^*(t)E_+(t)\rangle &= \langle(\frac{1}{2}\int_0^t e^{-\frac{\lambda_+(t-t')}{2}}[F^*(t') - F(t')]dt')(\frac{1}{2}\int_0^t e^{-\frac{\lambda_-(t-t'')}{2}}[F(t'') + F^*(t'')]dt'')\rangle, \\
&= \frac{1}{4}e^{\frac{-\lambda_+ t}{2}}e^{\frac{-\lambda_- t}{2}}\int_0^t e^{\frac{\lambda_+ t'}{2}}[\int_0^t e^{\frac{\lambda_- t''}{2}}\langle F^*(t')F(t'')\rangle dt'' + \int_0^t e^{\frac{\lambda_- t''}{2}}\langle F^*(t')F^*(t'')\rangle dt'' \\
&- \int_0^t e^{\frac{\lambda_- t''}{2}}\langle F(t')F(t'')\rangle dt'' - \int_0^t e^{\frac{\lambda_- t''}{2}}\langle F(t')F^*(t'')\rangle dt'']dt', \\
&= \frac{1}{4}e^{\frac{-\lambda_+ t}{2}}e^{\frac{-\lambda_- t}{2}}\int_0^t e^{\frac{\lambda_+ t'}{2}}[\frac{AC}{B}\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'' + (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'' \\
&- (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'' - \frac{AC}{B}\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'']dt', \\
&= 0, \quad (4.6)
\end{aligned}$$

$$
\begin{aligned}
\langle E_+^*(t)E_-(t)\rangle &= \langle(\frac{1}{2}\int_0^t e^{-\frac{\lambda_-(t-t')}{2}}[F^*(t')+F(t')]dt')(\frac{1}{2}\int_0^t e^{-\frac{\lambda_+(t-t")}{2}}[F(t'')-F^*(t'')]dt'')\rangle,\\
&= \frac{1}{4}e^{\frac{-\lambda_- t}{2}}e^{\frac{-\lambda_+ t}{2}}\int_0^t e^{\frac{\lambda_- t'}{2}}[\int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F^*(t')F(t'')\rangle dt'' - \int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F^*(t')F^*(t'')\rangle dt"\\
&+ \int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F(t')F(t'')\rangle dt'' - \int_0^t e^{\frac{\lambda_+ t''}{2}}\langle F(t')F^*(t'')\rangle dt'']dt',\\
&= \frac{1}{4}e^{\frac{-\lambda_- t}{2}}e^{\frac{-\lambda_+ t}{2}}\int_0^t e^{\frac{\lambda_- t'}{2}}[\frac{AC}{B}\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'' - (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt''\\
&+ (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'' - \frac{AC}{B}\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'']dt',\\
&= 0,
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
\langle E_+^*(t)E_+(t)\rangle &= \langle(\frac{1}{2}\int_0^t e^{-\frac{\lambda_-(t-t')}{2}}[F^*(t')+F(t')]dt')(\frac{1}{2}\int_0^t e^{-\frac{\lambda_-(t-t'')}{2}}[F(t'')+F^*(t'')]dt'')\rangle,\\
&= \frac{1}{4}e^{-\lambda_- t}\int_0^t e^{\frac{\lambda_- t'}{2}}[\int_0^t e^{\frac{\lambda_- t''}{2}}\langle F^*(t')F(t'')\rangle dt'' + \int_0^t e^{\frac{\lambda_- t''}{2}}\langle F^*(t')F^*(t'')\rangle dt''\\
&+ \int_0^t e^{\frac{\lambda_- t''}{2}}\langle F(t')F(t'')\rangle dt'' + \int_0^t e^{\frac{\lambda_- t''}{2}}\langle F(t)F^*(t)\rangle dt'']dt',\\
&= \frac{1}{4}e^{-\lambda_- t}\int_0^t e^{\frac{\lambda_- t'}{2}}[\frac{AC}{B}\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'' + (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt''\\
&+ (-\frac{AF}{B})\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'' + \frac{AC}{B}\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt'']dt',\\
&= \frac{1}{4}e^{-\lambda_- t}[-\frac{2AF}{B}+\frac{2AC}{B}]\int_0^t e^{\frac{\lambda_- t'}{2}}\int_0^t e^{\frac{\lambda_- t''}{2}}\delta(t'-t'')dt''dt',\\
&= \frac{1}{2}[-\frac{AF}{B}+\frac{AC}{B}]e^{-\lambda_- t}\int_0^t e^{\lambda_- t'}dt'\\
&= \frac{1}{2}[-\frac{AF}{B}+\frac{AC}{B}]e^{-\lambda_- t}\frac{1}{\lambda_-}(e^{\lambda_- t}-1),\\
&= \frac{A}{2B\lambda_-}[-F+C](1-e^{-\lambda_- t})
\end{aligned} \tag{4.8}
$$

By substituting Eqs. (4.5), (4.6), (4.7) and (4.8) in to Eq. (4.4) we get

$$
\overline{n} = \frac{A}{2B\lambda_+}[F+C](1-e^{-\lambda_+ t}) + \frac{A}{2B\lambda_-}[-F+C](1-e^{-\lambda_- t}). \tag{4.9}
$$

At steady state this becomes

$$
\overline{n} = \frac{A}{2B\lambda_+}[F+C] - \frac{A}{2B\lambda_-}[F-C] \tag{4.10}
$$

In order to put Eq. (4.10) in a more convenient manner, we express the initial atomic coherence of the top and bottom levels as

$$\rho_{ac}(0) = |\rho_{ac}(0)|e^{\imath\theta}, \tag{4.11}$$

where θ is the phase factor. It can also be easily checked that

$$|\rho_{ac}(0)| = \sqrt{\rho_{aa}(0)\rho_{cc}(0)}. \tag{4.12}$$

Moreover, upon introducing a new parameter η defined by [2], as described in Eq. (2.7), Eq. (2.10), Eq. (2.11), and Eq. (2.12),

$$\rho_{aa}(0) = \frac{1-\eta}{2}, \tag{4.13}$$

with $-1 \leq \eta \leq 1$,we see that

$$\rho_{cc}(0) = \frac{1+\eta}{2}, \tag{4.14}$$

$$\rho_{ac}(0) = \frac{\sqrt{1-\eta^2}}{2}e^{\imath\theta}. \tag{4.15}$$

Therefore, on the basis of Eqs.(2.55), (2.56), (2.57), (2.58), (2.59), (3.9), (3.10), (3.52), (4.13), (4.14), and Eq. (4.15), finally for the case $\theta = 0$, the form

$$\begin{aligned} B\lambda_{\mp} &= B(\mu \mp 2\beta), \\ &= B(\frac{A}{B}(D-C)) + \kappa \mp 2(\frac{A}{2B}(E-F)), \\ &= B\kappa + A(D-C) \mp A(E-F). \end{aligned} \tag{4.16}$$

$$
\begin{aligned}
D-C &= \rho_{aa}(0)\frac{3\Omega^2}{4\gamma^2}+\rho_{ac}(0)\frac{3\Omega}{2\gamma}+\rho_{cc}(0)(1+\frac{\Omega^2}{4\gamma^2}) \\
&- (\rho_{aa}(0)(1+\frac{\Omega^2}{4\gamma^2})-\rho_{ac}(0)\frac{3\Omega}{2\gamma}+\rho_{cc}(0)\frac{3\Omega^2}{4\gamma^2}), \\
&= \frac{3\Omega^2}{4\gamma^2}(\rho_{aa}(0)-\rho_{cc}(0))+(1+\frac{\Omega^2}{4\gamma^2})(\rho_{cc}(0)-\rho_{aa}(0))+2\rho_{ac}(0)\frac{3\Omega}{2\gamma}, \\
&= \frac{3\Omega^2}{4\gamma^2}(2\rho_{aa}(0)-1)+(1+\frac{\Omega^2}{4\gamma^2})(1-2\rho_{aa}(0))+2\rho_{ac}(0)\frac{3\Omega}{2\gamma}, \\
&= 2\rho_{aa}(0)\frac{3\Omega^2}{4\gamma^2}-\frac{3\Omega^2}{4\gamma^2}+(1+\frac{\Omega^2}{4\gamma^2})-2\rho_{aa}(0)(1+\frac{\Omega^2}{4\gamma^2})+2\rho_{ac}(0)\frac{3\Omega}{2\gamma}, \\
&= (1-\eta)\frac{3\Omega^2}{4\gamma^2}-\frac{3\Omega^2}{4\gamma^2}+(1+\frac{\Omega^2}{4\gamma^2})-(1-\eta)(1+\frac{\Omega^2}{4\gamma^2})+\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}, \\
&= -\frac{3}{4}\frac{\Omega^2}{\gamma^2}\eta+\eta+\frac{\Omega^2}{4\gamma^2}\eta+\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}, \\
&= (1-\frac{\Omega^2}{2\gamma^2})\eta+\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma},
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
E-F &= -\rho_{aa}(0)\frac{\Omega}{2\gamma}(1-\frac{\Omega^2}{2\gamma^2})-\rho_{ac}(0)(1-\frac{\Omega^2}{2\gamma^2})+\rho_{cc}(0)\frac{\Omega}{\gamma}(1+\frac{\Omega^2}{4\gamma^2}) \\
&- (-\rho_{aa}(0)\frac{\Omega}{\gamma}(1+\frac{\Omega^2}{4\gamma^2})-\rho_{ac}(0)(1-\frac{\Omega^2}{2\gamma^2})+\rho_{cc}(0)\frac{\Omega}{2\gamma}(1-\frac{\Omega^2}{2\gamma^2})), \\
&= -\frac{\Omega}{2\gamma}(1-\frac{\Omega^2}{2\gamma^2})(\rho_{aa}(0)+\rho_{cc}(0))+\frac{\Omega}{\gamma}(1+\frac{\Omega^2}{4\gamma^2}(\rho_{aa}(0)+\rho_{cc}(0)), \\
&= -\frac{\Omega}{2\gamma}(1-\frac{\Omega^2}{2\gamma^2})+\frac{\Omega}{\gamma}(1+\frac{\Omega^2}{4\gamma^2}), \\
&= -\frac{\Omega}{2\gamma}+\frac{\Omega^3}{4\gamma^3}+\frac{\Omega}{\gamma}+\frac{\Omega^3}{4\gamma^3}, \\
&= \frac{\Omega}{2\gamma}+\frac{\Omega^3}{2\gamma^3}, \\
&= \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{2\gamma^2}).
\end{aligned}
\tag{4.18}
$$

By substituting Eqs. (2.55), (4.17), and (4.18) in to Eq.(4.16), we have

$$
\begin{aligned}
\chi_\pm &= B\lambda_\mp \\
&= \kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+A[(1-\frac{\Omega^2}{2\gamma^2})\eta+\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}\mp\frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{2\gamma^2})],
\end{aligned}
\tag{4.19}
$$

and also

$$
\begin{aligned}
F \pm C &= -\rho_{aa}(0)\frac{\Omega}{\gamma}(1+\frac{\Omega^2}{4\gamma^2}) - \rho_{ac}(0)(1-\frac{\Omega^2}{2\gamma^2}) + \rho_{cc}(0)\frac{\Omega}{2\gamma}(1-\frac{\Omega^2}{2\gamma^2}) \\
&\pm (\rho_{aa}(0)(1+\frac{\Omega^2}{4\gamma^2}) - \rho_{ac}(0)\frac{3\Omega}{2\gamma} + \rho_{cc}(0)\frac{3\Omega^2}{4\gamma^2}), \\
&= \frac{\Omega}{2\gamma}(-2\rho_{aa}(0)+\rho_{cc}(0)) - \frac{\Omega^3}{4\gamma^3}(\rho_{aa}(0)+\rho_{cc}(0)) - \rho_{ac}(0)(1-\frac{\Omega^2}{2\gamma^2}) \\
&\pm (\rho_{aa}(0) + \rho_{aa}(0)\frac{\Omega^2}{4\gamma^2} + \frac{3\Omega^2}{4\gamma^2} - \rho_{aa}(0)\frac{3\Omega^2}{4\gamma^2} - \rho_{ac}(0)\frac{3\Omega}{2\gamma}), \\
&= \frac{\Omega}{2\gamma}(1-3\rho_{aa}(0)-\frac{\Omega^2}{2\gamma^2}) - \rho_{ac}(0)(1-\frac{\Omega^2}{2\gamma^2}) \\
&\pm (\rho_{aa}(0) - \rho_{aa}(0)\frac{\Omega^2}{2\gamma^2} + \frac{3\Omega^2}{4\gamma^2} - \rho_{ac}(0)\frac{3\Omega}{2\gamma}), \\
&= \frac{\Omega}{2\gamma}(1-3\frac{1-\eta}{2}-\frac{\Omega^2}{2\gamma^2}) - \frac{\sqrt{1-\eta^2}}{2}(1-\frac{\Omega^2}{2\gamma^2}) \\
&\pm (\frac{1-\eta}{2} - \frac{1-\eta}{2}\frac{\Omega^2}{2\gamma^2} + \frac{3\Omega^2}{4\gamma^2} - \frac{\sqrt{1-\eta^2}}{2}\frac{3\Omega}{2\gamma}), \\
&= \frac{\Omega}{2\gamma}(\frac{2-3+3\eta}{2}-\frac{\Omega^2}{2\gamma^2}) - \frac{\sqrt{1-\eta^2}}{2}(1-\frac{\Omega^2}{2\gamma^2}) \\
&\pm (\frac{1-\eta}{2} + \frac{\Omega^2}{2\gamma^2}(2+\eta) - \frac{\sqrt{1-\eta^2}}{2}\frac{3\Omega}{2\gamma}), \\
&= \frac{\Omega}{2\gamma}(\frac{-1+3\eta-\frac{\Omega^2}{\gamma^2}}{2}) - \frac{\sqrt{1-\eta^2}}{2}(1-\frac{\Omega^2}{2\gamma^2}) \\
&\pm (\frac{1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta)}{2} - \frac{\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}}{2}). \qquad (4.20)
\end{aligned}
$$

By substituting Eqs.(4.19) and (4.20) in to Eq.(4.10), at steady state we get

$$
\begin{aligned}
\overline{n} &= \frac{A}{2\chi_-}[\frac{\Omega}{2\gamma}(\frac{-1+3\eta-\frac{\Omega^2}{\gamma^2}}{2}) - \frac{\sqrt{1-\eta^2}}{2}(1-\frac{\Omega^2}{2\gamma^2}) \\
&+ (\frac{1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta)}{2} - \frac{\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}}{2})] - \frac{A}{2\chi_+}[\frac{\Omega}{2\gamma}(\frac{-1+3\eta-\frac{\Omega^2}{\gamma^2}}{2}) - \frac{\sqrt{1-\eta^2}}{2}(1-\frac{\Omega^2}{2\gamma^2}) \\
&- (\frac{1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta)}{2} - \frac{\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}}{2})], \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
\overline{n} = & -\frac{A[\frac{\Omega}{2\gamma}(1-3\eta)+\frac{\Omega^3}{2\gamma^3}-(1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta))]}{4\chi_-} \\
& + \frac{A\sqrt{1-\eta^2}(\frac{\Omega^2}{2\gamma^2}-1-\frac{3\Omega}{2\gamma})}{4\chi_-} \\
& + \frac{A[\frac{\Omega}{2\gamma}(1-3\eta)+\frac{\Omega^3}{2\gamma^3}+(1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta))]}{4\chi_+} \\
& - \frac{A\sqrt{1-\eta^2}(\frac{\Omega^2}{2\gamma^2}-1+\frac{3\Omega}{2\gamma})}{4\chi_+},
\end{aligned} \tag{4.22}
$$

in which

$$
\chi_{\pm} = \kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A[(1-\frac{\Omega^2}{2\gamma^2})\eta + \sqrt{1-\eta^2}\frac{3\Omega}{2\gamma} \mp \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2})]. \tag{4.23}
$$

In the absence of deriving coherent light($\Omega = 0$), Eq. (4.22) reduces to

$$
\overline{n} = \frac{A(1-\eta)}{2(A\eta+\kappa)}, \tag{4.24}
$$

which is the same as the result obtained by Fesseha [8] in the absence of the driving radiation. We see from Eq. (4.24) that the mean photon number would be zero when there is no driving light and all atoms are initially in the bottom level. Moreover one gets the most intense light when all atoms are initially in the upper level ($\eta = -1$) as expected.

We clearly see from Fig. (4.1) that the mean photon number decreases when the amplitude of the driven coherent light increases.

In addition, when the atoms are initially in the lower level ($\eta = 1$), Eq. (4.22) takes the form

$$
\overline{n} = -\frac{A[-\frac{\Omega}{\gamma}+\frac{\Omega^3}{2\gamma^3}-\frac{3\Omega^2}{2\gamma^2}]}{4\chi'_-} + \frac{A[-\frac{\Omega}{\gamma}+\frac{\Omega^3}{2\gamma^3}+\frac{3\Omega^2}{2\gamma^2}]}{4\chi'_+}, \tag{4.25}
$$

with

$$
\chi'_{\pm} = \kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A[1-\frac{\Omega^2}{2\gamma^2} \mp \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2})]. \tag{4.26}
$$

For the atoms initially in the upper level ($\eta = -1$), Eq. (4.22) reduces to

$$
\overline{n} = -\frac{A[-2+2\frac{\Omega}{\gamma}+\frac{\Omega^3}{2\gamma^3}-\frac{\Omega^2}{2\gamma^2}]}{4\chi''_-} + \frac{A[2+2\frac{\Omega}{\gamma}+\frac{\Omega^3}{2\gamma^3}+\frac{\Omega^2}{2\gamma^2}]}{4\chi''_+}, \tag{4.27}
$$

with

$$
\chi''_{\pm} = \kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A[-1+\frac{\Omega^2}{2\gamma^2} \mp \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2})]. \tag{4.28}
$$

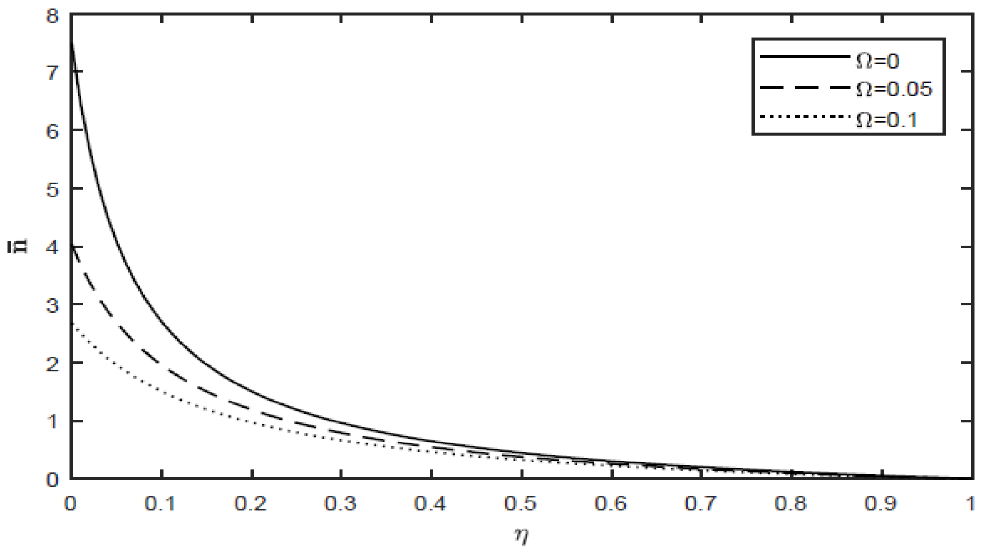

Figure 4.1: Plots of the mean photon number of the cavity radiation at steady state for [Eq. (4.22)] for $\kappa = 0.2, \theta = 0, \gamma = 1, A = 3$, and different values of Ω.

As indicated in Fig. (4.2), the intensity of the light subsequently decreases if we keep on increasing the strength of the driving light.

On the other hand, for $\eta = 0$, we readily get from Eq. (4.22) that

$$\overline{n} = -\frac{A[\frac{\Omega}{2\gamma}(2 + \frac{\Omega^2}{\gamma^2} - \frac{3\Omega}{\gamma})]}{4\chi_{-}^{'''}} + \frac{A[\frac{\Omega}{2\gamma}(-2 + \frac{\Omega^2}{\gamma^2} + \frac{\Omega}{\gamma}) + 2]}{4\chi_{+}^{'''}}, \tag{4.29}$$

in which

$$\chi_{\pm}^{'''} = \kappa(1 + \frac{\Omega^2}{\gamma^2})(1 + \frac{\Omega^2}{4\gamma^2}) + A[\frac{3\Omega}{2\gamma} \mp \frac{\Omega}{2\gamma}(1 + \frac{\Omega^2}{\gamma^2})]. \tag{4.30}$$

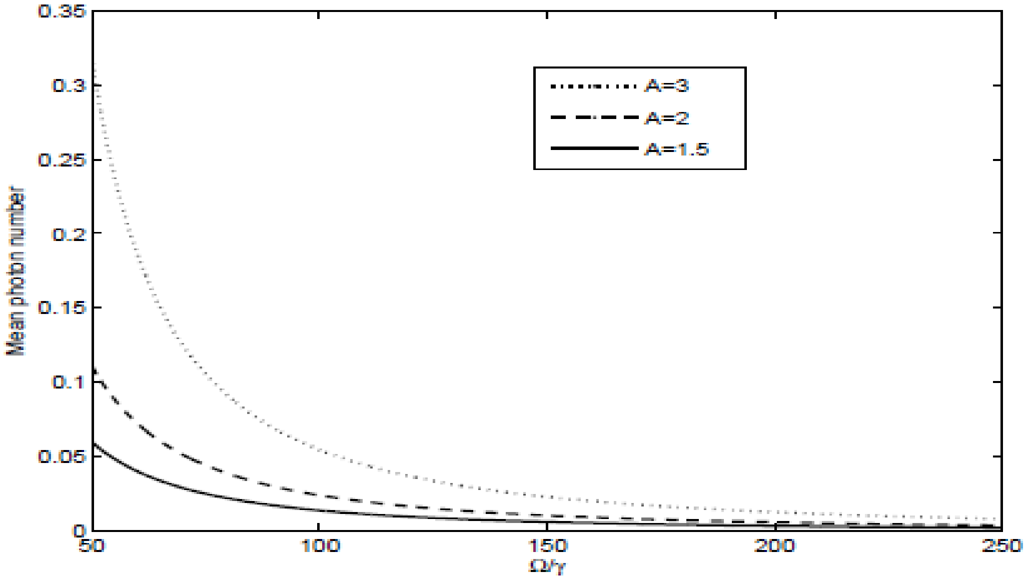

Figure 4.2: Plots of the mean photon number of the cavity radiation at steady state for [Eq. (4.25)] for $\kappa = 0.2, \theta = 0, \eta = 1$, and different values of A.

We clearly see from Figs. (4.2) and (4.3) that the mean photon number increases with the linear gain coefficient.

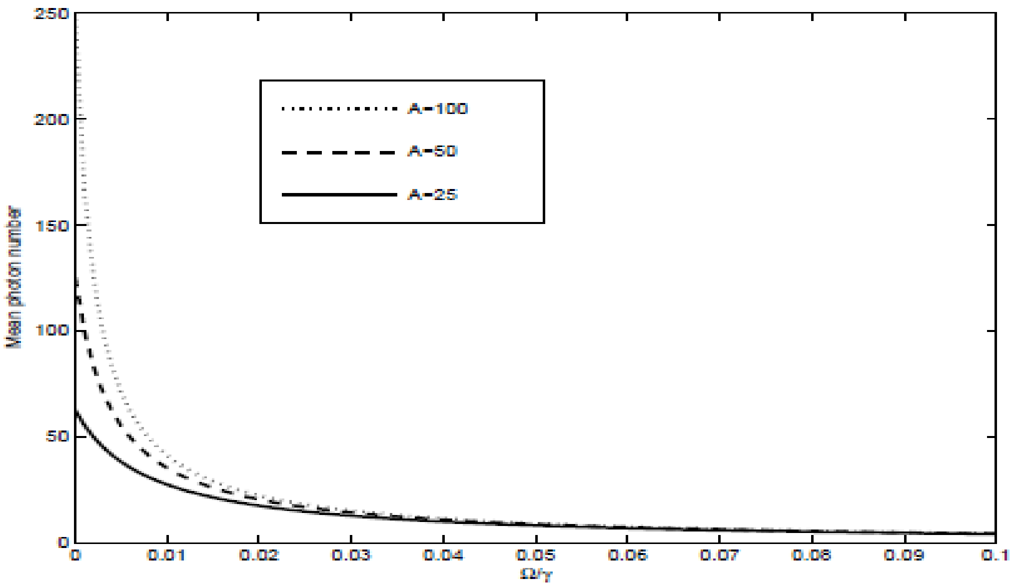

Figure 4.3: Plots of the mean photon number of the cavity radiation at steady state [Eq. (4.27)] for $\kappa = 0.2, \theta = 0, \eta = 0$, and different values of A.

4.2 Variance of the photon number

The variance of the photon number is defined as

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{4.31}$$

If $\hat{a}(t)$ is Guassian variable with a vanishing mean,

$$\begin{aligned} \langle \hat{n}^2 \rangle &= \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle, \\ &= \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle, \\ &= \bar{n}^2 + \langle \alpha^{*2} \rangle \langle \alpha^2 \rangle + \bar{n}(\bar{n}+1), \\ &= 2\bar{n}^2 + \bar{n} + \langle \alpha^{*2} \rangle \langle \alpha^2 \rangle. \end{aligned} \tag{4.32}$$

Therefore Eq. (4.31) becomes

$$\begin{aligned} (\Delta n)^2 &= 2\bar{n}^2 + \bar{n} + \langle \alpha^{*2} \rangle \langle \alpha^2 \rangle - \bar{n}^2 \\ &= \bar{n}^2 + \bar{n} + \langle \alpha^{*2} \rangle \langle \alpha^2 \rangle. \end{aligned} \tag{4.33}$$

Furthermore, by taking Eq. (3.59) we can find

$$\begin{aligned}\langle\alpha^2(t)\rangle =& a_+^2(t)\langle\alpha^2(0)\rangle + a_+(t)a_-(t)\langle\alpha(0)\alpha^*(0)\rangle + a_+(t)\langle\alpha(0)E_-(t)\rangle + a_+(t)\langle\alpha(0)E_+(t)\rangle\\ &+ a_-(t)a_+(t)\langle\alpha^*(0)\alpha(0)\rangle + a_-^2(t)\langle\alpha^{*2}(0)\rangle + a_-(t)\langle\alpha^*(0)E_-(t)\rangle + a_-(t)\langle\alpha^*(0)E_+(t)\rangle\\ &+ a_+(t)\langle E_-(t)\alpha(0)\rangle + a_-(t)\langle E_-(t)\alpha^*(0)\rangle + \langle E_-^2(t)\rangle + \langle E_-(t)E_+(t)\rangle + a_+(t)\langle E_+(t)\alpha(0)\rangle\\ &+ a_+(t)\langle E_+(t)\alpha^*(0)\rangle + \langle E_+(t)E_-(t)\rangle + \langle E_+^2(t)\rangle. \qquad (4.34)\end{aligned}$$

On account of Eq. (3.25) along with the fact that the noise forces F(t) and $F^*(t)$ at time t should not affect system variables at earlier times and for the cavity mode assumed initially to be in a vacuum state, we obtain

$$\langle\alpha^2(t)\rangle = \langle E_-^2(t)\rangle + \langle E_-(t)E_+(t)\rangle + \langle E_+(t)E_-(t)\rangle + \langle E_+^2(t)\rangle. \qquad (4.35)$$

According to Eqs. (3.37), (3.47) and (3.61) we have

$$\begin{aligned}\langle E_-^2(t)\rangle &= \langle E_-(t)E_-(t)\rangle\\ &= \langle(\frac{1}{2}\int_0^t e^{-\frac{\lambda_+(t-t')}{2}}[F(t') - F^*(t')]dt')(\frac{1}{2}\int_0^t e^{-\frac{\lambda_+(t-t'')}{2}}[F(t'') - F^*(t'')]dt')\rangle\\ &= \frac{1}{4}e^{-\lambda_+t}[\int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\langle F(t')F(t'')\rangle dt''dt' - \int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\langle F(t')F^*(t'')\rangle dt''dt'\\ &- \int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\langle F^*(t')F(t'')\rangle dt''dt' + \int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\langle F^*(t')F^*(t'')\rangle dt''dt']\\ &= \frac{1}{4}e^{-\lambda_+t}[\frac{-AF}{B}\int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\delta(t'-t'')dt''dt' - \frac{AC}{B}\int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\delta(t'-t'')dt''dt'\\ &- \frac{AC}{B}\int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\delta(t'-t'')dt''dt' + \frac{-AF}{B}\int_0^t e^{\frac{-\lambda_+t'}{2}}\int_0^t e^{\frac{\lambda_+t''}{2}}\delta(t'-t'')dt''dt']\\ &= \frac{1}{4}e^{-\lambda_+t}[\frac{-2AF}{B}\int_0^t e^{-\lambda_+t'}dt' - \frac{2AC}{B}\int_0^t e^{-\lambda_+t'}dt']\\ &= \frac{1}{4}e^{-\lambda_+t}[\frac{-2AF}{B}\frac{1}{\lambda_+}(e^{\lambda_+t}-1) - \frac{2AC}{B}\frac{1}{\lambda_+}(e^{\lambda_+t}-1)]\\ &= \frac{-A}{2B\lambda_+}[F+C](1-e^{-\lambda_+t}). \qquad (4.36)\end{aligned}$$

In a similar manner, one can establish that

$$\langle E_+^2(t)\rangle = \frac{-A}{2B\lambda_-}[F-C](1-e^{-\lambda_-t}), \qquad (4.37)$$

and

$$\langle E_-(t)E_+(t)\rangle = \langle E_+(t)E_-(t)\rangle = 0. \qquad (4.38)$$

By substituting Eqs. (4.36), (4.37) and (4.38) in to Eq. (4.35) we get

$$\langle\alpha^2(t)\rangle = \frac{-A}{2B\lambda_+}[F+C](1-e^{-\lambda_+ t}) + \frac{-A}{2B\lambda_-}[F-C](1-e^{-\lambda_- t}). \tag{4.39}$$

Which at steady state become

$$\langle\alpha^2(t)\rangle_{ss} = -\frac{A}{2B}(\frac{F+C}{\lambda_+} + \frac{F-C}{\lambda_-}), \tag{4.40}$$

and

$$\langle\alpha^{*2}(t)\rangle_{ss} = -\frac{A}{2B}(\frac{F+C}{\lambda_+} + \frac{F-C}{\lambda_-}). \tag{4.41}$$

Therefore by substituting Eqs. (4.40) and (4.41) in to Eq. (4.33) we get

$$(\Delta n)^2 = \bar{n}^2 + \bar{n} + [-\frac{A}{2B}(\frac{F+C}{\lambda_+} + \frac{F-C}{\lambda_-})]^2 \tag{4.42}$$

From this we observe that

$$(\Delta n)^2 > \bar{n}. \tag{4.43}$$

This means the photon statistics is super-Poissonian for the degenerate three level laser.

Chapter 5

Quadrature Variance

In this chapter we discuss the variance of the quadrature operators and quadrature squeezing of the cavity light. The quadrature operators of a single-mode cavity radiation are defined as

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a}, \tag{5.1}$$

and

$$\hat{a}_{-} = \imath(\hat{a}^{\dagger} - \hat{a}). \tag{5.2}$$

The variance of the plus and minus quadrature operators is

$$(\Delta a_{+})^2 = \langle \hat{a}_{+}^2 \rangle - \langle \hat{a}_{+} \rangle^2, \tag{5.3}$$

and

$$(\Delta a_{-})^2 = \langle \hat{a}_{-}^2 \rangle - \langle \hat{a}_{-} \rangle^2. \tag{5.4}$$

By substituting Eq.(5.1) in to Eq.(5.3) we have

$$\begin{aligned}
(\Delta a_{+})^2 &= \langle (\hat{a}^{\dagger} + \hat{a})(\hat{a}^{\dagger} + \hat{a}) \rangle - \langle (\hat{a}^{\dagger} + \hat{a}) \rangle^2 / \\
&= \langle (\hat{a}^{\dagger 2} + \hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger} + \hat{a}^2) \rangle - \langle (\hat{a}^{\dagger} + \hat{a}) \rangle \langle (\hat{a}^{\dagger} + \hat{a}) \rangle, \\
&= \langle \hat{a}^{\dagger 2} + 1 + 2\hat{a}^{\dagger}\hat{a} + \hat{a}^2 \rangle - (\langle \hat{a}^{\dagger} \rangle^2 + \langle \hat{a}^{\dagger} \rangle \langle \hat{a} \rangle + \langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle + \langle \hat{a} \rangle^2), \\
&= 1 + 2\langle \hat{a}^{\dagger}\hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - \langle \hat{a}^{\dagger} \rangle^2 - \langle \hat{a} \rangle^2 - 2\langle \hat{a}^{\dagger} \rangle \langle \hat{a} \rangle.
\end{aligned} \tag{5.5}$$

By substituting Eq.(5.2) in to Eq.(5.4) we have

$$
\begin{aligned}
(\Delta a_-)^2 &= \langle \imath(\hat{a}^\dagger - \hat{a})\imath(\hat{a}^\dagger - \hat{a})\rangle - \langle \imath(\hat{a}^\dagger - \hat{a})\rangle^2, \\
&= \imath^2\langle(\hat{a}^{\dagger 2} - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2)\rangle - \imath^2\langle(\hat{a}^\dagger - \hat{a})\rangle\langle(\hat{a}^\dagger - \hat{a})\rangle, \\
&= -\langle \hat{a}^{\dagger 2} - 1 - 2\hat{a}^\dagger\hat{a} + \hat{a}^2\rangle + (\langle\hat{a}^\dagger\rangle^2 - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle - \langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle^2), \\
&= 1 + 2\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}^2\rangle + \langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle. \qquad (5.6)
\end{aligned}
$$

Therefore Eq.(5.5) and Eq.(5.6) can be written as

$$(\Delta a_\pm)^2 = 1 + 2\langle\hat{a}^\dagger\hat{a}\rangle \pm \langle\hat{a}^{\dagger 2}\rangle \pm \langle\hat{a}^2\rangle \mp \langle\hat{a}^\dagger\rangle^2 \mp \langle\hat{a}\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle. \qquad (5.7)$$

The variance of these quadrature operators can be expressed in terms of the corresponding c-number variables associated with the normal ordering as

$$(\Delta a_\pm)^2 = 1 + 2\langle\alpha^*\alpha\rangle \pm \langle\alpha^{*2}\rangle \pm \langle\alpha^2\rangle \mp \langle\alpha^*\rangle^2 \mp \langle\alpha\rangle^2 - 2\langle\alpha^*\rangle\langle\alpha\rangle, \qquad (5.8)$$

$$(\Delta a_\pm)^2 = 1 \pm \big(\pm 2\langle\alpha^*\alpha\rangle + \langle\alpha^{*2}\rangle + \langle\alpha^2\rangle\big) \mp \big(\langle\alpha^*\rangle^2 + \langle\alpha\rangle^2 \pm 2\langle\alpha^*\rangle\langle\alpha\rangle\big), \qquad (5.9)$$

$$(\Delta a_\pm)^2 = 1 \pm \langle\alpha_\pm^2(t)\rangle \mp \langle\alpha_\pm(t)\rangle^2. \qquad (5.10)$$

Next we proceed to evaluate various correlations involved in Eq. (5.10). We see from Eq. (3.59) that

$$\langle\alpha(t)\rangle = a_+(t)\langle\alpha(0)\rangle + a_-(t)\langle\alpha^*(0)\rangle + \langle E_-(t)\rangle + \langle E_+(t)\rangle. \qquad (5.11)$$

From Eq.(3.61) and taking in to account Eq.(3.25) we get

$$\langle E_\pm(t)\rangle = \frac{1}{2}\int_0^t e^{-\frac{\lambda_\mp(t-t')}{2}}[\langle F(t')\rangle \pm \langle F^*(t')\rangle]dt' = 0, \qquad (5.12)$$

and for the cavity mode initially in a vacuum state,

$$
\begin{aligned}
\langle\hat{a}(0)\rangle &= Tr(\hat{\rho}\hat{a}) \\
&= Tr(|0\rangle\langle 0|\hat{a}) \\
&= \langle 0|\hat{a}|0\rangle \\
&= 0, \qquad (5.13)
\end{aligned}
$$

and

$$\langle \hat{a}^{\dagger}(0) \rangle = 0. \tag{5.14}$$

This means for the cavity mode initially in a vacuum state,

$$\langle \alpha^{*}(0) \rangle = 0, \tag{5.15}$$

and

$$\langle \alpha(0) \rangle = 0. \tag{5.16}$$

By substituting Eqs.(5.12), (5.15) and (5.16) in to Eq.(5.11), we have

$$\langle \alpha(t) \rangle = 0, \tag{5.17}$$

and

$$\langle \alpha^{*}(t) \rangle = 0. \tag{5.18}$$

Therefore, from $\alpha_{\pm}(t) = \alpha^{*}(t) \pm \alpha(t)$ and Eqs. (5.17) and (5.18), we note that

$$\langle \alpha_{\pm}(t) \rangle = 0, \tag{5.19}$$

as a result of which Eq. (5.10) becomes

$$(\Delta a_{\pm})^{2} = 1 \pm \langle \alpha_{\pm}^{2}(t) \rangle. \tag{5.20}$$

Furthermore using Eq. (3.51)

$$\begin{aligned} \langle \alpha_{\pm}^{2}(t) \rangle &= \langle (\alpha^{*}(t) \pm \alpha(t))(\alpha^{*}(t) \pm \alpha(t)) \rangle, \\ &= \langle \alpha^{*2}(t) \rangle + \langle \alpha^{2}(t) \rangle \pm 2\langle \alpha^{*}(t)\alpha(t) \rangle, \\ &= \langle \alpha^{*2}(t) \rangle + \langle \alpha^{2}(t) \rangle \pm 2\bar{n}. \end{aligned} \tag{5.21}$$

By combining Eqs. (4.10), (4.40), and (4.41) with Eq. (5.21) we obtain

$$\begin{aligned} \langle \alpha_{\pm}^{2} \rangle &= -\frac{A}{2B}(\frac{F+C}{\lambda_{+}} + \frac{F-C}{\lambda_{-}}) - \frac{A}{2B}(\frac{F+C}{\lambda_{+}} + \frac{F-C}{\lambda_{-}}) \pm 2(\frac{A}{2B\lambda_{+}}[F+C] - \frac{A}{2B\lambda_{-}}[F-C]), \\ &= \frac{-2A}{B\lambda_{\mp}}[F \mp C]. \end{aligned} \tag{5.22}$$

From Eq. (4.20)

$$F \mp C = \frac{\Omega}{2\gamma}(\frac{-1+3\eta-\frac{\Omega^2}{\gamma^2}}{2}) - \frac{\sqrt{1-\eta^2}}{2}(1-\frac{\Omega^2}{2\gamma^2}) \mp (\frac{1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta}{2}) - \frac{\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}}{2}), \tag{5.23}$$

and using Eqs. (4.19) and (5.23) in Eq. (5.22) we get

$$\langle \alpha_{\pm}^2(t)\rangle_{ss} = \frac{A[\frac{\Omega}{2\gamma}(1-3\eta+\frac{\Omega^2}{\gamma^2})+\sqrt{1-\eta^2}(1-\frac{\Omega^2}{2\gamma^2})]}{\chi_\pm} \pm \frac{A[1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta)-\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma^2}]}{\chi_\pm}, \tag{5.24}$$

on account of which, Eq. (5.20) becomes

$$(\Delta a_{\pm})^2 = 1 \pm \frac{A[\frac{\Omega}{2\gamma}(1-3\eta+\frac{\Omega^2}{\gamma^2})+\sqrt{1-\eta^2}(1-\frac{\Omega^2}{2\gamma^2})]}{\chi_\pm} + \frac{A[1-\eta+\frac{\Omega^2}{2\gamma^2}(2+\eta)-\sqrt{1-\eta^2}\frac{3\Omega}{2\gamma}]}{\chi_\pm}, \tag{5.25}$$

where $\chi_{\pm}$ is given by Eq. (4.19).

In order to study the dependence of the squeezing on the amplitude of the driving radiation and initially injected atomic coherence closely, we consider various cases of interest. In this respect, it is not difficult to check for $\Omega = 0$ that

$$(\Delta a_{\pm})^2 = \frac{\kappa + A(1 \pm \sqrt{1-\eta^2})}{A\eta + \kappa}. \tag{5.26}$$

The same result has been obtained by Fesseha [8].

From Fig. (5.1), we observe that the degree of squeezing increases when the amplitude of the driving coherent light increases and a substantial degree of squeezing is found for small values of η. Furthermore, we obtain the intersection point for $\Omega = 0$ and $\Omega = 0.03$ is $\eta = 0.22$, and for $0 \le \eta \le 0.22$ the quadrature variance for $\Omega = 0$ is greater than for $\Omega = 0.03$ and for $0.22 \le \eta \le 1$ the quadrature variance for $\Omega = 0$ is less than for $\Omega = 0.03$.

We clearly see from Fig. (5.2) that the degree of squeezing increases with the linear gain coefficient and a substantial degree of squeezing is found for small values of η. This indicates that the more atoms are injected into the cavity at a time the more the degree of the squeezing of the cavity radiation would be.

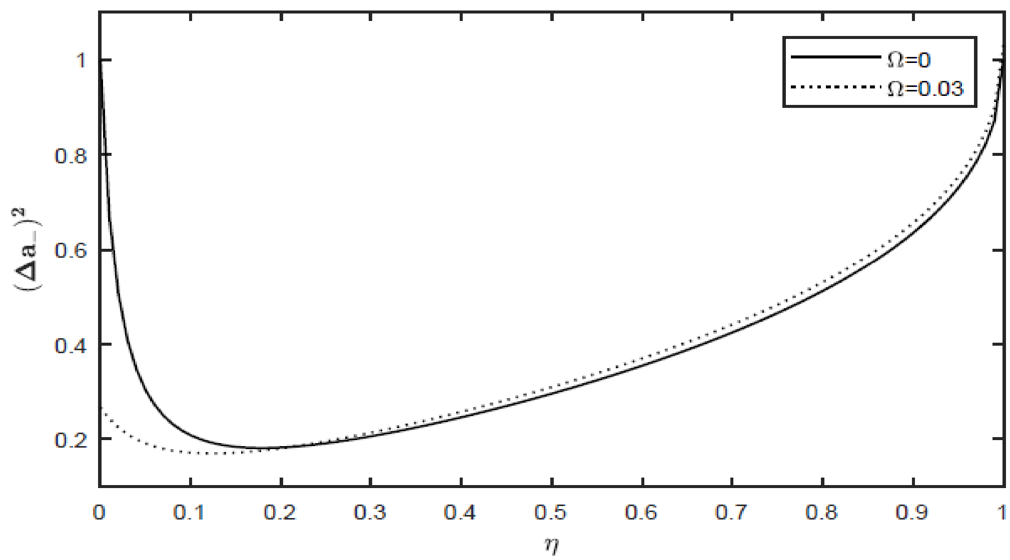

Figure 5.1: Plots of the quadrature variance $(\Delta a_-)^2$ of the cavity radiation versus η at steady state [Eq. (5.26)] for $\kappa = 0.2, \theta = 0, A = 10, \gamma = 1$, and different values of Ω

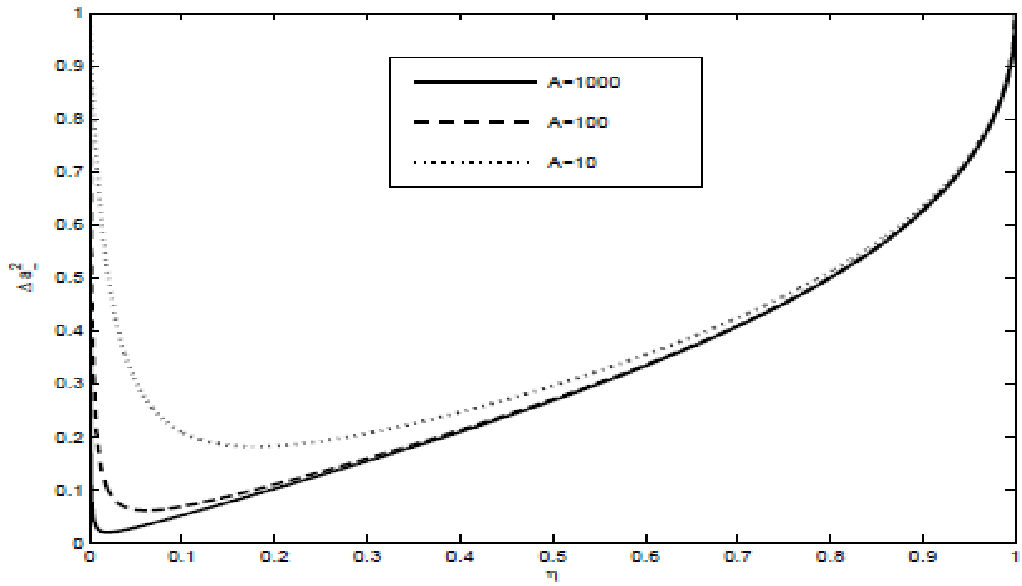

Figure 5.2: Plots of the quadrature variance $(\Delta a_-)^2$ of the cavity radiation versus η at steady state [Eq. (5.26)] for $\kappa = 0.2, \theta = 0, \Omega = 0$, and different values of A.

Chapter 6

Conclusion

In this project we have studied the statistical and squeezing properties of the light produced by the degenerate three-level laser coupled to a vacuum reservoir via one of the coupler mirrors and driven by an external resonant coherent radiation from the other. Applying the solutions of the c-number Langevin equations, we have calculated the mean photon number, the variance of the photon number and the quadrature variance. The mean photon number would be zero when there is no driving light and all atoms are initially in the bottom level, and the most intense light is generated when all atoms are initially in the upper level. When all atoms are initially in the bottom level, the mean photon number decreases if we keep on increasing the strength of the driving light. We found the variance of the photon number is greater than the mean photon number. Hence the photon statistics is super-Poissonian for the cavity radiation.

Driving the atoms with an external coherent radiation affects both the degree of squeezing and intensity of the generated light. We found that the degree of squeezing increases with the linear gain coefficient and a substantial degree of squeezing is found for small values of η. This indicates that the more atoms are injected into the cavity at a time the more the degree of the squeezing of the cavity radiation would be. We found the squeezing increases as the quadrature variance decreases.

Bibliography

[1] Sintayehu Tesfa, *Driven degenerate three-level cascade laser*,(Addis Ababa University, Ethiopia, 2013).

[2] Tewodros Yirgashewa, *Coherently Driven Three-Level Laser with Parametric Amplifier*,(Addis Ababa University, Ethiopia, 2010).

[3] C. Saaverda, J. C. Retamal, and C. H. Keitel, *Phys.*, Rev. A **55**, 3802 (1997).

[4] M. A. G. Martinez, P. R. Herczfeld, C. Samuels, L. M. Narducci, and C. H. Keitel, *Phys.*, Rev. A **55**, 4483 (1997).

[5] Y. Zhu, *Phys.*, Rev. A **55**, 4568 (1997).

[6] N. A. Ansari, J. G. Banacloche, and M. S. Zubairy, *Phys.*, Rev. A **41**, 5179 (1990).

[7] H. Xiong, M. O. Scully, and M. S. Zubairy, *Phys.*, Rev. Lett. **94** 023601 (2005).

[8] K. Fesseha, *Phys.*, Rev.A **63**, 033811 (2001).

[9] N. A. Ansari, *Phys.*,Rev. A **46**, 1560 (1992).

[10] M. O. Scully, K. Wodkiewicz, M. S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Van, *Phys.*, Rev. lett. **60**, 1832 (1988).

[11] J. Anwar and M. S. Zubairy, *Phys.*, Rev. A **49**, 481 (1994).

[12] N. A. Ansari, *Phys.*, Rev. A **48**, 4686 (1993).

DECLARATION

I, hereby declare that this project is a review of previous works and that all sources of materials have been duly acknowledged.

Name: Gebresenbet Entehabu

Signature:——————————————.

e-mail:senaye12@gmail.com

Place and time of submission:

Addis Ababa University

Department of physics

December 04, 2018

This Project has been submitted for examination with my approval as University advisor.

Name: Dr. Deribe Hirpo

Signature:———————————————.

e-mail:-deribeh@gmail.com