GRADUATE SEMINAR REPORT

ON

SINGULAR PERTURBATION THEORY

submitted in partial fulfillment as a requirement for the M.Sc. degree in mathematics

BY

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PREFACE

This paper is produced as a requirement in partial fulfillment of the Master of Science degree in Mathematics at Addis Ababa University. It complies all the presentations I made in the examiners board.

The paper comprises of four main chapters, on singular perturbation theory, organized in order of dependence of one on the other.

The first chapter introduces basic definitions and examples on ordering, asymptotic sequences and asymptotic expansions.

The second chapter introduces the concept of multiscale, or two-timing technique, which give good approximate solution as we want, uniformly valid for all time.

The third chapter deals about initial value problems II on singular perturbation theory and its application on dynamics and genetics.

And the last chapter, Boundary value problems on singular perturbation theory, provides two useful matching principles to match the boundary layer solution and the outer solution.

I want to express my heartfelt appreciation to my advisor and instructor Prof. Dr. S.N. Murthy who cultivated me not only academically but also in all other social aspects. I respect him very much and without him this seminar would be impossible.

I am grateful to all my friends and relatives for their unreserved moral, material and financial support that encouraged and made fruitful my study.

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Chapter 1

Introduction

1.1 Ordering

In this section we will use the conventional order symbols as a mathematical measure of the relative order of magnitude of various quantities. Though generalizations are straightforward, we need only be concerned with scalar functions of real variables.

In the definitions which follow Φ, ψ, etc. are scalar functions of the variable x (which may be a vector) and the scalar parameter ε. The variable x ranges over some domain D and ε belongs to some interval I.

Definition 1.1.1 (Large O)

Let x be fixed. We say $\Phi = O(\psi)$ in I if there exists a k(x) such that $|\Phi| \leq k(x)\,|\psi|$ for all ε in I.

Similarly, If $|\Phi(x)|\psi(x)|$ is bounded, we

write $\Phi(x) = O\{\psi(x)\}$, $(x \to \infty)$

or $\Phi = O(\psi)$; again, in words, Φ is of order not exceeding ψ.

Note that, the alternative definition above holds if Φ doesn't vanish in I.

Definition 1.1.2 (Small o)

Let x be fixed we say that $\Phi = o(\psi)$ as $\varepsilon \to \varepsilon_0$ if given any $\delta(x) > 0$, there exists a neighborhood N_δ of ε_0 such that $|\Phi| \leq \delta(x)\,|\psi|$ for all ε in N_δ. Here also the definition simplifies to the statement that $(\Phi|\psi) \to 0$ if $\psi \neq 0$ in I. Equivalent rotation:

$\Phi << \psi$, meaning Φ is much much smaller than ψ.

Definition 1.1.3 (Uniformity)

In the above definitions if the quantities k and δ can be found in dependently of the Variable x we say that the order relations holds uniformly in D.

Now let's illustrate the above ideas by considering the following examples.

Example 1.1.1:- Consider x a real variable, the domain D half-open interval $0 < x \leq 1$ and I the half-open interval $0 < \varepsilon \leq \mu < 1$ with

$\varepsilon_0 = 0$. Then show that $x + \varepsilon = O(1)$ in I, uniformly in D.

(I) To show $x + \varepsilon = O(1)$ in I

Using our large 0 definition $x + \varepsilon = O(1)$ in I if there exist k(x) such that $|x + \varepsilon| \leq k(x)\,|1|$ for all ε is I.

But $0 < \varepsilon \leq \mu < 1 \Rightarrow \varepsilon < 1$

$$\Rightarrow |x + \varepsilon| \leq |x + 1| \text{ for all } \varepsilon \text{ in I.}$$

Now put $k(x) = |x + 1|$

Then $|x + \varepsilon| \leq |x \mid 1|\,|1|$ for all ε in I

Hence, $x + \varepsilon = O(1)$ in I holds true.

(II) To show order relation above hold uniformly in D.

Here, we can make k(x) independent of x as follows:

from part (I) we have $|x + | \leq |x + 1|$ for all ε in I; but $x \in (0,1]$

Therefore, $|x + \varepsilon| \leq |1 + 1|$ for all ε in I.

Then it follows, $|x + \varepsilon| \leq 2$ for all ε in I.

Put $k(x) = 2$, independent of x.Hence, the order relation hold uniformly in D.

Example 1.1.2: - Show that $\frac{1}{x+\varepsilon} = O(1)$ in I. Is the order relation hold uniformly in D?

Clearly for any given x and for all ε in I, we have

$$\frac{1}{x+\varepsilon} < \frac{1}{x}$$

i.e., $\frac{1}{x+\varepsilon} < \frac{1}{x}$. (1)

Put $k(x) = \frac{1}{x}$

Thus $\frac{1}{x+\varepsilon} = O(1)$ in I.

To answer the second part, consider $k(x) = \frac{1}{x}$.

Here there is no finite constant k for which the required inequality holds for all x in D so long as x is allowed to approach the origin.

Therefore, k(x) can not be found independent of the variable x.

Thus the order relation is not uniformly hold in D.

Example 1.1.3:-

Show that $\frac{1}{x^2} = o\left(\frac{1}{x}\right)$

By using our small o definition

$$\frac{1}{x^2} = o\left(\frac{1}{x}\right) \text{ if } \frac{1/x^2}{1/x} \to 0$$

Now $\frac{1/x^2}{1/x} = \frac{1}{x^2} \cdot \frac{x}{1} = \frac{1}{x}$, $x \neq 0$; then as $x \to \infty$, $\frac{1}{x} \to 0$.

There fore, $\frac{1}{x^2} = o\left(\frac{1}{x}\right)$ holds true.

1.2 Asymptotic Sequences and Expansions

DEFINITION 1.2.1 A sequence $\{\Phi_n(\varepsilon)\}$ n = 1, 2, … of functions of ε is called an asymptotic sequence if $\Phi_{n+1}(\varepsilon) = o(\Phi_n(\varepsilon))$ as $\varepsilon \to \varepsilon_0$ for each n = 1, 2, ….

Example 1.2.1:- some examples of asymptotic sequences are:

$$\Phi_n(\varepsilon) = (\varepsilon - \varepsilon_0)^n \quad \ldots \quad , \text{as } \varepsilon \to \varepsilon_0, \qquad (\Delta)$$

and $\Phi_n(\varepsilon)\, e^{\varepsilon} e^{-\alpha n}$, as $\varepsilon \to \infty$, $\alpha_{n+1} > \alpha_n$ (*)

Because in (Δ)

$$\Phi_n(\varepsilon) = (\varepsilon - \varepsilon_0)^n \Rightarrow \Phi_{n+1}(\varepsilon) = (\varepsilon - \varepsilon_0)^{n+1}$$

and $$\frac{\Phi_{n+1}(\varepsilon)}{\varepsilon \to \varepsilon_0 \Phi_n} = \frac{(\varepsilon - \varepsilon_0)^{n+1}}{\varepsilon \to \varepsilon_0 (\varepsilon - \varepsilon_0)^n} = \underset{\varepsilon \to \varepsilon_0}{\varepsilon - \varepsilon_0} = 0$$

and in (*)

$$\Phi_n(\varepsilon)\, e^{\varepsilon} e^{-\alpha n} \Rightarrow \Phi_{n+1}(\varepsilon) = e^{\varepsilon} \varepsilon^{-\alpha_{n+1}} \text{ and}$$

$$\frac{\Phi_{n+1}(\varepsilon)}{\varepsilon \to \infty \Phi_n(\varepsilon)} = \frac{e^{\varepsilon} \varepsilon^{-\alpha_{n+1}}}{\varepsilon \to \varepsilon_o e^{\varepsilon} \varepsilon^{-\alpha_n}} = \underset{\varepsilon \to \infty}{\varepsilon^{\alpha_{n-\alpha_{n+1}}}},$$

but $\alpha_{n+1} > \alpha_n$, therefore $\alpha_n - \alpha_{n+1} < 0$.

Hence $\underset{\varepsilon \to \infty}{\varepsilon^{\alpha_{n-\alpha_{n+1}}}} = 0$.

DEFINITION 1.2.2 A sum of terms of the form $\sum_{n=1}^{N} a_n(x)\Phi_n(\varepsilon)$ is called an asymptotic expansion of the function f(x, ε) to N terms (N may be infinite) as $\varepsilon \to \varepsilon_0$ with respect it's the sequence $\{\Phi_n(\varepsilon)$ if

$$f(x, \boldsymbol{\varepsilon}) - \sum_{n=1}^{M} a_n(x)\,\Phi_n(\boldsymbol{\varepsilon}) = o(\Phi_M) \text{ as } \boldsymbol{\varepsilon} \to \boldsymbol{\varepsilon}_0 \text{ for each } M = 1, 2, \ldots N.$$

If $N = \infty$, the following notation is generally used;

$$f(x, \boldsymbol{\varepsilon}) \sim \sum_{n=1}^{\infty} a_n(x)\,\Phi_n(\boldsymbol{\varepsilon}) \text{ as } \boldsymbol{\varepsilon} \to \boldsymbol{\varepsilon}_0.$$

An equivalent definition for an asymptotic expansion is that

$$f(x, \boldsymbol{\varepsilon}) - \sum_{n=1}^{M-1} a_n(x)\,\Phi_n(\boldsymbol{\varepsilon}) = 0(\Phi_M) \text{ as } \boldsymbol{\varepsilon} \to \boldsymbol{\varepsilon}_0 \text{ for each } M = 2, 3, \ldots,.$$

Given a function f(x, **ε**) and an asymptotic sequence $\{\Phi_n(\boldsymbol{\varepsilon})\}$, one can uniquely calculate each of $a_n(x)$ defining the asymptotic expansion of f(x, **ε**) by repeated application of definition 1.2.2.

CHAPTER 2

Initial value problems I on singular perturbation theory

2.1 Introduction

The applied mathematician, attempting to understand or solve a physical problem very often uses a perturbation procedure. In essence, a perturbation procedure consists of constructing the solution for a problem involving a small parameter ε, when the solution for the limiting case $\varepsilon = 0$ is known.

The aim of this chapter is to introduce these perturbation methods, especially in connection with differential equations on singular perturbation problems which are very common in physical applications and which require special techniques.

The main mathematical tool used is asymptotic expansions with respect to a suitable asymptotic sequence of functions of ε.

Definition 2.1.1: The singular perturbation model of dynamic system is state-space model in which the derivatives of some of the states are multiplied by a small positive parameter ε; where setting $\varepsilon = 0$ causes of fundament and abrupt change in the dynamic properties of the system.

The solution of initial value problems is the first example of a singular perturbation problem that we will consider.

Consider the following linear second order differential equation

$$\frac{d^2u}{dt^2} + 2\,\varepsilon\,\frac{du}{dt} + u = 0,\ u(0) = a,\ u'(0) = 0 \qquad (2.1.1)$$

which describes the vibrations of a slightly damped mass-spring system. We suppose that ε is a small, positive number.

Clearly, the exact solution of equation 2.1.1 is given by

$$u(t) = Ae^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}\,t + \Phi\right) \qquad (2.1.2)$$

$$\text{where } \tan\Phi = \frac{-\varepsilon}{\sqrt{1-\varepsilon^2}},\ A = \frac{a}{\sqrt{1-\varepsilon^2}}$$

Now we will see methods of approximating a power series representation of the solution in powers of ε.

There are two ways to proceed:

Method I:- We could expand the known solution in its power series. But this method will not work for problems where the answer is not known beforehand.

Method II: - (We can use the following idea) Assuming that u(t, ε) has a power series representation in powers of ε, and expand the governing equation.

Let's approximate a power series representation of the solution in powers of ε using the second Method:

Suppose that

$$u(t, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j u_j(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots \qquad (2.1.3)$$

where the functions $u_i(t)$ $(i = 0, 1, \ldots)$ are independent of ε. Substituting equation 2.1.3 in equation 2.1.1, gives

$$(\ddot{u}_0(t) + \varepsilon\ddot{u}_1(t) + \varepsilon^2\ddot{u}_2(t) + \cdots) + 2\varepsilon(\dot{u}_0(t) + \varepsilon\dot{u}_1(t) + \varepsilon^2\dot{u}_2(t) + \cdots) + (u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots) = 0$$

extracting like powers of ε, We find

$$\varepsilon^0 :- \ddot{u}_0(t) + u_0(t) = 0$$

$$\varepsilon^1 :- \ddot{u}_1(t) + 2\dot{u}_0(t) + u_1(t) = 0$$

$$\varepsilon^2 :- \ddot{u}_2(t) + 2\dot{u}_1(t) + u_2(t) = 0$$

$$\vdots$$

$$\varepsilon^n :- \ddot{u}_n(t) + 2\dot{u}_{n-1}(t) + u_n(t) = 0$$

When we generalize the above we find that

$$\ddot{u}_0 + u_0 = 0, \; u_0(0) = a, \; \dot{u}_0(0) = 0 \qquad (2.1.4)$$

$$\ddot{u}_j + u_j + 2\dot{u}_{j-1} = 0, \; u_j(0) = \dot{u}_j(0) = 0 \text{ for } j = 1, 2, \ldots, \qquad (2.1.5)$$

Note:- $u_0(t)$ is the solution of the equation (equation 2.1.1) obtained by letting $\varepsilon = 0$.

Definition 2.1.2 The solution $u_0(t)$ is referred to as the zero – order approximation, or the generating solution of equation 2.1.1.

Clearly, from equation 2.1.4, the solution $u_0(t) = a\cos t$.

Then using the above value of $u_0(t)$ and equation 2.1.5, the equation of u_1 becomes

$$\ddot{u}_1 + u_1 = 2a\sin t, \quad u_1(0) = 0 = \dot{u}_1(0) \quad (\text{Since } \dot{u}_0 = -a\sin t)$$

Then the solution $u_1(t)$ is

$$u_1(t) = -at\cos t + a\sin t$$

Therefore, at this stage, we have the approximate solution

$$\begin{aligned} u(t, \varepsilon) &= u_0\varepsilon^0 + u_1\varepsilon + u_2\varepsilon^2 + \ldots. \\ &= a\cos t + \varepsilon(-at\cos t + a\sin t) + o(\varepsilon^2) \\ &= a\cos t - a\varepsilon t\cos t + \varepsilon a\sin t + o(\varepsilon^2) \end{aligned} \qquad (2.1.6)$$

Now, we see the problems

* Although this approximate representation is valid for any fixed t and ε very small, it is not useful for large t. Because for large t with ε fixed, this representation is not indicative of what really happens.

As $t \to \infty$, exact solution $\to 0$ but approximate representation goes to $-\infty$.

Note that the terms of the form εt, which are growing in t, are not uniformly bounded.

Definition 2.1.3:- The terms like εt, which are growing in t (or which are not uniformly bounded) are called secular term or troublesome term.

Another way to explain the dilemma is to notice that $e^{-\varepsilon t}$ has an essential singularity at $\varepsilon = 0$, $t = \infty$. That is, in the limit $\varepsilon \to 0$, $t \to \infty$, the limiting value of $e^{-\varepsilon t}$ is not uniquely defined. Therefore, the above example is singular for large t as we can see from power series representation of the solution.

Now, let us proceed to the remedy of this problem. To avoid this dilemma we should make two observations. First, we don't want to expand $e^{-\varepsilon t}$ in powers of εt. Second, We notice that the solution should behave like a periodic oscillator that is changing only very slowly. That is the solution should be represent as,

$$u(t) = A(\varepsilon t) \cos (\omega(\varepsilon)t + \Phi)$$

Where $A(\varepsilon t)$ varies slowly for ε small, and the function $\omega(\varepsilon)$ shows that ε dependence of the frequently of oscillation.
Having this hint, our aim is to exploit the asymptotic solution representation, which is valid for all time.

The technique we are going to use is the so called the multiscale, or two-timing, technique.

2.2 The Multiscale, or two-timing, technique

The method works well to give as good approximate solution as we want, uniformly valid for all time.

To introduce the essential ideas of the multiple timescales method, we will consider a situation we understand well: damped simple harmonic motion as modeled by equation 2.1.1.

$$\frac{d^2u}{dt^2} + 2\,\varepsilon\frac{du}{dt} + u = 0$$

As we have seen in the previous section the solution should be represented as

$$u(t) = A(\varepsilon t)\cos\left(\omega(\varepsilon)t + \Phi\right)$$

The method is as follows:-

Suppose $u = u(s, \tau, \varepsilon)$, where $s = \omega(\varepsilon)t$ is the fast variable, $\tau = \varepsilon t$ is the slow variable. With this as our guess, we seek a power series solution of the form

$$u(t) = u(s, \tau, \varepsilon) = \sum_{j=0}^{\infty} u_j(s,\tau)\,\varepsilon^j$$

$$\omega(\varepsilon) = \sum_{j=0}^{\infty} \omega_j\varepsilon^j;\ \omega_0 = 1.$$

Now to utilize these new variables in the governing equation

$$\frac{d^2u}{dt^2} + 2\varepsilon\frac{du}{dt} + u = 0$$

we must use the chain rule

$$\frac{du}{dt} = \frac{\partial u}{\partial s}\frac{ds}{dt} + \frac{\partial u}{\partial \tau}\frac{d\tau}{dt} = \omega(\varepsilon)\frac{\partial u}{\partial s} + \varepsilon\frac{\partial u}{\partial \tau}$$

and

$$\frac{d^2u}{dt^2} = \omega^2(\varepsilon)\frac{\partial^2 u}{\partial s^2} + 2\varepsilon\omega(\varepsilon)\frac{\partial^2 u}{\partial s \partial \tau} + \varepsilon^2\frac{\partial^2 u}{\partial \tau^2}$$

By using the D-operator $D \equiv \frac{d}{dt}$, and by defining similar operators $D1 \equiv \frac{\partial}{\partial s}$ and $D_2 = \frac{\partial}{\partial \tau}$, we can rewrite the above equations as:

$$D = \omega(\varepsilon)D_1 + \varepsilon D_2$$

$$D^2 = \omega^2(\varepsilon)D_1^2 + 2\varepsilon\omega(\varepsilon)D_1D_2 + \varepsilon^2D_2^2$$

Substituting these expressions in to our governing equation gives:

$$(\omega(\varepsilon)D_1 + \varepsilon D_2)^2 (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) +$$

$$2\varepsilon (\omega(\varepsilon)D_1 + \varepsilon D_2) (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) +$$

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) = 0$$

Finally, by extracting like powers of ε and by substituting $\omega(\varepsilon) = \sum_{j=0}^{\infty} \omega_j \varepsilon^j$ we obtain the following hierarchy of equations:

$$\omega_0^2 \frac{\partial^2 u_0}{\partial s^2} + u_0 = 0,\ u_0(0, 0) = a,\ \frac{\partial^2 u_0}{\partial s^2}(0, 0) = 0 \qquad (2.2.1)$$

$$\omega_0^2\frac{\partial^2 u_1}{\partial s^2} + u_1 + 2\omega_0^2\frac{\partial^2 u_0}{\partial s \partial \tau} + 2\omega_0\omega_1\frac{\partial^2 u_0}{\partial s^2} + 2\omega_0\frac{\partial u_0}{\partial s} = 0 \qquad (2.2.2)$$

$$u_1(0, 0) = 0$$

From equation 2.2.1, The solution u_0 is given by

$$u_0 = A(\tau) \cos s$$

$$\omega_0 = 1$$

Where now, since the equation for u_0 is a partial differential equation, A is a function of variable τ, with A(0) = a.

From equation 2.2.2 (letting $\omega_0 = 1$). We look for u_1, which is the solution of

$$\frac{\partial^2 u_1}{\partial s^2} + u_1 = -2\frac{\partial^2 u_0}{\partial s \partial \tau} - 2\omega_1 \frac{\partial^2 u_0}{\partial s^2} - 2\frac{\partial u_0}{\partial s}$$

$$= -2\left(-\frac{\partial A}{\partial \tau}\sin s\right) - 2\omega_1(-A\cos) - 2(-A\sin s)$$

$$= 2\frac{\partial A}{\partial \tau}\sin s + 2A\sin s + 2\omega_1 A \cos s$$

$$= 2\left(\frac{\partial A}{\partial \tau} + A\right)\sin s + 2\omega_1 A \cos s$$

Now if we solve this equation for u_1 without further restrictions we would find secular terms. In order to remove the several terms, we invoke the Fredholm alternative theorem, namely if u_1 is to be periodic in s the right hand side must not contain terms proportional to sins and $\cos s$. As a consequence we set

$$\frac{\partial A}{\partial \tau} + A = 0$$

$$\omega_1 = 0$$

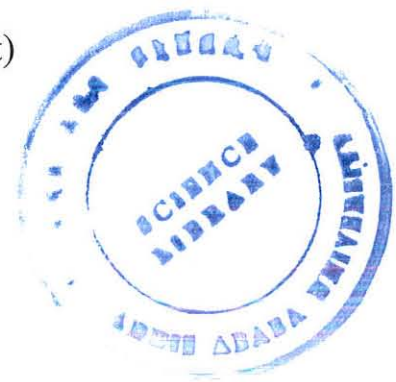
From this it follows that $A = ce^{-\tau}$ and from A(0) = a, we have c = a

Therefore,

$$A = ae^{-\tau}$$

$$= ae^{-\varepsilon t} \text{ (since } \tau = \varepsilon t)$$

and at this stage we have that



$$u(t) = u_0(s, \tau) + o(\varepsilon)$$
$$= ae^{-\varepsilon t} \cos s + o(\varepsilon)$$
$$= ae^{-\varepsilon t} \cos \omega(\varepsilon)t + o(\varepsilon) \text{ (since } s = \omega(t)t)$$
$$= ae^{-\varepsilon t} \cos t + o(\varepsilon) \text{ (since } \omega_0(t) = 1)$$

Now we notice that this solution representation has the correct qualitative behavior for large t.

2.3 Van der pol's equation

In the 1930's, the Dutch engineers Van der pol and Van der Mark invented an electric circuit exhibiting periodic oscillators that they proposed as a model for the pacemaker of the heart.

The Van der pol oscillator is a device with three parallel circuits, one a capacitor, the second a resistor, inductor and voltage source in series and the last some non-linear device, such as a tunnel diode.

Using kirkhoff's law for this circuit the resulting equation,

$$\ddot{u} + \varepsilon \dot{u}(u^2-1) + u = 0 \qquad (2.3.1)$$

is called van der pol equation.

In this section we will develop the multiscale, or two-timing, technique; to find the solution of van der pol equation.

Proceeding as we did for the example in the previous section, for small ε, the solution should be a slowly varying oscillation.
We guess that there are two independent time like variables

$\tau = \varepsilon t$ being the slow line and

$s = \omega(\varepsilon)\, t$ being the fast time.

We again seek a solution of the form

$$u(t) = \sum_{j=0}^{\infty} u_j(s, \tau)\varepsilon^j$$

$$\omega(\varepsilon) = \sum_{j=0}^{\infty} \omega_j\varepsilon^j \,, \omega_0 = 1.$$

To utilize these new variables in the governing equation (equation 2.3.1), the chain rule implies that

$$u' = \omega(\varepsilon)\frac{\partial u}{\partial s} + \varepsilon\frac{\partial u}{\partial \tau} \qquad (2.3.2)$$

$$u'' = \omega^2(\varepsilon)\frac{\partial^2 u}{\partial s^2} + 2\varepsilon\omega(t)\ \frac{\partial^2 u}{\partial s} + \varepsilon^2\frac{\partial^2 u}{\partial \tau^2} \qquad (2.3.3)$$

substituting equations 2.3.2 and 2.3.3 in to our governing equation and extracting for the same powers of ε, we obtain a hierarchy of equations:

$$\frac{\partial^2 u_0}{\partial s^2} + u_0 = 0 \qquad (2.3.4)$$

$$\frac{\partial^2 u_1}{\partial s^2} + u_1 + 2\omega_1\ \frac{\partial^2 u_0}{\partial s^2} + \frac{\partial u_0}{\partial s}\ (u_0^2\text{-}1) + 2\frac{\partial^2 u_0}{\partial s \partial \tau} = 0 \qquad (2.3.5)$$

And from equation 2.3.4, we find that

$$u_0 = A\cos(s + \Phi),$$

where both A and Φ are functions of τ.

For computations reasons, it is preferable to express the solution u_0 interms of complex exponentials

$$u_0 = Ae^{is} + \bar{A}\ e^{-is}$$

where A is possible complex function of τ.

From equation 2.3.5, with u_0 known, the equation for u_1 becomes

$$\frac{\partial^2 u_1}{\partial s^2} + u_1 = \{[2\omega_1 A + i\,(A - A|A|^2 - 2A_\tau)]e^{is} - iA^3 e^{3is}\} + c.c$$

where c.c. denotes the complex conjugate of the previous term.

Now by applying Fedholm alternative theorem to remove secular terms, if u_1 is to be periodic in s, we can't have terms proportional to e^{is} and e^{-is} on the right hand side of the equation for u_1.

The it follows that we must require,

$$A\tau = A\left(\frac{1}{2} - \frac{1}{2}|A|^2 - iw_1\right)$$

Taking A in the form $A = \rho e^{i\Phi}$, we require $\rho_\tau = \frac{1}{2}\rho\,(1 - p^2)$, $\Phi_\tau = -\omega_1$

The number ω_1 is arbitrary, but Φ_τ exactly compensates for ω_1, so we might as well take $\omega_1 = \Phi_\tau = 0$. The function $\rho(\tau)$ can be found explicitly, since the equation for ρ^2 is the logistic equation

$$(\rho^2)_\tau = \rho^2(1 - \rho^2)$$

Then the full solution at this point is given by

$$u(t) = 2\left(\frac{a}{a+e^{-\epsilon t}}\right)^{1/2}\cos(t+\Phi_0) + 0(\epsilon),$$

Where a and Φ_0 are determined from initial data.

2.4 Averaging

It would be a serious omission to end this chapter without mentioning the averaging method, which is due to krylov and Bogoliubov.

The results of averaging method are exactly the same as that of the multiscale method.

The averaging method calculates the amplitude of a slowly varying oscillation in an average sense. The starting point of the averaging method is always polar coordinates.

We will try to introduce this method by applying to the equation of the form

$$\ddot{x} + \omega^2 x + \varepsilon f(x, \dot{x}) = 0 \qquad (2.4.1)$$

where ε is small.

For ε = 0, equation 2.4.1 becomes $\ddot{x}+\omega^2 x = 0$. Here we may apply linear theory to obtain the solution

$$x = A\cos(\omega t + \Phi)$$

where A and Φare arbitrary constants.

And differentiating for x gives

$$\dot{x} = -A\,\omega \sin(\omega t + \Phi).$$

Krylov and Bogoliubov suggested that, for small ε, the solution of equation 2.4.1 takes the form

$$x = A(t)\cos(\omega t + \Phi(t)) \qquad (2.4.2)$$

$$\dot{x} = -A(t)\,\omega \sin(\omega t + \Phi(t)) \qquad (2.4.3)$$

where A(t) and Φ(t) are no longer constants but a slowly varying functions of t.

Now Let

$$y = \dot{x} \qquad (2.4.4)$$

Therefore, 2.4.1. becomes

$$\dot{y} = -\omega^2 x - \varepsilon f(x, y) \ldots. \qquad (2.4.5)$$

and we look for a solution of the form

$$\left.\begin{aligned} x &= A(t)\cos(\omega t + \Phi(t)) \\ y &= -A(t)w\sin(\omega t + \Phi(t)) \end{aligned}\right\} \qquad (2.4.6)$$

Substituting 2.4.6 in the equation 2.4.4 gives

$$-A(t)\omega\sin(\omega t + \Phi(t)) = \dot{A}\cos(\omega t + \Phi(t)) - A(\omega + \Phi)\sin(\omega t + \Phi(t))$$

Thus

$$\dot{A}\cos(\omega t + \Phi(t)) - A\dot{\Phi}\sin(\omega t + \Phi(t)) = 0\ldots \qquad (2.4.7)$$

and substituting equation 2.4.6 in to 2.4.5 gives

$$-\dot{A}\omega\sin(\omega t + \Phi(t)) - A(t)\omega(\omega + \dot{\Phi})\cos(\omega t + \Phi(t)) =$$

$$-\omega^2 A(t)\cos(\omega t + \Phi(t)) - \varepsilon f(x, y)$$

$$\Rightarrow -\dot{A}\omega\sin(\omega t + \Phi(t)) - A(t)\omega\,\dot{\Phi}\cos(wt + \Phi(t)) = -\varepsilon f(x, y)$$

multiplying both sides by (-1) and dividing by ω we've

$$\dot{A}\sin(\omega t + \Phi(t)) + A\dot{\Phi}\cos(\omega t + \Phi(t)) = \frac{\varepsilon}{\omega} f(x, y) \qquad (2.4.8)$$

Now solving equation 2.4.7 and equation 2.4.8 simultaneously we get

$$\dot{A} = \frac{\varepsilon}{\omega}\sin(\omega t + \Phi(t)).\ f(A\cos(\omega t + \Phi), -A\omega\sin(\omega t + \Phi)) \qquad (2.4.9)$$

$$\dot{\Phi} = \frac{\varepsilon}{\omega}\cos(\omega t + \Phi(t)).\ f(A\cos(\omega t + \Phi), -A\omega\sin(\omega t + \Phi)) \qquad (2.4.10)$$

Krylov and Bogoliubov's approximation is to replace $\dot{A}$ and $\dot{\Phi}$ in the equations 2.4.9 and 2.4.10 by their average values over one period, $\frac{2\pi}{\omega}$. A is regarded as a constant in taking the average. This procedure (known as a method of averaging) leads to

$$\dot{A} = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega}\frac{\varepsilon}{\omega}\sin(\omega t + \Phi(t)).f(A\cos(\omega t + \Phi), -A\omega\sin(\omega t + \Phi))dt$$

and

$$\dot{\Phi} = \frac{1}{\frac{2\pi}{\omega}} \int_0^{2\pi/\omega} \frac{\varepsilon}{A\omega} \cos(\omega t + \Phi(t)).f\,(A\cos(\omega t + \Phi), -A\,\omega\sin(\omega t + \Phi))dt$$

substituting $\theta = \omega t + \Phi$ gives the final averaging result

$$\dot{A} = \frac{\in}{2\pi\omega} \int_0^{2\pi} \sin\theta.f\,(A\cos\theta, -A\,\omega\sin\theta)d\theta \qquad (2.4.11)$$

and

$$\dot{\Phi} = \frac{\in}{2\pi A\omega} \int_0^{2\pi} \cos\theta.f(A\cos\theta, -A\,\omega\sin\theta)d\theta \qquad (2.4.12)$$

Note:- In the averaging method

(i) The exact equations (2.4.9) and (2.4.10) are replaced by approximate equations (2.4.11) and (2.4.12)

(ii) Evaluation of these integrals is simplified by making use of the following reduction formula, which may be verified using integration by parts

If we let $I_{m,n}$ denote $\int_0^{2\pi} \sin^m\theta\,\cos^n\theta\,d\theta$ then $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

and $I_{m-n} = \frac{n-1}{m+n} I_{m,n-2}$, where the equations may be used until we reach either $I_{o,o} = 2\pi$ or $I_{o,1} = I_{1,0} = I_{1,1} = 0$, and

(iii) Once the integrals have been evaluated we have first order differential equations to solve for A and Φ.

Example:- Apply averaging method to solve van der pol equation

$$\ddot{x} + \varepsilon (x^2 - 1)\, \dot{x} + x = 0$$

Solution:- Here $\omega = 1$, $f(x, y) = f(x, \dot{x}) = (x^2 - 1)\, \dot{x}$ and

$$x = A \cos \theta,\ y = \dot{x} = -A \sin \theta$$

then $f(x, \dot{x}) = f(A \cos \theta, -A \sin \theta)$

$$= A \sin \theta - A^3 \cos^2\theta \sin \theta$$

There fore.

$$\dot{A} = \frac{\varepsilon}{2\pi} \int_0^{2\pi} \sin\theta\, (A \sin \theta - A^3 \cos^2\theta \sin \theta)\, d\theta$$

and $\dot{\Phi} = \frac{\varepsilon}{2\pi A} \int_0^{2\pi} \cos\theta\, (A \sin \theta - A^3 \cos^2 \theta \sin \theta)\, d\theta$

Now let's solve for A:

$$\dot{A} = \frac{\varepsilon}{2\pi} \int_0^{2\pi} \sin\theta\, (A \sin \theta - A^3 \cos^2 \theta \sin \theta)\, d\theta$$

$$= \frac{\varepsilon}{2\pi} \int_0^{2\pi} (A \sin^2\theta - A^3 \cos^2 \theta \sin^2\theta)\, d\theta$$

$$= \frac{\varepsilon A}{2\pi} \left[\int_0^{2\pi} \sin^2 \theta\, d\theta - A^2 \left(\frac{2-1}{2+2} \int_0^{2\pi} \sin^2 \theta\, d\theta\right)\right]$$

$$= \frac{\varepsilon A}{2\pi} \left[1 - \frac{A^2}{4}\right) \int_0^{2\pi} \sin^2 \theta\, d\theta \right]$$

but $\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{2-1}{2+0}\int_0^{2\pi} d\theta = \frac{1}{2}(2\pi) = \pi$

Thus $\dot{A} = \frac{\varepsilon A}{2}\left(1-\frac{A^2}{4}\right) = \frac{A\varepsilon}{8}(4 - A^2)$

This may be written as

$$\int \frac{dA}{A(2-A)(2+A)} = \frac{\varepsilon}{8}\int dt$$

and using partial fractions

$$\int\left[\frac{1}{4A}+\frac{1}{8(2-A)}-\frac{1}{8(2+A)}\right]dA = \frac{\varepsilon}{8}\int dt$$

Multiplying both sides by 8 and performing the integration gives

$$2\ln A - \ln(2- A) - \ln(2 + A) = \varepsilon t + \ln c,$$

or

$$\ln\left[\frac{A^2}{c(4-A^2)}\right] = \varepsilon t.$$

Then from this we have

$$\frac{A^2}{c(4-A^2)} = e^{\varepsilon t}$$

and solving for A we get

$$A = \sqrt{\frac{4ce^{\varepsilon t}}{1+ce^{\varepsilon t}}} = 2\left(\frac{ce^{\varepsilon t}}{1+ce^{\varepsilon t}}\right)^{1/2}$$

$$= 2\left(\frac{c}{1+ce^{-\varepsilon t}}\right)^{1/2}$$

where c is a constant to be determined from the initial conditions.

Similarly, we can solve for $\dot{\Phi}$ as follows:

$$\dot{\Phi} = \frac{\varepsilon}{2\pi A}\int_0^{2\pi}\cos\theta\,(A\sin\theta - A^3\cos^2\theta\sin\theta)d\theta$$

$$= \frac{\varepsilon}{2\pi A}\left(\int_0^{2\pi} A\cos\theta\sin\theta - \int_0^{2\pi} A^3\cos^3\theta\sin\theta d\theta\right)$$

$$= \frac{\varepsilon}{2\pi}\left[\int_0^{2\pi}\cos\theta\sin\theta d\theta - A^2\left(\frac{3-1}{3+1}\int_0^{2\pi}\cos\theta\sin\theta d\theta\right)\right]$$

$$= \frac{\varepsilon}{2\pi}\left[\left(1-\frac{A^2}{2}\right)\int_0^{2\pi}\cos\theta\sin\theta d\theta\right]$$

but $\int_0^{2\pi}\cos\theta\sin\theta d\theta = \int_0^{2\pi}\frac{\sin 2\theta}{2} = \left[-\frac{1}{4}\cos 2\pi\right]_0^{2\pi} = 0.$

$$\Rightarrow \dot{\Phi} = 0$$

Thus $\Phi = \Phi_0 = $ constant, where Φ_0 is the value of Φ when $t = 0$.

Therefore, the solution is given approximately by

$x = A(t)\cos(\omega t + \Phi)$, and so

$$x \approx 2\left(\frac{c}{c + e^{-\varepsilon t}}\right)^{1/2}\cos(t + \Phi_0).$$

Chapter 3.

Initial value problems II on singular perturbation theory.

3.1 Introduction

Let's consider the equation of the mass spring system

$$\varepsilon u'' + u' + u = f(t), \qquad (3.1.1)$$

with specified initial data, $u(0) = a$, $u'(0) = \frac{b}{\varepsilon}$, where $\varepsilon << 1$ indicates that the system is over damped, having a small mass and order one damping.

Notice that the multiscale analysis of the previous section works when there is a slowly varying oscillation, and it will not work for the above system.

The naive approach to this problem is to set $\varepsilon = 0$ and see what happens. With $\varepsilon = 0$, we have the reduced equation"

$$u' + u = f(t), \qquad (3.1.2)$$

whose solution we can find exactly as

$$u(t) = Ae^{-t} + \int_0^t e^{s-t} f(s)ds \qquad (3.1.3)$$

This is a good approximation because as long as $u''(t)$ is of order one, $\varepsilon u''$ is small. However, unless we are unreasonably lucky, we cannot satisfy both pieces of initial data with one free parameter A. So we have no idea (yet) how best to pick A.

But some thing more useful also happened. In setting $\varepsilon = 0$, we replaced the characteristic equation $\varepsilon\lambda^2 + \lambda + 1 = 0$ by the reduced characteristic equation $\lambda + 1 = 0$.

That is, we replaced the exponential solutions

$u(t) = e^{\lambda_1 t}, e^{\lambda_2 t}$ where $\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$ by $u(t) = e^{-t}$ and $e^{-\infty} = 0$

Here we have discarded terms of the order of $e^{-t/\varepsilon}$, which for small ε and finite t are vanishing small, but at $t = 0$ contribute in a significant way to the solution.

The occurrence of the term $e^{-t/\varepsilon}$ is the reason this problem is singular. There is no expansion of $e^{-t/\varepsilon}$ which is valid in a neighborhood of $t = \varepsilon = 0$. We refer to the term $e^{-t/\varepsilon}$ as a boundary layer term because it undergoes a change of order one in a temporal region of order ε and this takes place at the boundary of the temporal domain.

Now to find what goes on in the boundary layer we make a change of timescale that gives the boundary layer some significance. We take $\tau = t/\varepsilon$ to be a "fast" time.

Hence, where $U(\tau) = u(\varepsilon \tau)$. We have

$$U'(\tau) = \varepsilon\, u'(\varepsilon \tau) \Rightarrow u'(\varepsilon \tau) = \frac{U'(\tau)}{\varepsilon}$$

$$U''(\tau) = \varepsilon^2 u''(\varepsilon \tau) \Rightarrow u''(\varepsilon \tau) = \frac{U''(\tau)}{\varepsilon^2}$$

Furthermore, $U(\tau) = u(\varepsilon \tau) \Rightarrow U(0) = u(0) = a$

and $U'(\tau) = \varepsilon\, u'(\varepsilon \tau) \Rightarrow U'(0) = \varepsilon\, u'(0) = \varepsilon \left(\dfrac{b}{\varepsilon}\right) = b.$

Therefore, interms of τ, the governing equation becomes

$$\ddot{U} + \dot{U} + \varepsilon U = \varepsilon f(\varepsilon \tau) \qquad (3.1.4)$$

$U(0) = a$, $\dot{U}(0) = b$; where $U(\tau) = u(\varepsilon \tau)$.

Setting $\varepsilon = 0$, we find the "boundary layer" equation

$$\ddot{U} + \dot{U} = 0 \qquad (3.1.5)$$

$U(0) = a$, $\dot{U}(0) = b$

Whose solution is $U(\tau) = b\,(1-e^{-\tau}) + a$ or interms of the original variable $t = \varepsilon \tau$

$$u(t) = b(1-e^{-t/\varepsilon}) + a$$

Now we notice that according to the boundary layer solution near $t = 0$, the solution starts with correct initial data and rapidly decays to $a + b$. Apparently, to make the two approximations match, we should take $A = a + b$.

Therefore, a solution that is valid, uniformly for all time, is the sum of the two approximations, with the common part subtracted out.

Note: - The common part is the contribution common to both approximations, that is $a + b$.

Hence $u(t) = -b e^{-t/\varepsilon} + (a + b)\, e^{-t} + \int_0^t e^{(s-t)} f(s)ds + 0(\varepsilon)$

3.2 Michaelis – Menten Enzyme kinetics

In this section we will consider the conversion of a chemical substrate S to a product P by enzyme catalysis. The reaction scheme

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} ES \xrightarrow{k_2} E + P \qquad (3.2.1)$$

was proposed by Michaelis and Menten in 1913. According to this reaction scheme, the enzyme E can combine at rate k_1 with S to form ES and ES can decompose in to E and S at rate k_{-1} or in to E and P at rate k_2.

Now, the law of mass action, which yields the following rate equations, governs the concentrations of E, S, ES and P:

$$\frac{d[E]}{d\tau} = -k_1[E][S] + k\,[ES] + k_2[ES] \qquad (3.2.2)$$

$$\frac{d[S]}{d\tau} = -k_1[E][S] + k_{-1}[ES] \qquad (3.2.3)$$

$$\frac{d[ES]}{d\tau} = k_1[E][S] - k_{-1}[ES] - k_2[ES] \qquad (3.2.4)$$

and $$\frac{d[P]}{d\tau} = k_2[ES] \qquad (3.2.5)$$

subjected to initial conditions $[E] = E_0$, $[S] = S_0$, $[ES] = [P] = 0$ at time $\tau = 0$; where the square brackets denote concentrations. From the above we notice the following facts:

1) [P] is found by direct integration once [ES] is known

2) From equations 3.2.2 and 3.2.4 we have

$$\frac{d[ES]}{d\tau} + \frac{d[E]}{d\tau} = 0 \text{ for all time}$$

$$\Rightarrow \quad \frac{[[ES]+[E]]}{d\tau} = 0, \text{ for all time } \tau.$$

$$\Rightarrow \quad [ES] + [E] = C, \text{ a constant, for all time.}$$

(But at time $\tau = 0$, $[ES] = 0$ and $[E] = E_0$)

Therefore, we have

$$[ES] + [E] = E_0 \text{ for all time } \tau. \qquad (3.2.6)$$

We now introduce the non dimensions at variables

$$u = [ES]/C_0\,,\ v = [S]/S_0\,,\ t = k_1 E_0\tau,\ k = (k_{-1} + k)/k_1 S_0,$$

$$\varepsilon = C_0/S_0\,,\ \lambda = \frac{k-1}{k_{-1}+k_2},\ C_0 = E_0/(1+k)$$

Using the above non-dimensional variables and what we have in equations 3.2.2 – 3.2.6, we obtain

$$\varepsilon \frac{du}{dt} = v - (v + k)\, u /(1+k) \qquad (3.2.7)$$

$$\frac{dv}{dt} = -v + (v + k\lambda)\, u/(1+k), \qquad (3.2.8)$$

subjected to initial conditions u(0) = 0, v(0) = 1.

It is typical that the initial concentration of substrate is large compared to the initial concentration of enzyme, so $\varepsilon > 0$ is small.

Notice also that λ and k/(1+k) are smaller than one, so that there are no big numbers to worry about other than $1/\varepsilon$.

The naive approximation is to set $\varepsilon = 0$ so that from equation (3.2.7) and equation (3.2.8) we have

$$u = \frac{v}{v+k}(1 + k) \qquad (3.2.9)$$

and

$$\frac{dv}{dt} = \frac{-kv}{v+k}(1 - \lambda) \qquad (3.2.10)$$

respectively.

Definition 3.2.1:- The above solution is called the QUASI – STEADY STATE APPROXIMATION (qss), and which is valid as long as $\varepsilon \dot{u}$ remains small.

Since we can't satisfy the two initial conditions with one first order equation, this cannot be uniformly valid for all time. The solution of this approximate equation is given implicitly by

$$v + k\ln v = A - k(1 - \lambda)t\,, \qquad (3.2.11)$$

where A is not yet known.

To determine A and find how u and v behave initially, we introduce a fast time $\delta = t/\varepsilon$.

Then in terms of δ, the original equations (3.2.7) and (3.2.8) becomes

$$\frac{dU}{d\delta} = V - (V + k)\, U/(1+k) \qquad (3.2.12)$$

$$\frac{dV}{d\delta} = \varepsilon\,(-V + (V + k\lambda)\, U/(1+k)) \qquad (3.2.13)$$

U(0) = 0, V(0) = 1: where U(δ) = u(εδ), V(δ) = V(εδ); which are the boundary layer equations. Now by setting ε = 0, we found the boundary layer equations reduced to

$$\frac{dU}{d\delta} = V - (V + k)\, U/(1+k) \qquad (3.2.14)$$

$$\frac{dV}{d\delta} = 0 \qquad (3.2.15)$$

U(0) = 0, V(0) = 1

And these equations imply that during a short initial time interval V doesn't change,

So $V(\delta) = 1$ (3.2.16)

while $U(\delta) = (1 - e^{-\delta})$ (3.2.17)

In original variables

$$u(t) = 1 - e^{-t/\varepsilon} \qquad (3.2.18)$$

The first order picture is now complete. In a short initial time segment U equilibrates rapidly to 1 while V remains fixed at v = 1. There after v satisfies,

$$v + k \ln v = 1 - k\,(1 - \lambda)t \qquad (3.2.19)$$

and $u = \frac{v(1+k)}{v+k}$. (3.2.20)

Therefore fore, a uniform valid representation of this is given by

$$v = k \ln v = 1 - k(1 - \lambda)t \qquad (3.2.21)$$

$$U = \frac{v(1+k)}{v+k} - e^{-t/\epsilon} \qquad (3.2.22)$$

to leading order in ε.

3.3 SLOW SELECTION IN POPULATION GENETICS

Suppose we have a population of organisms, one of whose characteristics is determined by some gene at a particular locus on the chromosome. We suppose that there are two alleles, denoted as type A or type B, and that the genotypes are the three diploids AA, AB and BB.

We model the population of genotypes by letting x_1, x_2 and x_3 denote the total number of AA, AB and BB respectively. The growth of each population is determined by

$$\frac{dx_i}{dt} = b_i p_i p - d_i x_i, \text{ i} = 1, 2, 3, \qquad (3.3.1)$$

where b_i and d_i are survival (at birth) and death rates, respectively, for population x_i *and* d_i is the probability that random mating will produce genotype i.

Simple probability arguments show that

$$p_1 = \left(\frac{2x_1 + x_2}{2p}\right)^2 \qquad (3.3.2)$$

$$p_2 = 2\left(\frac{2x_1 + x_2}{2p}\right)\left(\frac{2x_3 + x_2}{2p}\right) \qquad (3.3.3)$$

$$p_3 = \left(\frac{2x_3 + x_2}{2p}\right)^2 \qquad (3.3.4)$$

Where the total population is $p = x_1 + x_2 + x_3$. Now we introduce the scaled variables

$D = \frac{x_1}{P}$, $2H = \frac{x_2}{P}$, $R = \frac{x_3}{P}$, where D, H and R represent the dominant, hetrozygote and recessive genotype proportions of the population.

Hence, $P_1 = (D + H)^2$, $P_2 = 2(D + H)(R + H)$ and

$$P_3 = (R + H)^2$$

Now interms of the above scaled variables, the total population satisfies the following equations:

$$\frac{1}{P}\frac{dp}{dt} = \sum_{i=1}^{3} b_i p_i - (d_1 D + 2d_2 H + d_3 R) \qquad (3.3.5)$$

$$\frac{dD}{dt} = b_1 p_1 - (d_1 + F(D, H.R))\, D \qquad (3.3.6)$$

$$\frac{dH}{dt} = b_1 p_2/2 - (d_2 + F(D, H.R))H \qquad (3.3.7)$$

$$\frac{dR}{dt} = b_3 p_3 - (d_3 + F(D, H, R))R \qquad (3.3.8)$$

Where $F(D, H, R) \equiv \sum_{i=1}^{3} b_i p_i - (d_1 D + 2d_2 H + d_3 R) = \frac{1}{P}\frac{dp}{dt}$.

Note:- since $D + 2H + R = \frac{x_1}{P} + \frac{x_2}{P} + \frac{x_3}{P} = \frac{P}{P} = 1$, one of the above four equations is superfluous.

We now suppose that the death rates for the three genotypes are only slightly different, and that the survival rates at birth are identical for the three genotypes. And we incorporate this assumption into the model by taking

$$b_i = b,\ d_i = d + \varepsilon\Delta_i,\ i = 1, 3 \text{ and } \varepsilon > 0 \text{ small.}$$

With this specification the model equations become

$$\frac{1}{P}\frac{dp}{dt} = b - d - \varepsilon\,[\Delta_1 D + \Delta_3 R] \qquad (3.3.9)$$

$$\frac{dD}{dt} = b\,(p^2 - D) + \varepsilon D[\Delta_1 D + \Delta_3 R - \Delta_1] \qquad (3.3.10)$$

$$\frac{dp}{dt} = \varepsilon\,[p(\Delta_1 D + \Delta_3 R) - \Delta_1 D] \qquad (3.3.11)$$

Where p= D + H, R = 1 + D – 2p

In this system of equations, the total population P can be determined by quadrature if D and p are known. Thus, it is only necessary to solve the last two equations for D and P. We can do this with $\varepsilon > 0$ very small using the two-scale method of this section.

First let's make some important observations:

Since ε is small, p is a slowly varying function. This suggests that D equilibrates rapidly to something close to p^2. The quasi - steady state approximation for this is to take $D = p^2$ and substitute in to the differential equation for p. But the qss approximation is valid only after some initial time interval of rapid transition.

So to find a solution representation that is valid for all time, we use two timescales.

We take $s = t$ and $\tau = \varepsilon t$ to be the fast and slow time scales, respectively, and take $r = D - p^2$.

Using that $\frac{d}{dt} = \frac{\partial}{\partial s} + \varepsilon\frac{\partial}{\partial \tau}$. We transform the equations for D and p into the equations

$$\frac{\partial p}{\partial s} = \varepsilon p(p\text{-}1)\,[\Delta_1 p + \Delta_3(p\text{-}1] + \varepsilon r\,[(\Delta_1 + \Delta_3)p - \Delta_1] - \varepsilon\frac{\partial p}{\partial \tau} \quad (3.3.12)$$

$$\frac{\partial r}{\partial s} = -br + \varepsilon\,[\Delta_1(r + p^2)\,(r - (p\text{-}1)^2) + \Delta_3(r\text{-}p^2)\,(r + (p\text{-}1)^2) - \frac{\partial r}{\partial \tau}] \quad (3.3.13)$$

To the lowest order in ε, the solutions are $p = p_0(\tau)$ and $r = r_0(\tau)e^{-bs}$ and the first order correction terms are governed by

$$\frac{\partial p_1}{\partial s} = p_0(p_0 - 1)[\Delta_1 p_0 + \Delta_3(p_0\text{-}1)] + r_0 e^{-bs}[\Delta_1 + \Delta_3)p_0 - \Delta_1] - \frac{\partial p_0}{\partial \tau} \quad (3.3.14)$$

and

$$\frac{\partial r_1}{\partial s} + br_1 = \Delta_1(r_0 e^{-bs} + p_0^2)(r_0 e^{-bs} - (p_0 - 1)^2 +$$

$$\Delta_3(r_0 e^{-bs} - p_0^2)(r_0 e^{-bs} + (p_0 - 1)^2) - \frac{\partial r_0}{\partial \tau} e^{-bs} \qquad (3.3.15)$$

From equation (3.3.14), we can see that p_1 is bounded for all s if and only if

$$\frac{\partial p_0}{\partial \tau} = p_0 (p_0 - 1)[\Delta_1 p_0 + \Delta_3 (p_0 - 1)] \qquad (3.3.16)$$

which determines the slow variation of p_0.

And from equation 3.3.15 we can see that r_1 has no secular terms if and only if

$$\frac{\partial r_0}{\partial \tau} = r_0 (\Delta_1 - \Delta_3)(p_0^2 - (p_0 - 1)^2) \qquad (3.3.17)$$

For our purposes, the more important equation is that governing the slow variation of p. This equation is nonlinear first order differential equation with cubic right hand side. It can be integrated exactly, but that is unnecessary to gain an understanding of its behavior.

And we see that in equation 3.3.16, the cubic has three roots at

$$p_0 = 0,\ 1 \text{ and } p_0 = \Delta_3 / (\Delta_1 + \Delta_3) \equiv \hat{p}\,.$$

Clearly, $\hat{p}$ is biologically relevant only if $0 < \hat{p} < 1$.

Therefore, the Δ_1, Δ_3 parameter plane can be divided in to four regions (depending on the sign of Δ_1 and Δ_3) in which p_0 has different behavior.

Region I: When $\Delta_1 > 0$, $\Delta_3 > 0$.

In this region, $p_0 \neq 0$ and $p_0 \neq 1$. Then p_0 approaches $\hat{p}$ as $\tau \to \infty$ so that the hetrozygote AB population is stable.

Region II When $\Delta_1 < 0$, $\Delta_3 > 0$.

In this region, $\hat{p}$ is not biologically relevant and p_0 approaches 1, so that homozygote AA is the sole survivor since it has a selective advantage over the other populations.

Region III. When $\Delta_3 < 0$, $\Delta_1 > 0$.

In this region, $\hat{p}$ is also not biologically relevant and p_0 approaches 0, so that the homozygote BB is the sole survivor.

Region IV: When $\Delta_3 < 0$, $\Delta_1 < 0$.

In this region, $\hat{p}$ is unstable and the steady states $p = 0, 1$ are both stable. The outcome of the dynamics depends on initial data. But the surviving population will be homozygous (either AA or BB or both), since they have a selective advantage over the hetrozygote.

Chapter 4

Boundary value problems on Singular Perturbation Theory

4.1: Matched Asymptotic Expansions

Boundary value problems often show the same features as initial value problems with multiple scales in operation, and resulting boundary layers. The added difficulty is that we do not know a priori where the boundary layer is located.

To introduce the matching principle, let us consider the following example below:

$$\varepsilon y'' + y' + y = 0 \qquad (4.1.1)$$

$$y(0) = 0,\ y(1) = 1$$

which we can solve exactly.

By setting $\varepsilon = 0$, we find the reduced equation

$$y' + y = 0 \qquad (4.1.2)$$

Which has the solution

$$y_0(x) = Ae^{-x}, \qquad (4.1.3)$$

where A is a constant to be determined.

It is immediate to see that this naive approximation solution can't satisfy the two conditions at a time. At most one of the boundary conditions can be satisfied by the solution $y_0(x) = Ae^{-x}$.

It must be that $\varepsilon y''$ is large some where and cannot be uniformly ignored.

To find the appropriate boundary layer, we introduce a scaled space variable $\xi = (x - x_0)/\varepsilon$ where x_0 is unknown but represents the location of the boundary layer.

Interms of the fast variable ξ, the original equation becomes

$$Y'' + Y' + \varepsilon Y = 0 \qquad (4.1.4)$$

where $Y(\xi) = y(\varepsilon\xi)$.

And with $\varepsilon = 0$, this equation reduced to $Y'' + Y' = 0$, which is the boundary layer equation. The solution of the boundary layer equation is

$$Y_b = Be^{-\xi} + C \qquad (4.1.5)$$

where B and C are as yet unknown.

Notice that Y is exponentially decaying for positive ξ, but exponentially growing for ξ negative. Since we hope to have a boundary layer that is influential only in a narrow region, we want it to be exponentially decaying. That is, we want $\xi > 0$. It then follows that $x_0 = 0$, and the boundary layer should be located at $x = 0$.

Thus, we take

$$Y_b = B\,(e^{-\xi} - 1) \qquad (4.1.6)$$

as the lowest order representation of the solution near $x = 0$.

And the lowest order representation away from the boundary layer (called the "outer expansion") is

$$y_0 = e^{-(x-1)} \qquad (4.1.7)$$

Where the naive approximation solution $y_0(x) = Ae^{-x}$ has been chosen to satisfy the boundary condition at $x = 1$.

Now we need to choose the remaining unknown constant B.

The heuristic idea is to pick B so that y_0 and Y_b match,

that is $\lim_{x \to o} y_0(x) = \lim_{\xi \to \infty} Y_b(\xi)$

$$\Rightarrow \quad \lim_{x \to 0} e^{-(x-1)} = \lim_{\xi \to \infty} B(e^{-\xi}-1)$$

$$\Rightarrow \quad e = -B.$$

Therefore, $B = -e$. It follows that

$$y_0 = e^{-(x-1)} \text{ and } Y_b = e - e^{1-x/\varepsilon} \qquad (4.1.8)$$

As we did before we form a composite approximation by summing the two approximations and subtracting the common part (the common part is "e" since that is the value at which the two solutions agree). Thus we find the approximate solution as

$$y = e^{-(x-1)} - e^{1-x/\varepsilon} \qquad (4.1.9)$$

This heuristic prescription to find B recognizes that in the limit as $\varepsilon \to 0$, the boundary layer as $\xi \to \infty$ should agree with the outer solution as $x \to 0$. That is, there should be some common domain in which the boundary layer and the outer solutions have the same value.

The above prescription to match the boundary layer with the outer solution works well for the leading order approximation, but needs to be made more precise to match higher order approximations. There are two ways to state the Matching principle, both give the same result.

Kaplan Matching principle

Suppose $Y_b^n(\xi)$ is the n^{th} order boundary layer expansion and $y_0^n(x)$ is the n^{th} order outer expansion. Let $\eta(x, \varepsilon)$ be an intermediate variable, that is $\lim_{\varepsilon \to 0} \xi/\eta = \infty$ and $\lim_{\varepsilon \to 0} x/\eta = 0$.

Require that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} (Y_b^n (\xi (\eta)) - y_0^n(x(\eta)) = 0$$

in dependent of η.

Van Dyke matching principle

Suppose $Y_b^n(\xi)$ is the n^{th} order boundary layer expansion and $y_0^n(x)$ is the n^{th} order outer expansion. Require that the power series expansion of $Y_b^n(\xi(x))$ for fixed x be the same as the power series expansion of $y_0^n(x(\xi))$ for fixed ξ. i.e.

Expand in powers of ε the boundary layer solution interms of the outer variable and the outer solution interms of the boundary layer variable and require that the two expansions agree.

To illustrate the matching principles above let us apply the principles on the example we consider above.

I. The Kaplan matching principle works well on our example if we take

$$\eta = x/\varepsilon^{1/2}.$$

Because in terms of intermediate variable η,

$$\xi = x/\varepsilon = x/\varepsilon^{1/2}\varepsilon^{1/2} = \eta/\varepsilon^{1/2} \text{ and } x = \eta\varepsilon^{1/2}.$$

Then it is true that

$$\lim_{\varepsilon\to 0} \xi/\mu = \lim_{\varepsilon\to o} \frac{1}{\varepsilon^{1/2}} = \infty \text{ and } \lim_{\varepsilon\to 0} x/\eta = \lim_{\varepsilon\to 0} \varepsilon^{1/2} = 0.$$

Further more, interms of η

$$Y_b = B(e^{-\xi} - 1) = B\left(e^{-\eta/\varepsilon^{1/2}} - 1\right)$$

and $y_0 = e^{-(x-1)} = e^{-\left(\eta\varepsilon^{1/2}-1\right)}$.

Finally take the limit $\varepsilon \to 0$ with η fixed as $\lim_{\varepsilon\to 0}(Y_b - y_0) = 0$

$\Rightarrow$ -B – e = 0

$\Rightarrow$ B = -e, which determines B.

II To use the Vandyke Matching principle, we express Y_b interms of x as:

$$Y_b = B(e^{-\xi} - 1)$$

$$= B\left(e^{-x/\varepsilon} - 1\right)$$

and y_0 interms of ξ as:

$$y_0 = e^{-(x-1)}$$

$$= e^{-(\varepsilon\xi - 1)} \text{ because } x = \varepsilon\xi$$

$$= e^{1-\xi\varepsilon}$$

then expanding both as powers of ε, we obtain

$$Y_b = B\left(e^{-x/\varepsilon} - 1\right) \sim -B + o(\varepsilon)$$

and $y_0 = e^{1-\xi\varepsilon} \sim e + o(\varepsilon)$

Hence the two expressions agree if $B = -e$.

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