Two-Mode Coherent and Subharmonic Output Light

A Thesis Submitted to the
School of Graduate Studies
Addis Ababa University

In Partial Fulfillment of the Requirement for
the Degree of Master of Science in Physics

By
Menwuyelet Melaku
Addis Ababa, Ethiopia
Feb, 2012

ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
FACULTY OF CHEMICAL AND PHYSICAL SCIENCE
DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommend to the School of Graduate Studies for acceptance a thesis entitled "**Two-Mode Coherent and Subharmonic Output Light**" by **Menwuyelet Melaku** in partial fulfillment of the requirements for the degree of **Master of Science in Physics**.

Dated: Feb, 2012

Approved by the Examination Committee

Dr. Fesseha Kassahun, Advisor ______________________

Dr. Gizaw Mengistu, Examiner ______________________

Dr. Tadelech Atomssa, Examiner ______________________

ADDIS ABABA UNIVERSITY

Date: **Feb, 2012**

Author: **Menwuyelet Melaku**

Title: **Two-Mode Coherent and Subharmonic Output Light**

Department: **Physics**

Degree: **M.Sc.** Convocation: **Feb,** Year: **2012**

Permission is herewith granted to Addis Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

Table of Contents

Table of Contents **iii**

List of Figures **iv**

Abstract **v**

Acknowledgements **vi**

1 Introduction **1**

2 Two-mode Coherent output light **3**

2.1 c-number Langevin equations 3

2.2 The Q-function for a two-mode coherent output light 10

2.3 Output photon statistics 14

2.3.1 The mean of the output photon number sum and difference 14

2.3.2 The variance of the output photon number sum and difference . . . 16

3 Two-mode subharmonic output light **20**

3.1 c-number Langevin equations 20

3.2 Q-function for a two-mode subharmonic output light 28

3.3 Output photon statistics 33

3.3.1 The mean of the output photon number sum and difference 34

3.3.2 The variance of the output photon number sum and difference . . . 36

4 Superposed two-mode Coherent and Subharmonic output light **40**

4.1 The Q-function 40

4.2 Output photon statistics 46

4.2.1 The mean of the output photon number sum and difference 46

4.2.2 The variance of the output photon number sum and difference . . . 48

5 Conclusion **53**

References **54**

List of Figures

3.1 Two-mode subharmonic generator 20

3.2 Two-mode subharmonic generator for output mode 33

4.1 Two-mode Coherent and Subharmonic cavity light beams 44

4.2 Two-mode Coherent and Subharmonic output light beams 46

Abstract

With the aid of c-number Langevin equations, we have calculated the Q-functions for the two-mode coherent and subharmonic cavity light beams. Applying the input-output relation, we have obtained the Q-functions for the two-mode coherent and subharmonic output light beams. We have then determined the Q-function for the superposition of these two output light beams. With the help of the pertinent Q-functions for the output light, we determined the output photon statistics. We have calculated the mean and variance of the photon number sum and difference for the two-mode coherent output light, the two-mode subharmonic output light, and the superposed two-mode coherent and subharmonic output light beams.

We have found that photon stastics of the two-mode coherent output light is Poissonian and that of the subharmonic output light is super Poissonian. And the output signal and idler modes are separately in a chaotic states. In addition, the mean photon number for the superposed output light turns out to be the sum of the mean photon number of the two-mode coherent output light and that of the two-mode subharmonic output light.

Acknowledgements

First of all, I would like to thank the almighty God for letting me accomplish this study. Secondly, I would like to express my heart felt gratitude to my instructor and advisor Dr. Fesseha Kassahun, for his invaluable advices, continuous support and friendly approach through out this thesis. Finally, I am deeply indebted to my families-especially to my wife Tirsit Adeto for her sincere encouragement and moral support.

Chapter 1

Introduction

Quantum optics deals mainly with the quantum properties of the light generated by various optical systems such as lasers and with the effect of light on the dynamics of atoms. The quantum properties of light are largely determined by the states of the light mode and the well known quantum states of light are the number state, the chaotic state, the coherent state and the squeezed state[1]. Of all states of light, the coherent states are the most important and arise frequently in quantum optics. Not only can they be an accurate representation of the light produced by a stabilished laser operating well above threshold, but also many of the techniques for studying the properties of the light rely on the properties of the coherent states [2-5]

A one-mode subharmonic generator, consisting of a non linear crystal pumped by a coherent light, is a prototype source of a single-mode squeezed light. Squeezing like photon antibounching or subPoissonian photon statistics is a non-classical feature of light. It is also worth mentioning that squeezed light has potential applications in the detection of weak signal and in low-noise communications [1-10].

A two-mode subharmonic generator is a typical source of a two-mode squeezed light. In a two-mode subharmonic generator, a pump photons of frequency ω_c is down convereted into highly correlated signal and idler photons of frequency ω_a and ω_b [1-10].

Yibeltal siyum [11] studied on the statistical properties of two-mode coherent and subharmonic cavity light beams. His study shows that the mean photon number for the superposed two-mode coherent and subharmonic cavity light is the sum of the mean photon number of the individual cavity light beams and the variance of the photon number for cavity mode a is the same as that of cavity mode b. He also found that the mean of the photon number difference vanishes. His study was on the cavity light, however, it is also very essential to determine the cavity output which is directly accessible to experimental observations.

Hence, in this thesis we seek to study the statistical properties of two-mode coherent and subharmonic output light. To this end, With the aid of c-number Langevin equations, we calculate the Q-function for the two-mode coherent and subharmonic cavity light. Applying the input-output relation we find the Q-function for the two-mode coherent and subharmonic output lights. We then determine the Q-function for the superposition of these two output light beams. With the help of the pertinent Q-functions for the output lights, we determine the output photon statistics. We calculate the mean and variance of the photon number sum and difference for the two-mode coherent output light, the two-mode subharmonic output light, and the superposed two-mode coherent and subharmonic output light.

Chapter 2

Two-mode Coherent output light

2.1 c-number Langevin equations

In this section we consider a cavity mode driven by a two-mode coherent light and coupled to a two-mode vacuum reservoir via a single port mirror. The interaction between a two-mode cavity light and a two-mode driving coherent light, can be described by the Hamiltonian

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \tag{2.1.1}$$

where $\hat{a}(\hat{b})$ is the annihilation operator for the cavity mode $a(b)$ and ε is proportional to the amplitude of the driving modes. Applying Eq. (2.1.1), the master equation for a cavity mode driven by a two-mode coherent light and coupled to a two-mode vacuum reservor, can be written as [1]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -\varepsilon(\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b} + \hat{\rho}\hat{b}^\dagger) \\ &+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}). \end{aligned} \tag{2.1.2}$$

Using the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{2.1.3}$$

along with Eq. (2.1.2), we have

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\varepsilon Tr(\hat{a}\hat{\rho}\hat{a} - \hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{2} + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{b}\hat{\rho}\hat{a} - \hat{b}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a} + \hat{\rho}\hat{b}^{\dagger}\hat{a})$$
$$+\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}) + \frac{\kappa}{2}Tr(2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a}). \quad (2.1.4)$$

Applying the cyclic property of trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1, \quad (2.1.5)$$

and

$$[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^{\dagger}] = [\hat{a}^{\dagger}, \hat{b}] = [\hat{a}^{\dagger}, \hat{b}^{\dagger}] = 0, \quad (2.1.6)$$

we readily find

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle + \varepsilon. \quad (2.1.7)$$

It can be shown in a similar procedure that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa}{2}\langle\hat{b}(t)\rangle + \varepsilon, \quad (2.1.8)$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle + \varepsilon(\langle\hat{a}(t)\rangle + \langle\hat{b}(t)\rangle), \quad (2.1.9)$$

$$\frac{d}{dt}\langle\hat{a}^{2}(t)\rangle = -\kappa\langle\hat{a}^{2}(t)\rangle + 2\varepsilon\langle\hat{a}(t)\rangle, \quad (2.1.10)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger}(t)\rangle + \langle\hat{a}(t)\rangle), \quad (2.1.11)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger}(t)\rangle + \langle\hat{b}(t)\rangle). \quad (2.1.12)$$

The c-number equations corresponding to the normally ordered operators described by Eqs. (2.1.7), (2.1.8), (2.1.9), (2.1.10), (2.1.11), and (2.1.12) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\kappa}{2}\langle\alpha(t)\rangle + \varepsilon, \quad (2.1.13)$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{\kappa}{2}\langle\beta(t)\rangle + \varepsilon, \quad (2.1.14)$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle + \varepsilon(\langle\alpha(t)\rangle + \langle\beta(t)\rangle), \quad (2.1.15)$$

$$\frac{d}{dt}\langle\alpha^{2}(t)\rangle = -\kappa\langle\alpha^{2}(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle, \quad (2.1.16)$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\kappa\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle), \tag{2.1.17}$$

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = -\kappa\langle\alpha^*(t)\beta(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\beta(t)\rangle). \tag{2.1.18}$$

On the basis of Eqs. (2.1.13) and (2.1.14), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) + \varepsilon + f_\alpha(t), \tag{2.1.19}$$

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) + \varepsilon + f_\beta(t), \tag{2.1.20}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces associated with the normal ordering.

We next seek to determine the properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$. We note that Eq. (2.1.13) and the expectation value of Eq. (2.1.19) as well as Eq. (2.1.14) and the expectation value of Eq. (2.1.20) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.1.21}$$

Moreover, on account of the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\frac{d\alpha(t)}{dt}\beta(t)\rangle + \langle\alpha(t)\frac{d\beta(t)}{dt}\rangle, \tag{2.1.22}$$

along with Eqs. (2.1.19) and (2.1.20), we find

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle + \varepsilon(\langle\alpha(t)\rangle + \langle\beta(t)\rangle) + \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle. \tag{2.1.23}$$

Comparison of Eqs. (2.1.15) and (2.1.23) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = 0. \tag{2.1.24}$$

The formal solution of Eqs. (2.1.19) and (2.1.20) can be written as

$$\alpha(t) = \alpha(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon + f_\alpha(t')]dt', \tag{2.1.25}$$

and

$$\beta(t) = \beta(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon + f_\beta(t')]dt'. \tag{2.1.26}$$

Then on account of Eqs. (2.1.25) and (2.1.26), one readily obtains

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\beta(t)\rangle + \langle f_\beta(t)f_\alpha(t')\rangle]dt', \tag{2.1.27}$$

and

$$\langle\beta(t)f_\alpha(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\alpha(t)\rangle + \langle f_\alpha(t)f_\beta(t')\rangle]dt'. \tag{2.1.28}$$

Since a noise operator at a certain time should not affect the system operator at an earlier time, we see that

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \tag{2.1.29}$$

$$\langle\beta(0)f_\alpha(t)\rangle = \langle\beta(0)\rangle\langle f_\alpha(t)\rangle = 0. \tag{2.1.30}$$

Thus on account of Eqs. (2.1.21), (2.1.27), (2.1.28), (2.1.29), and (2.1.30), we get

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle dt', \tag{2.1.31}$$

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt'. \tag{2.1.32}$$

Therefore, in view of Eqs. (2.1.24), (2.1.31), and (2.1.32) and assuming

$$\langle f_\beta(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\beta(t')\rangle, \tag{2.1.33}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt' = 0. \tag{2.1.34}$$

Now on the basis of the relation [1]

$$\int_0^t e^{-a(t-t')/2}\langle f(t)g(t')\rangle dt' = b, \tag{2.1.35}$$

we assert that

$$\langle f(t)g(t')\rangle = 2b\delta(t-t'), \tag{2.1.36}$$

where a is a constant and b is a constant or some function of time t. We then see that

$$\langle f_\beta(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\beta(t')\rangle = 0. \tag{2.1.37}$$

Furthermore, using the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \langle\frac{d\alpha(t)}{dt}\alpha(t)\rangle + \langle\alpha(t)\frac{d\alpha(t)}{dt}\rangle, \tag{2.1.38}$$

along with Eq. (2.1.19), we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{2.1.39}$$

Comparison of Eqs. (2.1.16) and (2.1.39) shows that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{2.1.40}$$

In view of Eq. (2.1.25), one readily obtains

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\alpha(t')\rangle dt' = 0, \tag{2.1.41}$$

from which follows

$$\langle f_\alpha(t)f_\alpha(t')\rangle = 0. \tag{2.1.42}$$

It can be estabilished in a similar manner that

$$\langle f_\beta(t)f_\beta(t')\rangle = 0. \tag{2.1.43}$$

Moreover, employing the relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\frac{d\alpha^*(t)}{dt}\alpha(t)\rangle + \langle\alpha^*(t)\frac{d\alpha(t)}{dt}\rangle, \tag{2.1.44}$$

along with Eq. (2.1.19) and its complex conjugate, we find

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\kappa\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle) + \langle\alpha(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle. \tag{2.1.45}$$

Comparison of Eqs. (2.1.17) and (2.1.45) indicates that

$$\langle\alpha(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = 0. \tag{2.1.46}$$

Hence using Eq. (2.1.25) and its complex conjugate, and assuming

$$\langle f_\alpha(t)f^*_\alpha(t')\rangle = \langle f^*_\alpha(t)f_\alpha(t')\rangle, \tag{2.1.47}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f_\alpha^*(t')\rangle dt' = 0. \tag{2.1.48}$$

Therefore on account of Eq. (2.1.35) and (2.1.36), we see that

$$\langle f_\alpha(t) f_\alpha^*(t')\rangle = \langle f_\alpha^*(t) f_\alpha(t')\rangle = 0. \tag{2.1.49}$$

It can also be estabilished in a similar manner that

$$\langle f_\beta(t) f_\beta^*(t')\rangle = \langle f_\beta^*(t) f_\beta(t')\rangle = 0. \tag{2.1.50}$$

Finally, using the relation

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\beta(t)\rangle + \langle \alpha^*(t)\frac{d\beta(t)}{dt}\rangle, \tag{2.1.51}$$

along with Eq. (2.1.20) and the complex conjugate of Eq. (2.1.19), we get

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = -\kappa\langle \alpha^*(t)\beta(t)\rangle + \varepsilon(\langle \alpha^*(t)\rangle + \langle \beta(t)\rangle) + \langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f_\alpha^*(t)\rangle. \tag{2.1.52}$$

Comparison of Eqs. (2.1.18) and (2.1.52), indicates that

$$\langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f_\alpha^*(t)\rangle = 0. \tag{2.1.53}$$

In view of Eq. (2.1.26) and the complex conjugate of Eq. (2.1.25), and assuming

$$\langle f_\beta(t) f_\alpha^*(t')\rangle = \langle f_\alpha^*(t) f_\beta(t')\rangle, \tag{2.1.54}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t) f_\alpha^*(t')\rangle dt' = 0, \tag{2.1.55}$$

from which follows

$$\langle f_\beta(t) f_\alpha^*(t')\rangle = \langle f_\alpha^*(t) f_\beta(t')\rangle = 0. \tag{2.1.56}$$

It can be estabilished in a similar fashion that

$$\langle f_\alpha(t) f_\beta^*(t')\rangle = \langle f_\beta^*(t) f_\alpha(t')\rangle = 0. \tag{2.1.57}$$

We see that Eqs. (2.1.21), (2.1.37), (2.1.42), (2.1.43), (2.1.49), (2.1.50), (2.1.56), and (2.1.57), describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

Now we proceed to find the solutions of Eqs. (2.1.19) and (2.1.20). To this end, we introduce a new variable defined by

$$z_\pm(t) = \alpha(t) \pm \beta^*(t). \tag{2.1.58}$$

Applying Eq. (2.1.19) and the complex conjugate of Eq. (2.1.20), we get

$$\frac{d}{dt} z_\pm(t) = -\frac{\kappa}{2} z_\pm(t) + \varepsilon \pm \varepsilon + f_\alpha(t) \pm f^*_\beta(t). \tag{2.1.59}$$

The solution of this equation can be written as

$$z_\pm(t) = z_\pm(0) e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2} [\varepsilon \pm \varepsilon + f_\alpha(t') \pm f^*_\beta(t')] dt', \tag{2.1.60}$$

from which we obtain

$$\alpha(t) = p(t)\alpha(0) + q(t) + F_+(t) + F_-(t), \tag{2.1.61}$$

$$\beta(t) = p(t)\beta(0) + q(t) + F^*_+(t) - F^*_-(t), \tag{2.1.62}$$

where

$$p(t) = e^{-\kappa t/2}, \tag{2.1.63}$$

$$q(t) = \frac{2\varepsilon}{\kappa}(1 - e^{-\kappa t/2}), \tag{2.1.64}$$

and

$$F_\pm(t) = \frac{1}{2} \int_0^t e^{-\kappa(t-t')/2} [f_\alpha(t') \pm f^*_\beta(t')] dt'. \tag{2.1.65}$$

Now Eqs. (2.1.61) and (2.1.62) can be rewritten as

$$\alpha(t) = \alpha'(t) + q(t), \tag{2.1.66}$$

$$\beta(t) = \beta'(t) + q(t), \tag{2.1.67}$$

with

$$\alpha'(t) = p(t)\alpha(0) + \int_0^t e^{-\kappa(t-t')/2} f_\alpha(t')dt', \tag{2.1.68}$$

$$\beta'(t) = p(t)\beta(0) + \int_0^t e^{-\kappa(t-t')/2} f_\beta(t')dt'. \tag{2.1.69}$$

2.2 The Q-function for a two-mode coherent output light

We now proceed to obtain the Q-function for a two-mode coherent output light. The Q-function for a two-mode coherent light can be expressed as

$$Q(\alpha, \beta, t) = \frac{1}{\pi^4} \int d^2z d^2\eta \phi_a(z, \eta, t) exp(z^*\alpha + \eta^*\beta - z\alpha^* - \eta\beta^*), \tag{2.2.1}$$

with the antinormally-ordered characterstic function $\phi_a(z, \eta, t)$ defined in the Heisenberg picture by

$$\phi_a(z, \eta, t) = Tr\left[\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\eta^*\hat{b}(t)}e^{\eta\hat{b}^\dagger(t)}\right]. \tag{2.2.2}$$

Employing the identity [1]

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{2.2.3}$$

the antinormally-ordered characterstic function can be expressed as

$$\phi_a(z, \eta, t) = exp(-z^*z - \eta^*\eta)Tr\left[\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*\hat{b}(t)}\right]. \tag{2.2.4}$$

Then this function can be written in terms of c-number variables associated with the normal order as

$$\phi_a(z, \eta, t) = exp(-z^*z - \eta^*\eta)\langle exp[z\alpha^* - z^*\alpha + \eta\beta^* - \eta^*\beta]\rangle. \tag{2.2.5}$$

Therefore, using Eqs. (2.1.66) and (2.1.67), we have

$$\phi_a(z, \eta, t) = exp[-z^*z - \eta^*\eta + (z - z^* + \eta - \eta^*)q]\langle exp[z\alpha'^* - z^*\alpha' + \eta\beta'^* - \eta^*\beta']\rangle. \tag{2.2.6}$$

Now we seek to show α' and β' are Gaussian variables. To this end, taking the expectation values of Eqs. (2.1.68), and (2.1.69), and differentiating with respect to t yields

$$\frac{d}{dt}\langle\alpha'(t)\rangle = -\frac{\kappa}{2}\langle\alpha'(t)\rangle, \tag{2.2.7}$$

and

$$\frac{d}{dt}\langle\beta'(t)\rangle = -\frac{\kappa}{2}\langle\beta'(t)\rangle. \tag{2.2.8}$$

Which are linear equations, we then see that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables. In addition, using Eqs. (2.1.68), and (2.1.69), and assuming that the cavity mode is initially in a two-mode vaccum state, one easily gets

$$\langle\alpha'(t)\rangle = \langle\beta'(t)\rangle = 0. \tag{2.2.9}$$

Hence $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables with vanishing means. Thus employing the relation [1]

$$\langle exp(\alpha+\beta)\rangle = exp\Big[\frac{1}{2}\langle(\alpha+\beta)^2\rangle\Big], \tag{2.2.10}$$

which holds for Gaussian variables α and β with vanishing mean, the antinormally-ordered characterstics function can be put in the form

$$\begin{aligned}\phi_a(z,\eta,t) \quad &= exp\Big[-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)q\Big]exp\Big[\frac{1}{2}\langle(z\alpha'^*-z^*\alpha'+\eta\beta'^*\\ &-\eta^*\beta')^2\rangle\Big],\end{aligned} \tag{2.2.11}$$

which can be rewritten as

$$\begin{aligned}\phi_a(z,\eta,t) \quad &= exp\Big[-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)q\Big]exp\Big[\frac{z^2}{2}\langle\alpha'^{*2}\rangle+\frac{z^{*2}}{2}\langle\alpha'^2\rangle+\frac{\eta^2}{2}\langle\beta'^{*2}\rangle\\ &+\frac{\eta^{*2}}{2}\langle\beta'^2\rangle - z^*z\langle\alpha'\alpha'^*\rangle + z\eta\langle\alpha'^*\beta'^*\rangle - z^*\eta\langle\alpha'\beta'^*\rangle - z\eta^*\langle\alpha'^*\beta'\rangle + z^*\eta^*\langle\alpha'\beta'\rangle\\ &-\eta^*\eta\langle\beta'\beta'^*\rangle\Big].\end{aligned} \tag{2.2.12}$$

With the help of Eqs. (2.1.68), we obtain

$$\begin{aligned}
\langle\alpha'^2\rangle &= \bigg\langle \Big(p(t)\alpha(0)+\int_0^t e^{-\kappa(t-t')/2}f_\alpha(t')dt'\Big)\Big(p(t)\alpha(0)+\int_0^t e^{-\kappa(t-t'')/2}f_\alpha(t'')dt''\Big)\bigg\rangle \\
&= p(t)^2\langle\alpha(0)^2\rangle + p(t)\int_0^t e^{-\kappa(t-t'')/2}\langle\alpha(0)f_\alpha(t'')\rangle dt'' \\
&+p(t)\int_0^t e^{-\kappa(t-t')/2}\langle\alpha(0)f_\alpha(t')\rangle dt' + \int_0^t e^{-\kappa(2t-t'-t'')/2}\langle f_\alpha(t')f_\alpha(t'')\rangle dt'dt'', \quad (2.2.13)
\end{aligned}$$

in view of the assumption that the cavity mode is initially in a two-mode vacuum state and on account of the fact that a noise operator at a certain time should not affect the cavity operator at an earlier time the above equation reduces to

$$\langle\alpha'^2\rangle = \int_0^t e^{-\kappa(2t-t'-t'')/2}\langle f_\alpha(t')f_\alpha(t'')\rangle dt'dt'', \quad (2.2.14)$$

now with the aid of Eq. (2.1.42), we get

$$\langle\alpha'^2\rangle = 0. \quad (2.2.15)$$

Similarly one can also show that

$$\langle\alpha'^{*2}\rangle = 0, \quad (2.2.16)$$

$$\langle\beta'^{*2}\rangle = \langle\beta'^2\rangle = 0, \quad (2.2.17)$$

$$\langle\alpha'\alpha'^*\rangle = \langle\beta'\beta'^*\rangle = 0, \quad (2.2.18)$$

$$\langle\alpha'^*\beta'^*\rangle = \langle\alpha'\beta'\rangle = 0, \quad (2.2.19)$$

$$\langle\alpha'\beta'^*\rangle = \langle\alpha'^*\beta'\rangle = 0. \quad (2.2.20)$$

On account of the above relations, the characterstic function becomes

$$\phi_a(z,\eta,t) = exp\Big[-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)q\Big]. \quad (2.2.21)$$

Then, introducing Eq. (2.2.21) into Eq. (2.2.1), we have

$$\begin{aligned}
Q(\alpha,\beta,t) &= \frac{1}{\pi^2}\int\frac{d^2zd^2\eta}{\pi^2}exp\Big[-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)q\Big] \\
&\quad exp\Big[z^*\alpha+\eta^*\beta-z\alpha^*-\eta\beta^*\Big], \quad (2.2.22)
\end{aligned}$$

which can be rewritten as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}\int \frac{d^2z d^2\eta}{\pi^2} exp\Big[-z^*z - \eta^*\eta + kz + lz^* + r\eta + s\eta^*\Big], \tag{2.2.23}$$

where

$$k = q - \alpha^*, \tag{2.2.24}$$

$$l = -(q - \alpha), \tag{2.2.25}$$

$$r = q - \beta^*, \tag{2.2.26}$$

$$s = -(q - \beta). \tag{2.2.27}$$

Furthermore, using the relation [1]

$$\int \frac{d^2z}{\pi} exp[-az^*z + bz + cz^* + Az^2 + Bz^{*2}]$$
$$= \left[\frac{1}{a^2 - 4AB}\right]^{1/2} exp\left(\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right), a > 0, \tag{2.2.28}$$

and performing the integration, one readily obtains

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2} exp\Big[-\alpha^*\alpha - \beta^*\beta + (\alpha + \beta + \alpha^* + \beta^*)q - 2q^2\Big]. \tag{2.2.29}$$

When calculating the expectation values of normally-ordered output operators, with the cavity mode coupled to a vacuum reservoir, the input-output relation can be written as

$$\alpha_{out}(t) = \sqrt{\kappa}\alpha(t), \tag{2.2.30}$$

$$\beta_{out}(t) = \sqrt{\kappa}\beta(t), \tag{2.2.31}$$

from which follows

$$\alpha(t) = \frac{\alpha_{out}(t)}{\sqrt{\kappa}}, \tag{2.2.32}$$

$$\beta(t) = \frac{\beta_{out}(t)}{\sqrt{\kappa}}. \tag{2.2.33}$$

On the basis of Eqs. (2.2.32), and (2.2.33), the Q-function for a two-mode coherent output light takes the form

$$Q(\alpha_{out},\beta_{out},t) = \frac{1}{\pi^2} exp\Big[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\beta^*_{out}\beta_{out}}{\kappa} + \Big(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}} + \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta^*_{out}}{\sqrt{\kappa}}\Big)q - 2q^2\Big]. \tag{2.2.34}$$

2.3 Output photon statistics

In this section we seek to calculate the mean and variance of the photon number sum and difference for the output modes a and b.

2.3.1 The mean of the output photon number sum and difference

Here we like to determine the mean of the output photon number sum and difference. To this end, upon integrating Eq. (2.2.34) over β_{out}, we get

$$\begin{aligned}\int \frac{d^2\beta_{out}}{\kappa} Q(\alpha_{out}, \beta_{out}, t) \quad &= \frac{1}{\kappa}\int \frac{d^2\beta_{out}}{\pi^2} exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\beta^*_{out}\beta_{out}}{\kappa} \\ &+\bigg(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}} + \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta^*_{out}}{\sqrt{\kappa}}\bigg)q - 2q^2\bigg], \end{aligned} \quad (2.3.1)$$

from which follows

$$Q(\alpha_{out}, t) = \frac{1}{\pi} exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} + \frac{\alpha_{out}}{\sqrt{\kappa}}q + \frac{\alpha^*_{out}}{\sqrt{\kappa}}q - q^2\bigg]. \quad (2.3.2)$$

We define the output photon number sum and difference by

$$\hat{n}^{out}_{\pm} = \hat{n}^{out}_a \pm \hat{n}^{out}_b, \quad (2.3.3)$$

where, $\hat{n}^{out}_a = \hat{a}^\dagger_{out}\hat{a}_{out}$ and $\hat{n}^{out}_b = \hat{b}^\dagger_{out}\hat{b}_{out}$ are the photon number operators for output mode a and output mode b. The mean of the output photon number sum and difference is expressed as

$$\bar{n}^{out}_{\pm} = \bar{n}^{out}_a \pm \bar{n}^{out}_b. \quad (2.3.4)$$

The mean photon number for output mode a, is given in terms of the Q-function as

$$\begin{aligned}\bar{n}^{out}_a \quad &= \frac{1}{\pi}\int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\kappa^2} Q(\alpha^*_{out}, \gamma_{out}) exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\gamma^*_{out}\gamma_{out}}{\kappa} + \frac{\alpha^*_{out}\gamma_{out}}{\kappa} \\ &+\frac{\gamma^*_{out}\alpha_{out}}{\kappa}\bigg]\gamma^*_{out}\alpha_{out}, \end{aligned} \quad (2.3.5)$$

so that on account of Eq. (2.3.2), we have

$$\bar{n}_a^{out} = \frac{1}{\pi^2}\int \frac{d^2\alpha_{out}d^2\gamma_{out}}{\kappa^2} exp\bigg[-\frac{\alpha_{out}^*\alpha_{out}}{\kappa} - \frac{\gamma_{out}^*\gamma_{out}}{\kappa} + \frac{\gamma_{out}^*\alpha_{out}}{\kappa} + \frac{\alpha_{out}^*}{\sqrt{\kappa}}q + \frac{\gamma_{out}}{\sqrt{\kappa}}q - q^2\bigg]\gamma_{out}^*\alpha_{out}, \tag{2.3.6}$$

which can be rewritten as

$$\bar{n}_a^{out} = \frac{1}{\kappa}e^{-q^2}\frac{d}{da}\int \frac{d^2\alpha_{out}d^2\gamma_{out}}{\pi^2} exp\bigg[-\frac{\alpha_{out}^*\alpha_{out}}{\kappa} - \frac{\gamma_{out}^*\gamma_{out}}{\kappa} + a\frac{\gamma_{out}^*\alpha_{out}}{\kappa} + \frac{\alpha_{out}^*}{\sqrt{\kappa}}q + \frac{\gamma_{out}}{\sqrt{\kappa}}q\bigg]_{a=1}, \tag{2.3.7}$$

so that performing the integration employing Eq. (2.2.28), one readily obtains

$$\bar{n}_a^{out} = e^{-q^2}\kappa\frac{d}{da}[exp(aq^2)]_{a=1}. \tag{2.3.8}$$

Thus carrying out the differentiation and applying the condition $a = 1$, we get

$$\bar{n}_a^{out} = \kappa q^2, \tag{2.3.9}$$

so that on account of Eq. (2.1.64), there follows

$$\bar{n}_a^{out} = \frac{4\varepsilon^2}{\kappa}(1 - e^{-\kappa t/2})^2. \tag{2.3.10}$$

We observe that at steady state the mean photon number for output mode a reduces to

$$(\bar{n}_a^{out})_{ss} = \frac{4\varepsilon^2}{\kappa}. \tag{2.3.11}$$

Following the same procedure, the mean photon number for output mode b is found to be

$$\bar{n}_b^{out} = \frac{4\varepsilon^2}{\kappa}(1 - e^{-\kappa t/2})^2, \tag{2.3.12}$$

and at steady state this takes the form

$$(\bar{n}_b^{out})_{ss} = \frac{4\varepsilon^2}{\kappa}. \tag{2.3.13}$$

Therefore, in view of Eqs. (2.3.10) and (2.3.12), we have

$$\bar{n}_\pm^{out} = \frac{4\varepsilon^2}{\kappa}(1 - e^{-\kappa t/2})^2 \pm \frac{4\varepsilon^2}{\kappa}(1 - e^{-\kappa t/2})^2. \tag{2.3.14}$$

Then the sum of the mean output photon number is found to be

$$\bar{n}_{+}^{out} = \frac{8\varepsilon^2}{\kappa}(1 - e^{-\kappa t/2})^2. \tag{2.3.15}$$

We see that at steady state, the mean of the output photon number sum reduces to

$$(\bar{n}_{+}^{out})_{ss} = \frac{8\varepsilon^2}{\kappa}. \tag{2.3.16}$$

And the mean of the output photon number difference turns out to be

$$\bar{n}_{-}^{out} = 0. \tag{2.3.17}$$

2.3.2 The variance of the output photon number sum and difference

We next proceed to obtain the variance of the output photon number sum and difference. The variance of the output photon number sum and difference can be expressed as

$$(\Delta n_{\pm}^{out})^2 = \langle(\hat{n}_{\pm}^{out})^2\rangle - \langle\hat{n}_{\pm}^{out}\rangle^2. \tag{2.3.18}$$

Applying Eq. (2.3.3), the variance of the output photon number sum and difference can be put in the form

$$(\Delta n_{\pm}^{out})^2 = (\Delta n_{a}^{out})^2 + (\Delta n_{b}^{out})^2 \pm 2(\langle\hat{n}_{a}^{out}\hat{n}_{b}^{out}\rangle - \bar{n}_{a}^{out}\bar{n}_{b}^{out}). \tag{2.3.19}$$

On the other hand, the photon number variance for output mode a can be written in the form

$$(\Delta n_{a}^{out})^2 = \langle(\hat{n}_{a}^{out})^2\rangle - (\bar{n}_{a}^{out})^2, \tag{2.3.20}$$

where

$$\langle(\hat{n}_{a}^{out})^2\rangle = \langle\hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}\rangle + \kappa\langle\hat{a}_{out}^{\dagger}\hat{a}_{out}\rangle. \tag{2.3.21}$$

Now we can write the expectation value of $\hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}$ in terms of the Q-function as

$$\begin{aligned}\langle\hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}\rangle \quad &= \frac{1}{\pi}\int\frac{d^2\alpha_{out}d^2\gamma_{out}}{\kappa^2}Q(\alpha_{out}^{*},\gamma_{out})exp\bigg[-\frac{\alpha_{out}^{*}\alpha_{out}}{\kappa}-\frac{\gamma_{out}^{*}\gamma_{out}}{\kappa}\\ &+\frac{\alpha_{out}^{*}\gamma_{out}}{\kappa}+\frac{\gamma_{out}^{*}\alpha_{out}}{\kappa}\bigg]\gamma_{out}^{*2}\alpha_{out}^{2},\end{aligned} \tag{2.3.22}$$

thus on substituting the Q-function for the output mode a in to the above equation, there follows

$$\begin{aligned}\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle &= \frac{1}{\kappa^2}\int \frac{d^2\alpha_{out}d^2\gamma_{out}}{\pi^2}exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa}-\frac{\gamma^*_{out}\gamma_{out}}{\kappa}+\frac{\gamma^*_{out}\alpha_{out}}{\kappa}\\ &+\frac{\alpha^*_{out}}{\sqrt{\kappa}}q+\frac{\gamma_{out}}{\sqrt{\kappa}}q-q^2\bigg]\gamma^{*2}_{out}\alpha^2_{out},\end{aligned} \tag{2.3.23}$$

which can be written in the form

$$\begin{aligned}\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle &= \frac{d^2}{da^2}\int \frac{d^2\alpha_{out}d^2\gamma_{out}}{\pi^2}exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa}-\frac{\gamma^*_{out}\gamma_{out}}{\kappa}+a\frac{\gamma^*_{out}\alpha_{out}}{\kappa}\\ &+\frac{\alpha^*_{out}}{\sqrt{\kappa}}q+\frac{\gamma_{out}}{\sqrt{\kappa}}q-q^2\bigg]_{a=1},\end{aligned} \tag{2.3.24}$$

so that performing the integration using the relation given by Eq. (2.2.28), we get

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = e^{-q^2}\kappa^2\frac{d^2}{da^2}[exp(aq^2)]_{a=1}. \tag{2.3.25}$$

Carrying out the differentiation and applying the condition $a=1$, one readily obtains

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = \kappa^2 q^4. \tag{2.3.26}$$

On the basis of Eq. (2.3.9), we see that

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = (\bar{n}^{out}_a)^2. \tag{2.3.27}$$

Hence on account of Eqs. (2.3.20), (2.3.21), and (2.3.27) the variance of the photon number for output mode a can be put in the form

$$\begin{aligned}(\Delta n^{out}_a)^2 &= \kappa\bar{n}^{out}_a\\ &= \kappa^2 q^2.\end{aligned} \tag{2.3.28}$$

Using Eq. (2.1.64), we have

$$(\Delta n^{out}_a)^2 = 4\varepsilon^2(1-e^{-\kappa t/2})^2, \tag{2.3.29}$$

and this takes at steady state the form

$$(\Delta n^{out}_a)^2_{ss} = 4\varepsilon^2. \tag{2.3.30}$$

Following a similar procedure, one can readily estabilish that the variance of the photon number for output mode b is

$$(\Delta n_b^{out})^2 = 4\varepsilon^2(1 - e^{-\kappa t/2})^2, \tag{2.3.31}$$

and at steady state we see that

$$(\Delta n_b^{out})^2_{ss} = 4\varepsilon^2. \tag{2.3.32}$$

In view of Eq. (2.3.28), we observe that the variance of the output photon number is equal to the mean output photon number for mode a, and the same holds true for mode b.

We now proceed to find the expectation value of $\hat{n}_a^{out}\hat{n}_b^{out}$, which can be written interms of the Q-function as

$$\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle = \int \frac{d^2\alpha_{out}d^2\beta_{out}}{\kappa^2} Q(\alpha_{out},\beta_{out})(\alpha^*_{out}\alpha_{out} - \kappa)(\beta^*_{out}\beta_{out} - \kappa), \tag{2.3.33}$$

where $\alpha^*_{out}\alpha_{out} - \kappa$ and $\beta^*_{out}\beta_{out} - \kappa$ are the c-number functions corresponding to the operators $\hat{n}_a^{out}$ and $\hat{n}_b^{out}$. Thus on account of the Q-function (2.2.34), we have

$$\begin{aligned}\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle \quad &= \frac{1}{\kappa^2} exp(-2q^2) \int \frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2} exp\bigg[- \frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\beta^*_{out}\beta_{out}}{\kappa} + \bigg(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}} \\ &+ \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta^*_{out}}{\sqrt{\kappa}}\bigg) q\bigg] (\alpha^*_{out}\alpha_{out}\beta^*_{out}\beta_{out} - \kappa\alpha^*_{out}\alpha_{out} - \kappa\beta^*_{out}\beta_{out} + \kappa^2).\end{aligned} \tag{2.3.34}$$

One can rewrite this as

$$\begin{aligned}\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle \quad &= exp(-2q^2)\bigg(\frac{d}{da}\frac{d}{db}\int \frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2} exp\bigg[- a\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - b\frac{\beta^*_{out}\beta_{out}}{\kappa} \\ &+ \bigg(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}} + \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta^*_{out}}{\sqrt{\kappa}}\bigg) q\bigg] + \frac{d}{da}\int \frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2} exp\bigg[- a\frac{\alpha^*_{out}\alpha_{out}}{\kappa} \\ &- b\frac{\beta^*_{out}\beta_{out}}{\kappa} + \bigg(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}} + \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta^*_{out}}{\sqrt{\kappa}}\bigg) q\bigg] \\ &+ \frac{d}{db}\int \frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2} exp\bigg[- a\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - b\frac{\beta^*_{out}\beta_{out}}{\kappa} + \bigg(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}} \\ &+ \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta^*_{out}}{\sqrt{\kappa}}\bigg) q\bigg]\bigg)_{a=b=1} + \kappa^2,\end{aligned} \tag{2.3.35}$$

thus carrying out the integration employing Eq. (2.2.28), one obtains

$$\begin{aligned}\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle &= exp(-2q^2)\bigg[\kappa^2\frac{d}{da}\frac{d}{db}[\frac{1}{ab}exp(\frac{q^2}{a}+\frac{q^2}{b})]+\kappa^2\frac{d}{da}[\frac{1}{ab}exp(\frac{q^2}{a}+\frac{q^2}{b})] \\ &+\kappa^2\frac{d}{db}[\frac{1}{ab}exp(\frac{q^2}{a}+\frac{q^2}{b})]\bigg]_{a=b=1}+\kappa^2. \end{aligned} \tag{2.3.36}$$

Differentiating and applying the condition $a = b = 1$, we find

$$\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle = \kappa^2 q^4. \tag{2.3.37}$$

On account of Eq. (2.1.64), we see that

$$\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle = \frac{16\varepsilon^4}{\kappa^2}(1-e^{-\kappa t/2})^4. \tag{2.3.38}$$

Introducing Eqs. (2.3.10), (2.3.12) and (2.3.38) into Eq. (2.3.19), we get

$$(\Delta n_{\pm}^{out})^2 = 2(\Delta n_a^{out})^2 = 2(\Delta n_b^{out})^2. \tag{2.3.39}$$

And in view of Eq. (2.3.28), we observe that

$$(\Delta n_{\pm}^{out})^2 = 2\kappa^2 q^2, \tag{2.3.40}$$

with the help of Eq. (2.1.64), one finally obtains

$$(\Delta n_{\pm}^{out})^2 = 8\varepsilon^2(1-e^{-\kappa t/2})^2. \tag{2.3.41}$$

From which we observe that the variance of the output photon number sum and difference takes at steady state the form

$$(\Delta n_{\pm}^{out})^2_{ss} = 8\varepsilon^2. \tag{2.3.42}$$

Chapter 3

Two-mode subharmonic output light

3.1 c-number Langevin equations

We first obtain c-number Langevin equations associated with the normal ordering for the signal-idler mode produced by two-mode subharmonic generator. Then using the solutions of the resulting equations, we determine the Q-function for the output signal-idler modes. In a two-mode subharmonic generator a pump-photon of frequency ω_c is down converted into highly correlated signal and idler photons with frequency ω_a and ω_b such that $\omega_c = \omega_a + \omega_b$.

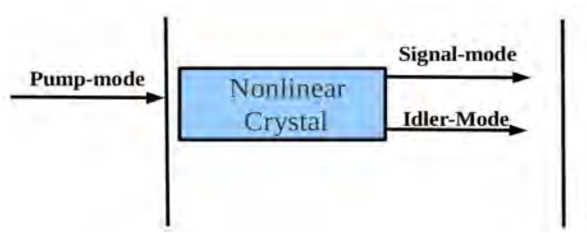

Figure 3.1: Two-mode subharmonic generator

The process of two-mode subharmonic generation is described by the Hamiltonian

$$\hat{H} = i\lambda(\hat{a}\hat{b}\hat{c}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{c}), \tag{3.1.1}$$

where $\hat{a}(\hat{b})$ is the annihilation operator for the signal(idler) mode and $\hat{c}$ is the annihilation opeartor for the pump mode, with λ being the coupling constant.

With the pump mode represented by a real and constant c-number μ, the Hamiltonian can be rewritten as

$$\hat{H} = i\gamma(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \tag{3.1.2}$$

with $\gamma = \lambda\mu$.

Applying Eq. (3.1.2) and taking into account the interaction of the signal-idler modes with a two-mode vacuum reservoir via a single-port mirror, the master equation for the cavity mode can be written as [1]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \gamma(\hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}) + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ &+\frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}),\end{aligned} \tag{3.1.3}$$

in which the cavity damping constant κ is assumed to be the same for both the signal and idler modes. Now employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{3.1.4}$$

one readily obtains

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= \gamma Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}) + \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &+\frac{\kappa}{2}Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}).\end{aligned} \tag{3.1.5}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \tag{3.1.6}$$

and

$$[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = [\hat{a}^\dagger, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = 0, \tag{3.1.7}$$

one gets

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle - \gamma\langle\hat{b}^\dagger(t)\rangle. \tag{3.1.8}$$

It can be shown in a similar manner that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa}{2}\langle\hat{b}(t)\rangle - \gamma\langle\hat{a}^{\dagger}(t)\rangle, \tag{3.1.9}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle - \gamma(\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle) - \gamma, \tag{3.1.10}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{a}(t)\rangle - 2\gamma\langle\hat{a}(t)\hat{b}^{\dagger}(t)\rangle, \tag{3.1.11}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \gamma(\langle\hat{a}(t)\hat{b}(t)\rangle + \langle\hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\rangle), \tag{3.1.12}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle - \gamma(\langle\hat{a}^{\dagger 2}(t)\rangle + \langle\hat{b}^{\dagger 2}(t)\rangle). \tag{3.1.13}$$

The c-number equations corresponding to the normally-ordered operators described by Eqs. (3.1.8), (3.1.9), (3.1.10), (3.1.11), (3.1.12), and (3.1.13) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\kappa}{2}\langle\alpha(t)\rangle - \gamma\langle\beta^{*}(t)\rangle, \tag{3.1.14}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{\kappa}{2}\langle\beta(t)\rangle - \gamma\langle\alpha^{*}(t)\rangle, \tag{3.1.15}$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle - \gamma(\langle\alpha^{*}(t)\alpha(t)\rangle + \langle\beta^{*}(t)\beta(t)\rangle) - \gamma, \tag{3.1.16}$$

$$\frac{d}{dt}\langle\alpha^{2}(t)\rangle = -\kappa\langle\alpha^{2}(t)\rangle - 2\gamma\langle\alpha(t)\beta^{*}(t)\rangle, \tag{3.1.17}$$

$$\frac{d}{dt}\langle\alpha^{*}(t)\alpha(t)\rangle = -\kappa\langle\alpha^{*}(t)\alpha(t)\rangle - \gamma(\langle\alpha(t)\beta(t)\rangle + \langle\alpha^{*}(t)\beta^{*}(t)\rangle), \tag{3.1.18}$$

$$\frac{d}{dt}\langle\alpha^{*}(t)\beta(t)\rangle = -\kappa\langle\alpha^{*}(t)\beta(t)\rangle - \gamma(\langle\alpha^{*2}(t)\rangle + \langle\beta^{*2}(t)\rangle). \tag{3.1.19}$$

On the basis of Eqs. (3.1.14) and (3.1.15), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) - \gamma\beta^{*}(t) + f_{\alpha}(t), \tag{3.1.20}$$

and

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) - \gamma\alpha^{*}(t) + f_{\beta}(t), \tag{3.1.21}$$

where $f_{\alpha}(t)$ and $f_{\beta}(t)$ are noise forces associated with the normal ordering.

We next seek to determine the properties of the noise forces $f_{\alpha}(t)$ and $f_{\beta}(t)$. We see that

Eq. (3.1.14) and the expectation value of Eq. (3.1.20) as well as Eq. (3.1.15) and the expectation value of Eq. (3.1.21) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.1.22}$$

Moreover, using the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\frac{d\alpha(t)}{dt}\beta(t)\rangle + \langle\alpha(t)\frac{d\beta(t)}{dt}\rangle, \tag{3.1.23}$$

along with Eqs. (3.1.20) and (3.1.21), we find

$$\begin{aligned}\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle \quad &= -\kappa\langle\alpha(t)\beta(t)\rangle - \gamma\langle\alpha^*(t)\alpha(t)\rangle - \gamma\langle\beta^*(t)\beta(t)\rangle \\ &+\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle.\end{aligned} \tag{3.1.24}$$

Comparison of Eqs. (3.1.16) and (3.1.24) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = -\gamma. \tag{3.1.25}$$

The formal solutions of Eqs. (3.1.20) and (3.1.21) can be written as

$$\alpha(t) = \alpha(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[f_\alpha(t') - \gamma\beta^*(t')]dt', \tag{3.1.26}$$

and

$$\beta(t) = \beta(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[f_\beta(t') - \gamma\alpha^*(t')]dt'. \tag{3.1.27}$$

In view of Eqs. (3.1.26) and (3.1.27), one readily obtains

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}\Big[\langle f_\beta(t)f_\alpha(t')\rangle - \gamma\langle\beta^*(t')f_\beta(t)\rangle\Big]dt', \tag{3.1.28}$$

$$\langle\beta(t)f_\alpha(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}\Big[\langle f_\alpha(t)f_\beta(t')\rangle - \gamma\langle\alpha^*(t')f_\alpha(t)\rangle\Big]dt'. \tag{3.1.29}$$

Since a noise operator at a certain time should not affect the system operator at an earlier time, we see that

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \tag{3.1.30}$$

$$\langle\beta(0)f_\alpha(t)\rangle = \langle\beta(0)\rangle\langle f_\alpha(t)\rangle = 0, \tag{3.1.31}$$

$$\langle \alpha^*(t') f_\alpha(t) \rangle = \langle \alpha^*(t') \rangle \langle f_\alpha(t) \rangle = 0, \tag{3.1.32}$$

$$\langle \beta^*(t') f_\beta(t) \rangle = \langle \beta^*(t') \rangle \langle f_\beta(t) \rangle = 0. \tag{3.1.33}$$

Thus on account of Eqs. (3.1.22), (3.1.30), (3.1.31), (3.1.32), and (3.1.33), Eqs. (3.1.28) and (3.1.29) reduce to

$$\langle \alpha(t) f_\beta(t) \rangle = \int_0^t e^{-\kappa(t-t')/2} \langle f_\beta(t) f_\alpha(t') \rangle dt', \tag{3.1.34}$$

$$\langle \beta(t) f_\alpha(t) \rangle = \int_0^t e^{-\kappa(t-t')/2} \langle f_\alpha(t) f_\beta(t') \rangle dt'. \tag{3.1.35}$$

Therefore, in view of Eqs. (3.1.25), (3.1.34), (3.1.35) and assuming

$$\langle f_\beta(t) f_\alpha(t') \rangle = \langle f_\alpha(t) f_\beta(t') \rangle, \tag{3.1.36}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2} \langle f_\beta(t) f_\alpha(t') \rangle dt' = \int_0^t e^{-\kappa(t-t')/2} \langle f_\alpha(t) f_\beta(t') \rangle dt' = -\frac{\gamma}{2}. \tag{3.1.37}$$

Now on the basis of the relation (2.1.35) and (2.1.36), we assert that

$$\langle f_\beta(t) f_\alpha(t') \rangle = \langle f_\alpha(t) f_\beta(t') \rangle = -\gamma \delta(t - t'). \tag{3.1.38}$$

Furthermore, using the relation

$$\frac{d}{dt} \langle \alpha(t) \alpha(t) \rangle = \langle \frac{d\alpha(t)}{dt} \alpha(t) \rangle + \langle \alpha(t) \frac{d\alpha(t)}{dt} \rangle, \tag{3.1.39}$$

along with Eq. (3.1.20), we obtain

$$\frac{d}{dt} \langle \alpha^2(t) \rangle = -\kappa \langle \alpha^2(t) \rangle - 2\gamma \langle \alpha(t) \beta^*(t) \rangle + 2 \langle \alpha(t) f_\alpha(t) \rangle. \tag{3.1.40}$$

Comparison of Eqs. (3.1.17) and (3.1.40) shows that

$$\langle \alpha(t) f_\alpha(t) \rangle = 0. \tag{3.1.41}$$

In view of Eq. (3.1.26), one readily obtains

$$\int_0^t e^{-\kappa(t-t')/2} \langle f_\alpha(t) f_\alpha(t') \rangle dt' = 0, \tag{3.1.42}$$

from which follows

$$\langle f_\alpha(t) f_\alpha(t')\rangle = 0. \tag{3.1.43}$$

It can be estabilished in a similar manner that

$$\langle f_\beta(t) f_\beta(t')\rangle = 0. \tag{3.1.44}$$

Moreover, employing the relation

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\alpha(t)\rangle + \langle \alpha^*(t)\frac{d\alpha(t)}{dt}\rangle, \tag{3.1.45}$$

along with Eq. (3.1.20) and its complex conjugate, we find

$$\begin{aligned} \frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle \quad &= -\kappa\langle \alpha^*(t)\alpha(t)\rangle - \gamma\langle \alpha(t)\beta(t)\rangle - \gamma\langle \alpha^*(t)\beta^*(t)\rangle \\ &+\langle \alpha(t) f^*_\alpha(t)\rangle + \langle \alpha^*(t) f_\alpha(t)\rangle. \end{aligned} \tag{3.1.46}$$

Comparison of Eqs. (3.1.18) and (3.1.46) indicates that

$$\langle \alpha(t) f^*_\alpha(t)\rangle + \langle \alpha^*(t) f_\alpha(t)\rangle = 0. \tag{3.1.47}$$

With the aid of Eq. (3.1.26) and its complex conjugate, we obtain

$$\langle \alpha(t) f^*_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t) f_\alpha(t')\rangle dt', \tag{3.1.48}$$

$$\langle \alpha^*(t) f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f^*_\alpha(t')\rangle dt'. \tag{3.1.49}$$

Now in view of Eqs. (3.1.47), (3.1.48), (3.1.49) and assuming

$$\langle f_\alpha(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\alpha(t')\rangle, \tag{3.1.50}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t) f^*_\alpha(t')\rangle dt' = 0, \tag{3.1.51}$$

from which one readily obtains

$$\langle f_\alpha(t) f^*_\alpha(t')\rangle = \langle f^*_\alpha(t) f_\alpha(t')\rangle = 0. \tag{3.1.52}$$

It can also be estabilished in a similar fashion that

$$\langle f_\beta(t) f^*_\beta(t') \rangle = \langle f^*_\beta(t) f_\beta(t') \rangle = 0. \tag{3.1.53}$$

Finally, introducing the relation

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = \langle \frac{d\alpha^*(t)}{dt}\beta(t)\rangle + \langle \alpha^*(t)\frac{d\beta(t)}{dt}\rangle, \tag{3.1.54}$$

along with Eq. (3.1.21) and the complex conjugate of Eq. (3.1.20), we obtain

$$\begin{aligned} \frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle \quad &= -\kappa\langle \alpha^*(t)\beta(t)\rangle - \gamma\langle \alpha^{*2}(t)\rangle - \gamma\langle \beta^2(t)\rangle \\ &+\langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle. \end{aligned} \tag{3.1.55}$$

Comparison of Eqs. (3.1.19) and (3.1.55), indicates that

$$\langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle = 0. \tag{3.1.56}$$

With the aid of Eq. (3.1.27) and the complex conjugate of Eq. (3.1.26), we find

$$\langle \beta(t) f^*_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t) f_\beta(t')\rangle dt', \tag{3.1.57}$$

$$\langle \alpha^*(t) f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t) f^*_\alpha(t')\rangle dt'. \tag{3.1.58}$$

Now taking into account Eqs. (3.1.56), (3.1.57), (3.1.58) and assuming

$$\langle f^*_\alpha(t) f_\beta(t')\rangle = \langle f_\beta(t) f^*_\alpha(t')\rangle, \tag{3.1.59}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t) f_\beta(t')\rangle dt' = 0. \tag{3.1.60}$$

Therefore, in view of the relations (2.1.35), and (2.1.36), one obtains

$$\langle f^*_\alpha(t) f_\beta(t')\rangle = \langle f_\beta(t) f^*_\alpha(t')\rangle = 0. \tag{3.1.61}$$

It can also be estabilished in a similar manner that

$$\langle f^*_\beta(t) f_\alpha(t')\rangle = \langle f_\alpha(t) f^*_\beta(t')\rangle = 0. \tag{3.1.62}$$

We see that Eqs. (3.1.22), (3.1.38), (3.1.43), (3.1.44), (3.1.52), (3.1.53), (3.1.61) and (3.1.62) describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal-ordering.

In order to obtain the solutions of Eqs. (3.1.20) and (3.1.21), we introduce a new variable defined by

$$\Omega_\pm(t) = \alpha(t) \pm \beta^*(t). \tag{3.1.63}$$

Applying Eq. (3.1.20) along with the complex conjugate of Eq. (3.1.21), we readily obtain

$$\frac{d}{dt}\Omega_\pm(t) = -\frac{1}{2}\lambda_\pm\Omega_\pm(t) + f_\alpha(t) \pm f_\beta^*(t), \tag{3.1.64}$$

where

$$\lambda_\pm = \kappa \pm 2\gamma. \tag{3.1.65}$$

According to Eq. (3.1.64) together with Eq. (3.1.65), the solution of $\Omega_-(t)$ does not have a wel-behaved solution for $\kappa < 2\gamma$. We then identify $\kappa = 2\gamma$ as the threshold condition. For $\kappa > 2\gamma$, the solution of Eq. (3.1.64) can be written as

$$\Omega_\pm(t) = \Omega_\pm(0)e^{-\lambda_\pm t/2} + \int_0^t e^{-\lambda_\pm(t-t')/2}[f_\alpha(t') \pm f_\beta^*(t')]dt'. \tag{3.1.66}$$

Now with the aid of Eqs. (3.1.63) and (3.1.66), we get

$$\alpha(t) = E_+(t)\alpha(0) + E_-(t)\beta^*(0) + F_+(t) + F_-(t), \tag{3.1.67}$$

$$\beta(t) = E_+(t)\beta(0) + E_-(t)\alpha^*(0) + F_+^*(t) - F_-^*(t), \tag{3.1.68}$$

where,

$$E_\pm(t) = \frac{1}{2}\left[e^{-\lambda_+ t/2} \pm e^{-\lambda_- t/2}\right], \tag{3.1.69}$$

$$F_\pm(t) = \frac{1}{2}\int_0^t e^{-\lambda_\pm(t-t')/2}[f_\alpha(t') \pm f_\beta^*(t')]dt'. \tag{3.1.70}$$

3.2 Q-function for a two-mode subharmonic output light

We next seek to obtain the Q-function for the output signal-idler modes. The Q-function for a two-mode light is expressible as

$$Q(\alpha,\beta,t)=\frac{1}{\pi^4}\int d^2zd^2\chi\phi_a(z,\chi,t)exp(z^*\alpha-\alpha^*z+\chi^*\beta-\beta^*\chi), \tag{3.2.1}$$

where the antinormally ordered characterstic function is defined in the Heisenberg picture by

$$\phi_a(z,\chi,t)=Tr\left[\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\chi^*\hat{b}(t)}e^{\chi\hat{b}^\dagger(t)}\right]. \tag{3.2.2}$$

This can be put employing the identity [1]

$$e^{\hat{A}}e^{\hat{B}}=e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{3.2.3}$$

in the form

$$\phi_a(z,\chi,t)=exp(z^*z-\chi^*\chi)Tr\left[\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\chi\hat{b}^\dagger(t)}e^{-\chi^*\hat{b}(t)}\right]. \tag{3.2.4}$$

The antinormally-ordered characterstics function can be expressed in terms of c-number variables associated with the normal ordering as

$$\phi_a(z,\chi,t)=exp(-z^*z-\chi^*\chi)\langle exp(z\alpha^*-z^*\alpha+\chi\beta^*-\chi^*\beta)\rangle. \tag{3.2.5}$$

Moreover, Eqs. (3.1.67) and (3.1.68) can be written as

$$\alpha(t)=E_+(t)\alpha(0)+E_-(t)\beta^*(0)+\mu_\alpha(t), \tag{3.2.6}$$

$$\beta(t)=E_+(t)\beta(0)+E_-(t)\alpha^*(0)+\mu_\beta(t), \tag{3.2.7}$$

where

$$\mu_\alpha(t)=\frac{1}{2}\left[\int_0^t e^{-\lambda_+(t-t')/2}[f_\alpha(t')+f_\beta^*(t')]dt'+\int_0^t e^{-\lambda_-(t-t')/2}[f_\alpha(t')-f_\beta^*(t')]dt'\right], \tag{3.2.8}$$

and

$$\mu_\beta(t) = \frac{1}{2}\bigg[\int_0^t e^{-\lambda_+(t-t')/2}[f_\alpha^*(t') + f_\beta(t')]dt' - \int_0^t e^{-\lambda_-(t-t')/2}[f_\alpha^*(t') - f_\beta(t')]dt'\bigg]. \quad (3.2.9)$$

Now in view of Eqs. (3.2.6) and (3.2.7), one readily obtains

$$\begin{aligned}\phi_a(z,\chi,t) \quad &= exp(-z^*z - \chi^*\chi)\bigg\langle exp\bigg[zE_+\alpha^*(0) + zE_-\beta(0) + z\mu_\alpha^* - z^*E_+\alpha(0)\\ &-z^*E_-\beta^*(0) - z^*\mu_\alpha + \chi E_+\beta^*(0) + \chi E_-\alpha(0) + \chi\mu_\beta^* - \chi^*E_+\beta(0)\\ &-\chi^*E_-\alpha^*(0) - \chi^*\mu_\beta\bigg]\bigg\rangle. \end{aligned} \quad (3.2.10)$$

It then follows that

$$\begin{aligned}\phi_a(z,\chi,t) \quad &= exp(-z^*z - \chi^*\chi)\bigg\langle exp\bigg[(\chi E_- - z^*E_+)\alpha(0) + (zE_+ - \chi^*E_-)\alpha^*(0)\\ &+(zE_- - \chi^*E_+)\beta(0) + (\chi E_+ - z^*E_-)\beta^*(0)\bigg]\bigg\rangle\langle exp[z\mu_\alpha^* - z^*\mu_\alpha\\ &+\chi\mu_\beta^* - \chi^*\mu_\beta]\rangle. \end{aligned} \quad (3.2.11)$$

Considering the cavity radiation to be initially in a two-mode vacuum state, we get

$$\begin{aligned}&\bigg\langle exp\bigg[(\chi E_- - z^*E_+)\alpha(0) + (zE_+ - \chi^*E_-)\alpha^*(0) + (zE_- - \chi^*E_+)\beta(0)\\ &+(\chi E_+ - z^*E_-)\beta^*(0)\bigg]\bigg\rangle = 1. \end{aligned} \quad (3.2.12)$$

In view of Eq. (3.2.12), we have

$$\phi_a(z,\chi,t) = exp(-z^*z - \chi^*\chi)\langle exp[z\mu_\alpha^* - z^*\mu_\alpha + \chi\mu_\beta^* - \chi^*\mu_\beta]\rangle. \quad (3.2.13)$$

We recall that Gaussian variables with vanishing mean satisfy the relation

$$\langle exp(z\alpha^* - z^*\alpha)\rangle = exp\bigg[\frac{1}{2}\langle(z\alpha^* - z^*\alpha)^2\rangle\bigg]. \quad (3.2.14)$$

Next we proceed to show that $\mu_\alpha(t)$ and $\mu_\beta(t)$ are Gausian variables. One can rewrite Eq. (3.2.8) as

$$\mu_\alpha = \mu_+ + \mu_-, \quad (3.2.15)$$

in which

$$\mu_+ = \frac{1}{2}\int_0^t e^{-\lambda_+(t-t')/2}[f_\alpha(t') + f_\beta^*(t')]dt', \tag{3.2.16}$$

and

$$\mu_- = \frac{1}{2}\int_0^t e^{-\lambda_+(t-t')/2}[f_\alpha(t') - f_\beta^*(t')]dt'. \tag{3.2.17}$$

Employing Eqs. (3.2.16) and (3.2.17), the time evolution for the expectation values of μ_+ and μ_- can be expressed as

$$\frac{d}{dt}\langle\mu_+\rangle = -\frac{\lambda_+}{4}\langle\mu_+\rangle, \tag{3.2.18}$$

and

$$\frac{d}{dt}\langle\mu_-\rangle = -\frac{\lambda_-}{4}\langle\mu_-\rangle. \tag{3.2.19}$$

Eqs. (3.2.18) and (3.2.19) indicates that μ_+ and μ_- are Gaussian variables. Thus, in view of Eq. (3.2.15), we see that μ_α is also Gaussian variable.

Similarly we rewrite Eq. (3.2.9) as

$$\mu_\beta = \Psi_+ - \Psi_-, \tag{3.2.20}$$

where

$$\Psi_+ = \frac{1}{2}\int_0^t e^{-\lambda_+(t-t')/2}[f_\alpha^*(t') + f_\beta(t')]dt', \tag{3.2.21}$$

and

$$\Psi_- = \frac{1}{2}\int_0^t e^{-\lambda_-(t-t')/2}[f_\alpha^*(t') - f_\beta(t')]dt'. \tag{3.2.22}$$

Now the time evolution for the expectation values of Ψ_+ and Ψ_- can be expressed as

$$\frac{d}{dt}\langle\Psi_+\rangle = -\frac{\lambda_+}{4}\langle\Psi_+\rangle, \tag{3.2.23}$$

and

$$\frac{d}{dt}\langle\Psi_-\rangle = -\frac{\lambda_-}{4}\langle\Psi_-\rangle. \tag{3.2.24}$$

In view of Eqs. (3.2.20), (3.2.23), and (3.2.24), we see that μ_β is Gaussian variable. In addition, using Eqs. (3.2.8) and (3.2.9), one easily gets

$$\langle\mu_\alpha(t)\rangle = \langle\mu_\beta(t)\rangle = 0. \tag{3.2.25}$$

Therefore, we note that μ_α and μ_β are Gaussian variables with vanishing mean. Then the antinormally-ordered characterstics function can be put in the form

$$\phi_a(z,\chi,t) = exp(-z^*z - \chi^*\chi)exp\bigg[\frac{1}{2}\langle(z\mu_\alpha^* - z^*\mu_\alpha + \chi\mu_\beta^* - \chi^*\mu_\beta)^2\rangle\bigg], \tag{3.2.26}$$

from which follows

$$\begin{aligned}\phi_a(z,\chi,t) \qquad &= exp(-z^*z - \chi^*\chi)exp\bigg[\frac{z^2}{2}\langle\mu_\alpha^{*2}\rangle + \frac{z^{*2}}{2}\langle\mu_\alpha^2\rangle + \frac{\chi^2}{2}\langle\mu_\beta^{*2}\rangle + \frac{\chi^{*2}}{2}\langle\mu_\beta^2\rangle \\ &-z'z^*\langle\mu_\alpha\mu_\alpha^*\rangle + z\chi\langle\mu_\alpha^*\mu_\beta^*\rangle - z^*\chi\langle\mu_\alpha\mu_\beta^*\rangle - z\chi^*\langle\mu_\alpha^*\mu_\beta\rangle + z^*\chi^*\langle\mu_\alpha\mu_\beta\rangle \\ &-\chi^*\chi\langle\mu_\beta\mu_\beta^*\rangle\bigg].\end{aligned} \tag{3.2.27}$$

On account of Eqs. (3.2.8) and (3.2.9), one can show that

$$\begin{aligned}\langle\mu_\alpha^2\rangle \qquad &= \frac{1}{4}\bigg\langle\bigg[\int_0^t e^{-\lambda_+(t-t')/2}[f_\alpha(t') + f_\beta^*(t')]dt' + \int_0^t e^{-\lambda_-(t-t')/2}[f_\alpha(t') - f_\beta^*(t')]dt'\bigg] \\ &\bigg[\int_0^t e^{-\lambda_+(t-t'')/2}[f_\alpha(t'') + f_\beta^*(t'')]dt'' + \int_0^t e^{-\lambda_-(t-t'')/2}[f_\alpha(t'') \\ &-f_\beta^*(t'')]dt''\bigg]\bigg\rangle,\end{aligned} \tag{3.2.28}$$

which leads to

$$\begin{aligned}\langle\mu_\alpha^2\rangle \qquad &= \frac{1}{4}\bigg(\int_0^t e^{-\lambda_+(2t-t'-t'')/2}\bigg[\bigg(\langle f_\alpha(t')f_\alpha(t'')\rangle + \langle f_\alpha(t')f_\beta^*(t'')\rangle + \langle f_\beta^*(t')f_\alpha(t'')\rangle \\ &+\langle f_\beta^*(t')f_\beta^*(t'')\rangle\bigg)dt'dt''\bigg] + \int_0^t e^{-(\lambda_++\lambda_-)t/2+(\lambda_+t'+\lambda_-t'')/2}\bigg[\bigg(\langle f_\alpha(t')f_\alpha(t'')\rangle \\ &-\langle f_\alpha(t')f_\beta^*(t'')\rangle + \langle f_\beta^*(t')f_\alpha(t'')\rangle - \langle f_\beta^*(t')f_\beta^*(t'')\rangle\bigg)dt'dt''\bigg] \\ &+\int_0^t e^{-(\lambda_++\lambda_-)t/2+(\lambda_+t'+\lambda_-t'')/2}\bigg[\bigg(\langle f_\alpha(t')f_\alpha(t'')\rangle + \langle f_\alpha(t')f_\beta^*(t'')\rangle - \langle f_\beta^*(t')f_\alpha(t'')\rangle \\ &-\langle f_\beta^*(t')f_\beta^*(t'')\rangle\bigg)dt'dt''\bigg] + \int_0^t e^{-\lambda_-(2t-t'-t'')/2}\bigg[\bigg(\langle f_\alpha(t')f_\alpha(t'')\rangle - \langle f_\alpha(t')f_\beta^*(t'')\rangle \\ &-\langle f_\beta^*(t')f_\alpha(t'')\rangle + \langle f_\beta^*(t')f_\beta^*(t'')\rangle\bigg)dt'dt''\bigg]\bigg).\end{aligned} \tag{3.2.29}$$

In view of Eqs. (3.1.43), (3.1.44) and (3.1.62), one readily obtains

$$\langle\mu_\alpha^2\rangle = 0. \tag{3.2.30}$$

Using a similar procedure as above, we can also show that

$$\langle \mu_\alpha^{*2} \rangle = \langle \mu_\beta^2 \rangle = \langle \mu_\beta^{*2} \rangle = 0, \tag{3.2.31}$$

$$\langle \mu_\alpha \mu_\beta^* \rangle = \langle \mu_\alpha^* \mu_\beta \rangle = 0, \tag{3.2.32}$$

$$\langle \mu_\alpha \mu_\alpha^* \rangle = \langle \mu_\beta \mu_\beta^* \rangle = -\frac{\gamma}{2}(S - T), \tag{3.2.33}$$

$$\langle \mu_\alpha \mu_\beta \rangle = \langle \mu_\alpha^* \mu_\beta^* \rangle = -\frac{\gamma}{2}(S + T), \tag{3.2.34}$$

where

$$S = \frac{1}{\lambda_+}(1 - e^{-\lambda_+ t}), \tag{3.2.35}$$

$$T = \frac{1}{\lambda_-}(1 - e^{-\lambda_- t}). \tag{3.2.36}$$

Thus on account of Eqs. (3.2.30), (3.2.31), (3.2.32), (3.2.33) and (3.2.34) Eq. (3.2.27), goes over into

$$\phi_a(z, \chi, t) = exp\bigg[- \Big(1 - \frac{\gamma}{2}(S - T)\Big)\Big(z^* z + \chi^* \chi\Big) - \frac{\gamma}{2}(S + T)\Big(\chi z + \chi^* z^*\Big)\bigg]. \tag{3.2.37}$$

Finally, we can also put the antinormally-ordered characterstics function in the form

$$\phi_a(z, \chi, t) = exp\bigg[- a(z^* z + \chi^* \chi) - b(\chi z + \chi^* z^*)\bigg], \tag{3.2.38}$$

in which

$$a = 1 - \frac{\gamma}{2}(S - T) = 1 - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}), \tag{3.2.39}$$

$$b = \frac{\gamma}{2}(S + T) = \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}). \tag{3.2.40}$$

Now upon introducing Eq. (3.2.38) into Eq. (3.2.1), we have

$$\begin{aligned} Q(\alpha, \beta, t) \quad &= \frac{1}{\pi^2} \int \frac{d^2 z d^2 \chi}{\pi^2} exp\bigg[- a(z^* z + \chi^* \chi) - b(\chi z + \chi^* z^*) + z^* \alpha - \alpha^* z \\ &\quad + \chi^* \beta - \beta^* \chi \bigg]. \end{aligned} \tag{3.2.41}$$

Thus on performing the integration employing Eq. (2.2.28), the Q-function for the signal-idler modes turns out to be

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}(u^2 - v^2)exp\Big[- u(\alpha^*\alpha + \beta^*\beta) - v(\alpha\beta + \alpha^*\beta^*)\Big], \tag{3.2.42}$$

in which

$$u = \frac{a}{a^2 - b^2}, \tag{3.2.43}$$

$$v = \frac{b}{a^2 - b^2}. \tag{3.2.44}$$

On account of the relations described by Eqs. (2.2.32) and (2.2.33), the Q-function for the output signal-idler mode takes the form

$$\begin{aligned} Q(\alpha_{out},\beta_{out},t) \quad &= \frac{1}{\pi^2}(u^2 - v^2)exp\Big[- u(\frac{\alpha^*_{out}\alpha_{out}}{\kappa} + \frac{\beta^*_{out}\beta_{out}}{\kappa}) - v(\frac{\alpha_{out}\beta_{out}}{\kappa} \\ &+ \frac{\alpha^*_{out}\beta^*_{out}}{\kappa})\Big]. \end{aligned} \tag{3.2.45}$$

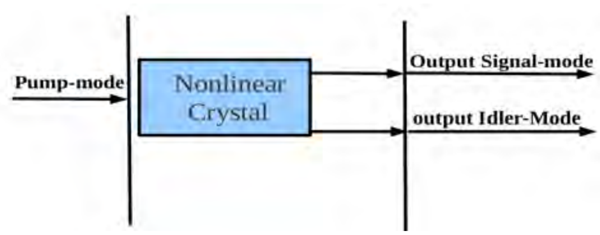

Figure 3.2: Two-mode subharmonic generator for output mode

3.3 Output photon statistics

In this section we seek to calculate the mean and variance of the photon number sum and difference for the output signal-idler mode.

3.3.1 The mean of the output photon number sum and difference

We now proceed to determine the mean of the photon number sum and difference for the output signal-idler mode. To this end, upon integrating Eq. (3.2.45) over β_{out}, we get

$$Q(\alpha_{out}, t) = \frac{u^2 - v^2}{\pi u} exp\bigg[-(\frac{u^2 - v^2}{u})\frac{\alpha^*_{out}\alpha_{out}}{\kappa}\bigg]. \tag{3.3.1}$$

The mean photon number of the output signal mode is expressible in terms of the Q-function as

$$\begin{aligned} \bar{n}_a^{out} &= \frac{1}{\pi}\int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\kappa^2} Q(\alpha^*_{out}, \gamma_{out}) exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\gamma^*_{out}\gamma_{out}}{\kappa} \\ &+ \frac{\gamma^*_{out}\alpha_{out}}{\kappa} + \frac{\alpha^*_{out}\gamma_{out}}{\kappa}\bigg] \gamma^*_{out}\alpha_{out}. \end{aligned} \tag{3.3.2}$$

Thus on substituting Eq. (3.3.1) into Eq. (3.3.2), we have

$$\begin{aligned} \bar{n}_a^{out} &= \frac{1}{\kappa^2}\bigg[\frac{u^2 - v^2}{u}\bigg]\int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\pi^2} exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\gamma^*_{out}\gamma_{out}}{\kappa} \\ &+ \frac{\gamma^*_{out}\alpha_{out}}{\kappa} + (\frac{v^2 - u^2 + u}{u})\frac{\alpha^*_{out}\gamma_{out}}{\kappa}\bigg] \gamma^*_{out}\alpha_{out}, \end{aligned} \tag{3.3.3}$$

which can be written in the form

$$\begin{aligned} \bar{n}_a^{out} &= \bigg[\frac{u^2 - v^2}{u}\bigg]\frac{1}{\kappa}\frac{d}{da}\int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\pi^2} exp\bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\gamma^*_{out}\gamma_{out}}{\kappa} \\ &+ a\frac{\gamma^*_{out}\alpha_{out}}{\kappa} + (\frac{v^2 - u^2 + u}{u})\frac{\alpha^*_{out}\gamma_{out}}{\kappa}\bigg]_{a=1}, \end{aligned} \tag{3.3.4}$$

so that performing the integration employing Eq. (2.2.28), one readily obtains

$$\bar{n}_a^{out} = \bigg[\frac{u^2 - v^2}{u}\bigg]\frac{d}{da}\bigg[\frac{u\kappa}{u - a(v^2 - u^2 + u)}\bigg]. \tag{3.3.5}$$

Thus carrying out the differentiation and applying the condition $a = 1$, we get

$$\bar{n}_a^{out} = \kappa\bigg[\frac{u}{u^2 - v^2} - 1\bigg]. \tag{3.3.6}$$

And in view of Eqs. (3.2.43) and (3.2.44), there follows

$$\bar{n}_a^{out} = \kappa(a - 1), \tag{3.3.7}$$

and hence on account of Eq. (3.2.39), we have

$$\bar{n}_a^{out} = \kappa\left[\frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t})\right]. \tag{3.3.8}$$

We see that at steady state the mean photon number of the output signal mode reduces to

$$(\bar{n}_a^{out})_{ss} = \frac{2\kappa\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.9}$$

One can also estabilish in a similar manner that the mean photon number of the output iddler mode is

$$\bar{n}_b^{out} = \kappa\left[\frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t})\right]. \tag{3.3.10}$$

Thus using the value of $\lambda_{\pm}$, we find at steady state

$$(\bar{n}_b^{out})_{ss} = \frac{2\kappa\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.11}$$

Furthermore, in view of Eqs. (3.3.8) and (3.3.10), the mean of the output photon number sum and difference can be written as

$$\begin{aligned}
\bar{n}_{\pm}^{out} \quad &= \bar{n}_a^{out} \pm \bar{n}_b^{out} \\
&= \kappa\left[\frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t})\right] \\
&\pm\kappa\left[\frac{\gamma}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1 - e^{-\lambda_+ t})\right].
\end{aligned} \tag{3.3.12}$$

Then the mean of the output photon number sum is

$$\bar{n}_{+}^{out} = \kappa\left[\frac{\gamma}{\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1 - e^{-\lambda_+ t})\right], \tag{3.3.13}$$

This can be put at steady state in the form

$$(\bar{n}_{+}^{out})_{ss} = \frac{4\kappa\gamma^2}{\kappa^2 - 4\gamma^2}. \tag{3.3.14}$$

On the other hand, in view Eqs. (3.3.8) and (3.3.10), the mean of the output photon number difference turns out to be

$$\bar{n}_{-}^{out} = 0. \tag{3.3.15}$$

3.3.2 The variance of the output photon number sum and difference

Here we seek to obtain the variance of the output photon number sum and difference. The variance of the output photon number sum and difference can be expressed as

$$(\Delta n_{\pm}^{out})^2 = \langle (\hat{n}_{\pm}^{out})^2 \rangle - \langle \hat{n}_{\pm}^{out} \rangle^2. \tag{3.3.16}$$

Applying Eq. (2.3.3), the variance of the output photon number sum and difference can be put in the form

$$(\Delta n_{\pm}^{out})^2 = (\Delta n_a^{out})^2 + (\Delta n_b^{out})^2 \pm 2(\langle \hat{n}_a^{out} \hat{n}_b^{out} \rangle - \bar{n}_a^{out} \bar{n}_b^{out}). \tag{3.3.17}$$

On the other hand, the photon number variance for the output signal mode can be written in the form

$$\begin{aligned}(\Delta n_a^{out})^2 &= \langle (\hat{n}_a^{out})^2 \rangle - (\bar{n}_a^{out})^2 \\ &= \langle \hat{a}_{out}^{\dagger 2} \hat{a}_{out}^2 \rangle + \kappa \langle \hat{a}_{out}^{\dagger} \hat{a}_{out} \rangle - (\bar{n}_a^{out})^2. \end{aligned} \tag{3.3.18}$$

We can write the expectation value of $\hat{a}_{out}^{\dagger 2} \hat{a}_{out}^2$ as

$$\begin{aligned}\langle \hat{a}_{out}^{\dagger 2} \hat{a}_{out}^2 \rangle &= \frac{1}{\pi} \int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\kappa^2} Q(\alpha_{out}^*, \gamma_{out}) exp\bigg[-\frac{\alpha_{out}^* \alpha_{out}}{\kappa} - \frac{\gamma_{out}^* \gamma_{out}}{\kappa} \\ &+ \frac{\gamma_{out}^* \alpha_{out}}{\kappa} + \frac{\alpha_{out}^* \gamma_{out}}{\kappa} \bigg] \gamma_{out}^{*2} \alpha_{out}^2. \end{aligned} \tag{3.3.19}$$

Thus on substituting the Q-function described in Eq. (3.3.1), there follows

$$\begin{aligned}\langle \hat{a}_{out}^{\dagger 2} \hat{a}_{out}^2 \rangle &= \frac{u^2 - v^2}{u} \frac{1}{\kappa^2} \int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\pi^2} exp\bigg[-\frac{\alpha_{out}^* \alpha_{out}}{\kappa} - \frac{\gamma_{out}^* \gamma_{out}}{\kappa} \\ &+ \frac{\gamma_{out}^* \alpha_{out}}{\kappa} + (\frac{u - u^2 + v^2}{u}) \frac{\alpha_{out}^* \gamma_{out}}{\kappa} \bigg] \gamma_{out}^{*2} \alpha_{out}^2, \end{aligned} \tag{3.3.20}$$

which can be written in the form

$$\begin{aligned}\langle \hat{a}_{out}^{\dagger 2} \hat{a}_{out}^2 \rangle &= \frac{u^2 - v^2}{u} \frac{d^2}{da^2} \int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\pi^2} exp\bigg[-\frac{\alpha_{out}^* \alpha_{out}}{\kappa} - \frac{\gamma_{out}^* \gamma_{out}}{\kappa} \\ &+ a\frac{\gamma_{out}^* \alpha_{out}}{\kappa} + (\frac{u - u^2 + v^2}{u}) \frac{\alpha_{out}^* \gamma_{out}}{\kappa} \bigg]_{a=1}, \end{aligned} \tag{3.3.21}$$

so that performing the integration using Eq. (2.2.28), one readily obtains

$$\langle \hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}\rangle = \frac{u^2-v^2}{u}\kappa^2\frac{d^2}{da^2}\left[\frac{u}{u-a(u-u^2+v^2)}\right]. \tag{3.3.22}$$

Thus carrying out the differentiation and applying the condition $a=1$, we get

$$\langle \hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}\rangle = 2\kappa^2\left[\frac{u}{u^2-v^2}-1\right]^2. \tag{3.3.23}$$

In view of Eqs. (3.2.43) and (3.2.44), we have

$$\langle \hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}\rangle = 2\kappa^2(a-1)^2 \tag{3.3.24}$$

and on account of Eq. (3.3.7), we see that

$$\langle \hat{a}_{out}^{\dagger 2}\hat{a}_{out}^{2}\rangle = 2(\bar{n}_a^{out})^2. \tag{3.3.25}$$

Thus Eq. (3.3.18) reduces to

$$(\Delta n_a^{out})^2 = (\bar{n}_a^{out})^2 + \kappa\bar{n}_a^{out} = \kappa^2(\bar{n}_a^2+\bar{n}_a). \tag{3.3.26}$$

Following the same procedure, one can readily estabilish that the variance of the photon number of the output idler mode is the same as that of the output signal mode.

Eq. (3.3.26) indicates that the output signal mode is in a chaotic state, and the same holds true for the output iddler mode.

Now on account of Eq. (3.3.8), we readily find

$$\begin{aligned}(\Delta n_a^{out})^2 \quad &= \kappa^2\left[\frac{\gamma^2}{4\lambda_-^2}(1-e^{-\lambda_- t})^2 + \frac{\gamma^2}{4\lambda_+^2}(1-e^{-\lambda_+ t})^2 - \frac{\gamma^2}{2\lambda_-\lambda_+}(1-e^{-\lambda_- t})(1-e^{-\lambda_+ t})\right]\\ &+\kappa^2\left[\frac{\gamma}{2\lambda_-}(1-e^{-\lambda_- t}) - \frac{\gamma}{2\lambda_+}(1-e^{-\lambda_+ t})\right].\end{aligned} \tag{3.3.27}$$

Thus at steady state this reduces to

$$(\Delta n_a^{out})^2_{ss} = 2\kappa^2\gamma^2\left[\frac{\kappa^2-2\gamma^2}{(\kappa^2-4\gamma^2)^2}\right]. \tag{3.3.28}$$

Similarly we have for the output idler mode at steady state

$$(\Delta n_b^{out})^2_{ss} = 2\kappa^2\gamma^2\left[\frac{\kappa^2-2\gamma^2}{(\kappa^2-4\gamma^2)^2}\right]. \tag{3.3.29}$$

We next seek to find the expectation value of $\hat{n}_a^{out}\hat{n}_b^{out}$: which is written, employing the Q-function (3.2.45), as

$$\begin{aligned}\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle &= (\frac{u^2-v^2}{\kappa^2})\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-u(\frac{\alpha^*_{out}\alpha_{out}}{\kappa}+\frac{\beta^*_{out}\beta_{out}}{\kappa})-v(\frac{\alpha_{out}\beta_{out}}{\kappa}\\ &+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})\bigg](\alpha^*_{out}\alpha_{out}\beta^*_{out}\beta_{out}-\kappa\alpha^*_{out}\alpha_{out}-\kappa\beta^*_{out}\beta_{out}+\kappa^2).\end{aligned} \tag{3.3.30}$$

This can be rewritten in the form

$$\begin{aligned}\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle &= (u^2-v^2)\bigg(\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-a\frac{u\alpha^*_{out}\alpha_{out}}{\kappa}+b\frac{u\beta^*_{out}\beta_{out}}{\kappa}\\ &-v(\frac{\alpha_{out}\beta_{out}}{\kappa}+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})\bigg]\\ &+\frac{1}{u}\frac{d}{da}\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-a\frac{u\alpha^*_{out}\alpha_{out}}{\kappa}+b\frac{u\beta^*_{out}\beta_{out}}{\kappa}-v(\frac{\alpha_{out}\beta_{out}}{\kappa}\\ &+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})\bigg]\\ &+\frac{1}{u}\frac{d}{db}\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-a\frac{u\alpha^*_{out}\alpha_{out}}{\kappa}+b\frac{u\beta^*_{out}\beta_{out}}{\kappa}-v(\frac{\alpha_{out}\beta_{out}}{\kappa}\\ &+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})\bigg]\bigg)_{a=b=1}+\kappa^2,\end{aligned} \tag{3.3.31}$$

so that performing the integration employing Eq. (2.2.28), we have

$$\begin{aligned}\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle &= (u^2-v^2)\bigg(\frac{\kappa^2}{u^2}\frac{d}{da}\frac{d}{db}\bigg[\frac{1}{abu^2-v^2}\bigg]+\frac{\kappa^2}{u}\frac{d}{da}\bigg[\frac{1}{abu^2-v^2}\bigg]\\ &+\frac{\kappa^2}{u}\frac{d}{db}\bigg[\frac{1}{abu^2-v^2}\bigg]\bigg)_{a=b=1}+\kappa^2.\end{aligned} \tag{3.3.32}$$

Thus carrying out the differentiation and applying the condition $a=b=1$, one readily obtains

$$\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle=\kappa^2\left[\frac{u^2+v^2}{(u^2-v^2)^2}-\frac{2u}{(u^2-v^2)}+1\right]. \tag{3.3.33}$$

Introducing Eqs. (3.2.43) and (3.2.44), we find

$$\begin{aligned}\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle &= \kappa^2(a^2+b^2-2a+1)\\ &= (\bar{n}_a^{out})^2+b^2\kappa^2.\end{aligned} \tag{3.3.34}$$

We found that the results of the mean and variance of the photon number for the output signal mode are the same as that of the output idler mode. Therefore the variance of the output photon number sum and difference defined by Eq. (3.3.17), can be rewritten in the form

$$(\Delta n_{\pm}^{out})^2 = 2(\Delta n_a^{out})^2 \pm 2[\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle - (\bar{n}_a^{out})^2]. \tag{3.3.35}$$

In view of Eqs. (3.3.26) and (3.3.34), we find

$$(\Delta n_{\pm}^{out})^2 = 2[(\bar{n}_a^{out})^2 + \kappa\bar{n}_a^{out} \pm b^2\kappa^2]. \tag{3.3.36}$$

Thus, the variance of the output photon number sum is

$$(\Delta n_{+}^{out})^2 = 2[(\bar{n}_a^{out})^2 + \kappa\bar{n}_a^{out} + b^2\kappa^2], \tag{3.3.37}$$

on account of Eqs. (3.2.40) and (3.3.8), one readily obtains

$$\begin{aligned}(\Delta n_{+}^{out})^2 \quad &= \kappa^2\left[\frac{\gamma^2}{\lambda_+^2}(1-e^{-\lambda_+ t})^2 + \frac{\gamma^2}{\lambda_-^2}(1-e^{-\lambda_- t})^2\right]\\ &+\kappa^2\left[\frac{\gamma}{\lambda_-}(1-e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1-e^{-\lambda_+ t})\right],\end{aligned} \tag{3.3.38}$$

the sum of the output photon number variance takes at steady state the form

$$(\Delta n_{+}^{out})^2_{ss} = 2\kappa^2\gamma^2\left[\frac{3\kappa^2 - 4\gamma^2}{(\kappa^2-4\gamma^2)^2}\right]. \tag{3.3.39}$$

And the variance of the output photon number difference is found to be

$$(\Delta n_{-}^{out})^2 = 2[(\bar{n}_a^{out})^2 + \kappa\bar{n}_a^{out} - b^2\kappa^2], \tag{3.3.40}$$

on account of Eqs. (3.2.40) and (3.3.8), one obtains

$$\begin{aligned}(\Delta n_{-}^{out})^2 \quad &= \kappa^2\left[\frac{\gamma}{\lambda_-}(1-e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1-e^{-\lambda_+ t})\right]\\ &-\kappa^2\left[\frac{2\gamma^2}{\lambda_+\lambda_-}(1-e^{-\lambda_+ t})(1-e^{-\lambda_- t})\right].\end{aligned} \tag{3.3.41}$$

Thus at steady state, this reduces to

$$(\Delta n_{-}^{out})^2_{ss} = \frac{2\kappa^2\gamma^2}{(\kappa^2-4\gamma^2)}. \tag{3.3.42}$$

Chapter 4

Superposed two-mode Coherent and Subharmonic output light

4.1 The Q-function

In chapter two and three of this thesis we have calculated the Q-functions for two-mode coherent and subharmonic output light beams, and with the help of these Q-functions we determined the output photon statistics. In this chapter we are interested to obtain the Q-function for the superposed two-mode coherent and subharmonic output light. To this end, we must first obtain the Q-function for the superposition of any two-mode light beams. Thus expanding the density operator for light beam-1 as a function of $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$ in the antinormal order we have

$$\hat{\rho}_1 = \hat{\rho}_1(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}) = \sum_{klmn} C_{klmn} \hat{a}^{\dagger k} \hat{b}^{\dagger l} \hat{a}^m \hat{b}^n. \tag{4.1.1}$$

Inserting the completeness relation for two-mode coherent states, with

$$I = \frac{1}{\pi^2} \int d^2\chi_A d^2\lambda_B |\chi_A, \lambda_B\rangle\langle\lambda_B, \chi_A|, \tag{4.1.2}$$

we readily find

$$\hat{\rho}_1 = \int d^2\chi_A d^2\lambda_B \frac{1}{\pi^2} \sum_{klmn} C_{klmn} |\chi_A, \lambda_B\rangle\langle\lambda_B, \chi_A| \hat{a}^{\dagger k} \hat{b}^{\dagger l} \hat{a}^m \hat{b}^n. \tag{4.1.3}$$

Applying the relation [1]

$$|\alpha\rangle\langle\alpha|\hat{a}^\dagger = \alpha^* |\alpha\rangle\langle\alpha|, \tag{4.1.4}$$

and

$$|\alpha\rangle\langle\alpha|\hat{a} = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|, \tag{4.1.5}$$

one readily obtains

$$\begin{aligned}\hat{\rho}_1 &= \int d^2\chi_A d^2\lambda_B \frac{1}{\pi^2}\sum_{klmn} C_{klmn}\chi_A^{*k}\left(\chi_A + \frac{\partial}{\partial\chi_A^*}\right)^m \lambda_B^{*l}\left(\lambda_B + \frac{\partial}{\partial\lambda_B^*}\right)^n \\ &\quad |\chi_A, \lambda_B\rangle\langle\lambda_B, \chi_A|. \end{aligned} \tag{4.1.6}$$

Which can be rewritten using the displacement operator as

$$\begin{aligned}\hat{\rho}_1 &= \int d^2\chi_A d^2\lambda_B \frac{1}{\pi^2}\sum_{klmn} C_{klmn}\chi_A^{*k}\lambda_B^{*l}\left(\chi_A + \frac{\partial}{\partial\chi_A^*}\right)^m \left(\lambda_B + \frac{\partial}{\partial\lambda_B^*}\right)^n \\ &\quad \hat{D}(\chi_A)\hat{D}(\lambda_B)\hat{\rho}_{0_A,0_B}\hat{D}(-\lambda_B)\hat{D}(-\chi_A), \end{aligned} \tag{4.1.7}$$

where $\hat{\rho}_{0_A,0_B} = |0_A, 0_B\rangle\langle 0_B, 0_A|$.

Moreover, using a similar procedure one can readily obtain the density operator for light beam-2 as

$$\hat{\rho}_2 = \hat{\rho}_2(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}) = \sum_{k'l'm'n'} C_{k'l'm'n'}\hat{a}^{\dagger k'}\hat{b}^{\dagger l'}\hat{a}^{m'}\hat{b}^{n'}, \tag{4.1.8}$$

from which follows

$$\begin{aligned}\hat{\rho}_2 &= \int d^2\gamma_A d^2\eta_B \frac{1}{\pi^2}\sum_{k'l'm'n'} C_{k'l'm'n'}\gamma_A^{*k'}\eta_B^{*l'}\left(\gamma_A + \frac{\partial}{\partial\gamma_A^*}\right)^{m'} \left(\eta_B + \frac{\partial}{\partial\eta_B^*}\right)^{n'} \\ &\quad \hat{D}(\gamma_A)\hat{D}(\eta_B)\hat{\rho}_{0_A,0_B}\hat{D}(-\eta_B)\hat{D}(-\gamma_A). \end{aligned} \tag{4.1.9}$$

Further more, on account of Eqs. (4.1.7) and (4.1.9), the density operator for the superposed two-mode light beams can be put in the form

$$\begin{aligned}\hat{\rho} &= \int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^4}\sum_{klmn} C_{klmn}\chi_A^{*k}\lambda_B^{*l}\left(\chi_A + \frac{\partial}{\partial\chi_A^*}\right)^m \left(\lambda_B + \frac{\partial}{\partial\lambda_B^*}\right)^n \\ &\quad \sum_{k'l'm'n'} C_{k'l'm'n'}\gamma_A^{*k'}\eta_B^{*l'}\left(\gamma_A + \frac{\partial}{\partial\gamma_A^*}\right)^{m'} \left(\eta_B + \frac{\partial}{\partial\eta_B^*}\right)^{n'}\hat{D}(\gamma_A)\hat{D}(\eta_B)\hat{D}(\chi_A) \\ &\quad \hat{D}(\lambda_B)|0_A, 0_B\rangle\langle 0_B, 0_A|\hat{D}(-\lambda_B)\hat{D}(-\chi_A\hat{D}(-\eta_B)\hat{D}(-\gamma_A). \end{aligned} \tag{4.1.10}$$

Using the results described in [1]

$$\hat{D}(\alpha)|\beta\rangle\langle\beta|\hat{D}(-\alpha) = |\alpha + \beta\rangle\langle\alpha + \beta|, \tag{4.1.11}$$

the density operator for the superposition can be put in the form

$$\begin{aligned}\hat{\rho} \quad &= \int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^4} \sum_{klmn} C_{klmn} \chi_A^{*k} \lambda_B^{*l} \left(\chi_A + \frac{\partial}{\partial \chi_A^*}\right)^m \left(\lambda_B + \frac{\partial}{\partial \lambda_B^*}\right)^n \\ &\sum_{k'l'm'n'} C_{k'l'm'n'} \gamma_A^{*k'} \eta_B^{*l'} \left(\gamma_A + \frac{\partial}{\partial \gamma_A^*}\right)^{m'} \left(\eta_B + \frac{\partial}{\partial \eta_B^*}\right)^{n'} |\gamma_A + \chi_A, \eta_B + \lambda_B\rangle \\ &\langle \eta_B + \lambda_B, \gamma_A + \chi_A|. \end{aligned} \tag{4.1.12}$$

We now proceed to obtain the Q-function for the superposition of two light beams, which is defined by

$$Q(\alpha, \beta) = \frac{1}{\pi^2} \langle \alpha, \beta | \hat{\rho} | \beta, \alpha \rangle, \tag{4.1.13}$$

the c-number function corresponding to the normally ordered density operator divided by π^2. Introducing Eq. (4.1.12) into Eq. (4.1.13) we readily find

$$\begin{aligned}Q(\alpha, \beta) \quad &= \int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^6} \sum_{klmn} C_{klmn} \chi_A^{*k} \lambda_B^{*l} \left(\chi_A + \frac{\partial}{\partial \chi_A^*}\right)^m \left(\lambda_B + \frac{\partial}{\partial \lambda_B^*}\right)^n \\ &\sum_{k'l'm'n'} C_{k'l'm'n'} \gamma_A^{*k'} \eta_B^{*l'} \left(\gamma_A + \frac{\partial}{\partial \gamma_A^*}\right)^{m'} \left(\eta_B + \frac{\partial}{\partial \eta_B^*}\right)^{n'} \langle \alpha, \beta | \gamma_A + \chi_A, \eta_B + \lambda_B\rangle \\ &\langle \eta_B + \lambda_B, \gamma_A + \chi_A | \beta, \alpha \rangle. \end{aligned} \tag{4.1.14}$$

Now using the relation [1]

$$|\langle \alpha | \beta \rangle|^2 = exp(-|\alpha - \beta|^2), \tag{4.1.15}$$

we find the results

$$\begin{aligned}&\langle \alpha, \beta | \gamma_A + \chi_A, \eta_B + \lambda_B \rangle \langle \eta_B + \lambda_B, \gamma_A + \chi_A | \beta, \alpha \rangle = |\langle \alpha | \gamma_A + \chi_A \rangle|^2 |\langle \beta | \eta_B + \lambda_B \rangle|^2 \\ &= exp(-\alpha^*\alpha - \beta^*\beta + \alpha^*\gamma_A + \alpha^*\chi_A + \beta^*\lambda_B + \beta^*\eta_B + \chi_A^*(\alpha - \gamma_A - \chi_A) \\ &+ \lambda_B^*(\beta - \eta_B - \lambda_B) + \gamma_A^*(\alpha - \gamma_A - \chi_A) + \eta_B^*(\beta - \eta_B - \lambda_B). \end{aligned} \tag{4.1.16}$$

In view of Eq. (4.1.16), Eq. (4.1.14) can be rewritten as

$$\begin{aligned}Q(\alpha, \beta) \quad &= \psi_1 \int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^6} \psi_2 \sum_{klmn} C_{klmn} \chi_A^{*k} \lambda_B^{*l} \left(\chi_A + \frac{\partial}{\partial \chi_A^*}\right)^m \\ &\psi_3 \left(\lambda_B + \frac{\partial}{\partial \lambda_B^*}\right)^n \psi_4 \sum_{k'l'm'n'} C_{k'l'm'n'} \gamma_A^{*k'} \eta_B^{*l'} \left(\gamma_A + \frac{\partial}{\partial \gamma_A^*}\right)^{m'} \\ &\psi_5 \left(\eta_B + \frac{\partial}{\partial \eta_B^*}\right)^{n'} \psi_6, \end{aligned} \tag{4.1.17}$$

where

$$\psi_1 = exp(-\alpha^*\alpha - \beta^*\beta), \tag{4.1.18}$$

$$\psi_2 = exp(\alpha^*\gamma_A + \alpha^*\chi_A + \beta^*\lambda_B + \beta^*\eta_B), \tag{4.1.19}$$

$$\psi_3 = exp[\chi_A^*(\alpha - \gamma_A - \chi_A)], \tag{4.1.20}$$

$$\psi_4 = exp[\lambda_B^*(\beta - \eta_B - \lambda_B)], \tag{4.1.21}$$

$$\psi_5 = exp[\gamma_A^*(\alpha - \gamma_A - \chi_A)], \tag{4.1.22}$$

$$\psi_6 = exp[(\eta_B^*(\beta - \eta_B - \lambda_B)]. \tag{4.1.23}$$

On the otherhand, using the binomial theorm

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k, \tag{4.1.24}$$

one finds

$$\begin{aligned}
&\left(\eta_B + \frac{\partial}{\partial \eta_B^*}\right)^{n'} exp\left[\eta_B^*(\beta - \eta_B - \lambda_B)\right] \\
&= \sum_{k=0}^{n'} \frac{n'!}{k!(n'-k)!} \eta_B^{n'-k} \frac{\partial}{\partial^k \eta_B^{*k}} exp\left[\eta_B^*(\beta - \eta_B - \lambda_B)\right] \\
&= (\beta - \lambda_B)^{n'} exp\left[\eta_B^*(\beta - \eta_B - \lambda_B)\right].
\end{aligned} \tag{4.1.25}$$

Using the same procedure one can show

$$\left(\eta_B + \frac{\partial}{\partial \eta_B^*}\right)^{n'} \psi_6 = (\beta - \lambda_B)^{n'} \psi_6, \tag{4.1.26}$$

$$\left(\gamma_A + \frac{\partial}{\partial \gamma_A^*}\right)^{m'} \psi_5 = (\alpha - \chi_A)^{m'} \psi_5, \tag{4.1.27}$$

$$\left(\lambda_B + \frac{\partial}{\partial \lambda_B^*}\right)^{n} \psi_4 = (\beta - \eta_B)^{n} \psi_4, \tag{4.1.28}$$

$$\left(\chi_A + \frac{\partial}{\partial \chi_A^*}\right)^{m} \psi_3 = (\alpha - \gamma_A)^{m} \psi_3. \tag{4.1.29}$$

Inserting Eqs. (4.1.26), (4.1.27), (4.1.28), (4.1.29) in Eq. (4.1.17), the Q-function for the superposed two-mode light beams is written in the form

$$
\begin{aligned}
Q(\alpha,\beta) &= \int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^2} Q(\chi_A^*,\lambda_B^*,\alpha-\gamma_A,\beta-\eta_B) \\
&\quad Q(\gamma_A^*,\eta_B^*,\alpha-\chi_A,\beta-\lambda_B) exp\Big[-|\alpha-\chi_A-\gamma_A|^2 \\
&\quad -|\beta-\lambda_B-\eta_B|^2\Big]. \qquad (4.1.30)
\end{aligned}
$$

We now proceed to obtain the Q-function for the superposed two-mode coherent and subharmonic light beams. To this end, in view of Eq. (2.2.29) and Eq.(3.2.42), we have

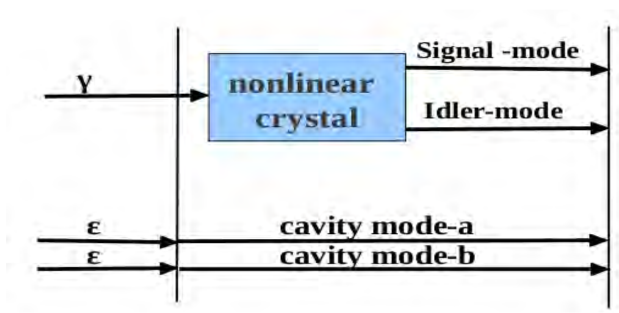

Figure 4.1: Two-mode Coherent and Subharmonic cavity light beams

$$
\begin{aligned}
Q(\chi_A^*,\lambda_B^*,\alpha-\gamma_A,\beta-\eta_B) &= \frac{1}{\pi^2} exp\Big[-\chi_A^*\alpha+\chi_A^*\gamma_A-\lambda_B^*\beta+\lambda_B^*\eta_B \\
&\quad +q(\alpha-\gamma_A+\chi_A^*+\beta-\eta_B+\lambda_B^*)-2q^2\Big], \qquad (4.1.31)
\end{aligned}
$$

and

$$
\begin{aligned}
Q(\gamma_A^*,\eta_B^*,\alpha-\chi_A,\beta-\lambda_B) &= \frac{u^2-v^2}{\pi^2} exp\Big[-u\gamma_A^*\alpha+u\gamma_A^*\chi_A-u\eta_B^*\beta+u\eta_B^*\lambda_B \\
&\quad -v\alpha\beta+v\alpha\lambda_B+v\chi_A\beta-v\chi_A\lambda_B-v\gamma_A^*\eta_B^*\Big]. \qquad (4.1.32)
\end{aligned}
$$

Introducing Eqs. (4.1.31) and (4.1.32) into Eq. (4.1.30) the Q-function for the superposed two-mode coherent and subharmonic light beams is expressed as

$$\begin{aligned} Q(\alpha,\beta,t) &= \frac{u^2-v^2}{\pi^2}\int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^4} exp\bigg[-\chi_A^*\alpha+\chi_A^*\gamma_A \\ &-\lambda_B^*\beta+\lambda_B^*\eta_B+q(\alpha-\gamma_A+\chi_A^*+\beta-\eta_B+\lambda_B^*)-2q^2\bigg] \\ &exp\bigg[-u\gamma_A^*\alpha+u\gamma_A^*\chi_A-u\eta_B^*\beta+u\eta_B^*\lambda_B-v\alpha\beta+v\alpha\lambda_B \\ &+v\chi_A\beta-v\chi_A\lambda_B-v\gamma_A^*\eta_B^*\bigg]exp\bigg[-|\alpha-\chi_A-\gamma_A|^2 \\ &-|\beta-\lambda_B-\eta_B|^2\bigg], \end{aligned} \tag{4.1.33}$$

which can be rewritten as

$$\begin{aligned} Q(\alpha,\beta,t) &= \frac{u^2-v^2}{\pi^2}\int \frac{d^2\chi_A d^2\lambda_B d^2\gamma_A d^2\eta_B}{\pi^4} exp\bigg[-\alpha^*\alpha-\beta^*\beta-\chi_A^*\chi_A \\ &-\lambda_B^*\lambda_B-\gamma_A^*\gamma_A-\eta_B^*\eta_B+\alpha\gamma_A^*+\alpha^*\chi_A-\chi_A\gamma_A^*+\gamma_A\alpha^*+\beta\eta_B^* \\ &+\lambda_B\beta^*-\lambda_B\eta_B^*+\eta_B\beta^*-u(\gamma_A^*\alpha-\gamma_A^*\chi_A-\eta_B^*\lambda_B+\eta_B^*\beta) \\ &-v(\alpha\beta+\chi_A\lambda_B+\gamma_A^*\eta_B^*-\alpha\lambda_B-\chi_A\beta)+q(\alpha-\gamma_A+\chi_A^* \\ &+\beta-\eta_B+\lambda_B^*)-2q^2\bigg]. \end{aligned} \tag{4.1.34}$$

Thus performing the integration employing the relation (2.2.28) one readily obtains

$$\begin{aligned} Q(\alpha,\beta,t) &= \frac{(u^2-v^2)}{\pi^2} exp\bigg[-u(\alpha^*\alpha+\beta^*\beta)-v(\alpha\beta+\alpha^*\beta^*) \\ &+q(u+v)(\alpha+\alpha^*+\beta+\beta^*)-2q^2(u+v)\bigg]. \end{aligned} \tag{4.1.35}$$

So, using the relations (2.2.32) and (2.2.33) the Q-function for the superposed two-mode coherent and subharmonic output light takes the form

$$\begin{aligned} Q(\alpha_{out},\beta_{out},t) &= \frac{(u^2-v^2)}{\pi^2} exp\bigg[-u\bigg(\frac{\alpha_{out}^*\alpha_{out}}{\kappa}+\frac{\beta_{out}^*\beta_{out}}{\kappa}\bigg)-v\bigg(\frac{\alpha_{out}\beta_{out}}{\kappa}+\frac{\alpha_{out}^*\beta_{out}^*}{\kappa}\bigg) \\ &+q(u+v)\bigg(\frac{\alpha_{out}}{\sqrt{\kappa}}+\frac{\alpha_{out}^*}{\sqrt{\kappa}}+\frac{\beta_{out}}{\sqrt{\kappa}}+\frac{\beta_{out}^*}{\sqrt{\kappa}}\bigg)-2q^2(u+v)\bigg]. \end{aligned} \tag{4.1.36}$$

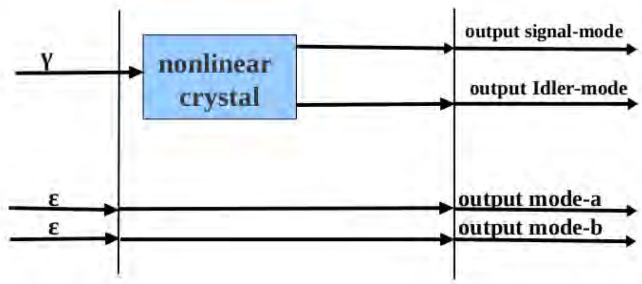

Figure 4.2: Two-mode Coherent and Subharmonic output light beams

4.2 Output photon statistics

In this section we will see about the mean of the photon number sum and difference and the variance of the photon number sum and difference for the superposed two-mode coherent and subharmonic output lights.

4.2.1 The mean of the output photon number sum and difference

We wish here to determine the mean of the photon number sum and difference for the superposed two-mode coherent and subharmonic output light. To this end, upon integrating Eq. (4.1.36) over β_{out}, one obtains

$$\begin{aligned} Q(\alpha_{out}, t) &= \left(\frac{u^2 - v^2}{\pi u}\right) exp\Bigg[- \left(\frac{u^2 - v^2}{u}\right) \frac{\alpha^*_{out}\alpha_{out}}{\kappa} + q(\frac{u^2 - v^2}{u})\left(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\alpha^*_{out}}{\sqrt{\kappa}}\right) \\ & -q^2\left(\frac{u^2 - v^2}{u}\right)\Bigg]. \end{aligned} \tag{4.2.1}$$

Now the mean photon number of mode a for the superposed two mode coherent and subharmonic output light is expressed as

$$\begin{aligned} \bar{n}_a^{out} &= \frac{1}{\pi}\int \frac{d^2\alpha_{out} d^2\gamma_{out}}{\kappa^2} Q(\alpha^*_{out}, \gamma_{out}) exp\Bigg[- \frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\gamma^*_{out}\gamma_{out}}{\kappa} \\ & + \frac{\alpha^*_{out}\gamma_{out}}{\kappa} + \frac{\gamma^*_{out}\alpha_{out}}{\kappa}\Bigg] \gamma^*_{out}\alpha_{out}, \end{aligned} \tag{4.2.2}$$

on substituting the Q-function described in Eq. (4.2.1), we have

$$\begin{aligned}\bar{n}_a^{out} \quad &= \frac{1}{\kappa}(\frac{u^2-v^2}{u})\frac{d}{da}\int\frac{d^2\alpha_{out}d^2\gamma_{out}}{\pi^2}exp\bigg[-\frac{\alpha_{out}^*\alpha_{out}}{\kappa}-\frac{\gamma_{out}^*\gamma_{out}}{\kappa}\\ &+(\frac{u-u^2+v^2}{u})\frac{\alpha_{out}^*\gamma_{out}}{\kappa}+a\frac{\gamma_{out}^*\alpha_{out}}{\kappa}+q(\frac{u^2-v^2}{u})(\frac{\gamma_{out}}{\sqrt{\kappa}}+\frac{\alpha_{out}^*}{\sqrt{\kappa}})\\ &-q^2(\frac{u^2-v^2}{u})\bigg]_{a=1},\end{aligned} \tag{4.2.3}$$

so that performing the integration using Eq. (2.2.28), we find

$$\begin{aligned}\bar{n}_a^{out} \quad &= \kappa(u^2-v^2)exp\bigg[-q^2(\frac{u^2-v^2}{u})\bigg]\frac{d}{da}\bigg[\frac{1}{[u-a(u-u^2+v^2)]}\bigg]\\ &exp\bigg[\frac{aq^2(\frac{(u^2-v^2)^2}{u})}{[u-a(u-u^2+v^2)]}\bigg].\end{aligned} \tag{4.2.4}$$

Thus carrying out the differentiation and applying the condition $a = 1$, we see that

$$\bar{n}_a^{out} = \kappa\left[\frac{u}{u^2-v^2}+q^2-1\right]. \tag{4.2.5}$$

And in view of Eqs. (3.2.43) and (3.2.44), we get

$$\bar{n}_a^{out} = \kappa(a-1+q^2). \tag{4.2.6}$$

Finally using Eqs. (3.2.39) and (2.1.64), we readily obtain

$$\bar{n}_a^{out} = \kappa\left[\frac{\gamma}{2\lambda_-}(1-e^{-\lambda_- t})-\frac{\gamma}{2\lambda_+}(1-e^{-\lambda_+ t})+\frac{4\varepsilon^2}{\kappa^2}(1-e^{-\kappa t/2})^2\right]. \tag{4.2.7}$$

We observe that at steady state the mean photon number for output mode a reduces to

$$(\bar{n}_a^{out})_{ss} = \frac{2\kappa\gamma^2}{\kappa^2-4\gamma^2}+\frac{4\varepsilon^2}{\kappa}. \tag{4.2.8}$$

Following the same procedure, the mean photon number for output mode b is found to be

$$\bar{n}_b^{out} = \kappa\left[\frac{\gamma}{2\lambda_-}(1-e^{-\lambda_- t})-\frac{\gamma}{2\lambda_+}(1-e^{-\lambda_+ t})+\frac{4\varepsilon^2}{\kappa^2}(1-e^{-\kappa t/2})^2\right], \tag{4.2.9}$$

and at steady state, it takes the form

$$(\bar{n}_b^{out})_{ss} = \frac{2\kappa\gamma^2}{\kappa^2-4\gamma^2}+\frac{4\varepsilon^2}{\kappa}. \tag{4.2.10}$$

From which we observe that the mean photon number for the superposed two-mode coherent and subharmonic output light is the sum of the mean photon number of the individual output light beams.

The mean of the output photon number sum and difference can be written as

$$\bar{n}_{\pm}^{out} = \bar{n}_{a}^{out} \pm \bar{n}_{b}^{out}. \tag{4.2.11}$$

Further more, in view of Eqs. (4.2.7) and (4.2.9), the mean of the output photon number sum can be written as

$$\bar{n}_{+}^{out} = \kappa \left[\frac{\gamma}{\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{8\varepsilon^2}{\kappa^2}(1 - e^{-\kappa t/2})^2 \right]. \tag{4.2.12}$$

This can be put at steady state in the form

$$(\bar{n}_{+}^{out})_{ss} = \frac{4\kappa\gamma^2}{\kappa^2 - 4\gamma^2} + \frac{8\varepsilon^2}{\kappa}. \tag{4.2.13}$$

On the other hand, in view of Eqs. (4.2.7) and (4.2.9) the mean of the output photon number difference turns out to be

$$\bar{n}_{-}^{out} = 0. \tag{4.2.14}$$

4.2.2 The variance of the output photon number sum and difference

We next proceed to obtain the variance of the photon number sum and difference for the superposed two-mode coherent and subharmonic output light. The variance of the output photon number sum and difference is defined by

$$(\Delta n_{\pm}^{out})^2 = (\Delta n_{a}^{out})^2 + (\Delta n_{b}^{out})^2 \pm 2(\langle \hat{n}_{a}^{out} \hat{n}_{b}^{out} \rangle - \bar{n}_{a}^{out} \bar{n}_{b}^{out}). \tag{4.2.15}$$

Further more, the variance of the photon number for output mode a, is given by

$$\begin{aligned} (\Delta n_{a}^{out})^2 \quad &= \langle (\hat{a}_{out}^{\dagger} \hat{a}_{out})^2 \rangle - (\bar{n}_{a}^{out})^2 \\ &= \langle \hat{a}_{out}^{\dagger 2} \hat{a}_{out}^{2} \rangle + \kappa \langle \hat{a}_{out}^{\dagger} \hat{a}_{out} \rangle - (\bar{n}_{a}^{out})^2. \end{aligned} \tag{4.2.16}$$

Using the Q-function described in Eq. (4.2.1) , we can write the expectation value of $\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}$ as

$$\begin{aligned}\langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle \quad &= \frac{1}{\kappa^2}(\frac{u^2-v^2}{u})\int\frac{d^2\alpha_{out}d^2\gamma_{out}}{\pi^2}exp\Bigg[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa}-\frac{\gamma^*_{out}\gamma_{out}}{\kappa}+\frac{\gamma^*_{out}\alpha_{out}}{\kappa}\\ &+q(\frac{u^2-v^2}{u})(\frac{\gamma_{out}}{\sqrt{\kappa}}+\frac{\alpha^*_{out}}{\sqrt{\kappa}})+\frac{\alpha^*_{out}\gamma_{out}}{\kappa}(1-\frac{u^2-v^2}{u})\\ &-q^2(\frac{u^2-v^2}{u})\Bigg]\gamma^{*2}_{out}\alpha^2_{out},\end{aligned} \tag{4.2.17}$$

we can rewrite the above equation as

$$\begin{aligned}\langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle \quad &= \frac{1}{\kappa^2}(\frac{u^2-v^2}{u})\int\frac{d^2\alpha_{out}}{\pi}exp\left[-\frac{\alpha^*_{out}\alpha_{out}}{\kappa}+q[\frac{u^2-v^2}{u}]\frac{\alpha^*_{out}}{\sqrt{\kappa}}-q^2[\frac{u^2-v^2}{u}]\right]\\ &\kappa^2\frac{d^2}{da^2}\int\frac{d^2\gamma_{out}}{\pi}exp\Bigg[-\frac{\gamma^*_{out}\gamma_{out}}{\kappa}+a\frac{\gamma^*_{out}\alpha_{out}}{\kappa}+\gamma_{out}(\frac{q}{\sqrt{\kappa}}[\frac{u^2-v^2}{u}]\\ &+\frac{\alpha^*_{out}}{\kappa}[1-\frac{u^2-v^2}{u}])\Bigg]_{a=1},\end{aligned} \tag{4.2.18}$$

so that performing the integration using Eq. (2.2.28), yields

$$\begin{aligned}\langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle \quad &= \frac{u^2-v^2}{u}\kappa^2 exp(-q^2[\frac{u^2-v^2}{u}])\\ &\frac{d^2}{da^2}\left[\frac{1}{(1-a[1-\frac{u^2-v^2}{u}])}\right]exp\left[\frac{q^2(\frac{u^2-v^2}{u})^2 a}{(1-a[1-\frac{u^2-v^2}{u}])}\right].\end{aligned} \tag{4.2.19}$$

Thus differentiating and applying the condition $a=1$, we obtain

$$\langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle = 2\kappa^2\left[\frac{u}{u^2-v^2}-1\right]^2+4\kappa^2q^2\left[\frac{u}{u^2-v^2}-1\right]+\kappa^2q^4, \tag{4.2.20}$$

and in view of Eqs. (3.2.43) and (3.2.44), we get

$$\langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle = 2\kappa^2(a-1)^2+4\kappa^2q^2(a-1)+\kappa^2q^4. \tag{4.2.21}$$

With the aid of Eq. (4.2.6), the above equation can be put in the form

$$\langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle = 2(\bar{n}^{out}_a)^2-\kappa^2q^4. \tag{4.2.22}$$

Now employing Eq. (4.2.22) along with (4.2.16), the variance of the photon number for output mode a, will reduces to

$$(\Delta n^{out}_a)^2 = (\bar{n}^{out}_a)^2+\kappa\bar{n}^{out}_a-\kappa^2q^4. \tag{4.2.23}$$

Following a similar procedure, one can readily estabilish that the variance of the photon number for output mode b is

$$(\Delta n_b^{out})^2 = (\bar{n}_b^{out})^2 + \kappa\bar{n}_b^{out} - \kappa^2 q^4. \tag{4.2.24}$$

On account of Eqs. (2.1.64) and (4.2.7), one finds

$$\begin{aligned}(\Delta n_a^{out})^2 \quad &= \frac{\kappa^2\gamma^2}{4\lambda_+^2}(1-e^{-\lambda_+ t})^2 + \frac{\kappa^2\gamma^2}{4\lambda_-^2}(1-e^{-\lambda_- t})^2 - \frac{\kappa^2\gamma^2}{2\lambda_+\lambda_-}(1-e^{-\lambda_+ t})(1-e^{-\lambda_- t})\\ &+\frac{4\gamma\varepsilon^2}{\lambda_-}(1-e^{-\lambda_- t})(1-e^{-\kappa t/2})^2 - \frac{4\gamma\varepsilon^2}{\lambda_+}(1-e^{-\lambda_+ t})(1-e^{-\kappa t/2})^2\\ &+\frac{\kappa^2\gamma}{2\lambda_-}(1-e^{-\lambda_- t}) - \frac{\kappa^2\gamma}{2\lambda_+}(1-e^{-\lambda_+ t}) + 4\varepsilon^2(1-e^{-\kappa t/2})^2,\end{aligned} \tag{4.2.25}$$

thus at steady state, we observe that

$$(\Delta n_a^{out})^2_{ss} = 2\kappa^2\gamma^2\left[\frac{\kappa^2-2\gamma^2}{(\kappa^2-4\gamma^2)^2}\right] + \frac{4\kappa^2\varepsilon^2}{\kappa^2-4\gamma^2}, \tag{4.2.26}$$

and

$$(\Delta n_b^{out})^2_{ss} = 2\kappa^2\gamma^2\left[\frac{\kappa^2-2\gamma^2}{(\kappa^2-4\gamma^2)^2}\right] + \frac{4\kappa^2\varepsilon^2}{\kappa^2-4\gamma^2}. \tag{4.2.27}$$

On the other hand employing the Q-function (4.1.36), we have for the expectation value of $\hat{n}_a^{out}\hat{n}_b^{out}$

$$\begin{aligned}\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle \quad &= \left[\frac{u^2-v^2}{\kappa^2}\right]exp[-2q^2(u+v)]\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-u(\frac{\alpha^*_{out}\alpha_{out}}{\kappa}\\ &+\frac{\beta^*_{out}\beta_{out}}{\kappa}) - v(\frac{\alpha_{out}\beta_{out}}{\kappa} + \frac{\alpha^*_{out}\beta^*_{out}}{\kappa}) + q(u+v)(\frac{\alpha_{out}}{\sqrt{\kappa}} + \frac{\alpha^*_{out}}{\sqrt{\kappa}} + \frac{\beta_{out}}{\sqrt{\kappa}}\\ &+\frac{\beta^*_{out}}{\sqrt{\kappa}})\bigg](\alpha^*_{out}\alpha_{out}\beta^*_{out}\beta_{out} - \kappa\alpha^*_{out}\alpha_{out} - \kappa\beta^*_{out}\beta_{out} + \kappa^2).\end{aligned} \tag{4.2.28}$$

One can rewrite this as

$$
\begin{aligned}
\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle \quad &= [u^2-v^2]exp[-2q^2(u+v)]\bigg(\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-a\frac{u\alpha^*_{out}\alpha_{out}}{\kappa}\\
&-b\frac{u\beta^*_{out}\beta_{out}}{\kappa}-v(\frac{\alpha_{out}\beta_{out}}{\kappa}+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})+q(u+v)(\frac{\alpha_{out}}{\sqrt{\kappa}}+\frac{\alpha^*_{out}}{\sqrt{\kappa}}+\frac{\beta_{out}}{\sqrt{\kappa}}+\frac{\beta^*_{out}}{\sqrt{\kappa}})\bigg]\\
&+\frac{1}{u}\frac{d}{da}\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-a\frac{u\alpha^*_{out}\alpha_{out}}{\kappa}-b\frac{u\beta^*_{out}\beta_{out}}{\kappa}-v(\frac{\alpha_{out}\beta_{out}}{\kappa}\\
&+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})+q(u+v)(\frac{\alpha_{out}}{\sqrt{\kappa}}+\frac{\alpha^*_{out}}{\sqrt{\kappa}}+\frac{\beta_{out}}{\sqrt{\kappa}}+\frac{\beta^*_{out}}{\sqrt{\kappa}})\bigg]\\
&+\frac{1}{u}\frac{d}{db}\int\frac{d^2\alpha_{out}d^2\beta_{out}}{\pi^2}exp\bigg[-a\frac{u\alpha^*_{out}\alpha_{out}}{\kappa}-b\frac{u\beta^*_{out}\beta_{out}}{\kappa}-v(\frac{\alpha_{out}\beta_{out}}{\kappa}\\
&+\frac{\alpha^*_{out}\beta^*_{out}}{\kappa})+q(u+v)(\frac{\alpha_{out}}{\sqrt{\kappa}}+\frac{\alpha^*_{out}}{\sqrt{\kappa}}+\frac{\beta_{out}}{\sqrt{\kappa}}+\frac{\beta^*_{out}}{\sqrt{\kappa}})\bigg]\bigg)_{a=b=1}+\kappa^2, \qquad (4.2.29)
\end{aligned}
$$

so that performing the integration employing Eq. (2.2.28), we obtain

$$
\begin{aligned}
\langle \hat{n}_a^{out}\hat{n}_b^{out}\rangle \quad &= [u^2-v^2]exp[-2q^2(u+v)]\bigg(\kappa^2\frac{1}{u^2}\frac{d}{da}\frac{d}{db}\bigg[\frac{1}{abu^2-v^2}\bigg]X\\
&exp\bigg[q^2(u+v)^2\frac{ua+ub+2v}{abu^2-v^2}\bigg]+\frac{\kappa^2}{u}\frac{d}{da}\bigg[\frac{1}{abu^2-v^2}\bigg]X\\
&exp\bigg[q^2(u+v)^2\frac{ua+ub+2v}{abu^2-v^2}\bigg]+\frac{\kappa^2}{u}\frac{d}{db}\bigg[\frac{1}{abu^2-v^2}\bigg]X\\
&exp\bigg[q^2(u+v)^2\frac{ua+ub+2v}{abu^2-v^2}\bigg]\bigg)_{a=b=1}+\kappa^2, \qquad (4.2.30)
\end{aligned}
$$

Thus carrying out the differentiation and applying the condition $a=b=1$, there follows

$$
\begin{aligned}
\langle \bar{n}_a^{out}\bar{n}_b^{out}\rangle \quad &= \kappa^2\bigg[\frac{2u^2}{(u^2-v^2)^2}+\frac{2q^2u}{u^2-v^2}-\frac{2q^2v}{u^2-v^2}-\frac{2u}{u^2-v^2}-\frac{1}{u^2-v^2}\\
&-2q^2+q^4+1\bigg]. \qquad (4.2.31)
\end{aligned}
$$

And in view of Eqs. (3.2.43) and (3.2.44), one obtains

$$\langle \bar{n}_a^{out}\bar{n}_b^{out}\rangle = \kappa^2\bigg[a^2+b^2+2q^2(a-b)-2q^2-2a+q^4+1\bigg], \qquad (4.2.32)$$

on the basis of Eq. (4.2.6), we can put the above equation as

$$\langle \bar{n}_a^{out}\bar{n}_b^{out}\rangle = (\bar{n}_a^{out})^2+\kappa^2[b^2-2bq^2]. \qquad (4.2.33)$$

We have obtained that, the mean and variance of the photon number for output mode a are the same as that of output mode b. Therefore, the variance of the output photon number sum and difference defined by Eq. (4.2.15), can be rewritten as

$$(\Delta n_{\pm}^{out})^2 = 2(\Delta n_a^{out})^2 \pm 2(\langle n_a^{out} n_b^{out}\rangle - (\bar{n}_a^{out})^2). \tag{4.2.34}$$

Introducing Eqs. (4.2.23) and (4.2.33), one obtains for the variance of the output photon number sum

$$(\Delta n_{+}^{out})^2 = 2[\bar{n}_a^{out2} + \kappa\bar{n}_a^{out} - \kappa^2 q^4 + \kappa^2 b^2 - 2\kappa^2 b q^2], \tag{4.2.35}$$

and in view of Eqs. (2.1.64), (3.2.40) and (4.2.7), we readily get

$$\begin{aligned}(\Delta n_{+}^{out})^2 \quad &= \kappa^2\left[\frac{\gamma^2}{\lambda_+^2}(1-e^{-\lambda_+ t})^2 + \frac{\gamma^2}{\lambda_-^2}(1-e^{-\lambda_- t})^2 - \frac{16\gamma\varepsilon^2}{\lambda_+\kappa^2}(1-e^{-\lambda_+ t})(1-e^{-\kappa t/2})^2\right] \\ &+\kappa^2\left[\frac{\gamma}{\lambda_-}(1-e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1-e^{-\lambda_+ t}) + \frac{8\varepsilon^2}{\kappa^2}(1-e^{-\kappa t/2})^2\right].\end{aligned} \tag{4.2.36}$$

Thus, at steady state we observe that

$$(\Delta n_{+}^{out})^2_{ss} = 2\kappa^2\gamma^2\left[\frac{3\kappa^2 - 4\gamma^2}{(\kappa^2-4\gamma^2)^2}\right] + \frac{8\kappa^2\varepsilon^2}{\kappa(\kappa+2\gamma)}. \tag{4.2.37}$$

On the otherhand, the variance of the output photon number difference is found to be

$$(\Delta n_{-}^{out})^2 = 2[\bar{n}_a^{out2} + \kappa\bar{n}_a^{out} - \kappa^2 q^4 - \kappa^2 b^2 + 2\kappa^2 b q^2], \tag{4.2.38}$$

and in view of Eqs. (2.1.64), (3.2.40) and (4.2.7), we see that

$$\begin{aligned}(\Delta n_{-}^{out})^2 \quad &= \kappa^2\left[\frac{16\gamma\varepsilon^2}{\lambda_-\kappa^2}(1-e^{-\lambda_- t})(1-e^{-\kappa t/2})^2 - \frac{2\gamma^2}{\lambda_-\lambda_+}(1-e^{-\lambda_- t})(1-e^{-\lambda_+ t})\right] \\ &+\kappa^2\left[\frac{\gamma}{\lambda_-}(1-e^{-\lambda_- t}) - \frac{\gamma}{\lambda_+}(1-e^{-\lambda_+ t}) + \frac{8\varepsilon^2}{\kappa^2}(1-e^{-\kappa t/2})^2\right].\end{aligned} \tag{4.2.39}$$

At steady state this reduces to

$$(\Delta n_{-}^{out})^2_{ss} = \frac{2\kappa^2\gamma^2}{\kappa^2-4\gamma^2} + \frac{8\kappa^2\varepsilon^2}{\kappa(\kappa-2\gamma)}. \tag{4.2.40}$$

Chapter 5

Conclusion

In chapter two and three of this thesis we have obtained c-number Langevin equations for the two-mode coherent and subharmonic cavity light beams, using the pertinent master equation following the procedure discussed in ref.[1]. With the help of the c-number Langevin equations we have determined the antinormally-ordered characterstics function and then we have determined the Q-functions for these cavity light beams.

With the aid of the input-output relation, we determined the Q-functions for the two-mode coherent and subharmonic output light. Then employing the pertinent Q-functions we determined the output photon statistics and we have found that the variance of the photon number for the two-mode coherent output light is equal to kappa square times the mean cavity photon number. And we have also found that the output signal and idler modes are separately in a chaotic states and the output photon number statistics for the two-mode coherent output light is Poissonian and that of the subharmonic output light is Super Poissonian.

Furthermore, we have determined the Q-function for the superposed two-mode coherent and subharmonic output light. With the aid of this Q-function we have calculated the output photon statistics and we have found that the mean of the output photon number for the superposed two-mode coherent and subharmonic output light is the sum of the individual mean output photon number. We have also seen that the mean of the output photon number difference vanishes.

References

[1] Fesseha Kassahun, Fundamentals of Quantum Optics (Lulu, North Carolina, 2008).

[2] D. F. Walls and G. J. Milburn, Quantum Optics (Springer-Verlege, Berlin, 1994).

[3] B.Daniel and K.Fesseha,opt.commun.384(1998)

[4] C. C. Gerry and P. L. Knight, Introductory Quantum Optics (Cambridge University press, Cambridge, 2004).

[5] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge University press, Cambridge, 1997).

[6] A. M. Fox, Quantum Optics: *An Introduction* (Oxford University Press, New York, 2006).

[7] P. Meystre and M.Sargent III, Elements of Quantum Optics (Springer-Verlag, Berlin, 1991).

[8] L. Manedel and E.Wolf, Optical coherence and quantum optics (Cambridge University press, Cambridge, 1995).

[9] R. R. Puri, Mathematical Methods of Quantum Optics (Springer, Berlin, 2001).

[10] W. Vogel and D. G Welsch, Quantum Optics (Wiley-VCH, New York, 2006).

[11] Yibeltal Siyum, MSc Thesis (Addis Ababa University, 2011).

Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

Name: Menwuyelet Melaku

Signature:– – – – – – – – – – – – –

This thesis has been submitted for examination with my approval as University advisor.

Name: Dr. Fesseha Kassahun

Signature:– – – – – – – – – – – –

Place and time of submission:

Department of Physics
Addis Ababa University
Feb, 2012