SOLUTION OF NONLINEAR WAVE EQUATIONS USING NATURAL DECOMPOSITION METHOD

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Declaration

In partial fulfillment of the requirements for the Master of Science in Mathematics degree, this thesis was submitted to the department of Mathematics at the College of Natural and Computational Sciences at Addis Ababa University. Consequently, I hereby declare that this thesis was written entirely by me and that no portion of it was submitted to any other location or institution in order to receive a degree or academic qualification.

Name of candidate: ……………………… Signature: …………… Date: …………………

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This is to Certify that this thesis was prepared by Henok Adane Adisu, entitled: on natural decomposition method for solution of nonlinear wave equations in Department of Mathematics, Addis Ababa University, under my supervision. I hear by also confirmed that I have read all the part of the thesis, and so it can be submitted for evaluation by examiners and eventual for defense.

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This is to certify that this thesis prepared by Henok Adane Adisu, entitled on natural decomposition method for solution of nonlinear wave equations and submitted in partial fulfillment for the requirements of the degree of Master of Science in Mathematics compiled with the regulations of the University and meets the accepted standards with respect to originality and quality

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Nomenclature

ODES	Ordinary differential equations
PDE	Partial differential equation
BVP	Boundary value problem
IVPS	Initial value problems
ADM	Adomian Decomposition method
NTM	Natural Transform method
NDM	Natural Decomposition method

ABSTRACT

In this thesis, the Natural Decomposition Method, a combination of the Natural Transform Method and the Adomian Decomposition Method is used to obtain solutions to nonlinear wave equations. The method yields an exact solution in a form of a rapidly convergent series with easily computable components. The results we obtained have been compared with the approximate solution of nonlinear wave equations with other solutions by the methods of NDM and ADM. The natural decomposition method works perfectly for nonlinear wave equations.

CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND

An equation that relates a function and its derivatives is known as a differential equation. The functions might be representations of wave equations, heat equations, or other physical phenomena.

The study of differential equations is important in many fields, such as biology, physics, engineering, and economics.

Either a partial differential equation or an ordinary differential equation is a differential equation. A partial differential equation is an equation that involves partial derivatives of a dependent variable with respect to two or more independent variables.

1.2 WAVE EQUATION

A wave is a disturbance of a field that transmits energy from one place to another. It is commonly used to model wind, water, and string vibrations. In mathematics, physical sciences, and related fields, a wave is defined as a disturbance of a field in which a physical attribute oscillates repeatedly at each point or propagates from each point to neighboring points, or appears to move through space.

Let a vibrating string's modest displacement at any point at location x at time t be defined by the function u(x,t). The equation for waves $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$ contains the term u_{tt} and u_{xx}. u_{xx} describes how the shape of the wave evolves along the horizontal axis (spatial dimension), and u_{tt}describes how the motion of the wave evolves over time.

In order to simulate a vibrating string, [8] introduced and examined the one-dimensional wave equation in. His work was expanded to include two- and three-dimensional wave equations .

The majority of problems in nearly all branches of science and engineering may be expressed as linear or nonlinear partial or ordinary differential equations, and solving these kinds of equations is very important.

Ordinary or partial differential equations can be used to explain the majority of phenomena that occur in mathematical physics and engineering domains. Differential equations used in physics to describe the phenomenon of wave propagation. Differential equations describe the dispersion of a chemically reactive substance. Furthermore, partial differential equations and/or ordinary differential equations govern the majority of physical processes in the fields of fluid dynamics, quantum mechanics, electricity, plasma physics, shallow water wave propagation, and many other models within their respective domains of validity.

Partial differential equations are a helpful tool for explaining natural events in scientific and engineering models [7]. As a result, it becomes more and more crucial to understand how to apply all of the established as well as cutting-edge approaches for solving partial differential equations. Numerous issues in real-world phenomena can be described using several kinds of differential equations. For instance, in order to simulate issues that arise in the domains of science and engineering, ordinary differential equations are extensively employed in classical mechanics.

Since most issues we come across are essentially nonlinear, most of the differential equations that describe them are likewise nonlinear; if a problem is linear, its corresponding differential equation is also linear; if the problem we are looking at is nonlinear, the differential equation that is derived is generally nonlinear.

The method of solving nonlinear problems has been a major area of focus. The study of nonlinear problems is very important in the fields of engineering and physics. Recently, analytical methods and approximate numerical methods have been applied to solve nonlinear problems. However, solving nonlinear problems analytically is not easy. Various methods have been applied, such as the Adomian decomposition method and others, to obtain the exact or approximate solutions for linear and nonlinear differential equations. Some of these methods use transformation to reduce the equations to more simple equations or even systems of equations, while others offer the solution in the form of a series that converges to the exact solution. In addition, some other methods which employ a trial function in an iterative scheme converging quickly.

Because linear differential equations are the primary objective of most mathematical approaches, George Adomian (1970) introduced an analytical method known as the Adomain decomposition method. A large class of linear and nonlinear differential and integral problems can be handled using the approach with ease.

There are several uses for the Adomian decomposition method of solution, one of which is the resolution of integral and algebraic differential equations. One of the most helpful methods for resolving mathematical differential equations is the integral transform approach.

The Laplace and Sumudu transform method is linked to the natural transform method[11]. The natural decomposition method, which combines the natural transform and Adomian decomposition methods, is utilized to solve ordinary and partial differential equations. The natural decomposition method yields dependable outcomes for nonlinear methods; the provided solution quickly approaches the exact solution [2].

1.3 LINEAR AND NONLINEAR DIFFERENTIAL EQUATION

- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ is considered a linear wave equation since the dependent variable's degree is one and its coefficient is constant.
- $\frac{\partial^2 u}{\partial t^2} = u\frac{\partial^2 u}{\partial x^2}$ is a nonlinear wave equation, the coefficient of the derivative of dependent variable is not constant, rather the coefficient is the dependent variable.
- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u^2$ is a nonlinear, u^2 is not in linear form.
- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + t^2$is linear,$t^2$ does not affect, as it is a product of independent variable.

1.4 ADOMIAN DECOMPOSITION METHOD

Ordinary and partial nonlinear differential equations are solved using the Adomian decomposition method (ADM). The method was developed by George Adomain from (1970)'s to (1990)'s.[9]

Consider the following equation

$$Lu + Nu + Ru = g$$

where, R is the remaining linear portion, N is a nonlinear operator, and L is a linear operator. Assuming that L^{-1} exists, we define the inverse operator of L as follows:

$$u = L^{-1}g - L^{-1}Nu - L^{-1}Ru$$

The Adomain decomposition method works under the assumption that an infinite series of the type can be used to describe the unknown function u.

$$u = \sum_{n=0}^{\infty} u_n$$

where a recursive determination of component u_n will be made.

More specifically, the ADM believes that an endless sequence of polynomials provided by can decompose the nonlinear operator N (u).

$$N(u) = \sum_{n=0}^{\infty} A_n$$

where A_n are Adomain polynomials.

When it comes to certain highly accurate terms, the ADM provides a convergent series of estimations. The decomposition methodology and consideration of numerous ordinary and partial differential equations provide a method for solving nonlinear BVPs in a limited number of measurements.

Instead of merely linearizing the system, this approach enables the convergence of the solution for the nonlinear part of the equation.

The ADM works by dividing the equation into its linear and nonlinear parts. The approximate solution to the problem is obtained by applying the highest order derivative of the linear portion to the corresponding integral. Depending on whether the problem involves an initial or boundary value, the initial or boundary condition determines the integration constant.

The nonlinear part of ADM method is calculating Adomain polynomials for nonlinear polynomials.

In this section, we could obtain the Adomian general formula for Adomian polynomials.

The decomposition method decomposes the solution $u(x)$ and the non-linearity $N(u)$ into series

$$u(x) = \sum_{n=0}^{\infty} u_n, \qquad N(u) = \sum_{n=0}^{\infty} A_n$$

where, A_n, n = 0, 1, 2 …are the Adomian Polynomials.

Among its benefits is its ability to solve nonlinear problems without the need for linearization or perturbation, as well as the fact that it involves fewer calculations than more conventional methods. Furthermore, it provides a series of analytical solutions that typically converge quite quickly for the majority of situations. The Adomian decomposition approach for nonlinear equations generates the polynomials, A_n for each non-linearity by substituting a unique series known as Adomian polynomials , A_n for the nonlinear term. [5]

1.5. STATEMENT OF THE PROBLEM

This study focuses on the Natural Decomposition Method for nonlinear wave equations. The researcher has a background idea on the influence of linear wave equations in different methods like separable equation and exact equation. Linear differential equation can be solved by elementary methods such as separable equation, exact equation, integrating factor etc. But most of nonlinear differential equations cannot be solved by such elementary methods. Recently, for the nonlinear models many researchers have been proposing different approximation methods like variation iteration method, perturbation methods, linearization methods etc. To study the solution of non-linear wave equations by the method of natural decomposition method, the present study aims to examine the solution of non-linear wave equations. Therefore, natural decomposition method is considered in this study for the solution of and nonlinear wave equations. The study also aims to find solution of non-linear wave equations using natural decomposition method. Natural decomposition method is used to solve linear and nonlinear differential equations.

1.6 OBJECTIVE OF THE STUDY

1.6.1 GENERAL OBJECTIVE

To apply the natural decomposition method to find the solution of wave equations

1.6.2 SPECIFIC OBJECTIVES

The specific objective of this study is: To find series form of solution for the non-linear wave equation.

1.7 SIGNIFICANCE OF STUDY

This research can be applied to the solution of nonlinear partial differential equations in several scientific and engineering domains. Nonlinear equations, and specifically nonlinear wave equations, can be solved with it. Additionally, the study's result might be used as a reference by researchers and students studying applied mathematics.

CHAPTER TWO

METHODOLOGY

Ordinary differential equations are used throughout engineering mathematics and science to describe how physical quantities change. Systems of ordinary differential equations may have many solutions, commonly a solution of linear and nonlinear components wave equation. A solution of interest is determined by specifying the solution of all its components at a single point. This point defined as nonlinear terms and linear terms, nonlinear terms determined by Adomian polynomial.

But the procedure focused on nonlinear components (terms). Numerical and Analytical methods that are designed to handle nonlinear differential equations have been developed. The problem is to determine the effectiveness of natural decomposition method.

The procedure that followed in solving the problem involves the developing ordinary and partial differential equations, identify linear wave equations or nonlinear wave equations after that using natural decomposition method, Adomian decomposition method, Natural Transforms and the Conversion to Sumudu and Laplace Transform.

In this part, we would take a brief look at differential equations and understand the basic terminologies that go with it. We take in-depth look at the algorithm derivation of the Adomian decomposition method, Natural Transform Methods and Natural Decomposition Methods. The approach computes the nonlinear heat and nonlinear wave equations.

The ADM is a semi analytical method for solving ordinary and partial linear and nonlinear differential equations.

The steps that followed in solving the problem involve formulating the mathematical systems of the flow problem. The formulated equations are nonlinear differential equations.

CHAPTER THREE

LITERATURE REVIEW

In compressible media like fluids and solids, waves exist. A restoring force is the primary requirement for the existence of the waves. Since there is only one restoring force in fluids in relation to volume changes, the waves are compression/expansion waves. These waves are referred to as longitudinal waves because the media displacement in them is co-linear with the wave's propagation direction. There is also a restoring force in solids when it comes to a media element's shape shift.

More significant mathematical models in quantum field theory with nonlinear optics and plasma physics are the solutions to linear and nonlinear wave equations. It can be found in both linear and nonlinear forms in physics, and the NDM approach has solved it. It is helpful in explaining the phenomenon of scattered waves.

The exact solutions, if applicable for nonlinear systems or the approximate closed form of the solution are found using the natural decomposition method. The approach is based on the Natural transform method and a modified version of Wazwaz's Adomian decomposition method from 2005. As demonstrated on a few nonlinear Klein-Gordon equations with quadratic non-linearity, this innovative method reduces the computational amount of the work compared to previous methods by giving precise solutions only in two iterations by computing a single Adomian polynomial [13].

Additionally, Adio (2016) suggested the use of the natural decomposition method to solve two-dimensional nonlinear Brussel at equations using initial conditions. The technique, which combines the Adomian Decomposition Method with the Natural Transform Method, yields an exact solution in the form of a quickly convergent series with easily calculable components. The application goes on to show how accurate and efficient it is in comparison to a few other well-known methods. The technique works well and can be applied to a wide range of additional mathematical physics NPDEs. This demonstrates unequivocally that the Natural Decomposition Method greatly advances the field compared to the current method.

In order to solve linear and nonlinear Schrodinger equations without the need for linearization, discretization, transformation, or the adoption of certain limiting assumptions, Maitama (2014) presented a novel computer methodology known as the natural decomposition method. Two illustrative instances have exact solutions determined, and the outcomes are contrasted using the Variation Iteration Method and the Adomian Decomposition method. This demonstrates how the new strategy is accurate, simple, and efficient. Therefore, many linear and nonlinear Schrödinger equations, as well as related applications in applied mathematics and engineering, can be solved with ease using the natural decomposition method.

NDM was utilized to solve partial differential equations, and it turned out to be a helpful tool for characterizing these natural occurrences in scientific and engineering models. As a result, it becomes more and more crucial to understand how to apply all of the established as well as cutting-edge approaches for solving partial differential equations.[12]

George Adomian invented the Adomian decomposition method in the 1980s, which [14] used to solve both linear and nonlinear differential equations. A large class of differential and integral equations, both linear and nonlinear, can be handled with ease by the algebraic decomposition approach.

CHAPTER FOUR

MATHEMATICAL FORMULATIONOF WAVE EQUATION

4.1 GENERAL DESCRIPTION OF ADM

Consider the general equation

$$\text{Lu} + \text{Nu} + \text{Ru} = \text{g} \quad (1)$$

If L is a higher order linear differential operator that is easily invertibl and its inverse is L^{-1} which implies integral, and u is the unknown function, then L is both differentiable and integral. Assume that g is a given function (source), N is the nonlinear operator, R is the remaining linear portion. . This would make it an integral operator.

Take L^{-1} to both sides of (1)to get:

$$L^{-1}(\text{Lu} + \text{Nu} + \text{Ru} = \text{g}) \quad (2)$$

$$L^{-1}\text{Lu} = L^{-1}\text{g} - L^{-1}\text{N (u)} - L^{-1}\text{R (u)}$$

Thus,

$$\text{u} - \phi = L^{-1}\text{g} - L^{-1}\text{N(u)} - L^{-1}\text{R(u)}$$

where, ϕ is presented from the initial conditions or from the boundary conditions or both, it depends on how we choose differential operator that solve the given problem. The ADM assumes that solution u of the functional equation can be decomposed into infinite series.

$$\text{u} = \sum_{n=0}^{\infty} u_n \quad (3)$$

and the nonlinear term N(u) can be written as infinit series

$Nu = \sum_{n=0}^{\infty} A_n$,where$A_n$ is Adomian polynomial

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2 \ldots \quad (4)$$

Where the A_n's are the Adomian polynomials. By substitution this gives

$$\sum_{n=0}^{\infty} u_n = \phi + L^{-1} g - L^{-1} \sum_{n=0}^{\infty} A_n - L^{-1} \sum_{n=0}^{\infty} R(u_n) \quad (5)$$

We can now derive the solution algorithm as follows from equation (5)

$$u_0 = \phi + L^{-1}g, u_{n+1} = -L^{-1}(A_n + Ru_n),\ n = 0, 1, 2, .. \tag{6}$$

given, u_0 the other terms of u can be determined respectively. If one term of u_n is equal zero then the following terms are all zeros.

The remaining terms of u can be found in accordance with u_0. The following terms are all zeros if one term in $u_n$equals zero.

Algorithm of Solution

It would be demonstrated how the solution algorithm (6) was selected. The nonlinear functional equation should be considered.

$$u - N(u) = f \tag{7}$$

where N is the nonlinear operator and f is a differential equation determined after applying L^{-1} to the source function g.Suppose that the solution of (7) is a family of

$$u = \sum_{n=0}^{\infty} u_n \tag{8}$$

where λ is a parameter. Suppose that the radius of convergent ρ of the series above is greater than one, so the series converges for | λ |< ρ, where ρ > 1. As we showed previously the nonlinear function N (u) can be expanded in infinite series

$$N(u) = \sum_{i=0}^{\infty} \alpha_i u^i \tag{9}$$

where α_i are the coefficients of non-linear term, with radius of convergence $\rho_0 > 1$, this implies that the series above converges for $|u| < \rho_0$. In general, it can suppose that $\rho_0 = \infty$ because in practical applications the nonlinear operator N (u) is a polynomial or a nonlinear function admitting an entire series converging for any u with $|u| < \infty$. Now, by substituting (8) in (9) we get

$$\text{(a)}\ N\left(\sum_{n=0}^{\infty} u_n\lambda^n\right) = \sum_{i=0}^{\infty} \alpha_i\left(\sum_{n=0}^{\infty} u_n\,\lambda^n\right)^i \tag{10}$$

Let $\sum_{i=0}^{\infty} \alpha_i\left(\sum_{n=0}^{\infty} u_n\right)^i = \sum_{i=0}^{\infty} A_i$, then the above series is equivalent to

$$\text{(b)}\ N(u) = \sum_{i=0}^{\infty} A_i\lambda^i = A_0 + A_1\lambda + A_2\lambda^2 + A_3\lambda^3 + \ldots \tag{11}$$

From above, we can conclude that the values of A_i's depend only on the values of u_i's.

By substituting (11) and (9) in (7) and make λ = 1, we obtain, because of the convergence of the two series:

$$\sum_{n=0}^{\infty} u_n - \sum_{i=0}^{\infty} A_i = f \tag{12}$$

This equation is satisfied if, $u_0 = f$

4.2 ADOMIAN POLYNOMIALS

ADM method's primary function is to compute Adomian polynomials for nonlinear polynomials.

For Adomian polynomials, we will find the Adomian general formula in this section.

In addition, a bifurcation pattern that is unique to the NLDE is shown in the solution. The non-linearity N (u) and the solution u = u(x) are broken down into series using the decomposition method.

$$u(x) = \sum_{n=0}^{\infty} u_n, \qquad N(u) = \sum_{n=0}^{\infty} A_n \tag{13}$$

where, A_n are the Adomian polynomials andu_n is an infinite series.

To compute A_n take $N(u) = f(u)$ to be a nonlinear function in u, where $u = u(x)$, and consider the Taylor series expansion $f(u)$ of aroundu_0.

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \frac{1}{2!} f''(u_0)(u - u_0)^2 + \frac{1}{3!} f'''(u_0)(u - u_0)^3 + \ldots \tag{14}$$

But $u = u_0 + u_1 + u_2 + \ldots$then,

$$f(u) = f(u_0) + f'(u_0)(u_1 + u_2 + u_3 + \ldots) + \frac{1}{2!} f''(u_0)(u_1 + u_2 + u_3 + \ldots)^2 + \frac{1}{3!} f'''(u_0)(u_1 + u_2 + u_3 + \ldots)^3 \tag{15}$$

We must first rearrange and reorder the terms before expanding each term to get the Adomian polynomials. In fact, it is necessary to ascertain the sequence in which each term appears in the aforementioned series, as this is contingent upon the subscripts and exponents of the $u_n's$. As an example, u_1 is of order 1, u_{12} is of order 2, u_{23} is of order 6, and so on. Generally speaking,u_{nk} is of order kn. If a given term involves multiplying $u_n's$, then the total of the terms of the $u_n's$ in each term determines the order of that term. As an illustration, $u_{23}u_{12}$is of order 8.

Consequently, by rearranging the terms in the expansion above in the specified order, we have.

$$f(u) = f(u_0) + f^{'}(u_0)u_1 + f^{'}(u_0)u_2 + \frac{1}{2!}f^{''}(u_0){u_1}^2 + f^{'}(u_0)u_3 + \frac{2}{2!}f^{''}(u_0)u_{1+}u_2\frac{1}{3!}f^{'''}(u_0){u_1}^3 + f^{'}(u_0)u_4 + \frac{1}{2!}f^{''}(u_0){u_2}^2 + \frac{2}{2!}f^{''}(u_0)u_1u_3 + \frac{3}{3!}f^{'''}(u_0){u_1}^2u_2 + \ \ldots \tag{16}$$

The polynomials Adomian all terms in the expansion of order 1 make up A_1, all terms in the expansion of order 2 make up A_2, and so on. Generally speaking, all terms of order 'n' comprise A_n. Consequently, the following is a list of the first five terms of Adomian polynomials:

$$A_0 = f(u_0),$$

$$A_1 = u_1f^{'}(u_0),$$

$$A_2 = u_2f^{'}(u_0) + \frac{1}{2!}{u_1}^2f^{''}(u_0),$$

$$A_3 = u_3f^{'}(u_0) + \frac{2}{2!}u_1u_2f^{''}(u_0) + \frac{1}{3!}{u_1}^3f^{'''}(u_0)$$

$$A_4 = u_4f^{'}(u_0) + \left[\frac{1}{2!}{u_2}^2 + u_1u_3\right]f^{''}(u_0) + \frac{1}{2!}{u_1}^2u_2f^{'''}(u_0) + \frac{1}{4!}{u_1}^4f^{''''}(u_0)$$

$$\vdots$$

Hence, $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0},\ n \geq 0.$ (17)

These polynomials will be computed as follows in order to determine A_n's using the Adomian general formula:

$$A_0 = N(u_0)$$

$$A_1 = N'(u_0)u_1 = \frac{d}{d\lambda}N(u_0 + \lambda u_1)_{\lambda=0}$$

$$A_2 = N^{'}(u_0)u_2 + \frac{1}{2!}N^{''}(u_0){u_1}^2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}N\,(u_0 + \lambda u_1 + \lambda^2u_2)_{\lambda=0}$$

$$A_3 = N^{'}(u_0)u_3 + \frac{2}{2!}N^{''}(u_0)u_1u_2 + \frac{1}{3!}N^{'''}(u_0){u_1}^3$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}N(u_0 + \lambda u_1 + \lambda^2u_2 + \lambda^3u_3)_{\lambda=0}$$

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$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n \geq 0.$$

4.3 ANALYSIS OF THE ADM

As well, the ADM consist of decomposing the unknown function u = $u(x, y)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u\,(x, y) = \sum_{i=0}^{\infty} u_n\,(x, y) \tag{18}$$

where, the components $u_n(x, y),\ n \geq\ 0$ are to be determined in a recursive manner.

These components' determinant can be obtained in a simple manner using a cursive relation, which typically requires basic integrals. In an abstract formulation, this technique is very simple, but the challenge lies in figuring out the Adomian polynomials and demonstrating the function's series' convergence.

The steps involved in the ADM are as follows: dividing the given equation into linear and nonlinear components; inverting the highest-order derivative operator present in the linear operator on both sides; determining the initial and/or boundary conditions; breaking down the unknown function into a series whose components must be ascertained; breaking down the nonlinear function into terms of special polynomials known as Adomian polynomials; and utilizing Adomian polynomials to find the successive terms of the series solution by recurrent relation..

Infinite series with easily determinable terms that converge rapidly to a precise solution is found to be the solution.

4.4 ADM FOR ORDINARY DIFFERENTIAL EQUATIONS

Considered here is a differential equation that provides a concise overview of ADM.

$$F\big(u(x)\big) = g(x)$$

where, g is a given function, and F is a general non-linear ordinary differential operator with both linear and nonlinear terms.

F(u(x))'s linear terms can be broken down into L(u)+R(u), where R is the linear operator's remainder and L is an easily invertible operator that is considered the highest order derivative.

Consequently, it can be expressed as:

$$Lu + Ru + Nu = g$$

where, Nu represents the non-linear terms in $F\big(u(x)\big)$.

Applying the inverse operator L^{-1} on both sides yields.

$$u = \varphi_0 + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu)$$

where, $\varphi_0$the constant of integration is satisfies the condition $L\varphi_0 \ = 0$.

Then substituting

$$\sum_{n=0}^{\infty} u_n = \varphi_0 + L^{-1}\,(\,g\,) - L^{-1}\Big(R\sum_{n=0}^{\infty} u_n\Big) - L^{-1}\Big(\sum_{n=0}^{\infty} A_n\Big) \qquad (19)$$

where,

$$\varphi_0 = \left\{ \begin{array}{cc} u(0) & if\ L = \dfrac{d}{dx} \\ u(0) + xu^{'}(0) & if\ \ L = \dfrac{d^2}{dx^2} \\ u(0) + xu^{'}(0) + \dfrac{x^2}{2!}u^{''}(0) & if\ \ L = \dfrac{d^3}{dx^3} \\ u(0) + xu^{'}(0) + \dfrac{x^2}{2!}u^{''}(0) + \ldots + \dfrac{x^n}{n!}u^{(n)} & if\ \ L = \dfrac{d^{n+1}}{dx^{n+1}} \end{array} \right\}$$

Therefore, each terms of series is given by the recurrent relation

$$\mathrm{u_0} = \varphi_0 + \mathrm{L}^{-1}(\,\mathrm{g})$$

$$\mathrm{u_1} = -\mathrm{L}^{-1}\mathrm{Ru_0} - \mathrm{L}^{-1}\mathrm{A_0}$$

$$\mathrm{u_2} =- \mathrm{L}^{-1}\mathrm{Ru_1}\_\mathrm{L}^{-1}\mathrm{A_1}$$

$u_{n+1} = - L^{-1}Ru_n - L^{-1}A_n$, $n \geq 0$ in general.

4.5 NATURAL TRANSFORM METHOD

4.6.1 DEFINITION AND BASIC THEORY

Definition: Let $f(t)$ be a function defined for all $t \geq 0$. The natural transform of the function f (t) for t ∈ (0, ∞) is defined by N [f (t)] = $R(u, s)$. Then,$R(u, s)$= N[f (t)] = $\int_0^\infty f(tu)e^{-st}$ dt, s, u ∈ (0, ∞).

$$\int_0^\infty f\,(tu)e^{-st}dt,\ s,\ u \in (0, \infty) = N[f(t)] = R(s, u), \quad [10] \tag{20}$$

where, N [f(t)] is the natural transformation of the time function f(t) and the variable s and u are the natural transform variables.

Table1: Some special Natural transforms and the Conversion to Sumudu and Laplace transform.

F(t)	N[f(t)]	τ[f(t)]	S[f(t)]
1	$\frac{1}{s}$	$\frac{1}{s}$	1
T	$\frac{u}{s^2}$	$\frac{1}{s^2}$	U
e^{Bt}	$\frac{1}{s - Bu}$	$\frac{1}{s - B}$	$\frac{1}{1 - Bu}$
$\frac{t^{n-1}}{(t-1)!} n = 1,2. \ldots$	$\frac{u^{n-1}}{s^n}$	$\frac{1}{s^n}$	u^{n-1}
Sin(t)	$\frac{u}{s^2 + u^2}$	$\frac{1}{1 + s^2}$	$\frac{u}{1 + u^2}$
$\mathbf{cos}\,(\boldsymbol{t})$	$\frac{s}{s^2 + u^2}$	$\frac{s}{1 + s^2}$	$\frac{1}{1 + u^2}$

4.6 N -TRANSFORM OF ELEMENTARY FUNCTION

4.6.1 EXPONENTIAL FUNCTION

Let f (t) =$\boldsymbol{e^{at}}$ when t ≥ 0 where a is constant, the N-Transform of this function can be written as

$$N\,(\boldsymbol{e^{at}})=\int_{\boldsymbol{0}}^{\infty} \boldsymbol{e^{aut}e^{-st}}dt = \lim_{\boldsymbol{b\to\infty}} \int_{\boldsymbol{0}}^{\boldsymbol{b}} \boldsymbol{e^{-(s-au)t}}dt$$

$$= \lim_{\boldsymbol{b\to\infty}} \left[\frac{\boldsymbol{e^{-(s-au)t}}}{\boldsymbol{s-au}}\right]\Big|_{\boldsymbol{0}}^{\boldsymbol{b}} = \frac{\boldsymbol{1}}{\boldsymbol{s-au}} \quad [11]$$

{$\frac{\boldsymbol{1}}{\boldsymbol{s-a}}$is the Laplace transform of e^b, (u=1)} (21)

{$\frac{\boldsymbol{1}}{\boldsymbol{1-bu}}$ Sumudu transform of e^{at}, (s=1)} (22)

4.6.2 PROPERTIES OF N-TRANSFORM

LINEARITY PROPERTY

Theorem4.1 If a and b are any constants and f and g are functions, then

$$N\{af(t) + bg(t)\} = aN\{f(t)\} + bN\{g(t)\} \quad (23)$$

Proof:

N (f) =$\int_0^\infty f(ut)e^{-st}\ dt$, and N (g) =$\int_0^\infty g(ut)e^{-st}\,dt$, then if a and b are any constants,

N {a f(t) + b g(t)} =$\int_0^\infty e^{-st}$ {a f(t) + b g(t)} dt = a$\int_0^\infty f(ut)e^{-st}$dt +b$\int_0^\infty g(ut)\ e^{-st}\ dt$

= aN (f) + bN (g)

Example4.1 N {3t + 4cos $2t$} = 3N(t) + 4(cos $2t$) = 3$\frac{u}{s^2}$+ 4 $\frac{s}{s^2+\ \ 4u^2}$

N-TRANSFORM OF DERIVATIVE:

We can write the natural transform of partial derivatives are:

$$N\left[\frac{\partial y(x,t)}{\partial t}\right] = \frac{1}{u}[sN[\,y\,(x,t)] - y\,(x,0)]. \quad (24)$$

$$N\left[\frac{\partial^2 y(x,t)}{\partial t^2}\right] = \frac{1}{u^2}[s^2N[\,y(x,t)] - sy\,(x,0)] - \frac{y'}{u}(x,0) \quad (25)$$

$$N\left[\frac{\partial^n y\,(x,t)}{\partial t^n}\right] = \frac{1}{u^n}\left[s^n\, N\,[\,y\,(x,t)] - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}}\, y^k\,(x,0)\right] \qquad (26)$$

Theorem 4.2 If $N\{f(t)\} = R(s,u)$ then $N\{f^{'}(t)\} = \frac{s}{u}R(s,u) - \frac{f(0)}{u}$. (27)

Proof:

$$\mathrm{N}\{f^{'}(t)\} = \int_0^{\infty} e^{-st} f^{'}(ut)d = \lim_{p\to\infty}\int_0^{p} e^{-st} f^{'}(ut)dt$$

$$= \lim_{p\to\infty}\left\{ \left| \frac{e^{-st}f(ut)}{u} \right|_0^p + \frac{s\int_0^p e^{-st}}{u} f(ut)dt \right\}$$

$$= \lim_{p\to\infty}\left\{ \left| \frac{e^{-sp}f(up)}{u} - \frac{f(0)}{u} + \frac{s\int_o^p e^{-st}}{u} f(ut)dt \right\}$$

$$= \frac{s\int_o^p e^{-st}}{u} f(ut)dt - \frac{f(0)}{u} = \frac{s}{u}\mathrm{R}\ (\mathrm{s,\ u}) - \frac{f(0)}{u}$$

Theorem4.3 If $N\{f(t)\} = R(s,u)$, then $N\{f^{''}(t)\} = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f^{'}(0)}{u}$ (28)

Proof:

$$\mathrm{N}\ \{G^{'}(t)\} = \frac{s}{u}\mathrm{N}\ \{\mathrm{G}\ (\mathrm{t})\} - \frac{f(0)}{u}, \text{ let G (t) } = f^{'}\text{(t) then}$$

$$\mathrm{N}\ \{f^{''}(\mathrm{t})\} = \frac{s}{u}\mathrm{N}\ \{f^{'}(\mathrm{t})\} - \frac{f^{'}}{u}(0) \frac{s}{u}\{\frac{s}{u}\mathrm{N}\{\mathrm{f(t)}\} - \frac{f(0)}{u}\} - \frac{f^{'}(0)}{u} = \frac{s^2\mathrm{N}\{\mathrm{f(t)}}{u^2} - \frac{s}{u^2}\frac{f(0)}{u} - \frac{f^{'}(0)}{u}$$

$$= \frac{s^2}{u^2}\mathrm{R(s,u)} - \frac{s}{u^2}\frac{f(0)}{u} - \frac{f^{'}(0)}{u}.$$

N-TRANSFORM OF INTEGRAL

Theorem4.4 If $N\{f(t)\} = R(s,u)$then $N\left\{\int_0^t f(p)\,dp\right\} = \frac{u}{s}R(s,u)$

Proof:

Let G (t) $=\int_0^t f(p)dp$, then$G^{'}$= f (t)and G (0) =0.

Taking the N−Transform of both sides, we have

$$N\{G'(t)\} = \frac{s}{u}N\{G(t)\} - \frac{f(0)}{u} = \frac{s}{u}N\{G(t)\} = R(s,u)$$

this is $\frac{s}{u}N\{G(t)\} = R(s,u) = N\{G(t)\} = \frac{u}{s}R(s,u)$

Then, $N\left\{\int_0^t f(p)\,dp\right\} = \frac{u}{s}R(s,u)$

4.7 NATURAL DECOMPOSITION METHOD

Consider the general nonlinear Non-homogeneous DEs of the form

$$\mathrm{Ly}(x,t) + \mathrm{Ry}(x,t) + \mathrm{Ny}(x,t) = h(x,t) \tag{29}$$

Subject to the initial condition

$$y(x,0) = f(x)\,;\frac{\partial y}{\partial t}(x,0) = g(x)$$

Here L represents the second order linear differential operator for $L_t = \frac{\partial^2 y}{\partial t^2}$, R is the linear differential operator of less order L, N is the general nonlinear differential operator and h(x, t) is the source term. We apply the N-Transform to Eq. (28) to get:

$$N\,[Ly(x,t)] + N\,[\,Ry(x,t)] + N[\,Ny(x,t)] = N\,[h(x,t)] \tag{30}$$

Using the Property Equation,(table1) and the properties:

$$\frac{s^2}{u^2}\,R(x,s,u) - \frac{sy(x,0)}{u^2} - \frac{y'(x,0)}{u} + N\,[Ry(x,t)] + N[Ny(x,t)] = N\,[h(x,t)]$$

Substitute Eq. (29) into Eq. (30) to get:

$$\frac{s^2}{u^2}R(x,s,u) - \frac{sf(x)}{u^2} - \frac{g(x)}{u} + N\,[Ry(x,t)] + N[Ny(x,t)] = N[h(x,t)] \tag{31}$$

From Eq. (31), we get

$$R(x,s,u) = \frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}N\,[h(x,t)] - \frac{u^2}{s^2}N\,[Ry(x,t)] - \frac{u^2}{s^2}N[Ny(x,t)] \tag{32}$$

We take the inverse Natural transform of Eq. (32)

$$N^{-1}R(x,s,u) = N^{-1}\left[\frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}N\,[h(x,t)]\right] - N^{-1}\left[\frac{u^2}{s^2}N[Ry(x,t)]\right] - N^{-1}\left[\frac{u^2}{s^2}N[Ny(x,t)]\right] \tag{33}$$

From Eq. (33) we can get

$$y(x,t) = H(x,t) - N^{-1}\left[\frac{u^2}{s^2}N[Ry(\,x,t)] + \frac{u^2}{s^2}N[Ny(x,t)]\right] \qquad (34)$$

Note $H(x,t)$ is the term arising from the source term and the prescribed initial conditions. Now to deal with nonlinear term, we represent the solution in an infinite series form:

$$y\,(x,t) = \sum_{n=0}^{\infty} y_n(x,t) \qquad (35)$$

Also, the nonlinear term can be written as:

$$Ny\,(x,t) = \sum_{n=0}^{\infty} A_n \qquad (36)$$

where, A_n's are the polynomials of $u_0, u_1, u_2, u_3, \ldots, u_n, \ldots$ and can be calculated by the following formula:

$$A_n = \frac{1}{n!}\frac{d^n}{dx^n}\left[F\left(\sum_{i=0}^{n} \lambda^i y_i\right)\right],\ n = 0,1,2,\ldots \qquad (37)$$

The general formula in Eq. (19), can be simplified as follows: We first assume that the nonlinear function is F (u), the Adomian polynomials are given by:

$$A_0 = F(y_0)$$

$$A_1 = u_1 F(y_0)$$

$$A_2 = u_2 F^{'}(y_0) + \frac{1}{2!}{y_1}^2 F^{'}(y_0)$$

And so on. Now, we substitute Eq. (35) and Eq. (36) into Eq. (34) to get:

$$\sum_{n=0}^{\infty} y_n(x,t) = H(x,t) - N^{-1}\left[\frac{u^2}{s^2}N\left[R\sum_{n=0}^{\infty} y_n(x,t)\right] + \frac{u^2}{s^2}N\left[\sum_{n=0}^{\infty} A_n\right]\right] \qquad (38)$$

Comparing both sides of Eq. (37) we conclude

$$y_0(x,t) = H(x,t)$$

$$y_1(x,t) =- N^{-1}\left[\frac{u^2}{s^2}N[Ry_0(x,t)] + A_0\right]$$

$$y_2(x,t) =- N^{-1}\left[\frac{u^2}{s^2}N[Ry_1(x,t)] + A_1\right]$$

The general recursive relation is given by:

$$y_{n+1}(x,t) =- N^{-1}\left[\frac{u^2}{s^2}N[Ry_n(x,t)] + A_n\right], n \geq 1$$

Hence, from the general recursive relation in Eq. (38) we can easily compute the remaining components of $y(x,t)$as $y_1(x,t)$, $y_2(x,t)$, . . . where $y_0(x,t)$ is the given initial condition. Finally, the exact solution is given by: $y(x,t) = \sum_{n=0}^{\infty} y_n(x,t)$

4.8 APPLICATIONS

4.9.1 Applications on natural decomposition method and Adomian decomposition method for nonlinear wave equation.

Example1: The Adomian decomposition method for solving nonlinear wave equation.

$\frac{\partial^2 u}{\partial x^2} = \frac{u\partial^2 u}{\partial t^2} + 2-2(t^2 + x^2)$ with initial condition $u(0, t) = t^2$, $u_x(0, t) = 0$.

$L_{xx}u - Nu = 2 - 2(t^2 + x^2)$, where, $L_{xx} = \frac{\partial^2}{\partial x^2}$ it is inverse $\int_0^x \int_0^x (.)dsds$ and $N(u) = uu_{tt}$

$L^{-1}_{xx}L_{xx} - L^{-1}_{xx}N(u) + L^{-1}_{xx}(2 - 2(t^2 + x^2))$

$\int_0^x \int_0^x \frac{\partial^2 u}{\partial s^2} dsds = L^{-1}_{xx}N(u) + L^{-1}_{xx}(2-2(t^2 + x^2))$

$\int_0^x \frac{\partial u(x,t)}{\partial s} - u_x(0, t)\, ds = L^{-1}_{xx}N(u) + L^{-1}_{xx}(2 - 2(t^2 + x^2))$

$u(x, t) - u(0, t) - xu_x(0,t) = L^{-1}_{xx}N(u) + L^{-1}_{xx}(2-2(t^2 + x^2))$

$u(x, t) = u(0,t) + xu_x(0,t) = L^{-1}_{xx}N(u) + L^{-1}_{xx}(2-2(t^2 + x^2))$

using the initial conditions

$u(0, t) = t^2$ and $xu_x(0,t) = 0$

Using that,

$u(x, t) = \sum_{n=0}^{\infty} u_n(x,t)$ and where, $Nu = \sum_{n=0}^{\infty} A_n$ where, A_n is Adomian polynomial

$\sum_{n=0}^{\infty} u_n(x,t) = u(0,t) + xu_x(0,t) + \sum_{n=0}^{\infty} A_n$ where $u(0,t) = t^2$ and $u_x(0,t) = 0$.

$\sum_{n=0}^{\infty} u_n(x,t) = t^2 + L^{-1}_{xx}\sum_{n=0}^{\infty} A_n + L^{-1}{}_{xx}(2 - 2(t^2 + x^2))$

Then it follows that,

$u_0(x, t) = t^2 + L^{-1}{}_{xx}((2 - 2(t^2 + x^2))$

$= t^2 + \int_0^x \int_0^x (2 - 2t^2 - 2x^2)\, dsds$

$= t^2 + \int_0^x (2x - 2t^2 x - \frac{2}{3}x^3)\, ds$

$u_0(x, t) = t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4$

$u_1(x, t) = L^{-1}_{xx}(A_0)$

where, $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}$, n = 0,1,2,3…

We write the first five Adomian polynomials,

$A_0 = N(u_0)$

$A_1 = u_1 N'(u_0)$,

$A_2 = u_2 N'(u_0) + \frac{1}{2!} u1^2 N''(u_0)$

$A_3 = u_3 N'(u_0) + \frac{2}{2!} u_1 u_2 N''(u_0) + \frac{1}{3!} u_1{}^3 N'''(u_0)$

$A_4 = u_4 N'(u_0) + \left[\frac{1}{2!} u_2{}^2 + u_1 u_3\right] N''(u_0) + \frac{1}{2!} u_1{}^2 u_2 N'''(u_0) + \frac{1}{4!} u_1{}^4 f N''''(u_0)$

Hence,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0},\ n \geq 0.$$

To Find A_n's by Adomian general formula, these polynomials would be computed as follow.

$A_0 = N(u_0)$

$A_1 = N'(u_0)u_1 = \frac{d}{d\lambda} N(u_0 + \lambda u_1)_{\lambda=0}$

$$A_2 = N'(u_0)u_2 + \frac{1}{2!} N''(u_0) u_1{}^2 = \frac{1}{2!}\frac{d^2}{d\lambda^2} N(u_0 + \lambda u_1 + \lambda^2 u_2)_{\lambda=0}$$

$A_3 = N'(u_0)u_3 + \frac{2}{2!} N''(u_0) u_1 u_2 + \frac{1}{3!} N'''(u_0) u_1{}^3 = \frac{1}{3!}\frac{d^2}{d\lambda^2} N(u_0 + \lambda u_1 +$
$\lambda^2 u_2 + \lambda^3 u_3)_{\lambda=0}$

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It follows that,

$u_0(x, t) = t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4$

$u_1(x, t) = L^{-1}_{xx}(A_0)$

where,

$A_0 = N(u_0) = (u_0 u_{0tt})$

$= (t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4)\,(t^2 + x^2 - \frac{1}{6}x^4 - t^2x^2)''$

$= (t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4)(2t - 2tx^2 - 0)'$

$= (t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4)(2 - 2x^2) = 2t^2 + 2x^2 - 2t^2x^2 - \frac{1}{3}x^4 - 2t^2x^2 - 2x^4 + 2t^2x^4 +$
$\frac{1}{3}x^6$

$A_0 = 2t^2 + 2x^2 - 4t^2x^2 - \frac{7}{3}x^4 + \frac{1}{3}x^6 + 2t^2x^4$

Then,

$u_1(x, t) = L^{-1}_{xx}(A_0)$

$$= \int_0^x \int_0^x \left(2t^2 + 2x^2 - 4t^2x^2 - \frac{7}{3}x^4 + \frac{1}{3}x^6 + 2t^2x^4\right) dsds$$

$$= \int_0^x \left(2t^2s + \frac{2}{3}s^3 - \frac{4}{3}t^2s^3 - \frac{7}{15}s^5 + \frac{1}{21}s^7 + \frac{2}{5}t^2s^5\right) ds$$

$$= t^2x^2 + \frac{2}{12}x^4 - \frac{4}{12}t^2x^4 - \frac{7}{90}x^6 + \frac{2}{30}t^2x^6 + \frac{1}{168}x^8$$

$$u_1(x, t) = t^2x^2 + \frac{1}{6}x^4 - \frac{1}{3}t^2x^4 - \frac{7}{90}x^6 + \frac{1}{15}t^2x^6 + \frac{1}{168}x^8$$

$u_2(x,t) = L^{-1}_{xx}(A_1)$ where $A_1 = N'(u_0)u_1 = \frac{d}{d\lambda}N(\sum_{n=0}^{1}(\lambda^n u_n)_{\lambda=0}$

$$A_1 = \frac{d}{d\lambda}(N(\lambda^0 u_0 + \lambda^1 u_1)]_{\lambda=0})$$

$$= \frac{d}{d\lambda}\left(u_0 + u_1\lambda\right)\frac{\partial^2(uo + u_1\lambda)}{\partial t^2}\Big) \lambda_{=0}$$

$$= \frac{d}{d\lambda}\left(u_0\frac{\partial^2(u_0+u_1\lambda)}{\partial t^2} + u_1\lambda\frac{\partial^2(u_0+u_1\lambda)}{\partial t^2}\right) \lambda_{=0}$$

$$= \frac{d}{d\lambda}\left(u_0\frac{\partial^2(u_0)}{\partial t^2} + u_0\frac{\partial^2(u_1\lambda)}{\partial t^2} + u_1\lambda\frac{\partial^2(u_0)}{\partial t^2} + u_1\lambda\frac{\partial^2(u_1\lambda)}{\partial t^2}\right) \lambda_{=0}$$

$$= \frac{d}{d\lambda}\left(u_0\frac{\partial^2(u_0)}{\partial t^2} + \lambda u_0\frac{\partial^2(u_1)}{\partial t^2} + u_1\lambda\frac{\partial^2(u_0)}{\partial t^2} + \lambda^2\frac{\partial^2(u_1)}{\partial t^2}\right) \lambda_{=0}$$

$$= \left(u_0\frac{\partial^2(u_1)}{\partial t^2} + u_1\frac{\partial^2(u_0)}{\partial t^2} + 2\lambda u_1\frac{\partial^2(u_1)}{\partial t^2}\right) \lambda_{=0}$$

$$= \left(u_0\frac{\partial^2(u_1)}{\partial t^2} + u_1\frac{\partial^2(u_0)}{\partial t^2} + 2 \times 0 \times u_1\frac{\partial^2(u1)}{\partial t^2}\right) \lambda_{=0}$$

$$= \left(u_0\frac{\partial^2(u_1)}{\partial t^2} + u_1\frac{\partial^2(u_0)}{\partial t^2} + 0\right) \lambda_{=0}$$

Therefore, $A_1 = u_o\frac{\partial^2(u_1)}{\partial t^2} + u_1\frac{\partial^2 u_0}{\partial t^2}$

$$= t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4\left(\frac{\partial^2(t^2x^2 + \frac{1}{6}x^4 - \frac{1}{3}t^2x^4 - \frac{7}{90}x^6 + \frac{1}{15}t^2x^6 + \frac{1}{168}x^8)}{\partial t^2}\right) + (t^2x^2 + \frac{1}{6}x^4 -$$

$$\frac{1}{3}t^2x^4 - \frac{7}{90}x^6 + \frac{1}{15}t^2x^6 + \frac{1}{168}x^8\left(\frac{\partial^2\left(t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4\right)}{\partial t^2}\right)$$

$= (t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4)(2x^2 - \frac{2}{3}x^4 + \frac{2}{15}x^6) + (\frac{1}{6}x^4)(2x^2 - \frac{2}{3}x^4 + \frac{2}{15}x^6) + (\frac{1}{6}x^4 + x^2t^2 - \frac{7}{90}x^6 - \frac{1}{3}x^4t^2 + \frac{1}{168}x^6 + \frac{1}{15}x^6t^2)(2-2x^2).$

$= \frac{1}{3}x^4 + 2x^2t^2 - \frac{7}{45}x^6 - \frac{2}{3}x^4t^2 + \frac{1}{84}x^8 + \frac{2}{15}x^6t^2 - \frac{1}{3}x^6 - 2x^4t^2 + \frac{7}{45}x^8 + \frac{2}{3}x^6t^2 - \frac{1}{84}x^{10} - \frac{2}{15}x^8t^2 - 2x^2t^2 - \frac{2}{3}x^4t^2 + \frac{2}{15}x^6t^2 + 2x^4 - \frac{2}{3}x^6 + \frac{2}{15}x^8 - \frac{1}{3}x^6 + \frac{1}{9}x^8 - \frac{1}{45}x^{10} - 2x^4t^2 + \frac{2}{3}x^6t^2 - \frac{2}{15}x^8t^2$

Therefore,

$A_1 = \frac{7}{3}x^4 + 4x^2t^2 - \frac{1}{3}x^6 - 2x^4t^2 - \frac{7}{45}x^6 - \frac{2}{3}x^4t^2 + \frac{1}{84}x^8 + \frac{2}{15}x^6t^2 + \frac{7}{45}x^8 + \frac{2}{3}x^6t^2 - \frac{1}{84}x^{10} - \frac{2}{15}x^8t^2 + \ldots,$

$u_2(x,t) = L^{-1}_{xx}(A_1)$

$= \int_0^x \int_0^x ((\frac{7}{3}x^4 + 4x^2t^2 - \frac{1}{3}x^6 - 2x^4t^2 - \frac{7}{45}x^6 - \frac{2}{3}x^4t^2 + \frac{1}{84}x^8 + \frac{2}{15}x^6t^2 + \ldots)dsds$

$= \int_0^x (\frac{7}{15}x^5 + \frac{4}{3}x^3t^2 - \frac{1}{21}x^7 - \frac{2}{5}x^5t^2 - \frac{1}{45}x^7 - \frac{2}{15}x^5t^2 + \frac{1}{765}x^9 + \frac{2}{105}x^7t^2 + \ldots)ds$

$= \frac{7}{90}x^6 + \frac{1}{3}x^4t^2 - \frac{1}{168}x^8 - \frac{2}{30}x^6t^2 - \frac{1}{360}x^8 - \frac{2}{90}x^6t^2 + \frac{1}{7650}x^{10} + \frac{2}{840}x^8t^2 + \ldots$

Hence,

$u_2(x,t) = \frac{7}{90}x^6 + \frac{1}{3}x^4t^2 - \frac{1}{168}x^8 - \frac{2}{30}x^6t^2 - \frac{1}{360}x^8 - \frac{2}{90}x^6t^2 + \frac{1}{7650}x^{10} + \frac{2}{840}x^8t^2 + \ldots$

$u(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t) + \ldots$

$= t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4 + t^2x^2 + \frac{1}{6}x^4 - \frac{1}{3}t^2x^4 - \frac{7}{90}x^6 + \frac{1}{15}t^2x^6 + \frac{1}{168}x^8 + \frac{7}{90}x^6 + \frac{1}{3}x^4t^2 - \frac{1}{168}x^8 - \frac{2}{30}x^6t^2 - \frac{1}{360}x^8 - \frac{2}{90}x^6t^2 + \frac{1}{7650}x^{10} + \frac{2}{840}x^8t^2 + \ldots$

And so on for other components. It can be easily observed that the self-canceling terms appear between various components. Canceling the third term in u_o and the first term in u_1 the fourth term in u_o and the first term in, u_1 in keeping the non-canceled terms in u_o yields the exact solution given by, u(x, t) $= x^2 + t^2$.

Example2. The Natural Decomposition Method (NDM) for solving nonlinear wave equation $\frac{\partial^2 u}{\partial x^2} - \frac{u\partial^2 u}{\partial t^2} = 2-2(t^2 + x^2)$ subject to the initial condition u(0, t) $= t^2$, u_x (0, t) $= 0$

$\frac{\partial^2 u}{\partial x^2} = u_{xx}$and $\left(\frac{u\partial^2 u}{\partial t^2}\right) = uu_{tt}$

Taking the N-Transform to both sides of the equation we obtain

$$N\left(\frac{\partial^2 u}{\partial x^2}\right) = N\ \left(\frac{U\partial^2 U}{\partial t^2}\right) + N(2-2(t^2+x^2))$$

$\frac{s^2}{u^2}$R (u, s, t) $-\frac{s}{u^2}$u (0, t) $-\frac{1}{u}u_x$ (0,t) $= N\left(\frac{u\partial^2 u}{\partial t^2}\right) + \left(\frac{2}{s}-\frac{2t^2}{s}-4\frac{u^2}{s^3}\right)$ from the initial condition

u (0, t) = t^2, u_{xo} (0, t) =0

Then,

$$\frac{s^2}{u^2}\text{R (u, s, t)} - \frac{s}{u^2}t^2 - \frac{0}{u} = N\left[\ \frac{u\partial^2 u}{\partial t^2}\ \right] + \frac{2}{s} - \frac{2t^2}{s} - 4\frac{u^2}{s^3}$$

$$\text{R (u, s, t)} - \frac{u^2}{s^2}\ \left(\frac{s}{u^2}t^2\right) - \frac{0}{u} = \frac{u^2}{s^2}\text{N}\left[\ \frac{U\partial^2 U}{\partial t^2}\ \right] + \frac{u^2}{s^2}\left(\ \frac{2}{s} - \ \frac{2t^2}{s} - 4\frac{u^2}{s^3}\right)$$

$$\text{R (u, s, t)} - \frac{1}{s}t^2 = \frac{u^2}{s^2}\text{N}\left[\ \frac{U\partial^2 U}{\partial t^2}\right] + \frac{u^2}{s^2}\left(\ \frac{2}{s} - \frac{2t^2}{s} - 4\frac{u^2}{s^3}\right)$$

$$N^{-1}\ [\text{R (u, s, t)}]\ - N^{-1}\left[\ \frac{1}{s}t^2\right] = N^{-1}\left[\ \frac{u^2}{s^2}\text{N}\left[\frac{U\partial^2 U}{\partial t^2}\right]\right] + N^{-1}\left(\ \frac{2u^2}{s^3} - \frac{2t^2u^2}{s^3} - 4\frac{u^4}{s^5}\right)$$

u(x, t) = $t^2 + x^2 - \ t^2x^2 - \frac{1}{6}x^4 + N^{-1}\left[\ \frac{u^2}{s^2}\text{N}\left(\frac{U\partial^2 U}{\partial t^2}\right)\right]$ where,

u_0(x, t) $=\sum_{n=0}^{\infty} u_n(x,t)\ \ \sum_{n=0}^{\infty} u_n(x,t)\ = t^2 + x^2 - \ t^2x^2 - \frac{1}{6}x^4 + N^{-}\left(\frac{u^2}{s^2}\ \text{N} + \left(\frac{U\partial^2 U}{\partial t^2}\right)\right)$

$$\sum_{n=0}^{\infty} u_n(x,t)\ = t^2 + x^2 - \ t^2x^2 - \frac{1}{6}x^4 + N^{-}\left[\ \frac{u^2}{s^2}\text{N}\left[\ \sum_{n=0}^{\infty} A_n\ \right]\right]$$

where, $A_n$is Adomian polynomial then,u_o (x, t) $=t^2 + x^2 - \ t^2x^2 - \frac{1}{6}x^4$

u_1(x, t) $=N^{-1}\left[\ \ \frac{u^2}{s^2}\text{N}\left[\ \sum_{n=0}^{\infty} A_o\ \right]\right]$

u_2 (x, t) = $\text{N}^{-1}\left[\ \ \frac{\text{u}^2}{\text{s}^2}\text{N}\left[\ \sum_{\text{n}=0}^{\infty}\text{A}_1\ \right]\right]$

u_3(x, t) $=\text{N}^{-1}\left[\ \ \frac{\text{u}^2}{\text{s}^2}\text{N}\left[\ \sum_{\text{n}=0}^{\infty}\text{A}_2\ \ \right]\right]$

$u_4(x, t) = N^{1-} [\frac{u^2}{s^2} N [\sum_{n=0}^{\infty} A_3]]$

$u_5(x, t) = N^{-1} [\frac{u^2}{s^2} N [\sum_{n=0}^{\infty} A_4]]$

$u_6(x, t) = N^{-1} [\frac{u^2}{s^2} N [\sum_{n=0}^{\infty} A_5]]$

$u_7(x, t) = N^{-}1 [\frac{u^2}{s^2} N [\sum_{n=0}^{\infty} A_6]]$

$u_8(x, t) = N^{-}1 [\frac{u^2}{s^2} N [\sum_{n=0}^{\infty} A_7]]$

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$u_{n+1}(x, t) = N^{-1} [\frac{u^2}{s^2} N [\sum_{n=0}^{\infty} A_n]$

$u_0(x,t) = t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4$

$u_1(x,t) = N^{-1} [\frac{u^2}{s^2} N [A_0]]$ where, $A_0 = N(u_0) = \frac{d}{d\lambda} N(\sum_{n=0}^{0} (\lambda^n u_n)_{\lambda=0}$

$= N^{-1}[\frac{u^2}{s^2} N(t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4) \frac{\partial^2}{\partial t^2} (t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4)]$

$= N^{-1}[\frac{u^2}{s^2} N(t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4) \frac{\partial u}{\partial t} (2t - 2tx^2)]$

$= N^{-1} [\frac{u^2}{s^2} N(t^2 + x^2 - t^2x^2 - \frac{1}{6}x^4)((2 - 2x^2)]$

$= N^{-1}[\frac{u^2}{s^2} N(2t^2 + 2x^2 - 2t^2x^2 - \frac{1}{3}x^4 - 2t^2x^2 - 2x^4 + 2t^2x^4 + \frac{1}{3}x^6)]$

$= N^{-1}[\frac{u^2}{s^2} N(2t^2 + 2x^2 - 4t^2x^2 - \frac{7}{3}x^4 + 2t^2x^4 + \frac{1}{3}x^6)]$

$= N^{-1}(\frac{u^2}{s^2}(\frac{2t^2}{s} + \frac{4u^2}{s^3} - \frac{8t^2u^2}{s^3} - \frac{7\times4!u^4}{3s^5} + \frac{2\times4!t^2u^4}{s^3} + \frac{6!u^6}{3s^7})$

$= N^{-1}(\frac{2t^2u^2}{s^3} + \frac{4u^4}{s^5} - \frac{8t^2u^4}{s^5} - \frac{7\times4!u^6}{3s^5} + \frac{2\times4!t^2u^6}{s^7} + \frac{6!u^8}{3s^9})$

$$u_1(x,t) = t^2x^2+\frac{1}{6}x^4 - \frac{1}{3}t^2x^4 - \frac{7}{90}x^6+\frac{1}{15}t^2x^6 + \frac{1}{8\times7\times}\frac{1}{3}x^8$$

u_2(x, t) $=N^{-1}[\ \frac{u^2}{s^2}$N $[A_1]$, where, $A_1 = N'(u_0)u_1 = \frac{d}{d\lambda}N(\sum_{n=0}^{1}(\lambda^n u_n)_{\lambda=0}$

$$A_1 = \frac{d}{d\lambda}\Big(N(\lambda^0 u_o + \lambda^1 u_1)_{\lambda=0}, A_1 = \frac{d}{d\lambda}(u_o + u_1\lambda)\frac{\partial^2(uo+u_1\lambda)}{\partial t^2}\Big)\ \lambda=0$$

$$= \frac{d}{d\lambda}\Big(u_o\frac{\partial^2(uo+u1\lambda)}{\partial t^2} + u_1\lambda\frac{\partial^2(uo+u1\lambda)}{\partial t^2})\Big)\ \lambda=0$$

$$= \frac{d}{d\lambda}\Big((u_o\frac{\partial^2(uo)}{\partial t^2} + u_o\frac{\partial^2(u1\lambda)}{\partial t^2} + u_1\lambda\frac{\partial^2(uo)}{\partial t^2} + u_1\lambda\frac{\partial^2(u1\lambda)}{\partial t^2})\Big)\ \lambda=0$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\mathrm{u_o}\frac{\partial^2(\mathrm{uo})}{\partial \mathrm{t}^2} + \lambda\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2} + \mathrm{u_1}\lambda\frac{\partial^2(\mathrm{uo})}{\partial \mathrm{t}^2} + \mathrm{u_1}\lambda^2\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2}\Big)\ \lambda=0$$

$$= \Big(0 + \mathrm{u_o}\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2} + \mathrm{u_1}\frac{\partial^2(\mathrm{uo})}{\partial \mathrm{t}^2} + 2\lambda\mathrm{u_1}\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2}\Big)\ \lambda=0$$

$$= \Big(\mathrm{u_o}\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2} + \mathrm{u_1}\frac{\partial^2(\mathrm{uo})}{\partial \mathrm{t}^2} + 2\times0\times\mathrm{u_1}\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2}\Big)\ \lambda=0$$

$$= [\mathrm{u_o}\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2}+\mathrm{u_1}\frac{\partial^2(\mathrm{uo})}{\partial \mathrm{t}^2} + 0]\ \lambda=0$$

Then, $\mathrm{A_1} = \mathrm{u_o}\frac{\partial^2(\mathrm{u1})}{\partial \mathrm{t}^2}+\mathrm{u_1}\frac{\partial^2(\mathrm{uo})}{\partial \mathrm{t}^2}$

$$\mathrm{A_1} = \mathrm{t^2} + \mathrm{x^2} - \mathrm{t^2x^2} - \frac{1}{6}\mathrm{x^4}\left(\frac{\partial^2(\mathrm{t^2x^2}+\frac{1}{6}\mathrm{x^4}-\frac{1}{3}\mathrm{t^2x^4}-\frac{7}{90}\mathrm{x^6}+\frac{1}{15}\mathrm{t^2x^6}+\frac{1}{168}\mathrm{x^8})}{\partial \mathrm{t}^2}\right)+(\mathrm{t^2x^2}+\frac{1}{6}\mathrm{x^4} - \frac{1}{3}\mathrm{t^2x^4} -$$

$$\frac{7}{90}\mathrm{x^6} + \frac{1}{15}\mathrm{t^2x^6} + \frac{1}{168}\mathrm{x^8}\left(\frac{\partial^2\left(\mathrm{t^2+x^2-t^2x^2}-\frac{1}{6}\mathrm{x^4}\right)}{\partial \mathrm{t}^2}\right)$$

$$= (\mathrm{t^2} + \mathrm{x^2}\text{-}\mathrm{t^2x^2}\text{-}\frac{1}{6}\mathrm{x^4})(2\mathrm{x^2} - \frac{2}{3}\mathrm{x^4} + \frac{2}{15}\mathrm{x^6})+(\frac{1}{6}\mathrm{x^4}+\mathrm{x^2t^2} - \frac{7}{90}\mathrm{x^6} - \frac{1}{3}\mathrm{x^4t^2}+\frac{1}{168}\mathrm{x^6}$$

$$+\frac{1}{15}\mathrm{x^6t^2})(2-2\mathrm{x^2}).$$

$$= \frac{1}{3}x^4 + 2x^2t^2 - \frac{7}{45}x^6 - \frac{2}{3}x^4A_1t^2 + \frac{1}{84}x^8+\frac{2}{15}x^6t^2 - \frac{1}{3}\mathrm{x^6} - 2\mathrm{x^4t^2} + \frac{7}{45}\mathrm{x^8}+\frac{2}{3}\mathrm{x^6t^2} - \frac{1}{84}\mathrm{x^{10}} -$$

$$\frac{2}{15}\mathrm{x^8t^2}$$

$$= 2\mathrm{x^2t^2} - \frac{2}{3}\mathrm{x^4t^2} +\frac{2}{15}\mathrm{x^6t^2} +2\mathrm{x^4} - \frac{2}{3}\mathrm{x^6} +\frac{2}{15}\mathrm{x^8} - \frac{1}{3}\mathrm{x^6} +\frac{1}{9}\mathrm{x^8} - \frac{1}{45}\mathrm{x^{10}} - 2\mathrm{x^4t^2} + \frac{2}{3}\mathrm{x^6t^2} -$$

$$\frac{2}{15}\mathrm{x^8t^2}$$

Therefore,

$$\mathrm{A_1}=\frac{7}{3}\mathrm{x^4} + 4\mathrm{x^2t^2} - \frac{1}{3}\mathrm{x^6} - 2\mathrm{x^4t^2} - \frac{7}{45}\mathrm{x^6} - \frac{2}{3}\mathrm{x^4t^2} + \frac{1}{84}\mathrm{x^8}+\frac{2}{15}\mathrm{x^6t^2} + \frac{7}{45}\mathrm{x^8}+\frac{2}{3}\mathrm{x^6t^2} - \frac{1}{84}\mathrm{x^{10}} -$$

$$\frac{2}{15}\mathrm{x^8t^2} +\ldots$$

$$u_2(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^{+}\left[\frac{7}{3}x^4+4x^2t^2-\frac{1}{3}x^6-2x^4t^2-\frac{7}{45}x^6-\frac{2}{3}x^4t^2+\frac{1}{84}x^8+\frac{2}{15}x^6t^2+\frac{7}{45}x^8+\frac{2}{3}x^6t^2-\frac{1}{84}x^{10}-\frac{2}{15}x^8t^2+\ldots\right]\right]$$

$$= N^{-1}\left[\frac{u^2}{s^2}\left[\frac{7(4!)}{3}\frac{u^4}{s^5}+4(2!)\frac{u^2}{s^3}t^2-\frac{1(6!)}{3}\frac{u^6}{s^7}-2(4!)\frac{u^4}{s^5}t^2-\frac{7(6!)}{45}\frac{u^6}{s^7}-\frac{2(4!)}{3}\frac{u^4}{s^5}t^2+\frac{(8!)}{84}\frac{u^8}{s^9}+\frac{2(6!)}{15}\frac{u^6}{s^7}t^2+\frac{7(8!)}{45}\frac{u^8}{s^9}+\frac{2(6!)}{3}\frac{u^6}{s^7}t^2-\frac{1(10!)}{84}\frac{u^{10}}{s^{11}}-\frac{2(8!)}{15}\frac{u^8}{s^9}t^2+\ldots\right]\right]$$

$$= N^{-1}\left[\left[\frac{7(4!)}{3}\frac{u^6}{s^7}+4(2!)\frac{u^4}{s^5}t^2-\frac{1(6!)}{3}\frac{u^8}{s^9}-2(4!)\frac{u^6}{s^7}t^2-\frac{7(6!)}{45}\frac{u^8}{s^9}-\frac{2(4!)}{3}\frac{u^6}{s^7}t^2+\frac{1(8!)}{84}\frac{u^{10}}{s^{11}}+\frac{2(6!)}{15}\frac{u^8}{9}t^2+\frac{7(8!)}{45}\frac{u^{10}}{s^{11}}+\frac{2(6!)}{3}\frac{u^8}{s^9}t^2-\frac{1(10!)}{84}\frac{u^{12}}{s^{13}}-\frac{2(8!)}{15}\frac{u^{10}}{s^{11}}t^2+\ldots\right]\right]$$

$$= \frac{7(4!)}{3}\frac{u^6}{6!}+4(2!)\frac{u^4}{4!}t^2-\frac{1(6!)}{3}\frac{u^8}{8!}-2(4!)\frac{u^6}{6!}t^2-\frac{7(6!)}{45}\frac{u^8}{8!}-\frac{2(4!)}{3}\frac{u^6}{6!}t^2+\frac{1(8!)}{84}\frac{u^{10}}{10!}+\frac{2(6!)}{15}\frac{u^8}{9}t^2+\frac{7(8!)}{45}\frac{u^{10}}{10!}+\frac{2(6!)}{3}\frac{u^8}{8!}t^2-\frac{1(10!)}{84}\frac{u^{12}}{12!}-\frac{2(8!)}{15}\frac{u^{10}}{10!}t^2+\ldots$$

$$= \frac{7x^6}{90}+\frac{x^4}{3}t^2-\frac{1}{168}x^8-\frac{x^6}{15}t^2-\frac{x^8}{360}+\frac{x^{10}}{7560}+\frac{x^8}{420}t^2+\frac{x^{10}}{4050}+\frac{x^8}{84}t^2-\frac{x^{12}}{11088}-\frac{x^{10}}{1350}t^2+\ldots$$

Therefore,

$$u_2(x,t) = \frac{7x^6}{90}+\frac{x^4}{3}t^2-\frac{1}{168}x^8-\frac{x^6}{15}t^2-\frac{x^8}{360}+\frac{x^{10}}{7560}+\frac{x^8}{420}t^2+\frac{x^{10}}{4050}+\frac{x^8}{84}t^2-\frac{x^{12}}{11088}-\frac{x^{10}}{1350}t^2+\ldots$$

And so on for other components. It can be easily observed that the self-canceling terms appear between various components. Canceling the third term in u_o and the first term in, u_1 the fourth term in u_o and the first term in, u_1 in keeping the non-cancelled terms in u_o yields the exact solution of given by u(x,t) = x^2+t^2.

$$u(x,t) = u_o(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+u_5(x,t)+\ldots$$

$$= t^2+x^2-t^2x^2-\frac{1}{6}x^4+t^2x^2+\frac{1}{6}x^4-\frac{1}{3}t^2x^4-\frac{7}{90}x^6+\frac{1}{15}t^2x^6+\frac{1}{168}x^8+\frac{7x^6}{90}+\frac{x^4}{3}t^2-\frac{1}{168}x^8-\frac{x^6}{15}t^2-\frac{x^8}{360}+\frac{x^{10}}{7560}+\frac{x^8}{420}t^2+\frac{x^{10}}{4050}+\frac{x^8}{84}t^2-\frac{x^{12}}{11088}-\frac{x^{10}}{1350}t^2+\ldots$$

Example3. The Natural Decomposition Method (NDM) for solving nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u^2 = x\cos(t)+x^2cos^2(t),$$

Subject to the initial condition u (x, 0) = x, u_t(x, 0) =0.

$\frac{\partial^2 u}{\partial x^2} = u_{xx}$and $\frac{\partial^2 u}{\partial t^2} = u_{tt}$, taking the N-Transform to both sides of the equation we obtain

$$\left[\frac{\partial^2 u}{\partial t^2}\right] - N\left[\frac{\partial^2 U}{\partial x^2}\right] + N[u^2] = N\left[-x\cos(t) + x^2 cos^2(t)\right]$$

Using the properties in table 1 and properties of the N−Transform we have

$$\frac{s^2}{u^2}\text{ R (x, s, u)} - \frac{s}{u^2}\text{ u (x,0)} - \frac{1}{u}u_t\text{(x,0)} - N\left(\frac{\partial^2 U}{\partial x^2}\right) + N[u^2] = N\left[-x\cos(t)\right] + N\left[x^2 cos^2(t)\right]$$

From the initial condition u (x, 0) =x, u_t(x, 0) =0

Then,

$$\frac{s^2}{u^2}\text{R (x, s,u)} - \frac{s}{u^2}x - \frac{0}{u}) - N\left(\frac{\partial^2 U}{\partial x^2}\right) + N\left[u^2\right] = N[-x\cos(t)] + N[x^2 cos^2(t)]$$

$$\text{R (x, s, u)} - \frac{u^2}{s^2}\left(\frac{s}{u^2}x\right) - \frac{0}{u} - \frac{u^2}{s^2}N\left(\frac{\partial^2 U}{\partial x^2}\right) + \frac{u^2}{s^2}N[u^2] = \frac{u^2}{s^2}N\ \ [-x\cos(t)] + \frac{u^2}{s^2}[x^2 cos^2(t)]$$

$$\text{R (x, s, u)} - \frac{x}{s} - \frac{u^2}{s^2}N\left(\frac{\partial^2 U}{\partial x^2}\right) + \frac{u^2}{s^2}N[u^2] = \frac{u^2}{s^2}N\ \ [-x\cos(t)] + \frac{u^2}{s^2}[x^2 cos^2(t)]$$

$$\text{R (x, s, u)} = \frac{x}{s} - \frac{u^2}{s^2}x\left(\frac{s}{s^2+u^2}\right) + \frac{x^2u^2}{s^2}\left(\frac{s}{s^2+u^2}\right)^2 + \frac{u^2}{s^2}N(u_{xx} - u^2)$$

Now taking the inverse of N−Transform

$$\text{N}^{-1}\text{ [R (x, s, u)]} = \text{N}^{-1}\left[\frac{x}{s}\right] - N^{-}\left[\frac{1}{s}\left(\frac{u^2 x}{s^2+u^2}\right)\right] + \text{N}^{-1}\left(\frac{u^2x^2}{(s^2+u^2)^2}\right) + N^{-1}\left[\frac{u^2}{s^2}N(u_{xx} - u^2)\right]$$

From table one

$$\text{u(x,t)} = \text{xcos}(t) + x^2 sin^2(t) + \text{N}^{-1}\left(\frac{u^2}{s^2}N(u_{xx} - u^2)\right)$$

$$= \sum_{n=0}^{\infty} u_n(x,t)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = \text{x}\cos(t) + x^2 sin^2(t) + N^{-1}\left[\frac{u^2}{s^2}N\left[\sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} A_n\right]\right]$$

Here, A_n is the Adomian polynomial which represent the nonlinear terms. So, we compute few components of A_n and some values ofB_n.

$A_0 = u_0{}^2$, $A_1 = 2u_0u_1$, $B_0 = u_{0xx}$, $B_1 = u_{1xx}$

$A_2 = 2u_0u_2 + u_1{}^2$,$B_2 = u_{2xx}$,

u_0(x,t) = $x\cos(t) + x^2 sin^2(t)$

u_1(x, t) = $N^{-1}[\frac{u^2}{s^2} N\left[\sum_{n=0}^{\infty} B_0 - \sum_{n=0}^{\infty} A_0\right]]$

u_2 (x, t) = N^{-1} [$\frac{u^2}{s^2}$ N[$\sum_{n=0}^{\infty} B_1 - \sum_{n=0}^{\infty} A_1$]]

u_3(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N [$\sum_{n=0}^{\infty} B_2 - \sum_{n=0}^{\infty} A_2$]]

u_4(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N[$\sum_{n=0}^{\infty} B_3 - \sum_{n=0}^{\infty} A_3$]]

u_5(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N [$\sum_{n=0}^{\infty} B_4 - \sum_{n=0}^{\infty} A_4$]]

u_6(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N [$\sum_{n=0}^{\infty} B_5 - \sum_{n=0}^{\infty} A_5$]]

u_7(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N [$\sum_{n=0}^{\infty} B_6 - \sum_{n=0}^{\infty} A_6$]]

u_8(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N [$\sum_{n=0}^{\infty} B_7 - \sum_{n=0}^{\infty} A_7$]].

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u_{n+1}(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N [$\sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} A_n$]],u_0(x,t) = $x\cos(t) + x^2 sin^2(t)$

u_1(x, t) =N^{-1} [$\frac{u^2}{s^2}$ N[$B_0 - A_0$]] where $A_0 = N(u_0) = \frac{d}{d\lambda} N(\sum_{n=0}^{0} (\lambda^n u_n)_{\lambda=0}$

u_1(x, t) = N^{-1} [$\frac{u^2}{s^2}$ N[$B_0 - A_0$]] but $B_0 = u_{0xx}$ and $A_0 = {u_0}^2$

Then $u_{0xx} = \frac{\partial^2 (uo)}{\partial x^2}] = \frac{\partial^2(-x\cos(t) + x^2 sin^2(t))}{\partial x^2} = \frac{\partial^1(-\cos(t) + 2x sin^2(t))}{\partial x^1} = 2sin^2(t)$

$u_{0xx} = 2sin^2(t)$ and

$A_0 = {u_0}^2 = (-x\cos(t) + (x^2 sin^2(t))^2$

$= (-x\cos(t) + x^2 sin^2(t)) \quad (-x\cos(t) + x^2 sin^2(t)) = x^2 cos^2(t) - x^3 sin^2 t \cos(t) - x^3 sin^2 t \cos(t) + x^4 sin^4(t)$

$A_0 = x^2 cos^2(t) - 2x^3 sin^2 t \cos(t) + x^4 sin^4(t)$

Then $B_0 - A_0 = 2sin^2(t) - x^2 cos^2(t) + 2x^3 sin^2 t \cos(t) - x^4 sin^4(t)$

$$u_1(x, t) = N^{-1}[\frac{u^2}{s^2} N[2sin^2(t) - x^2 cos^2(t) + 2x^3 sin^2 t \cos(t) - x^4 sin^4(t)]$$

$$= N^{-1}[\frac{u^2}{s^2} N[- x^2 cos^2(t)] + \text{---} = - x^2 N^{-1}\left[\left(\frac{u}{s^2+u^2}\right)^2\right] + \text{---}$$

$$= -x^2 sin^2(t) + \text{-----}$$

Add $u_0(x,t)$ and $u_1(x,t) = x\cos(t) + x^2 sin^2(t) + (-x^2 sin^2(t)) = x\cos(t)$. It can be easily observed that the self-canceling terms appear between various components one can see that the non-canceled term is $x\cos(t)$,then $u(x, t) = x\cos(t)$

CHAPTER FIVE
CONCLUSIONS AND RECOMMENDATION

5.1 CONCLUSION

In this thesis, we studied definitions, examples and certain properties of natural decomposition method, Adomian decomposition method and natural transform.To some wave exact solution using NDM to nonlinear wave equations is obtained. The natural decomposition method is applied to differential equation by separating in to linear and nonlinear components. The nonlinear component is replaced by the Adomian polynomials, the linear part is replaced by infinite series and the solution rapidly converges approximately to the exact solution.Natural decomposition method is more efficient and more accurate method than other methods.Then natural decomposition method (NDM) is an excellent mathematical tool for solving linear and nonlinear wave equations.

5.2 RECOMMENDATION

These study analysis on one and two dimensional nonlinear non-linear wave equations by using natural decomposition method and Adomian decomposition method.

For the future work the researchers may study:

The solution of four and five-dimensional nonlinear wave equations can be obtained using natural decomposition method.

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