ON CONE D.C OPTIMIZATION AND CONJUGATE DUALITY

Addis Ababa University
Department of Mathematics

A Project submitted to Addis Ababa University Department of Mathematics in partial fulfillment of the requirements for the Degree of Master of Science in Mathematics (Optimization)

Prepared by: Gashaw Alemye
Advisor: Semu Mitiku(Ph.D)

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **On Cone D.C Optimization and Conjugate Duality** by Gashaw Alemye in partial fulfillment of the requirements for the degree of master of Science.

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Advisor: ______________________
Dr.Semu Mitiku

Examining committee: ______________________

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Abstract

This Project is about the study of first order necessary and sufficient conditions for unconstrained cone d.c. programming problems where the underlined space is partially ordered with respect to a cone.These conditions are given in terms of directional derivatives and sub differentials of the component functions.Moreover,conjugate duality for cone d.c.optimization is discussed and weak duality theorem is proved in a more general partially ordered linear topological vector space.

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Notations And Symbols

- R the set of real numbers.
- $\bar{R}$ the extended set of real numbers.
- R^m_+ the non-negative orthant of R^m.
- D^* the dual cone of the cone$D$.
- $\geqq_D$ the partial ordering induced by the cone D.
- $\geqq_{D^*}$ the partial ordering induced by the dual cone D^*.
- f^* the conjugate of the function f.
- X_G the indicator function of the set G.
- D_L Lagrange dual problem.
- D_F Fenchel dual problem.
- D_{FL} Fenchel-Lagrange dual problem.
- $dom(f)$ the domain of the function f.
- $inf(p)$ the optimal objective value of the scalar minimum optimization problem(p).
- F^* conjugate map of the set-valued map.
- ∂f the subdifferential of the function f.
- $\partial F(x, y)$ subdifferential of the point-to-set maps F at (x, y).

Introduction

Convexity is one of the well developed mathematical structures since 1950s and it plays an important role in optimization. An increasing number of problems arise from application that can not be solved successfully by standard methods of linear or nonlinear programming. But a great many of these can be described by using difference of two convex functions called d.c. functions for short. One can utilize the concepts and tools of convex analysis to solve such problems. Many practical methods have been developed by several researchers to solve a d.c. optimization problem in a scalar case.
The class of D.C. functions is a remarkable subclass of locally lipschitz functions that is of interest both in analysis and optimization. It appears very naturally as the smallest vector space containing all continuous convex functions on a given set. A natural generalization of this concept to vector optimization is named simply as cone d.c.programming. (where cone refers here the ordering cone of the image set). A function is said to be cone d.c. if it can be described as the difference of two cone convex functions. The class of cone d.c. functions is very rich. For example, cone convex, cone concave functions and the combination of them fall under this category. On finite dimensional spaces one can easily verify that the extension of Hartman's theorem holds true also for vector valued functions, i.e. every locally cone d.c. function is globally cone d.c. Since convexity is present twice (in the reverse sense) in the decomposition of d.c. function, many concepts and results from convex analysis will be used also for cone d.c. programming.

The generalization of the results for scalar d.c. programming to cone d.c. Optimization was studied by Yin Zhiwen and Li Yuanxi. However their result is valid only for the case where the underlined space is an order-complete, finite dimensional vector space. Moreover, they used difference to give the minimality conditions. But this difference produces a set which is empty too often and may not give any information when the point is not efficient.
In this project, we study the extension of scalar D.C programming to a more general case and present some minimality conditions also using the

conjugate duality. The duality case, here, is discussed on a more general partially ordered linear topological vector space extending Toland's duality theorem for scalar d.c. problems.

The project is organized as follows. In chapter 1 scalar D. C function and their duality are discussed. In the second chapter preliminaries for vector optimization, which are useful in the next part are discussed. In the third chapter the necessary and sufficient local weak optimality conditions are discussed. It also contains the conjugate duality for cone d.c. programming, which gives some generalizations of Toland's duality theorem, which was given for scalar d.c. programming case.

Chapter 1

Scalar D C functions and Their Duality

1.1 D C Function

Definition 1.1.1. . *[5] Let f be a real valued function mapping* R^n *to* R. *Then f is a D .C. function if there exist convex functions* $g, h : R^n \to R$ *such that f can be decomposed as the difference between g and h.*
$f(x) = g(x) - h(x) \ \ \forall x \in R^n$.

Consider the D.C. programming problem

$$(DCP) \qquad \min_{x \in R^n} f_0(x)$$

subject to $f_i(x) \leq 0, \ \ i = 1, \dots m$. where $f_i : R^n \to R$ is a differentiable DC function for $i = 0, \dots m$.

Before we continue with the discussion of the solution to the above D.C Programming Problem, we develop some intuition regarding DC functions. Recall that a function $f : R^n \to R$ is convex if for every $x_1, x_2 \in R^n$ and every $\alpha \in [0, 1], f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1 + (1 - \alpha)f(x_2)$.

In particular, as we may recall, if f is a twice-differentiable function, then it is convex if and only if its Hessian matrix is positive semi- definite.

Example 1.1.1. *.[2] (a)*$f_1(x) = abs(x)$. *(b)* $f_2(x) = -log(x)$ *(c)* $f(x) = abs(x) + log(x)$.

Notice that while in these examples, the minimum is easy to find by inspection in the convex functions, it is less clear in the resulting D.C function. Then, having DC functions as part of an optimization problem adds a level of complexity to the problem that we did not encounter in convex functions.

1.2 Subgradient of convex functions

Differentiability facilitates the analysis of optimization problems. For convex functions, even when they are not differentiable, we might consider subgradients instead of nonexistent gradients.

Definition 1.2.1. . *[5] (one-sided directional Derivative)*
Let f be a function from R^n to $\{-\infty, +\infty\}$, and let x be a point where f is finite. The one-sided directional derivative of f at x with respect to a vector d is defined to be the limit $f'(x; d) = lim_{t\downarrow 0}\{\frac{[f(x+td)-f(x)]}{t}\}$, *if it exists.*

If f is actually differentiable at x, then $f'(x; d) = \langle \nabla f(x), d\rangle$ for any d, where $\nabla f(x)$ is the gradient of$f$ at x.

Definition 1.2.2. . *[5] (sub gradient)*
Let f be a convex function from R^n to $[-\infty, +\infty]$. A vector $x^* \in R^n$ is said to be a subgradient of f at x if $f(x') \geqq f(x) + \langle x^*, x' - x\rangle$ for any $x' \in R^n$.

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$. If $\partial f(x)$ is not empty, f is said to be subdifferentiable at x. If f is a continuously differentiable function defined on an open convex set $C \subset R^n$ then we denote its gradient at $x \in C$, as usual, by $(\nabla f(x).)$ The excess function
$E(x, y) := f(y) - f(x) - \langle \nabla f(x, y - x\rangle$
is a measure of the discrepancy between the value of f at the point y and the value of the tangent approximation at x to f at the point y.

Now we introduce the notation of a monotone definition

Definition 1.2.3. .*[5] The map $x \to \nabla f(x)$ is said to be monotone on C provided*
$\langle \nabla f(y) - \nabla f(x), y - x\rangle \geq 0, \forall x, y \in C.$

Theorem 1.2.1. .*[4] Let f be a continuously differentiable function defined on an open convex set $C \subset R^n$. Then the following are equivalent:*

- *(a). $E(x, y) \geq 0$ for all $x, y \in C$;*
- *(b).the map $x \to \nabla f(x)$ is monotone in C;*
- *(c).the function f is convex on C.*

Proof: Suppose that (a) holds, i.e.$E(x, y) \geq 0$ on $C \times C$. Then we have both
$f(y) - f(x) \geq \langle \nabla f(x), y - x\rangle,$

and $f(x) - f(y) \geq \langle f(y), x - y \rangle = -\langle \nabla f(y), y - x \rangle$.

Then, from the second inequality, $f(y) - f(x) = \langle \nabla f(y), x - y \rangle$, and so $\langle \nabla f(y) - \nabla f(x), y - x \rangle = \langle \nabla f(y), y - x \rangle - \langle \nabla f(x), y - x \rangle$
$\geq (f(y) - f(x)) - (f(y) - f(x)) \geq 0$.

Hence, the map $x \to \nabla f(x)$ is monotone in C.

Now suppose the map $x \to \nabla f(x)$ is monotone in C, and choose $x, y \in C$. Define a function $\varphi : [0, 1] \to R$ by $\varphi(t) : f(x+t(y-x))$. We observe,first,that if φ is convex on $(0, 1)$ then f is convex on C. To see this , let $u, v \in [0, 1]$ be arbitrary. on the one hand,

$\varphi((1 - \lambda)u + \lambda v) = f(x + [(1 - \lambda)u + \lambda v](y - x)) = f((1 - [(1 - \lambda)u + \lambda v])x + ((1 - \lambda)u + \lambda v)y)$,

while, on the other hand,

$\psi((1 - \lambda)u + \lambda v) \leq (1 - \lambda)f(x + u(y - x)) + f(x + v(y - x))$.

setting $u = 0$ and $v = 1$ in the above expressions yields

$f((1 - \lambda)x) + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$.

So the convexity of $\psi on[0, 1]$ implies the convexity of f on C.

Now, choose any $\alpha, \beta, 0 \leq \alpha \leq \beta \leq 1$. Then

$\varphi'(\beta) - \varphi'(\alpha) = \langle (\nabla f(x + \beta(y - x)) - \nabla f(x + \alpha(y - x)), y - x \rangle$.

setting $u := x+\alpha(y-x) andv := x+\beta(y-x)^5 wehavev-u = (\beta-\alpha)(y-x)$ and so

$\varphi'(\beta)-\varphi'(\alpha) = \langle (\nabla f(v)-\nabla f(u), v-u \rangle \geq 0$. Hence φ' is non-decreasing,so that the function φ is convex.

Finally, if f is convex on C, then for fixed $x, y \in C$ define

$h(y) := (1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)$.

Then $\lambda \to h(\lambda)$ is a non-negative,differentiable function on $[0, 1]$ and attains its minimum at $\lambda = 0$. Therefore $0 \leq h'(0) = E(x, y)$, and the proof is complete.

$$f'(x; d) = sup\{\langle x^*, d \rangle : x^* \in \partial f(x)\} = \delta^*(d/\partial f(x)). \tag{1.1}$$

Moreover, $\partial f(x)$ is a non empty bounded set if and only if $x \in int(domf)$, in which case $f'(x; d)$ is finite for every d.

If the convex sets $ri(domf_i)(i = 1, \ldots, m)$ have a point in common. then actually

$$\partial f(x) = \partial f_1(x) + \partial f_2(x) + \ldots + \partial f_m(x) \text{ for any } x. \tag{1.2}$$

1.3 Conjugate Function

Definition 1.3.1. *.[4] (Linear functional) Let V be a vector space over a field F the mapping $T : V \to F$ that satisfy the following condition*

$(LF_1) T(x+y) = T_x + T_y \ \forall x, y \in V$. $(LF_2) T(\alpha x) = \alpha T_x \ \forall x \in V, \alpha \in F$. *is called a linear functional on V.*

Definition 1.3.2. . *[4] (Linear continuous functional) If a linear functional $f : V \to F$ has real or complex number values, then f is continuous for each $x \in V$ if and only if $\exists M$ such that*
$\frac{|f(x)|}{||x||} \leq M$.

Definition 1.3.3. . *[4] The topological vector space $L(X, R)$ which is the set of all linear functionals on X is said to be the topological dual space of X being denoted by X^*.*

Further, we refer with dual to topological space not to the dual in algebraical space, unless otherwise specified. Analogously to vector space by $\langle x^*, x \rangle$, we denote the value taken at $x \in X$ by the linear continuous functional $x^* \in X^*$. Conjugate function plays an important role in the duality theory. Let U be a linear normed space $(e.g. R^n), U^*$ is topological dual space,while $\langle x^*, x \rangle$ denotes the value of the linear continuous functional $x^* \in U^*$ at the point $x \in U$.

Definition 1.3.4. . *[4] Let $f : U \to \bar{R}$ be a given function. Then the function $f^* : U^* \to \bar{R}$ given by*

$$f^*(x^*) = \sup_{x \in U} \{\langle x^*, x \rangle - f(x)\}$$

is called the conjugate function of f.

Proposition 1.3.1. . *[4] (Some elementary properties of conjugate functions).*

i. (Young's inequality)

$$f(x) + f^*(x^*) \geq \langle x^*, x \rangle, \forall x \in U, \forall x^* \in U^*.$$

ii.

$$\inf_{x \in U} f(x) = -f^*(0).$$

iii.

$$f \leq g \Rightarrow f^* \geq g^*.$$

iv.

$$(\lambda.f)^*(x^*) = \lambda.f^*(\frac{x^*}{\lambda}), \forall \lambda \geq 0.$$

proof.
i. follows immediately from the definition.
ii.

$$f^*(0) = \sup_{x\in U}\{\langle 0, x\rangle - f(x)\}$$

$$= -\inf_{x\in U} f(x)$$

This implies

$$\inf_{x\in U} f(x) = -f^*(0)$$

iii. for any $x \in U$ and $x^* \in U^*$ we have

$$\langle x^*, x\rangle + (g(x)) \leq \langle x^*, x\rangle + (-f(x))$$

and considering the supremum after $x \in U$ it holds

$$\sup_{x\in U}\{\langle x^*, x\rangle + (-g(x))\} \leq \sup_{x\in U}\{\langle x^*, x\rangle + (-f(x))\}$$

which actually means

$$g^*(x^*) \leq f^*(x^*)$$

This implies

$$g^* \leq f^*$$

iv.

$$(\lambda.f)^*(x^*) = \sup_{x\in U}\{\langle x^*, x\rangle - (\lambda.f(x))\}$$

$$= \sup_{x\in U}\{\lambda.[\langle \frac{x^*}{\lambda}, x\rangle - f(x)]\}$$

$$= \lambda.\sup_{x\in U}\{\langle \frac{x^*}{\lambda}, x\rangle - f(x)\}$$

$$= \lambda.f^*(\frac{x^*}{\lambda})$$

This implies $(\lambda.f)^*(x^*) = \lambda.f^*(\frac{x^*}{\lambda})$

Definition 1.3.5. . *[4] Let $f^* : U^* \to \bar{R}$ be a given function. Then $f^{**} : U \to \bar{R}$*
given by
$f^{**}(x) = Sup\{\langle x^*, x\rangle - f^*(x^*)\}$ is called the bi-conjugate function to f.

1.4 Conjugate Duality in Scalar Optimization problem

We consider three different conjugate dual problems the well-known lagrange and fenchel dual problems (D_L) and (D_F), respectively and a combination of the above two, which we shall call the Fenchel-Lagrange dual problem $(D_F L)$.

1.4.1 Formulation of the Dual Problem

Before we discuss a formulation of the dual problem, let us first define a cone, convex cone,dual cone and closed convex cone.

Definition 1.4.1. . *[5] Let C be any nonempty subset of a vector space V. The set C is said to be a cone if* $x \in C$ *implies* $\lambda x \in C, \forall \lambda \geq 0$.

Definition 1.4.2. . *[5] A subset C of a vector space V is a convex cone if* $\alpha x + \beta y$ *belongs to C for any positive scalars* α, β *and* $x, y \in C$.

Definition 1.4.3. . *[5] Let K be a cone. The set* $K^* = \{y \mid x^T y \geq 0, \forall\ x \in K\}$ *is called the dual cone of K. As the name*
suggests K^* *is always convex even when the original cone K is not.*

Definition 1.4.4. . *[4] Let* $K \subseteq V$ *be a nonempty closed set. K is called a closed convex cone if the following two properties hold.*
(i) For all $x \in K$ *and all non negative real numbers* λ, *we have* $\lambda x \in K$.
(ii) For all $x, x' \in K$, *we have* $x + x' \in K$

Let $X \subseteq R^n$ be a non empty set and $K \subseteq R^p$ a nonempty closed convex cone with $int(K) \neq \emptyset$. The set $K^* := \{k^* \in R^p : k^{*T} x \geq 0\ \forall x \in K\}$ is the dual cone of K. Consider the partial order " $\leq$ " induced by K in R^p, namely for
$y, z \in R^p$ we have that $y \leq z$ if and only if $z - y \in K$. Let $f : R^n \to \bar{R} = R \bigcup \pm\infty$ and
$g = (g_1, \ldots, g_k)^T : R^n \to R^p$. The optimization problem we investigate in this section is

$$(P) \qquad \inf_{x \in G} f(x)$$

Where $G = \{x \in X : g(x) \leq 0, X \subseteq R^n\}$, where the inequality here means point-wise inequality.

In the following we suppose that the feasible set G is nonempty. Assume further that $dom(f) = X$, where $dom(f) := \{x \in R^n : f(x) < +\infty\}$.

The optimal objective value of problem P is denoted by $\inf(p)$.

Definition 1.4.5. . *[4] An element $\bar{x} \in G$ is said to be an optimal solution for (P) if $f(\bar{x}) \in \inf(P)$.*

The aim of this section is to construct different dual problems to (P). Now let us first consider the general optimization problem without constraints

$$(PG) \qquad\qquad \inf_{x \in R^n} F(x)$$

where F is a mapping from R^n into $\bar{R}$. and $\bar{R}$ is the extended set of real number.

Remark 1.4.1. *By the assumption we made for f, we have*

$$f^*(p^*) = \sup_{x \in R^n} \{p^{*T}x - f(x)\} = \sup_{x \in R^n} \{p^{*T}x - f(x)\}. \tag{1.3}$$

The approach is based on the construction of a so called perturbation function $\Phi : R^n \times R^m \to \bar{R}$ with the property that $\Phi(x, 0) = F(x)$ for each $x \in R^n$. Here $,R^m$ is the space of the perturbation variables. For each $p \in R^m$ we obtain a new optimization problem

$$(PG_p) \qquad\qquad \inf_{x \in R^n} \Phi(x, p)$$

For $p \in R^m$ the problem (PG_p) is called the perturbed problem of (PG)

By definition 1.3.4. the conjugate of Φ is the function $\Phi^* : R^n \times R^m \to \bar{R}$ given by

$$\Phi^*(x^*, p^*) = \sup_{x \in R^n, p \in R^m} \{(x^*, p^*)^T(x, p) - \Phi(x, p)\} = \sup_{x \in R^n, p \in R^m} \{x^{*T}x + p^{*T}p - \Phi(x, p)\} \tag{1.4}$$

Now at $x^* = 0$ we can define the following optimization problem

$$(DG) \qquad\qquad \sup_{p^* \in R^m} \{p^{*T}p - \Phi^*(0, p^*)\}$$

The problem (DG) is called the dual problem to (PG) and its optimal objective value is denoted by $\sup(DG)$.

Theorem 1.4.1. . *[4] The relation $-\infty \leq \sup(DG) \leq \inf(PG) \leq +\infty$ always holds.*

Proof.

Let $p^* \in R^m$ from (1.4)

$$\Phi^*(0, p^*) = \sup_{x \in R^n, p \in R^m} \{0^T x + p^{*T}p - \Phi(x, p)\}$$

$$= \sup_{x \in R^n, p \in R^m} \{p^{*T} p - \Phi(x,p)\}$$

$$\geq \sup_{x \in R^n} \{p^{*T} 0 - \Phi(x,0)\}$$

$$= \sup_{x \in R^n} \{-\Phi(x,0)\}$$

This means that, for each $p^* \in R^m, x \in R^n$ it holds

$$-\Phi^*(0,p^*) \leq \Phi(x,0) = F(x).$$

which implies that

$$\sup(DG) \leq \inf(PG).$$

Our next aim is to show how we can apply this approach to the constrained optimization problem (P). Therefore, let $F : R^n \to \bar{R}$ be the function given by

$$F(x) = \begin{cases} f(x), if \quad x \in G \\ +\infty, otherwise. \end{cases}$$

The primal problem (P) is then equivalent to

$$(PG) \qquad \inf_{x \in R^n} F(x)$$

and , since the perturbation function $\Phi : R^n \times R^m \to \bar{R}$ satisfies $\Phi(x,0) = F(x)$ for each $x \in R^n$, we obtain that

$$\Phi(x,0) = f(x) \ \ \forall x \in G \tag{1.5}$$

and

$$\Phi(x,0) = +\infty, \forall x \in R^n \backslash G. \tag{1.6}$$

1.4.2 The Lagrange Dual Problem

Let the function $\Phi_L : R^n \times R^k \to \bar{R}$ be defined by

$$\Phi_L(x,q) = \begin{cases} f(x), if x \in X, g(x) \leqq_k q \\ +\infty, otherwise \end{cases}$$

With the perturbation variable $q \in R^k$. The relations (1.5) and (1.6) are fulfilled. For the conjugate of Φ_L we have

$$\Phi_L^*(x^*, q^*) = \sup_{x \in R^n, q \in R^k} \{x^{*T}x + q^{*T}q - \Phi_L(x, q)\}$$

$$= \sup_{x \in X, q \in R^k, g(x) \leqq_k q} \{x^{*T}x + q^{*T}q - f(x)\}$$

In order to calculate this expression we introduce the variable s instead of q by $s = q - g(x)$. This implies

$$\Phi_L^*(x^*, q^*) = \sup_{x \in X, s \in K} \{x^{*T}x + q^{*T}[s + g(x)] - f(x)\}$$

$$= \sup_{x \in X} \{x^{*T}x + q^{*T}g(x) - f(x)\} + \sup_{s \in K} q^{*T}s$$

$$\Phi_L^*(x^*, q^*) = \begin{cases} sup_{x \in X}\{x^{*T}x + q^{*T}g(x) - f(x)\}, if \quad q^* \in -K^* \\ +\infty, otherwise. \end{cases}$$

As we have seen , the dual of (P) obtained by the perturbation function Φ_L is

$$(D_L) \qquad \sup_{q^* \in R^k} \{-\Phi_L^*(0, q^*)\}$$

and since

$$\sup_{q^* \in -K^*} \{-\sup_{x \in X}[q^{*T}g(x) - f(x)]\} = \sup_{q^* \in -K^*} \{\inf_{x \in X}[-q^{*T}g(x)] + f(x)\}$$

the dual has the following form

$$(D_L) \qquad \sup_{q^* \geqq_{k^*} 0} \inf_{x \in X} [f(x) + q^{*T}g(x)] \tag{1.7}$$

The problem (D_L) is actually the well known Lagrange dual problem . Its optimal objective value is denoted by $sup(D_L)$ and theorem 1.4.1. implies

$sup(D_L) \leq inf(P)$.

Example 1.4.1. *.[3] Let* $K = R_+, X = [0, +\infty] \subseteq R, f : R \to \bar{R}, g : R \to R$, *be defined by*

$$f(x) = \begin{cases} -x^2, if x \in X \\ +\infty, otherwise. \end{cases}$$

and $g(x) = x^2 - 1$.

Now the optimal objective value of the Lagrange dual is

$$sup(D_L) = \sup_{q^* \geq 0} \inf_{x \geq 0} [f(x) + q^* g(x)]$$

$$= \sup_{q^* \geq 0} \inf_{x \geq 0} [-x^2 + q^*(x^2 - 1)]$$

$$= \sup_{q^* \geq 0} \inf_{x \geq 0} [(q^* - 1)x^2 - q^*]$$

$$= \sup_{q^* \geq 1} (-q^*)$$

$$= -1$$

This implies that the optimal objective value of D_L is -1.

We are now interested to obtain dual problem for (P) , different from the classical Lagrange problem .

1.4.3 Fenchel Dual Problems

Let us consider the perturbation function $\Phi_F : R^n \times R^n \to \bar{R}$ given by

$$\Phi_F(x,p) = \begin{cases} f(x+p), if \quad x \in G \\ +\infty, otherwise. \end{cases}$$

With the perturbation variable $p \in R^n$. The relations (1.5) and (1.6) are also fulfilled and it holds

$$\Phi_F^*(x^*, p^*) = \sup_{x \in R^n, p \in R^n} \{x^{*T} x + p^{*T} p - \Phi_F(x,p)\}$$

$$= \sup_{x \in X, p \in R^n, g(x) \leqq_k 0} \{x^{*T} x + p^{*T} p - f(x+p)\}$$

Introducing a new variable $r = x + p \in R^n$, we have

$$\Phi_F^*(x^*, p^*) = \sup_{x \in X, r \in R^n, g(x) \leqq_k 0} \{x^{*T} x + p^{*T}(r - x) - f(r)\}$$

$$= \sup_{r \in R^n} \{p^{*T} r - f(r)\} + \sup_{x \in X, g(x) \leqq_k 0} \{(x^* - p^*)^T x\}$$

$$= f^*(p^*) - \inf_{x \in X, g(x) \leqq_k 0} \{(p^* - x^*)^T x\}$$

$$= f^*(p^*) - \inf_{x \in G}\{(p^* - x^*)^T x\}$$

Now the dual of (P) will be

$$(D_F) \qquad \sup_{p^* \in R^n}\{-\Phi_F^*(0, P^*)\}$$

and can be written in the form

$$D_F \qquad \sup_{p^* \in R^n}\{-f^*(p^*) + \inf_{x \in X, g(x) \leqq_k 0} p^{*T} x\}$$

Denoting by

$$X_G(x) = \begin{cases} 0, if \quad x \in G \\ +\infty, otherwise. \end{cases}$$

The indicator function of the set G we have that

$$X_G^*(-p^*) = \inf_{x \in G} p^{*T} x.$$

The dual D_F becomes then

$$D_F \qquad \sup_{p^* \in R^n}\{-f^*(p^*) - X_G^*(-p^*)\}$$

Let us call (D_F) the Fench dual problem and denote its optimal objective value by Sup (D_F). The weak duality

$$\sup(D_F) \leq \inf(P)$$

Example 1.4.2. *.[4] Let $K = R_+, X = [0, +\infty) \subseteq R, f : R \to \bar{R}, g : R \to R$, be defined by*

$$f(x) = \begin{cases} x, if \quad x \in X \\ +\infty, otherwise. \end{cases}$$

and

$$g(x) = 1 - x^2$$

Now the optimal objective value of the Fenchel dual problem is

$$\sup(D_F) = \sup_{p^* \in R}\{-\sup_{x \geq 0}[p^* x - x] + \inf_{x \geq 0, 1 - x^2 \leq 0} p^* x\}$$

$$= \sup_{p^* \in R} \{ \sup_{x \geq 0} (1-p^*)x + \inf_{x \geq 1} p^* x \}$$

$$= \sup_{0 \leq p^* \leq 1} (p^*)$$

$$= 1$$

This implies that the optimal objective value of D_F is 1 .

1.4.4 Fenchel- Lagrange Dual Problems

Another dual problem different from (D_L) and (D_F) can be obtained by considering the perturbation function as a combination of the functions Φ_L and Φ_F . Let this be defined by

$\Phi_{FL} : R^n \times R^n \times R^k \to \bar{R}$

$$\Phi_{FL}(x,p,q) = \begin{cases} f(x+p), if \quad x \in X, g(x) \leqq_k q \\ +\infty, otherwise. \end{cases}$$

With the perturbation variables $p \in R$ and $q \in R, \Phi_{FL}$ satisfies the relation (1.5) and (1.6) and its conjugate is

$$\Phi^*_{FL}(x^*,p^*,q^*) = \sup_{x \in R^n, p \in R^n, q \in R^k} \{ (x^*,p^*,q^*)^T (x,p,q) - \Phi_{FL}(x,p,q) \}$$

$$= \sup_{x \in R^n, p \in R^n, q \in R^k} \{ x^{*T} x + p^{*T} p + q^{*T} q - \Phi_{FL}(x,p,q) \}$$

$$= \sup_{x \in X, g(x) \leqq_k q, p \in R^n, q \in R^k} \{ x^{*T} x + p^{*T} p + q^{*T} q - f(x+p) \}$$

Like in the previous subsection we introduce new variables $r = x+p \in R^n$ and $s = q - g(x) \in K$. Then we have

$$\Phi^*_{FL}(x^*,p^*,q^*) = \sup_{r \in R^n, s \in K, x \in X} \{ x^{*T} x + p^{*T}(r-x) + q^{*T}[s+g(x)] - f(r) \}$$

$$= \sup_{r \in R^n} \{ p^{*T} r - f(r) \} + \sup_{x \in X} \{ (x^* - p^*)^T x + q^{*T} g(x) \} + \sup_{s \in K} \{ q^{*T} s \}$$

Computing the first supremum we get

$$\sup_{r\in R}\{p^{*T}r - f(r)\} = f^*(p^*$$

While for the last it holds

$$\sup_{s\in k} q^{*T}s = \begin{cases} 0, if \quad q^* \in -K \\ +\infty, otherwise. \end{cases}$$

In this case the dual problem

$$(D_{FL}) \qquad \sup_{p^*\in R^n, q^*\in R^k}\{-\Phi^*_{FL}(0,p^*,q^*)\}$$

becomes

$$(D_{FL} \qquad \sup_{p^*\in R^n, q^*\in -K}\{-f^*(p^*) - \sup_{x\in X}[-p^{*T}x + q^{*T}g(x)]\}$$

or equivalently

$$(D_{FL}) \qquad \sup\{-f^*(p^*) + \inf[-p^{*T}x + q^{*T}g(x)]\}$$

We will call (D_{FL}) the fenchel - Lagrange dual problem and denote its optimal objective value by $\sup(D_{FL})$.By theorem 1.4.1
the weak duality

$$\sup(D_{FL}) \leq \inf(P)$$

also holds.

Example 1.4.3. *.[4] Let* $K = R_+, X = [0,+\infty) \subseteq R, f : R \to \bar{R}, g : R \to R,$ *be defined by*

$$f(x) = \begin{cases} -x^2, if \quad x \in X \\ +\infty, otherwise. \end{cases}$$

and

$$g(x) = x^2 - 1.$$

now the optimal objective value of the Fenchel- Lagrange dual problem is

$$\sup(D_{FL}) = \sup_{p^*\in R, q^*\geq 0}\{-\sup_{x\geq 0}[p^*x + x^2] + \inf_{x\geq 0}[p^* + q^*(x^2-1)]\}$$

$$= \sup_{p^*\in R, q^*\geq 0}\{-\infty + \inf_{x\geq 0}[p^*x + q^*(x^2-1)]\}$$

$$= -\infty$$

Chapter 2

Preliminaries for Vector optimization

2.1 Some Definitions and Preliminary Concepts

Order Relations and Ordering Cones

order relations

Given a nonempty set M, by $M \times M$ we represent the set of ordered pairs of elements of M,that is

$M \times M := \{(x_1, x_2) \mid x_1, x_2 \in M\}$.

The following definition gives the notion of an order relation on a set M.

Definition 2.1.1. *.[2] Let M be a nonempty set and let $\Re$ be a nonempty subset of $M \times M$. Then $\Re$ is called a binary relation on M and the pair $(M, \Re)$ is called a set M with binary relation $\Re$.*

The containment $(x_1, x_2) \in \Re$ will be denoted by $x_1 \Re x_2$.

The binary relation $\Re$ is called:

(a) reflexive if for every $x \in M$, we have $x \Re x$;

(b) transitive if for all $x_1, x_2, x_3 \in M$, the relations $x_1 \Re x_2$ and $x_2 \Re x_3$ imply that $x_1 \Re x_3$;

(c) antisymmetric if for all $x_1, x_2 \in M$, the relations $x_1 \Re x_2$ and $x_2 \Re x_1$ imply that $x_1 = x_2$.

Moreover, a binary relation $\Re$ is called a pre-order on M if $\Re$ is transitive, a quasiorder if $\Re$ is reflexive and transitive and a partial order on M if $\Re$ is reflexive, transitive, and antisymmetric. In all the three cases. The containment $(x_1, x_2) \in \Re$ is also denoted by $x_1 \leq_{\Re} x_2$. The binary relation $\Re$ is called a linear or total order if $\Re$ is a partial order and any two elements of M are comparable, that is for all $x_1, x_2 \in M$ either $x_1 \leq_{\Re} x_2$ or $x_2 \leq_{\Re} x_1$.

Furthermore, if each nonempty subset M' of M has a first element x' (that means $x' \in M'$ and $x' \leq_{\Re} x \quad \forall x \in M'$) then M is called well-ordered.

Definition 2.1.2. *.[2] Let $\Re$ be an order relation on the nonempty set M and let $M_0 \subseteq M$ be nonempty. An element $x_0 \in M_0$ is called a maximal (minimal) element of M_0 relative to $\Re$ if for every $x \in M_0$,*
$x_0 \Re x \Rightarrow x \Re x_0$ *(respectively, $x \Re x_0 \Rightarrow x_0 \Re x$).*

The collection of all maximal (minimal) elements of M_0 with respect to $\Re$ is denoted by $Max(M_0, \Re)(respecyively, Min(M_0, \Re))$.

Note that x_0 is a maximal element of M_0 with respect to $\Re$ if and only if x_0 is a minimal element of M_0 with respect to $\Re^{-1}$ and hence $Max(M_o, \Re) = -Min(M_0, \Re^{-1})$.

Where $\Re^{-1}$ is the inverse of the relation $\Re \subset M \times M$ defined by

$$\Re^{-1} := \{(x_1, x_2) \in M \times M \mid (x_2, x_1) \in \Re\}$$

Definition 2.1.3. *.[2] Let $\emptyset \neq M_0 \subset M$and let $\Re$ be a binary relation on M. Then :*

M_0 is lower (upper) bounded with respect to $\Re$ if there exists $a \in M$ such that $a \Re x (x \Re a)$ for every $x \in M_0$. In this case, the element a is called a lower (upper) bound of M_0 with respect to $\Re$.

We recall that given a linear space X, a nonempty set $M \subset X$ is affine (or a linear manifold) if $\lambda x + (1 - \lambda) y \in M$ for all $x, y \in M$ and $\lambda \in R$.

Definition 2.1.4. *.[5] A nonempty set $C \subset X$ is called convex if $[x, y] := \{\lambda x + (1 - \lambda) y \mid \lambda \in [0, 1]\} \subset C$ for all $x, y \in C$.*

The set $C \subset R^n$ is called convex, if all convex combinations of any two points $x^1, x^2 \in C$ are again in C. In other words, the line segment connecting two arbitrary points of a convex set is contained in the set. The intersection of (possibly infinitely many) convex sets is again a convex set.

Definition 2.1.5. *.[5] A function $f : C \to R$ defined on a convex set C is called convex if for all $x^1, x^2 \in C$ and $0 \leq \lambda \leq 1$ one has $f(\lambda x^1 + (1 - \lambda) x^2) \leq \lambda f(x^1) + (1 - \lambda) f(x^2)$.*

A parabola $f(x) = ax^2 + bx + c$ with $a > 0$ is a familiar example of a convex function.

$f : R^n \rightarrow R$, defined by $f(x) = ||x||$ for some norm $||.||$ is a convex function

$f(x) = sinx, f(x) = e^{-x^2}, f(x) = x^3$ are not convex functions.

Definition 2.1.6. *.[5] The convex set $\varepsilon \subseteq C$ is an extremal set of the convex set C if for all $x^1, x^2 \in C$ and $0 < \lambda < 1$, one has $\lambda x^1 + (1 - \lambda)x^2 \in \varepsilon$ only if $x^1, x^2 \in \varepsilon$.*

An extremal set consisting of only one point is called an extremal point. observe that extremal sets are convex by definition, and the convex set C itself is always an extremal set of C.

Example 2.1.1. *.[5] Let C be the cube $\{x \in R^3 \mid 0 \leq x_i \leq 1, i = 1, 2, 3, \}$, then the vertices are extremal points, the edges are 1 -dimensional extremal sets, the faces are 2 - dimensional extremal sets, and the whole cube is a 3-dimensional extremal set.*

By convention the empty set $\emptyset$ is considered to be affine and convex. A linear subspace is affine and an affine set is convex. Moreover, any intersection of linear subspace, affine sets, or convex sets is a linear subspace, an affine set, or a convex set respectively. These properties allow us to introduce the linear hull, the affine hull and the convex hull of a nonempty set $A \subset X$ as being, respectively.

$linA := \bigcap\{Y \subset X \mid A \subset Y, \text{Y is a linear subspace}\}$

$affA := \bigcap\{M \subset X \mid A \subset M, \text{M is a linear manifold}\}$

$convA := \bigcap\{C \subset X \mid A \subset C, \text{C is a convex set}\}$

Definition 2.1.7. *.[2] Let $\Re$ be a binary relation on the linear space X; we say that $\Re$ is compatible with the linear structure of X if*

$\forall x_1, x_2 \in X, \forall \lambda \in R : x_1 \Re x_2, 0 \leq \lambda \Rightarrow \lambda x_1 \Re \lambda x_2.$

and $\forall x_1, x_2, x \in X : x_1 \Re x_2 \Rightarrow (x_1 + x) \Re (x_2 + x) hold.$

In linear spaces, a large number of relations $\Re$ can be defined by cones which are compatible with the linear structure of the space. For this we first give the following definition.

Definition 2.1.8. *.[5] A nonempty set $D \subset X$ is a cone if for every $x \in D$ and for every $\lambda \in R_+$ (where R_+ is the set of non-negative real numbers) we have $\lambda x \in D$., if D is a cone, then $0 \in D$. The cone D is called*

(a). convex if for all $x_1, x_2 \in D$ we have $x_1 + x_2 \in D$.

(b). nontrivial or proper if $D \neq \{0\}$ and $D \neq X$.

(c). reproducing if $D - D = X$.

(d). pointed if $D \cap (-D) = 0$.

Example 2.1.2. *.[5] The set $C = \{(x_1, x_2) \in R^2 \mid x_2 \geq 2x_1, x_2 \geq -0.5x_1\}$ is a convex cone in R^2.*

The set $C' = \{(x_1, x_2, x_3) \in R^3 \mid x_1^2 + x_2^2 \leq x_3^2\}$ is a convex cone in R^3.

Note: A set C is convex cone if and only if $\alpha C = C$ and $C + C = C$ for any positive scalar α. The empty set, the space V, and any linear subspace of V are convex cones.

Note that, A convex cone is pointed if it does not contain any sub space except the origin.

A pointed convex cone could be defined equivalently as a convex cone that does not contain any line. A convex cone D is pointed if and only if the origin O is an extremal point of D.

If a convex cone $D \in R^2$ is not pointed, then it is either, a line through the origin or a half space or R^2

2.2 Cone Properties Related to the Topology and Order

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We discuss now the connections between topology and order. Unlike the notion of an ordered linear space, the notion of an ordered topological linear space does not demand for any direct relation between the order and the involved topology. However, because a compatible reflexive pre order on a linear space is defined by a convex cone. It is customary to require that the cone defining the order be closed, have nonempty interior, or be normal. Before introducing the notion of a normal cone, we recall that a nonempty sub set A of a linear topological space X is full with respect to the convex cone $D \subset X$ if $A = [A]_c$, where

$[A]_c := (A + C) \cap (A - C)$.

Note that $[A]_C$ is full with respect to C for every set $A \subset X$.

Definition 2.2.1. *.[2]*

Let Y be a Hausdorff topological vector space and $C \subseteq Y$ be a proper convex cone.

(i). D is based if there exists a nonempty convex subset B of D such that $D = R_+B(where R_+B := \{\lambda b \mid b \in B \ \ and \ \ \lambda \geq 0\})$ *and* $0 \notin clB$ *; the set B is called a base for D.*

(ii). D is called well-based if D has a bounded base.

(iii). Let the topology of Y be defined by a family $\wp$ *of semi norms. D is called supernormal or nuclear if for each* $p \in \wp$ *there exists* $y^* \in Y^*$ *such that* $p(y) \leq \langle y, y^* \rangle \forall y \in D$.

(iv). D is said to be regular if any decreasing (increasing) net which has a lower bound (upper bound) is convergent.

The following are some basic properties of cones.

- (a). The intersection $\bigcap_{i \in I} C_i$ of a collection $\{C_i | i \in I\}$ of cones is a cone.
- (b). The cartesian product $C_1 \times C_2$ of two cones $C_1 and C_2$ is a cone.
- (c). Thevector sum $C_1 + C_2$ of two cones $C_1 and C_2$ is a cone.
- (d). The image and the inverse image of a cone under a linear transformation is a cone.
- (e). A subset C is a convex cone if and only if it is closed under addition and positive scalar multiplication. i.e $C + C \subset C$, and $\gamma C \subset C$ for all $\gamma > 0$,

Let Y be a real topological vector space which is partially ordered by a pointed, closed, convex cone D with nonempty interior in Y. The following notations will be used in the sequel.

$$y \leq_D y' \text{ iff } y' - y \in D \quad and \quad y <_D y' \text{ iff } y' - y \in intD. \tag{2.1}$$

The next definition of weak Supremum and Infimum of a set in partially ordered linear topological vector spaces was given by Tanino in [9]. This definition, which is on the basis of weak efficiency, is used to obtain the strong conjugate duality result in vector optimization (see [9] and [6]). Let $\bar{Y}$ denote the extended space of $Y (i.e, \bar{Y} := Y \bigcup \{\pm\infty\}$. Given a set $Z \subset \bar{Y}$, we define the set $A(Z)$ of $\bar{Y}$ by $A(Z) = \{y \in \bar{Y} : y' <_D y$ for some $y' \in Z\}$ which is the set of all points above Z and the set $B(Z)$ of $\bar{Y}$ by $B(Z) = \{y \in \bar{Y} : y' >_D y$ for some $y' \in Z\}$ which is the set of all points below Z. Clearly $A(Z) \subseteq Y \bigcup \{+\infty\}$ and $B(Z) \subseteq Y \bigcup \{-\infty\}$. A point $z \in Z$ is said to be a D-minimal point of Z if there is no $y \in Z$ with $y <_D z$. The set of all D-minimal points of Z is called the D-minimum of Z and is denoted by D-Min Z. The D-maximum of Z, D-Max Z, is defined analogously.

Definition 2.2.2. *.[5] Let Z be a nonempty subset of* $\bar{Y}$ *such that* $Z \neq \{+\infty\}$*. A point* $p \in \bar{Y}$ *is said to be a D-infimal point of a set Z, if there is no* $y \in Z$ *such that* $y <_D p$ *and if the relation* $y' >_D p$ *implies the existence of some* $z \in Z$ *such that* $y' >_D z$*.*

The set of all D-infimal points of Z is called a D-infimum of Z and is denoted by D-InfZ. If $Z = \emptyset$ and $Z = \{+\infty\}$, we define $D - InfZ = +\infty$. The D-supremum of Z, D-SupZ is defined similarly.

As an easy consequence from the definition we deduce the following

$$(i) - D - Max(-Z) = D - MinZ \quad and \quad -D - Inf(-Z) = D - SupZ) \tag{2.2}$$

$$(ii) D - Max\ \emptyset = \emptyset \ and \ D - Sup\ \emptyset = -\infty \tag{2.3}$$

One observes that in the above definitions the D-infimum and the D-minimum are defined for the weak pareto minimality (by using the order $<_D$. The following proposition which is proved in [9, Proposition 2.2.1] shows how a set can be partitioned by using the above defined concepts.

Proposition 2.2.1. $\bar{Y} = (D - SupZ) \bigcup A(D - SupZ) \bigcup B(D - SupZ)$ *and the above three sets in the right-hand side are disjoint.*

Definition 2.2.3. *.[5] Let* $a \in R^n$ *and* $b \in R$ *and assume* $a \neq 0$*. Then the set*

$H := \{x \in R^n | \langle a, x \rangle = b\}$*, is called a hyperplane with normal vector a.*

Definition 2.2.4. *.[5] Let* $S, T \in R^n$ *and let H be a hyperplane.Then H is said to separate S from T if S lies in one closed half- space determined by H while T lies in the other closed half-space. In this case H is called a separating hyperplane.*

If S and T lie in the open half-spaces, then H is said to strictly separate S and T. Let X and Y be real locally convex topological vector spaces and let L(X,Y) denote the space of all linear continuous operators from X into Y. Then we define the conjugate mapping and the subdifferential of a mapping $F : X \to \bar{Y}$. Hereafter by domain of F, domF, we mean the effective domain of F which is given by $domF = \{x \in X \mid F(x) \neq \emptyset, F(x) \neq +\infty\}$ for a set valued mapping F and by

$domf = \{x \in X \mid f(x) <_D +\infty$ for a vector valued function f }.

Definition 2.2.5. *.[5] For a vector valued function* $f : X \to \bar{Y}$*, its conjugate map is a set valued mapping* $f^* : L(X, Y) \to \bar{Y}$ *defined by* $f^*(T) = D - SupTx - f(x) \mid x \in XforT \in L(X, Y)$*.*

Moreover, its bi conjugate mapping is a set-valued mapping $f^{**} : X \to \bar{Y}$ defined by

$$f^{**}(x) = D - Sup \bigcup_{T \in L(X,Y)} [Tx - f^*(T)] \text{for} x \in dom f.$$

Definition 2.2.6. *.[5] For a set-valued mapping $F : X \to \bar{Y}$, its conjugate map is a set valued mapping $F^* : L(X,Y) \to \bar{Y}$ defined by*

$$F^*(T) = D - Sup \bigcup_{x \in X} [Tx - F(T)] for T \in L(X,Y).$$

Moreover, its bi conjugate mapping is a set valued mapping $F^{**} : X \to \bar{Y}$ defined by

$$F^{**}(x) = D - Sup \bigcup_{T \in L(X,Y)} [Tx - F(T)] \text{for} x \in dom F.$$

Assume that $dom f \neq \emptyset$.

Definition 2.2.7. *.[5] (Subgradients and Subdifferentials). Let $x_o \in X$. An operator $T \in L(X,Y)$ is said to be a sub gradient of a vector valued function f at x_o if*

$f(x) \not<_D f(x_o) + T(x - x_o)$ for all $x \in X$,

or equivalently

$$f(x_o) - Tx_o \in \{D - Min\{f(x) - T(x) \mid x \in X\}.$$

The set of all subgradients of f at x_o is called the subdifferential of f at x_0 and is denoted by $\partial f(x_0)$. More precisely

$\partial f(x_0) = \{T \in L(X,Y) \mid f(x) - f(x_o) \not<_D T(x - x_0)$ *for all* $x \in X\}$.

For a set-valued mapping $F : X \to \bar{Y}$, let $x_o \in X$ and $y_0 \in F(x_o)$. An operator $T \in L(X,Y)$ is called a subgradient of F at (x_0, y_0) if

$$Tx_0 - y_0 \in D \ Max \bigcup_{x \in X} [Tx - F(x)].$$

The subdifferential of F at (x_0, y_0) is denoted by $\partial F(x_o, y_o)$. Moreover, we let

$\partial F(x_0) = \bigcup \partial F(x_0, y)$.

Unlike to the scalar case, the subdifferential of a vector-valued function may not be a closed convex set even when f is a finite D-convex function .

If the subdifferential of f at x_0 is nonempty (or if $\partial F(x_0, y_0) \neq \emptyset$ for every $y_0 \in F(x_0)$. when F is a set-valued mapping), then f is said to be subdifferentiable at x_0.

From the definitions of conjugate maps and subgradients that $T \in \partial f(x)$ iff $Tx - f(x) \in f^*(T)$ and immediately from the definition of the subgradients one can see that $f(x^*) \in D{-}Min\{f(x) \mid x \in X\}$ *if and only if* $0 \in \partial f(x^*)$.

Moreover it is well known that $f(x_0) \in f^{**}(x_0)$ for every $x_0 \in X$.

A vector valued function $f : X \to \bar{Y}$ is said to be D-convex iff its $D-$ *epigraph*, $D - epif = \{(x, y) \in X \times Y \mid y \in f(x) + D\}$ is a convex set in $X \times \bar{Y}$, or iff for every $x^1, x^2 \in X$ and for any $\alpha \in [0, 1]$,

$\alpha f(x^1) + (1 - \alpha)f(x^2) - f(\alpha x^1) + (1 - \alpha)x^2) \in D.$

A set-valued mapping $F : X \to \bar{Y}$ is said to be D-convex iff its D-epigraph, $D - epiF = \{(x, y) \in X \times Y \mid y \in F(x) + D\}$ is convex, or iff for all $t \in [0, 1]$ *and* $x^1, x^2 \in X$,

$tF(x^1) + (1 - t)F(x^2) \subseteq F(tx^1 + (1 - t)x^2) + D.$

Definition 2.2.8. *.[5] A subset $X of R^n$ is said to be convex if $\alpha x^1 + (1 - \alpha)x^2 \in X$ for any $x^1, x^2 \in X$ and any $\alpha \in [0, 1]$.*

The intersection of all the convex sets containing a given subset X of R^n is called the convex hull of X and is denoted by co X.

Definition 2.2.9. *.[5] (convex function)*

A function f from $X \subset R^n$ to $[-\infty, +\infty]$ is said to be convex function on X if epi f is convex as a subset of R^{n+1}. A concave function on X is a function whose negative is convex. An affine function on X is a function which is finite ,convex, and concave.

Definition 2.2.10. *.[5] Let $f : A \to B$, where A and B are non-empty sets.Then the graph of f is defined by*

$Gr(f) := \{(a, b) : a = f(b), a \in A, b \in B\}.$

Definition 2.2.11. *Let $S \subset R^n$ be a non-empty set. Then the indicator function of S is defined as the function Ψs given by*

$$\Psi s(x) := \begin{cases} 0, if x \in S \\ +\infty, if x \notin S. \end{cases}$$

Definition 2.2.12. *.[5] (proper convex function)*

A convex function f on X is said to be proper if $f(x) < +\infty$ for at least one $x \in X$ and if $f(x) > -\infty$ everywhere.

Definition 2.2.13. *.[5] (Cone and Convex Cone)*

A subset K of R^n is called a cone if $\alpha x \in K$ whenever $x \in K$ and $\alpha > 0$. Moreover,a cone K is said to be a convex cone when it is also convex.

Definition 2.2.14. *.[5] (pointed Cone and Acute Cone)*

A cone $K \subset R^n$ is said to be pointed if $-x \notin K$ when $x \neq 0$ and $x \in K$. In other words K is pointed if it does not contain any nontrivial sub spaces. It is said to be acute if there is an open half space

$\tilde{H}^+ = x \in R^n : \langle x, x^* \rangle > 0, x^* \neq 0.$

such that $cLK \subset H^+ \bigcup \{0\}$.

Definition 2.2.15. *.[5] (cone convex function)*

Let X be a convex set in R^n , f be a function from X into R^p and D be a convex cone in R^p.Then f is said to be D-convex if for any $x^1, x^2 \in X$ and for any $\alpha (o \leqq \alpha \leqq 1)$,

$\alpha f(x^1) + (1-\alpha) f(x^2) - f(\alpha x^1 + (1-\alpha) x^2) \in D.$

2.3 Vector optimization and solution concepts

We consider a linear topological vector space Y, partially ordered by a proper pointed convex closed cone C and a nonempty set $A \subset Y$.

Definition 2.3.1. *.[2] (Pareto Minimal (Maximal) points). Consider*

$Min(A, C) := \{\bar{y} \in A \mid A \cap (\bar{y} - C) = \{\bar{y}\}\}.$

An element $\bar{y} \in Min(A, C)$ is called a pareto minimal point of A with respect to C.

Furthermore, consider

$Max(A, C) := \{\bar{y} \in A \mid A \cap (\bar{y} + C) = \{\bar{y}\}\}.$

An element $\bar{y} \in Max(A, C)$ is called a pareto maximal point of A with respect to C.

Many solution procedures for vector optimization problems generate weakly minimal elements.

Definition 2.3.2. *.[2] (Weakly Minimal(Maximal) points. Suppose that $intC \neq \emptyset$.*

Consider

$WMin(A, C) := \{\bar{y} \in A \mid A \cap (\bar{y} - intC) = \emptyset\}.$

An element $\bar{y} \in WMin(A, C)$ is called a weakly minimal point of A with respect to C.

Furthermore, consider

$WMax(A, C) := \{\bar{y} \in A \mid A \cap (\bar{y} + intC) = \emptyset\}.$

An element $\bar{y} \in WMax(A, C)$ is called a weakly maximal point of A with respect to C.

Moreover, we introduce the concept of strongly minimal points:

Definition 2.3.3. *.[2] Consider*
$strMin(A, C) := \{\bar{y} \in A \mid A \subseteq \bar{y} + C\}$.
An element $\bar{y} \in strMin(A, C)$ is called a strong minimal point of A with respect to C.

Definition 2.3.4. *.[5] Assume that $D \subset Y$ is a proper cone with nonempty interior and put $Q := intD$. We say that $\bar{y}$ is a Q-minimal point of $A(\bar{y} \in QMin(A, C))$ if*
$A \cap (\bar{y} - Q) = \emptyset$
or, equivalently,
$(A - \bar{y}) \cap (-Q) = \emptyset$.

We assume that X, Y are topological vector spaces and S is a subset of X. For the following notations and results the lineality of C, defined by
$l(C) := C \cap (-C)$
is very important. Of course, C is pointed if $l(C) = \{0\}$. We use the notations $y \leqq y'$ if $y' - y \in C; y < y' if y' - y \in intC$ and furthermore, $y \leq y'$ if $y' - y \in C \backslash l(C)$.

In the following we consider a proper vector-valued objective function $f : X \to Y, S \subseteq X$ and use the notation
$f(S) := \{f(x) \mid x \in S \cap domf\} \subset Y$.
Consider now the vector optimization problem
minimze f(x) subject to $x \in S$.

Definition 2.3.5. *.[2] $\bar{x} \in S \cap domf$ is*
. strongly l(C)- minimal if $f(\bar{x}) \leqq f(x)$ for $x \in S$, or equivalently
$f(S) \subseteq f(\bar{x}) + C$.
. pareto l(C)minimal if $f(x) \leqq f(\bar{x}) \Rightarrow f(\bar{x}) \leqq f(x) \forall x \in S$, or equivalently
$f(S) \cap (f(\bar{x}) - (C \backslash l(C))) = \emptyset$.
. weakly l(C)minimal if $f(x) \not< f(\bar{x}) \forall x \in S$ or equivalently
$f(S) \cap (f(\bar{x} - intC) = \emptyset$.
. l(C)-properly minimal if there exists $D \subset Y$ a proper convex cone with $C \backslash l(C) \subseteq intD$ such that $\bar{x}$ is efficient with respect to D,
$f(S) \cap (f(\bar{x} - D \backslash l(D)) = \emptyset$.

Solution concepts and some properties of solutions for multi objective optimization problems

Solution concepts

The concept of optimal solutions to multi objective optimization problems is not trivial and in it self debatable.It is closely related to the preference attitudes of the decision makers. The most fundamental solution concept is that of efficient solutions (also called non dominated solutions or noninferior solution) with respect to the domination structure of the decision maker.

Consider the multi objective optimization problem

(P) minimize $f(x) = (f_1(x), f_2(x), ...f_p(x))$ subject to $x \in X \subset R^n$.

Let $Y = f(X) = \{y : y = f(x), x \in X\}$.

A domination structure representing a preference attitude of the decision maker is supposed to be given as a point-to-set map from Y into R^p.

Definition 2.3.6. *.[5] (Efficient Solution) A point $\hat{x} \in X$ is said to be an efficient solution to the multi objective optimization problem (P) with respect to the domination structure D if $f(\hat{x}) \in \delta(Y, D)$; where $\delta(Y, D)$ is efficient set of the set Y with respect to the cone D. That is , if there is no $x \in X$ such that $f(\hat{x}) \in f(x) + D(f(x)) \setminus \{0\}$.*

Many interesting cases of efficient solutions are obtained when D is a constant point- to- set map whose value is a constant (convex) cone. In such cases, we identify the map (domination structure) with the cone D. Then $\hat{x} \in X$ is an efficient solution to the problem (P) if and only if there is no $x \in X$ such that $f(\hat{x}) - f(x) \in D \setminus \{0\}$; namely , $\hat{x}$ is efficient if and only if $(f(x) - f(\hat{x}) \cap (-D) = \{0\}$. If D is an open cone, it does not contain 0. However, we consider $D(y) = D \cup \{0\}$ and call D it self the domination structure in this case.It does no matter so much whether D contains 0 or not since the set $D \setminus \{0\}$ is used for the definition of efficient solutions.

Given a closed convex cone D,$\check{x}$ a weakly efficient solution to the problem (P) if $f(\check{x}) \in \delta(Y, intD)$, i.e.if $(f(X) - f(\check{x})) \cap (-intD) = \emptyset$. Weakly efficient solutions are often useful. Since they are completely characterized by scalarization.

Proposition 2.3.1. *.[5] Let D be a nonempty cone containing 0, then*

$\delta(Y, D) \supset \delta(Y + D, D)$.

with equality holding if D is pointed and convex.

Proof.

The result is trivial if Y is empty , so we assume otherwise. First suppose $y \in \delta(Y+D, D)$ but $y \notin \delta(Y, D)$, if $y \notin Y$, there exist $y' \in Y$ and nonzero $d \in D$ such that $y = y' + d$. Since $0 \in D, Y \subset Y + D$. Hence, $y \notin \delta(Y+D, D)$, which is a contradiction. If $y \in Y$, we directly have a similar contradiction.

Next suppose that D is pointed and convex, $y \in \delta(Y, D)$ but $y \notin \delta(Y + D, D)$. Then there exists a $y' \in Y + D$ with $y - y' = d' \in D \setminus \{0\}$. Then $y' = y'' + d''$ with $y'' \in Y, d'' \in D$. Hence, $y = y'' = (d' + d'')$ and $d' + d'' \in D$, since D is convex and pointed cone. Since D is pointed, $d' + d'' \neq 0$ and so $y \notin \delta(Y, D)$, which leads to a contradiction. This completes the proof of the proposition.

Remark 2.3.1. *The pointedness of D can not be eliminated in the* $\delta(Y, D) \subset \delta(Y + D, D)$.

The convexity of D is also essential. In fact, let $Y = \{(y_1, y_2) : 0 \leqq y_1 = y_2 \leqq 1\} \subset R^2$

and

$D = \{(d_1, d_2) : d_2 \geqq 2d_1 \geqq 0\} \cup \{(d_1, d_2) : d_1 \geqq 2d_2 \geqq 0\}$,

which is pointed but not convex. Then $(1, 1) \in \delta(Y, D)$. However, $(1, 1) = (0, 0) + (\frac{1}{2}, 0) + (\frac{1}{2}, 1) \in (Y + D + D$. Hence $(1, 1) \notin \delta(Y + D, D)$.

Proposition 2.3.2. *.[5] Let* Y_1 *and* Y_2 *be two sets in* R^p, *and let D be a constant domination structure on* R^p *(a constant cone, for example). Then*

$\delta(Y_1 + Y_2, D) \subset \delta(Y_1, D) + \delta(Y_2, D)$.

Proof.

Let $\hat{y} \in \delta(Y_1 + Y_2, D)$.Then $\hat{y} = y^1 + y^2$ for some $y^1 \in Y_1$ and $y^2 \in Y_2$. We show that $y^1 \in \delta(Y_1, D)$. If we suppose the contrary, then there exist $y \in Y_1$ and nonzero $d \in D$ such that $y^1 = y + d$. Then $\hat{y} = y^1 + y^2 = y + y^2 + d$ and $y + y^2 \in Y_1 + Y_2$, which contradicts the assumption $\hat{y} \in \delta(Y_1 + Y_2, D)$. Similarly we can prove that $y^2 \in \delta(Y_2, D)$. Therefore, $\hat{y} \in \delta(Y_1, D) + \delta(Y_2, D)$.

Remark 2.3.2. *The converse inclusion of proposition* 2.3.2. *does not always hold. For example, let* $Y_1 = Y_2 = \{(y_1, y_2) : (y_1)^2 + (y_2)^2 \leqq 1\} \subset R^2$ *and* $D = R^2_+$. *Then*

$y^1(-1, 0) \in \delta(Y_1, D)$ *and* $y^2 = (0, -1) \in \delta(Y_2, D)$.

However,

$y^1 + y^2 = (-1, -1) > (-\sqrt{2}, -\sqrt{2})$

$=(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}) + (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}) \in Y_1 + Y_2$.

The most fundamental kind of efficient solution is obtained when D is the nonnegative orthant $R^p_+ = \{y \in R^p : y \geqq 0\}$ and is usually called a pareto optimal solution or non inferior solution.

Definition 2.3.7. *.[5] (Pareto optimal solution) A point $\hat{x} \in X$ is said to be a pareto optimal solution (or noninferior solution (Zadeh [Z1])) to the problem (P) if there is no $x \in X$ such that $f(x) \leq f(\hat{x})$.*

A number of theoretical papers concerning multi objective optimization are related to the pareto optimal solution. In some cases a slightly weaker solution concept than pareto optimality is often used. It is called weak Pareto optimality, which corresponds to the case in which the domination cone $D \setminus \{0\}$ is equal to the positive orthant$R^p_+ = \{y \in R^p : y > 0\}$.

Definition 2.3.8. *.[5] (Weak Pareto optimal solution)*
A point $\hat{x} \in X$ is said to be a weak pareto optimal solution to the problem (p) if there is no $x \in X$ such that $f(x) < f(\hat{x})$.

Properly Efficient Solutions

This subsection is devoted to another slightly strengthened solution concept, proper efficiency. The domination cone D is assumed to be a nontrivial closed convex cone in R^p unless and otherwise noted.

Definition 2.3.9. *.[4] (Tangent Cone) Let $S \subset R^p$ and $y \in S$. The tangent cone to S at y, denoted by $T(S, y)$, is the set of limits of the form $h = lim_{t_k}(y^k - y), where\{t_k\}$ is a sequence of nonnegative real numbers and $\{y^k\}$ is a sequence in S with limit y.*

Remark 2.3.3. *The tangent cone $T(S, Y)$ is always a closed cone.*

Definition 2.3.10. *.[5] (Borwein's Proper Efficiency)*
A point $\hat{x} \in X$ is said to be a properly efficient solution of the multi objective optimization problem (P) if
$T(Y + D, f(\hat{x})) \cap (-D) = \{0\}$.

Proposition 2.3.3. *.[5] If a point $\hat{x} \in X$ is a properly efficient solution of (P) by the definition of Borwein, then it is also an efficient solution of (P).*

Proof.
If $\hat{x}$ is not efficient, there exists a nonzero vector $d \in D$ such that $d = f(\hat{x}) - y$ for some $y \in Y$. Let $d^k = (1 - \frac{1}{k}d \in D$ and $t_k = k$ for $k = 1, 2, \ldots\ldots$ Then
$y + d^k = f(\hat{x} - d + (1 - \frac{1}{k})d = f(\hat{x}) - (\frac{1}{k})d \to f(\hat{x})$ as $k \to \infty$, and
$t_k(y + d^k - f(\hat{x})) = k(-d + (1 - \frac{1}{k}d) = -d \to -d$ as $k \to \infty$.
Hence,$T(Y + D, f(\hat{x})) \cap (-D) \neq \{0\}$, and $\hat{x}$ is not properly efficient in the sense of Borwein.

Example 2.3.1. *.[5] if* $X = \{(x_1, x_2) : (x_1)^2 + (x_2)^2 \leqq 1\} \subset R^2$,

$f_1(x) = x_1, f_2(x) = x_2$, *and* $D = R^2_+$. *Then* $(-1, 0)$ *and* $(0, -1)$ *are efficient solutions but not properly efficient solutions (in the sense of Borwein).*

Definition 2.3.11. *.[5](Projecting cone).*

Let $S \subset R^p$. *The projecting cone of S, denoted by* $P(S)$, *is the set of all points h of the form* $h = \alpha y$, *where* α *is a nonnegative real number and* $y \in S$. *The projecting cone is also known as the cone generated by S.*

Definition 2.3.12. *.[5] (Benson's Proper Efficiency). A point* $\hat{x} \in X$ *is said to be a properly efficient solution of the problem* (P) *if*

$clP(Y + D - f(\hat{x})) \cap (-D) = \{0\}$.

Since $T(Y + D, f(\hat{x})) \subset clP(Y + D - f(\hat{x}))$ Benson's proper efficiency strengthens Borwein's proper efficiency. The converse, however, does not always hold.

Example 2.3.2. *.[5] [Borwein, but not Benson]*

Let $X = \{(x_1, x_2) \in R^2 : x_1 + x_2 \geqq 0\} \cup \{(x_1, x_2) \in R^2_+ : x_1 \geqq 1\} \cup \{(x_1, x_2) \in R^2 : x_2 \geqq 1\}$,

$f_1(x_1, x_2) = x_1$,

$f_2(x_1, x_2) = x_2$,

$D = R^2_+$.

Then $\hat{x} = (0, 0)$ is a properly efficient solution according to Borwein's definition. However, it is not properly efficient in the sense of Benson.

Lemma 2.3.1. *.[5] Let S be a convex set and* $y \in S$. *Then*

$T(S, y) = clP(S - y)$,

which is a closed convex cone.

Proof.

$clP(S-y)$ is a closed convex cone. So it suffices to show that $clP(S-y) \subset T(S, y)$, since the converse follows directly from the definitions. Since $T(S, y)$ is closed, we need only to prove that $P(S - y) \subset T(S, y)$. Let $h \in P(S - y)$. Then $h = \beta(y' - y)$ for some $\beta \geqq 0$ and $y' \in S$.

Let $y^k = [1 - \frac{1}{k}]y + (\frac{1}{k})y'$

and $t_k = \beta_k \geqq 0$. Then $t_k(y^k - y) = \beta(y' - y)$. Hence $y^k \in S$ from the convexity of S, and

$y^k \to y$ and $t_k(y^k - y) \to h \quad as \quad k \to \infty$.

Thus $h \in T(S, y)$, and the proof is completed.

Definition 2.3.13. *Let X be a convex set in* R^n, f *be a function from* R^n *into* R^p *, and D be a convex cone in* R^p. *If the function f is D-convex, then the set* $f(x)$ *is D-convex.*

Theorem 2.3.1. *.[5] If $\hat{x}$ is a properly efficient solution of the problem (P) in the sense of Benson, it is also a properly efficient solution of (P) in the sense of Borwein. If X is a convex set and if f is a D-convex function on X(see definition 2.3.13), then the converse also holds ; that is, Borwein's proper efficiency is equivalent to Benson's proper efficiency.*

Proof.

If D is a convex cone and f is D-convex on the convex set X, the set Y + D is a convex set. Hence ,$clP(Y + D - f(\hat{x})) = T(Y + D, f(\hat{x}))$ by lemma 2.3.1. Therefore, the proper efficiency in the sense of Borwein is equivalent to that in the sense of Benson.

The following definition of proper efficiency by Henig does not require the domination cone D to be closed.

Definition 2.3.14. *.[5] (Henig's Proper efficiency)*

(i). A point $\hat{x} \in X$ is said to be a global properly efficient solution of (P) if,

$f(\hat{x}) \in \delta(Y + D')$, *for some convex cone D' with $D \setminus \{0\} \subset intD'$.*

(ii). A point $\hat{x} \in X$ is a local properly efficient solution of (P) if for every $\epsilon > 0$, there exists a convex cone D' with $D \setminus \{0\} \subset intD'$, such that

$f(\hat{x}) \in \delta((Y + D) \cap (f(\hat{x} + \epsilon B), D')$, *where B is the closed unit ball in R^p.*

These definitions are essentially the same as Benson's and Borwein's, respectively.

Definition 2.3.15. *.[5] (Geoffrion's proper efficiency) When $D = R^p_+$, a point $\hat{x}$ is said to be a properly efficient solution of (P) if it is efficient and if there is some real $M > 0$ such that for each i and for each $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$, there exists at least one j such that $f_j(\hat{x}) < f_j(x)$ and $(f_i(\hat{x}) - f_i(x))/(f_j(x) - f_j(\hat{x})) \leqq M$.*

Chapter 3

On cone D. C. Optimization And Conjugate duality

3.1 Problem Formulation

Before we discuss a problem formulation let us first discuss some points about convex cone and convex function.

Definition 3.1.1. *.[5] The set $C \subset R^n$ is a convex cone if it is a convex set and for all $x \in C$ and $0 \leq \lambda$ one has $\lambda x \in C$.*

Example 3.1.1. *(i). The set $C = \{(x_1, x_2) \in R^2 \mid x_2 \geq 2x_1, x_2 \geq \frac{-1}{2}x_1\}$ is a convex cone in R^2*

(ii). The set $C^1 = \{x_1, x_2, x_3 \in R^3 \mid x_1^2 + x_2^2 \leq x_3^2\}$ is a convex cone in R^3

Definition 3.1.2. *.[5] A convex cone is called pointed if it does not contain any subspace except the set containing only the origin.*

A pointed convex cone could be defined equivalently as a convex cone that does not contain any line through the origin.

Example 3.1.2. *If a convex cone $C \in R^2$ is not pointed, then it is either*

- *a line through the origin.*
- *a half space, or*
- R^2.

Definition 3.1.3. *.[5] (Cone convex function) Let X be a convex set in R^n, f be a function from X into R^p, and C be a convex cone in R^p. Then f is said to be C-convex if for any $x^1, x^2 \in X$ and $\lambda \in [0, 1], \lambda f(x^1) + (1 - \lambda)f(x^2) - f(\lambda x^1 + (1 - \lambda)x^2) \in C$.*

Proposition 3.1.1. *.[5] Let $X \subset R^n$ be a nonempty convex set, a function $f : X \to R^p$ and C be a convex cone in R^p. If the function f is C-convex then the set $f(X)$ is C-convex.*

Proof.

For $y^1, y^2 \in f(X) + C$ there exist $x^1, x^2 \in X$ and $c^1, c^2 \in C$ such that $Y^i = f(x^i) + c^i$ for $i = 1, 2$. Hence, we can drive for $\alpha \in [0,1], x^3 \in X$ and $c^3 \in C$ such that $\alpha y^1 + (1-\alpha)y^2 = \alpha(f(x^1) + c^1) + (1-\alpha)(f(x^2) + c^2) = \alpha f(x^1) + (1-\alpha)f(x^2) + \alpha c^1 + (1-\alpha)c^2 \leq f(\alpha x^1 + (1-\alpha)x^2) + c^3$, since C is convex and f is C-convex $= f(x^3) + C^3$, since X is convex $\subset f(X) + C$.

Definition 3.1.4. *.[5] Let $C \subseteq R^n$ be a convex cone, the dual cone C^* is defined by*

$C^* := \{z \in R^n \mid x^T \geq 0 \text{ for all } x \in C\}$

Every convex cone C has an associated dual cone.

3.2 Duality for the cone constrained convex optimization problem

In order to formulate the general cone constrained convex optimization problem we want to deal with the following sets and functions. Let X be a nonempty convex subset of R^n, C a nonempty closed convex cone in R^m, $f : X \to R$ a convex function and $g : X \to R^m$ a C-convex function. The problem we consider is

$$(P) \qquad \inf_{x \in X, g(x) \in -C} f(x)$$

The dual problem may be obtained by perturrbations.

The Lagrange dual problem to (P) is

$$(D^L) \qquad \sup_{q \in C^*} \inf_{x \in X} [f(x) + q^T g(x)].$$

where by $q^T g$ we denote the function defined on X whose value at any $x \in X$ is equal to $\sum_{j=1}^m q_j g_j(x)$, with $q = (q_1, \ldots q_m)^T$.

Now let us write the Fenchel dual problem to the inner infimum in (D^L). For $q \in C^*$, both f and $q^T g$ are real valued convex functions defined on X, so in order to apply rigourously Fenchel's duality theorem we have to consider their convex extensions to R^n, say $\bar{f}$ and $\widetilde{q^T} g$, which take the value $+\infty$ outside X. As $dom(\tilde{f}) = dom(q^T g) = X$ and $ri(X) \neq \emptyset$, due to the onvexity of the nonempty set X, we have

$$\inf_{x\in X}[f(x)+q^Tg(x)] = \inf_{x\in R^n}[\bar{f}(x)+\tilde{q^T}g(x)] = \sup_{p\in R^n}\{-\bar{f}^*(p)-\tilde{q^T}^* g(-p)\}$$

The Fenchel-Lagrange dual problem is obtained and it is

$$(D) \qquad \sup_{q\in C^*, p\in R^n}\{-f_x^*(p)-(q^Tg)_x^*(-p)\},$$

3.3 Optimality Conditions

Let X be a real linear topological vector space and we assume for this section that $Y = R^P$. Let $D \subset Y$ be a pointed, closed,convex cone which has a nonempty interior in Y. Assume that Y is partially ordered by the cone D.

Now consider the following unconstrained vector minimization problem

$$(P_1) \qquad minimize \;\; f(x) \quad s.t \quad x\in X.$$

Definition 3.3.1. *.[5] A function $f: X \to Y$ is said to be locally D-Lipschitz if for every $x \in X$, there exist a neighborhood $\beta(x,\delta)$ of x with radius $\delta > 0$ and such that $-L||x-y|| \leq_D f(x)-f(y) \leq_D L||x-y||, \forall x, y \in \beta(x,\delta)$.*

Here, L is called Lipschitz constant.

For a vector valued function $f : X \to \bar{Y}$ the directional derivative of f at a point $x_0 \in X$ in the direction $u \in X$ is given by

$$f'(x_0,u) := \; lim_{t\to 0^+}\frac{f(x_0+tu)-f(x_0)}{t}$$

if the limit exists.

It is clear that for D-convex functions this is always the case.

Lemma 3.3.1. *.[5] Suppose that f is directionally differentiable vector-valued function. Then $T \in \partial f(x_0)$ iff $f'(x_0,u) \not<_D T_U$ for any $u \in X$.*

Proof.

Let $T \in \partial f(x_0)$. Then we have $tf'(x_0,u)+0(t) = f(x_0+tu)-f(x_0) \not<_D T(tu)\forall t > 0$ and for all $u \in X$, where $0(t)/t \to 0ast \to 0$, or $f'(x_0,u) + 0(t)/t \not<_D T_u\forall t > 0$, which in turn implies that $f'(x_0,u) \not<_D T_u\forall u \in X$. The proof of the other side is obvious.

Note that for a D-convex function f, we have a stronger condition

$$f'(x_0;u) = \; lim_{t\to 0^+}\frac{f(x_0+tu)-f(x_0)}{t} \leq Df(x_0+u)-f(x_0)\;\forall u\in X \quad (3.1)$$

which together with the above lemma gives us

$$f'(x_0, u) \in D - MaxT_U \mid T \in \partial f(x_0) \tag{3.2}$$

A point $x_0 \in domf$ is said to be a local efficient point for (p_1) if there exists a neighborhood of x_0 such that $f(x) \not<_D f(x_0)$ for all x in the neighborhood.

Using the above two definitions we give the following optimality condition for (P_1) which is a necessary condition for local weak efficiency.

Proposition 3.3.1. *.[5] Let f be directionally differentiable at $x_0 \in X$. If x_0 is a local efficient point for (P_1), then $f'(x_0, u) \not<_D 0 \; \forall \; u \in X$.*

Proof.

since f is directionally differentiable at the point x_0, we have $f(x_0 + tu) = f(x_0) + tf'(x_0, u) + 0(t), \forall t > 0, \forall u \in X$, where $0(t) := 0(t, x_0, u)$ and $\lim t^{-1}[0(t, x_0, u)] = 0, t \to 0^+, \forall u \in X$. Then $f'(x_0, u) = t^{-1}[f(x_0 + tu) - f(x_0)] - 0(t)/t$. since x_0 is a local minimum point of f, the assertion of the proposition follows.

A slight modification (i.e, adding a local $D - Lipschitz$ criterion) to the above necessary condition will make it also sufficient as the following proposition shows.

Proposition 3.3.2. *.[5] Let f be a function which is directionally differentiable at a point $x_0 \in X$. If f is locally $D - Lipschitz$ in a neighborhood of x_0 and if $f'(x_0, u) \not\leq_D 0 \; \forall \; u \in X$ and $u \neq 0$, then x_0 is a local efficient point of f.*

Proof.

Suppose the contrary.Then there exists a net of vectors u_i with $||u_i|| = 1, \forall i$ in a neighborhood of x_0 and a sequence t_i with $t_i \geq 0, t_i \to 0$ and $u_i \to u$ as $i \to \infty$ such that $f(x_0, +t_i u_i) <_D f(x_0)$. But $f(x_0 + t_i u_i) - f(x_0) = f(x_0 + t_i u) - f(x_0 + f(x_0 + t_i u_i) - f(x_0 + t_i u) <_D 0$. Since f is locally $D - Lipschitz$, there exists $L \in D$ such that for sufficiently large i,

$-Lt_i||u_i - u|| \leq_D f(x_0 + t_i u_i) - f(x_0 + t_i u) \leq_D Lt_i||u_i - u||$.

Then $[f(x_0 + t_i u) - f(x_0)]/t_i \leq_D L||u_i - u||$ for sufficiently large i. Hence

$$f'(x_0, u) = \lim_{i\to\infty} \frac{f(x_0 + t_i u) - f(x_0)}{t_i} \leq \lim_{i\to\infty} L \mid\mid U_i - u \mid\mid = o, i.e. f'(x_0; u) \leq_{D_0},$$

contradicting the assumption. Therefore, the conclusion of the proposition is true.

A $D - convex$ function f is said to be proper if $f(x) >_D -\infty \; \forall x \in X$.

Definition 3.3.2. *.[5] A vector valued function $f : X \to \bar{Y}$ is said to be a $D-d.c.$ function if and only if it can be written as a difference of two proper cone-convex functions, i.e, $f(x) = g(x) - h(x)$, where g and h are D-convex and proper vector valued functions.*

Let $g, h : X \to \bar{Y}$ be D-convex proper vector valued functions. Then the function $f : X \to \bar{Y}$ given by f(x):= g(x)- h(x) is a D-d.c. function on X and f is locally $D - Lipschitz$ at each points of X and is directionally differentiable on X with $f'(x_0, u) = g'(x_0, u) - h'(x_0, u), \forall u, x_0 \in X$.

Consider the following $D - d.c.$ optimization problem

$$(P) \qquad\qquad minimize f(x) \qquad\qquad s.t. \;\; x \in X,$$

where $f = g - h$ and g and h are as above. We adopt in the sequel the convention $+\infty - (+\infty) = +\infty$. To state the necessary condition for minimality we first define the strong subdifferential of a vector valued function f at a point x_0, denoted by

$\partial_s f(x_0)$, and defined by

$$\partial_s f(x_0) := \{T \in L(X, Y) \mid T(x - x_0) \leq_D f(x) - f(x_0) \;\; \forall x \in X\}.$$

and

$T \in \partial_s f(x_0) \qquad iff \qquad T_u \leq_D f'(x_0, u) \forall u \in X.$

Applying the condition (3.1), for D-convex $f, \partial_s f(x_0)$ is nonempty as it at least contains the directional derivative of f at the point x_0. Note also that $\partial_s f(x_0) \subseteq \partial f(x_0)$ for any x_0 at which f is sub differentiable.

Theorem 3.3.1. *.[5](Necessary condition). For $f = g - h$ to attain its local D-minimal value at a point $x_0 \in X$, it is necessary that $\partial_s h(x_0) \subseteq g(x_0)$.*

Proof.

If x_0 is a local minimum point for f, then there exists a neighborhood υ of x_0 such that $f(x) \not<_D f(x_0) \forall x \in \upsilon$, or $g(x) - g(x_0) \not<_D h(x) - h(x_0), \forall x \in \upsilon$.

But for $T \in \partial_s h(x_0)$ we have $T(x - x_0) \leq_D h(x) - h(x_0), \forall x \in X$. Then one can conclude that $g(x) - g(x_0) \not<_D T(x - x_0)$ or $T \in \partial g(x_0)$.

For otherwise, if $g(x) - g(x_0) <_D T(x - x_0)$, together with the relation $T(x - x_0) \leq_D h(x) - h(x_0)$ we will have

$g(x) - g(x_0) <_D h(x) - h(x_0)$, which is a contradiction. Hence the theorem is proved.

Theorem 3.3.2. *.[5]*

(sufficient condition). If $\partial_s h(x_0) \subseteq int \partial g(x_0)$, then the criterion vector x_0 is a local efficient point for (P).

Proof.

Let $f(x) := g(x) - h(x) \ \forall x \in X$. Then f is directionally differentiable on X and it is locally $D - Lipschitz$. From the assumption of the theorem,we have $\partial_s h(x_0) \subseteq int\partial g(x_0)$, and from the relation (3.2) it follows immediately that

$g'(x_0, u) \nleq_D h'(x_0, u), \forall u \in X, u \neq 0.$

Hence invoking proposition 3.3.2 we have the conclusion of the theorem.

3.4 Conjugate Duality in cone D.C.Optimization.

In this section we develop a conjugate duality in cone D.C. Optimization. Conjugate duality was fully developed in scalar optimization. Some concepts such as conjugate maps and subgradients are introduced for vector valued, point to set maps. These concepts enable us to develop the conjugate duality in vector optimization.

3.4.1 conjugate maps

First we discus the concepts of point-to-set maps

- Point-to-set maps

A point-to-set map F from a set X into a set Y is a map that associates a subset of Y with each point of X. Equivalently, F can be viewed as a function from the set X into the power set 2^Y. In a multiobjective optimization problem,it is rather difficult to obtain a unique optimal solution. Solving the problem often leads to a solution set. Thus if the problem has a parameter,the solution set defines a point-to-set maps from the parameter space into the objective (decision) space. When f is an extended real-valued function on R^n (i.e. when $f : R^n \to \bar{R}$) its conjugate function f^* is defined by

$f^*(x^*) = \sup \langle x^*, x\rangle - f(x) : x \in R^n$, for $x^* \in R^n$

In order to consider the vector-valued case we must define a paired space of R^n with respect to R^P. A most natural paired space is the set of all $p \times n$ matrices which is denoted by $R^{p\times n}$. However,its dimension $p \times n$ is often too large. Another idea is to take $(R^n)^* = R^n$ as paired space as in the scalar.

Definition 3.4.1. *.[5] Let F be a point -to-set maps from R^n into R^p. The point -to- set map $F^* : R^{p\times n} \to R^p$ defined by*

$$F^*(T) = \sup \bigcup_{x\in R^n} [Tx - F(x)], \ for \ \ T \in R^{p\times n}$$

is called the conjugate map of F.

Moreover, A point - to- set maps F^{**} defined by $F^{**}(x) = \sup \bigcup [Tx - F^*(T)]$, for $x \in R^n, T \in R^{p\times n}$ is called the bi conjugate map of F.

When f is a vector-valued function from R^n to $R^p \cup \{+\infty\}$, let $domf = \{x \in R^n : f(x) \neq +\infty\}$ and define the conjugate map f^* of f by

$$f^*(T) = \sup Tx - f(x) : x \in domf$$

Here $+\infty$ is the imaginary point whose every component is $+\infty$. We identify the function f as the point-to-set map that is equal to $f(x)$ for $x \in domf$ and is empty otherwise. The bi-conjugate map f^{**} can be defined as the conjugate map of f^*.

Proposition 3.4.1. *.[5] Let F be a point- to -set map from R^n into R^p and $x \in R^n$. If we define another point-to -set map G by $G(x) = F(x+\bar{x}) \forall x \in X$, then*

$(i) G^*(T) = F^* - T\bar{x}, \forall T \in R^{p\times n}$

$(ii) G^{**}(x) = F^{**}(x + \bar{x}), \forall x \in R^n$.

Proof.(i)

$$G^*(T) = \sup \bigcup_x [Tx - G(x)]$$

$$= \sup \bigcup_x [Tx - F(x + \bar{x})]$$

$$= \sup \bigcup_{x'} \{[Tx' - F(x')] - T\bar{x}\}$$

$$= \sup \bigcup_{x'} \{[Tx' - F(x')] - T\bar{x}\}$$

$$= F^*(T) - T\bar{x}$$

(ii)

$$G^{**}(x) = \sup \bigcup_T [Tx - G^*(T)]$$

$$= \sup \bigcup_T [Tx - F^*(T) + T\bar{x}]$$

$$= F^{**}(x + \bar{x})$$

Proposition 3.4.2. *.[5] Let F be a point- to- set map From R^n into R^p and $\bar{Y} \in R^p$. Then*

$(i).(F + \bar{Y})^*(T) = F^*(T) - \bar{Y}$

$(ii).(F + \bar{Y})^{**}(x) = F^{**}(x) + \bar{Y}$.

Proof. (i)

$$(F+\bar{Y})^*(T) = \sup \bigcup_x [Tx - F(x) - \bar{Y}]$$

$$= \sup \bigcup_x [Tx - F(x)] - \bar{Y}$$

$$= F^*(T) - \bar{Y}$$

(ii)

$$(F+\bar{Y})^{**}(x) = \sup \bigcup_T [Tx - F(T) + \bar{y}]$$

$$= \sup \bigcup_T [Tx - F(T)] + \bar{Y}$$

$$= F^{**}(x) + \bar{Y}$$

Lemma 3.4.1. *.[5] [8] Let* $\inf F$ *be another point-to-set-maps from* R^n *to* R^p *defined by* $(\inf F)(x) = \inf F(x)\ \forall x \in R^n$*. Then* $F^*(T) = (\inf F)^*(T)$ *and* $F^{**}(x) = (\inf F)^{**}(x)$

Proof.

$$(\inf F)^*(T) = \sup \bigcup_{x \in R^n} [Tx - (\inf F)(x)]$$

$$= \sup \bigcup_{x \in R^n} [Tx - \inf F(x)]$$

$$= \sup \bigcup_{x \in R^n} \sup[Tx - F(x)]$$

$$= \sup \bigcup_{x \in R^n} [Tx - F(x)]$$

$$= F^*(T)$$

$F^{**}(x) = (\inf F)^{**}(x)$ follows directly from the above relation.

Proposition 3.4.3. *.[5] Let* F *be a point-to-set map from* R^n *to* R^p*. If* $y \in F(x)$ *and* $y' \in F^*(T)$*, then*

$$y + y' \nleq T\hat{x}(i.e. T\hat{x} - (y+y') \notin R^p_+ \backslash 0.$$

Proof.

Since $y \in F(\hat{x}), T\hat{x} - y \in \cup_x[Tx - F(x)]$. Hence, if $y' \leq T\hat{x} - y$, it contradicts the assumption

$$y' \in F^*(T) = \sup \cup_x[Tx - F(x)].$$

Lemma 3.4.2. *.[5] Let F_1 and F_2 be point-to-set maps from R^n into R^p. Then*

$$Max \bigcup_x [F_1(x) + F_2(x)] \subset Max \bigcup_x [F_1(x) + MaxF_2(x)]$$

If $MaxF_2(x)$ is externally stable (i.e.$F_2 \subset MaxF_2(x) - R^p_+$) for every $x \in R^n$, then the converse inclusion also holds.

Proof.

Let $\hat{y} \in Max \bigcup_x [F_1(x) + F_2(x)]$. Then there exists $\hat{x} \in R^n$ such that $\hat{y} = y^1 + y^2$ for some $y^1 \in F_1(\hat{x})$ and $y^2 \in F_2(\hat{x})$. If we suppose that $y^2 \notin MaxF_2(\hat{x})$, there exists $\bar{y}^2 \in F_2(\hat{x})$ such that $y^2 \leq \bar{y}^2$. Then $\hat{y} = y_1 + y_2 \leq y_1 + \bar{y}^2$ which is contradiction. Therefore, $y^2 \in MaxF_2(\hat{x})$, since $\bigcup_x [F_1(x) + F_2(x)] \supset \bigcup_x [F_1(x) + MaxF_2(x)]$, then

$\hat{y} \in Max \bigcup_x [F_1(x) + MaxF_2(x)]$

Next, suppose that $MaxF_2(x)$ is externally stable for every x, then

$$F_2(x) - R^p_+ = MaxF_2(x) - R^p_+, for\ every\ x.$$

Thus

$$F_1(x) + F_2(x) - R^p_+ = F_1(x)MaxF_2(x) - R^p_+, for\ every\ x.$$

$$\bigcup [F_1(x)F_2(x)] - R^p_+ - \bigcup_x [F_1(x) + MaxF_2(x)] - R^p_+$$

Taking the Max of both sides, we have

$$Max \bigcup_x [F_1(x) + F_2(x)] = Max \bigcup_x [F_1(x) + MaxF_2(x)]$$

3.4.2 sub gradients

First define the sub gradient at a point

Definition 3.4.2. *.[5] Let f be a convex function from R^n to R. A vector $x^* \in R^n$ is said to be a sub gradient of f at x if*

$f(x') \geq f(x) + \langle x^*, x' - x \rangle, \forall x' \in R^n.$

The set of all sub gradients of f at x is called the sub differential of f at x and is denoted by

$\partial f(x)$. If $\partial f(x)$ is not empty, f is said to be sub differential at x.
In this subsection we introduce the concepts of subgradients of vector-valued functions and point-to-set maps. A subgradient x^* of a scalar-valued function f at x is defined in definition 3.4.2. The definition can be formally extended to a non convex function, though it essentially requires convexity (least locally). We may extend the definition of subgradients to the vector-valued case. Which is also an intuitively direct extension of Definition 3.4.2.

Definition 3.4.3. *.[5] (i). Let f be a function from R^n to $R^p \cup \{+\infty\}$. A $p \times n$ matrix T is said to be a subgradient of f at $\hat{x} \in domf$ if*

$$f(x) \nleq f(\hat{x}) + T(x + \hat{x}) \forall x \in R^n$$

i.e, if

$f(\hat{x}) - T_{\hat{x}} \in \min f(x) - T_x \in R^p : x \in R^n = \min f(x) - T_x : x \in domf.$

The set of all subgradients of f at $\hat{x}$ is called the subdifferential of f at $\hat{x}$ and is denoted by $\partial f(\hat{x})$. If $\partial f(\hat{x})$ is not empty, then f is said to be subdifferentiable at $\hat{x}$.
(ii). Let F be a point-to-set map from R^n into R^p and $\hat{y} \in F(\hat{x}).p \times n$ matrix is said to be a subgradient of F at $(\hat{x}, \hat{y})$ if

$$\hat{y} - T_{\hat{x}} \in max \bigcup_{x \in R^n} [T_x - F(x)]$$

The set of all subgradients of F at $(\hat{x}, \hat{y})$ is called the subdifferential of F at $(\hat{x}, \hat{y})$ and is denoted by $\partial F(\hat{x}, \hat{y})$. When $\partial F(\hat{x}, \hat{y}) \neq \emptyset$ for every $\hat{y} \in F(\hat{x})$, F is said to be subdifferentiable at $\hat{x}$.

Let X be a real linear topological vector space and Y be a locally convex linear topological vector space. Assume that Y is partially ordered by a pointed, closed, convex cone D which has a nonempty interior. Let g and h be vector valued D-convex functions from X to $\bar{Y}$. In the next part we assume that the functions g and h are proper (note that, a D-convex function P is said to be proper in X iff $P(x) >_D -\infty \forall x \in X$).

Now consider the $D - d.c.$ Optimization problem

$$(P) \qquad\qquad minimize f(x) \quad s.t. \;\; x \in X.$$

Solving this problem means to find the set

$D - \inf(p) = D - \inf g(x) - h(x) \mid x \in X.$

Let $U \subseteq X$ be another locally convex linear topological vector space and U^* be its dual space. We introduce a special perturbation function

$\varphi : X \times U \to \bar{Y}$ such that $\varphi(x,u) = h(x+u) - g(x) \quad \forall (x,u) \in X \times U$. Then $\varphi(x,0) = -f(x) \forall x \in X$. For $\Lambda \in M := L(U,Y)$, the space of all linear continuous operators from U to Y, let the Lagrangian of problem (P) be given by $\Gamma(x,\Lambda) = D - Sup \Lambda u - \varphi(x,u) \mid u \in U$

$$= D - Sup \Lambda(x,u) + g(x) - h(x+u) - \Lambda_x \mid u \in U$$

$$= h^*(\Lambda) + g(x) - \Lambda_x,$$

where $h^*(\Lambda)$ denotes the conjugate map of h. Now we put $-J(\Lambda) := D - Sup \bigcup \Gamma(x,\Lambda), x \in X$ which is equal to $g^*(\Lambda) - h^*(\Lambda)$. Then the dual optimization problem for (P) is written as

$(Dual)$ $\quad\quad$ $minimize \;\; h^*(\Lambda) - g^*(\Lambda), \Lambda \in M,$

which is equivalent to the formulation

$(Dual)$ $\quad\quad$ $D - Inf \bigcup \; [h^*(\Lambda) - g^*(\Lambda)], \Lambda \in M.$

We can observe the symmetry between the primal problem and the dual one. But since both $h^*(.)$ and $g^*(.)$ are set valued maps the dual problem $(Dual)$ is not a usual vector optimization problem. However, it can be understood as determining the set

$$D - Inf \bigcup [h^*(\Lambda) - g^*(\Lambda)], \Lambda \in M$$

On the other hand,

$$D - \sup \bigcup [-\Gamma(x,\Lambda)], x \in X = D - Sup \bigcup D - sup \Lambda u - \varphi(x,u) \mid u \in U, x \in X.$$

$$= D - Sup \bigcup \Lambda u + 0x - \varphi(x,u), (x,u) \in X \times U.$$

$$= \varphi^*(0,\Lambda).$$

Therefore, $\varphi(0,\Lambda) = h^*(\Lambda) - g^*(\Lambda)$. This will help us in proving the following weak duality results. In the next theorem it is proved that any feasible value of the primal problem is not above any feasible value of the dual problem.

Theorem 3.4.1. *.[5] For any $x \in X$ and $\Lambda \in M, f(x) \notin A(h^*(\Lambda) - g^*(\Lambda))$ and thus*

$D - Inf(P) \cap A(D - Inf(Dual)) = \emptyset.$

Proof.

Suppose the contrary. Then there exists $y \in h^*(\Lambda) - g^*(\Lambda)$ such that $y <_D f(x)$. But since

$$h^*(\Lambda) - g^*(\Lambda) = D - Sup \bigcup \Lambda u + 0x - \varphi(x,u), (x,u) \in X \times U.$$

$y \not<_D \Lambda u - \varphi(x,u) \forall u \in U$. In particular, if we put $u = 0$ and noting that$f(x) = -\varphi(x,0)$, it follows that $y \not< D - \varphi(x,0) = f(x), \forall y \in h^*(\Lambda) - g^*(\Lambda)$, which contradicts our assumption. Hence the theorem is proved.

The above theorem assures us that for any $x, f(x) \in D - Inf \bigcup [h^*(\Lambda) - g^*(\Lambda)], \Lambda \in M$. If we can find some $\Lambda_0 \in M$ such that $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$ for some x_0, then it means that Λ_0 solves (Dual). The next theorem reflects this fact.

Theorem 3.4.2. *.[5] If x_0 solves (P), then there exists some $\Lambda_0 \in M$ which solves $(Dual)$.*

Proof.

If x_0 solves (p), then since $f(x) = -\varphi(x,0), the same x_0$ solve the problem (P') $\qquad min - \varphi(x_0,0), x \in X.$

Then there exists some $\Lambda_0 \in M such that (0, \Lambda_0 \in \partial\varphi(x_0,0)$. But this in turn implies that $(0,\Lambda_0)(x_0,0)^T - \varphi(x_0,0) \in \varphi^*(0,\Lambda_0)$, which means that $f(x_0) = -\varphi(x_0,0) \in \varphi^*(0,\Lambda_0) = h^*(\Lambda_0) - g^*(\Lambda_0)$.

Now assume that Λ_0 does not solve $(Dual)$. then there exists $\Lambda \in M$ such that $(h^*(\Lambda) - g^*(\Lambda)) \cap B(h^*(\Lambda_0) - g^*(\Lambda_0) \neq \emptyset$. since $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$, there exists $y \in h^*(\Lambda) - g^*(\Lambda)$ such that $y < Df(x_0)$. But this contradicts the statement in theorem 3.4.1. Hence Λ_0 solves $(Dual)$.

Corollary 3.4.1. *.[5] If x_0 solves (P) and $\Lambda_0 \in \partial_s h(x_0)$, then Λ_0 solves $(Dual)$.*

Proof.

From the assumption we have $\partial_s h(x_0) \subseteq \partial g(x_0)$, and the relation $T \in \partial f(x_0)$ iff $Tx_0 - f(x_0) \in f^*(T)$ gives

$g(x_0) - h(x_0) = (\Lambda_0 x_0 - h(x_0)) - (\Lambda_0 x_0 - g(x_0)) \in h^*(\Lambda_0) - g^*(\Lambda_0)$. That is, $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$. Hence Λ_0 solves $(Dual)$.

Proposition 3.4.4. *.[5] Let $x_0 \in dom f$. If $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$ for some $\Lambda_0 \in M$, then x_0 solves (P) at least locally.*

Proof.

By the assumption we have

$$-\varphi(x_0, 0) = f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0) = \varphi^*(0, \Lambda_0),$$

which is equivalent to

$$(0, \Lambda_0) \in \partial\varphi(x_0, 0) \qquad (*)$$

Then from the relation $-\varphi(x_0, 0) = f(x_0)$ and $(*)$ we can see that $0 \in \partial f(x_0, or$ x_0 is a local minimum of $f(x)$.

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