SUPERPOSITION OF TWO-LEVEL LASER LIGHT BEAMS

By

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Abstract

We study the quantum properties of the light generated by a two-level laser in which two-level atoms available in a closed cavity are pumped to the upper level by means of electron bombardment. We present a brief discussion of the number and coherent states for an arbitrary commutator of the annihilation and creation operators.

Moreover, taking into account the interaction of the two-level atoms with a resonant cavity mode and the damping of the cavity mode by a vacuum reservoir, we obtain the equations of evolution of atomic and cavity mode operators. The solutions of these equations is used to determine the Q function for the light produced by a two-level laser. Employing this Q function, we compute the mean and variance of photon number, and the quadrature variance. The two-level laser generates coherent light when operating well above threshold and chaotic light when operating at threshold.

On the other hand, we obtain the Q function for the superposition of a pair of two-level laser light beams. Applying the resulting Q function, we have calculated the mean and variance of the photon number, and the quadrature variance. The mean photon number of the superposed light beams turns out to be a simple sum of the mean photon numbers of the two light beams. In addition, it is found that the variance of the photon number of the superposed light beams is four times the variance of the photon number of a single light beam produced by the two-level laser.

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Chapter 1

Introduction

The quantum description of radiation is one of the central topic in quantum optics. Various quantum distribution functions are used for this purpose, because they provide convenient means of describing the quantum properties of light modes. Among the most widely used distribution functions is the Q function [1,2]. Q functions are c-number functions which corresponds to the normal ordering of the density operator. Making use of the advantages offered by these functions can ease the rigor of obtaining various expectation values and the superposition of two or more light beams. To this effect, it turns out to be very useful to express the Q-function in terms of the anti-normally ordered characteristic function.

A two-level laser is a source of coherent or chaotic light emitted by two-level atoms inside a cavity coupled to a vacuum reservoir via a single-port mirror. In one model of a laser, two-level atoms initially in the upper level are injected at a constant rate into a cavity and removed after they have decayed due to spontaneous emission [3,4]. In another model, the two-level atoms available in a cavity are pumped to the upper level by some convenient means such as electron bombardment [4, 5].

There has been a considerable interest to study the quantum properties of the light generated by a two-level laser [6-13]. It is found that the light generated by this laser operating well above threshold is coherent and the light generated by the same laser operating at threshold is chaotic.

We seek here to analyze the quantum properties of the light emitted by two-level atoms available in a closed cavity and pumped to the upper level at a constant rate.

Thus taking into account the interaction of the two-level atoms with a resonant cavity mode and the damping of the cavity mode by a vacuum reservoir, we obtain the equations of evolution of atomic and cavity operators. Employing the solutions of these equations, we determine the Q function of the light produced by the two-level laser. Using this Q function we calculate the mean photon number, the variance of photon number and the quadrature variance. Furthermore, using the resulting Q function, we obtain the Q function for the superposition of a pair of two-level laser light beams. Employing this Q-function, we determine the mean photon number, the variance of the photon number and the quadrature variance of the superposed two-level laser light beams. We carry out this analysis by putting the noise operators associated with the vacuum reservoir in normal order and without considering the interaction of the two-level atoms with the vacuum reservoir outside the cavity.

Chapter 2

Quantum analysis with arbitrary commutator

2.1 The number and coherent states

We consider here a single mode light available in a closed cavity that can be represented by the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfying the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \lambda, \tag{2.1.1}$$

where λ is a real constant.

The eigenvalue equation for the number operator is expressible as

$$\hat{n}|n\rangle = n|n\rangle, \tag{2.1.2}$$

in which $|n\rangle$ is the number state. In order to determine the action of the operator $\hat{a}$ on the number state $|n\rangle$, one can write [2, 3]

$$\hat{a}|n\rangle = c|m\rangle. \tag{2.1.3}$$

We next proceed to fix the value of c and m. To this end, we note that

$$\langle n|\hat{a}^\dagger = c^*\langle m|. \tag{2.1.4}$$

Thus on taking the inner product of Eq. (2.1.3) and Eq. (2.1.4), we get

$$n\langle n|n\rangle = |c|^2\langle m|m\rangle. \tag{2.1.5}$$

Now assuming that $\langle n|n\rangle$ and $\langle m|m\rangle$ to be normalized and taking c to be real, we find

$$c = \sqrt{n}. \tag{2.1.6}$$

Hence in view of this result, Eq. (2.1.3) can be rewritten as

$$\hat{a}|n\rangle = \sqrt{n}|m\rangle. \tag{2.1.7}$$

Furthermore, multiplying Eq. (2.1.7) by $\hat{a}^\dagger$, we have

$$\hat{a}^\dagger\hat{a}|n\rangle = \sqrt{n}\hat{a}^\dagger|m\rangle. \tag{2.1.8}$$

We then see that

$$\langle n|\hat{a}^\dagger\hat{a} = \sqrt{n}\langle m|\hat{a}. \tag{2.1.9}$$

Now on taking the inner product of Eqs. (2.1.8) and (2.1.9) and using the commutation relation given by Eq. (2.1.1), we arrive at

$$m = n - \lambda. \tag{2.1.10}$$

Finally on account of this result, Eq. (2.1.7) takes the form

$$\hat{a}|n\rangle = \sqrt{n}|n-\lambda\rangle. \tag{2.1.11}$$

It can also be established in a similar manner that

$$\hat{a}^\dagger|n\rangle = \sqrt{n+\lambda}|n+\lambda\rangle. \tag{2.1.12}$$

Furthermore, we see that

$$\hat{a}^\dagger|0\rangle = \sqrt{\lambda}|\lambda\rangle,$$

$$\hat{a}^{\dagger 2}|0\rangle = \sqrt{\lambda^2 \times 2!}|2\lambda\rangle,$$

$$\hat{a}^{\dagger 3}|0\rangle = \sqrt{\lambda^3 \times 3!}|3\lambda\rangle,$$

$$\hat{a}^{\dagger n}|0\rangle = \sqrt{\lambda^n n!}|n\lambda\rangle$$

or

$$|n\lambda\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{\lambda^n n!}}|0\rangle \tag{2.1.13}$$

with $n = 0, 1, 2....$

A coherent state $|\alpha\rangle$ is defined in terms of the displacement operator

$$\hat{D}(\alpha) = exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \tag{2.1.14}$$

as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \tag{2.1.15}$$

where α is a complex number. Besides this, the power series expansion of a function $f(\hat{a}^\dagger, \hat{a})$ in the anti-normal order, is expressible as

$$\begin{aligned}
[\hat{a}, f(\hat{a}^\dagger, \hat{a})] &= \sum_{lm} C_{lm}[\hat{a}, \hat{a}^l\hat{a}^{\dagger m}] \\
&= \sum_{lm} C_{lm}\hat{a}^l(\hat{a}\hat{a}^{\dagger m} - \hat{a}^{\dagger m}\hat{a}) \\
&= \sum_{lm} C_{lm}\hat{a}^l[\hat{a}, \hat{a}^{\dagger m}].
\end{aligned}$$

We see that

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \lambda, \\
[\hat{a}, \hat{a}^{\dagger 2}] &= [\hat{a}, \hat{a}^\dagger\hat{a}^\dagger] \\
&= [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger + \hat{a}\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] \\
&= 2\lambda\hat{a}^\dagger,
\end{aligned}$$

and

$$\begin{aligned}
[\hat{a}, \hat{a}^{\dagger 3}] &= [\hat{a}, \hat{a}^\dagger\hat{a}^{\dagger 2}] \\
&= [\hat{a}, \hat{a}^\dagger]\hat{a}^{\dagger 2} + \hat{a}^\dagger[\hat{a}, \hat{a}^{\dagger 2}] \\
&= 3\lambda\hat{a}^{\dagger 2}.
\end{aligned}$$

In general, one can write

$$[\hat{a}, \hat{a}^{\dagger m}] = m\lambda\hat{a}^{\dagger m-1},$$

which can be rewritten as

$$[\hat{a}, \hat{a}^{\dagger m}] = \lambda\frac{\partial}{\partial\hat{a}^\dagger}\hat{a}^{\dagger m}.$$

It then follows that

$$[\hat{a}, f(\hat{a}^\dagger, \hat{a})] = \lambda \frac{\partial}{\partial \hat{a}^\dagger} f(\hat{a}^\dagger, \hat{a}). \tag{2.1.16}$$

One can also show in a similar manner that

$$[\hat{a}^\dagger, f(\hat{a}^\dagger, \hat{a})] = -\lambda \frac{\partial}{\partial \hat{a}} f(\hat{a}^\dagger, \hat{a}). \tag{2.1.17}$$

Since $\hat{D}^\dagger(\alpha)=\hat{D}(-\alpha)$, the displacement operator is unitary. Employing the commutation relation given by Eq. (2.1.16), we find

$$[\hat{a}, \hat{D}(\alpha)] = \lambda\alpha\hat{D}(\alpha). \tag{2.1.18}$$

Applying this result, we readily obtain

$$\hat{D}(-\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \lambda\alpha. \tag{2.1.19}$$

In addition, using Eq. (2.1.17), one can establish that

$$\hat{D}(-\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \lambda\alpha^*. \tag{2.1.20}$$

Furthermore, we note that

$$\langle\alpha|\alpha\rangle = \langle 0|\hat{D}(-\alpha)\hat{D}(\alpha)|0\rangle, \tag{2.1.21}$$

so that on account of the unitary property of $\hat{D}(\alpha)$ and the fact that $\langle 0|0\rangle = 1$, we have

$$\langle\alpha|\alpha\rangle = 1, \tag{2.1.22}$$

which shows that the coherent state is normalized. Moreover, we see that

$$\hat{a}|\alpha\rangle = \hat{a}\hat{D}(\alpha)|0\rangle \tag{2.1.23}$$

and taking into account the unitary property of the displacement operator, we have

$$\hat{a}|\alpha\rangle = \hat{D}(\alpha)\hat{D}(-\alpha)\hat{a}\hat{D}(\alpha)|0\rangle. \tag{2.1.24}$$

On account of Eq. (2.1.19), we rewrite Eq. (2.1.24) as

$$\hat{a}|\alpha\rangle = \hat{D}(\alpha)(\hat{a} + \lambda\alpha)|0\rangle. \tag{2.1.25}$$

It then follows that

$$\hat{a}|\alpha\rangle = \lambda\alpha|\alpha\rangle. \tag{2.1.26}$$

With the aid of the Baker-Hausdorff identity [14]

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}, \tag{2.1.27}$$

the displacement operator can be put in the normal order as

$$\hat{D}(\alpha) = e^{-\frac{\lambda}{2}\alpha^*\alpha}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \tag{2.1.28}$$

and hence using the fact that

$$e^{-\alpha^*\hat{a}}|0\rangle = 0, \tag{2.1.29}$$

one can also express a coherent state in terms of the displacement operator as

$$|\alpha\rangle = e^{-\frac{\lambda}{2}\alpha^*\alpha}e^{\alpha\hat{a}^\dagger}|0\rangle. \tag{2.1.30}$$

Now expanding $e^{\alpha\hat{a}^\dagger}$ in power series and taking into account Eq. (2.1.13), we get

$$|\alpha\rangle = e^{-\frac{\lambda}{2}\alpha^*\alpha}\sum_{n=0}^{\infty}\frac{\alpha^n\sqrt{\lambda^n}}{\sqrt{n!}}|n\lambda\rangle, \tag{2.1.31}$$

which is an expression for a coherent state in terms of the number state. Since the number states are eigenstates of a Hermitian operator, they are orthogonal. Moreover, we consider the number states to be normalized so that they become orthonormal. Employing Eq. (2.1.13), one can write

$$\langle n\lambda|m\lambda\rangle = \frac{1}{\sqrt{\lambda^{n+m}n!m!}}\langle 0|\hat{a}^n\hat{a}^{\dagger m}|0\rangle. \tag{2.1.32}$$

It then follows that

$$\langle n\lambda|m\lambda\rangle = \delta_{nm}, \tag{2.1.33}$$

where

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{2.1.34}$$

Therefore, using the fact that a set of orthonormal eigenstates is complete [2], the number states must satisfy the completeness relation

$$\hat{I} = \sum_{n=0}^{\infty} |n\lambda\rangle\langle n\lambda|. \tag{2.1.35}$$

We also note that

$$\begin{aligned} \langle n\lambda|\alpha\rangle &= e^{-\frac{\lambda}{2}\alpha^*\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m \lambda^{\frac{m}{2}}}{\sqrt{m!}} \langle n\lambda|m\lambda\rangle \\ &= e^{-\frac{\lambda}{2}\alpha^*\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m \lambda^{\frac{m}{2}}}{\sqrt{m!}} \delta_{nm}, \end{aligned} \tag{2.1.36}$$

from which follows

$$\langle n\lambda|\alpha\rangle = e^{-\frac{\lambda}{2}\alpha^*\alpha} \frac{\alpha^n \lambda^{\frac{n}{2}}}{\sqrt{n!}}. \tag{2.1.37}$$

We now wish to obtain the explicit form of the inner product of two coherent states. Expressing $\langle\alpha|$ and $|\beta\rangle$ as

$$\langle\alpha| = e^{-\frac{\lambda}{2}\alpha^*\alpha} \sum_{n=0}^{\infty} \frac{\alpha^{*n} \lambda^{\frac{n}{2}}}{\sqrt{n!}} \langle n\lambda| \tag{2.1.38}$$

and

$$|\beta\rangle = e^{-\frac{\lambda}{2}\beta^*\beta} \sum_{m=0}^{\infty} \frac{\beta^m \lambda^{\frac{m}{2}}}{\sqrt{m!}} |m\lambda\rangle, \tag{2.1.39}$$

we easily find

$$\begin{aligned} \langle\alpha|\beta\rangle &= e^{-\frac{\lambda}{2}\alpha^*\alpha} e^{-\frac{\lambda}{2}\beta^*\beta} \sum_{n,m} \frac{\alpha^{*n}\beta^m \lambda^{\frac{n}{2}} \lambda^{\frac{m}{2}}}{\sqrt{n!}\sqrt{m!}} \langle n\lambda|m\lambda\rangle \\ &= exp\left[-\frac{\lambda}{2}\alpha^*\alpha - \frac{\lambda}{2}\beta^*\beta\right] \sum_n \frac{\alpha^{*n}\beta^n\lambda^n}{n!} \\ &= exp\left[\lambda\alpha^*\beta - \frac{\lambda}{2}\alpha^*\alpha - \frac{\lambda}{2}\beta^*\beta\right]. \end{aligned} \tag{2.1.40}$$

We see that the coherent states are not orthogonal.

On the basis of Eq. (2.1.31), the state $|\beta + \gamma\rangle$ takes the form

$$|\beta + \gamma\rangle = e^{-\frac{\lambda}{2}|\beta+\gamma|^2} \sum_{n=0}^{\infty} \frac{(\beta+\gamma)^n \lambda^{\frac{n}{2}}}{\sqrt{n!}} |n\lambda\rangle. \tag{2.1.41}$$

Hence the explicit form of the inner product of two states $\langle\alpha|$ and $|\beta+\gamma\rangle$ can be written as

$$\begin{aligned}\langle\alpha|\beta+\gamma\rangle &= e^{-\frac{\lambda}{2}\alpha^*\alpha}e^{-\frac{\lambda}{2}|\beta+\gamma|^2}\sum_{n,m}\frac{\alpha^{*n}(\beta+\gamma)^m\lambda^{\frac{n}{2}}\lambda^{\frac{m}{2}}}{\sqrt{n!}\sqrt{m!}}\langle n\lambda|m\lambda\rangle \\ &= exp\left[-\frac{\lambda}{2}\alpha^*\alpha-\frac{\lambda}{2}|\beta+\gamma|^2\right]\sum_n\frac{\alpha^{*n}(\beta+\gamma)^n\lambda^n}{n!} \\ &= exp\left[\lambda\alpha^*(\beta+\gamma)-\frac{\lambda}{2}|\alpha|^2-\frac{\lambda}{2}|\beta+\gamma|^2\right]. \qquad (2.1.42)\end{aligned}$$

Moreover, one can write

$$\hat{D}(\gamma)|\beta\rangle = exp(\gamma\hat{a}^\dagger-\gamma^*\hat{a})\hat{D}(\beta)|0\rangle. \qquad (2.1.43)$$

With the aid of Eq. (2.1.14), this is expressible as

$$\hat{D}(\gamma)|\beta\rangle = exp(\gamma\hat{a}^\dagger-\gamma^*\hat{a})exp(\beta\hat{a}^\dagger-\beta^*\hat{a})|0\rangle. \qquad (2.1.44)$$

Employing the identity given by Eq. (2.1.27), Eq.(2.1.44) can be rewritten as

$$\begin{aligned}\hat{D}(\gamma)|\beta\rangle &= exp\left[\frac{\lambda}{2}\gamma\beta^*-\frac{\lambda}{2}\gamma^*\beta\right]exp[(\gamma+\beta)\hat{a}^\dagger-(\gamma^*+\beta^*)\hat{a}]|0\rangle \\ &= exp\left[\frac{\lambda}{2}\gamma\beta^*-\frac{\lambda}{2}\gamma^*\beta\right]|\gamma+\beta\rangle. \qquad (2.1.45)\end{aligned}$$

We now proceed to determine the action of $\frac{\partial}{\partial\beta}$ on $|\beta\rangle\langle\beta|$. To this end, one can write

$$\frac{\partial}{\partial\beta}|\beta\rangle\langle\beta| = \left(\frac{\partial}{\partial\beta}|\beta\rangle\right)\langle\beta| + |\beta\rangle\left(\frac{\partial}{\partial\beta}\langle\beta|\right). \qquad (2.1.46)$$

With the help of Eq. (2.1.15), we have

$$\frac{\partial}{\partial\beta}|\beta\rangle = \frac{\partial}{\partial\beta}\hat{D}(\beta)|0\rangle. \qquad (2.1.47)$$

In view of Eq. (2.1.28), this can be rewritten as

$$\frac{\partial}{\partial\beta}|\beta\rangle = \frac{\partial}{\partial\beta}\left(e^{-\frac{\lambda}{2}\beta^*\beta}e^{\beta\hat{a}^\dagger}e^{-\beta^*\hat{a}}|0\rangle\right). \qquad (2.1.48)$$

Performing the differentiation, we get

$$\frac{\partial}{\partial\beta}|\beta\rangle = -\frac{\lambda}{2}\beta^*|\beta\rangle + \hat{a}^\dagger|\beta\rangle. \qquad (2.1.49)$$

Following similar procedure, one can readily establish that

$$\frac{\partial}{\partial\beta}\langle\beta| = -\frac{\lambda}{2}\beta^*\langle\beta|. \tag{2.1.50}$$

Upon introducing Eqs. (2.1.49) and (2.1.50) into Eq. (2.1.46), we obtain

$$\hat{a}^\dagger|\beta\rangle\langle\beta| = \left(\lambda\beta^* + \frac{\partial}{\partial\beta}\right)|\beta\rangle\langle\beta|, \tag{2.1.51}$$

from which follows that

$$|\beta\rangle\langle\beta|\hat{a} = \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)|\beta\rangle\langle\beta|, \tag{2.1.52}$$

and thus

$$|\beta\rangle\langle\beta|\hat{a}^l = \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)^l|\beta\rangle\langle\beta|. \tag{2.1.53}$$

We next seek to obtain the completeness relation for coherent states. To this end, we note that

$$\begin{aligned}\int d^2\alpha|\alpha\rangle\langle\alpha| &= \sum_{n,m}\int d^2\alpha|n\lambda\rangle\langle n\lambda|\alpha\rangle\langle\alpha|m\lambda\rangle\langle m\lambda\rangle \\ &= \sum_{n,m}\frac{|n\lambda\rangle\langle m\lambda|}{\sqrt{n}\sqrt{m}}\lambda^{\frac{n}{2}}\lambda^{\frac{m}{2}}\int d^2\alpha e^{-\lambda\alpha^*\alpha}\alpha^n\alpha^{*m}.\end{aligned} \tag{2.1.54}$$

One can write

$$\begin{aligned}\int d^2\alpha e^{-\lambda\alpha^*\alpha}\alpha^n\alpha^{*m} &= \int d^2\alpha\ exp\left[-\lambda\alpha^*\alpha + a\alpha + b\alpha^*\right]\alpha^n\alpha^{*m}|_{a=b=0} \\ &= \frac{\partial^n}{\partial a^n}\frac{\partial^m}{\partial b^m}\int d^2\alpha\ exp\left[-\lambda\alpha^*\alpha + a\alpha + b\alpha^*\right]|_{a=b=0}.\end{aligned} \tag{2.1.55}$$

Using the relation

$$\int d^2\alpha e^{-a\alpha^*\alpha + b\alpha + c\alpha^*} = \frac{\pi}{a}e^{\frac{bc}{a}}, \tag{2.1.56}$$

it then follows that

$$\begin{aligned}\int d^2\alpha e^{-\lambda\alpha^*\alpha}\alpha^n\alpha^{*m} &= \frac{\pi}{\lambda}\frac{\partial^n}{\partial a^n}\frac{\partial^m}{\partial b^m}e^{\frac{ab}{\lambda}}|_{a=b=0} \\ &= \frac{\pi}{\lambda}\sum_l\frac{1}{l!}\frac{\partial^n}{\partial a^n}\frac{\partial^m}{\partial b^m}\left[\frac{a^l b^l}{\lambda^l}\right]_{a=b=0}.\end{aligned} \tag{2.1.57}$$

In addition, using the identity

$$\frac{\partial^n}{\partial x^n}x^m = \frac{n!}{(n-m)!}x^{n-m} \tag{2.1.58}$$

and applying the condition $a = b = 0$, one obtains

$$\int d^2\alpha e^{-\lambda\alpha^*\alpha}\alpha^n\alpha^{*m} = \frac{\pi}{\lambda}\sum_l \frac{1}{l!}\frac{l!}{(l-n)!}\frac{l!}{(l-m)!}\frac{1}{\lambda^l}\delta_{ln}\delta_{lm}$$

$$= \frac{\pi}{\lambda^{n+1}}\frac{n!}{(n-m)!}\delta_{nm}. \tag{2.1.59}$$

Now in view of Eq. (2.1.54) and Eq. (2.1.59), there follows

$$\int d^2\alpha|\alpha\rangle\langle\alpha| = \frac{\pi}{\lambda}\sum_{n,m}\frac{|n\lambda\rangle\langle m\lambda|}{\sqrt{n!}\sqrt{m!}}\frac{n!}{(n-m)!}\frac{1}{\lambda^n}\lambda^{\frac{n}{2}}\lambda^{\frac{m}{2}}\delta_{nm}$$

$$= \frac{\pi}{\lambda}\sum_{n=0}^{\infty}|n\lambda\rangle\langle n\lambda|, \tag{2.1.60}$$

so that on account of Eq. (2.1.35), we have

$$\hat{I} = \frac{\lambda}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|. \tag{2.1.61}$$

2.2 The Q-function

The Q function is an important tool in quantum optics. Knowing this function, provide convenient means of describing the quantum properties of light modes. Here we will confine our discussion to the Q function which is best suited to the evaluation of the expectation value of anti-normally ordered operators. To this effect, the Q function is expressible in terms of the anti-normally ordered characteristic function defined in Heisenberg picture by [15, 16]

$$\phi_a(z) = Tr\left(\hat{\rho}e^{-z^*\hat{a}}e^{z\hat{a}^\dagger}\right). \tag{2.2.1}$$

Applying the completeness relation for coherent states given by Eq. (2.1.61) and performing the trace operation, we find

$$\phi_a(z) = \lambda\int d^2\alpha Q(\lambda\alpha^*, \lambda\alpha)exp(\lambda z\alpha^* - \lambda z^*\alpha), \tag{2.2.2}$$

where $Q(\lambda\alpha^*, \lambda\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle$. Upon introducing new variables

$$\lambda\alpha = \beta, \tag{2.2.3}$$

and

$$\lambda\alpha^* = \beta^*, \tag{2.2.4}$$

Eq. (2.2.2) can be rewritten as

$$\phi_a(z) = \int \frac{d^2\beta}{\lambda} Q(\beta^*, \beta) exp[z\beta^* - z^*\beta]. \tag{2.2.5}$$

Since $\frac{1}{\lambda}Q(\beta^*, \beta)$ is the inverse Fourier transform of this characteristic function, we see that

$$Q(\beta^*, \beta) = \frac{\lambda}{\pi^2} \int d^2z \phi_a(z) exp[z^*\beta - z\beta^*]. \tag{2.2.6}$$

Following a procedure of analysis given by Fesseha [2], we obtain the Q-function for the superposition of two light beams with the same frequency. Suppose $\hat{\rho}'(\hat{a}, \hat{a}^\dagger)$ is the density operator for a certain light beam. Upon expanding this density operator in normal order and employing the completeness relation for coherent state, one finds

$$\hat{\rho}' = \frac{\lambda}{\pi} \int d^2\beta \sum_{kl} C_{kl} (\lambda\beta^*)^k |\beta\rangle\langle\beta| \hat{a}^l \tag{2.2.7}$$

and on account of the relation Eq. (2.1.53), there follows

$$\hat{\rho}' = \frac{\lambda}{\pi} \int d^2\beta \sum_{kl} C_{kl} (\lambda\beta^*)^k \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)^l |\beta\rangle\langle\beta|. \tag{2.2.8}$$

We now realize that the density operator for the superposition of the first light beam and another one, is expressible as

$$\hat{\rho} = \frac{\lambda}{\pi} \int d^2\gamma \sum_{nm} C_{mn} \, (\lambda\gamma^*)^m \left(\lambda\gamma + \frac{\partial}{\partial\gamma^*}\right)^n \hat{D}(\gamma) \hat{\rho}' \hat{D}(-\gamma), \tag{2.2.9}$$

so that in view of Eq. (2.2.8), we have

$$\hat{\rho} = \frac{\lambda^2}{\pi^2} \int d^2\beta d^2\gamma \sum_{kl} C_{kl} \, (\lambda\beta^*)^k \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)^l$$

$$\times \sum_{nm} C_{mn} \, (\lambda\gamma^*)^m \left(\lambda\gamma + \frac{\partial}{\partial\gamma^*}\right)^n \hat{D}(\gamma) |\beta\rangle\langle\beta| \hat{D}(-\gamma). \tag{2.2.10}$$

Employing Eq. (2.1.45) and its complex conjugate, we find

$$\hat{\rho} = \frac{\lambda^2}{\pi^2} \int d^2\beta d^2\gamma \sum_{kl} C_{kl} \, (\lambda\beta^*)^k \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)^l$$

$$\times \sum_{nm} C_{mn} (\lambda\gamma^*)^m \left(\lambda\gamma + \frac{\partial}{\partial\gamma^*}\right)^n |\beta + \gamma\rangle\langle\gamma + \beta|. \tag{2.2.11}$$

Using Eq. (2.2.11) the Q function for the superposition of two light beams can be written as

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{\lambda^2}{\pi^3} \int d^2\beta d^2\gamma \sum_{kl} C_{kl} (\lambda\beta^*)^k \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)^l$$
$$\times \sum_{nm} C_{mn} (\lambda\gamma^*)^m \left(\lambda\gamma + \frac{\partial}{\partial\gamma^*}\right)^n$$
$$\times \langle\alpha|\beta + \gamma\rangle\langle\gamma + \beta|\alpha\rangle. \tag{2.2.12}$$

Substituting Eq. (2.1.42) and its complex conjugate into this equation, we get

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{\lambda^2}{\pi^3} \int d^2\beta d^2\gamma \sum_{kl} C_{kl} (\lambda\beta^*)^k \left(\lambda\beta + \frac{\partial}{\partial\beta^*}\right)^l exp\left[(\lambda\alpha - \lambda\beta - \lambda\gamma)\beta^*\right]$$
$$\times \sum_{nm} C_{mn} (\lambda\gamma^*)^m \left(\lambda\gamma + \frac{\partial}{\partial\gamma^*}\right)^n \; exp\left[(\lambda\alpha - \lambda\beta - \lambda\gamma)\gamma^*\right]$$
$$\times\, exp\left[-\lambda\alpha^*\alpha + \lambda\alpha^*\beta + \lambda\alpha^*\gamma\right]. \tag{2.2.13}$$

Finally, applying the binomial theorem, we readily arrive at

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{\lambda^2}{\pi} \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda(\alpha-\gamma))Q(\lambda\gamma^*, \lambda(\alpha-\beta))$$
$$\times exp\left[-\lambda\alpha^*\alpha - \lambda\beta^*\beta - \lambda\gamma^*\gamma + \lambda\alpha\beta^*\right]$$
$$\times exp\left[\lambda\alpha\gamma^* + \lambda\alpha^*\beta + \lambda\alpha^*\gamma - \lambda\beta\gamma^* - \lambda\beta^*\gamma\right] \tag{2.2.14}$$

where

$$Q(\lambda\beta^*, \lambda\alpha - \lambda\gamma) = \frac{1}{\pi} \sum_{kl} C_{kl} (\lambda\beta^*)^k (\lambda\alpha - \lambda\gamma)^l \tag{2.2.15}$$

and

$$Q(\lambda\gamma^*, \lambda\alpha - \lambda\beta) = \frac{1}{\pi} \sum_{nm} C_{mn} (\lambda\gamma^*)^m (\lambda\alpha - \lambda\beta)^n \tag{2.2.16}$$

are the Q functions associated with the two light beams.

We now proceed to determine the expectation value of operator $\hat{A}$ in terms of the Q-function. To this end, the expectation value of $\hat{A}$ is expressible as

$$\langle\hat{A}\rangle = Tr(\hat{\rho}\hat{A}). \tag{2.2.17}$$

Expanding the operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ in the anti-normal order, this can be rewritten as

$$\langle \hat{A} \rangle = \sum_{lm} C_{lm} Tr(\hat{\rho}\hat{a}^l \hat{a}^{\dagger m}), \tag{2.2.18}$$

so that introducing the completeness relation for coherent states, we have

$$\langle \hat{A} \rangle = \frac{\lambda}{\pi} \int d^2\alpha \sum_{lm} C_{lm} Tr(\hat{\rho}\hat{a}^l |\alpha\rangle\langle\alpha| \hat{a}^{\dagger m}) \tag{2.2.19}$$

Performing the trace, this can be rewritten as

$$\langle \hat{A} \rangle = \lambda \int d^2\alpha Q(\lambda\alpha^*, \lambda\alpha) A_a(\lambda\alpha^*, \lambda\alpha). \tag{2.2.20}$$

Using new variables defined by Eqs. (2.2.3) and (2.2.4), we arrive at

$$\langle \hat{A} \rangle = \int \frac{d^2\beta}{\lambda} Q(\beta^*, \beta) A_a(\beta^*, \beta), \tag{2.2.21}$$

where

$$A_a(\beta^*, \beta) = \sum_{lm} C_{lm} \beta^l \beta^{*m} \tag{2.2.22}$$

is the c-number function corresponding to the operator function $A(\hat{a}^\dagger, \hat{a})$ in the anti-normal order.

Chapter 3

Two-Level Laser Dynamics

In this chapter we analyze the quantum properties of the light emitted by two-level atoms available in a closed cavity and pumped to the upper level at a constant rate. Following the discussion given in Fesseha [2], and taking into account the interaction of the two-level atoms with a resonant cavity mode and the damping of the cavity mode by a vacuum reservoir, we obtain the equations of evolution of atomic and cavity operators. Then using the solutions of these eqautions, we determine the anti-normally ordered characteristic function which is used to find the Q-function of light produced by a two-level laser. Employing the Q-function we compute the mean photon number, the variance of photon number and the quadrature variance.

3.1 Operator dynamics

We consider here the case in which N two-level atoms are available in a closed cavity. Then the interaction of the cavity mode with one of the atoms can be described at resonance by the Hamiltonian

$$\hat{H} = ig\left[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k\right], \tag{3.1.1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{kk}\langle a| \tag{3.1.2}$$

is lowering atomic operator, $\hat{a}$ is the annihilation operator for the cavity mode, and g is the coupling constant between the atom and the cavity mode. We assume that the cavity light is coupled to a vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators. We can therefore drop the noise operator and write the quantum Langevin equation for the operator $\hat{a}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}], \tag{3.1.3}$$

where κ is the cavity damping constant. Therefore, with the aid of Eq. (3.1.1), we easily find

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k. \tag{3.1.4}$$

Furthermore, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = -i\langle[\hat{A}, \hat{H}]\rangle \tag{3.1.5}$$

along with Eq. (3.1.1), one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g\langle(\hat{\eta}_b^k - \hat{\eta}_a^{\dagger k})\hat{a}\rangle, \tag{3.1.6}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + g\langle\hat{a}^\dagger\hat{\sigma}_a{}^k\rangle, \tag{3.1.7}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -g\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - g\langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle, \tag{3.1.8}$$

where

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \tag{3.1.9}$$

and

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|. \tag{3.1.10}$$

We see that Eqs. (3.1.6), (3.1.7), and (3.1.8) are nonlinear differential equations and hence it is not possible to obtain exact time-dependent solutions of these equations. Thus applying the large-time approximation scheme [17], we obtain from Eq. (3.1.4) the approximately valid relation

$$\hat{a}(t) = -\frac{2g}{\kappa}\hat{\sigma}_a^k(t). \tag{3.1.11}$$

Now substitution of (3.1.11) into the aforementioned equations yields

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\frac{1}{2}\gamma_c\langle\hat{\sigma}_a^k\rangle, \tag{3.1.12}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma_c\langle\hat{\eta}_a^k\rangle, \tag{3.1.13}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = \gamma_c\langle\hat{\eta}_a^k\rangle, \tag{3.1.14}$$

where

$$\gamma_c = \frac{4g^2}{\kappa}. \tag{3.1.15}$$

We wish to call the parameter defined by Eq. (3.1.15) the stimulated emission decay constant. Based on the definition of this decay constant, we infer that an atom in the upper level and inside a closed cavity emits a photon due to its interaction with the cavity light. We certainly identify this process to be stimulated photon emission [18].

In order to include the contribution of all the atoms to the dynamics of the two-level laser, we sum Eqs. (3.1.12), (3.1.13), and (3.1.14) over the N two-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\frac{1}{2}\gamma_c\langle\hat{m}_a\rangle, \tag{3.1.16}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -\gamma_c\langle\hat{N}_a\rangle, \tag{3.1.17}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = \gamma_c\langle\hat{N}_a\rangle, \tag{3.1.18}$$

in which

$$\hat{m_a} = \sum_{k=1}^{N}\hat{\sigma}_a^k, \tag{3.1.19}$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \tag{3.1.20}$$

$$\hat{N}_b = \sum_{k=1}^{N}\hat{\eta}_b^k, \tag{3.1.21}$$

with the operators $\hat{N}_a$ and $\hat{N}_b$ representing the number of atoms in the upper and lower levels. Furthermore, with the aid of the identity

$$\hat{\eta}_a^k + \hat{\eta}_b^k = \hat{I}, \tag{3.1.22}$$

we see that

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle = N. \tag{3.1.23}$$

In addition, using the definition given by Eq. (3.1.2) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{3.1.24}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{3.1.25}$$

We therefore observe that

$$\hat{m}_a^\dagger \hat{m}_a = N\hat{N}_a, \tag{3.1.26}$$

in which

$$\hat{N}_a = N|a\rangle\langle a|. \tag{3.1.27}$$

Following the same procedure, one can also establish that

$$\hat{m}_a \hat{m}_a^\dagger = N\hat{N}_b, \tag{3.1.28}$$

with

$$\hat{N}_b = N|b\rangle\langle b|. \tag{3.1.29}$$

In the presence of N two-level atoms, we rewrite Eq. (3.1.4) as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \mu\hat{m}_a. \tag{3.1.30}$$

We now proceed to determine the value of the constant μ. Thus applying the steady-state solution of Eq. (3.1.4), we easily find

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{\gamma_c}{\kappa}(\hat{\eta}_b^k - \hat{\eta}_a^k) \tag{3.1.31}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}(\hat{N}_b - \hat{N}_a), \tag{3.1.32}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^\dagger]_k \tag{3.1.33}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^\dagger$ when the cavity mode is interacting with all the N two-level atoms. It proves to be convenient to rewrite Eq.(3.1.32) with the atomic operators replaced by their expectation value as

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}(\langle \hat{N}_b \rangle - \langle \hat{N}_a \rangle). \tag{3.1.34}$$

On the other hand, using the steady-state solution of Eq.(3.1.30), one can easily establish that

$$[\hat{a}, \hat{a}^\dagger] = N \left(\frac{2\mu}{\kappa}\right)^2 (\langle \hat{N}_b \rangle - \langle \hat{N}_a \rangle). \tag{3.1.35}$$

Hence from Eqs. (3.1.34) and (3.1.35), we get

$$\mu = \pm \frac{g}{\sqrt{N}} \tag{3.1.36}$$

and on account of this result, Eq. (3.1.30) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a. \tag{3.1.37}$$

The two-level atoms available in the cavity are pumped from the lower to the upper level by means of electron bombardment [4, 5]. The pumping process must certainly affect the time evolution of the atomic operators. Hence we take into account the effect of the pumping process on the time evolution of the operators $\hat{N}_a$ and $\hat{N}_b$ by rewriting Eqs. (3.1.17) and (3.1.18) as

$$\frac{d}{dt}\langle \hat{N}_a \rangle = -\gamma_c \langle \hat{N}_a \rangle + r_a \langle \hat{N}_b \rangle \tag{3.1.38}$$

and

$$\frac{d}{dt}\langle \hat{N}_b \rangle = -r_a \langle \hat{N}_b \rangle + \gamma_c \langle \hat{N}_a \rangle, \tag{3.1.39}$$

where r_a is the rate at which a single atom is pumped to the upper level. Now taking into account Eq. (3.1.23), one can put Eq. (3.1.38) in the form

$$\frac{d}{dt}\langle \hat{N}_a \rangle = -(\gamma_c + r_a)\langle \hat{N}_a \rangle + r_a N. \tag{3.1.40}$$

We immediately see that the steady-state solution of this equation is

$$\langle \hat{N}_a \rangle = \frac{r_a N}{\gamma_c + r_a} \tag{3.1.41}$$

and the steady-state solution of Eq. (3.1.39) turns out to be

$$\langle \hat{N}_b \rangle = \frac{\gamma_c}{r_a} \langle \hat{N}_a \rangle. \tag{3.1.42}$$

Finally, we seek to determine the effect of the pumping process on the dynamics of the atomic operator $\hat{m}_a$. To this end, we rewrite Eq. (3.1.16) as

$$\frac{d}{dt}\hat{m}_a = -\frac{1}{2}\eta\hat{m}_a + \hat{F}_a(t), \tag{3.1.43}$$

where $\hat{F}_a(t)$ is a noise operator with a vanishing mean and η is a parameter whose value remains to be fixed. Employing the relation

$$\frac{d}{dt}\left\langle \hat{m}_a^\dagger \hat{m}_a \right\rangle = \left\langle \frac{d\hat{m}_a^\dagger}{dt}\hat{m}_a \right\rangle + \left\langle \hat{m}_a^\dagger \frac{d\hat{m}_a}{dt} \right\rangle \tag{3.1.44}$$

along with Eqs. (3.1.43) and (3.1.26), we get

$$\frac{d}{dt}\langle \hat{N}_a \rangle = -\eta \langle \hat{N}_a \rangle + \frac{1}{N}\langle \hat{F}_a^\dagger(t)\hat{m}_a(t) \rangle + \frac{1}{N}\langle \hat{m}_a^\dagger(t)\hat{F}_a(t) \rangle. \tag{3.1.45}$$

Now comparison of Eqs. (3.1.40) and (3.1.45) shows that

$$\eta = \gamma_c + r_a \tag{3.1.46}$$

and

$$\langle \hat{F}_a^\dagger(t)\hat{m}_a(t) \rangle + \langle \hat{m}_a^\dagger(t)\hat{F}_a(t) \rangle = r_a N^2. \tag{3.1.47}$$

This result implies that

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a(t') \rangle = r_a N^2 \delta(t-t'). \tag{3.1.48}$$

Following a similar procedure, one can also easily establish that

$$\langle \hat{F}_a(t)\hat{F}_a^\dagger(t') \rangle = \gamma_c N^2 \delta(t-t'). \tag{3.1.49}$$

It is certainly interesting to consider different regimes of the laser operation. We then wish to call the regime of laser operation with more atoms in the upper level than in the lower level above threshold, and the regime of laser operation with equal number of atoms in the upper and lower levels threshold. Thus inspection of Eq. (3.1.41) shows that for the laser operating at threshold $\gamma_c = r_a$ and for the laser operating above threshold $\gamma_c < r_a$. Applying the solution of Eq.(3.1.43) and considering the atoms to be initially in the lower level, we easily find

$$\langle \hat{m}_a(t) \rangle = 0. \tag{3.1.50}$$

Moreover, the expectation value of the solution of Eq. (3.1.37) is expressible as

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\frac{\kappa t}{2}} + \frac{g}{\sqrt{N}} e^{-\frac{\kappa t}{2}} \int_0^t e^{\frac{\kappa t'}{2}} \langle \hat{m}_a(t') \rangle dt'. \tag{3.1.51}$$

On account of (3.1.50) and the assumption that the cavity light is initially in a vacuum state, Eq. (3.1.51) reduces to

$$\langle \hat{a}(t) \rangle = 0. \tag{3.1.52}$$

We note from Eqs. (3.1.37) and (3.1.52) that $\hat{a}$ is a Gaussian variable with zero mean.

At steady-state the solution of Eq. (3.1.37) can be put in the form

$$\hat{a}(t) = \frac{2g}{\kappa\sqrt{N}} \hat{m}_a. \tag{3.1.53}$$

3.2 The Q-function for the cavity light

Here we proceed to obtain the Q-function of the light produced by a two-level laser. To this end, we first determine the anti-normally ordered characteristic function defined in Heisenberg picture by [15, 16]

$$\phi_a(z) = Tr\left(\hat{\rho} e^{-z^* \hat{a}(t)} e^{z \hat{a}^\dagger(t)}\right). \tag{3.2.1}$$

With the aid of the identity given by Eq. (2.1.27), the anti-normally ordered characteristic function can be put in the form

$$\begin{aligned} \phi_a(z) &= e^{-\frac{\lambda}{2} z^* z} Tr\left(\hat{\rho} e^{z \hat{a}^\dagger(t) - z^* \hat{a}(t)}\right) \\ &= e^{-\frac{\lambda}{2} z^* z} \left\langle e^{z \hat{a}^\dagger(t) - z^* \hat{a}(t)} \right\rangle. \end{aligned} \tag{3.2.2}$$

Now observing that the operator $\hat{a}(t)$ is a Gaussian variable with vanishing mean, we have

$$\left\langle e^{z\hat{a}^{\dagger}(t)-z^{*}\hat{a}(t)}\right\rangle = exp\left(\frac{1}{2}\langle(z\hat{a}^{\dagger}(t)-z^{*}\hat{a}(t))^{2}\rangle\right). \tag{3.2.3}$$

Which can be rewritten as

$$\left\langle e^{z\hat{a}^{\dagger}(t)-z^{*}\hat{a}(t)}\right\rangle = exp\left(\frac{1}{2}\left[z^{2}\langle\hat{a}^{\dagger 2}(t)\rangle + z^{*2}\langle\hat{a}^{2}(t)\rangle\right]\right)$$
$$\times\, exp\left(\frac{1}{2}\left[-z^{*}z\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - z^{*}z\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle\right]\right). \tag{3.2.4}$$

Employing Eq. (3.1.53) and its complex conjugate together with Eqs. (3.1.26) and (3.1.28), it can be readily verified that

$$\langle\hat{a}^{2}(t)\rangle = 0, \tag{3.2.5}$$

$$\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = \frac{\gamma_c}{\kappa}\langle\hat{N}_b\rangle, \tag{3.2.6}$$

and

$$\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle. \tag{3.2.7}$$

Introducing Eqs. (3.2.5), (3.2.6) and (3.2.7) into Eq. (3.2.4), we have

$$\left\langle e^{z\hat{a}^{\dagger}(t)-z^{*}\hat{a}(t)}\right\rangle = exp\left(-\frac{\gamma_c}{2\kappa}[\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle]z^{*}z\right). \tag{3.2.8}$$

With this result and Eq. (3.1.34), the characteristic function takes the from

$$\phi_a(z) = exp\left(-\frac{\gamma_c}{\kappa}\langle\hat{N}_b\rangle z^{*}z\right). \tag{3.2.9}$$

Using Eq. (3.2.9) into Eq. (2.2.6), the Q function can be expressed as

$$Q(\beta^{*},\beta) = \frac{\lambda}{\pi^{2}}\int d^{2}z exp\left(-\frac{1}{a}z^{*}z + z^{*}\beta - z\beta^{*}\right). \tag{3.2.10}$$

Integrating Eq. (3.2.10) over z using the relation given by Eq. (2.1.56), we get

$$Q(\beta^{*},\beta) = \frac{\lambda a}{\pi}\, exp(-a\beta^{*}\beta), \tag{3.2.11}$$

where

$$a = \frac{\kappa}{\gamma_c\langle\hat{N}_b\rangle}. \tag{3.2.12}$$

One can check that the Q function is normalized to λ.

3.3 Photon statistics

The statistical properties of a light beam can be described in terms of the mean and the variance of the photon number. Here we wish to calculate the mean and the variance of the photon number for the cavity light generated by two-level laser employing the Q function.

3.3.1 The mean photon number

According to Eq. (2.2.21) the mean photon number for the cavity light can be expressed as

$$\bar{n} = \int \frac{d^2\beta}{\lambda} Q(\beta^*, \beta) A_a(\beta^*, \beta), \tag{3.3.1}$$

where

$$A_a(\beta^*, \beta) = \beta^*\beta - \lambda, \tag{3.3.2}$$

is the c-number function corresponding to the operator function $A(\hat{a}^\dagger, \hat{a})$ in the anti-normal order.

On account of Eqs. (3.2.11) and (3.3.2), expression (3.3.1) can be rewritten as

$$\bar{n} = \frac{a}{\pi} \int d^2\beta \quad e^{-a\beta^*\beta} (\beta^*\beta - \lambda). \tag{3.3.3}$$

This can be put in the form

$$\bar{n} = \frac{a}{\pi} \left(\frac{d}{dc}\frac{d}{db} - \lambda \right) \int d^2\beta \quad e^{-a\beta^*\beta + b\beta + c\beta^*}|_{b=c=0}, \tag{3.3.4}$$

so that on carrying out the integration over β using the relation (2.1.56), we obtain

$$\bar{n} = \left(\frac{d}{dc}\frac{d}{db} - \lambda \right) exp\left(\frac{bc}{a} \right) |_{b=c=0}. \tag{3.3.5}$$

Performing the differentiation and applying the condition $b = c = 0$, we get

$$\bar{n} = \frac{1}{a} - \lambda. \tag{3.3.6}$$

Upon introducing Eqs. (3.1.34) and (3.2.12) into Eq. (3.3.6), we easily find

$$\bar{n} = \frac{\gamma_c}{\kappa} \langle \hat{N}_a \rangle, \tag{3.3.7}$$

so that in view of Eq. (3.1.41), the mean photon number has at steady state the form

$$\bar{n} = \frac{\gamma_c}{\kappa}\left(\frac{r_a N}{\gamma_c + r_a}\right). \tag{3.3.8}$$

We note that for the two-level laser operating well above threshold ($\gamma_c \ll r_a$), Eq. (3.3.8) reduces to

$$\bar{n} = \frac{\gamma_c}{\kappa}N \tag{3.3.9}$$

and for the same laser operating at threshold ($\gamma_c = r_a$), we have

$$\bar{n} = \frac{\gamma_c}{2\kappa}N. \tag{3.3.10}$$

3.3.2 The variance of the photon number

The variance of the photon number can be expressed as

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{3.3.11}$$

With the help of Eq. (2.1.1), one can check that

$$\langle (\hat{a}^\dagger \hat{a})^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\lambda \langle \hat{a}\hat{a}^\dagger \rangle + \lambda^2. \tag{3.3.12}$$

According to Eq. (2.2.21) the expectation value of $\hat{n}^2$ can be written as

$$\langle \hat{n}^2 \rangle = \int \frac{d^2\beta}{\lambda} Q(\beta^*, \beta) A_a(\beta^*, \beta), \tag{3.3.13}$$

where $A_a(\beta^*, \beta)$ is the c-number function corresponding to the operator function $A(\hat{a}^\dagger, \hat{a})$ in the anti-normal order. Using the fact that

$$A_a(\beta^*, \beta) = \beta^2 \beta^{*2} - 3\lambda\beta\beta^* + \lambda^2, \tag{3.3.14}$$

along with Eq. (3.2.11), we can express Eq. (3.3.13) as

$$\langle \hat{n}^2 \rangle = \frac{a}{\pi} \int d^2\beta e^{-a\beta^*\beta} (\beta^2 \beta^{*2} - 3\lambda\beta\beta^* + \lambda^2), \tag{3.3.15}$$

which can be rewritten in the form

$$\langle \hat{n}^2 \rangle = \frac{a}{\pi}\left[\frac{d^2}{dc^2}\frac{d^2}{db^2} - 3\lambda\frac{d}{dc}\frac{d}{db} + \lambda^2\right] \int d^2\beta e^{-a\beta^*\beta + b\beta + c\beta^*}|_{b=c=0}, \tag{3.3.16}$$

so that carrying out the integration using the relation given by Eq. (2.1.56), we get

$$\langle \hat{n}^2 \rangle = \left(\frac{d^2}{dc^2} \frac{d^2}{db^2} - 3\lambda \frac{d}{dc} \frac{d}{db} + \lambda^2 \right) exp \left(\frac{bc}{a} \right) |_{b=c=0}. \tag{3.3.17}$$

Performing the differentiation and applying the condition $b = c = 0$, one easily finds

$$\langle \hat{n}^2 \rangle = \frac{2}{a^2} - \frac{3\lambda}{a} + \lambda^2. \tag{3.3.18}$$

Substituting Eqs. (3.1.34) and (3.2.12) into Eq. (3.3.18), we have

$$\langle \hat{n}^2 \rangle = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_b \rangle \langle \hat{N}_a \rangle + \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_a \rangle^2. \tag{3.3.19}$$

Now in view of Eqs. (3.3.7) and (3.3.19), one can write Eq. (3.3.11) as

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa} \right)^2 \langle \hat{N}_b \rangle \langle \hat{N}_a \rangle. \tag{3.3.20}$$

We see from Eq. (3.1.41) that for the two-level laser operating well above threshold

$$\langle \hat{N}_a \rangle = N, \tag{3.3.21}$$

so that on account of this and Eq. (3.1.23), we arrive at

$$\langle \hat{N}_b \rangle = 0. \tag{3.3.22}$$

Therefore, for $\gamma_c \ll r_a$, the variance of the photon number turns out to be

$$(\Delta n)^2 = 0. \tag{3.3.23}$$

This represents the normally-ordered variance of the photon number for coherent light. On the other hand, we observe from Eq.(3.1.41) that for the same laser operating at threshold,

$$\langle \hat{N}_a \rangle = \frac{N}{2}, \tag{3.3.24}$$

so that in view of this and Eq. (3.1.23), we get

$$\langle \hat{N}_b \rangle = \frac{N}{2}. \tag{3.3.25}$$

As a result, for $\gamma_c = r_a$, the the variance of the photon number is

$$(\Delta n)^2 = \bar{n}^2, \tag{3.3.26}$$

which represents the normally-ordered variance of the photon number for chaotic light.

3.4 Quadrature variance

We next proceed to calculate the quadrature variance for the plus and minus quadrature operators defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{3.4.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{3.4.2}$$

The quadrature variance can be expressed as

$$(\Delta a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2. \tag{3.4.3}$$

Furthermore, the expression for plus and minus quadratures can also be written in terms of the oparators $\hat{a}$ and $\hat{a}^\dagger$ as

$$(\Delta a_\pm)^2 = \lambda + 2\langle \hat{a}^\dagger \hat{a} \rangle \pm \langle \hat{a}^2 \rangle \pm \langle \hat{a}^{\dagger 2} \rangle \mp \langle \hat{a} \rangle^2 \mp \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle. \tag{3.4.4}$$

Now we are in a position to determine the expectation values in Eq. (3.4.4). Thus according to Eq. (2.2.21) the expectation value of $\hat{a}$ can be expressed as

$$\langle \hat{a} \rangle = \int \frac{d^2\beta}{\lambda} Q(\beta^*, \beta) A_a(\beta^*, \beta). \tag{3.4.5}$$

In view of Eq.(3.2.11) and the fact that $A_a(\beta^*, \beta) = \beta$, Eq. (3.4.5) takes the form

$$\langle \hat{a} \rangle = \frac{a}{\pi} \int d^2\beta e^{-a\beta^*\beta} \beta. \tag{3.4.6}$$

This can be put in the form

$$\langle \hat{a} \rangle = \frac{a}{\pi} \frac{d}{db} \int d^2\beta e^{-a\beta^*\beta + b\beta + c\beta^*} \,|_{b=c=0}, \tag{3.4.7}$$

so that integrating Eq. (3.4.7) over β using the relation defined by (2.1.56), we obtain

$$\langle\hat{a}\rangle = \frac{d}{db} e^{\frac{bc}{a}} \mid_{b=c=0} . \tag{3.4.8}$$

Differentiating Eq. (3.4.8) with respect to b and applying the condition $b = c = 0$, we get

$$\langle\hat{a}\rangle = 0. \tag{3.4.9}$$

In a similar manner, it can be established that

$$\langle\hat{a}^2\rangle = 0. \tag{3.4.10}$$

On account of Eq. (3.4.9) and Eq. (3.4.10) along with their complex conjugates, Eq. (3.4.4) reduces to

$$(\Delta a_{\pm})^2 = \lambda + 2\langle\hat{a}^+\hat{a}\rangle. \tag{3.4.11}$$

Moreover, taking into account Eqs. (3.1.34) and (3.3.7), one easily obtains

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{\kappa}\left(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\right). \tag{3.4.12}$$

It can be readily established that the quadrature operators satisfy the commutation relation

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_b). \tag{3.4.13}$$

Thus with the aid of this result, we find

$$\Delta a_+ \Delta a_- \geq \frac{\gamma_c}{\kappa}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle\right|. \tag{3.4.14}$$

Now in view of Eqs. (3.3.21) and (3.3.22) together with Eq. (3.4.12), we see that for the laser operating well above threshold

$$(\Delta a_+)^2 = (\Delta a_-)^2 = \bar{n}, \tag{3.4.15}$$

with $\bar{n}$ given by Eq. (3.3.9). In addition, on account of Eqs. (3.3.24), (3.3.25) and (3.4.12), we see that for the laser operating at threshold

$$(\Delta a_+)^2 = (\Delta a_-)^2 = 2\bar{n}, \tag{3.4.16}$$

in which $\bar{n}$ is given by Eq. (3.3.10). This represents the normally-ordered quadrature variance for chaotic light.

We define coherent light to be a light mode in which the uncertainties in the two quadratures are equal and satisfy the minimum uncertainty relation. On account of Eq. (3.4.14), we note that for the laser operating well above threshold, $\Delta a_+ \Delta a_- \geq \bar{n}$. Hence on the basis of this result and Eq. (3.4.15), we assert that the light generated by the two-level laser operating well above threshold is coherent. We have seen that the quadrature variance for this case takes the relatively small value described by Eq. (3.4.15). However, this value markedly differs from the corresponding value for an ideal coherent light [19]. This must be due to the fact that the coherent light generated by the two-level laser is represented by operators with vanishing mean. Perhaps it is also worth mentioning that whenever the laser coherent light is used as a driving or pump mode, we may replace the operators representing this light mode by the square root of the steady state mean photon number. Such replacement transforms the laser coherent light into an ideal coherent light.

Chapter 4

Superposition of Two-Level Laser Light Beams

4.1 The Q-function

We proceed here to obtain the Q function for the superposition of two identical cavity light beams produced by a two-level laser. We recall that the Q function for the superposition of the two light beams can be expressed as

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{\lambda^2}{\pi}\int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda(\alpha-\gamma))Q(\lambda\gamma^*, \lambda(\alpha-\beta))$$
$$\times\, exp\left[-\lambda\alpha^*\alpha - \lambda\beta^*\beta - \lambda\gamma^*\gamma + \lambda\alpha\beta^* + \lambda\alpha\gamma^*\right]$$
$$\times\, exp\left[\lambda\alpha^*\beta + \lambda\alpha^*\gamma - \lambda\beta\gamma^* - \lambda\beta^*\gamma\right]. \tag{4.1.1}$$

On account of Eq. (3.2.11), one can write

$$Q(\lambda\beta^*, \lambda\alpha - \lambda\gamma) = \frac{a\lambda}{\pi}exp(-a\lambda^2\alpha\beta^* + a\lambda^2\beta^*\gamma), \tag{4.1.2}$$

and

$$Q(\lambda\gamma^*, \lambda\alpha - \lambda\beta) = \frac{a\lambda}{\pi}exp(-a\lambda^2\alpha\gamma^* + a\lambda^2\beta\gamma^*). \tag{4.1.3}$$

Upon substituting Eqs. (4.1.2) and (4.1.3) into Eq. (4.1.1), we have

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{\lambda^4 a^2}{\pi^3}e^{-\lambda\alpha^*\alpha}\int d^2\gamma exp(-\lambda\gamma^*\gamma + \lambda\alpha^*\gamma + \lambda\alpha\gamma^* - a\lambda^2\alpha\gamma^*)$$
$$\times \int d^2\beta exp\left[-\lambda\beta^*\beta + \lambda(\alpha - \gamma^* + a\lambda\gamma^*)\beta + \lambda(\alpha - a\lambda\alpha + a\lambda\gamma - \gamma)\beta^*\right], \tag{4.1.4}$$

so that integrating Eq. (4.1.4) over β using the relation given by Eq. (2.1.56), we get

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{\lambda^3 a^2}{\pi^2} e^{-a\lambda^2\alpha^*\alpha} \int d^2\gamma exp(-a\lambda^2(2-a\lambda)\gamma^*\gamma + a\lambda^2\alpha^*\gamma + a\lambda^2(1-a\lambda)\alpha\gamma^*) \tag{4.1.5}$$

Performing the integration over γ using the relation defined by (2.1.56), we easily find

$$Q(\lambda\alpha^*, \lambda\alpha) = \frac{a\lambda}{\pi(2-a\lambda)} exp\left(-\frac{a\lambda^2}{2-a\lambda}\alpha^*\alpha\right), \tag{4.1.6}$$

and employing new variables defined by Eqs .(2.2.3) and (2.2.4), the Q function turns out to be

$$Q(\beta^*, \beta) = \frac{a\lambda}{\pi(2-a\lambda)}\; exp\left(-\frac{a}{2-a\lambda}\beta^*\beta\right). \tag{4.1.7}$$

One can check that the Q function is normalized to λ.

4.2 Photon statistics

Here we seek to determine the mean and the variance of the photon number for the superposed light beams employing the Q function.

4.2.1 The mean photon number

The expectation values of normally ordered operators using the Q function can be expressed as

$$\langle \hat{A} \rangle = \frac{1}{\pi\lambda^2} \int d^2\alpha d^2\beta Q(\alpha^*, \beta) |\langle \alpha|\beta\rangle|^2 A_n(\beta^*, \alpha), \tag{4.2.1}$$

in which $A_n(\beta^*, \alpha)$ is the c-number function corresponding to the operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ in normal order and

$$|\langle \alpha|\beta\rangle|^2 = exp\left(-\frac{\alpha^*\alpha}{\lambda} - \frac{\beta^*\beta}{\lambda} + \frac{\alpha^*\beta}{\lambda} + \frac{\alpha\beta^*}{\lambda}\right). \tag{4.2.2}$$

With the help of Eq. (4.2.1) and the fact that

$$A_n(\beta^*, \alpha) = \beta^*\alpha, \tag{4.2.3}$$

the mean photon number of the superposed light can be evaluated as

$$\langle \hat{a}^{\dagger}\hat{a}\rangle = \frac{1}{\pi\lambda^{2}}\int d^{2}\alpha d^{2}\beta Q(\alpha^{*},\beta)|\langle\alpha|\beta\rangle|^{2}\beta^{*}\alpha. \tag{4.2.4}$$

Now applying Eqs. (4.1.7) and (4.2.2), one can put Eq. (4.2.4) in the form

$$\begin{aligned}\bar{n} = &\frac{a}{\pi^{2}\lambda(2-a\lambda)}\int d^{2}\alpha exp\left(-\frac{\alpha^{*}\alpha}{\lambda}\right)\alpha \\ &\times\int d^{2}\beta exp\left(-\frac{\beta^{*}\beta}{\lambda}+\frac{2(1-a\lambda)\alpha^{*}\beta}{\lambda(2-a\lambda)}+\frac{\alpha\beta^{*}}{\lambda}\right)\beta^{*}.\end{aligned} \tag{4.2.5}$$

So that carrying out the integration over β using the relation given by Eq. (2.1.56), there follows

$$\bar{n} = \frac{2a(1-a\lambda)}{\pi(2-a\lambda)^{2}}\frac{d}{dc}\frac{d}{db}\int d^{2}\alpha exp\left(-\frac{a}{2-a\lambda}\alpha^{*}\alpha+b\alpha+c\alpha^{*}\right)|_{b=c=0}. \tag{4.2.6}$$

Further, integrating Eq. (4.2.6) over α using the relation given by Eq. (2.1.56), we obtain

$$\bar{n} = 2\left(\frac{1-a\lambda}{2-a\lambda}\right)\frac{d}{dc}\frac{d}{db}\; exp\left(\frac{2-a\lambda}{a}bc\right)|_{b=c=0}. \tag{4.2.7}$$

Performing the differentiation and applying the condition $b=c=0$, we get

$$\bar{n} = \frac{2}{a}-2\lambda. \tag{4.2.8}$$

Substituting Eqs .(3.1.34) and (3.2.12) into Eq. (4.2.8), we arrive at

$$\bar{n} = 2\frac{\gamma_{c}}{\kappa}\langle\hat{N}_{a}\rangle, \tag{4.2.9}$$

so that in view of Eq. (3.1.41), the mean photon number has the form

$$\bar{n} = 2\frac{\gamma_{c}}{\kappa}\left(\frac{r_{a}N}{\gamma_{c}+r_{a}}\right). \tag{4.2.10}$$

This result shows that the mean photon number of the superposed light beams is a simple sum of the mean photon number of the separate light beams.

It then follows that for $\gamma_{c}\ll r_{a}$, Eq. (4.2.10) reduces to

$$\bar{n} = 2\frac{\gamma_{c}}{\kappa}N \tag{4.2.11}$$

which is twice the mean photon number found in Eq. (3.3.9) and for $\gamma_{c}=r_{a}$, we have

$$\bar{n} = \frac{\gamma_{c}}{\kappa}N, \tag{4.2.12}$$

which is also twice the mean photon number given by Eq. (3.3.10).

These results shows that, the mean photon number of the superposed light beams for each of the cases, $\gamma_c \ll r_a$ and $\gamma_c = r_a$ is the linear sum of the mean photon number of the individual light beams respectively.

4.2.2 The variance of the photon number

The variance of the photon number for the superposed light can be expressed as

$$(\Delta n)^2 = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + 2\lambda\langle \hat{a}^\dagger \hat{a}\rangle - \langle \hat{n}\rangle^2. \tag{4.2.13}$$

Here according to Eq. (4.2.1), the expectation value of $\hat{a}^{\dagger 2}\hat{a}^2$ can be written as

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \frac{1}{\pi\lambda^2}\int d^2\alpha d^2\beta Q(\alpha^*,\beta)|\langle\alpha|\beta\rangle|^2 A_n(\beta^*,\alpha) \tag{4.2.14}$$

where

$$A_n(\beta^*,\alpha) = \beta^{*2}\alpha^2, \tag{4.2.15}$$

is the c-number function corresponding to the operator function $A(\hat{a}^\dagger, \hat{a})$ in normal order. Employing Eqs. (4.1.7), (4.2.2) and (4.2.15), one can easily establish that

$$\begin{aligned}\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = &\frac{a}{\pi^2\lambda(2-a\lambda)}\int d^2\alpha exp\left(-\frac{\alpha^*\alpha}{\lambda}\right)\alpha^2 \\ &\times \int d^2\beta exp\left(-\frac{\beta^*\beta}{\lambda} + \frac{2(1-a\lambda)\alpha^*\beta}{\lambda(2-a\lambda)} + \frac{\alpha\beta^*}{\lambda}\right)\beta^{*2}.\end{aligned} \tag{4.2.16}$$

So that integrating Eq. (4.2.16) over β using the relation given by Eq. (2.1.56), we find

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \frac{4a(1-a\lambda)^2}{\pi(2-a\lambda)^3}\frac{d^2}{d^2c}\frac{d^2}{d^2b}\int d^2\alpha exp\left(-\frac{a}{2-a\lambda}\alpha^*\alpha + b\alpha + c\alpha^*\right)|_{b=c=0}. \tag{4.2.17}$$

Furthermore, integrating Eq. (4.2.17) over α using the relation given by Eq. (2.1.56), one easily gets

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \left(\frac{2(1-a\lambda)}{2-a\lambda}\right)^2\frac{d^2}{d^2c}\frac{d^2}{d^2b}\ exp\left(\frac{2-a\lambda}{a}bc\right)|_{b=c=0}. \tag{4.2.18}$$

Performing the differentiation and applying the condition $b = c = 0$, we obtain

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = 8\lambda^2 + \frac{8}{a^2} - \frac{16\lambda}{a}. \tag{4.2.19}$$

Thus taking into account Eqs. (3.1.34) and (3.2.12), Eq. (4.2.19) can be expressed as

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = 2\bar{n}^2 \tag{4.2.20}$$

and

$$2\lambda\langle \hat{a}^{\dagger}\hat{a}\rangle = 2\frac{\gamma_c}{\kappa}\bar{n}\langle \hat{N}_b\rangle - \bar{n}^2. \tag{4.2.21}$$

With the aid of Eqs. (4.2.20) and (4.2.21), expression (4.2.13) turns out to be

$$(\Delta n)^2 = 2\left(\frac{\gamma_c}{\kappa}\right)\bar{n}\langle \hat{N}_b\rangle \tag{4.2.22}$$

On account of Eqs. (3.3.21) and (3.3.22), for $\gamma_c \ll r_a$, we easily get

$$(\Delta n)^2 = 0. \tag{4.2.23}$$

On the other hand, we see from Eqs. (3.3.24), (3.3.25) and (4.2.22) that for $\gamma_c = r_a$, the variance of the photon number becomes

$$(\Delta n)^2 = 4\bar{n}^2. \tag{4.2.24}$$

This represent four times the normally ordered variance of the photon number for a single chaotic light.

4.3 Quadrature variance

We know proceed to calculate the quadrature variance. To this end, we note that the variance of the plus and minus quadratures is expressible as

$$(\Delta a_{\pm})^2 = 2\lambda + 2\langle \hat{a}^{\dagger}\hat{a}\rangle \pm \langle \hat{a}^2\rangle \pm \langle \hat{a}^{\dagger 2}\rangle \mp \langle \hat{a}\rangle^2 \mp \langle \hat{a}^{\dagger}\rangle^2 - 2\langle \hat{a}\rangle\langle \hat{a}^{\dagger}\rangle. \tag{4.3.1}$$

According to Eq. (4.2.1), the expectation value of $\hat{a}^2$ can be written as

$$\langle \hat{a}^2\rangle = \frac{1}{\pi\lambda^2}\int d^2\alpha d^2\beta Q(\alpha^*,\beta)|\langle\alpha|\beta\rangle|^2\alpha^2. \tag{4.3.2}$$

Employing Eqs. (4.1.7) and (4.2.2), one can write

$$\begin{aligned}\langle \hat{a}^2\rangle &= \frac{a}{\pi^2\lambda(2-a\lambda)}\int d^2\alpha exp\left(-\frac{\alpha^*\alpha}{\lambda}\right)\alpha^2 \\ &\times \int d^2\beta exp\left(-\frac{\beta^*\beta}{\lambda} + \frac{2(1-a\lambda)\alpha^*\beta}{\lambda(2-a\lambda)} + \frac{\alpha\beta^*}{\lambda}\right).\end{aligned} \tag{4.3.3}$$

Integrating Eq. (4.3.3) over β using the relation given by Eq. (2.1.56), one finds

$$\langle\hat{a}^2\rangle = \frac{a}{\pi(2-a\lambda)}\frac{d^2}{d^2b}\int d^2\alpha exp\left(-\frac{a}{2-a\lambda}\alpha^*\alpha + b\alpha + c\alpha^*\right)|_{b=c=0}. \tag{4.3.4}$$

Further, integrating Eq. (4.3.4) over α using the relation given by Eq. (2.1.56), we obtain

$$\langle\hat{a}^2\rangle = 2\left(\frac{1-a\lambda}{2-a\lambda}\right)\frac{d^2}{d^2b}exp\left(\frac{2-a\lambda}{a}bc\right)|_{b=c=0}. \tag{4.3.5}$$

Moreover, performing the differentiation and applying the condition $b = c = 0$, the expectation value of $\hat{a}^2$ turns out to be

$$\langle\hat{a}^2\rangle = 0. \tag{4.3.6}$$

Following similar procedure, we readily obtain

$$\langle\hat{a}\rangle = 0. \tag{4.3.7}$$

Employing Eqs. (4.3.6) and (4.3.7) together with their complex conjugates into expression (4.3.1), we get

$$(\Delta a_\pm)^2 = 2\lambda + 2\langle\hat{a}^+\hat{a}\rangle, \tag{4.3.8}$$

and thus taking into account Eq. (4.2.9), one can easily verify that

$$(\Delta a_\pm)^2 = 2\frac{\gamma_c}{\kappa}\left(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\right). \tag{4.3.9}$$

It can be readily established that these quadrature operators satisfy the commutation relation

$$[\hat{a}_-, \hat{a}_+] = 4i\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_b). \tag{4.3.10}$$

Thus on account of this result, we obtain

$$\Delta a_+ \Delta a_- \geq 2\frac{\gamma_c}{\kappa}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle\right|. \tag{4.3.11}$$

Now in view of Eqs. (3.3.21), (3.3.22) and (4.3.9), we see that for $\gamma_c \ll r_a$

$$(\Delta a_+)^2 = (\Delta a_-)^2 = \bar{n}, \tag{4.3.12}$$

with $\bar{n}$ given by Eq. (4.2.11).

On account of Eq. (4.3.11), we note that for $\gamma_c \ll r_a$, the uncertainties in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation, $\Delta a_+ \Delta a_- \geq \bar{n}$. Hence on the basis of this result and Eq. (4.3.12), we assert that the light generated by a pair of superposed two-level laser light beams for $\gamma_c \ll r_a$ is coherent. Furthermore, for $\gamma_c = r_a$, we obtain from Eqs. (3.3.24), (3.3.25) and (4.3.9) that

$$(\Delta a_+)^2 = (\Delta a_-)^2 = 2\bar{n}, \tag{4.3.13}$$

in which $\bar{n}$ is given by Eq. (4.2.12). This is twice the normally-ordered quadrature variance for a single chaotic light. This result shows that for $\gamma_c = r_a$, the light generated by a pair of superposed two-level laser light beams is chaotic.

Chapter 5

Conclusion

Taking into account the interaction of the two-level atoms with a cavity mode, we obtained the equations of evolution of atomic and cavity operators. Then using the solutions of these eqautions, we determined the Q-function of the light produced by a two-level laser. Employing this Q-function, we have computed the mean photon number, the variance of photon number and the quadrature variance.

We have found that for the two-level laser operating well above threshold, the uncertainties in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation. In view of this, we have identified the light generated by the two-level laser operating well above threshold to be coherent. Moreover, we have observed that the light generated by the two-level laser when operating at threshold is chaotic.

On the other hand, we have obtained the Q function for the superposition of a pair of two-level laser light beams. Applying the resulting Q function, we have calculated the mean and variance of the photon number. Our result shows that the mean photon number of the superposed light beams is a simple sum of the mean photon number of the separate light beams. In addition, we have found that the variance of the photon number for the superposed light beams is four times the variance of the photon number of a single chaotic light.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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