ADDIS ABABA UNIVERSITY
SCHOOL OF GRADUATE STUDIES
DEPARTMENT OFMATHEMATIC
GRADUATE SEMINAR ON FINITE ELEMENT METHOD
SUBMITTED IN PARTIAL FULFILLMENT OF M.SC. DEGREE

COMPILED
BY
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ADVISED
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Preface

The rapid development of high speed digital computers and the increasing desire for numerical answers to applied problems have led to enhanced demands in the course dealing with methods and techniques of numerical analysis. Although numerical methods have always been useful, their role in the present-day scientific research is fundamental importance. One reason for this is that numerical methods can give the solution when ordinary analytical methods fail, for example in finding the roots of transcendental equations, or in solving nonlinear differential equations.

Almost all physical phenomena in nature can be described in terms of differential equations. Differential equations also can be solved analytically or numerically. In case where both the given equation and the given domain are simple, the solution can be obtained analytically. However, methods of exact solution to general Partial differential equations are failed for irregular and geometrically complicated domains and the geometric and material nonlinearities of most practical problems. In addition driving the governing dynamics of physical processes is a complicated task in itself; finding exact solution to the governing partial differential equations is usually even more formidable. When trying to solve such equations, approximate methods of analysis provide a convenient, alternative method for finding solutions.

Although there are different approximate methods of solving differential equations, in this seminar we deal with the Finite Element Method of solving differential equations. Moreover ,two such methods, Rayleigh-Ritz Method and Galerkin Method, are included in the seminar .The purpose of the later two methods is to introduce with variational approximation of Differential Equations whose knowledge is mandatory for the understanding of the Finite Element Method in the succeeding sections and these methods are referred as classical variational method.

1. Introduction

The rapid development of high speed computers and the increasing desire for numerical answers to applied problems have lead to enhanced demands in courses dealing with the methods and techniques of numerical analysis. Although numerical methods have always been useful, their role in the present-day scientific research is of fundamental importance. One reason for this is that numerical methods can give the solution when ordinary analytical method fails, for example in finding the roots of transcendental equations, in solving nonlinear differential equations and in finding solutions of integral equations etc.

Many physical problems in applied science and engineering are formulated in terms of mathematical relations involving certain known and unknown quantities and their derivatives. The analytical method can be applied to solve only a selected class of differential equations. Those equations which govern physical systems do not possess, in general, closed form solutions, and hence recourse must be made to numerical methods for solving such differential equations. Some of the numerical methods for solving differential equations are Finite difference method, Shooting method, Variational approximation, Finite element method etc. From the methods mentioned above, we are going to discuss only **Finite Element Method.**

Finite Element method is a numerical method for obtaining approximate solutions of ordinary and partial differential equations. It is especially powerful when dealing with boundary conditions defined over complex geometries that are common in practical applications for which **equilibrium** or **conservation law** can be easily stated in terms of physical quantities one want to obtain. Some of the practical applications of FEM are **Structural analysis**, **Thermal system analysis**, **Flow analysis, Energy conservation, Mass conservation, Plane stress, Plane strain** etc. However, this seminar report presents the FEM using simple problems. Therefore, it must be understood that it is for sake of easy introduction that I use relatively simple problems.

Historical Background of Finite Element

Although the method was first developed in 1956 for the analysis of aircraft structural problems, the concept has been used several centuries back. For example ancient mathematicians found the circumference of a circle by approximating it by the parameter of a polygon. In terms of the present day notation, each side of polygon can be called a **'finite element'.**

The basic idea behind Finite Element Method is to replace a continuous function by piecewise polynomials, such an approximation is called piecewise polynomial approximation. But, before the discussion of Finite element Method let us discuss some basic concepts like Boundary conditions, Variational Principle and Variational approximation, then we discusses Finite Element Method for solving boundary value problems.

1.1. Boundary Value Problem

Definitions:

i) A differential equation is said to describe a **boundary value problem** if the dependent variable and possibly its derivatives are required to take specified values on the boundary.

ii) The general second order differential equation of the form

$$y'' = f(x, y, y'), \quad a \le x \le b \tag{a}$$

together with either of the following boundary conditions

$$y(a) = \alpha \text{ and } y(b) = \beta$$

$$y'(a) = \alpha \text{ and } y'(b) = \beta$$

$$\propto_1 y(a) - \beta_1 y'(a) = \alpha \text{ and } \alpha_2 y(b) - \beta_2 y'(b) = \beta$$

Where $|\alpha_1| + |\beta_1| \neq 0$ and $|\alpha_2| + |\beta_2| \neq 0$

Table 1.1 Varitional and Proper Name of Boundary Conditions

Boundary Condition		Variational Name	Proper Name
Homogeneous	Non-Homogeneous		
V=0	V=$f(s)$	Essential	Drichlet
V'=0	V'=g(s)	Natural	Neumann
V+βV'=0	V+βV'=h(s)	Mixed	Robin

Boundary conditions are classified as homogeneous and non- homogeneous boundary conditions. A boundary condition is said to be **homogeneous** if it is specified as being equal to zero, otherwise it is called **non homogeneous.**

The conditions that guarantee that a solution to (a) exists should be checked before any numerical scheme is applied; otherwise a list of meaningless output may be generated. The general conditions are stated, without proof, in the following theorem.

Theorem:

Assume that $f(x, y, y')$ is continuous on the region $\mathbf{R}=\{(\mathbf{x}, \mathbf{y}, \mathbf{y}'): \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}, -\infty < y < \infty \ and\ \infty < y' < \infty\}$ and that $\frac{\partial f}{\partial y} = f_y(x, y, y')$ *and* $\frac{\partial f}{\partial y'} = f_{y'}(x, y, y')$ are continuous on R. If there exist a constant M > 0 for which $\frac{\partial f}{\partial y} = f_y(x, y, y') > 0$ **and** $|f_{y'}(x, y, y')| < M$ for all (x, y, y') ∈ **R**

Then the boundary value problem $y'' = f(x, y, y')$ $with\ y(a) = \alpha\ and\ y(b) = \beta$ has a unique solution y= y(x) $for\ a \leq x \leq b$.

1.2. Variational Principle

The Mathematical formulation of variational principle is that the integral of a typical function say F(x, u(x), u'(x)) has an **extreme value** for actual performance subject to the general conditions of the system. The problem of determining the minimum and maximum of the integral often leads to one or more partial differential equations together with the appropriate boundary conditions.

1.3. List of some important formulae

The lists of some important formulae which are frequently used latter in the variational formulation of Differential Equations are given below.

a) $\delta d(x)=d(\delta x)$,where "δ" is a variational operator

b) $\delta \int_a^b f(x)dx = \int_a^b \delta f(x)dx$

c) $\int_a^b \frac{dv}{dx}u(x)dx = v(x)u(x)|^b_a - \int_a^b v(x)\frac{du}{dx}dx = v(b)u(b)-v(a)u(a) - \int_a^b v(x)\frac{du}{dx}dx$

d) $\int_a^b v(x)\frac{d^2u}{dx^2}dx = \frac{du}{dx}v(x)|^b_a - \int_a^b \frac{du}{dx}\frac{dv}{dx}dx = \frac{du}{dx}|_b v(b) - \frac{du}{dx}|_a v(a) - \int_a^b \frac{du}{dx}\frac{dv}{dx}dx$

1.4. Definition:

1) Functional: It is a mapping which assigns a definite real number to each function of some set. Alternatively it is defined as a function which transforms a given function to a real number.

Example: The arc length L of a plane curve connecting two given points A $(x_{0,}\ y_0)$ and $B(x_{1,}\ y_1)$ is functional. The quantity L may be computed if the equation of the curve y= y(x) is given; then the functional is given by

$$L[y(x)] = \int_{x_0}^{x_1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

2) The extremum of a functional is a function at which the functional attains its **maximum or minimum**.

3) A problem in which it requires determination of the maxima or minima of a functional is called **Variational Problem.**

Example: Find the Functionals of the following DE

a) $u_{xx}= F(x)$ where $u(a) = u(b) = 0$ (1.4.1)

Solution: let v(x) be a test function which satisfies the boundary conditions.

Then, $u_{xx}v(x) =F(x)\, v(x)$ multiplying (1.4.1) by v(x)

$\Rightarrow v_{xx}v(x)-F(x)\, v(x) =0$ (1.4.2)

$\Rightarrow \int_a^b u_{xx}v(x)dx - \int_a^b F(x)\ v(x)dx = 0$, Integrating (1.4.2) over the domain,

$\Rightarrow u_x v(x)|_a^b - \int_a^b u_x\, v_x dx - \int_a^b F(x)\ v(x)dx = 0$ Integrating by parts

$\Rightarrow - \int_a^b u_x\ v_x dx - \int_a^b F(x)\ v(x)dx = 0$ $(\because v(a) = 0 = v(b))$

$$\Rightarrow \int_a^b u_x\ v_x dx + \int_a^b F(x)\ v(x)dx = \int_a^b \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx + \int_a^b F(x)\ v(x)dx = 0$$

Now let $I(v)= \int_a^b \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx + \int_a^b F(x)\ v(x)dx$

$$= \int_a^b [\left(\frac{dv}{dx}\right)^2 + 2F(x)\, v(x)\ ]dx \qquad (1.4.3)$$

Then the function I (v) in (1.4.3) is the functional of (1.4.1) and from calculus of variation, the necessary condition for the integral I (v) to have an extreme value is that v(x) must satisfy the Euler Lagrange Differential Equation.

Let the integrand in (1.4.3) be G, then G is a function of x, v, v' hence (1.4.3) can be written as,

$$I(v)=\int_a^b G(x,v,v')\,dx \tag{1.4.4}$$

Then the Euler Lagrange Equation associated to (1.4.4) is

$$\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial v'}\right)-\frac{\partial G}{\partial v}=0$$

b) $\frac{d^2y}{dx^2}=f(x)$ Where y(a) = y(b)= 0 (1.4.5)

Solution: In a Similar approach as we have done in problem **a**, the associated functional is given by, $I(v)=\int_a^b v\left(2f-v''\right)dx$ (1.4.6)

And the Euler Lagrange Equation associated to (1.4.6) is

$$\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v'}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right)=0 \qquad \text{Where } F=F(x,v,v',v'').$$

c) $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=k$, (1.4.7)

Where u= 0 on the boundary **C** of a region R.

Solution: Let v(x, y) be a test function such that v= 0 on the boundary **C** of region R. Then multiplying (1.4.7) by v(x, y) ,we get

$$u_{xx}v+u_{yy}v-kv=0 \tag{1.4.8}$$

Integrating (1.4.8) over the region R, we get

$$\iint_R \left(vu_{xx}+vu_{yy}-kv\right)dx\,dy=0$$

$$\Rightarrow \iint_R (vu_{xx})\,dx\,dy + \iint_R (vu_{yy})\,dx\,dy - \iint_R (k\,v)\,dx\,dy = 0$$

$$\Rightarrow \iint_R \frac{1}{2}(vv_{xx})\,dxdy + \iint_R \frac{1}{2}(vv_{yy})\,dx\,dy - \iint_R (kv)\,dx\,dy = 0$$

$$\Rightarrow \iint_R v\left(v_{xx} + v_{yy} - 2k\right)\,dx\,dy = 0$$

Then, $I(v) = \iint_R v\left(v_{xx} + v_{yy} - 2k\right)\,dx\,dx$ (1.4.9)

is the functional of problem (1.4.7).

Remark: The Euler-Lagrange equation for higher derivatives can have several solutions and the one which satisfies the given boundary conditions is selected.

1.5. Base Function

Suppose we want to approximate a real valued function f(x) over a finite Interval [a, b].We divide [a, b] in a number of sub-intervals $[x_i, x_{i+1}]$, i=0, 1, 2…, n-1 where x_0= a and x_n= b, and interpolate linearly between the values of f(x) at the end points of each sub interval. For $[x_i, x_{i+1}]$,then we have

$$l_i(x) = \frac{x - x_i}{x_i - x_{i+1}} f(x_i) + \frac{x - x_{i+1}}{x_{i+1} - x_i} f(x_{i+1}) \tag{1.5.1}$$

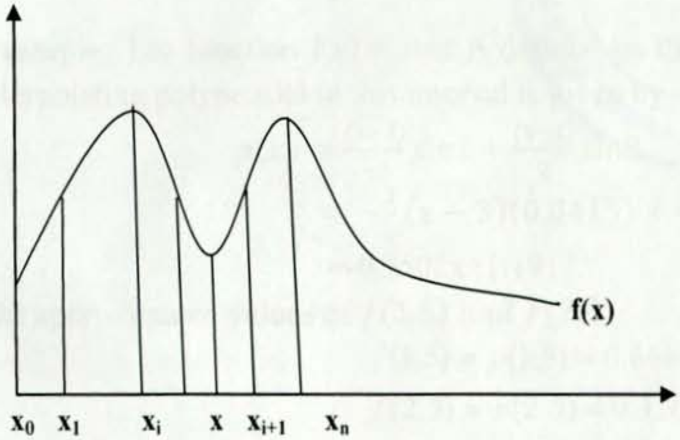

fig 1.5.1

Where $\ell_i(x)$ is Lagrange interpolating polynomial of f(x) on the interval $[x_i, x_{i+1}]$, from this we construct the Lagrange interpolating function of f(x) over $[x_0, x_n]$, which is given by the formula

$$P(x)=\sum_{i=o}^{n}\phi_i(x)f(x_i) \qquad (1.5.2)$$

$$where \quad \phi_0=\begin{cases}\dfrac{x-x_1}{x_0-x_1} & for \quad x_0\le x\le x_1\\ 0 & for \quad x_1\le x\le x_n\end{cases} \qquad (1.5.3)$$

$$\phi_i=\begin{cases}\dfrac{x-x_{i-1}}{x_i-x_{i-1}} & for \ \ x_{i-1}\le x\le x_i\\ \dfrac{x_{i+1}-x}{x_{i+1}-x_i} & for \ x_i\le x\le x_{i+1}\\ 0 & for \ x\ge x_{i+1}\end{cases} \qquad 1.5.4)$$

$$\phi_n=\begin{cases}0 & for \quad x_0\le x\le x_{n-1}\\ \dfrac{x-x_{n-1}}{x_n-x_{n-1}} & for \quad x_{n-1}\le x\le x_n\end{cases} \qquad (1.5.5)$$

The functions $\phi_i(x), \forall(\ i=0,1,2\ldots,n)$ are called Lagrange fundamental polynomials. Sometimes also named as Base Functions or Shape Functions and have the properties

$$\phi_i(x_i)=\begin{cases}1 & for\, x\in\left[x_{i-1},x_{i+1}\right]\\ 0 & otherwise\end{cases} \qquad (1.5.6)$$

Example: The function f(x) = sinx is defined on the interval [1, 3].Then the Lagrange interpolating polynomial in this interval is given by

$$\begin{aligned}p(x) &= \tfrac{(x-3)}{-2}\sin 1+\tfrac{(x-1)}{2}\sin 3\\ &= -\tfrac{1}{2}(x-3)(0.8415)+\tfrac{1}{2}(x-1)(0.1411)\\ &= -0.3502x+1.1917.\end{aligned}$$

The approximated values of $f(1.5)$ *and* $f(2.5)$

$$f(1.5)\approx p(1.5)=0.6664$$

$$f(2.5)\approx p(2.5)=0.3162$$

2. Variational Method of Approximation

In this section we discuss two classical variational Methods, the Rayleigh-Ritz and Galerkin methods of approximation for solving differential equations. The former method is based on the existence of functional where as the latter belongs to a wider class of methods called weighted residual method. An advantage of the Galerkin method is that it does not require functionals. Both the methods have a common feature in that they seek an approximate solution in the form of a linear combination of suitable approximate or base functions $\phi_i(x)$, *and undetermined parameter* α_i *i.e.* $\sum_{i=1}^{n}\alpha_i\phi_i$.

2.1: Rayleigh-Ritz Method of Approximation

As mentioned earlier this method is based on the existence of a functional which is then minimized. In this method we don't get the actual minimum but only an approximate one as close to the actual solution as the base function allows. To get a good approximation, therefore, the choice of the base function is important and to improve the approximation, the number of base functions should be increased.

We explain this by considering a second order boundary value problem defined by

$$y'' + p(x)y' + q(x) \qquad (2.1.1)$$

Where $y(a) = y(b) = 0$, $a<x<b$,

The functional for problem (2.1.1) is given by

$$I(v) = \int_a^b [\left(\frac{dv}{dx}\right)^2 - pv^2 - 2qv]dx \qquad (2.1.2)$$

Problem (2.1.1) has a unique solution at the minimum value of I (v).

Now we try with an approximate solution and determine the parameters of approximation so that the integral is a minimum. This is the central idea of Rayleigh - Ritz method.

$$\text{Let } v(x) = \sum_{i=1}^{n}\alpha_i\phi_i(x) \qquad (2.1.3$$

be an approximate solution where $\phi_i(x), \forall(i=1,2,..,n)$ are linearly independent base functions and satisfy the boundary conditions i.e. $\phi_i(a)=\phi_i(b)=0, \forall(i=1,2,3...,n)$.

Substituting equation (2.1.3) in equation (2.1.2) we get

$$I(\alpha_1,\alpha_2,...,\alpha_n) = \int_a^b\left[\left(\frac{d}{dx}\left(\sum_{i=1}^{n}\alpha_i\phi_i(x)\right)\right)^2 - p\left(\sum_{i=1}^{n}\alpha_i\phi_i\right)^2 - 2q\left(\sum_{i=1}^{n}\alpha_i\phi_i\right)\right]dx \qquad (2.1.4)$$

To find the minimum, we have $\frac{\partial I}{\partial\alpha_1}\delta\alpha_1 + \frac{\partial I}{\partial\alpha_2}\delta\alpha_2 + ... + \frac{\partial I}{\partial\alpha_n}\delta\alpha_n = 0$ (2.1.5)

Since $\delta\alpha_i$, is arbitrary $\forall(i = 1,2,\ldots,n)$, (2.1.5) implies that $\frac{\partial I}{\partial \alpha_i} = 0$, $\forall(i = 1,2,\ldots,n)$.

Now we are going to state, without proof, that the Rayleigh-Ritz method converges to the actual solution of the problem provided that the functions $\phi_i(x)$ $i = 1,2,\ldots n$ are linearly independent and satisfy at least the essential boundary condition of the problem.

Example: Consider the two point boundary value problem defined by

$$y'' + x = 0 \tag{2.1.6}$$

Where 0<x<1 and y(0) = y(1) = 0

Exact solution:

$$\frac{d^2y}{dx^2} = -x \textit{ where } 0 < x < 1 \textit{ and } y(0) = y(1) = 0$$

$$\Rightarrow y(x) = -\iint x dx^2 = \frac{-x^3}{6} + k_1 x + k_2, \textit{but } y(0) = 0 \Rightarrow k_2 = 0 \textit{ and } y(1) = 0 \Rightarrow k_1 = \frac{1}{6}$$

$$\textit{threfore}, \; y(x) = \frac{1}{6}x(1 - x^2) \textit{ is the exact solution.}$$

The Rayleigh-Ritz approximate solution:

Let v(x) be a test function which satisfies the boundary conditions, then the corresponding functional for problem (2.1.6) is given by

$$\mathrm{I(v)} = \int_0^1 v(-2x - v'')\,dx \tag{2.1.7}$$

Where v (0) = v (1) = 0

$$\text{Again , } \textit{Let } \; v(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x) \tag{2.1.8}$$

be anapproximate solution of problem(2.1.6)

Where $\phi_i(0) = \phi_i(1) = 0$, $\forall(i = 1,2,3\ldots,n)$ *and* $0 < x < 1$ Substituting (2.1.8) in (2.1.7), we obtain

$$I(V) = -\int_0^1 \left[\sum_{i=1}^{n} \alpha_i \phi_i \left(2x + \sum_{j=1}^{n} \alpha_j \phi_j'' \right) \right] dx$$

$$= -\sum_{i=1}^{n} 2\alpha_i \int_0^1 x\phi_i(x)dx - \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \int_0^1 \phi_i \phi_j'' \,dx \tag{2.1.9}$$

$$let \quad P_i = \int_0^1 x\phi_i dx \quad and$$

$$q_{ij} = \int_0^1 \phi_i \phi_j'' dx = \phi_i \phi' j \Big|_0^1 - \int_0^1 \phi_i' \phi_j' dx = -\int_0^1 \phi_i' \phi_j' dx \left(\because \phi_i(0) = \phi_i(1) = 0\right) \qquad (2.1.10)$$

Substituting (2.1.10) in (2.1.9), we get

$$\text{I(v)}= -\sum_{i=1}^{n} 2\alpha_i p_i - \sum_{i=1}^{n}\sum_{i=1}^{n} \alpha_i \alpha_j q_{ij} \qquad (2.1.11)$$

$$Minimizing\ I(v) \Rightarrow \quad \frac{\partial I}{\partial \alpha_i} = 0$$

$$\Rightarrow -2p_i - 2\sum_{j=0}^{n} \alpha_j q_{ij} = 0\ ,\ \forall (i = 1,2,...,n) \qquad (2.1.12)$$

Now let us find the approximate solution with n=1 and n=2.

For n=1,

Let $\phi_1(x) = x(1-x)$ so that the boundary conditions are satisfied.

Then using (2.1.10) , we have

$$P_1 = \int_0^1 x\phi_1 dx = \int_0^1 x^2(1-x)\,dx = \frac{1}{12} \quad and$$

$$q_{11} = -\int_0^1 \phi_i' \phi_j' dx = -\int_0^1 (1-2x)^2 dx = \frac{-1}{3} \qquad (2.1.13)$$

Substituting (2.1.13) in (2.1.12), we get

$$2p_1 - 2\alpha_1 q_{11} = 0 \Rightarrow \frac{-1}{6} + \frac{2}{3}\alpha_1 = 0 \Rightarrow \alpha_1 = \frac{1}{4}$$

$$\therefore \frac{1}{4}x(1-x)\, is\, the\ first\ approximat e\ solution\ of\ problem (2.1.6)$$

For n=2,

Let $\phi_1(x) = x(1-x)$ and $\phi_2(x) = x^2(1-x)$, so that the boundary conditions are satisfied. Similarly, we have

$$P_1 = \int_0^1 x\phi_1 dx = \int_0^1 x^2(1-x)\,dx = \frac{1}{12}$$

$$q_{11} = -\int_0^1 \phi_i' \phi_j' dx = -\int_0^1 (1-2x)^2 dx = -\frac{1}{3}$$

$$P_2 = \int_0^1 x\phi_2 dx = \int_0^1 x^3(1-x)\,dx = \frac{1}{20} \qquad (2.1.14)$$

$$q_{12} = -\int_0^1 \phi_1^{'}\phi_2^{'}\,dx = -\int_0^1 (1-2x)(2x-3x^2)\;dx = -\frac{1}{6} = q_{21} \qquad (by\;symetry)$$

$$q_{22} = -\int_0^1 \phi_2^{'}\phi_2^{'}\,dx = -\int_0^1 (2x-3x^2)^2\,dx = -\frac{2}{15}$$

Then substituting (2.1.14) in (2.1.12) ,we obtain

$$2p_1 + 2(\alpha_1 q_{11} + \alpha_2 q_{12}) = 0 \;\; and$$

$$2p_1 + 2(\alpha_1 q_{21} + \alpha_2 q_{22}) = 0$$

$$\Rightarrow 4\alpha_1 + 2\alpha_2 = 1 \;\; and \;\; 10\alpha_1 + 8\alpha_2 = 3$$

$$\Rightarrow \alpha_1 = \alpha_2 = \frac{1}{6}$$

$$Hence, \frac{1}{6}x(1-x) + \frac{1}{6}x^2(1-x) = \frac{1}{6}x(1-x^2)$$

is the second approximate solution of problem (2.1.6)
which is the same as the exact solution.

2.2. The Galerkin Method

The Rayleigh-Ritz method discussed above is a powerful technique to solve boundary value problems (BVP), however, it has disadvantage of requiring the existence of functional which is not always possible to obtain. The fact that it requires functional is one of its short coming because most of physical and engineering problems are expressed in terms of certain governing equations and boundary conditions, and not in terms of functionals.

Galerkin method uses an approximating function called trial function (which satisfies all the boundary conditions) and is substituted in the given differential equation to give what is called the **residual** which will not be zero since we have substituted an approximating function. The residual is then weighted (multiplying the residue by an arbitrary weight function) and the integral of the product, taken over the domain, is then set to zero.

To explain Galerkin's method consider the following second order boundary value problem

$$y'' + py' + qy = f(x) \quad ,a<x<b \tag{2.2.1}$$

with the boundary conditions

$$p_0 y(a) + q_0 y'(a) = \ell_o$$

$$p_1 y(b) + q_1 y'(b) = \ell_1 \quad \textit{where } p \textit{ and } q \textit{ are functions of } x.$$

To find an approximate solution of the problem in (2.2.1), we choose base function $\phi_i(x)$ in the way we have done in Rayleigh-Ritz method .Then an approximated solution v(x) which is a linear combination of the base functions $\phi_i(x)$ is taken where $\alpha_i s$ are unknown parameters,

$$v(x) = \sum_{i=1}^{n} \alpha_i \phi_i \tag{2.2.2}$$

$v(x)$ will not in general satisfy (2.2.1) but produce a residual

$$R(v) = v'' + p(x)v' + q(x)v \tag{2.2.3}$$

Taking the weight function $\varphi_i(x)$, the weighted integral equation is written as

$$\int_a^b \varphi_i(x) R(v)\,dx = 0 \tag{2.2.4}$$

The integral in (2.2.4) leads to a system of equations form which we can solve the unknown parameters α_i.

Notice: In Galerkin Method usually we take $\boldsymbol{Base\ function\ \emptyset_i = weight\ function \varphi_i}$

Example: Consider $y'' + y = -x, \quad 0<x<1$ (2.2.5)

Where $y(0) = y(1) = 0$

Solution: Choose $v(x) = \sum_{i=0}^{n} \alpha_i \phi_i$ to be an approximate solution of (2.2.5) which satisfies the boundary conditions.

For n=1,

Let $v(x)=\alpha_1\phi_1$ such that $\phi_1(0)=\phi_1(1)=0$ and let $\phi_1(x)=x(1-x) \Rightarrow v(x)=\alpha_1 x(1-x)$

Substituting for v(x) in (2.2.5), we get

$$R(v) = v''+v = -x \text{, but } v' = \alpha_1(1-2x) \Rightarrow v'' = -2\alpha_1 \qquad (2.2.6)$$

Hence using (2.2.4) we have

$$\int_0^1 (v''+v+x)\,\varphi_1(x)dx = 0 \text{, where } \varphi_1(x) \text{ is weight function but } \phi_1(x) = \varphi_1(x)$$

$$\Rightarrow \int_0^1 (v''+v+x)\,\phi_1(x)dx = 0$$

$$\Rightarrow \int_0^1 (v''+v+x)x(1-x)dx = 0$$

$$\Rightarrow \int_0^1 \left[\alpha_1(-2+x(1-x))+x\right]x(1-x)dx = 0$$

$$\Rightarrow -\int_0^1 2x(1-x)\,dx + \alpha_1 \int_0^1 x^2(1-x)dx = 0 \;\Rightarrow \alpha_1 = \frac{5}{18}$$

$$\Rightarrow \text{The first approximate solution for (2.2.5) is } v(x) = \frac{5}{18}x(1-x).$$

For n=2,

Let $v(x) = \alpha_1 x(1-x) + \alpha_2 x^2(1-x)$, *then* $v'' = -2\alpha_1 + \alpha_2(2-6x)$ *Substituting in* (2.2.4)

we get $\int_0^1 (v''+v+x)\varphi_1(x)dx = \int_0^1 (v''+v+x)x(1-x)dx = 0$ (*)

$\int_0^1 (v''+v+x)\varphi_2(x)dx = \int_0^1 (v''+v+x)x^2(1-x)dx = 0$ (**)

where φ_1 *and* φ_2 *are weight functions*

Then substituting v and v'' on $(*)$ and $(**)$, we obtain

$$2\alpha_1 + \alpha_2 = \frac{5}{9} \text{ and}$$

$$\frac{3}{20}\alpha_1 + \frac{13}{105}\alpha_2 = \frac{1}{20}$$

$$\Rightarrow \alpha_1 = \frac{71}{369} \text{ and } \alpha_2 = \frac{7}{41}$$

$$\Rightarrow v(x) = \frac{71}{369}x(1-x) + \frac{7}{41}x^2(1-x)$$

is the second approximate solution of (2.2.5).

2.3. Application of Rayleigh-Ritz and Galerkin Methods to two Dimensional Problems:

Although the application of Rayleigh-Ritz and Galerkin Methods to two Dimensional Problems is more complicated because of the increase in the number of parameters to be determined, we illustrate it using the following simple example.

Example: Consider the poison equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=4\,,0<x,y<1 \tag{2.3.1}$$

Where u= 0 on the boundary C of a region R.

i) **Rayleigh-Ritz method Solution:**

Let v(x, y) be a test function which vanishes on the boundary C of a region R. Then, using Equation (1.2.16), the functional of (2.3.1) is given by

$$\mathrm{I}\,(\mathrm{v})=\iint_R v\left(v_{xx}+v_{yy}-8\right)\,dx\,dy \tag{2.3.2}$$

For n=1,

Let $\mathrm{v}(\mathrm{x})=\alpha\mathrm{xy}(\mathrm{x}-1)(\mathrm{y}-1)$ be the first approximate solution for (2.3.1)

Clearly v(x, y) satisfies the boundary condition i.e. v= 0 on the boundary C of region R.

Then the derivatives are given by

$v_{xx}= 2\alpha y(y-1)$ and $v_{yy}= \alpha 2x(x-1)$ (2.3.4)

Substituting in (2.3.2), we get

$$I(v)=\iint_R \alpha xy(x-1)(y-1)\left[8-2\alpha y(y-1)-2\alpha x(x-1)\right]\,dx\,dy \tag{2.3.5}$$

$$Let\ a=\iint_R xy(x-1)(y-1)^2 dxdy=\frac{1}{36}$$

$$b=\iint_R x\ y^2(x-1)^2(y-1)^2 dxdy=-\frac{1}{180} \tag{2.3.6}$$

$$c=\iint_R x^2 y(x-1)^2(y-1)^2 dxdy=-\frac{1}{180}$$

Equation (2.3.5) simplifies to $I(v)= 8\alpha a-2\alpha^2 b-2\alpha^2 c$

and min*imizing* $I(v)$, *we get,*

$$\frac{\partial I}{\partial \alpha} = 0 \Rightarrow 8a - 4\alpha b - 4\alpha c = 0$$

$$\Rightarrow \alpha = \frac{4a}{2(b+c)} = -5\ u\sin g \quad (2.3.6)$$

The Rayleigh-Ritz approximate solution for (2.3.1) is given by $v(x, y) =-5xy(x-1)(y-1)$.

ii) Galerkin's Method Solution:

Let $v(x, y) = \alpha xy(x-1)(y-1)$ be a trial solution where $v(x, y) = 0$ at the boundary C of a region R. Then, we have

$R(v) = v_{xx}+v_{yy}-4$, but $v_{xx} = 2\alpha y^2- 2\alpha y$ and $v_{yy}= 2\alpha x^2-2\alpha x$ 2.3.7)

Using equation (2.2.4),

$$\iint_R \varphi_i(x,y)R(v)dxdy = 0 \qquad where\ \varphi_i(x,y) = \alpha xy(y-1)(x-1)$$

$$\Rightarrow \iint_R (v_{xx}+v_{yy}-4)\alpha xy(x-1)(y-1)dxdy = 0 \qquad (2.3.8)$$

But we have the following

$$\iint_R v_{xx}\alpha xy(x-1)(y-1)dxdy = -\frac{\alpha^2}{90} \quad and$$

$$\iint_R v_{yy}\,\alpha xy(x-1)(y-1)\ dxdy = -\frac{\alpha^2}{90} \qquad (2.3.9.)$$

$$\iint_R 4\alpha xy(x-1)(y-1)dxdy = \frac{4\alpha}{36}$$ Hence, equation (2.3.8) becomes

$$\Rightarrow -\frac{\alpha^2}{90} - \frac{\alpha^2}{90} - \frac{4\alpha}{36} = 0$$

Since $\alpha \neq 0$, we have $\alpha = -5$

Thus, the Galerkin's approximation is given by

$v(x,y) = -5xy(x-1)(y-1)$

which is the same as that of Rayleigh – Ritz approximation.

Summary on Rayleigh-Ritz and Galerkin's Method

The variational methods of approximation discussed above, the **Rayleigh- Ritz** and **Galerkin's method,** both have a common feature in that they seek an approximate solution in the form of a linear combination of base functions. Nevertheless, they differ from each other is that the Galerkin's method begins with the weighted-integral form of the dynamic equation as opposed to the weak form.

The Rayleigh Ritz method is powerful technique for the solution of boundary value problem .It has, however, the disadvantage of requiring the existence of functional which is not always possible to obtain. Since most engineering problems are expressed in terms of certain governing equations and boundary conditions, and not in terms of a functional, it is advantageous to use **Galerkin's method** than **Rayleigh- Ritz method.**

Although both of the above methods are applicable in solving different applied science and engineering problems, it is difficult to apply them in their classical form. The two basic reasons not to apply them in their variational form are:

- ✓ The difficulty associated with the choice of trial functions satisfying the given boundary conditions. Particularly for complicated boundaries and boundary conditions defined over complex geometries that are common in practical application.
- ✓ Very high order polynomials have to be used to obtain global solutions with reasonable accuracy.

3: Finite Element Method of Approximation

The variational methods we discussed above are powerful numerical techniques. However due to the formerly mentioned two basic reasons ,Rayleigh-Ritz and Galerkin methods cannot be applied directly for obtaining global approximate solutions to applied science and engineering problems . Finite element method is one of the most important numerical applications of Rayleigh-Ritz and Galerkin methods. In this method the idea of both Rayleigh-Ritz and Galerkin are used in such a way that the above two types of problems are avoided. In this method the region of interest is subdivided into a finite number of sub regions, called **elements**, and over each element the variational formulation of the differential equation is constructed using simple function for approximation. The individual element is then assembled and the equations for the whole problem are formed by a piecewise approximation of the variational method. Instead of increasing the order of the approximating functions, a finer mesh is used to obtain a result with a better accuracy. This is the way how the problems mentioned earlier are avoided.

As outlined by Reddy (1993), there are three main features of finite element method that make it superior over the classical variational methods. First, extremely complex domains are broken down into a collection of geometrically simple sub-domains which we refer to as **finite elements**. Secondly, over the domain of each finite element, the approximation functions are derived under the assumption that continuous functions can be well approximated as a linear combination of algebraic polynomials and finally, the undetermined coefficients are obtained by satisfying the governing equations over each element. Now we illustrate the basic steps involved in the finite element method through simple examples involving boundary value problems in one and two dimensions respectively.

3.1: Finite Element Method of approximation for One Dimensional Problem

Consider the two point boundary value problem defined by

$$\frac{d}{dx}\left[a(x)\frac{dy}{dx}\right] = -f(x) \ ,0 < x < 1 \qquad (3.1.1)$$

with the boundary conditions,

$$y(0) = 0 \ \ and \ \ a(x)\frac{dy}{dx}\bigg|_{x=1} = 0 \qquad (*)$$

Solution: The basic steps involved in the finite element method are given below.

Step1: Discretization of the region of interest into a set of finite elements.

The region of interest in problem (3.1.1) is the x-axis i.e. from x= 0 to x=1.Suppose [0, 1] is divided into a set of finite and non overlapping sub-intervals **D_e**, called elements, of arbitrary length.

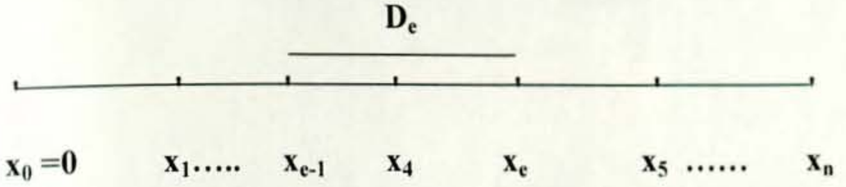

fig 3.1a

Where **e** ∈ **N** and $x_{e-1} \leq x \leq x_e$

$x_0, x_1, x_2, \ldots, x_n$ are called nodal points. $y(x_i) = y_i$'s $(i=0,1,2,\ldots,n)$ are called nodal values. For simplicity, let us take intervals of equal length h_e and consider a typical element being e^{th} of length h_e from node e-1 to node e. Let x_{e-1} and x_e be the values of x at the nodes e and e-1 and $y^{(e-1)}$and $y^{(e)}$ be the values of y at these nodes ,respectively.In general, $y^{(e)}$ satisfies the condition that;

$$y^{(e)}(x) \equiv 0 \ , \forall x \notin D_e = \left[x_{e-1,} x_e\right] \ and \ \forall e \tag{3.1.2}$$

For example in (fig 3.1a)

$y^{(e)}(x_4)$ is non zero whereas $y^{(e)}(x_0) = y^{(e)}(x_1) = y^{(e)}(x_5) = y^{(e)}(x_n) = 0.$

Now using (3.1.2) the global approximate solution y(x), can be written as

$$y(x) = \sum_e y^{(e)}(x) \tag{3.1.3}$$

where, the summation is taken over all elements.

Step 2: variational formulation over the element "e"

Choose an arbitrary element from the elements in step 1, and then construct variational formulation of the differential equation over the element e.

Let v(x) be a function satisfying the boundary conditions given in (*)

From (3.1.1), we have

$$\int_{x_{e-1}}^{x_e} v \frac{d}{dx}\left(a(x)\frac{dy}{dx}\right)dx = -\int_{x_{e-1}}^{x_e} vf dx \tag{3.1.4}$$

multiplying (3.1.1) by v and integrating it over the domain D_e.

$$\Rightarrow \int_{x_{e-1}}^{x_e} v\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right)dx + \int_{x_{e-1}}^{x_e} vfdx = 0$$

$$\Rightarrow va(x)\frac{dy}{dx}\Big|_{x_{e-1}}^{x_e} - \int_{x_{e-1}}^{x_e} a(x)v'\frac{dy}{dx}dx + \int_{x_{e-1}}^{x_e} vfdx = 0$$

$$\Rightarrow -\int_{x_{e-1}}^{x_e} a(x)v'\frac{dy}{dx}dx + \int_{x_{e-1}}^{x_e} vfdx + v(x_e)D_2^e + v(x_{e-1})D_1^e = 0 \qquad (3.1.5)$$

$$where \quad D_1^e = -a(x)\frac{dy}{dx}\Big|_{x_{e-1}} \quad and \quad D_2^e = a(x)\frac{dy}{dx}\Big|_{x_e} \qquad (3.1.6)$$

Step 3:

Rayleigh-Ritz approximation over the element e

Let $y_e(x)$ be an approximate solution to y(x) over the element e, so that

$$y_e(x) = \sum_{j=1}^{n} \alpha_j^e \phi_j(x) \qquad (3.1.7)$$

where α_j's are parameters to be determined and $\phi_j(x)s$ are approximate functions to be chosen. Substituting (3.1.7) in (3.1.5), we get

$$-\sum_{j=1}^{n} \alpha_j^e \int_{x_{e-1}}^{x_e} a(x)v'\phi_j'(x)dx + \int_{x_{e-1}}^{x_e} vfdx + v(x_e)D_2^e + v(x_{e-1})D_1^e = 0$$

$$now \quad v(x) = \phi_i(x) \Rightarrow v' = \phi_i'(x) \ as\ base\ \ function$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j^e \int_{x_{e-1}}^{x_e} a(x)\phi_i'(x)\phi_j'(x)dx = \int_{x_{e-1}}^{x_e} \phi_i(x) fdx + \phi_i(x_e)D_2^e + \phi_i(x_{e-1})D_1^e \qquad (3.1.8)$$

$$Let \quad K_{ij}^e = \int_{x_{e-1}}^{x_e} a(x)\phi_i'(x)\phi_j'(x)dx$$

Then

$$and \quad F_i^e = \int_{x_{e-1}}^{x_e} \phi_i f\, dx + \phi_i(x_e)D_2^e + \phi_i(x_{e-1})D_1^e, i = 1,2,...n \qquad (3.1.9)$$

equation (3.1.8) can be written in the matrix form

$$K_{ij}^e \alpha_j^e = F_i^e \qquad (3.1.10)$$

Where K_{ij} and F_i are called stiffness matrix and force vector respectively.

Remark: In the Rayleigh-Ritz and Galerkin's methods, the system of equations is obtained in terms of arbitrary parameters α_j. In finite element method; on the other hand, the unknown values of the dependent variable y at the nodes are taken as parameters. This is illustrated in the following way:

Let $y(x) = \alpha_1 + \alpha_2 x$ (3.1.11)

Be an approximation in the element e. Then we have,

$$y(x_{e-1}) = \alpha_1 + \alpha_2 x_{e-1} = y_1^e$$
$$y(x_e) = \alpha_1 + \alpha_2 x_e = y_2^e \tag{3.1.12}$$

Solving for α_i (i=1,2) in (3.1.11) ,we obtain

$$\alpha_1 = \frac{y_1^e x_e - y_2^e x_{e-1}}{x_e - x_{e-1}}$$
$$\alpha_2 = \frac{y_2^e - y_1^e}{x_e - x_{e-1}} \tag{3.1.13}$$

Equation (3.1.11) becomes

$$y(x) = \frac{y_1^e x_e - y_2^e x_{e-1}}{x_e - x_{e-1}} + \frac{y_2^e - y_1^e}{x_e - x_{e-1}} x$$
$$= \frac{x_e - x}{x_e - x_{e-1}} y_1^e + \frac{x - x_{e-1}}{x_e - x_{e-1}} y_2^e$$
$$= \sum_{i=1}^{2} y_i^e \phi_i^e(x) \tag{3.1.14}$$

where $\phi_2^e(x) = \dfrac{x - x_{e-1}}{x_e - x_{e-1}}$ and $\phi_1^e(x) = \dfrac{x_e - x}{x_e - x_{e-1}}$ (3.1.15)

with $x_1 = x_{e-1}$ and $x_2 = x_e$ such that the function ϕ_i^e have the property

$$\phi_i^e(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{3.1.16}$$

Now from (3.1.10) we have,

$$K_{ij}^e \alpha_j^e = K_{ij}^e y_j^e = F_i^e \tag{3.1.17}$$

Step 4: Assembly of the element equations to obtain the global system

We construct or assemble the equations to the overall system of equations (i.e. the systematic description of the summation of the solution y(x) of the given problem from y^e).The procedure for constructing the system equations is the same regardless of the type of the problem and number and type of elements used. Let us first define

$Y^T = (y_0, y_1, y_2, \ldots, y_n)$ (3.1.18)

where y_i(i=1,2,…,n) are the nodal values

Let us set $M_e = b_{kj}^e$ *for* $k = 1,2$ *and* $j = 1,2,\ldots,n.$ *to be the* $2\times(n+1)$ *matrix defined by*

$$b_{ij}^e = \begin{cases} 1\ ,(k=1 \text{ and } j=e) \text{ or } (k=2 \text{ and } j=e+1) \\ 0, \text{ otherwise} \end{cases}$$

with this observe that

$$y_j^e = M_e Y \tag{3.1.19}$$

Then (3.1.17) can be written as

$$K_{ij}^e M_e Y = F_i^e \tag{3.1.20}$$

$\Rightarrow M_e^T K_{ij}^e M_e Y = M_e^T F_i^e$ *multiplying both sides of* (3.1.20) *from the left by* M_e^T*.Adding these over the* $(n+1)$ *elements,*

$$\sum_{e=1}^{n+1} M_e^T K_{ij}^e M_e Y = \sum_{e=1}^{n+1} M_e^T F_i^e \tag{3.1.21}$$

which is the system of equations fortheglobal system.

Step 5: Imposition of the boundary conditions

To describe the enforcement of essential boundary conditions, assumes as a trial case first that $y_0 = 0$, then the first equation in (3.1.21) should be zero. Moreover, since $y_0 = 0$ all terms in (3.1.21) that contain $y_0=0$ are affected by deleting the first row and the first column in $\sum_{e=1}^{n+1} M_e^T K_{ij}^e M_e$ and the first entry $\sum_{e=1}^{n+1} M_e^T F_i^e$. Finally after the nodal values are determined, the finite element solution y(x) of the given problem is given by

$$y(x) = \sum_{e=1}^{n} y_e(x) \tag{3.1.22}$$

We demonstrate the computation of K_{ij}^e *and* F_i^e *with the choice of* $\phi_i^e(x)$ *in* (3.1.15) .

For simplicity, let $h_e = x_e - x_{e-1}$, then we have the following

$$\frac{d\phi_1^e}{dx} = \frac{d}{dx}\left(\frac{x_e - x}{x_e - x_{e-1}}\right) = -\frac{1}{x_e - x_{e-1}} = -\frac{1}{h_e}$$

$$\frac{d\phi_2^e}{dx} = \frac{d}{dx}\left(\frac{x - x_{e-1}}{x_e - x_{e-1}}\right) = \frac{1}{x_e - x_{e-1}} = \frac{1}{h_e} \tag{3.1.23}$$

But from (3.1.9), we have that

$$K_{ij}^e = \int_{x_{e-1}}^{x^e} a(x)\phi_i'(x)\phi_j'(x)dx \text{ and } F_i^e = \int_{x_{e-1}}^{x_e} \phi_i f dx + \phi_i(x_e)D_2^e + \phi_i(x_{e-1})D_1^e \qquad \forall i(1,2,\ldots n) \qquad (i)$$

Remark : $K_{ij}^e = K_{ji}^e$ *by symetry*

In particular, we choose a(x) = 5 and f(x) = 4, then using (3.1.23) and (i) we have

$$K_{11}^e = \int_{x_{e-1}}^{x^e} 5\phi_1'(x)\phi_1'(x)dx = 5\int_{x_{e-1}}^{x_e}\left(-\frac{1}{h_e}\right)^2 dx = \frac{5}{h_e}$$

$$K_{12}^e = \int_{x_{e-1}}^{x^e} 5\phi_1'(x)\phi_2'(x)dx = 5\int_{x_{e-1}}^{x_e}\left(\frac{1}{h_e}\right)^2 dx = -\frac{5}{h_e} = K_{21}^e \qquad (3.1.24)$$

$$K_{22}^e = \int_{x_{e-1}}^{x^e} 5\phi_2'(x)\phi_2'(x)dx = 5\int_{x_{e-1}}^{x_e}\left(\frac{1}{h_e}\right)^2 dx = \frac{5}{h_e}$$

Since $\phi_1(x_e) = 0$ and $\phi_1(x_{e-1}) = 1$, then we have

$$F_1^e = \int_{x_{e-1}}^{x_e} 4\frac{x_e - x}{h_e}dx + D_1^e = \frac{4}{h_e}\left[x_e x - \frac{x^2}{2}\right]_{x_{e-1}}^{x_e} + D_1^e = 2h_e + D_1^e \qquad (3.1.25)$$

In a Similar approach, we find

$$F_2^e = \int_{x_{e-1}}^{x_e} 4\frac{x - x_{e-1}}{h_e}dx + D_2^e = 2h_e + D_2^e \qquad (3.1.26)$$

Example: consider the problem defined by

$$\frac{1}{2}\frac{d^2y}{dx^2} = -8, \; 0 < x < 1 \qquad (3.1.27)$$

Where y (0) = 0 and $y'(1) = 0$

Solution: The exact solution of (3.1.27) is
$y(x) = 16x - 8x^2$

Approximate solution: Comparison with (3.1.1) shows that $a(x) = \frac{1}{2}$ *and* $f(x) = -8$

i)To show the steps involved using finite element method , we divided [0, 1] into two equal sub intervals with $h_e = \frac{1}{2}$

Step: 1-3:

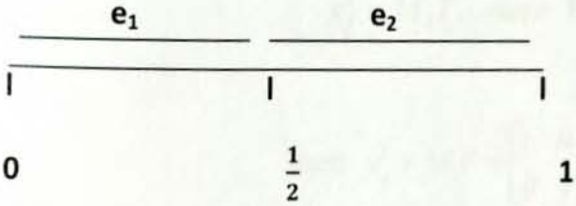

fig 3.1b

a) **Element e₁**, the nodal points are $x_{e-1} = 0$ *and* $x_e = \frac{1}{2}$

then , using (3.1.24),(3.1.25) and (3.1.26),we have

$$K_{11}^{1}=\frac{1}{2h_e}=1 \qquad K_{12}^{1}=-\frac{1}{2h_e}=-1=K_{21}^{1}\ ,by\ symetry$$

$$and \quad K_{22}^{1}=\frac{1}{2h_e}=1$$

$$F_1^1=2+D_1^1\ and \quad F_2^1=2+D_2^1$$

Now let

$$K^1=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ and \quad F^1=\begin{bmatrix}2+D_1^1\\ 2+D_2^1\end{bmatrix} \tag{3.1.28}$$

b) **Element e₂**, the nodal points are

$$x_{e-1}=\frac{1}{2}\ and \quad x_e=1$$

Then in a similar approach, we obtain

$$K^2=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ and \quad F^2=\begin{bmatrix}2+D_1^2\\ 2+D_2^2\end{bmatrix} \tag{3.1.29}$$

Step 4: Assembly of element equations

In this case the two elements are connected at node two. Since the function y(x) is continuous, it follows that y_2 of element e_1 should be the same as y_1 of element e_2 or the correspondence can be shown mathematically as follows:

Consider M_e where M_e is 2x3 matrix introduced in step 4

$$\text{then}, M_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\ and \quad M_2\ \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Using (3.1.19), we have

$$y_j^e=M_eY, where\ Y^T=(y_0,y_1,y_2)$$

$$\text{then}\ y_j^1=M_1Y=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}y_0\\ y_1\\ y_2\end{bmatrix}=\begin{bmatrix}y_0\\ y_1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1^1 \\ y_2^1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \Rightarrow y_1^1 = y_0 \text{ and } y_2^1 = y_1 \qquad (3.1.29)$$

$$\text{again, we have } y_j^2 = M_2 Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1^2 \\ y_2^2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow y_1^2 = y_1 \text{ and } y_2^2 = y_2$$

Then, what we have

$$y_1^1 = y_0 , y_2^1 = y_1 \quad , y_1^2 = y_1 \text{ and } y_2^2 = y_2 .$$

Again using (3.1.21), we obtain

$$\sum_{e=1}^{2} M_e^T K_{ij}^e M_e Y = \sum_{e=1}^{2} M_e^T F_i^e \quad \text{and} \quad 1 \le i, j \le 2 \qquad (3.1.30)$$

$$But\ M_1^T K_{ij} M_1 Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$$M_2^T K_{ij} M_2 Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{11} & k_{12} \\ 0 & k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \qquad (3.1.31)$$

$$M_e^T F_i^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1^1 \\ F_2^1 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix}$$

$$M_e^T F_i^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix} \qquad (3.1.32)$$

Then combining (3.1.31) and (3.1.32), we have the following system matrices

For e

$$\begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix} \qquad (*)$$

For e_2

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{11} & k_{12} \\ 0 & k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix} \qquad (**)$$

Using (3.1.30), adding up $(*)$ *and* $(**)$, we obtain

$$\begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22}+k_{11} & 0 \\ 0 & k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1+F_1^2 \\ F_2^2 \end{bmatrix}$$

Using the above relation, the global finite element model of the given boundary value problem in (3.1.1) is given by

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2+D_1^1 \\ 2+D_2^1+D_1^2 \\ 2+D_2^2 \end{bmatrix} \qquad (3.1.33)$$

Step 5: Imposition of boundary conditions:

The homogeneous boundary condition in (3.1.1) gives;

$y_0 = 0$ in addition $D_2^1 = \left[\frac{1}{2}\frac{dy}{dx}\right]_{x=\frac{1}{2}}$ *and* $D_1^2 = \left[-\frac{1}{2}\frac{dy}{dx}\right]_{x=\frac{1}{2}}$ *(i)*

Moreover, from natural boundary condition, we have $D_2^2 = 0$ *(ii)*

Then substituting (i) and (ii) in (3.1.33) and simplifying it, we get

$2y_1 - y_2 = 4$ and $-y_1 + y_2 = 2$ (3.1.34)

Solving (3.1.34) simultaneously, we obtain $y_1 = 6$ and $y_2 = 8$.

Finally using (3.1.14), the approximate solution throughout the interval [0, 1] can now be found by

$$y(x) = \sum_{i=1}^{2} y_i^e \phi_i^e = \begin{cases} y_1^1\phi_1^1(x) + y_2^1\phi_2^1(x), & 0 \le x \le \frac{1}{2} \\ y_1^2\phi_1^2(x) + y_2^2\phi_2^2(x), & \frac{1}{2} \le x \le 1 \end{cases}$$

Thus using (3.1.29), we have

$$y(x) = \begin{cases} y_0\phi_1^1(x) + y_1\phi_2^1(x), & 0 \le x \le \frac{1}{2} \\ y_1\phi_1^2(x) + y_2\phi_2^2(x), & \frac{1}{2} \le x \le 1 \end{cases} \tag{3.1.35}$$

$$\text{since } y_0 = 0, y_1 = 6 \text{ and } y_2 = 8$$

$$\text{and } \phi_1^1(x) = \frac{x_e - x}{x_e - x_{e-1}} = 1 - 2x$$

$$\phi_2^1(x) = \frac{x - x_{e-1}}{x_e - x_{e-1}} = 2x$$

$$\phi_1^2(x) = \frac{x_e - x}{x_e - x_{e-1}} = 2(1 - x) \tag{3.1.36}$$

$$\phi_2^2(x) = \frac{x - x_{e-1}}{x_e - x_{e-1}} = 2x - 1$$

Substituting (3.1.36) in (3.1.35), we get

$$y(x) = \begin{cases} 12x, & 0 \le x \le \frac{1}{2} \\ 4x + 4, & \frac{1}{2} \le x \le 1 \end{cases}$$

From the above, we obtain

Table 3.1a

Value of x	Approximate value of y	Exact value of y
$\frac{1}{4}$	3	$\frac{7}{2} = 3.5$
$\frac{2}{3}$	$\frac{20}{3} = 6.67$	$\frac{64}{9} = 7.11$

ii) For better accuracy, let us take four elements of length $\frac{1}{4}$

Steps 1-3:

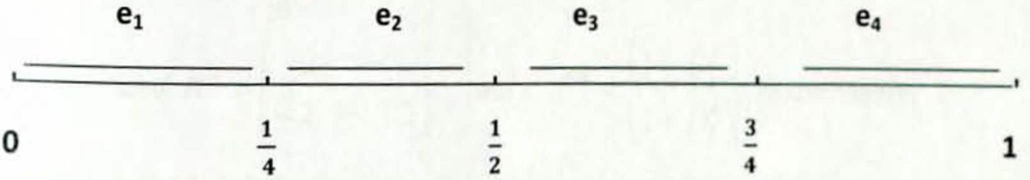

fig3.1c

For e_1, the nodal points are

$$x_{e-1} = 0 \ \ and \ x_e = \frac{1}{4}$$

Then the element matrices become

$$K_{11}^1 = 2 \quad K_{21}^1 = -2 = K_{12}^1 \ \ by\ symetry\ and \ \ K_{22}^1 = 2$$

$$F_1^1 = 1 + D_1^1 \ and \quad F_2^1 = 1 + D_2^1$$

$$K^1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad and \quad F^1 = \begin{bmatrix} 1 + D_1^1 \\ 1 + D_2^1 \end{bmatrix}$$

For e_2, the values of x at the nodes are

$$x_{e-1} = \frac{1}{4} \ and \ x_e = \frac{1}{2}$$

Then the element matrices become

$$K^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad and \quad F^2 = \begin{bmatrix} 1 + D_1^2 \\ 1 + D_2^2 \end{bmatrix}$$

For e_3, the values of x at the nodes are

$$x_{e-1} = \frac{1}{2} \ and \ x_e = \frac{3}{4}$$

and the element matrices become

$$K^3 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad and \quad F^3 = \begin{bmatrix} 1 + D_1^3 \\ 1 + D_2^3 \end{bmatrix}$$

similarly the nodal points and element matrices for e_4 are given by

$$x_{e-1} = \frac{3}{4} \ and \ x_e = 1$$

$$and \ K^4 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad and \quad F^4 = \begin{bmatrix} 1 + D_1^4 \\ 1 + D_2^4 \end{bmatrix} \ respectively.$$

Step4 assembly of element equation:

Applying the same technique as that of part (i) the element matrices for each element are

$$e_1,\ \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1+D_1^1 \\ 1+D_2^1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$e_2,\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1+D_1^2 \\ 1+D_2^2 \\ 0 \\ 0 \end{bmatrix}$$

$$e_3,\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1+D_1^3 \\ 1+D_2^3 \\ 0 \end{bmatrix}$$

$$and\ e_4,\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1+D_1^4 \\ 1+D_2^4 \end{bmatrix}$$

Adding up the above, we obtain,

$$\begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 2+2 & 2 & 0 & 0 \\ 0 & -2 & 2+2 & 2 & 0 \\ 0 & 0 & -2 & 2+2 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1+D_1^1 \\ 2+D_2^1+D_1^2 \\ 2+D_2^2+D_1^3 \\ 2+D_2^3+D_1^4 \\ 1+D_2^4 \end{bmatrix} \qquad (3.1.37)$$

From the boundary conditions we have D_2^1 and D_1^2, D_2^2 and D_1^3, D_2^3 and D_1^4 cancel each other in addition from natural condition $D_2^4 = 0$

then the system in (3.1.37) become

$$\begin{aligned}
&4y_1-y_2=2\\
&-2y_1+4y_2-2y_3=2\\
&-2y_2+4y_3-2y_4=2\\
&-2y_3+2y_4=1
\end{aligned}\tag{3.1.38}$$

Solving (3.1.38), we get the approximate solution for the problem valid for [0, 1]

$$y(x)=\begin{cases} 14x, & 0\le x\le \frac{1}{4}\\ 10x+1, & \frac{1}{4}\le x\le \frac{1}{2}\\ 6x+\frac{9}{3}, & \frac{1}{2}\le x\le \frac{3}{4}\\ 24-22x, & \frac{3}{4}\le x\le 1\end{cases}$$

From the above, we obtain

Table 3.1d

Value of x	Approximate value of y	Exact value of y
$\frac{1}{4}$	$\frac{7}{2}$	$\frac{7}{2}$
$\frac{2}{3}$	7	$\frac{64}{9}$

Therefore from table 3.1b and 3.1c we can conclude that the finer the mesh the better the approximate solution.

3.2: Application of the Finite Element Method to a two-dimensional Problem

The finite element described in the previous section can be extended to two-dimensional problems. Since the two dimensional problems are modeled by partial differential equations, the analysis will be complicated and we demonstrate this application by considering the Poisson equation. Also, we consider triangular elements only, although the geometric shapes chosen could be rectangles or quadrilateral.

We now consider the poison equation

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f(x,y), \qquad on\ D \tag{3.2.1}$$

With the condition u(x, y) = 0

Step1: Descritization of the Domain in to a set of finite elements

The domain D is sub divided in to non-over lapping sub domains. For simplicity we consider triangular elements only. In which the nodes are numbered in the counter-clock wise direction from node 1, which is arbitrarily specified. Let the coordinates of the nodes and nodal values in the nodes 1, 2, 3 be given by

(x_1, y_1),(x_2, y_2) , (x_3, y_3) and $u(x_1, y_1)= u_1$,$u(x_2, y_2) = u_2$,$u(x_3, y_3)= u_3$ respectively.

Accordingly, we assume

$$u(x,y)=\alpha_1+\alpha_2 x+\alpha_3 y \tag{3.2.2}$$

as the required approximation in the element D_e where (x, y) is inside $\Delta 123$.

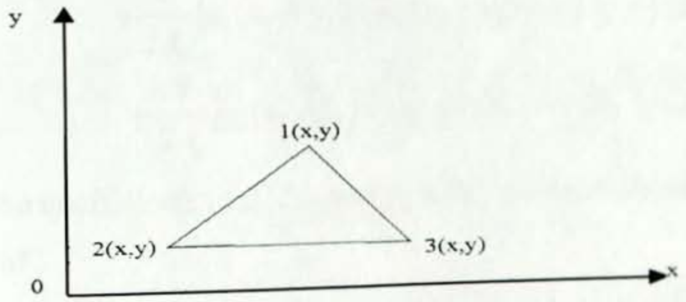

fig 3.2.1

Where i(x, y) = (x_i, y_i) and i=1,2,3 denotes the three vertices of the above triangle.

We also set u(x_i, y_i))= u_i where i=1,2,3 (3.2.3)

Substituting (3.2.3) in (3.2.2) we obtain

$$\begin{aligned} u_1&=\alpha_1+\alpha_2x_1+\alpha_3y_1\\ u_2&=\alpha_1+\alpha_2x_2+\alpha_3y_2\\ u_3&=\alpha_1+\alpha_2x_3+\alpha_3y_3 \end{aligned} \qquad (3.2.4)$$

Solving equation (3.2.4), we obtain

$$\alpha_1=\frac{1}{2A_e}\begin{vmatrix}u_1 & u_2 & u_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}$$

$$\alpha_2=\frac{1}{2A_e}\begin{vmatrix}u_1 & u_2 & u_3\\ y_1 & y_2 & y_3\\ 1 & 1 & 1\end{vmatrix} \qquad (3.2.5)$$

$$\alpha_3=\frac{1}{2A_e}\begin{vmatrix}u_1 & u_2 & u_3\\ 1 & 1 & 1\\ x_1 & x_2 & x_3\end{vmatrix}$$

Where A_e is area of triangle Δ123 and is given by

$$A_e=\frac{1}{2}\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix} \qquad (3.2.6)$$

Substituting α_1, α_2 and α_3 in (3.2.2) and simplifying, we obtain

$$\begin{aligned} u(x,y)=\frac{1}{2A_e}&\left[u_1(x_2y_3-x_3y_2)+u_2(x_3y_1-x_1y_3)+u_3(x_1y_2-x_2y_1)\right]\\ &+\frac{1}{2A_e}\left[u_1(y_2-y_3)+u_2(y_3-y_1)+u_3(y_1-y_2)\right]x\\ &+\frac{1}{2A_e}\left[u_1(x_3-x_2)+u_2(x_1-x_3)+u_3(x_2-x_1)\right]y \end{aligned} \qquad (3.2.7)$$

Collecting the coefficients of u_1 , u_2 and u_3 in the above, equation (3.2.7) can be written in the form of

$$u(x,y)=\sum_{i=1}^{3}u_i\phi_i^e(x,y) \tag{3.2.8}$$

Where the ϕ_i^es are the linear interpolating functions for the triangular elements under consideratio and are given by

$$\phi_1^e(x,y)=\frac{1}{2A_e}\begin{vmatrix}1 & x & y\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}$$

$$\phi_2^e(x,y)=\frac{1}{2A_e}\begin{vmatrix}1 & x & y\\ 1 & x_3 & y_3\\ 1 & x_1 & y_1\end{vmatrix} \tag{3.2,9}$$

$$and\ \phi_3^e(x,y)=\frac{1}{2A_e}\begin{vmatrix}1 & x & y\\ 1 & x_1 & y_1\\ 1 & x_2 & y_2\end{vmatrix}$$

From formula (3.2.9), evaluating the shape function $\varphi_1(x, y)$ at nodes (x_1, y_1), (x_2, y_2), (x_3, y_3) gives

$$\phi_1(x_1,y_1)=\frac{1}{2A_e}\left[x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1\right]$$

$$but\ A_e=\frac{1}{2}\left[x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1\right]$$

$$\Rightarrow 2A_e=\left[x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1\right]$$

$$\Rightarrow \phi_1(x_1,y_1)=1$$

$$\phi_1(x_2,y_2)=\frac{1}{2A_e}\left[x_2y_3-x_3y_2+x_3y_2-x_2y_3+x_2y_2-x_2y_2\right]=0$$

$$\phi_1(x_3,y_3)=\frac{1}{2A_e}\left[x_2y_3-x_3y_2+x_3y_3-x_3y_3+x_3y_2-x_2y_3\right]=0$$

Similarly the shape functions ϕ_2 *and* ϕ_3 have a value of 1 at nodes 2 and 3 respectively and 0 at other nodes. Hence

$$\left.\begin{aligned}\phi_i^e(x_j,y_j)&=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases}\\ \text{and}\ \ &\sum_{i=1}^{3}\phi_i^e=1\end{aligned}\right\} \qquad (3.2.10)$$

We also have

$$\begin{aligned}\frac{\partial\phi_1^e}{\partial x}&=\frac{y_2-y_3}{2A_e},\frac{\partial\phi_1^e}{\partial y}=\frac{x_3-x_2}{2A_e}\\ \frac{\partial\phi_2^e}{\partial x}&=\frac{y_3-y_1}{2A_e},\frac{\partial\phi_2^e}{\partial y}=\frac{x_1-x_3}{2A_e}\\ \frac{\partial\phi_3^e}{\partial x}&=\frac{y_1-y_2}{2A_e},\frac{\partial\phi_3^e}{\partial y}=\frac{x_2-x_1}{2A_e}\end{aligned} \qquad (3.2.11)$$

Step 2: Variational Formulation

To find the variational form of equation (3.2.1), we multiply it with the test function v(x, y) and integrate the result using formula (1. 2.16) 0ver a typical element R_e and we obtain,

$$\iint_{R_e}\left[\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)-vf\right]dxdy-\int_{c_e}vq_n ds \qquad (3.2.12)$$

where $q_n=\eta_x\frac{\partial u}{\partial x}+\eta_y\frac{\partial u}{\partial y}$, η_x *and* η_y are the cosines of aunit normal $\hat{n}$ on the boundary $c_e$and ds is an arc length of an infinisimal element along the boundary.

Step 3: Approximation over the element e

Let $u^e(x,y)$ be an approximation to u(x, y) over the element e so that

$$\phi_i(x_j,y_j)=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases} \qquad (3.2.14)$$

Using (3.2.13) in (3.2.12), we get

$$\sum_{j=1}^{n} \iint_{R_e}\left[\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}+\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\right)u_j^e dxdy\right]-\iint_{R_e} f\phi_i dxdy-\int_{c_e}\phi_i q_n ds \qquad (3.2.15)$$

where $i=1,2,..,n.$ $v(x,y)=\phi(x,y)$ as basebase functions

hence, equation (3.2.15) can be written in the matrix form

$$\sum_{j=1}^{n} K_{ij}^e u_j^e = F_i^e \qquad (3.2.16)$$

$$where \quad K_{ij}^e = \iint_{R_e}\left[\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}+\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\right)dxdy\right]-$$

$$F_i^e = \iint_{R_e} f\phi_i dxdy + \int_{c_e}\phi_i q_n ds \qquad (3.2.17)$$

Using equation (3.2.11) ,the element matrices K^e_{ij} and F^e_i (in 3.2.16) can be computed easily. These computations will be demonstrated using simple example.

Example: we consider a particular case of the problem defined by equation (3.2.1) that is the poison equations

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=2, \qquad 0 \le x,y \le 1 \qquad (3.2.18)$$

with condition $u=0$ *on the boundary of the square*

$0 \le x \le 1, 0 \le y \le 1$

Solution: Comparing (3.2.1) and (3.2.18) we have f(x, y) =-2

Step1-3 is given below

We divided the square region along a line of symmetry x=y and then consider only the lower triangular part. We again sub divided the lower triangular part in to four triangular elements as shown in Fig 3.2.2. Let the elements be numbered as shown in the figure and it is seen that elements a, b and d are symmetrical. Hence, the element matrices for these elements will be of the same type.

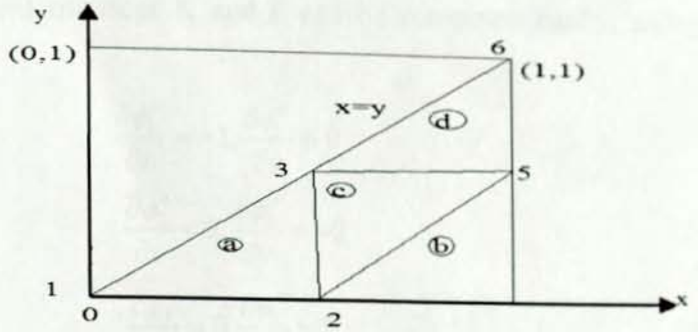

Fig 3.2.2

Now assume $u^e(x, y)$ can be expressed

$u^e(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$ where α_1, α_2 and α_3 are given in equation (3.2.5).

Then we have the following table

Table 3.2.3: Information needed for the computations of the base functions

e	x_1	x_2	x_3	y_1	y_2	y_3	Area= Ae
a	0	1/2	1/2	0	0	1/2	1/8
b	1/2	1	1	0	0	1/2	1/8
c	1/2	1	1/2	0	1/2	1/2	1/8
d	1	1	1/2	1/2	1	1/2	1/8

The derivation of the element matrices and vectors of the elements:

Element a

The vertices 1, 2 and 3 of element "a" are given by (0, 0), (1/2, 0) and (1/2, 1/2) respectively and area of this element is

$$A_a = \iint_{\Delta 123} dxdy = \frac{1}{8}$$

Using equations (3.2.9)

$$\begin{aligned}
\phi_1^1(x,y) &= 4\left[\frac{1}{4} - 0 + \left(-\frac{1}{2}\right)x\right] = 1 - 2x \\
\phi_2^1(x,y) &= 4\left[0 + \frac{1}{2}x - \frac{1}{2}y\right] = 2(x-y) \qquad (3.2.19) \\
\text{and} \quad \phi_3^1(x,y) &= 4\left[0 + \frac{1}{2}y\right] = 2y
\end{aligned}$$

It can be shown that

$$\emptyset_1^1 + \emptyset_2^1 + \emptyset_3^1 = 1$$

The element matrices K and F can be computed easily, using equation (3.2.11) we obtain

$$\begin{aligned}
&\frac{\partial \phi_1^a}{\partial x} = -2, \frac{\partial \phi_1^a}{\partial y} = 0\\
&\frac{\partial \phi_2^a}{\partial x} = 2, \frac{\partial \phi_2^a}{\partial y} = -2\\
&\frac{\partial \phi_3^a}{\partial x} = 0, \frac{\partial \phi_3^a}{\partial y} = 2
\end{aligned} \tag{3.2.20}$$

Equation (3.2.17) gives

$$\begin{aligned}
&K_{11}^a = \iint_{R_e} [4dxdy] = \frac{1}{2}, K_{12}^a = \iint -4dxdy = -\frac{1}{2}, similarly\ K_{13}^a = 0\\
&K_{21}^a = -\frac{1}{2}, K_{22}^a = 1,\ K_{23}^a = -\frac{1}{2}\\
&K_{31}^a = K_{32}^a = -\frac{1}{2},\ and\ \ K_{33}^a = \frac{1}{2}
\end{aligned} \tag{3.2.21}$$

$$\begin{aligned}
\text{Simlarly, } F_1^a &= \iint_{\Delta 123} 2(1-2x)dxdy + \int_{C_{123}} q_n(1-2x)ds\\
&= \int_0^{\frac{1}{2}}\int_y^{\frac{1}{2}} 2(1-2x)dxdy + l_1^a\\
&= \frac{1}{12} + l_1^a\ wher\ l_1^{1a} = \int_{C_{123}} q_n(1-2x)ds\\
F_2^a &= \iint_{\Delta 123} 2(2x-2y)dxdy + \int_{C_{123}} q_n(2x-2y)ds\\
&= \frac{1}{12} + l_2^a\ wher\ l_2^a = \int_{C_{123}} q_n(2x-2y)ds\\
and\ \ F_3^a &= \iint_{\Delta 123} 4ydxdy + \int_{C_{123}} 2yds\\
&= \frac{1}{12} + l_2^a\ wher\ l_3^a = \int_{C_{123}} q_n 2yds
\end{aligned} \tag{3.2.22}$$

$$K^a = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad and \quad F^a = \begin{bmatrix} \frac{1}{12} + l_1^a \\ \frac{1}{12} + l_2^a \\ \frac{1}{12} + l_3^a \end{bmatrix}$$

But elements **a**, **b** and **c** are symmetrical. Hence, in a similar fashion we can compute for the elements **b** and **c**.

<u>Element b</u>

The vertices with respect to the local nodes 2, 4, and 5 of element "b" are given by (1/2, 0), (1, 0) and (1, 1/2) respectively. Then,

$$K^b = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad and \quad F^b = \begin{bmatrix} \frac{1}{12} + l_1^a \\ \frac{1}{12} + l_2^a \\ \frac{1}{12} + l_3^a \end{bmatrix}$$

<u>Element d</u>

The vertices with respect to the local nodes 5, 6, and 3 of element "c" are given by (1,1/2), (1, 1) and (1/2, 1/2) respectively.

Then

$$K^c = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad and \quad F^c = \begin{bmatrix} \frac{1}{12} + l_1^a \\ \frac{1}{12} + l_2^a \\ \frac{1}{12} + l_3^a \end{bmatrix}$$

<u>Element c</u>

The vertices with respect to the local nodes 2, 5 and 3 of element "c" are given by (½, 0), (1, ½) and (½, ½) respectively. Then using equation (3.2.9) ,we obtain

$$\begin{aligned} \phi_1^d(x,y) &= 1-2x+2y \\ \phi_2^d(x,y) &= 1-2y \\ \phi_3^d(x,y) &= 2x-1 \end{aligned} \qquad (3.2.23)$$

Again from equation (3.2.23) we have the following

$$\begin{aligned} \frac{\partial \phi_1^d}{\partial x} &= -2, \frac{\partial \phi_1^d}{\partial y} = 2 \\ \frac{\partial \phi_2^d}{\partial x} &= 0, \frac{\partial \phi_2^d}{\partial y} = -2 \\ \frac{\partial \phi_3^d}{\partial x} &= 2, \frac{\partial \phi_3^d}{\partial y} = 0 \end{aligned} \qquad (3.2.24)$$

From equation (3.2.17), we get that

$$\text{where} \quad K_{11}^d = \iint_{\Delta 253} \left[\left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_1}{\partial x} \right)^2 \right] dxdy = 8 \iint_{\Delta 253} dxdy = 1$$

$$K_{12}^d = \iint_{\Delta 253} \left[\left(\frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} \right) + \left(\frac{\partial \phi_1}{\partial y} \frac{\partial \phi_2}{\partial y} \right) \right] dxdy = -4 \iint_{\Delta 253} dxdy = -\frac{1}{2} = K_{21}^d \text{ by symetry}$$

$$K_{13}^d = \iint_{\Delta 253} \left[\left(\frac{\partial \phi_1}{\partial x} \frac{\partial \phi_3}{\partial x} \right) + \left(\frac{\partial \phi_1}{\partial y} \frac{\partial \phi_3}{\partial y} \right) \right] dxdy = -4 \iint_{\Delta 253} dxdy = -\frac{1}{2} = K_{31}^d \text{ by symetry}$$

$$K_{22}^d = \iint_{\Delta 253} \left[\left(\frac{\partial \phi_2}{\partial x} \right)^2 + \left(\frac{\partial \phi_2}{\partial x} \right)^2 \right] dxdy = 4 \iint_{\Delta 253} dxdy = \frac{1}{2}$$

$$K_{23}^d = \iint_{\Delta 253} \left[\left(\frac{\partial \phi_2}{\partial x} \frac{\partial \phi_3}{\partial x} \right) + \left(\frac{\partial \phi_2}{\partial y} \frac{\partial \phi_3}{\partial y} \right) \right] dxdy = 0 \iint_{\Delta 253} dxdy = 0 = K_{32}^d \text{ by symetry}$$

$$K_{33}^d = \iint_{\Delta 253} \left[\left(\frac{\partial \phi_3}{\partial x} \right)^2 + \left(\frac{\partial \phi_3}{\partial x} \right)^2 \right] dxdy = 4 \iint_{\Delta 253} dxdy = \frac{1}{2}$$

$$F_1^d = \iint_{\Delta 253} 2\phi_1 dxdy + \int_{c_{\Delta 253}} \phi_1 q_n ds =$$

$$= 2 \iint\limits_{\Delta 253} (1 - 2x + 2y)dxdy + l_1^d$$

$$= 2 \int_{\frac{1}{2}}^{1} \int_{x-\frac{1}{2}}^{\frac{1}{2}} (1 - 2x + 2y)dxdy + l_1^d$$

$$= \frac{1}{12} + l_1^d \qquad \text{where} l_1^d = \int_{c_{\Delta 253}} (1 - 2x + 2y)q_n ds$$

Similarly we can compute for

$$F_2^d = \frac{1}{12} + l_1^d \qquad where\, l_2^d = \int_{c_{\Delta 253}} (1 - 2y)q_n ds$$

$$F_3^d = \frac{1}{12} + l_3^d \qquad where\, l_3^d = \int_{c_{\Delta 253}} (2x - 1)q_n ds$$

Then

$$K^d = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad and \quad F^d = \begin{bmatrix} \frac{1}{12} + l_1^d \\ \frac{1}{12} + l_2^d \\ \frac{1}{12} + l_3^d \end{bmatrix}$$

Let the global nodes be U_1, U_2, U_3, U_4, U_5 and U_6 corresponding to the local nodes u_1, u_2, u_3, u_4, u_5 and u_6 at the respective vertices. As there are six nods, the corresponding matrix will be of order six. Now, we write down the complete system for each element.

Element e=a

$$K^a = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad and \quad F^a = \frac{1}{12}\begin{bmatrix} 1+12l_1^a \\ 1+12l_2^a \\ 1+12l_3^a \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{(i)}$$

Since the elements b and d are similar to element a, the element matrices will be the same type as that of element a. Thus **for element e=b**

$$K^b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad and \quad F^b = \frac{1}{12}\begin{bmatrix} 0 \\ 1+12l_1^b \\ 0 \\ 1+12l_2^b \\ 1+12l_3^b \\ 0 \end{bmatrix} \qquad \textbf{(ii)}$$

Similarly, **for element e=d**

$$K^d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad and \quad F^d = \frac{1}{12}\begin{bmatrix} 0 \\ 0 \\ 1+12l_1^d \\ 0 \\ 1+12l_2^d \\ 1+12l_3^d \end{bmatrix} \qquad \textbf{(iii)}$$

Finally, **for element c**

$$K^c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad and \quad F^c = \frac{1}{12}\begin{bmatrix} 0 \\ 1+12l_3^c \\ 1+12l_2^c \\ 0 \\ 1+12l_1^c \\ 0 \end{bmatrix} \qquad \textbf{(iv)}$$

Step 4: Assembly of the element equations to get the global system

Assembling the elements matrices in (i),(ii),(iii)and(iv) we get

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} l_1^a \\ l_2^a + l_1^b + l_3^c \\ l_3^a + l_2^c + l_1^d \\ l_2^b \\ l_3^b + l_1^c + l_2^d \\ l_3^d \end{bmatrix} \qquad \textbf{(v)}$$

From the boundary conditions, we have that

$U_1 = U_2 = U_4 = U_5 = U_6 = 0$

Hence, equation (v) gives

$$-U_3 = \frac{1}{4} + l_1^b + l_2^a + l_3^c \qquad \text{(vi)}$$

$$2U_3 = \frac{1}{4} + l_3^a + l_2^c + l_1^d \qquad \text{(vii)}$$

$$U_3 = \frac{1}{4} + l_3^b + l_1^c + l_2^d \qquad \text{(v)}$$

From (vii) we have that $U_3 = \frac{1}{8} + \frac{1}{2}(l_3^a + l_2^c + l_1^d)$ (ix)

$$\text{But} \quad l_3^a = \int_{C_{123}} q_n^a 2y ds$$

$$= \int_0^{0.5} [q_n^a .2y]_{y=0} dx + \int_0^{0.5} \left[q_n^a .2y\right]_{x=0.5} dy + \int_{0.5}^{0} \left[q_n^a .2y\right]_{y=x} dx = 0,$$

Since the first integral vanishes and the remaining two integrals cancel each other, in a similar manner, it can be shown that

$$l_2^c = l_1^d = 0$$

Hence it follows that $U_3 = \frac{1}{8} = 0.125$ from the analytical solution the exact value is 0.14734 and hence the finite element solution obtained above has an error of about 15%.The accuracy of the finite element solution can be improved by using finer meshes.

Summary on Finite Element Method

Finite element is the most applicable numerical method for obtaining global or integral approximation to differential equations especially when dealing with boundary conditions defined over complex geometries that are common in practical applications. Furthermore FEM can be viewed as a special form of the well known Rayleigh –Ritz and Galrkin's methods of numerical approximations. The important reason is that it overcomes the basic problems encountered in applying these methods in their classical form.

Conclusion on the seminar

In the seminar we have introduced with Rayleigh-Ritz, Galerkin and Finite Element approximation methods and used them in solving BVPs.

The Rayleigh-Ritz and Galerkin methods are powerful approximation techniques under certain ideal conditions. The geometry of the system under analysis has to be simple otherwise it is extremely difficult to determine what approximate function should be used in the solution. The drawback of both methods is that they don't have symmetric procedure for finding global approximation functions especially for 2-D and 3-D cases. Further the resulting matrices are fully populated, making computation of larger system matrices fairly intensive.

Finite Element Method uses the idea of Rayleigh-Ritz and Galerkin, but has the powerful benefit of a symmetric procedure for finding approximation function over each element. Extremely complex geometries can be broken up into finely-meshed elements of simple geometry and consequently the assembling of all system matrices is an easy task through the use of computer. Once a general computer program is written, it can be used for the solution of any problem simply by changing the input data. Although the finite element method requires much greater matrix sizes, the method leads to banded, symmetric matrices that are more computationally efficient.

The seminar is, therefore, intended to be an introduction to the terminologies, methodologies of the three methods of approximation. The reader who wishes to pursue these methods and understand the analysis of these methods is recommended to refer other references cited in the bibliography.

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