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DEPARTMENT OF MATHEMATICS

Critical Curve and Singular Zeros of Harmonic Quadrinomial

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Abstract

In this thesis, we show that the critical curve of the quadrinomial $P_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ is the part of the unit circle. Moreover, we prove that there are at most two singular zeros for the quadrinomial $P_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ we are considering.

Keywords: Analytic functions, harmonic polynomials, harmonic quadrinomials, sense preserving, sense reversing, critical curves, zeros, singular, and nonsingular zeros.

Chapter 1

Introduction

It is well known that a holomorphic function on an open set D ($D \subset \mathbb{C}$) is a complex function that is differentiable at every point in its domain D . A harmonic function f on D is a function that satisfies the Laplace equation on D . In this thesis, we study critical curves and zeros of harmonic functions of the form $f(z) = h(z) + \overline{g(z)}$ where h and g are nonconstant analytic functions of one complex variable. We focus on the complex harmonic family of polynomials of the form $Q_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ which are subfamilies of what we call harmonic quadrinomial of the form $Q_{b,c} = bz^k + \bar{z}^n + c\bar{z}^m + z$ where $k, m, n \in \mathbb{N}$ with $n \geq k > m \geq 1$. The zero inclusion regions and the possible maximum number of the zeros of quadrinomial of this type were studied in [2] for real parameters b and c .

The critical curve of such harmonic quadrinomials was partly studied in the zero inclusion regions and the possible maximum number of the zeros of quadrinomials of this type were studied in [2] for real parameters b and c but only for the special case $n = k$, $m = 1$ and z^{k-1} is purely imaginary. In [21] the critical curves for $p_c(z) = z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n$ with $c > 0$, have been studied. It is found that the critical curve of such a quadrinomial is a unit circle. Motivated by [2] and [21] we are interested in exploring a family of such polynomials which have unit circles as their critical curves and study the zeros of such polynomials. In these thesis, we focus on the polynomial of $P_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ where $k, n \in \mathbb{N}$ with $n \geq k \geq 2$ and $b \in \mathbb{R}$ and explore their critical curves and find zeros of such polynomials.

In doing these we found that the unit circle is always part of the critical curve and there are at most two zeros on the critical curve. Actually, the zeros on the critical curves are called singular zeros. The main theorems in this thesis are theorem 3.1, 3.2, and 3.3.

This thesis is structured as follows. Chapter 2 introduces fundamental concepts related to complex harmonic functions of the form $f = h + \bar{g}$, where h and g are analytic. We will examine the properties of complex-valued harmonic functions, including critical curves, the argument principle, and the harmonic analog of Rouché's theorem. These results will be used to explore critical curves and analyze singular zeros in the subsequent chapter. In Chapter 3, we investigate critical curves and the zeros of the quadrimomial. Chapter 4 provides a summary and conclusion.

Chapter 2

Preliminaries

In this chapter, we review some important concepts and definitions that we will use later to prove our results. We begin by stating the well-known results of Cauchy's theorem, argument principles, Rouché's theorem, and some other useful results.

2.1 Holomorphic and harmonic functions

A holomorphic function $f(z)$ is a function defined on an open subset of the complex plane that is differentiable at every point in its domain. More precisely, we have the following. If $f(z)$ is analytic at z_0 , then the power series expansion is given by $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, where the coefficients a_n are complex numbers determined by the derivatives of $f(z)$ at z_0 .

Theorem 2.1. [8] *Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous partial derivatives. Then $f : G \rightarrow \mathbb{C}$ defined by*

$$f(z) = u(z) + iv(z)$$

is analytic if and only if u and v satisfy Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Definition 2.1. For a complex variable $z = x + iy$, we have the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (2.1)$$

For a complex-valued function $f(z)$, the equation $\frac{\partial f}{\partial \bar{z}} = 0$ is just another way of writing

the Cauchy-Riemann equations. The operators $f_z = \frac{\partial f}{\partial z}$ and $f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}$ are linear and they have the usual properties of differential operators.

Lemma 2.1. [21] *The Cauchy-Riemann equations are equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$.*

Proof. Let $f = u + iv$ be a complex function. Then the definition of $\frac{\partial f}{\partial \bar{z}}$ yields

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + iv) \\ &= \frac{1}{2} \left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right).\end{aligned}$$

Thus, $\frac{\partial f}{\partial \bar{z}} = 0$ if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

□

Definition 2.2. [11] A real valued function $\phi = \phi(x, y)$ defined on a domain $G \subset \mathbb{R}^2$ is said to be harmonic on G if it possesses continuous second order partial derivatives and satisfies Laplace's equation

$$\Delta \phi := \phi_{xx}(x, y) + \phi_{yy}(x, y) = 0. \quad (2.2)$$

Theorem 2.2. [8] *Let G be either the whole plane \mathbb{C} or some open disk. If $u : G \rightarrow \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.*

Theorem 2.3. [8] *If a function $f(x, y) = u(x, y) + iv(x, y)$ is analytic throughout the domain G , then both $u(x, y)$ and $v(x, y)$ are harmonic.*

Definition 2.3. [11] A continuous function $f(x, y) = u(x, y) + iv(x, y)$ defined in G is a complex-valued harmonic function in G if $u(x, y)$ and $v(x, y)$ are real harmonic (but not necessarily harmonic conjugates) in G .

Example 2.1. Consider the complex-valued harmonic function: $f(x, y) = e^x \cos y + ie^x \sin y$. The function $u(x, y) = e^x \cos y$ is harmonic because:

$$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0.$$

The function $v(x, y) = e^x \sin y$ is harmonic because:

$$v_{xx} + v_{yy} = e^x \sin y - e^x \sin y = 0.$$

Again, compute $u_x = e^x \cos y$ and $v_y = e^x \cos y$, showing that $u_x = v_y$. Compute $u_y = -e^x \sin y$ and $-v_x = -e^x \sin y$, showing that $u_y = -v_x$.

Therefore, $u(x, y)$ and $v(x, y)$ are harmonic conjugates, and the function $f(x, y) = e^x \cos y + ie^x \sin y$ is analytic.

Example 2.2. Let $u(x, y) = x^2 - y^2$ and $v(x, y) = 4xy$. Both are harmonic because

$$u_{xx} + u_{yy} = 2 - 2 = 0,$$

$$v_{xx} + v_{yy} = 0 + 0 = 0.$$

Now, let's check the Cauchy-Riemann equations again:

$$u_x = 2x \quad \text{and} \quad v_y = 4x.$$

Clearly, $u_x \neq v_y$ in general since $2x \neq 4x \neq 0$. Thus, $u(x, y)$ and $v(x, y)$ are harmonic functions, but they are not harmonic conjugates. These shows the function $f(x, y) = (x^2 - y^2) + i(4xy)$ is a complex-valued harmonic function that is not analytic.

Theorem 2.4. [11] *If $f(x, y) = u(x, y) + iv(x, y)$ is harmonic in the simply connected domain G , then it can be written in the form of $f(z) = h(z) + \overline{g(z)}$ where h and g are analytic functions in G .*

Proof. Assume $f(x, y) = u(x, y) + iv(x, y)$ is harmonic in simply connected domain G . Then by definition, $u(x, y)$ and $v(x, y)$ are real-valued harmonic on G . Then there exist analytic functions $a(z)$ and $b(z)$ such that $u = \text{Re}a(z)$ and $v = \text{Im}b(z)$. Hence,

$$\begin{aligned} f(x, y) &= u(x, y) + iv(x, y) = \text{Re}a(z) + i\text{Im}b(z) \\ &= \frac{a(z) + \overline{a(z)}}{2} + i \frac{b(z) - \overline{b(z)}}{2i} \\ &= \frac{a(z) + b(z)}{2} + \frac{\overline{a(z) - b(z)}}{2}. \end{aligned}$$

This implies $h(z) = \frac{a(z)+b(z)}{2}$ and $g(z) = \frac{a(z)-b(z)}{2}$.

Therefore, f can decompose as $f(z) = h(z) + \overline{g(z)}$ where h and g are analytic functions in G . □

Definition 2.4. [2] The dilatation of a complex-valued harmonic function $f(z) = h(z) + \overline{g(z)}$ is defined to be $\omega(z) = \frac{g'(z)}{h'(z)}$.

Note that the dilatation of the analytic function is zero. In general, the dilatation is a measure of how far a harmonic function is from being analytic. There is a connection between the dilatation of a harmonic function and its locally univalent and sense-preserving nature.

Lemma 2.2. [21] *If f is harmonic with continuous second partial derivatives then f_z is analytic.*

Proof. Let f be harmonic, so $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. By definition 2.1 we have,

$$\begin{aligned} f_z &= \frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + iv) \\ &= \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{i}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\ &= \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right). \end{aligned}$$

Applying the differential operator $\frac{\partial}{\partial \bar{z}}$ yields:

$$\begin{aligned} f_{z\bar{z}} &= \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \\ &= \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2} - i \frac{\partial^2 f}{\partial x \partial y} + i \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2} \right). \end{aligned}$$

Because partial derivatives commute, we have

$$f_{z\bar{z}} = \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

Therefore, $f_{z\bar{z}} = 0$ because, f is harmonic. Then by Lemma 2.1 f_z is analytic. \square

Definition 2.5. [1] A function $f(z)$ is said to be locally univalent in the domain G if there is a small neighborhood around each point $z_0 \in G$, such as a smaller disk centered at z_0 and the function is one-to-one in that neighborhood.

Theorem 2.5. [8] *If $f : G \rightarrow \mathbb{C}$ is analytic then f preserves angles at each point z_0 of G where $f'(z_0) \neq 0$.*

A function $f : G \rightarrow \mathbb{C}$ which has the angle preserving property and also has

$$\lim_{z \rightarrow a} \frac{|f(z) - f(a)|}{|z - a|}$$

existing is called a conformal map. If f is analytic and $f'(z) \neq 0$ for any z then f is conformal. The converse of these statement is also true.

Proposition 2.1. [12] The Jacobian of complex-valued function $f(z) = u + iv$ is defined by

$$J_f = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x. \quad (2.3)$$

Theorem 2.6. [12] If f is analytic, it's Jacobian takes the form $J_f(x) = (u_x)^2 + (v_x)^2 = |f'|^2$.

Theorem 2.7. [20] If f is a complex-valued harmonic function that is locally univalent in a domain $D \subset \mathbb{C}$, then its Jacobian $J_f(z)$, does not vanish for all $z \in D$.

Note that, univalent function is complex valued one-to-one function.

Theorem 2.8. For harmonic function $f = h + \bar{g}$, $J_f = |h'|^2 - |g'|^2$.

Proof. Let us look at a harmonic function $f(z) = h(z) + \overline{g(z)}$ where $h(z) = u(x, y) + iv(x, y)$ and $g(z) = s(x, y) + it(x, y)$. Then

$$\begin{aligned} f(x + iy) &= u(x, y) + iv(x, y) + \overline{s(x, y) + it(x, y)} \\ &= u(x, y) + iv(x, y) + s(x, y) - it(x, y) \\ &= (u(x, y) + s(x, y)) + i(v(x, y) - t(x, y)). \\ J_f(z) &= \begin{vmatrix} u_x + s_x & v_x - t_x \\ u_y + s_y & v_y - t_y \end{vmatrix} = (u_x^2 + v_x^2) - (s_x^2 - t_x^2) = |h'|^2 - |g'|^2. \end{aligned}$$

□

It is a well-established fact that analytic functions are conformal when their derivatives are non-zero, meaning they preserve the sense (or orientation) of angles. Since complex harmonic polynomials are formed by adding an analytic function to the conjugate of another analytic function, certain regions of the complex plane will preserve the sense, while others will reverse it.

Definition 2.6. [1] A complex-valued harmonic function $f = h(z) + \overline{g(z)}$ is said to be sense-preserving at z_0 if $J_f(z_0) > 0$ and is sense-reversing at z_0 if $J_f(z_0) < 0$. If f is neither sense-preserving nor sense-reversing at z_0 then f is said to be singular at z_0 .

Proposition 2.2. [21] A complex function f is sense-preserving when $|\omega(z)| < 1$ and sense-reversing when $|\omega(z)| > 1$. The curve that divides the sense-preserving region from the sense-reversing region is known as the critical curve.

Definition 2.7. The critical curve of $f(z) = h(z) + \overline{g(z)}$ is the set of all points $z \in \mathbb{C}$ such that $|\omega(z)| = 1$.

Definition 2.8. [8] A mapping of the form $S(z) = \frac{az+b}{cz+d}$ is called linear fractional transformation. If $a, b, c,$ and d is also satisfies $ad - bc \neq 0$, then $S(z)$ is called Möbius transformation.

Theorem 2.9. [20] The function $f = h + \bar{g}$ is locally univalent and sense-preserving if and only if $|\omega(z)| < 1$ for all $z \in G$.

Definition 2.9. [5] The roots that lies on the unit circle are referred to as uni-modular roots.

Definition 2.10. [1] The roots of the quadrinomial $q(z) = bz^k + \bar{z}^n + c\bar{z}^m + z$ that lies on the circle of radius $M_{b,c}$ are said to be the $M_{b,c}$ -modular roots.

Theorem 2.10. [1] Let $q(z) = bz^k + \bar{z}^n + c\bar{z}^m + z$ be a two-parameter family of quadrinomials with $b, c \in \mathbb{R}^+ \setminus \{1\}$ where $k, m, n \in \mathbb{N}$. Then for this family of quadrinomials, they determined the curve $\Gamma_{b,c} = \left\{ z \in \mathbb{C} : |z| = M_{b,c} = \left(\frac{c^2-1}{k^2(b^2-1)} \right)^{\frac{1}{2k-2}} \right\}$ which separates the zeros in the sense-preserving region from the zeros in the sense-reversing region by considering the relation $k = n > m = 1$.

Theorem 2.11. [21] For the complex harmonic family of the polynomial $p_c(z) = z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n$ where $n \geq 3$ and $c > 0$, they determined the unit circle $|z| = 1$ is always part of the critical curve of $p_c(z)$.

Theorem 2.12. [4] Let $p_c(z) = z^n + c\bar{z}^k - 1$ where $1 \leq k \leq n-1, n \geq 3, c \in \mathbb{R}^+$ and $\gcd(n, k) = 1$. Then the critical circle is $\Gamma_c = \left\{ z \in \mathbb{C} : |z| = R_c = \left(\frac{ck}{n} \right)^{\frac{1}{n-k}} \right\}$. They determined $p_c(z)$ being sense-reversing on the interior of this circle and sense-preserving on its exterior.

Theorem 2.13 ([19]Bezout's Theorem In the Plane). Let f and g be relatively prime polynomials in the real variables x and y with real coefficients, and let $\deg(h) = n$ and $\deg(g) = m$. Then the two algebraic curves $f(x, y) = 0$ and $g(x, y) = 0$ have at most mn points in common.

Bezout's theorem is one of the most fundamental results about the degrees of polynomial surfaces and it bounds the size of the intersection of polynomial surfaces.

2.2 Order of a zeros

Definition 2.11. [21] Let $f(z)$ be an analytic function at a point z_0 . The function can be expanded into a Taylor series around z_0 as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n = \frac{f(z_0)^{(n)}}{n!}$ are the coefficients of the series . If z_0 is a zero of $f(z)$, then $f(z_0) = 0$. The order of the zero at z_0 is the smallest integer m such that the coefficient a_m is nonzero. Now, consider a complex harmonic function of the form $f(z) = h(z) + \overline{g(z)}$, where both h and g are analytic functions. Suppose f has a zero at some point $z_0 \in \mathbb{C}$. We can represent h and g using their Taylor series expansions around z_0 as follows:

$$h(z) = a_0 + \sum_{j=r}^{\infty} a_j (z - z_0)^j,$$

$$g(z) = b_0 + \sum_{j=s}^{\infty} b_j (z - z_0)^j,$$

where $r > 0$, $s > 0$, $a_r \neq 0$, and $b_s \neq 0$. Given that $f(z_0) = 0$, it follows that $b_0 = -\overline{a_0}$. Then, we define the order of the zero at z_0 as r if z_0 is in a sense-preserving region, or $-s$ if it is in a sense-reversing region. Zeros that lie within sense-preserving or sense-reversing regions are known as non-singular zeros. However, zeros located on the critical curve are referred to as singular zeros, and their order is not defined.

Theorem 2.14. (*Argument Principle For Analytic Function* [8]) Let $f(z)$ be a meromorphic function in the interior of some simply connected region G with poles $P_1, P_2, P_3, \dots, P_m$ and zeros $Z_1, Z_2, Z_3, \dots, Z_n$ counted according to multiplicity. If γ is closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $P_1, P_2, P_3, \dots, P_m$ and $Z_1, Z_2, Z_3, \dots, Z_n$. Suppose K is the number of poles minus the number of zeros. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = K.$$

Proof. Let $f(z)$ have zeros at $(z - z_0)$ of order m . $f(z) = (z - z_0)^m g(z)$, where $g(z) \neq 0$ and analytic. Then

$$f'(z) = m(z - z_0)^{m-1} g(z) + g'(z)(z - z_0)^m.$$

Dividing both side by $f(z)$ yields,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)} \\ &= \frac{m(z-z_0)^m g(z)}{(z-z_0)^m g(z)(z-z_0)} + \frac{(z-z_0)^m g'(z)}{(z-z_0)^m g(z)} \\ &= \frac{m(z-z_0)^m g(z)}{(z-z_0)^{m+1} g(z)} + \frac{(z-z_0)^m g'(z)}{(z-z_0)^m g(z)} \\ &= \frac{m}{(z-z_0)} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Since $\frac{m}{(z-z_0)}$ is singularity part and $\frac{g'(z)}{g(z)}$ is analytic part with $g'(z) \neq 0$

$$\text{Res}\left(\frac{f'}{f}, z_0\right) = m,$$

where m is total number of zeros.

Similarly, if $f(z)$ have poles of order m at $z = z_0$, we have

$$\begin{aligned}f(z) &= \frac{g(z)}{(z-z_0)^{m'}} \\ &= (z-z_0)^{-m'} g(z) \\ f'(z) &= -m(z-z_0)^{-m'-1} g(z) + (z-z_0)^{-m'} g'(z) \\ &= \frac{-m(z-z_0)^{-m'} g(z)}{(z-z_0)} + (z-z_0)^{-m'} g'(z).\end{aligned}$$

Simplifying gives,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-m(z-z_0)^{-m'} g(z)}{(z-z_0)(z-z_0)^{-m'} g'(z)} + \frac{(z-z_0)^{-m'} g'(z)}{(z-z_0)^{-m'} g'(z)} \\ &= \frac{-m}{(z-z_0)} + \frac{g'(z)}{g(z)}\end{aligned}$$

Also $\frac{-m}{(z-z_0)}$ is singularity part,

$$\begin{aligned}\text{Res}\frac{f'(z)}{f(z)} &= -m \\ \int_{\gamma} \frac{f'(z)}{f(z)} dz &= 2\pi i\end{aligned}$$

(the sum of the residue)

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = 2K\pi i.$$

This implies that,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = K.$$

Therefore,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = K = \frac{1}{2\pi} \Delta_C \arg f(z).$$

□

Recall to Argument principle for harmonic functions in [13]. Let f be a complex-valued harmonic function in a Jordan domain G with boundary Γ . Suppose that f is continuous in \overline{G} and $f \neq 0$ in Γ . Suppose that f has no singular zeros in G , and let $N = N_+ + N_-$, where N_+ and N_- are the number of zeros in sense-preserving region and sense-reversing region of f in G , respectively. Then, $\Delta \Gamma \arg f(z) = 2\pi N$.

Theorem 2.15. (*Rouché's theorem for Analytic Function [21]*)

Suppose f and g are Meromorphic in a neighborhood of $B(a;R)$ with no zeros or poles on the circle $\gamma = z : |z - a| = R$. If $Z_f, Z_g (P_f, P_g)$ are the number of zeros (poles) of f and g inside γ counted according to their multiplicity and if

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ , then

$$Z_f - P_f = Z_g - P_g.$$

Corollary 2.1. (*Rouché's Theorem for Complex Harmonic Functions [21]*)

Let p and $p+q$ be harmonic functions in D that have no singular zeros with in \overline{D} . If $|q(z)| < |p(z)|$ on C , then the total sum of the orders of the zeros of p and the total sum of the orders of the zeros of $p+q$ are equal in G .

Now, we are ready to prove our results in the next chapter.

Chapter 3

Some Results Concerning Critical Curve and Singular Zeros

In this chapter, we explore the critical curve and zeros for the complex-valued harmonic quadrinomial of the type $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$, where $b \in \mathbb{R}$ and $k, n \in \mathbb{N}$ with $n > k \geq 2$ and $\gcd(n, k) = 1$.

3.1 Properties of Critical Curves of Quadrinomials

For the quadrinomial $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$, we have analytic part $h = bz^k + \frac{1}{k-1}z^{k-1}$ and co-analytic part $g = \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$. For our analysis, we write the dilatation function of the complex-valued harmonic quadrinomial p_b as the product of Möbius transformations. The complex dilatation function for p_b is

$$\omega(z) = \frac{g'(z)}{h'(z)} = \frac{k b z^{n-2} + z^{n-1}}{k b z^{k-1} + z^{k-2}} = \frac{z^{n-2}(k b + z)}{z^{k-2}(k b z + 1)}.$$

Simplifying yields,

$$\omega(z) = z^{n-k} \left(\frac{k b + z}{k b z + 1} \right) \quad (3.1)$$

for $z \neq 0$. Now from this construction, we begin by analyzing properties of Möbius transformations $\Phi(z) = \frac{k b + z}{k b z + 1}$.

Lemma 3.1. *Let $\Phi(z) = \frac{k b + z}{k b z + 1}$ for $b \in \mathbb{R}$ and $k b \neq \pm 1$. Then $|\Phi(z)| = 1$ if and only if $|z| = 1$.*

Proof. Suppose $|\Phi(z)| = 1$. Then

$$\left| \frac{k b + z}{k b z + 1} \right| = 1.$$

Squaring both side

$$\left| \frac{kb+z}{kbz+1} \right|^2 = \frac{(kb+z)(kb+\bar{z})}{(kbz+1)(kb\bar{z}+1)} = 1.$$

This equation is equivalent to

$$k^2b^2 + kb(z+\bar{z}) + |z|^2 = k^2b^2|z|^2 + kb(z+\bar{z}) + 1$$

This equation is equivalent to

$$0 = |z|^2 + 2kb\text{Re}(z) + k^2b^2 - k^2b^2|z|^2 - 2kb\text{Re}(z) - 1.$$

Simplifying yields

$$k^2b^2|z|^2 - |z|^2 = k^2b^2 - 1.$$

This equation is equivalent to

$$|z|^2[k^2b^2 - 1] = k^2b^2 - 1.$$

This equation is equivalent to

$$|z|^2 = \frac{k^2b^2 - 1}{k^2b^2 - 1},$$

Because $kb \neq \pm 1$. This implies that

$$|z| = 1.$$

Thus, $|\Phi(z)| = 1$ if and only if $|z| = 1$. □

Lemma 3.2. *When $0 < kb < 1$, the function $\Phi(z)$ is an automorphism of the unit disc with inverse*

$$\Phi^{-1}(z) = \frac{z - kb}{-kbz + 1}.$$

Proof. This is a standard fact about Möbius transformations, and we include its proof for completeness.

Notice that Φ is holomorphic which means, Φ is complex differentiable every where with in the unit disc, because $0 < bk < 1$ implies $1 < \frac{1}{bk}$. According to Lemma 3.1, $|\Phi(z)| = 1$ if and only if $|z| = 1$. This means Φ maps the unit disc into itself. Therefore, if $|z| = 1$ we have $|\Phi(z)| = 1$, which means by the Maximum Modulus Principle, $|\Phi(z)|$ can not exceed 1 for any z inside the unit disc, thus $|\Phi(z)| < 1$ for $|z| < 1$. Thus Φ maps the unit disc into the unit disc. The inverse function Φ^{-1} is of the same form as Φ , so it also maps the unit disc into

the unit disc by the same argument as for Φ .

Observe that Φ^{-1} is in fact the inverse to Φ :

$$\Phi(\Phi^{-1}(z)) = \frac{\frac{z-kb}{-kbz+1} + kb}{kb\left(\frac{z-kb}{-kbz+1}\right) + 1} = \frac{z - kb - k^2b^2z + kb}{kbz - k^2b^2 - kbz + 1} = \frac{[-k^2b^2 + 1]z}{-k^2b^2 + 1} = z.$$

And

$$\Phi^{-1}(\Phi(z)) = \frac{\frac{z+kb}{kbz+1} - kb}{-kb\left(\frac{z+kb}{kbz+1}\right) + 1} = \frac{z + kb - k^2b^2z - kb}{-kbz - k^2b^2 + kbz + 1} = \frac{[-k^2b^2 + 1]z}{-k^2b^2 + 1} = z.$$

Thus, $\Phi(\Phi^{-1}(z)) = \Phi^{-1}(\Phi(z)) = z$ for all z . Therefore, Φ and Φ^{-1} are inverses and Φ is an automorphism of the unit disc. \square

Theorem 3.1. Let $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ where $n, k \in \mathbb{N}$ such that $n > k \geq 2$ and $b \in \mathbb{R}$. Then $|\omega(z)| = 1$ if and only if $|z| = 1$.

Proof. If $|z| = 1$, then $|z|^{n-k} = 1$ and $|\Phi(z)| = 1$ by Lemma 3.1.

Consequently,

$$|\omega(z)| = |z|^{n-k} |\Phi(z)| = 1.$$

Now, we need to proof the forward implication: Suppose $|\omega(z)| = 1$. Then

$$1 = |z|^{2(n-k)} \frac{|z + kb|^2}{|kbz + 1|^2}$$

Squaring both side

$$1 = |z|^{2(n-k)} \frac{|z + kb|^2}{|kbz + 1|^2} = |z|^{2(n-k)} \frac{|z|^2 + 2kb\operatorname{Re}(z) + k^2b^2}{k^2b^2|z|^2 + 2kb\operatorname{Re}(z) + 1}.$$

Letting $z = re^{i\theta}$ for some $r > 0$ and $\theta \in [0, 2\pi)$, the above equation becomes

$$\begin{aligned} 1 &= |re^{i\theta}|^{2(n-k)} \frac{|re^{i\theta}|^2 + 2kb\operatorname{Re}(z) + k^2b^2}{k^2b^2|re^{i\theta}|^2 + 2kb\operatorname{Re}(z) + 1} \\ &= r^{2n-2k} \frac{r^2 + 2kbr \cos(\theta) + k^2b^2}{k^2b^2r^2 + 2kbr \cos(\theta) + 1} \\ &= \frac{r^{2n-2k+2} + 2kbr^{2n-2k+1} \cos(\theta) + k^2b^2r^{2n-2k}}{k^2b^2r^2 + 2kbr \cos(\theta) + 1} \end{aligned}$$

This implies that

$$k^2 b^2 r^2 + 2kbr \cos(\theta) + 1 = r^{2n-2k+2} + 2kbr^{2n-2k+1} \cos(\theta) + k^2 b^2 r^{2n-2k}$$

$$\Rightarrow 0 = r^{2n-2k-2} + 2kbr^{2n-2k+1} \cos(\theta) + k^2 b^2 r^{2n-2k} - k^2 b^2 r^2 - 2kbr \cos(\theta) - 1$$

Factoring out yields:

$$0 = k^2 b^2 r^2 (r^{2n-2k+2} - 1) + 2kbr \cos(\theta) (r^{2n-2k} - 1) + (r^{2n-2k+2} - 1).$$

Rearranging and factoring out $r - 1$ leaves us with:

$$0 = (r - 1) \left[k^2 b^2 r^2 (r^{2n-2k+1} + r^{2n-2k} + \dots + r + 1) + 2kbr \cos(\theta) (r^{2n-2k-1} \right.$$

$$\left. + r^{2n-2k-2} + \dots + r + 1) + (r^{2n-2k+1} + r^{2n-2k} + \dots + r + 1) \right].$$

Thus, the above equation is only satisfied when $r = 1$. This means when z is on the unit circle. \square

Corollary 3.1. *The critical curve of the complex-valued harmonic quadrinomial $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ with $k > n \geq 2$ and $b \in \mathbb{R} \setminus \{\pm 1\}$ is given by*

$$\Gamma_b = \{z \in \mathbb{C} : |z| = 1\}.$$

Definition 3.1. The roots of the quadrinomial p_b that lie on the circle of radius Γ_b are said to be Γ_b -modular roots.

Proposition 3.1. *For $0 < bk < 1$, the critical curve of p_b consists only of the unit circle.*

Proof. We applied the properties of the Möbius function Φ for Our proof. Let $0 < bk < 1$. If $|z| < 1$, then $|z|^{n-k} < 1$, and $|\Phi(z)| < 1$ by the proof of Lemma 3.2, which implies $|\omega(z)| = |z|^{n-k}|\Phi(z)| < 1$. If $|z| = 1$, then $|z|^{n-k} = 1$ and $|\Phi(z)| = 1$, by Lemma 3.1 consequently $|\omega(z)| = 1$. If $|z| > 1$, then $|z|^{n-k} > 1$ and $|\Phi(z)| > 1$, since Φ is an automorphism of the unit disc when $0 < bk < 1$, as shown in Lemma 3.2. Thus, $|\omega(z)| = |z|^{n-k}|\Phi(z)| > 1$ when $|z| > 1$. Hence, $|\omega(z)| = 1$ if and only if $|z| = 1$. Therefore, the critical curve of p_b consists only of the unit circle when $0 < bk < 1$. \square

3.2 Singular Zeros of Quadrinomial

Theorem 3.2. *Suppose the critical curve for $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{bk}{n-1}\bar{z}^{n-1}$ is the unit circle. Then, $p_b(z)$ has exactly one singular zero provided that $n = k + 1$.*

Proof. First let us analyze singular zeros of $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{bk}{n-1}\bar{z}^{n-1}$. Suppose $p_b(z) = 0$ where $|z| = 1$ and $\omega(z) = 1$. Then $|z^{n-k} \frac{z+bk}{bkz+1}| = 1$, so $|z^{n-k}||z+bk| = |bkz+1|$. For some $\beta \in \mathbb{C}$, $|\beta| = 1$, $z^{n-k}(z+bk) = \beta(bkz+1)$. Thus,

$$\bar{z}^{n-k} = \beta \left(\frac{bkz+1}{z+bk} \right). \quad (3.2)$$

Taking $p_b(z) = 0$ gives:

$$bz^k + \frac{z^{k-1}}{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1} = 0.$$

Next, let us have the following substitutions from equation 3.2 :

$$bz^k + \frac{z^{k-1}}{k-1} + \frac{\beta}{n} \frac{bkz+1}{bk+z} \bar{z}^k + \frac{bk\beta}{n-1} \frac{bkz+1}{bk+z} \bar{z}^{k-1} = 0.$$

Separating analytic part and co-analytic part yields:

$$bz^k + \frac{z^{k-1}}{k-1} = - \left(\frac{\beta}{n} \frac{bkz+1}{bk+z} \bar{z}^k + \frac{bk\beta}{n-1} \frac{bkz+1}{bk+z} \bar{z}^{k-1} \right).$$

Taking common factor from both side yields:

$$z^{k-1} \left(bz + \frac{1}{k-1} \right) = -\bar{z}^{k-1} \left[\left(\beta \frac{bkz+1}{bk+z} \right) \left(\frac{\bar{z}}{n} + \frac{bk}{n-1} \right) \right].$$

Taking the modulus of both side:

$$\left| z^{k-1} \left(bz + \frac{1}{k-1} \right) \right| = \left| -\bar{z}^{k-1} \left(\beta \frac{bkz+1}{bk+z} \right) \left(\frac{\bar{z}}{n} + \frac{bk}{n-1} \right) \right|.$$

$$\left| bz + \frac{1}{k-1} \right| = \left| \frac{\bar{z}}{n} + \frac{bk}{n-1} \right|.$$

Taking the squared modulus of both side:

$$\left(bz + \frac{1}{k-1} \right) \overline{\left(bz + \frac{1}{k-1} \right)} = \left(\frac{\bar{z}}{n} + \frac{bk}{n-1} \right) \overline{\left(\frac{\bar{z}}{n} + \frac{bk}{n-1} \right)}.$$

Multiplying yields

$$b|z|^2 + \frac{bz}{k-1} + \frac{b\bar{z}}{k-1} + \frac{1}{(k-1)^2} = \frac{|z|^2}{n^2} + \frac{bk\bar{z}}{n(n-1)} + \frac{bkz}{n(n-1)} + \frac{b^2k^2}{(n-1)^2}$$

If $|z| = 1$, we have

$$\begin{aligned} b^2 + \frac{b}{k-1}(z + \bar{z}) + \frac{1}{(k-1)^2} &= \frac{1}{n^2} + \frac{kb}{n(n-1)}(z + \bar{z}) + \frac{b^2k^2}{(n-1)^2} \\ \Rightarrow \frac{2b}{k-1}Re(z) - \frac{2bk}{n(n-1)}Re(z) &= \frac{1}{n^2} + \frac{b^2k^2}{(n-1)^2} - \frac{1}{(k-1)^2} - b^2. \end{aligned}$$

Simplifying the LHS gives

$$2bRe(z) \left(\frac{1}{k-1} - \frac{k}{n(n-1)} \right) = \frac{1}{n^2} + \frac{k^2b^2}{(n-1)^2} - \frac{1}{(k-1)^2} - b^2.$$

A simple calculation then yields

$$2bRe(z) \left[\frac{n(n-1) - k(k-1)}{(k-1)n(n-1)} \right] = \frac{1}{n^2} - \frac{1}{(k-1)^2} + \frac{b^2k^2}{(n-1)^2} - b^2.$$

This implies

$$2bRe(z) = \left[\frac{(k-1)n(n-1)}{n(n-1) - k(k-1)} \right] \left[\frac{1}{n^2} - \frac{1}{(k-1)^2} + \frac{b^2k^2}{(n-1)^2} - b^2 \right].$$

From this we get :

$$Re(z) = \frac{1}{2b} \left[\frac{(k-1)n(n-1)}{n(n-1) - k(k-1)} \right] \left[\frac{1}{n^2} - \frac{1}{(k-1)^2} + \frac{b^2k^2}{(n-1)^2} - b^2 \right]. \quad (3.3)$$

Now, it may not be simple to solve for parameter b explicitly in this equation due to the complexity of the expression. Let us take the particular case for k and n . For instance, suppose $n = k + 1$:

Substituting in the above expression yields:

$$Re(z) = \frac{1}{2b} \left[\frac{(k-1)k(k+1)}{k(k+1) - k(k-1)} \right] \left[\frac{1}{(k+1)^2} - \frac{1}{(k-1)^2} + \frac{b^2k^2}{k^2} - b^2 \right].$$

and simplifying yields:

$$Re(z) = \frac{1}{2b} \left[\frac{k^2 - 1}{2} \right] \left[\frac{1}{(k+1)^2} - \frac{1}{(k-1)^2} \right].$$

Combine and simplfying the terms inside the bracket yields:

$$Re(z) = -\frac{k}{b(k^2 - 1)}. \quad (3.4)$$

Now, using equation 3.4 and taking the case for the value of $Re(z)$, we can solve for parameter b:

Case 1: $Re(z) = 1$: If and only if

$$1 = -\frac{k}{b(k^2 - 1)}$$

If and only if

$$b = -\frac{k}{(k^2 - 1)}.$$

Case 2: $Re(z) = -1$ If and only if

$$-1 = -\frac{k}{b(k^2 - 1)}$$

$$\Leftrightarrow b = \frac{k}{(k^2 - 1)}.$$

Hence, the quadrinomial $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ has exactly one singular zero, if $n = k + 1$ and either

$$b = -\frac{k}{(k^2 - 1)} \text{ or } b = \frac{k}{(k^2 - 1)}.$$

□

Corollary 3.2. *If $-1 < Re(z) < 1$, and $n = k + 1$ then $p_b(z)$ has exactly two singular zeros.*

Proof. From the theorem, we have derived that for the specific values of $Re(z) = 1$ or $Re(z) = -1$, the polynomial $p_b(z)$ has exactly one singular zero. To prove the corollary, we need to consider the case when $-1 < Re(z) < 1$. Recall that:

$$Re(z) = \frac{k}{b(k^2 - 1)}.$$

For $-1 < Re(z) < 1$, we analyze the behavior of the equation:

$$-1 < \frac{k}{b(k^2 - 1)}.$$

This inequality suggests that there are two distinct values of z on the unit circle that satisfy

this condition. Since $|z| = 1$, the values of $Re(z)$ represent points on the unit circle in the complex plane. Given that $Re(z)$ takes values strictly between -1 and 1, it indicates the presence of two distinct points (or singular zeros) on the unit circle where the real part of z lies within this interval. These two points correspond to the two singular zeros of the polynomial $p_b(z)$. Therefore, $p_b(z)$ has exactly two singular zeros when $-1 < Re(z) < 1$. \square

The next theorem is more general than the above and can be applied to any circle in the complex plane, not just the unit circle. This is particularly helpful in fields like control theory, where the stability of systems, which may be analyzed using polynomials, could depend on the scale of a system's state space. Modifying (r) , gives insight into how the scale affects the behavior of the polynomial and hence into phenomena that are dependent on the scale.

Theorem 3.3. *Assume the critical curve for the quadrinomial $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ is not the unit circle. Then the quadrinomial $p_b(z)$ will have precisely one singular zero, provided that $n = k + 1$.*

Proof. Suppose z satisfies $|\omega(z)| = 1$ and $p_b(z) = 0$. Then, $\omega(z)$ is given by $\omega(z) = z^{n-k} \frac{z+bk}{bkz+1}$.

$$|\omega(z)| = 1 \Rightarrow |z^{n-k}| \left| \frac{z+bk}{bkz+1} \right| = 1.$$

Implies,

$$|z^{n-k}| = \left| \frac{bkz+1}{z+bk} \right|.$$

Implies,

$$\bar{z}^{n-k} = \beta \frac{bkz+1}{z+bk}$$

where $\beta \in \mathbb{C}$ such that $|\beta| = 1$. This implies,

$$bz^k + \frac{z^{k-1}}{k-1} = -\frac{\bar{z}^n}{n} - \frac{bk}{n-1}\bar{z}^{n-1}$$

$$\Rightarrow z^{k-1} \left(bz + \frac{1}{k-1} \right) = -\bar{z}^{k-1} \left((\bar{z}^{n-k}) \left(\frac{bk}{n-1} + \frac{\bar{z}}{n} \right) \right) = -\bar{z}^{k-1} \left(\beta \frac{bkz+1}{z+bk} \right) \left(\frac{bk}{n-1} + \frac{\bar{z}}{n} \right)$$

Taking the modulus of both side:

$$\left| z^{k-1} \left(bz + \frac{1}{k-1} \right) \right| = \left| -\bar{z}^{k-1} \left(\beta \frac{bkz+1}{z+bk} \right) \left(\frac{bk}{n-1} + \frac{\bar{z}}{n} \right) \right|$$

$$\Rightarrow \left| bz + \frac{1}{k-1} \right| = \left| \frac{bkz+1}{z+bk} \right| \left| \frac{bk}{n-1} + \frac{\bar{z}}{n} \right|. \quad (3.5)$$

Now, suppose $|z| = r$ is critical curve. Then, $|\omega(z)| = r^{n-k} \left| \frac{z+bk}{bkz+1} \right| = 1$.

Thus,

$$r^{n-k} = \left| \frac{bkz+1}{z+bk} \right|. \quad (3.6)$$

Equation 3.5 and 3.6 yields:

$$\left| bz + \frac{1}{k-1} \right| = r^{n-k} \left| \frac{bk}{n-1} + \frac{z}{n} \right|.$$

Squaring both side and simplifying yields:

$$\begin{aligned} b^2|z|^2 + \frac{2b}{k-1}Re(z) + \frac{1}{(k-1)^2} &= r^{2(n-k)} \left[\frac{b^2k^2}{(n-1)^2} + \frac{2bk}{n(n-1)}Re(z) + \frac{|z|^2}{n^2} \right] \\ \Rightarrow \frac{2b}{k-1}Re(z) - \frac{2bk}{n(n-1)}r^{2(n-k)}Re(z) &= r^{2(n-k)} \left[\frac{b^2k^2}{(n-1)^2} + \frac{|z|^2}{n^2} \right] - \frac{1}{(k-1)^2} - b^2|z|^2. \end{aligned}$$

But we have $|z| = r$:

$$\begin{aligned} \frac{2b}{k-1}Re(z) - \frac{2bk}{n(n-1)}r^{2(n-k)}Re(z) &= \left(\frac{bk}{(n-1)}r^{n-k} \right)^2 + \frac{r^2}{n^2}r^{2(n-k)} - \frac{1}{(k-1)^2} - (br)^2 \\ \Rightarrow \left(\frac{n(n-1) - k(k-1)r^{2(n-k)}}{(k-1)n(n-1)} \right) 2bRe(z) &= \left(\frac{bk}{(n-1)}r^{n-k} \right)^2 + \left(\frac{r^{n+1-k}}{n} \right)^2 - \frac{1}{(k-1)^2} - (br)^2 \end{aligned}$$

Simplifying yields:

$$Re(z) = \frac{1}{2b} \left(\frac{(k-1)n(n-1)}{n(n-1) - k(k-1)r^{2(n-k)}} \right) \left[\left(\frac{bk}{n-1}r^{n-k} \right)^2 + \left(\frac{r^{n+1-k}}{n} \right)^2 - \frac{1}{(k-1)^2} - (br)^2 \right]$$

Now, it may not be simple to solve for parameter b explicitly in this equation due to the complexity of the expression. Let us take the particular case for k and n . Consider the specific case $n = k + 1$ to further simplify the expression for $Re(z)$:

$$\begin{aligned} Re(z) &= \frac{1}{2b} \left(\frac{(k-1)(k+1)k}{(k+1)k - k(k-1)r^2} \right) \left(\frac{bk}{k}r \right)^2 + \left(\frac{r^2}{k+1} \right)^2 - \frac{1}{(k-1)^2} - (br)^2 \\ &= \frac{1}{2b} \left(\frac{k(k-1)}{k[k(1-r^2) + (1+r^2)]} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \end{aligned}$$

Simplifying yields:

$$Re(z) = \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right). \quad (3.7)$$

Now, using equation 3.7 and taking the case for the value of $Re(z)$, we can solve for parameter b :

Case 1: Let $Re(z) = r$:

then

$$\begin{aligned} r &= \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \\ 2br &= \left(\frac{(k^2 - 1)}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \\ b &= \frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right). \end{aligned}$$

Case 2: $Re(z) = -r$:

$$\begin{aligned} -r &= \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \\ \Leftrightarrow b &= -\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \end{aligned}$$

Therefore, the solution for parameter b is:

$$b = \pm \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right).$$

Hence, if we take the particular case $n = k + 1$, the general $|z| = r$

we get the parameter

$$b = \pm \frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right).$$

Thus, the quadrinomial $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$ has exactly one singular zero. \square

Assuming $|z| = r$ provides a more general result applicable to any radius. The choice between $|z| = r$ and $|z| = 1$ depends on the objectives of the analysis. To infer general, scalable properties of the behavior of the polynomial, $|z| = r$ is preferable. For insights related to the

unit circle, $|z| = 1$ will be enough. Our desired result then follows as a corollary.

Corollary 3.3. *If*

$$\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) > r$$

or

$$-\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < -r$$

then, p_b has no singular zero.

Proof. From the theorem, we know that $p_b(z)$ has exactly one singular zero if:

$$b = \pm \frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right).$$

We need to show that if:

$$\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) > r$$

or

$$-\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < -r,$$

then, p_b has no singular zeros. Let's consider the two cases separately.

Case 1:

$$\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) > r$$

This implies:

$$b > \frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right).$$

Recall from the theorem that the singular zero z satisfies:

$$Re(z) = \pm \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right)$$

If b is greater than the given bound, the right-hand side of the equation becomes less than r :

$$\left| \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \right| < r$$

This contradicts the condition that $Re(z) = r$ or $Re(z) = -r$. Hence, no such z can exist,

implying that p_b has no singular zeros.

Case 2:

$$-\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < -r.$$

This implies:

$$b < -\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right).$$

Similar to case 1, if b is less than the given bound, the right-hand side of the equation becomes greater than $-r$:

$$\left| -\frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) \right| < r$$

This contradicts the condition that $Re(z) = -r$. Hence, no such z can exist, implying that p_b has no singular zeros. Therefore, if b lies outside these critical bounds, p_b has no singular zeros. \square

Corollary 3.4. *If $-r < -\frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < r$, then p_b has exactly two singular zeros.*

Proof. From the theorem, we know:

$$Re(z) = \pm \frac{1}{2b} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right)$$

We need to show that if:

$$-r < -\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < r$$

then, p_b has exactly two singular zeros.

This inequality implies:

$$-r < -\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < r$$

This means the parameter b falls within the range:

$$-\frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right) < r$$

and

$$r > \frac{1}{2r} \left(\frac{k^2 - 1}{k - kr^2 + 1 + r^2} \right) \left(\frac{r^4}{(k+1)^2} - \frac{1}{(k-1)^2} \right)$$

Therefore, for this range of b , the polynomial $p_b(z)$ satisfies conditions that are critical for $Re(z)$ to lie within $([-r, r])$. This means, we have two distinct values for $Re(z)$ within this range. These values imply the existence of two singular zeros, one for each $Re(z)$. Thus, when b lies within the bounds specified, p_b has exactly two singular zeros. \square

Chapter 4

Summary and Conclusion

In this thesis, we found that the critical curve under complex-valued harmonic quadrinomial $p_b(z) = bz^k + \frac{1}{k-1}z^{k-1} + \frac{1}{n}\bar{z}^n + \frac{kb}{n-1}\bar{z}^{n-1}$, where $n, k \in \mathbb{N}$ with $n > k \geq 2$ and $b \in \mathbb{R}$, was shown that $|\omega(z)| = 1$ if and only if $|z| = 1$, which means that the critical curve of p_b comprises only the unit circle. We also investigated properties of the critical curve for p_b , under the appropriate conditions on the parameters b , k , and n , by using its dilatation function, $\omega(z)$, expressed as a Möbius transformation, $\Phi(z)$. We demonstrated that $|\Phi(z)| = 1$ if and only if $|z| = 1$ and when $0 < kb < 1$, the Möbius transformation $\Phi(z)$ defines an automorphism of the unit disc mapping the unit disc into itself. We also proved that $p_b(z)$ has exactly one singular zero, located on the unit circle for the special case, (where $n = k + 1$,) and determine the behavior of nonsingular zeros of the quadrinomial p_b by considering different cases.

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