COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON COEXISTENCE OF DISTRIBUTIONAL AND RATIONAL SOLUTION TO SOME LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Submitted in partial fulfilment of requirements for
the Degree of Master of Science in Mathematics

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The undersigned here by certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **On coexistence of distributional and rational solution to some linear ordinary differential equations** by Gashawbeza Tadiyos in partial fulfillment of the requirements for the degree of master of Science.

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Notation

C	The space of continuous function
C^k	The space of k times continuously differentiable functions
C^∞	The space of infinitely differentiable functions
N^n	The set of n-tuple of Natural numbers
R^n	The set of n-tuple of Real numbers
$\mathbb{C}$	The set of complex number
$supp$	Support
$L(f)$	Laplace transform of f
$D(\Omega)$	The space of test functions
$S(\Omega)$	Schwartz test functions
$D^{'}(\Omega)$	The space of distributions
$S^{'}(\Omega)$	The space of tempered distribution
$\delta(t)$	Dirac delta function
$\delta^{(n)}(t)$	n^{th} derivative of the dirac delta function
$H(t)$	Heaviside function
$(a)_n$	Pochhammer's symbol
ODE	Ordinary differential equations

Abstract

In this paper, we visit a distributional solution of the differential equation

$$ty^{(n)}(t) + my^{(n-1)}(t) + ty(t) = 0$$

where m and n are any integers with $n \geq 2$ and $t \in \mathbb{R}$ using Laplace transform technique. We find that the types of Laplace transformable solution in the space of right-sided expiration depend on the relationship between m and n. Precisely, we have a distributional solution provided $m \geq n$, and a weak solution otherwise.

Introduction

Kanwal in his book [4] classifies types of solution of the linear homogenous ordinary differential equation

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + ... + a_1(t)x'(t) + a_0(t)x(t) = 0 \qquad (1)$$

where $a_i(t), i = 1, 2, ...n$, are infinitely differentiable function. If a solution is smooth enough so that the differentiation in (1) can be carried out in the usual sense and the resulting equation becomes an identity, then it is a classical solution. If the solution is not smooth enough so that the differentiation in (1) can not be carried out in the usual sense, but it satisfies (1) in the sense of distribution, then it is a weak solution. If the solution is a singular distribution, and it satisfies (1) in the sense of distribution, then it is a distributional solution. All these solution are said to be generalized solution. It is widely known that only classical solution satisfy the normal linear homogeneous ordinary differential equations with infinitely differentiable coefficients. No distributional solution satisfies such an equation; however, it may arise for the equations with singularities in the coefficients. For example,the Dirac delta function, $\delta(t)$, is a solution of the following equations.
(i). The confluent hypergeometric equation

$$tx''(t) + (2 - t)x'(t) - x(t) = 0$$

Applying Laplace transform $L\{x(t)\} = x(s)$ to the equation

$$L\{tx''(t)\} + L\{(2 - t)x'(t)\} - L\{x(t)\} = 0$$

We obtained $x(s) = c$ where c is constant.The Laplace inverse of x(s) is

$$x(t) = \delta(t)$$

is a distributional solution.
(ii) The differential equation

$$tx''(t) + 2x'(t) + tx(t) = 0$$

Applying Laplace transform $L\{x(t)\} = x(s)$ to the equation

$$L\{tx''(t)\} + L\{2x'(t)\} + L\{tx(t)\} = 0$$

We obtained $x(s) = c$ where c is constant.The Laplace inverse of x(s) is

$$x(t) = \delta(t)$$

is a distributional solution.The question concerning to the existence of solutions to differential equations in various spaces of distributions has long been studied. Wiener [2] in 1982 established an existence criterion of finite-order solution written as a finite sum of a Dirac delta function and its derivatives to any linear ordinary differential equations with application to some third-order equation. Wiener and Shah [3] in 1983 provided an overview of contemporary results related to distributional and entirely solution of ordinary differential equations, particularly on the linear equations with polynomial coefficients They claimed that a unified approach is possible to study both types of solutions in certain classes of linear ordinary differential equation via integral transformations.Littlejohn and Kanwal [4] in 1987 presented the distributional solution to the confluent hypergeometric differential equation. The solutions are given in terms of infinite-order solutions expressed as an infinite series of the Dirac delta function and its derivatives.Their incentive idea for studying this infinite series came from the works of Morton and Krall,[9] Krall,[10] and Littlejohn[8]. These collective works indicated that weight distributions for a certain class of orthogonal polynomials are in the form of infinite series of the Dirac delta function and its derivatives, and at the same time satisfy a system of ordinary differential equations. In this paper, our objective is to find the generalized solutions in the space of right-sided expiration of the differential equation

In Kanwal and Littlejohn [4], during the presentation of results on the distributional solution of the hypergeometric differential equation that appear in the form of infinite series of the Dirac delta functions and their derivatives, noted some interesting observations about their coexistence, Wiener and Cooke [7]

The ordinary differential equation

$$t(1-t)x''(t) + (1-3t)x'(t) - x(t) = 0,$$

has the distributional solution,

$$\delta(t-1)$$

and the rational function solution,

$$\frac{1}{1-t}$$

Both solutions exhibit intriguing similarities: $\frac{1}{1-t}$ has a pole of at $t = 1$ and similarly, the distributional solution has a simple pole $t = 1$.

The ordinary differential equation

$$t(1-t)x''(t) + (1-5t)x'(t) - 4x(t) = 0$$

has the distributional solution

$$\delta'(t-1) - \delta''(t-1)$$

its pole at $t = 1$ and order 3 and the rational function solution

$$\frac{1+t}{(1-t)^3}$$

a solution have a pole of order 3 at $t = 1$.
Wiener and Cooke [7] gave a necessary and sufficient condition for the simultaneous existence of solutions to ordinary differential equations with polynomial coefficients in the form of finite order linear combination of the Dirac delta function and its derivatives and the rational function solutions by using the Laplace transform and functional differential equations techniques.

Chapter 1

Preliminaries

If f is a distribution, then its value at the test function φ is denoted by $< f, \varphi >$. We sometimes denote this by $\int_{-\infty}^{\infty} f(t)\varphi(t)dt$. The associated derivative to the tempered distribution (general distribution) f is defined by

$$< f', \varphi >= \int_{-\infty}^{\infty} f'(t)\varphi(t)dt = -\int_{-\infty}^{\infty} f(t)\varphi'(t)dt = - < f, \varphi' >$$

and higher order k-th derivatives are defined by

$$< f^{(k)}, \varphi >= \int_{-\infty}^{\infty} f^{(k)}(t)\varphi(t)dt = (-1)^{(k)} \int_{-\infty}^{\infty} f(t)\varphi^{(k)}(t)dt = (-1)^{k} < f, \varphi^{(k)} > .$$

We also in this paper to find coexistence of distributional and rational solution of first order differential equation, second order differential equation and n^{th} order differential equation with polynomial coefficients by using laplace transform and its inverse.

1.1 DISTRIBUTION

1.1.1 Test Functions

Definition 1.1.1. *A test function $\varphi : \Omega \to \mathbb{C}$ is a smooth function with compact support. We shall let $D(\Omega)$ denote the space of test functions.Let $\Omega \subset \mathbb{R}^n$ be an open set. Smooth means that φ is continuous and infinitely differentiable on Ω, so that all its derivatives $\varphi', \varphi'', \varphi'''$ ....are continuous. One often says that φ is class of $C^{\infty}(\Omega)$. A function φ has compact support if there is a real number R such that $\varphi(t) = 0$ for all $|t| > R$*
More precisely, if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function, we can define its support supp(φ) to be the closure of the set of points on which φ does not vanish:

$$supp(\varphi) = \overline{\{t : t \in \Omega : \quad \varphi(t) \neq 0\}}.$$

Alternatively, supp(φ) is the complement of the largest open set on which φ. In other words, a function will always vanish outside its support, but may also vanish at some points at the boundary of its support.

A useful class of test functions are the so-called bump functions. To construct them, consider first the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} exp(\frac{-1}{t}), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

This function is clearly smooth everywhere except possibly at $t = 0$; but L'Hôpital's rule removes that doubt: h is everywhere smooth. Clearly, $supp(h) = [0, \infty)$.
Define the function

$$\varphi(t) = h(t)h(1-t)$$

It is clearly smooth, being the product of two smooth functions. Its support is now $[0, 1]$, so that it is a test function. Notice that $\varphi(t) \geq 0$. If you plot it, you find that it is a small bump between 0 and 1.

1.1.2 Properties of Test Functions

1. Differentiation:
   The operation of differentiation $\varphi^{(k)}(t)$ is continuous from $D(\Omega)$ to $D(\Omega)$ if $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ in $D(\Omega)$, then $\varphi_n^{(k)}(t) \to \varphi^{(k)}(t)$.

2. Multiplication by a function $a(t) \in C^\infty(R^n)$:
   Let $\varphi(t) \in D(R^n)$ and $a(t)\varphi(t)$, are continuous from $D(R^n)$ into $D(R^n)$.

3. $D(\Omega)$ is a real vector space; so that if $\varphi_1, \varphi_2 \in D(\Omega)$ and $c_1, c_2 \in \mathbb{R}$,then $c_1\varphi_1 + c_2\varphi_2 \in D(\Omega)$.

4. If f is smooth and $\varphi \in D(\Omega)$, then $f\varphi \in D(\Omega)$.

5. If $\varphi \in D(\Omega)$, then $\varphi' \in D(\Omega)$. Hence all the derivatives of a test function are test functions.

We are only considering real-valued functions of a real variable, but mutatis mutandis everything we say also holds for complex-valued functions of a real variable.

Example 1.1.1.

$$f(t) = \begin{cases} exp\frac{1}{(1-|t|)^2}, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

We show that f is a test function over R.

$$supp(f) = \overline{\{t : f(t) \neq 0\}} = [-1, 1]$$

since $[-1, 1]$ is closed and bounded.Thus f has a compact support in $[-1, 1]$ and $f(t)$ is infinitely continuity differentiable in this region.Hence f is a test function in $[-1, 1]$

Definition 1.1.2. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called absolutely integrable if*

$$\int_{\mathbb{R}} |f(t)|dt < \infty$$

We say that $f : \mathbb{R} \to \mathbb{R}$ *is locally integrable if*

$$\int_a^b |f(t)|dt < \infty$$

for any finite interval [a,b]. A special class of locally integrable functions are the (piecewise) continuous functions.

Proposition:1 If f is locally integrable and φ is a test function,then the following integral is finite:

$$\int_{\mathbb{R}} f\varphi := \int_{\mathbb{R}} f(t)\varphi(t)dt$$

Proof. Since $\varphi \in D$, it vanishes outside some closed interval [-R,R]. Moreover since it is continuous, it is bounded:$|\varphi(t)| \leq M$ for some M. Therefore

$$\int_{\mathbb{R}} f(t)\varphi(t)dt = \int_{-\mathbb{R}}^{\mathbb{R}} f(t)\varphi(t)dt$$

whence we can estimate

$$|\int_{-\mathbb{R}}^{\mathbb{R}} f(t)\varphi(t)dt| \leq \int_{-\mathbb{R}}^{\mathbb{R}} |f(t)||\varphi(t)|dt \leq M \int_{-\mathbb{R}}^{\mathbb{R}} f(t)dt < \infty$$

where in the last inequality we have used that f is locally integrable. $\square$

Definition 1.1.3. *A sequence of test function φ_m is said to converge to zero (written $\varphi_m \to 0$) if the following two conditions are satisfied:*

1. *All the test functions φ_m vanish outside some common finite interval [-R,R] (in other words, R does not depend on m);and*

2. *For a fixed k,the k-th derivatives $\varphi_m^{(k)}$ converge uniformly to zero.*

We can also talk about test functions converging to a function $\varphi_m \to \varphi$ in the same way. One can show that φ is itself a test function. In other words, the space of test functions is complete.

1.1.3 Distribution in $D^{'}(\Omega)$

Definition 1.1.4. *A distribution is a continuous linear functional on the space $D(\Omega)$ of test functions. The space of distributions is denoted by $D^{'}(\Omega)$. In other words, a distribution T_f is a linear map which, acting on a test function φ gives a complex number denoted by $\langle T_f, \varphi\rangle$ Linearity means that*

$$\langle T_f, c_1\varphi_1 + c_2\varphi_2\rangle = c_1\langle T_f, \varphi_1\rangle + c_2\langle T_f, \varphi_2\rangle$$

For all $\varphi_1, \varphi_2 \in D(\Omega)$ and $c_1, c_2 \in \mathbb{C}$ Continuity simply means that if a sequence φ_m of test functions converges to zero, then so does the sequence of complex numbers $\langle T_f, \varphi_m\rangle$

$$\varphi_m \to 0 \Rightarrow \langle T, \varphi_m\rangle \to 0$$

i.e*:A function $T_f : D(\Omega) \to \mathbb{C}$*

Proposition 2: Every locally integrable function f defines a distribution T_f defined by

$$\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(t)\varphi(t)dt$$

for all $\varphi \in D(\Omega)$

Proof. The map T_f is clearly linear. To show that it is also continuous, suppose that $\varphi_m \to 0$. By the first convergence property in Definition 1.1.3 there exist finite R such that all φ_m vanish outside of [-R,R]. Let C be the result of integrating $|f(t)|$ on [-R,R], and let M_m be such that $|\varphi_m(t)| \leq M_m$ then

$$|\langle T_f, \varphi_m\rangle| = |\int_{-R}^{R} f(t)\varphi_m(t)dt| \leq \int_{-R}^{R} |f(t)||\varphi_m(t)|dt \leq CM_m$$

But now notice that $\varphi_m \to 0 \Rightarrow M_m \to 0 \Rightarrow \langle T_f, \varphi_m\rangle \to 0$ □

A distribution of the form T_f for some locally integrable function is called regular. A famous example of a regular distribution is the one associated to the Heaviside step function.Let H be the following function:

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

It is clearly locally integrable, although it is not continuous. It gives rise to a regular distribution T_H defined by

$$\langle T_H, \varphi \rangle = \int_{-\infty}^{\infty} H(t)\varphi(t)dt = \int_{0}^{\infty} \varphi(t)dt$$

The value at $t = 0$ of the Heaviside step function is a matter of convention. It is an intrinsic ambiguity in its definition.Notice however that the regular distribution T_H does not depend on this choice. This simply reiterates the fact that we should think of the Heaviside step function not as a function, but as a distribution.If all distributions were regular, there would be little point to this theory. In fact, the raison of the theory of distributions is to provide a rigorous framework for the following distribution.

Definition 1.1.5. *The Dirac δ-distribution is the distribution δdefined by*

$$\langle \delta, \varphi \rangle = \varphi(0) \qquad \forall \varphi \in D(\Omega)$$

This distribution cannot be regular: indeed,if there were a function $\delta(t)$ such that $\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0)$ it would have to satisfy that $\delta(t) = 0$ for all $t \neq 0$ but then such a function could not possibly have a nonzero integral with any test function. Nevertheless it is not uncommon to refer to this distribution as the Dirac δ function. Distributions which are not regular are called singular. Distributions obey properties which are analogous to those obeyed by the test functions.

1.1.4 Properties of distribution in $D'(\Omega)$

1. $D'(\Omega)$ is a real vector space; indeed if $T_{f_1}, T_{f_2} \in D'(\Omega)$ and $c_1, c_2 \in \mathbb{C}$ then $c_1 T_{f_1} + c_2 T_{f_2} \in D'(\Omega)$ defined by

$$\langle c_1 T_{f_1} + c_2 T_{f_2}, \varphi \rangle = c_1 \langle T_{f_1}, \varphi \rangle + c_2 \langle T_{f_2}, \varphi \rangle \qquad \forall \varphi \in D(\Omega)$$

is a distribution

2. If g is smooth and $T_f \in D'(\Omega)$,Then gT_f defined by

$$\langle gT_f, \varphi \rangle = \langle T_f, g\varphi \rangle \qquad \forall \varphi \in D(\Omega)$$

   is a distribution.

3. If $T_f \in D'(\Omega)$, then T_f' defined by

$$\langle T_f', \varphi \rangle = -\langle T_f, \varphi' \rangle \qquad \forall \varphi \in D(\Omega)$$

   is a distribution.

4. Let g be a smooth function and T_f be a distribution.Then gT_f is a distribution,

$$(gT_f)' = gT_f' + g'T_f$$

   φ is any basic function,

$$\begin{aligned}
\langle (gT_f)', \varphi \rangle &= -\langle gT_f, \varphi' \rangle \\
&= -\langle T_f, g\varphi' \rangle \\
&= -\langle T_f, (g\varphi)' - \varphi g' \rangle \\
&= -\langle T_f, (g\varphi)' \rangle + \langle T_f, \varphi g' \rangle \\
&= \langle T_f', g\varphi \rangle + \langle T_f g', \varphi \rangle \\
&= \langle gT_f', \varphi \rangle + \langle T_f g', \varphi \rangle \\
&= \langle gT_f' + T_f g', \varphi \rangle \\
&\Longrightarrow (gT_f)' = gT_f' + T_f g'
\end{aligned}$$

Definition 1.1.6. *We say that a sequence of distributions T_{f_n} converges to to a distribution T_f, written $T_{f_n} \to T_f$, if*

$$\langle T_{f_n}, \varphi \rangle = \langle T_f, \varphi \rangle \qquad \forall \varphi \in D(\Omega)$$

We will often say that T_n converges to T in the distributional sense to mean this type of convergence. It is in this sense that singular distributions can be approximated by regular distributions. In fact, shows how to construct sequences of regular distributions which converge to δ.

Test Functions in $S(\Omega)$

Definition 1.1.7. *The set of test functions $S(\Omega)$ all functions the class $C^\infty(\Omega)$ which decreases as $|x| \to \infty$, together with all their derivatives, faster than any power of $|x|^{-1}$.*

Definition 1.1.8. *The Schwartz space $S(\Omega)$ of rapidly decreasing function is the set of infinitely differentiable functions $f : \Omega \to \mathbb{C}$ such that $\forall\beta, \beta \in N_0^n$,*

$$sup \mid x^{\alpha}\frac{\partial^{\beta}}{\partial x^{\beta}}\varphi(x) \mid < +\infty, \forall x \in \Omega.$$

1.1.5 Distribution in $S^{'}(\Omega)$

Definition 1.1.9. *Each linear functional over the space of test functions $S(\Omega)$ is known as a distribution of rapidly decreasing functions (tempered distribution). That is*

$$S^{'}(\Omega) = \{T_f : S(\Omega) \to \mathbb{C}\}$$

. It is the set of all distribution of rapidly decreasing functions. ***i.e****: denoted by $S'(\Omega)$.*

Definition 1.1.10. *The sequence of distribution $T_{f_1}, T_{f_2}, T_{f_3}, ..., T_{f_n}$, belonging to $S'(\Omega)$ converges to the distribution $T_f \in S'(\Omega)$, $T_{f_n} \to T_f$ as $n \to \infty$ $S'(\Omega)$, if $\forall\varphi \in S'(\Omega)$,*

$$\langle T_{f_n}, \varphi\rangle \longrightarrow \langle T_f, \varphi\rangle, as\ n \to \infty.$$

if $T_f \in S'(\Omega)$, then $T_f \in D^{'}(\Omega)$. since $D(\Omega) \subseteq S(\Omega)$ and the convergence in $S(\Omega)$ follows from the convergence in $D(\Omega)$.Further, if $T_{f_n} \longrightarrow T_f$ as $n \longrightarrow \infty$ in $S^{'}(\Omega)$ then $\langle T_{f_n}, \varphi\rangle \longrightarrow \langle T_f, \varphi\rangle$ as $n \to \infty, \forall\varphi \in D(\Omega) \subseteq S(\Omega)$ and consequently,$T_{f_n} \to T_f$ as $n \to \infty$ in $D^{'}(\Omega)$

Properties of Functions in $S^{'}(\Omega)$

1. Differentiation:
   If $T_f \in S^{'}(\Omega)$, then each derivative $T_f^{(k)} \in S^{'}(\Omega)$. since the operation of differentiation $T_f^{(k)}$ is continuous from $S^{'}(\Omega)$ into $S^{'}(\Omega)$ using integration by parts, the right-hand side of the equation

   $$\langle T_f^{(k)}, \varphi\rangle = (-1)^{|k|}\langle T_f, \varphi^{(k)}\rangle$$

   is a linear continuous functionals over $S^{'}(\Omega)$.

2. Multiplication:
   If $T_f \in S^{'}(\Omega)$ and $a(t) \in C^{\infty}(\Omega)$, then $aT_f \in S^{'}(\Omega)$.
   since the operation of multiplication by the function $a(t)$ belonging to

$C^\infty(\Omega)$ is continuous from $S^{'}(\Omega)$ into $S^{'}(\Omega)$,the right-hand side of the equation

$$\langle aT_f, \varphi\rangle = \langle T_f, a\varphi\rangle$$

is a linear continuous over $S^{'}(\Omega)$

1.1.6 Distributional derivatives and ODEs.

Let f be a locally integrable function. It need not be continuous, and certainly not differentiable. So there is little point in thinking of its derivative f' as a function: it may not even be well defined at points. Nevertheless, we can make sense of f' as a distribution: T_f'. If f is differentiable, then $T_f' = T_{f'}$.On the other hand, if f is not differentiable, then T_f' need not be a regular distribution at all. Let us illustrate this with an important example The derivative H' of the step function is not a function: it would not be defined for $t = 0$. Nevertheless, the distributional derivative T_H' is well defined

$$\langle T_H', \varphi\rangle = -\langle T_H, \varphi'\rangle = -\int_0^\infty \varphi' dt = \varphi(0) \ \forall \varphi \in D(\Omega)$$

where we have used the fact that φ has compact support. Comparing Definition 1.1.5. we see that δ is the distributional derivative of the step function:

$$T_H' = \delta$$

Let us give some examples of distributional derivatives. We write them as if they were functions, but it is important to keep in mind that they are not. Hence, when we write f' what we really mean is the distribution T_f' and so on.

1. Let $f(t) = tH(t)$. Then $f' = H$ and $f'' = \delta$

2. Let $f(t) = |t|$. Then $f'(t) = H(t) - H(-t)$ and $f'' = 2\delta$

3. Let f(t) be k times continuous differentiable (i.e ,, of class C^k). Then

$$f\delta^{(k)}(t) = \sum_{n=0}^{k} \binom{k}{n} (-1)^n f^{(k-n)}(0)\delta^{(n)}(t) \tag{1.1}$$

4. In particular we have that

$$t^m \delta^{(n)} = \begin{cases} 0, & m > n \\ (-1)^m m! \delta, & m = n \\ (-1)^m \dfrac{n!}{(n-m)!} \delta^{(n-m)} & m < n \end{cases} \tag{1.2}$$

Our present interest in distributions is to be able to solve a larger class of differential equations.We will focus solely on linear ODEs.Therefore we start by showing how differential operators act on distributions. Since the derivative of a distribution is a distribution, we can take any finite number of derivatives of a distribution and still get a distribution for any distribution T_f and any test function φ:

Chapter 2

The Delta function and Laplace transform

Laplace transforms are very useful in solving differential equations.They give the solution directly without the necessity of evaluating arbitrary constants separately. Suppose that f is a real or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter.We define the Laplace transform of f as

$$F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}dt = lim_{\top \to \infty} \int_0^\top f(t)e^{-st}dt \tag{2.1}$$

whenever the limit exists (as a finite number).When it does, the integral (2.1) is said to converge. If the limit does not exist, the integral is said to diverge and there is no Laplace transform defined for f.

Properties of the Laplace Transform

1. The Linear Property
   Let f(t) and g(t) be functions whose Laplace transforms exist for c_1 and c_2 any constants

   $$L\{c_1 f(t)+c_2 g(t)\} = \int_0^\infty e^{-st}\{c_1 f(t)+c_2 g(t)\}dt = c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_0^\infty e^{-st} g(t)dt$$

   $$= c_1 L\{f(t)\} + c_2 L\{g(t)\}$$

   This means the the Laplace transform is a linear operator.

2. The Laplace Transform of derivatives
   Suppose that $f(t), f'(t), f''(t), ...f^{(n-1)}(t)$ are continuous on $(0, \infty)$ and

of exponential order, while $f^{(n)}(t)$ is piecewise continuous on $(0,\infty)$ then

$$L(f^{(n)}(t)) = s^n f(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{n-1}(0)$$

3. First Translation Property
   Suppose $L\{f(t)\} = F(s)$ exists for $Re(s) > c$.Then,for any constant c,

$$F(s-c) = L\{e^{ct}f(t)\} \qquad for \;\; Re(s) > c$$

Proof.

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$$

Replacing s by $s-c$,

$$F(s-c) = \int_0^\infty e^{-(s-c)t}f(t)dt = \int_0^\infty e^{-st}[e^{ct}f(t)]dt = L\{e^{ct}f(t)\}$$

□

4. Multiplication by t^n
   Suppose $L\{f(t)\} = f(s)$ exists for $s > a$. Then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[f(s)]$$

2.1 Laplace transform of distributional derivatives

We will now try to find a possible definition for the Laplace transform of a distribution.First we recall that a function f(t) can be considered as a distribution T_f by means of the rule $\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(t)\varphi(t)dt$ for $\varphi \in S(\Omega)$. Because we take $\varphi = e^{-st}$ a function e^{-st} is rapidly decreasing function it has no compact support. If we now take for φ a function e^{-st}, then its follows for causal function f(t) that

$$L(T_f) = \langle T_f, \varphi\rangle = \int_0^\infty f(t)e^{-st}dt = F(s) \tag{2.2}$$

For a function f(t) that have a support only for positive value of t, so that $L(f'(t)) = sf(s)$ continuous in this fashion we find $L(f^{(n)}(t)) = s^n f(s)$, and also $L\{t^k f(t)\} = \dfrac{(-1)^k d^k}{ds^k}F(s)$

2.2 The Laplace Transform of the Delta Function

Since the Laplace transform is given by an integral, it should be easy to compute the following laplace of the delta function.
(i). $L(\delta(t)) = 1$
(ii). $L(\delta(t-a)) = e^{-as}, \qquad a > 0$
(iii). $L(\delta^{(k)}(t)) = s^k$ where k is a positive intiger
As expected, proving these formulas is straightforward as long as we use the precise form of the Laplace transform of distributional. For (i) we have:

$$L(\delta(t)) = \int_0^\infty \delta(t)e^{-st}dt = \langle \delta(t), e^{-st} \rangle = 1$$

As we saw in a previous session, integrating e^{-st} against $\delta(t)$ amounts to evaluating e^{-st} at $t = 0$, and $e^0 = 1$. Similarly for the shifted version (ii), integrating e^{-st} against $\delta(t-a)$ amounts to evaluating e^{-st} at $t = a$:

$$L(\delta(t-a)) = \int_0^\infty \delta(t-a)e^{-st}dt = \langle \delta(t-a), e^{-st} \rangle = e^{-sa}$$

Notice that the two formulas are consistent: if we set $a = 0$ in formula (ii) then we recover formula (i).

2.3 Distributional solution of first order ODE

Theorem 2.3.1. *consider the first order ordinary differential equation with polynomial coefficient of the form*

$$tx'(t) + ax(t) = 0 \tag{2.3}$$

where a is any integer and $t \in \mathbb{R}$ *the laplace transformable solution of (2.3) depend on the value of a as follows*

1. *If* $a > 1$*, then the solutions of (2.3) have distributional solutions;*
2. *If* $a = 1$*, then the solutions of (2.3) have distributional solutions in the form* $C\delta(t)$*, where C is a constant;*
3. *If* $a < 1$*, then the solutions of (2.3) have weak solutions.*

Proof. Applying the Laplace transform $L\{x(t)\} = x(s)$ to (2.3) and using formulas of laplace inverse in section 2.2 we obtained

$$tx'(t) + ax(t) = 0$$

$$\frac{-d}{ds}[sx(s)] + ax(s) = 0$$

$$-sx'(s) - x(s) + ax(s) = 0$$

$$-sx'(s) + (a-1)x(s) = 0$$

with the general solution

$$x(s) = Cs^{a-1} \tag{2.4}$$

Now consider in case(1) $a > 1$. Observe that x(s) is an analytic function over the entire s-plane. By applying the inverse laplace transform directly to (2.4) and using the formulas in the laplace transform of the delta function (i) and (iii) we obtain the distributional solution of (2.3) as

$$x(t) = C\delta^{(a-1)}(t) \tag{2.5}$$

For case(2) $a = 1$.Then (2.4) is reduced to $x(s) = C$, where C is a constant. By applying the inverse Laplace transform to x(s), we obtain the distributional solution of (2.3) as

$$x(t) = C\delta(t) \tag{2.6}$$

For case(3) $a < 1$.Observe that x(s) is not an analytic function over the entire s-plane at $s = 0$. so we obtain the weak solution of ordered $a + 1$.

$$x(t) = \frac{Ct^a H(t)}{a!} \tag{2.7}$$

where H(t) is a heaviside function. □

The classical solution of equation (2.3) is simply give by

$$x(t) = Ct^{-a} \tag{2.8}$$

If $a > 0$, then the solution is rational solution. And if $a < 0$, then the solution is polynomial solution.

Example 2.3.1. *The equation*

$$tx'(t) + 2x(t) = 0$$

the equation has a distributional solution by (2.5) $x(t) = \delta'(t)$. It has also a rational solution $x(t) = t^{-2}$.

Example 2.3.2. *The equation*

$$tx'(t) + x(t) = 0$$

has a distributional solution by (2.6) $x(t) = \delta(t)$ and a rational solution $x(t) = t^{-1}$.

Example 2.3.3. *The equation*

$$tx'(t) - 2x(t) = 0$$

is $x(t) = \dfrac{Ct^2H(t)}{2!}$ *is weak solution given by in equation (2.7) and a classical solution* t^2

Example 2.3.4. *The equation*

$$(1 + t)x'(t) + x(t) = 0$$

The classical solution is $\dfrac{1}{1+t}$ *and the distributional solution is* $x(t) = \delta(t+1)$

To find general solution of the equation

$$t^m x'(t) = 0 \quad , m > 1$$

$$x(t) = c_1 + c_2H(t) + c_3\delta(t) + c_4\delta'(t) + ...c_{m+1}\delta^{(m-2)}(t)$$

is the general solution. For $m = 1$ the solution reduces to $x(t) = c_1 + c_2H(t)$ is a weak solution. For $m \geq 2$ is the distributional solution.

2.4 Second Order Linear Equation

The general second order linear differential equation is

$$A(t)x''(t) + P(t)x'(t) + Q(t)x(t) = R(t) \tag{2.9}$$

where A(t), P(t), Q(t) and R(t) are functions of t alone or constant. If $R(t) = 0$, then equation reduce to

$$A(t)x''(t) + P(t)x'(t) + Q(t)x(t) = 0 \tag{2.10}$$

and it is called homogeneous linear equation. If $R(t) \neq 0$, then the equation is called non homogeneous.

2.5 Distributional and classical solution of second order linear ODE

Consider the second order differential equation

$$tx''(t) + (a-t)x'(t) - bx(t) = 0 \tag{2.11}$$

This is Known as the confluent hypergeometric equation its classical solution of (2.11) is clear that $t = 0$ is the only singular points of this differential equation (2.11). Since

$$p(t) = \lim_{t\to 0} \frac{(a-t)t}{t} = a \quad and \quad q(t) = \lim_{t\to 0} \frac{-bt^2}{t} = 0$$

We conclude that $t = 0$ is a regular singular point of (2.11). Note that $p_0 = a$ and $q_0 = 0$ lead to the indicia equation

$$r(r-1) + ar = 0 \quad \Leftrightarrow \quad r(r-1+a) = 0$$

. This implies that there exist solutions of the form

$$x_1(t) = \sum_{n=0}^{\infty} a_n t^n \qquad and \qquad x_2(t) = t^{1-a} \sum_{n=0}^{\infty} a_n t^n$$

for the confluent hypergeometric differential equation (2.11). These two solutions are linearly independent at least for $a \neq t$. so that we have

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow x'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1} \quad and \quad x''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

Substitution into the differential equation (2.11) leads to

$$\sum_{n=0}^{\infty} n(n-1)t^{n-1}a_n + a\sum_{n=0}^{\infty} n a_n t^{n-1} - \sum_{n=0}^{\infty} n a_n t^n - b\sum_{n=0}^{\infty} a_n t^n = 0$$

hence

$$\sum_{n=0}^{\infty} [\{n(n+1) + a(n+1)\}a_{n+1} - \{n+b\}a_n]t^n = 0$$

This leads to the recurrence relation

$$(n+a)(n+1)a_{n+1} = (b+n)a_n \qquad n = 0, 1, 2, 3...$$

for the coefficients $\{a_n\}_{n=o}^{\infty}$. The solution can be written as

$$a_{n+1} = \frac{(b)_n}{(a)_n n!} a_0$$

Where $(a)_n$ is the pochhammer's symbol:

$$(a)_n = a(a+1)(a+2)(a+3)(a+4)(a+5)....(a+n-1), n > 0, (a)_0 = 1$$

hence

$$x(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} \frac{(b)_n}{(a)_n n!} a_0 t^n$$

is the solution Similarly,we obtained

$$x_2(t) = t^{1-a} \sum_{n=0}^{\infty} a_n t^n = a_0 t^{1-a} \sum_{n=0}^{\infty} \frac{(b-a+1)_n}{(2-a)_n n!} t^n$$

Example 2.5.1. *consider the second order differential equation*

$$tx''(t) + (1-t)x'(t) - x(t) = 0$$

The classical solution is given by using the above formula if $a = b = 1$*, then*

$$x(t) = \sum_{0}^{\infty} \frac{(1)_n t^n}{(1)_n n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3}!...$$

which is equal to in Taylor series

$$\sum_{0}^{\infty} \frac{t^n}{n!} = e^t$$

Theorem 2.5.1. *Consider the second order differential equation with polynomial coefficient of the form*

$$tx''(t) + (a-t)x'(t) - bx(t) = 0 \tag{2.12}$$

This is known as confluent hyper geometric differential equation. Where a *and* b *are positive integer and* $t \in \mathbb{R}$ *,the Laplace transformable solution of (2.12) depend on the value of* a *and* b *as follows*
(i) If $a = 2$ *and* $b = 1$*, then the solutions of (2.12) have a distributional solution in the form*

$$x(t) = C\delta(t)$$

where C is a constant

(ii) If $b = 1$ and $a > 2$, then solutions of (2.12) are a distributional solutions of the form

$$x(t) = C\sum_{i=0}^{a-2} \binom{a-2}{i} (-1)^i \delta^{(a-2-i)}(t)$$

(iii) If $b > 1$ and $a > b + 1$, then the solutions of (2.12) are distributional solutions of the form

$$x(t) = C \sum_{i=b-1}^{a-2} \binom{a-b-1}{i-b+1} (-1)^i \delta^{(i)}(t)$$

(iv) If $a < b + 1$, then the solution of (2.12) are a distributional solutions of the form

$$x(t) = (-1)^{b+1-a}[\sum_{n=b-1}^{\infty} \frac{(b+1-a)_{(n-b+1)}}{(n-b+1)!}\delta^n(t)]$$

Proof. applying the Laplace transform $L(x(t)) = x(s)$ to (2.12) then

$$L(tx'') + L((a-t)x') - L(bx) = 0$$

$$-[\frac{d}{ds}s^2x(s)] + asx(s) + [\frac{d}{ds}sx(s)] - bx(s) = 0$$

$$-s(s-1)x'(s) + [(a-2)s + (1-b)]x(s) = 0$$

with the general solution

$$x(s) = C(s-1)^{(a-b-1)}s^{b-1} \tag{2.13}$$

where C is a constant of integration.

Now, we consider for case(i) if $a = 2$ and $b = 1$, then (2.13) reduces to $x(s) = C$ where C is constant. By applying the inverse Laplace transform to x(s), we obtain the distributional solution of (2.12) as

$$x(t) = C\delta(t) \tag{2.14}$$

For case(ii) if $b = 1$ and $a > 2$, then (2.13) reduces to $x(s) = C(s-1)^{a-2}$ where C is constant. Then the right side of the expression by using binomial formula we obtained

$$x(s) = C[s^{a-2} - (a-2)s^{a-3} + \frac{1}{2!}(a-2)(a-3)s^{a-4} + ... + (-1)^{a-2}]$$

$$x(s) = C\sum_{i=0}^{a-2}(-1)^i\binom{a-2}{i}s^{(a-2-i)}$$

Observe that x(s) is an analytic function over the entire s-plane. By applying the Laplace transform directly to obtain distributional solution to (2.13)

$$x(t) = C\sum_{i=0}^{a-2}(-1)^i\binom{a-2}{i}\delta^{(a-2-i)}(t) \tag{2.15}$$

For case(iii) if $a > b+1$ and $b > 1$, then (2.13) reduce to $x(s) = C(s-1)^{(a-b-1)}s^{b-1}$ then by using binomial formula. let $k = a-b-1$ is positive integer

$$x(s) = C(s-1)^k s^{b-1}$$

then the right side can be expressed as

$$x(s) = Cs^{b-1}[s^k - ks^{k-1} + \frac{1}{2!}k(k-1)s^{k-2} - \frac{1}{3!}k(k-1)(k-2)s^{k-3} + ... + (-1)^k]$$

and multiplying by s^{b-1}.We obtained

$$x(s) = C[s^{k+b-1} - ks^{k+b-2} + \frac{1}{2!}k(k-1)s^{k+b-3} - \frac{1}{3!}k(k-1)(k-2)s^{k+b-4} + ... + (-1)^k s^{b-1}]$$

Observe that x(s) is an analytic function over the entire s-plane.By applying the inverse laplace transform directly to obtain distributional solution to (2.13)

$$x(s) = C[\delta^{(k+b-1)}(t) - k\delta^{(k+b-2)}(t) + \frac{1}{2!}k(k-1)\delta^{(k+b-3)}(t) + ... + (-1)^k\delta^{b-1}(t)]$$

which is equal to

$$x(t) = \sum_{i=b-1}^{a-2}(-1)^{i-b+1}\binom{a-b-1}{i-b+1}\delta^{(i)}(t) \tag{2.16}$$

For case(iv) if $a < b+1$ then $a-b-1 < 0$ and the exponent of $(s-1)$ in equation (2.13) became negative by using negative binomial exponents of (2.13) we obtained

$$x(s) = s^{b-1}(s-1)^{a-b-1}$$

let $-k = a-b-1$

$$x(s) = s^{b-1}(s-1)^{-k}$$

$$x(s) = s^{b-1}\frac{(-1)^{b+1-a}}{(1-s)^k}$$

$$x(s) = s^{b-1}(-1)^{b+1-a}(\sum_{n=0}^{\infty} \frac{(b+1-a)_n}{n!} s^n)$$

$$x(s) = (-1)^{b+1-a}(\sum_{n=b-1}^{\infty} \frac{(b+1-a)_{(n-b+1)}}{(n-b+1)!} s^n)$$

then by taking the inverse laplace we obtained the distributional solution is

$$x(t) = (-1)^{b+1-a}[\sum_{n=b-1}^{\infty} \frac{(b+1-a)_{(n-b+1)}}{(n-b+1)!} \delta^{(n)}(t)]$$

Lastly if $a = b + 1$ then the distributional solution is simply given by

$$x(t) = C\delta^{(b-1)}(t)$$

□

Example 2.5.2. *The equation*

$$tx''(t) + (2-t)x'(t) - x(t) = 0$$

The distributional solution in above case(i) and equation (2.14) is

$$x(t) = C\delta(t)$$

Example 2.5.3. *The equation*

$$tx''(t) + (4-t)x'(t) - x(t) = 0$$

The distributional solution in above case(ii) and equation (2.15) is

$$x(t) = C(\delta''(t) - 2\delta'(t) + \delta(t))$$

Example 2.5.4. *The equation*

$$tx''(t) + (10-t)x'(t) - 5x(t) = 0$$

The distributional solution in above case(iii) and equation (2.16) is

$$x(t) = C(\sum_{i=4}^{8} \binom{4}{i-4} (-1)^i \delta^{(i)}(t))$$

Example 2.5.5. *For $tx''(t) + (3-t)x'(t) - 5x(t) = 0$, its distributional solution is*

$$(-1)^3[\sum_{n=4}^{\infty} \frac{(3)_{n-4}}{(n-4)!} \delta^{(n)}(t)]$$

2.6 The Hypergeometric differential equation

The hypergeometric differential equation is given by

$$t(1-t)x'' + \{c-(a+b+1)t\}x' - abx = 0 \quad a, b, c \in \mathbb{C} \tag{2.17}$$

It is clear that $t = 0$ and $t = 1$ are the only singular points of this differential equation. Since

$$tp(t) = \frac{c-(a+b+1)t}{1-t} \quad \textit{and} \quad t^2q(t) = \frac{-abt}{1-t}$$

are both analytic at $t = 0$, we conclude that $t = 0$, is a regular singular point of (2.17). Note that $p_0 = c$ and $q_0 = 0$ which leads to the indicia equation

$$r(r-1) + cr = 0 \quad \Leftrightarrow \quad r(r-1+c) = 0$$

This implies that there exists solutions of the form

$$x_1(t) = \sum_{n=0}^{\infty} a_n t^n \;\; \textit{and} \;\; x_2(t) = t^{1-c}\sum_{n=0}^{\infty} a_n t^n$$

for the hypergeometric differential equation (2.17). These two solutions are linearly independent at least for $c \notin t$. so that we have

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \quad \Rightarrow \quad x'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1} \;\; \textit{and} \;\; x''(t) = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}$$

Substitution into the differential equation (2.17) leads to

$$\sum_{n=0}^{\infty} n(n-1)t^{n-1}a_n - \sum_{n=0}^{\infty} n(n-1)a_n t^n + c\sum_{n=0}^{\infty} n a_n t^{n-1} - (a+b+1)\sum_{n=0}^{\infty} n a_n t^n - ab\sum_{n=0}^{\infty} a_n t^n = 0$$

hence

$$\sum_{n=0}^{\infty} [\{n(n+1) + c(n+1)\}a_{n+1} - \{n(n-1) + (a+b+1)n + ab\}a_n]t^n = 0$$

This leads to the recurrence relation

$$(n+c)(n+1)a_{n+1} = (a+n)(b+n)a_n \qquad n = 0, 1, 2, 3...$$

for the coefficients $\{a_n\}_{n=o}^{\infty}$ The solution can be written as

$$a_n = \frac{(a)_n(b)_n}{(c)_n n!}a_0$$

hence

$$x(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} a_0 t^n$$

is the solution Similarly,we obtained

$$x(t) = t^{1-c} \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} b_n t^{n+1-c} = b_0 t^{1-c} \sum_{n=0}^{\infty} \frac{(a+1-c)_n (b+1-c)_n}{(2-c)_n n!} t^n$$

let us see some examples

$$(1): \qquad t(1-t)x''(t) + [1-3t]x'(t) - x(t) = 0.$$

from the equation we have $a = b = c = 1$,then using the formula

$$x(t) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n t^n}{(1)_n n!} = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$$

is the classical solution and we observe the intriguing similarities.The function $\frac{1}{1-t}$ has a simple pole at $t = 1$ the distributional solution is also given by

$$x(t) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x) = \delta(t-1)$$

is distributional solution $\delta(t-1)$ with a simple pole at $t = 1$ and $t = 0$ order

$$(2): \qquad t(1-t)x''(t) + [1-5t]x'(t) - 4x(t) = 0.$$

has the distributional solution

$$x(t) = \delta'(t-1) - \delta''(t-1)$$

and the rational function solution

$$x(t) = \frac{1+t}{(1-t)^3}$$

each of the solutions have a pole of order 3 at $t = 1$.

Chapter 3

Distributional solution of certain n^{th} order differential equations with polynomial coefficients

In this section, we propose the distributional solutions of the differential equation

$$tx^{(n)}(t) + mx^{(n-1)}(t) + tx(t) = 0$$

where m and n are any integers with $n \geq 2$ and and $t \in \mathbb{R}$. using Laplace transform technique. We find that the types of Laplace transformable solutions in the space of right-sided distributions depend on the relationship between m and n. Precisely, we have a distributional solution provided $m \geq n$, and a weak solution otherwise.

Theorem 3.0.1. *Consider the n^{th}-order differential equations of the form*

$$tx^{(n)}(t) + mx^{(n-1)}(t) + tx(t) = 0 \tag{3.1}$$

where m and n are any integers with $n \geq 2$ and $t \in \mathbb{R}$ The types of Laplace transformable solutions of (3.1) depend on the values of m and n, as follows:

1. *If $m > n$, then the solutions of (3.1) are distributional solutions;*
2. *If $m = n$, then the solutions of (3.1) are distributional solutions of the form $C\delta(t)$, where C is a constant;*
3. *If $m < n$,then the solutions of (3.1) are weak solutions.*

Proof. Applying the Laplace transform $L\{x(t)\} = x(s)$ to (3.1) and using formulas we obtained

$$-[\frac{d}{ds}s^n x(s)] + ms^{n-1}x(s) - x'(s) = 0$$

that is,

$$s^n x'(s) + ns^{n-1}x(s) - ms^{n-1}x(s) + x'(s) = 0$$

with the general solution

$$x(s) = C(s^n + 1)^{\frac{m-n}{n}} \tag{3.2}$$

where C is a constant of integration. for $s \in \mathbb{C}$ such that the right-hand side of (3.2) can be expressed as

$$x(s) = c[s^{m-n} + (\frac{m-n}{n})s^{m-2n} + \frac{1}{2!}(\frac{m-n}{n})(\frac{m-2n}{n})s^{m-3n} + ...] \tag{3.3}$$

Now, we consider the case(1) $m > n$. First, we suppose that $m - n \geq n$ and $m - n = kn$ where $k = 1, 2, 3, ...$ then (3.3) reduces to

$$x(s) = c[s^{kn} + ks^{(k-1)n} + \frac{k(k-1)}{2!}s^{(k-2)n} + ... + ks^n + 1] \tag{3.4}$$

Observe that x(s) is an analytic function over the entire s-plane. By applying the inverse Laplace transform directly to (3.4) and using the formulas in the Laplace transform of the delta function (i) and (iii) we obtain the distributional solution of (3.1) as

$$x(t) = c[\delta^{kn}(t) + k\delta^{kn-n}(t) + \frac{k(k-1)}{2!}\delta^{kn-2n}(t) + ...k\delta^n(t) + \delta(t)]$$

or

$$x(t) = c\Sigma_{i=0}^{k}\binom{k}{i}\delta^{(in)}(t) \tag{3.5}$$

where $\delta(t)$ is the Dirac delta function and the superscript n stands for the n^{th} -order derivatives. Next, we suppose that $m - n \geq n$ but $m - n$ is not divisible by n. Let $p = m - n$. Then (3.3) becomes

$$
\begin{aligned}
x(s) = c[s^{(p)} + \frac{p}{n}s^{p-n} + \frac{p}{2!n}(\frac{p}{n}-1)s^{p-2n} + \\
... + \frac{1p}{k!n}(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-k+1)s^{p-kn} \\
+\frac{1}{(k+1)!}\frac{p}{n}(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-k)s^{p-(k+1)n}]
\end{aligned}
\tag{3.6}
$$

where $p-(k+1)n < 0 < p-kn$ for some positive integers $k \geq 1$. By applying the inverse Laplace transform to (3.6) and using formulas in the Laplace transform of the delta function (i) and (iii) we obtain the distributional solution of (3.1)

$$
\begin{aligned}
x(t) = C[\delta^{p}(t) + \frac{p}{n}\delta^{(p-n)}(t) + \frac{p}{2!n}(\frac{p}{n}-1)\delta^{(p-2n)}(t) + ... \\
\frac{p}{2!n}(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-k+1)\delta^{(p-kn)}] \\
+CH(t)[\frac{1}{(k+1)!}(\frac{p}{n})(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-k)\frac{t^{(k+1)n-p-1}}{[(k+1)n-p-1]!} \\
+\frac{1}{(k+2)!}(\frac{p}{n})(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-k-1)\frac{t^{(k+2)n-p-1}}{[(k+2)n-p-1]!} + ...]
\end{aligned}
$$

where H(t) is the Heaviside function. Thus, we obtain the distributional solution of (3.1) as

$$
x(t) = C\sum_{i=1}^{k} a_i\delta^{(p-in)}(t) + CH(t)\sum_{j=1}^{\infty} b_j t^{(k+j)n-p-1} \tag{3.7}
$$

where

$$
a_0 = 1 \quad and \quad a_i = \frac{p}{i!n}(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-i+1) \qquad for \;\; i = 1,2,3...k
$$

and

$$
b_j = \frac{1}{(k+j)![(k+j)n-p-1]!}\frac{p}{n}(\frac{p}{n}-1)(\frac{p}{n}-2)...(\frac{p}{n}-j+1) \qquad for \;\; j = 1,2,3...
$$

Finally, we suppose that $0 < m-n < n$ that is $n < m < 2n$ let $q = m-n$. Then (3.3) becomes

$$
x(s) = C[s^{q} + \frac{q}{n}s^{q-n} + \frac{1q}{2!n}(\frac{q}{n}-1)s^{q-2n} + ...] \tag{3.8}
$$

By applying the inverse Laplace transform to (3.8) we obtain the distributional solution of (3.1) as

$$x(t) = C\delta^{(q)}(t) + CH(t)[\frac{q}{n}\frac{t^{n-q-1}}{(n-q-1)!} + \frac{q}{2!n}(\frac{q}{n}-1)\frac{t^{2n-q-1}}{(2n-q-1)!} + ...]$$

or

$$x(t) = C\delta^{(q)}(t) + CH(t)\sum_{j=1}^{\infty} b_j t^{jn-q-1} \tag{3.9}$$

where

$$b_j = \frac{1}{j!(jn-q-1)!}\frac{q}{n}(\frac{q}{n}-1)(\frac{q}{n}-2)...(\frac{q}{n}-j+1)$$

For case(2) $m = n$ then (3.2) is reduced to $x(s) = C$, where C is a constant. By applying the inverse Laplace transform to x(s), we obtain the distributional solution of (3.1) as

$$x(t) = C\delta(t) \tag{3.10}$$

Finally, for case(3) if $m < n$ and $m - n = -kn$ for some positive integer $k \geq 1$. Then (3.2) is just a rational function $x(s) = \frac{C}{(s^n+1)^k}$. Using a partial fraction expansion we have

$$x(s) = C\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{c_{ij}}{(s-\gamma_i)^j}$$

where γ_i are the finite poles of x(s) and c_{ij} are the coefficients of the expansion. Therefore, the inverse Laplace transform implies that the weak solution is

$$x(t) = CH(t)\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{c_{ij}}{(j-1)!}t^{j-1}e^{\gamma_i t} \tag{3.11}$$

The rest case can be expressed by using generalized binomial series (3.3) as

$$x(s) = C[\frac{1}{s^{n-m}} + (\frac{m-n}{n})\frac{1}{s^{2n-m}} + \frac{1}{2!}(\frac{m-n}{n})(\frac{m-2n}{n})\frac{1}{s^{3n-m}} + ...] \tag{3.12}$$

By applying the inverse Laplace transform to (3.12), we obtain the weak solution of (3.1) as

$$x(t) = CH(t)[\frac{t^{n-m-1}}{(n-m-1)!} + (\frac{m-n}{n})\frac{t^{2n-m-1}}{(2n-m-1)!} +$$

$$\frac{1}{2!}(\frac{m-n}{n})(\frac{m-2n}{n})\frac{t^{3n-m-1}}{(3n-m-1)!}...]$$

or

$$x(t) = CH(t)\sum_{i=1}^{\infty}[\frac{a_i t^{in-m-1}}{(in-m-1)!}] \tag{3.13}$$

where $a_1 = 1$ and $a_i = (\frac{m-n}{n})(\frac{m-2n}{n})....(\frac{m-(i-1)n}{n})$ for $i = 1, 2, 3, 4...$ This completes the proof. □

3.1 Examples on theorem 3.0.1

The results of theorem 3.0.1 are very beneficial to a consideration of the solutions of many well known second-order differential equations.

Example 3.1.1. *For $n = 2$,and $m = -2$. Equation (3.1) becomes*

$$tx''(t) - 2x'(t) + tx(t) = 0$$

Different values of m leads to different types of solutions as described in theorem 3.0.1. For this case have weak solution by (3.13) can be written in short as

$$x(t) = CH(t)[\frac{1}{2}(sin(t) - tcos(t)]$$

For $m = 1$,is just the particular Bessel differential equation

$$tx''(t) + x'(t) + tx(t) = 0$$

Its weak solution, according to (3.13), is $x(t) = CJ_0H(t)$ where

$$J_0(t) = \sum_{v=0}^{\infty}\frac{(-1)^v}{(v!)^2}(\frac{t}{2})^{2v}$$

is the Bessel function of the first kind of order zero. Actually $x = j_o(t)$ satises the Bessel differential equation for all values of t. However, by..... precisely, $J_0(t)$ and $J_0(t)H(t)$ are both solutions of the Bessel differential equation where $J_0(t)$ is not Laplace transformable, but $J_0(t)H(t)$ is. This particular example tells us that not all solutions of an ordinary differential equation are Laplace-transformable. Nevertheless, if the solution x(t) of the transformed equation is a Laplace transform, then its inverse transform x(t) will be a solution of the original equation.For $m = 2$, we have,

$$tx''(t) + 2x'(t) + tx(t) = 0$$

Following (3.10), we obtain the distributional solution

$$x(t) = C\delta(t)$$

Let us verify our claim. By distributional derivatives and ODEs in 4 we find that $t\delta''(t) = -2\delta'(t)$ and $t\delta(t) = 0$ and we have an identity. For $m = 6$,the equation becomes

$$tx''(t) + 6x'(t) + tx(t) = 0$$

Its distributional solution is

$$x(t) = c[\delta(t) + \delta''(t) + \delta^{(4)}(t)] \quad by \ \ (3.5)$$

Example 3.1.2. *For $n = 3$, Equation (3.1) is*

$$tx'''(t) + mx''(t) + tx(t) = 0 \quad where \ \ t \in \mathbb{R} \tag{3.14}$$

for $m = -3$. Equation (3.14) becomes

$$tx'''(t) - 3x''(t) + tx(t) = 0$$

Its weak solution is

$$x(t) = CH(t)\{\frac{1}{9}(2+t)e^{-t} - \frac{1}{9}e^{\frac{t}{2}}[(2+t)cos(\frac{\sqrt{3}}{2}t) - (2-t)\sqrt{3}sin(\frac{\sqrt{3}}{2}t)]\}$$

For $m = 2$. Equation (3.14) becomes

$$tx'''(t) + 2x''(t) + tx(t) = 0$$

Solution formula (3.13) leads to its weak solution

$$x(t) = CH(t)\sum_{i=1}^{\infty}[\frac{a_i t^{3i-3}}{(3i-3)!(i-1)!}]$$

where

$$a_1 = 1 \quad and \ \ a_i = (\frac{-1}{3})(\frac{-4}{3})...(-\frac{(3i-3)-2}{3}) \qquad for \ \ i = 1, 2, 3, ...$$

For $m = 3$, Equation (3.14) becomes

$$tx'''(t) + 3x''(t) + tx(t) = 0$$

It follows from (3.10) that the distributional solution is

$$x(t) = C\delta(t)$$

. For $m = 12$, Equation (3.15) now becomes

$$tx'''(t) + 12x''(t) + tx(t) = 0$$

Its distributional solution derived from (3.5) is

$$x(t) = C[\delta(t) + 3\delta^{(3)}(t) + 3\delta^{(6)}(t) + \delta^{(9)}(t)]$$

Chapter 4

Conclusions

We use the Laplace transform to find the distributional solutions of the first order differential equation the form

$$tx'(t) + ax(t) = 0$$

where $a \in \mathbb{Z}$ and the second order differential equation the form

$$tx''(t) + (a - t)x'(t) - bx(t) = 0$$

and

$$t(1 - t)x''(t) + [c - (a + b + 1)t]x'(t) - abx(t) = 0$$

And also to find the distributional solution of certain n^{th} order differential equation the form

$$tx^{(n)}(t) + mx^{(n-1)}(t) + tx(t) = 0$$

We found that if $m \geq n$, then the Laplace-transformable solution in the space of right-sided distributions solution is a distributional solution. If $m < n$, then the solution is a weak solution. It should be noted here that there are solutions to the considered equation which can not be obtained by Laplace transform as shown in the example. Furthermore, in our study we concern only the solution in the space of right-sided distribution which is the one widely used in application. However, it would be of interest to find the generalized solutions in the space of distributions having unbounded supports.

Chapter 5

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