ARTINIAN RINGS

GRADUATE SEMINAR REPORT

BY

BERHANU SYMENGNE

May __, 1985.

ACKNOWLEDGEMENT

In taking care to remove misprints and to correct all sorts of errors, I am obliged to Dr. Wolter and Dr. Yismaw Alemu, who having gone through the whole paper, have pointed out to me the misprints and errors crept therein. I have incorporated most of their suggestion about the presentation of certain propositions.

I am extremely grateful to Dr. Wolter, being my advisor, whose contribution towards the betterment of this paper is of paramount significance. Moreover, I am deeply thankful to him for his constant inspiration and suggestions had a great influence on this paper.

Finally, it is a pleasure to thank W/t zenebech W. Tsadik, Secretary, Department of Statistics, Addis Ababa University for typing the final draft so carefully.

Berhanu Symengne

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ARTINIAN RINGS

The objective of this paper is to characterize Artinian rings and thereby prove their structure theorem. In fact, we will show that an Artinian ring is necessarily Noetherian and of a special kind.

In the following "ring" always means "commutative ring with unity."

I. DEFINITIONS AND REMARKS

Def. 1

For any ring R,

i) An element $a \in R$ is said to be nilpotent if $a^n = 0$ for some positive integer n.

ii) A (right, left, two-sided) ideal I of R is said to be a nilpotent ideal if $I^n = 0$ for some positive integer n.

iii) Two ideals I, J of R are said to be comaximal (coprime) if $I+J = R$.

Def. 2

Let R be a commutative ring with unity. Then,

i) η is said to be the nilradical of R iff η is the set of all nilpotent elements of R.

ii) $\mathfrak{r}$ is called the Jacobson radical of R iff $\mathfrak{r}$ is the intersection of all maximal ideals of R.

Def. 3

A ring R is said to satisfy

i) the 'descending chain condition', denoted by d.c.c., if, whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq$ --- is a descending infinite sequence of ideals, there always exists a positive integer n such that $I_m = I_n$ for all $m \geq n$.

ii) the 'minimal condition' if every non-empty collection of all ideals of R, partially ordered by inclusion relation, has a minimal element.

REMARK 1

For a ring R, R satisfies the d.c.c. iff it satisfies the minimal condition

Proof: (⇒) suppose R satisfies the d.c.c. for ideals. Let Ω be a non-empty set of ideals of R and let $I_1 \in \Omega$. If I_1 is not a minimal element of Ω, then there exists an ideal I_2 of R such that $I_1 \supset I_2$. Now, if Ω has no minimal element, this process can be repeated indefinitely giving rise to an infinite strictly descending chain

$$I_1 \supset I_2 \supset I_3 \supset \text{---}$$

of ideals of R. Thus, contradicting the hypothesis that R satisfies the d.c.c for ideals. Consequently, Ω has a minimal element.

(⇐) conversly, suppose R satisfies the minimal condition for ideals. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq$ --- be a descending chain of ideals of R. Next, consider the set $\Omega = \{I_i : i = 1,2,\ldots\}$. Then, $\Omega \neq \emptyset$ since $I_1 \in \Omega$. Hence, Ω has a minimal element say I_n, for some positive integer n. Consequently, $I_n \supseteq I_m$ for all $m \geq n$. But $I_m \neq I_n$ implies that $I_m \notin \Omega$ as I_n is a minimal element of Ω. Thus, $I_m = I_n$ for all $m \geq n$; and hence, R satisfies the d.c.c. for ideals.

Def. 4

i) Any ring R satisfying any (hence both) condition/s of Def. 3 is called an Artinian ring.

ii) An ideal I of R is said to be right (left) Artinian if every descending chain of right (left) ideals of R, which are contained in I, terminates.

Remark 2

One can also define an Artinian ring as a ring in which every strictly descending chain of ideals is finite.

Def, 5

i) By a 'chain' of prime ideals of a ring R we mean a finite strictly increasing sequence $P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_n$, where P_i is a prime ideal in R for each $i = 0,1,2,\ldots,n$; the 'length' of the chain is defined to be n.

ii) The 'dimension' of a ring R with unity is defined to be the supremum of the length of all chains of prime ideals in R.

Def. 6

Let R be a commutative ring with unity. Then,

i) An ideal q of R is said to be primary if $q \neq (1)$ and for $x,y \in R$, $x \cdot y \in q$ and $y \notin q$ implies $x^n \in q$ for some positive integer n.

ii) An ideal q of R is said to be P-primary if q is a primary ideal and $P = r(q)$ (radical of q) where $r(q) = \{a \in R\colon a^n \in q$ for a positive integer $n\}$. P is called the associated prime ideal of q.

iii) A prime ideal P is called a 'minimal prime ideal' of an ideal q, if $q \subseteq P$ and there exists no smaller prime ideal with this property.

iv) A minimal prime ideal of the zero ideal in R is known as a minimal prime ideal of the ring R, i.e. a prime ideal P is a minimal prime ideal of its ring R if it does not strictly contain any other non-zero prime ideal of the ring R.

v) The unique associated prime ideal of an ideal q which contains no other non-zero prime ideal of q is called the isolated prime ideal of q; otherwise it is called the imbedded prime ideal of q.

Def 7

Let R be a commutative ring with unity and α be an ideal in R. Then,

i) A representation $\alpha = \bigcap_{i=1}^{n} q_i$ of an ideal α as an intersection of primary ideals q_i is said to be minimal (or reduced, or irredundant) if it satisfies the following conditions:

a) $q_j \not\supseteq \bigcap_{\substack{i=1 \\ i\neq j}}^{n} q_i$ b) $r(q_i) \neq r(q_j)$ where $i \neq j$.

ii) If $\alpha = \bigcap_{i=1}^{n} q_i$ is an irredundant primary representation of α, the ideals q_i are said to be the primary components of α, and q_i is called isolated or imbedded according as its associated prime ideal is isolated or imbedded.

Def. 8

Let R be a commutative Artinian ring with unity. Then,

i) R is called a primary Artin ring if it possesses exactly one prime ideal.

ii) R is called an Artin local ring if it has exactly one maximal ideal.

REMARK 3

In fact, (i) and (ii) in Def. 8 are the same since every prime ideal in an Artinian ring is maximal (see prop. 4 below).

II EXAMPLES

i) Every division ring D is right (left) Artinian as its only right (left) ideals are (0) and D, itself.

ii) Every finite ring is both right and left Artinian.

iii) Z - the ring of integers is not Artinian. In fact, for any positive integer n, the strictly descending chain

$$(n) \supset (2n) \supset (4n) \supset \dots$$

of ideals of Z is infinite.

iv) Let F be a field. Then, F is Artinian by (i) being a commutative division ring with unity. But in F[x], the strictly descending chain

$$(x) \supset (x^2) \supset (x^3) \supset \dots$$

of ideals never terminates. Hence F[x] is <u>not</u> Artinian.

v) Q - the set of rationals, being a field, is Artinian. But Z, a subring of Q, is not Artinian. Thus, a subring of an Artinian ring is not necessarily Artinian.

vi) Let F be any field and $F[x_1, x_2, \ldots]$ the ring of polynomials in infinitely many indeterminates. Then $F[x_1, x_2, \ldots]$ is not Artinian since $(x_1) \supset (x_1^2) \supset \ldots$ is a strictly descending chain of infinite ideals in $F[x_1, x_2, \ldots,]$. But $F[x_1, x_2, \ldots]$ is an integral domain. Then we can form its field of fractions, namely $F(x_1, x_2, \ldots)$. Consequently, being a field, $F(x_1, x_2, \ldots)$ is Artinian. But $F[x_1, x_2, \ldots]$ is a subring of $F(x_1, x_2, \ldots)$. Thus again a subring of all Artinian ring need not be Artinian.

III FACTS ASSUMED

F1

For k-vector spaces (i.e. modules over a field k) the following conditions are equivalent:

i) finite dimension, ii) finite length, iii) the a.c.c., iv) the d.c.c.

F2

Let R be a commutative ring with unity, and $I_1, I_2, \ldots I_n$, J be ideals in R.

Then

i) $J + I_i = R$ for each i, $1 \le i \le n$, implies $J + \prod_{i=1}^{n} I_i = J + \bigcap_{i=1}^{n} I_i = R$;

ii) $I_i + I_j = R$, $1 \le i, j \le n$, implies $\prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$, whenever $i \neq j$

iii) the map $\psi : R \to \prod_{i=1}^{n} (R/I_i)$ defined by

$\psi(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_n(x))$ for $x \in R$, where

$\psi_k : R \to R/I_k$ defined by $\psi_k(x) = x + I_k$ is the cannonical epimorphism,

for each k, $1 \le k \le n$, is

a) an epimorphism iff $I_i + I_j = R$ whenever $i \neq j$, $1 \le i, j \le n$.

b) A monomorphism iff $\bigcap_{i=1}^{n} I_i = (0)$.

F3

Let R be a commutative ring with unity and I, J be ideals in R such that $r(I) + r(J) = R$. Then, $I + J = R$.

F4

In a Noetherian ring R, every ideal has a primary decomposition.

F5

In a Noetherian ring R, the nilradical is nilpotent.

F6 (NAKAYAMA'S LEMMA)

Let R be a commutative ring with unity, I be an ideal in R, and M be a finitely generated R-module. Suppose I is contained in the Jacobson radical $\mathfrak{r}$ of R. Then $IM = M$ implies $M = 0$.

F7

Suppose R is any local ring and $\mathfrak{m}$ is its maximal ideal. Let x_i, $1 \leq i \leq n$, be elements of an R-module M whose images in the vector-space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$ form a basis of this vector-space. Then, x_i, $1 \leq i \leq n$ generate M.

F8

The isolated primary components (i.e. primary components q_i corresponding to minimal primary ideals P_i) of I are uniquely determined by I.

F9

Let R be a commutative ring with unity. Then the following are equivalent:

i) the set of non-units of R form an ideal of R.

ii) R has a unique maximal ideal.

F10

Let R be a commutative ring with unity and $\mathfrak{m} \neq (1)$ an ideal of R such that every $x \in R - \mathfrak{m}$ is a unit in R. Then, R is a local ring and $\mathfrak{m}$ is its unique maximal ideal.

Fli

Let R be a commutative ring with unity and $I_1, I_2, \ldots, I_n$ be ideals in R such that $\bigcap_{i=1}^{n} I_i \subseteq P$ for some prime ideal P in R. Then, $I_i \subseteq P$ for some i, $1 \leq i \leq n$.

REMARK 4

'Ring' means 'commutative ring with unity' in all the following propositions.

IV BASIC PROPERTIES

Prop. 1

Every ideal of an Artinian ring is Artinian.

Proof:

Let R be an Artinian ring and I be any ideal in R. Suppose $I_1 \supset I_2 \supset I_3 \supset$ --- is a strictly descending chain of ideals in I. Then, this is a strictly descending chain of ideals in R; and hence it is finite since R is an Artinian ring by hypothesis. Consequently, I is Artinian.

Prop. 2

A homomorphic image of an Artinian ring is Artinian.

Proof:

Suppose R' is a homomorphic image of an Artinian ring R, i.e. $R' \cong R/I$ for some ideal I of R. Thus, it suffices to show R/I is Artinian. To this end, suppose

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \text{---} \qquad \ldots(*)$$

is a descending chain of ideals of R/I. Then, each J_i is of the form k_i/I where ki is an ideal in R containing I, for each i. Moreover, $J_i \supseteq J_{i+1}$ for each i; and hence $k_i \supseteq k_{i+1}$ for each i. Thus, the descending chain (*) gives rise to a descending chain

$$k_1 \supseteq k_2 \supseteq k_3 \supseteq \text{---}$$

of ideals in R containing I. But then, as R is Artinian by hypothesis, there exists a positive integer n such that $k_m = k_n$, for all $m \geq n$. Consequently, $J_m = J_n$ for all $m \geq n$; and hence R/I is Artinian.

Prop. 3

Let I be an ideal in R. Then, R is Artinian if and only if I and R/I are Artinian.

Proof:

($\Rightarrow$) Suppose R is Artininan and I is an ideal in R. Then, I is Artinian, by Prop. 1, and R/I is Artinian by Prop. 2.

($\Leftarrow$) Conversely, suppose I and R/I are Artinian, and let

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$$

be a descending chain of ideals in R. Then,

$$J_1 \cap I \supseteq J_2 \cap I \supseteq J_3 \cap I \supseteq \cdots$$

is a descending chain of ideals of R contained in I. But I is Artinian by hypothesis. Hence, there exists a positive integer n such that $J_m \cap I = J_n \cap I$ for all $m \geq n$. On the other hand,

$$(J_1 + I)/I \supseteq (J_2 + I)/I \supseteq (J_3 + I)/I \supseteq \cdots$$

is a descending chain of ideals of R contained in R/I. Again, as R/I is Artinian by hypothesis, there exists a positive integer t such that $(J_m + I)/I = (J_t + I)/I$ for all $m \geq t$. Let $r = \max\{n, t\}$. Then, $J_m \cap I = J_r \cap I$ and $(J_m + I)/I = (J_r + I)/I$ for all $m \geq r$.

Claim: $J_m = J_r$ for all $m \geq r$.

To see this,

$$J_m = J_m \cap (J_m + I) = J_m \cap (J_r + I) \quad \text{for all } m \geq r$$

$$= J_r + (J_m \cap I) \quad \text{by the modular law}$$

$$= J_r + (J_r \cap I) \quad \text{since } J_m \cap I = J_r \cap I \text{, for all } m \geq r$$

$= J_r \cap (J_r + I)$ again by the modular law.

$J_m = J_r$ for all $m \geq r$.

Hence, R is Artinian.

Prop. 4

In an Artinian ring, every prime ideal is maximal.

Proof:-

Let R be an Artinian ring and P be a prime ideal in R. Then, $R' = R/P$ is an Artinian integral domain by Prop. 2. Let $0 \neq x \in R'$ and $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \ldots$ be a descending chain of ideals in R'. Then, there exists a positive integer n such that $(x^n) = (x^{n+m})$ for all $m \geq n$. Especially, $(x^n) = (x^{n+1})$; and hence $x^n = x^{n+1}y$ for some $y \in R$.

Thus, $x^n(1-xy) = 0$. Consequently, $xy = 1$ since $x^n \neq 0$ and R' is an integral domain. Hence, x is a unit; and therefore R' is a field. As a result, P is a maximal ideal.

Coro. 1

In an Artinian ring, the nilradical is equal to the Jacobson radical.

Proof:-

Let R be an Artinian ring and η, γ be the nilradical and the Jacobson radical, respectively. But

$$\eta = \bigcap_{\substack{P \text{ is prime} \\ P \neq 0}} P$$

Moreover, every prime ideal in R is maximal, by Prop. 4. Thus, $\eta = \bigcap_{m \text{ is max}} m$; and therefore, Def. 2 (ii) obtains

$$\eta = \bigcap_{m \text{ is max}} m = \gamma$$

Prop. 5

An Artinian ring R, has only a finite number of maximal ideals.

Proof:-

Let Ω be the set of all ideals of R obtained as finite intersections $m_1 \cap m_2 \cap \ldots \cap m_r$, where m_i, $1 \leq i \leq r$ are maximal ideals of R. $\Omega \neq \phi$ since a ring with unity has atleast one maximal ideal. But R is an Artinian ring; and thus satisfies the minimal condition for ideals. Hence, Ω has a minimal element, say $m_1 \cap m_2 \cap \ldots \cap m_n$. suppose m is an arbitrary maximal ideal of R.

Claim: $m = m_i$ for some i, $1 \leq i \leq n$.

But $m \cap m_1 \cap m_2 \cap \ldots \cap m_n \subseteq m_1 \cap m_2 \cap m_3 \cap \ldots \cap m_n$ and $m \cap m_1 \cap m_2 \cap \ldots \cap m_n \in \Omega$ being a finite intersection.

Then, $m \cap m_1 \cap m_2 \cap \ldots \cap m_n = m_1 \cap m_2 \cap \ldots \cap m_n$ by the minimality of $m_1 \cap m_2 \cap \ldots \cap m_n$. Now, it follows that $m_1 \cap m_2 \cap \ldots \cap m_n \subseteq m$ implying that $m_i \subseteq m$ for some i, $1 \leq i \leq n$ by F11 since m is a prime ideal. But m_i is a maximal ideal in R for each i. Then, $m_i = m$ for some i. Hence, $m_1, m_2, \ldots, m_n$ are the only distinct maximal ideals in R.

Prop. 6

In an Artinian ring, the nilradical is nilpotent.

Proof:

Let R be an Artinian ring and η be its nilradical. Since η is also an ideal in R, and R is an Artinian ring the finite descending chain

$$\eta \supseteq \eta^2 \supseteq \eta^3 \supseteq \cdots$$

of ideals in R terminates. Then, there exists a positive integer m such that $\eta^m = \eta^{m+1} = \ldots = I$, say.

Claim: $I = 0$

Suppose $I \neq 0$. Let $\Omega = \{J: J \text{ is an ideal in } R \text{ such that } JI \neq 0\}$

Then, $\Omega \neq \phi$ since $II = I^2 = I \neq 0$; and hence $I \in \Omega$. Consequently, Ω has a minimal element, say A; i.e. A is a minimal ideal of R such that $AI \neq 0$. Now, if $aI \neq 0$ for some $0 \neq a \in A$, then $(a)I \neq 0$, for $(a) \subseteq A$ and $(a) \in \Omega$. But then $(a) = A$ by the minimality of A.

On the other hand,

$$(aI)I = aI^2 = aI \neq 0 \Rightarrow aI \in \Omega.$$

Since $aI \subseteq (a) = A$, $aI = A$, again by the minimality of A. Then, $a = ab$, for some $b \in I$. Consequently,

$$a = ab = ab^2 = \ldots = ab^n = \ldots$$

for some positive integer n. But $b \in I = \eta^m \subseteq \eta$; and hence $b \in \eta$ i.e. b is nilpotent. Thus, $a = ab^n = a.0 = 0$ contradicting the choice of a. Hence, $\eta^m = I = 0$ for some positive integer m i.e. the nil-radical η of R is nilpotent.

Prop. 7

Suppose R is a ring in which the zero ideal is a product $m_1 \circ m_2 \circ m_3 \circ \ldots \circ m_n$ (not necessarily distinct) of maximal ideals of R. Then, R is Artinian if and only if R is Noetherian.

Proof:-

Let m_i, $1 \leq i \leq n$ be the distinct maximal ideals of R. Consider the chain

$$R \supset m_1 \supseteq m_1 \circ m_2 \supseteq m_1 \circ m_2 \circ m_3 \supseteq \ldots \supseteq m_1 \circ m_2 \circ \ldots \circ m_n = 0$$

of ideals in R. Each factor $m_1 \cdot m_2 \cdot \ldots \cdot m_{i-1} / m_1 \cdot m_2 \ldots \cdot m_i$ for $1 \leq i \leq n$ is annihilated by m_i. Then, each factor may be considered as a module over the field R/m_i. Thus, each factor is a vector space over the field R/m_i. By F1, the a-c-c holds for each factor if and only if the d.c.c holds for each factor. But then the d.c.c. holds for each factor if and only if it holds for R by the repeated application of Prop. 3. Thus, the d.c.c. holds for R if and only if the a.c.c holds for R. Consequently, R is Artininan if and only if R is Noetherian.

Prop. 8

A ring R is Artinian if and only if R is Noetherian and dim R = 0

Proof:-

(⟹) Suppose R is an Artininan ring. Then, every prime ideal in R is maximal, by Prop. 4. Consequently, dim R = 0. Furthermore, R has a finite number of distinct maximal ideals, by Prop. 5. Let m_i, $1 \leq i \leq n$, be the distinct maximal ideals in R. Then, for all positive integer k,

$\prod_{i=1}^{n} m_i^k \subseteq (\bigcap_{i=1}^{n} m_i)^k$. But $\bigcap_{i=1}^{n} m_i = \eta$ by coro. 1 where η is the nilradiacal of R. Thus,

$$\prod_{i=1}^{n} m_i^k \subseteq (\bigcap_{i=1}^{n} m_i)^k = \eta^k, \text{ for all integer } k>0$$

But, by Prop. 6, $\eta^k = 0$ for some positive integer k.

Hence, $\prod_{i=1}^{n} m_i^k = 0$. Consequently, R is Noetherian by Prop. 7

(⟸) Conversly, suppose R is Noetherian and dim R = 0. Then, the zero ideal in R, has a primary decomposition by F4. Thus, R has only a finite number of minimal prime ideals. But as the dim R = 0, these minimal prime ideals in R are all maximal. Consequently, the nilradical η in R is given by:

$$\eta = \bigcap_{i=1}^{n} m_i$$

where m_i are the distinct maximal ideals in R. But $\eta^k = 0$ for some positive integer k by F5. Then, we have

$$\prod_{i=1}^{n} m_i^k \subseteq (\bigcap_{i=1}^{n} m_i)^k = \eta^k = 0,$$

for some positive integer k.

Hence, $\prod_{i=1}^{n} m_i^k = 0$ for some k.

Consequently, R is an Artinian ring by Prop. 7.

REMARK 5:

Suppose R is an Artinian local ring with the maximal ideal m. Then, m is the only prime ideal in R, by Prop. 4. Then, by the definition of the nilradical η of R, $\eta = \cap m = m$. Thus, m is the nilradical of R, and every element of m is nilpotent. Consequently, m itself is nilpotent i.e. $m^n = 0$ for some positive integer n. Hence, every element $\gamma \varepsilon R$ is either nilpotent or unit according as $\gamma \varepsilon m$ or $\gamma \notin m$.

For example, consider the ring $R = Z / (P^n)$, where P is a prime number and $n \geq 1$ in Z.

Claim: R is an Artinian local ring.

Let $a \varepsilon Z$ and $(a, P^n) = 1$. Then, there exists $x, y \varepsilon Z$ such that $xa + yP^n = 1$. But $yP^n = 0$ in R. Thus, $xa = 1$ in R; and therefore $a + (P^n)$ is a unit in R. Consequently, (a) = R; and hence (a) is not a maximal ideal in R. Since a non-zero ring with unity has atleast one maximal ideal, R, being a non-zero commutative ring with unity, has a maximal ideal, say $\mathfrak{m}$, whose elements are non-units in R. But then, $\mathfrak{m}$ is a unique maximal ideal of R by F9. Consequently, because every $x \in R - \mathfrak{m}$ is a unit in R, R is a local ring by F10.

Finally, to show R is Artinian, we will show that $R = Z/(P^n)$ has only a finite number of distinct ideals. Since (P^n) is a proper ideal in Z, and the ideals in $Z/(P^n)$ correspond to the ideals in Z which contain (P^n), using the fact that Z is a PID and writing $(P^n) = (x)$ for some $0 \neq x \in Z$ obtains that these distinct ideals correspond to the classes of associated divisors of x. But by the unique factorization property, these classes are finite in number. Hence, so are the ideals in $R = Z/(P^n)$; and therefore R is an Artinian ring. Consequently, combining this fact that R is Artinian with the first part of the claim yields that $R = Z/(P^n)$ is an Artinian local ring. Furthermore, to put it explicitely, the unique maximal ideal of $R = Z/(P^n)$ is $(P)/_{(P^n)}$. (See Remark 6. below).

REMARK 6:

Since Z is a PID and a prime number $P \in Z$ is an irreducible element of Z, $Z/(P^n)$ is a PIR, with $m = (P)/_{(P^n)}$ as its unique prime ideal and $m^n = 0$ for some integer $n > 0$. Moreover, when the 'index of nilpotency'; $n = 1$, $Z/(P^n) = Z/(P)$ will be a field with (P) as a unique maximal ideal. In all other cases, $Z/(P^n)$ has proper zero divisors. At any rate $m = (P)/(P^n)$ is maximal. But then m is a unique maximal ideal in $Z/(P^n)$. Hence, $Z/(P^n)$ is a local ring.

Finally, to obtain the ideals in $Z/(P^n)$ explicitly, we know that the ideals in $Z/(P^n)$ correspond to the ideals in Z containing (P^n). Next, put m = (P), where p is a prime number in Z and denote by n the smallest positive integer such that $P^n = 0$. Then, every non-zero element $x \varepsilon Z$ may obviously be written in the form

$$x = uP^k, \text{ where } 0 \leq k \leq n-1, \; u \notin (P) = m$$

For, either x is a unit, in which case $x \notin (P)$ and so k = 0; or x is non-unit in which case x must be contained in the unique maximal ideal (P) of Z, and if k is the highest power of P which divides x, then $x = uP^k$ where $k \leq n-1$ since $x \neq 0$ and $u \notin (P)$. Furthermore, the integer k in the representation $x = uP^k$ is uniquely determined by x, since

$$uP^k = u'P^{k'} \text{ and } 0 \leq k < k' < n$$

implies that $P^{k'-k} = 0$ contradicting the minimality of n. Similarly, the unit u is uniquely determined mod. (P^{n-k}). Consequently, the only ideals in Z which contain (P^n) are (P^k), where $0 \leq k < n$, and these ideals are all distinct. Further more, these distinct ideals in $Z/(P^n)$ which are finite in number are related as follows:

$$Z/(P^n) \supset (P)/(P^n) \supset (P^2)/(P^n) \supset (P^3)/(P^n) \supset \ldots \supset (P^{n-1})/(P^n) \supset (P^n)/(P^n) = 0$$

Hence, $Z/(P^n)$ is an Artinian ring, too. Thus, $Z/(P^n)$ is an Artinian local ring with the unique maximal ideal $m = (P)/(P^n)$, where p is a prime in Z and $n \geq 1$.

In general, let $n \geq 2$ be in Z, and consider the ring R = Z/(n). Then, (n) is a proper ideal in Z, and the ideals in Z/(n) correspond to the ideals in Z which contain (n). Since Z is a PID, these ideals correspond to the classes of associated divisors of n. But then by the unique factorization property, these classes are finite in number, and hence so are ideals in Z/(n).

Moreover, since Z is a PID, the proper prime ideals in R are those generated by irreducible (prime) elements, and they are all maximal. Then, depending on the associated divisors of n we have the following cases:

Case 1. Suppose n is prime (irreducible) in Z. Then, Z/(n) will be a field; and therefore Artinian with a maximal ideal (n). Moreover, Z/(n) is a primary Artinian ring with a unique prime ideal (n); and hence an Artinian local ring since every prime ideal is maximal.

Case 2. Suppose $n = P_1^{k_1} \cdot P_2^{k_2} \cdot \ldots \cdot P_m^{k_m}$, where $p_i^{k_i}$, $1 \leq i \leq m$ are distinct prime power divisors of n. Then, being a finite ring Z/(n) is Artinian, but may have more than one distinct maximal ideals. Hence, Z/(n) is <u>not</u> an Artinian local ring.

Case 3. Suppose $n = P^m$, p is a prime in Z and $m \geq 1$. Then, $Z/(n) = Z/(P^m)$; and hence by arguments similar to Remark 6, Z/(n) is an Artinian local ring.

Prop. 9

Let R be a local ring, and m be its maximal ideal. Then, one of the following two statements holds:

i) $m^n \neq m^{n+1}$ for all n.

ii) $m^n = 0$ for some n, in which case R is an Artinian local ring.

Proof:-

Suppose $m^n = m^{n+1}$ for some n. Since a local ring is Noetherian, and every ideal of a Noetherian ring is finitely generated, m^n is a finitely generated R-module and m is an ideal in R contained in the Jacobson radical γ of R. Taking I = m and M = m^n in F6 (NAKAYAMA'S Lemma) obtains, $m^n = 0$. Thus, R* is an Artinian local ring by Prop. 7.

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*One can also prove Prop. 9 directly as follows: Suppose $m^n = m^{n+1}$ for some n. Then, taking M= m^n and I = m in NAKAYAMA's Lemma obtains $m^n = 0$. Let P be any prime ideal of R. Then, $m^n = 0 \subseteq P$. Taking radicals and using the fact that m is maximal and the radical of a prime ideal is itself yields m = P. Thus, since P is an arbitrary prime ideal of R, it follows that m is the only prime ideal of R. Consequently, R is an Artinian local ring.

THE STRUCTURE THEOREM

An Artinian ring R is uniquely given (upto isomorphism) as a finite direct product of Artinian local rings.

Proof:-

Let R be an Artinian ring. Then R has a finite number of distinct maximal ideals by Prop. 5. So, let m_i, $1 \le i \le n$ be these distinct maximal ideals of R. But then

$$\prod_{i=1}^{n} m_i^k \subseteq \left(\bigcap_{i=1}^{n} m_i\right)^k = \eta^k \text{ , for all } k > 0 \text{ by Coro. 1}$$

where η is the nilradical of R. Moreover, $\eta^k = 0$, for some positive integer k by Prop. 6 Hence,

$$\prod_{i=1}^{n} m_i^k = 0$$

for some positive integer k. Since $r(m_i^k) = m_i$ and $m_i + m_j = R$ whenever $i \neq j$, for $1 \le i, j \le n$ by F3 we have $m_i^k + m_j^k = R$ whenever $i \neq j$, $1 \le i, j \le n$ consequently,

$$\prod_{i=1}^{n} m_i^k = \bigcap_{i=1}^{n} m_i^k = (0)$$

by F2 (ii) and the cannonical mapping

$$\psi : R \longrightarrow \prod_{i=1}^{n} (R/m_i^k)$$

is an isomorphism by F2 (ii) i.e.

$$R \simeq \prod_{i=1}^{n} (R/m_i^k) .$$

Now, it remains to show that R/m_i^k, for each i, is an Artinian local ring. By Prop. 2, R/m_i^k is an Artinian ring for each i, $1 \le i \le n$. Furthermore, the maximal ideals of R/m_i^k for each i correspond to the maximal ideals of R which contain m_i^k. But m_i is the only maximal ideal of R with this property since the m_i^k's and their corresponding radicals i.e. m_i's are pairwise comaximal. Then, m_i/m_i^k is the only maximal ideal of R/m_i^k for each i, $1 \le i \le n$. Consequently, R/m_i^k for each i, $1 \le i \le n$, is an Artinian local ring.

Conversly, suppose

$$R \cong \prod_{i=1}^{n} R_i ,$$

where R_i is an Artinian local ring for each i, $1 \le i \le n$. Since

$$\psi : R \longrightarrow \prod_{i=1}^{n} R_i$$

is an isomorphism, the natural mapping $\psi_i : R \longrightarrow R_i$, for each i, $1 \le i \le n$, is an epimorphism (i.e. a projection on the i^{th} factor). Let $I_i = \ker(\psi_i)$ for each i. Then, the I_i's are pairwise comaximal and

$$\bigcap_{i=1}^{n} I_i = (0)$$

by F3 (iii). Suppose q_i is the unique prime ideal of R_i, and let $P_i = \psi_i^{-1}(q_i)$ (i.e. the contraction of q_i in R). But then, P_i, for each i, is a prime ideal in R since the contraction of a prime ideal is prime. Then P_i is maximal for each i, $1 \le i \le n$, by Prop. 4. Moreover, q_i is the nilradical of R_i, for each i, and therefore, nilpotent by Prop.6. Hence, I_i is P_i - primary and

$$(0) = \bigcap_{i=1}^{n} I_i$$

is the primary decomposition of the zero ideal in R. Since I_i are pairwise comaximal, P_i are also pairwise comaximal. Thus, the P_i are isolated prime ideals of (0). Then, all the primary components I_i are isolated, and therefore, uniquely determined by R, by F8. Consequently, the rings $R_i \cong R/I_i$, $1 \le i \le n$, are uniquely determined by R.

REMARK 7.

Since every prime ideal in an Artinian ring is maximal, and direct product and direct sum are the same in the finite case, one can also express the structure theorem using the direct sum of Artinian primary rings. This version of the structure theorem is given below, for the sake of completeness, without proof since its proof is completely analogous to the preceeding one except for slight changes.

STRUCTURE THEOREM

An Artinian ring R is uniquely given (upto isomorphism) as a finite direct sum of primary Artinian rings, i.e.

$$R \cong \bigoplus_{i=1}^{n} (R/P_i^k)$$

where P_i, for each i, $1 \le i \le n$ is a prime ideal in R and k is some positive integer.

REMARK 8

Suppose R is a local ring, and let m be its unique maximal ideal. Then, $k = R/m$ is the residual field of R and the R - module m/m^2 is annihilated by m. Hence, m/m^2 has the structure of a k - vector space. Now, if R is Noetherian, then m is finitely generated and the images in m/m^2 of a set of generators of m will span m/m^2 as a vector space over the field k by F7. Thus, the $\dim_k(m/m^2)$ is finite.

Prop. 10

Suppose R is an Artinian local ring. Then, the following are equivalent:

i) Every ideal in R is principal;

ii) The maximal ideal m is principal;

iii) $\dim_k(m/m^2) \leq 1$, $k = R/m$

Proof:-

(i) $\Longrightarrow$ (ii) suppose every ideal in R is principal. Then, the maximal ideal m of R is also principal, obviously.

(ii) $\Longrightarrow$ (iii) suppose the maximal ideal m of R is principal, i.e. m is singly generated. But the images in m/m^2 of a set of generators of m will span m/m^2 as a vector space over the field $k = R/m$ by F7. Hence, $\dim_k(m/m^2) \leq 1$ since m is singly generated.

(iii) $\Longrightarrow$ (i) suppose $\dim_k(m/m^2) \leq 1$. There are two cases to consider.

Case 1**.

Let $\dim_k(m/m^2) = 0$. Then, $m = m^2$; and hence $m = 0$ by F7. Consequently, R is a field; and therefore, every ideal of R is principal.

**In case 1, above instead of F7 one can use the fact that in a ring R with only one maximal ideal m, every idempotent e is either 0 or 1. To show this fact, let $e \neq 0$, $e \neq 1$ be an elt in R. Then $e^2 = e$ if and only if $e(1-e) = 0$. Thus, e and 1-e are zero divisors in R; and therefore, cannot be units in R. Consequently, Re and R(1-e) are proper ideals in R. But proper ideals are always contained in maximal ideals. Hence, $Re \subseteq m$ and $R(1-e) \subseteq m$ since m is the only maximal ideal in R by hypothesis. It follows that $e \in m$ and $(1-e) \in m$. Then, $1 = e+(1-e) \in m$ contradicting the properness of m in R. Consequently, e = 0 or 1.

Now, returning to case 1, suppose $\dim_k(m/m^2) = 0$. Then, $m = m^2$, i.e. m is idempotent. But then, R being a local ring having only one maximal ideal m, m = 0 or m = (1). But $m \neq (1)$ since m is a proper ideal in R by definition. Consequently, m = 0 i.e. the zero-ideal in R; and therefore, R is a field. Thus, every ideal in R is principal since the only ideals in R are the zero ideal and (1) = R.

Case 2

Suppose $\dim_k(m/m^2) = 1$. Then, taking $M = m$ in F7 obtains that m is a principal ideal in R since m/m^2 is singly generated. So, let $m = (x)$, $x \in R$. Suppose I is an ideal of R such that $I \neq (0)$ or $I \neq (1)$. But since R is an Artinian local ring, the nil radical $n = m$. Thus, m is nilpotent by Prop. 6. Then, $m^n = 0$, for some positive integer n. Since I is an ideal of R other than (0) or (1),

$$I \subsetneq m^n \text{ and } I \not\subseteq m^{n+1}$$

Then, there exists $y \in I$ such that

$$y = a\,x^n \text{ and } y \notin (x^{n+1})$$

Thus, $a \notin (x) = m$; and therefore, a is a unit in R. Consequently, $x^n \in I$ since $I \neq (1)$ and $m^n = (x)^n = (x^n) \subsetneq I$. Hence, $I = m^n = (x^n)$, i.e. I is a principal ideal in R.

EXAMPLE

Consider the ring $R = Z/(P^n)$, where p is prime and $n \geq 1$. By Remarks 5 and 6, R is an Artinian local ring with the unique prime (and hence maximal) ideal $m = (P)/(P^n)$. Moreover, m/m^2 is a vector space over the residue field $k = R/m = Z/(P^n)\,/\,(P)/(P^n) \simeq Z/(P)$. Since every ideal in R is principal by Remark 6, the maximal ideal $(P)/(P^n)$ is also principal. Furthermore, $\dim_k(m/m^2) \leq 1$ because $(P)/(P^n)$ is also a unique prime ideal in this PIR - $Z/(P^n)$. Consequently, $Z/(P^n)$ satisfies Prop. 10.

Prop. 11

An Artinian local ring has only a finite number of ideals.

Proof:-

Let R be an Artininan local ring and $\mathfrak{m}$ be its unique maximal ideal. Then, $\mathfrak{m}$ is the only proper prime ideal of R since every proper prime ideal of R is maximal by Prop. 4. But, because $\mathfrak{m} \cdot (R:\mathfrak{m}) = R$, there exist finite families $\{m_i\}$ and $\{m_i'\}$ of elements of $\mathfrak{m}$ and $(R:\mathfrak{m})$, respectively, such that

$$\sum_{i=1}^{n} m_i \cdot m_i' = 1 .$$

Then, since $\mathfrak{m}$ is proper there exists some product $m_j \cdot m_j' \notin \mathfrak{m}$, $1 \le j \le n$; i.e. there exists $m \in \mathfrak{m}$ and $m' \in (R:\mathfrak{m})$ such that $m \cdot m' = 1$ for $\mathfrak{m}$ contains only non-units being a unique maximal ideal of R. Consequently, for each $x \in \mathfrak{m}$, $x = (xm')m \in Rm = (m)$; and hence, $\mathfrak{m} = (m)$ i.e. $\mathfrak{m}$ is a principal ideal. Equivalently, every ideal in R is principal and $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \le 1$ by Prop 10. But, being a local ring, R is Noetherian by def. Then, $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$, for all n, by Prop. 9. Thus, $\mathfrak{m} \neq \mathfrak{m}^2$; and therefore, $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 1$. Then, by Prop. 9 Case 2, for any ideal $\mathfrak{a} \neq (0), (1)$ in R, $\mathfrak{a} = \mathfrak{m}^n$ for some $n>0$. Consequently, $\mathfrak{a} = \mathfrak{m}^n = (x^n)$ for some $n>0$ and $x \in \mathfrak{m}$. But since $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ we have $(x^n) \neq (x^{n+1})$ for for all $n>0$ and $x \in \mathfrak{m}$. Hence, for each $0 \neq r \in R$, $(r) = (x^n)$ for exactly one n. Then, the distinct ideals in R are

$$\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \ldots\ldots \qquad (*)$$

But R, being Artinian by hypothesis, satisfies the d.c.c. Hence, the strictly descending chain (*) is finite. So, the number of distinct ideals in R is finite.

BIBLIOGRAPHY

1. M.F. ATIYAH AND I.G. MACDONALD: Introduction to Commutative Algebra - Addison-Wesley Series in Math., University of Oxford, London, 1969.

2. H. MATSUMARA: Commutative Algebra - 2nd edition - Nagoya University, Nagoya, Japan, 1980.

2. M. NAGATA: Local Rings - Tracts in Math. #13 - Kyoto University, Kyoto, Japan - 1962.

4. D.G. NORTHCOTT: Lessons on Rings, Modules and Multiplicities - Cambridge University Press, London, 1968.

5. O. ZARISKI AND P. SAMUEL: Commutative Algebra - Vol. I and II - Van Nostrand Company, INC., Princeton, 1958.