MAXIMUM MODULUS THEOREM OF HOLOMORPHIC FUNCTION

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Declaration

I declare that this project has been composed by me and that no part of the project has ormed the basis for the award of any Degree, Diploma, associate ship, fellowship or any other imilar title to me.

Binyam Zigta Teferi

PERMISSION

This is to certify that this project is compiled by Binyam Zigta Teferi in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Seid Mohammed (Ph.D)

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Seid Mohammed (Ph.D)

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What shall I render to the Lord for all his benefits towards me? PSALM 116:12

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NOTATIONS AND SYMBOLS

/ is homotopic to zero	$\gamma \sim 0$
$Cis\ \theta$	$cos\ \theta + isin\theta$
$approaches\ to\ a$	$z \to a$
he extended boundary of G	$\partial_\infty G$
ontinuous on the closure of G	$C(G^-)$
he extended complex plane	$\mathbb{C}_\infty$

ABSTRACTS

The purpose of this project is to understand and analyze the Maximum Modulus Theorem ılso called Maximum Principle) which shows that a function which is analytic in a compact ɔmain D assumes its Maximum Modulus on the boundary. In general, if we consider ıbounded domains, the theorem no longer holds. For example, $f(z) = e^z$ is analytic and ıbounded in the right half plane despite the fact that on the boundary $|e^z| = |e^{iy}| = 1$. evertheless, given certain restrictions on the growth of the function, we can conclude that it tains its Maximum Modulus on the boundary. The most natural such condition is that the nction remains bounded throughout D and we will discuss its application.

Next, we will discuss some of the applications and related underlying results of the aximum Modulus Theorem such as the Schwarz's lemma, the Phragmen-Lindelöf Theorem hich extend the Maximum Modulus Principle of complex analysis and the Hadamard's Three ircles Theorem and we will see some examples on these theorems.

TABLE OF CONTENTS

INTRODUCTION

One of the most important results of the Cauchy integral formula is the Maximum Modulus Theorem. It states that if an analytic function has a local maximum at a point, then it must be a constant near that point. In other words if f is a non constant analytic function on a region G, then $|f|$ cannot have a local maximum any where inside G. It can attain a maximum only on the boundary of G. Contextually, the Maximum Modulus Principle in complex analysis does not apply to unbounded regions. That is, an analytic functions on an unbounded region may be bounded on edges but be unbounded in the interior.

The Schwarz lemma is one of the simplest results in all of complex function theory. A direct application of the Maximum Modulus Principle, it is merely a statement about the rate of growth of analytic functions on the unit disk. The Schwarz lemma has a powerful role in complex geometry. Almost any result in geometric theory of analytic functions has the Schwarz lemma lurking in the background.

One useful interpretation of the Schwarz lemma is that an analytic function f from the disk to the disk must take each disk $D(0,r), 0 < r < 1$ in to (not necessarily onto) the image of that disk under the linear functional map

$$z \to \frac{z+\alpha}{1+\bar{\alpha}z} \quad where \quad f(0) = \alpha$$

This image is in fact (in the case $-1 < \alpha < 1$) a standard Euclidean disk with center on the real axis at α and diameter (in case $0 < \alpha < 1$) given by the interval $[\frac{\alpha-r}{1-\alpha r}, \frac{\alpha+r}{1+\alpha r}]$.

In 1908, L.E. Phragmen and E.L. Lindelöf extended the Maximum Modulus Principle to unbounded domains. Their result is referred as "The Phragmen-Lindelöf Theorem" which places a growth restriction on analytic function $f: G \to \mathbb{C}$ as z nears a point on the extended boundary. Nevertheless, the conclusion, like that of the Maximum Modulus Theorem is that f is bounded.

CHAPTER 1

THE MAXIMUM MODULUS THEOREM

1.1 Topology and analysis in complex plane

In this section we will discuss different versions of the Maximum Modulus Theorem and we will see how to apply these versions using examples. Throughout this section we shall use the notations, $B(x;r)$ and $\bar{B}(x;r)$ are called the **open** and **closed** balls, respectively with center x and radius r, if x and $r > 0$ are fixed then we denote

$$B(x;r) = \{y \in X: |y - x| < r\}$$

$$\bar{B}(x;r) = \{y \in X: |y - x| \leq r\}$$

We denote the boundary of Ω by $\partial\Omega$ and defined by $\partial\Omega = \Omega^- \setminus \Omega$ or $\partial\Omega = \Omega^- \cap (X \setminus \Omega)^-$. Furthermore the extended boundary of G is denoted by $\partial_\infty G$, plus optionally the point at infinity if in fact G is unbounded. Furthermore we introduce the extended plane by $\mathbb{C}_\infty = \mathbb{C} \cup \infty$.

A metric space is a pair (X, d) where X is a set and $d: X \times X \to \mathbb{R}$ is called **Metric** or **the distance function** satisfying the following

(1) $d(x, y) \geq 0 \; \forall (x, y) \in X$

(2) $d(x, y) = 0$ *if and only if* $x = y$ (positive definiteness)

(3) $d(x, y) = d(y, x)$ (symmetry)

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

If X and Ω are metric spaces and $f: X \to \Omega$ has the property that $f(U)$ is open in Ω whenever U is open in X, then f is called an open map. If f is a one to one and onto map then we can define the inverse map $f^{-1}: \Omega \to X$ by $f^{-1}(w) = x$ where $f(x) = w$. It follows that f^{-1} is continuous exactly when f is open.

A metric space (X, d) is **connected** if the only subsets of X which are both open and closed are ϕ and X. That is, an open subset of $\mathbb{C}$ is connected if it cannot be expressed as a union of two nonempty disjoint open sets, and a closed subset of $\mathbb{C}$ is connected if it cannot be expressed as a union of two non empty disjoint closed sets. For example single points are connected they cannot be separated in to two nonempty sets. The empty set is connected.

For a metric space (X,d) a subset G of $\mathbb{C}$ is **open** if for each $x\ in\ G$ there is $\epsilon > 0$ such that $B(x;\epsilon) \subset G$. Thus, a set in $\mathbb{C}$ is open if it has no "edge." For example $G = \{z \in \mathbb{C}: a < Rez < b\}$ is open but $\{z: Rez \leq 0\}$ is not open because $B(0;\epsilon)$ is not contained in this set no matter how small we choose ϵ. A subset G of $\mathbb{C}$ is called **closed** if its complement $G^c = \mathbb{C} \setminus G = \{z \in \mathbb{C}: z \notin G\}$ is open. The smallest closed set, obtained by taking the intersection of all closed sets that contain set E, is called the **closure** of E, and is denoted by E^-. Thus E is closed if and only if $E^- = E$. A set which is both bounded and closed is called **Compact.**

If G is an open set in $\mathbb{C}$ and $f: G \to \mathbb{C}$ then f is differentiable at a point z_0 in G if $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ or $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists; the value at this point is denoted by $f'(z_0)$ and is called the derivative of f at z_0 .If f is differentiable at each point of G we say that f is differentiable on G then $f'(z_0)$ defines a function $f': G \to \mathbb{C}$. If f' is continuous then we say that f is **continuously differentiable**. If f' is differentiable then f is twice differentiable; continuing, a differentiable function such that each successive derivative is a gain differentiable is called infinitely differentiable. A function $f: G \to \mathbb{C}$ (G is open) is **analytic** if f is continuously differentiable on G.

The next proposition is taken from John B. Conway [1], page 70 and we will use to prove Theorem 1.1.2.

Proposition 1.1.1

Let $f: G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G (r > 0)$.If $\gamma(t) = a + re^{it}$

$0 \leq t \leq 2\pi, then\ f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw\ \ for\ \ |z-a| < r.$

Theorem 1.1.2 (Maximum Modulus Theorem)

If G is a region and $f: G \to \mathbb{C}$ is an analytic function such that there is a point a in G with $|f(a)| \geq |f(z)|\ for\ all\ \ z\ \ in\ \ G$, then f is constant.

(Another way to state this is that $|f(z)|$ cannot have a maximum in G, unless f is constant).

Proof

Suppose $|f(a)| \geq |f(z)|$ for all $z \in G$ we need to show f is constant. Let $r > 0$ then there exist $B(a,R) \subseteq G$. Let $0 < r < R$ consider $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$ then by proposition 1.1.1 we have $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw$ for $z \in B(a,r)$ this implies $f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a} dw$ write this out in terms of a parameterization $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$.

If we substitute $w - a = re^{it}$ then $dw = ire^{it}dt$

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} i\, re^{it} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt$$

Hence $|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)| = |f(a)|$ since the two ends are equal, we must have equality everywhere since $|f(a + re^{it})| \leq |f(a)|$ $for\ all\ t$ this implies $\int_0^{2\pi} [|f(a)| - |f(a + re^{it})|\,] dt = 0$. Thus the integrand is non-negative it follows that $|f(a)| = |f(a + re^{it})|$ $for\ all\ t \in [0,2\pi]$ since r was arbitrary,

$|f(a)| = |f(z)|$ for $|z - a| < R$, thus $f = C\ on\ B(a,R)$ for some C, $C = f(a)$ thus $g(z) = C$ is analytic in G therefore $f(z) = g(z)\ in\ B(a,R)$ which has a limit point $B(a,R) \subseteq \{z \in G: f(z) = g(z)\}$ has a limit point in G therefore $f(z) = C\ for\ all\ z$.

Remark 1.1.3

According to the Maximum Modulus Theorem, a non constant analytic function on a region cannot assume its maximum modulus; this fact is far from obvious even in the case of polynomials.

Remark 1.1.4

The next section continues about analytic functions that are already seen in Theorem1.1.2 in this section the theorem is again presented with a second proof and other versions are given. In Theorem 1.1.2 we prove by applying the Cauchy Integral formula here we apply the open mapping theorem to prove theorem 1.2.1.

1. 2 The Maximum principle

Let Ω be any subset of $\mathbb{C}$ and suppose α is in the interior of Ω. We can, therefore, choose a positive number ρ such that $B(\alpha,\rho) \subset \Omega$ it readily follows that there is a point ξ in Ω with $|\xi| > |\alpha|$ to state another way, if α is a point in Ω with $|\alpha| \geq |\xi|$ for each ξ in the set Ω then α belongs to $\partial\Omega$.

A non empty open connected subset of $\mathbb{C}$ is called a **region** (some authors use the term "domain" instead of "region").

To prove Theorem1.2.1 we use the **open mapping theorem** which is stated and proved in John B. Conway [1], page 99.

Theorem 1.2.1 Maximum Modulus Theorem (First version)

If f is analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|\, for\ all\ z\ in\ G$ then f must be a constant function.

Proof

Let $\Omega = \mathrm{f(G)}$ and put $\alpha = f(a)$ .If $|\alpha| \geq |\xi|$ for each $\xi\ in\ \Omega$ then α belongs to $\partial\Omega$ and if there is a point $\xi\ in\ \Omega$ with $|\xi| > |\alpha|$ then we can choose a positive number ρ such that $B(\alpha,\rho) \subset \Omega$ therefore α is in $\partial\Omega \cap \Omega$. In particular the set Ω cannot be open (since $\partial\Omega \cap \Omega = \phi$). Hence by open mapping theorem f must be constant.

A subset G of $\mathbb{C}$ is called **bounded** if it is contained in some disk.

Theorem1.2.2 Maximum Modulus Theorem (second version)

Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on G^- which is analytic in $G\ then\ max\{|f(z)|: z \in G^-\} = max\{|f(z)|: z \in \partial G\}$.

(Another way to state this is that a function $f(z)$ analytic on a bounded region G and continuous on G^- assumes its maximum on ∂G).

Proof

Since G is bounded the set G^- is bounded and closed, so on G^- the continuous function $|f|$ is bounded and attains its supermum M at some point of G^-. Now assume that $|f|$ does not attain the value on ∂G. Then $|f(a)| = M$ for some $a \in G$ .If f is not constant then by Maximum Modulus Theorem (first version) $|f|$ attains its maximum on the ∂G. Hence the theorem is proved.

Remark 1.2.3

In Theorem1.2.2 we didn't assume that G is connected as in Theorem1.2.1

Example 1.2.1

Let $G = \left\{z = x + iy : \frac{-\pi}{2} < y < \frac{\pi}{2}\right\}$ and put $f(z) = \exp\,(exp^z)$ .Then f is continuous on G^- and analytic on G. If $z \in \partial G$ then $z = x \pm \frac{\pi i}{2}$

So $|f(z)| = |\exp\,(exp^z)| = |expexp^{\left(x \pm \frac{\pi i}{2}\right)}| = |exp(exp^x exp^{\pm\frac{\pi i}{2}})|$

$$= |exp(exp^x(cos\tfrac{\pi}{2} \pm isin\tfrac{\pi}{2})|$$

$$= |exp(\pm ie^x)|$$

$$= |cose^x \pm i\;sine^x|$$

$$= 1$$

Hence $|f(z)| = |\exp(\pm ie^x)| = 1$

However, as x goes to infinity through the real numbers, $f(x) \longrightarrow \infty$.This does not contradict the Maximum Modulus Theorem because G is not bounded.

In the light of the above example it is impossible to drop the assumption of boundedness of G in Theorem1.2.2; however it can be replaced. The substitute is a growth condition on $|f(z)|$ *as* z approaches infinity. In fact, it is also possible to omit the condition that f is defined and continuous on G^-. To do this, the following definitions are needed.

Consider the sequence $\{a_n\}$ in the set of real numbers, and then the following two sequences need to be considered

$$b_n = sup_{k\geq n}^{a_k} \; and \;\; c_n = inf_{k\geq n}^{a_k}$$

Here $\{b_n\}$ is a decreasing sequence where as $\{c_n\}$ is an increasing sequence and hence their limit exist in the extended real number $[-\infty, \infty]$. Hence the limit $\{b_n\}$ is called limit superior where as the limit $\{c_n\}$ is called the limit inferior and are computed

$$\lim_{n\to\infty} b_n = \overline{\lim}\; b_n = inf_n^{[sup_{k\geq n}^{b_k}]} and \; \lim_{n\to\infty} c_n = \underline{\lim}\; c_n = sup_n^{[inf_{k\geq n}^{c_k}]}$$

But here one needs to understand that the limit superior and the limit inferior are only computed whenever the sequence $\{a_n\}$ is in $\mathbb{C}$ both values exist in the set of extended real numbers we use those definitions most of the time so we should have a good understanding of those definitions.

Definition 1.2.1

If $f: G \longrightarrow \mathbb{R}$ and $a \in G^-$ or $a = \infty$ then the **limit superior** of $f(z)$ *as z approaches a,* denoted by $\lim_{z\to a} \sup f(z)$, is defined by

$$\limsup_{z\to a} f(z) = \lim_{r\to 0+} sup\{f(z): z \in G \cap B(a;r)\}$$

(If $a = \infty$, $B(a;r)$ is the ball in the metric of $\mathbb{C}_\infty$)

Similarly, the **limit inferior** of $f(z)$ *as z approaches a,* is denoted by $\lim_{z\to a} inf f(z)$, is defined by

$$\lim_{z\to a} inf f(z) = \lim_{r\to 0^+} inf\{f(z): z \in G \cap B(a;r)\}$$

If $\lim_{z\to a} f(z)$ exists and equals β then

$\lim_{z\to a} \sup f(z) = \beta = \lim_{z\to a} \inf f(z)$

Example 1.2.2

Let G be a region and suppose that $f: G \to \mathbb{C}$ is analytic and $a \in G$ such that

$|f(a)| \leq |f(z)|$ *for all z in G*. Show that either $f(a) = 0$ *or f is constant.*

Proof

Let $f: G \to \mathbb{C}$ *be analytic where G is a region. Let* $a \in G$ *such that*

$|f(a)| \leq |f(z)|$ *for all z in G* (1.2.2.1) *obviously;* $f(a)$ *satisfies*

(1.2.2.1) *since* $0 \leq |f(z)|$ *for all z in G is true. Now suppose that*

$f(a) \neq 0$ *for all a in G*.Define $g(z) = \frac{1}{f(z)}$ *since* $f(a) \neq 0$ *for all a in G and*

$|g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{|f(a)|} = |g(a)|$ *for all z in G*. From (the definition and (1.2.2.1)) so according to the Maximum Modulus Theorem g is constant and thus f is constant. Thus we conclude either $f(a) = 0$ *or f is constant*.

Example 1.2.3

Prove the following Minimum Principle. If f is a non constant analytic function on a bounded open set G and is continuous on G^-, then either f has a zero in G or $|f|$ assumes its minimum value on ∂G.

Solution

since f belongs to $C(G^-)$ *we have* $|f|$ *also belongs to* G^- *hence there exists a an element of* G^- *such that* $|f(a)| \leq |f(z)|$. If $a \in \partial G$ then $|f|$ assumes its minimum value on ∂G. Otherwise, if $a \notin \partial G$ then $a \in G$ and we can write $G = \sqcup A_i$ where A_i are the components of G that is $a \in A_i$ for some i. But each A_i is a region, so we can use example (1.2.2) which yields either $f(a) = 0$ *or f is constant* but f is assumed to be non constant, so f has to have a zero in G, therefore either f has a zero in G *or* $|f|$ assumes its minimum value on ∂G.

Example1.2.4

Let G be a bounded region and suppose f is continuous on G^- and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c \ for \ all \ z \ on \ the \ boundary \ of \ G$ then either f is a constant function or f has a zero in G.

Solution

Suppose there is a constant $c \geq 0 \ such \ that \ |f(z)| = c \ for \ all \ z \in \partial G$. According to the Maximum Modulus Theorem (version 2),

$we \ get \ max\{\,|f(z)|: z \in G^-\} = max\{|f(z)|: z \in \partial G\} = c \ \ so \ |f(z)| \leq c \ \ for \ all$

$z \in G^-$ (1.2.4.1). Since f belongs $C(G)$ which implies $|f|$ belongs to $C(G^-)$ and there exists an $a \in G \ such \ that \ |f(a)| \leq |f(z)| \leq c$ from (1.2.4.1) then by example (1.2.2) either f is a constant or f has a zero in G.

Remark 1.2.4

If $G \subset \mathbb{C}$ then let $\partial_\infty G$ denote the boundary of G in $\mathbb{C}_\infty$ and call it **the extended boundary** of G. If G is bounded then $\partial_\infty G = \partial G$ and if G is unbounded then $\partial_\infty G = \partial G \cup \{\infty\}$.

After these preliminaries the final version of the Maximum Modulus Theorem can be stated.

Theorem 1.2.5 Maximum Modulus Theorem (third version)

Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\lim_{z\to a} \sup|f(z)| \leq M \ for \ all \ a \ in \ \partial_\infty G$. Then $|f(z)| \leq M \ for \ \ all \ z \ \ in \ \ G$.

Proof

Let $\delta > 0$ be arbitrary and set $H = \{z \in G: |f(z)| > M + \delta\}$ we need to show H is empty. Since $|f|$ is continuous, H is open. Since $\lim_{z\to a} \sup|f(z)| \leq M$ for each a in $\partial_\infty G$ by definition there is a ball $B(a;r)$ such that $|f(z)| < M + \delta$ for all z in $G \cap B(a;r)$. Thus $H^- \subset G$. But we know that this condition also holds if G is unbounded and $a = \infty$, now H must be bounded. Therefore H^- is compact (That is H^- is both closed and bounded). Hence by the second version of the Maximum Modulus Theorem if $z \in \partial H, |f(z)| = M + \delta$ since $H^- \subset \{z: |f(z)| \geq M + \delta\}$; Therefore, $H = \phi$ or f is constant. But the hypothesis implies if f is constant then $H = \phi$.

Example 1.2.5

$If\ G = \left\{z: |Im\ z| < \frac{\pi}{2}\right\}, then\ f(z) = \exp(exp^z)\ satisfies\ the\ condition$

$\lim_{z \to a} sup|f(z)| \leq 1$ for all a in ∂G but not for $a = \infty$.

Example 1.2.6

Suppose G is a region, $f: G \to \mathbb{C}$ is analytic, and M is a constant such that whenever z is on $\partial_\infty G$ and $\{z_n\}$ is a sequence in G with $z = \lim z_n$ we have $\lim sup|f(z_n)| \leq M$ Show that $|f(z)| \leq M$, for each z in G.

Solution

Here we need to show that $\lim_{z \to a} sup|f(z)| \leq M\ for\ all\ a \in \partial_\infty G$. Then we can use the Maximum Modulus Theorem (version 3). Instead of showing $\lim\ sup|f(z_n)| \leq M \Rightarrow \lim_{z \to a} sup|f(z)| \leq M\ we\ show\ the\ contrapositive\ that\ is$ $\lim_{z \to a} sup|f(z)| > M \Rightarrow \lim\ sup|f(z_n)| > M$. Assume $\lim_{z \to a} sup|f(z)| > M \Longrightarrow \lim\ sup|f(z_n)| > M$ since $z_n \to z$ as $n \to \infty$, that is z_n gets arbitrarily close to z and since f is analytic, that is continuous, we have $f(z_n)$ gets arbitrarily close to $f(z)$. Therefore

$\text{Lim}\ sup|f(z_n)| \leq M \Rightarrow |f(z)| \leq M$ for each $z\ in\ G$.

CHAPTER 2

THE SCHWARZ'S LEMMA AND ITS CONSEQUENCES

First let us set the following standard notations: we denote the disk with center z_0 and radius $(r > 0)$ by $D(z_0 : r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and the corresponding closed disk, and the circle will be denoted by $\overline{D}(z_0; r)$ and $C(z_0; r)$, respectively.

$$\overline{D}(z_0; r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

$$C(z_0; r) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

For simplicity, the unit disk will be denoted by $D: D = D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

2.1 Schwarz's Lemma

The Schwarz Lemma, named after Hermann Amandus Schwarz, is a result in complex analysis about analytic functions defined on the open unit disk. The present section explores the Schwarz's lemma and how to apply Schwarz lemma to characterize the conformal maps of the open unit disk onto itself.

Theorem 2.1.1 (Schwarz's Lemma)

Let $D = \{z : |z| < 1\}$ be an open unit disk in the complex plane and suppose f is analytic on D which satisfies

(a) $|f(z)| \leq 1$ *for z in D*; and

(b) $f(0) = 0$.

Then

$$|f'(0)| \leq 1 \text{ and } |f(z)| \leq |z| \text{ for all z in the disk D.}$$

Moreover, if $|f'(0)| = 1$ *or* $|f(z)| = |z|$ *for some* $z \neq 0$, then there is *a constant c*, $|c| = 1$, such that

$$f(w) = cw \text{ for all w in D}$$

Proof

To proof the Lemma, one applies the Maximum Modulus Principle to the function $\frac{f(z)}{z}$. Define $g: D \to \mathbb{C}$ by $g(z) = \begin{cases} \frac{f(z)}{z}; \text{if } z \neq 0 \\ f'(0); \text{if } z = 0 \end{cases}$ then g is analytic in D.

Using the Maximum Modulus Theorem, $|g(z)| \leq r^{-1}$ for $|z| \leq r$ and $0 < r < 1$. Letting r approach1 gives $|g(z)| \leq 1$ for all $z\ in\ D$. That is, $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$ .If $|f(z)| = |z|$ for some $z\ in\ D,\ z \neq 0$, or $|f'(0)| = 1$ then $|g|$ assumes its maximum value inside D. Thus, a gain applying the Maximum Modulus Theorem, $g(z) \equiv c$ for some constant c with $|c| = 1$. This gives $f(z) = cz$ and completes the proof of the theorem.

A mapping of the form $S(z) = \frac{az+b}{cz+d}$ is called **a linear fractional transformation**. If $a, b, c\ and\ d$, also satisfy $ad - bc \neq 0$ then $S(z)$ is called **a Mobius transformation.**

Next, we will see how to apply Schwarz's lemma to characterize the conformal maps of the open unit disk on to itself. First we introduce a class of such maps. If $|a| < 1$ define the Mobius transformation

$$\varphi_{a(z)} = \frac{z-a}{1-\bar{a}z}$$

φ_a *is analytic for* $|z| < |a|^{-1}$. So that it is analytic in an open disk containing *the closure of* $D = \{z: |z| < 1\}$. For $|z| < 1$, we get

$$\varphi_a(\varphi_{-a^{(z)}}) = \varphi_a\left(\frac{z+a}{1+\bar{a}z}\right) = \frac{\frac{z+a}{1+\bar{a}z} - a}{1-\bar{a}\left(\frac{z+a}{1+\bar{a}z}\right)}$$

$$= \frac{z-a\bar{a}z}{1-a\bar{a}}$$

$$= \frac{z(1-a\bar{a})}{1-a\bar{a}} = \frac{z(1-|a|^2)}{1-|a|^2}$$

$$= z$$

and

$$\varphi_{-a}\left(\varphi_a(z)\right) = \varphi_{-a}\left(\frac{z-a}{1-\bar{a}z}\right) = \frac{\left(\frac{z-a}{1-\bar{a}z}\right)+a}{1+\bar{a}\left(\frac{z-a}{1-\bar{a}z}\right)}$$

$$= \frac{z-aaz}{1-aa} \quad = \frac{z(1-aa)}{1-aa}$$

$$= \frac{z(1-|a|^2)}{1-|a|^2}$$

$$= z.$$

Therefore, $\varphi_a(\varphi_{-a}(z)) = z = \varphi_{-a}\big(\varphi_a(z)\big)\ for\ \ |z| < 1$. Hence φ_a maps D onto itself in a one to one fashion.

Let θ be a real number then

$$|\varphi_a(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right| \quad = \frac{|e^{i\theta}-a|}{|e^{i\theta}||e^{-i\theta}-\bar{a}|} \quad = \frac{|e^{i\theta}-a|}{|e^{i\theta}-\bar{a}|}.$$

Since $cos^2\theta + sin^2\theta = 1$, we have $|e^{i\theta}| = 1$ for all real θ.

Hence, since $|\overline{e^{i\theta}} - \bar{a}| = |e^{i\theta} - a|$,

$$\frac{|e^{i\theta} - a|}{|\overline{e^{i\theta}} - \bar{a}|} = \frac{|e^{i\theta} - a|}{|e^{i\theta} - a|} = 1$$

This says that $\varphi_a(\partial D) = \partial D$.

These facts, and other pertinent information which can be easily checked, are summarized as follows.

Proposition 2.1.2

If $|a| < 1$ *then* φ_a is a one to one map of $D = \{z\colon |z| < 1\}$ on to itself; the inverse of φ_a *is* φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a{}'(0) = 1 - |a|^2$ and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Now let us see how these functions φ_a can be used in applying Schwarz's lemma. Suppose f is analytic on D with $|f(z)| \leq 1$. Also suppose $|a| < 1$ and $f(a) = \alpha$ (so $|\alpha| < 1$ unless f is constant).Among all functions having these properties what is the maximum possible value of $|f'(a)|$? to solve this problem let $g = \varphi_\alpha o f o \varphi_{-a}$. Then $g: D \to D$ and $g(0) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = 0$. Now we can apply Schwarz's lemma to obtain that $|g'(0)| \leq 1$. Applying the chain rule

$$g'(0) = (\varphi_\alpha o f)'(\varphi_{-a}(0))\varphi'_{-a}(0)$$

$$= (\varphi_\alpha o f)'(a)(1 - |a|^2)$$

$$= \varphi'_\alpha(f(a))f'(a)(1 - |a|^2)$$

$$= \varphi'_\alpha(\alpha)f'(a)(1 - |a|^2) \quad (\text{since } f(a) = \alpha)$$

$$= \frac{1-|a|^2}{1-|\alpha|^2} f'(a) \quad (\text{ since } \varphi'_\alpha(\alpha) = (1 - |\alpha|^2)^{-1} = \frac{1}{1-|\alpha|^2})$$

Hence

$$|f'(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2} \quad (2.1.2.1)$$

Moreover equality will occur exactly when $|g'(0)| = 1$, or, by virtue of Schwarz's lemma, when there is a constant c with $|c| = 1$ and

$$f(z) = \varphi_{-\alpha}(c\varphi_a(z)) \; for \; |z| < 1 \quad (2.1.2.2)$$

We are now ready to state and prove one of the main consequences of Schwarz's lemma. Note that if $|c| = 1$ and $|a| < 1$ then $f = c\varphi_a$ defines a one to one analytic map of the open unit disk D onto itself. The next result says that the converse is also true.

Theorem 2.1.3

Let $f: D \to D$ be a one to one analytic map of D onto itself and suppose $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = c\varphi_a$.

Proof

Since f is one-to-one and onto, there is an analytic function $g: D \to D$ such that

$g(f(z)) = z \, for \; |z| < 1$.Now since $g(f(z)) = z$ then $g(f(a)) = a$ for $|a| < 1$

$(g(f(a))' = g'(f(a))f'(a) = 1$ this implies $g'(0)f'(a) = 1$

Since we have $f(a) = \alpha \; this \; implies \; f(a) = 0 \; hence \; \alpha = 0$ and

Now using inequality(2.1.2.1) to both f and g gives

$|f'(a)| \leq (1 - |a|^2)^{-1} \; and \; |g'(0)| \leq 1 - |a|^2$ (since $g(0) = a = \alpha$)

Now since $1 = g'(0)f'(a) \leq (1 - |a|^2)f'(a)$, we obtain

$$(1 - |a|^2)^{-1} \leq |f'(a)|.$$

Therefore, $|f'(a)| = (1 - |a|^2)^{-1}$. Applying formula(2.1.2.2), we have

$$f(z) = \varphi_{-\alpha}(c\varphi_a(z))$$

It follows that φ_0 is the identity mapping. Thus

$$f(z) = \varphi_{-\alpha}(c\varphi_a(z)) = \varphi_{-0}(c\varphi_a(z)) = \varphi_a(z)$$

Therefore, $f = c\varphi_a$

2.2 Schwarz's Lemma and its application

The **Schwarz's Lemma** is one of the simplest results in all of complex function theory. A direct application of the Maximum Modulus Principle, it is merely a statement about the rate of growth of analytic functions on the unit disk.

But there is hardly any result that has been quite so influential. Thanks in part to Lars Ahlfors's geometrization of the proof (he showed that the Schwarz's lemma can be interpreted in terms of curvature), the Schwarz's lemma has assumed a central and powerful role in complex geometry.

Example2.2.1

Prove that $||z|-|w|| \leq |z-w|$ for all $z, w \in \mathbb{C}$?

Proof

$|z-w|^2 = (z-w)(\overline{z-w}) = (z-w)(\bar{z}-\bar{w})$

$$= |z|^2 + |w|^2 - (\bar{w}z - w\bar{z})$$

$$\Rightarrow |z-w|^2 = |z|^2 + |w|^2 - (\bar{w}z - w\bar{z}) = |z|^2 + |w|^2 - 2Re(z\bar{w})$$

$$\geq |z|^2 + |w|^2 - 2|z||w|$$

$$= (|z|-|w|)^2$$

$\Rightarrow |z-w|^2 \geq (|z|-|w|)^2$. This implies $||z|-|w|| \leq |z-w|$

Example2.2.3

Suppose $|f(z)| \leq 1$ *for* $|z| < 1$ and f *is a non constant analytic function* by considering the function $g: D \to D$ defined by

$$g(z) = \frac{f(z) - a}{1 - \bar{a}f(z)}$$

where $a = f(0)$, prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|} \quad for \quad |z| < 1$$

Proof

Since $g(0) = \frac{f(0) - a}{1 - \bar{a}f(0)} = \frac{a - a}{1 - \bar{a}a} = 0$. Hence $g(0) = 0$. Now we claim $|g(z)| \leq 1$

We also have

$$|g(z)| = \frac{|f(z) - a|}{|1 - \bar{a}f(z)|} \leq 1$$

Since $|f(z)| \leq 1$ for $|z| \leq 1$ now we can apply Schwarz lemma $|g(z)| \leq 1$

$\Leftrightarrow |f(z) - a| \leq |z||1 - \bar{a}f(z)|$ this implies

$||f(z)| - |a|| \leq |f(z) - a| \leq |1 - \bar{a}f(z)| \leq |z||a||f(z)|$

$\Rightarrow ||f(z)| - |a|| \leq |z| + |z||a||f(z)|$

$\Rightarrow -|z| - |z||a||f(z)| \leq |f(z)| - |a| \leq |z| + |z||a||f(z)|$

$\Rightarrow -|z| - |z||a||f(z)| \leq |f(z)| - |a| \quad and \quad |f(z)| - |a| \leq |z| + |z||a||f(z)|$

$\Rightarrow |a| - |z| \leq |f(z)| + |z||a||f(z)| \ and \ |f(z)| - |z||a||f(z)| \leq |a| + |z|$

$\Rightarrow |a| - |z| \leq |f(z)|(1 + |z||a|) \ and \ |f(z)|(1 - |z||a|) \ \leq |a| + |z|$

$\Rightarrow \frac{|a|-|z|}{1+|z||a|} \leq |f(z)| \ and \ |f(z)| \ \leq \frac{|a|+|z|}{1-|z||a|}$

$\Rightarrow \frac{|a|-|z|}{1+|z||a|} \leq |f(z)| \leq \frac{|a|+|z|}{1-|z||a|}$

If we set $a = f(0)$ then we obtain

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$

Therefore this is the desired result.

CHAPTER 3

CONVEX FUNCTIONS AND HADAMARD'S THREE CIRCLES THEOREM

3.1 Convex and Logarithmically Convex Functions

In this section we will study convex functions and logarithmically convex functions and show that such functions appear in connection with study of analytic functions.

Definition 3.1.1

If $[a, b]$ is an interval in the real line, a function $f: [a, b] \to \mathbb{R}$ is convex if for any two points x_1 and x_2 in $[a, b]$

$$f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1)$$

whenever $0 \leq t \leq 1$. A subset $A \subset \mathbb{C}$ is *convex* if whenever z and w are in A, $tz + (1-t)w\ is\ in\ A\ for\ \ 0 \leq t \leq 1$; that is, A is convex when for any two points in A the line segment joining the two points is also in A.

Example 3.1.1

If $x \in \mathbb{R}^n$ and $r > 0$, the open (closed) ball B with center at x and radius r is defined to be the set of all $y \in \mathbb{R}^n$ such that $|y - x| < r$ $(or\ |y - x| \leq r)$.We call a set $E \subseteq \mathbb{R}^n$ convex if $\lambda x + (1-\lambda)y \in E$ whenever $x, y \in E$ and $0 < \lambda < 1$.

For example, balls are convex. For if $|y - x| < r$; $|z - x| < r$ and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda y + (1-\lambda)z - x| &= |\lambda(y - x) + (1-\lambda)(z - x)| \\ &\leq \lambda|y - x| + (1-\lambda)|z - x| \\ &\leq \lambda r + (1-\lambda)r \\ &= r \end{aligned}$$

Remark 3.1.1

A function is convex if and only if the portion of the plane lying above the graph of the function is a convex set. That is, if we look at the graph of f in $\mathbb{R}^2$, this condition can be formulated geometrically by saying that each point on the chord between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is above the graph of f.

Remark 3.1.2

If f is convex on $[a,b]$ and if x, y, x_1, y_1 are points of $[a,b]$ with $x < x_1 < y_1$ and $x < y < y_1$, then the chord over (x_1, y_1) has larger slope than the chord over (x, y):

That is, $\frac{f(y)-f(x)}{y-x} \leq \frac{f(y_1)-f(x_1)}{y_1-x_1}$

Proposition3.1.3

A function $f: [a,b] \to \mathbb{R}$ *is convex if and only if the set*

$$A = \{(x,y): a \leq x \leq b \text{ and } f(x) \leq y\}$$

is convex.

Proof

($\Rightarrow$) *Suppose* $f: [a,b] \to \mathbb{R}$ is a convex function and let (x_1, y_1) and (x_2, y_2) be points in A. If $0 \leq t \leq 1$ then, by the definition of convex function,

$$f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1) \leq ty_2 + (1-t)y_1$$

Thus $t(x_2, y_2) + (1-t)(x_1, y_1) = (tx_2 + (1-t)x_1, ty_2 + (1-t)y_1)$ is in A; so A is convex

($\Leftarrow$) Suppose A is a convex set and let x_1, x_2 be two points in $[a,b]$. Then

$\left(tx_2 + (1-t)x_1,\ tf(x_2) + (1-t)f(x_1)\right)$ is in A if $0 \leq t \leq 1$ by virtue of its convexity. Thus the definition of A gives that

$f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1)$

Therefore f is convex.

Proposition 3.1.4

(a) *A function* $f:\{a,b] \to \mathbb{R}$ *is convex iff for any points* $x_1, \cdots, x_n$ *in* $[a,b]$ *and real numbers* $t_1, \cdots, t_n \geq 0$ *with* $\sum_{k=1}^{n} t_k = 1$,

$$f(\sum_{k=1}^{n} t_k x_k) \leq \sum_{k=1}^{n} t_k f(x_k)$$

(b) *A set* $A \subset \mathbb{C}$ *is convex iff for any points* $z_1, \cdots, z_n$ *in* A *and real numbers* $t_1, \cdots, t_n \geq 0$ *with* $\sum_{k=1}^{n} t_k = 1, \sum_{k=1}^{n} t_k z_k$ *belongs to* A.

Proof

(a)

($\Rightarrow$) *suppose a function* $f:[a,b] \to \mathbb{R}$ *is convex.*

We need to show for any points $x_1, \cdots, x_n$ in $[a,b]$ and real number $t_1, \cdots, t_n \geq 0$ with $\sum_{k=1}^{n} t_k = 1$

$$f(\sum_{k=1}^{n} t_k x_k) \leq \sum_{k=1}^{n} t_k f(x_k)$$

This is proved by induction, for $k=2$. let $x_1, x_2 \in [a,b]$ then

$f(t_1x_1 + t_2x_2) = f(t_1x_1 + (1-t_1)x_2) \leq t_1f(x_1) + (1-t_1)f(x_2)$ *where* $t_1 + t_2 = 1$

(by definition of convex function).

Suppose it is true for $k = n-1$, i.e.. $f(\sum_{k=1}^{n-1} t_k x_k) \leq \sum_{k=1}^{n-1} t_k f(x_k)$ (where $\sum_{k=1}^{n-1} t_k = 1$) We need to prove for $k = n$ is true. Let $x_1, \cdots, x_n \in [a,b]$ and $t_1, \cdots, t_n \geq 0$ with $\sum_{k=1}^{n} t_k = 1$. Then

$$\sum_{k=1}^{n} t_k x_k = \sum_{k=1}^{n-1} t_k x_k + t_n x_n.$$

Now let $\alpha = t_1 + \cdots + t_{n-1}$. For $\alpha = 0$, the statement holds trivially. Assume that $\alpha \neq 0$. Let $x = \sum_{k=1}^{n-1} \frac{t_k}{\alpha} x_k$. Then, by the induction assumption,

$$f(x) = f(\sum_{k=1}^{n-1} \frac{t_k}{\alpha} x_k) = \sum_{k=1}^{n-1} \frac{t_k}{\alpha} f(x_k) = \frac{1}{\alpha} \sum_{k=1}^{n-1} t_k f(x_k). \quad (3.1.4.1)$$

Since f is convex,

$$f(\alpha x + t_n x_n) \leq \alpha f(x) + t_n f(x_n). \quad (3.1.4.2)$$

Thus it follows from (3.1.4.1) and (3.1.4.2) that

$$f(\alpha x + t_n x_n) \leq \alpha \sum_{k=1}^{n-1} \frac{t_k}{\alpha} f(x_k) + t_n x_n = \sum_{k=1}^{n} t_k f(x_k).$$

Therefore,

$$f\left(\sum_{k=1}^{n} t_k x_k\right) \leq \sum_{k=1}^{n} t_k f(x_k).$$

(⇐) Suppose for any points $x_1, \cdots, x_n$ in $[a, b]$ and real numbers $t_1, \cdots, t_n \geq 0$ with $\sum_{k=1}^{n} t_k = 1$,

$f(\sum_{k=1}^{n} t_k x_k) \leq \sum_{k=1}^{n} t_k f(x_k)$ We need to show $f: [a, b] \to \mathbb{R}$ is convex.

Let $x_1, x_2 \in [a, b]$ with $0 < \alpha < 1$

Now since $\alpha + (1 - \alpha) = 1$ where $t_1 = \alpha$ and $t_2 = 1 - \alpha$

$f(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha f(x_1) + (1 - \alpha) f(x_2)$ (by hypothesis)

Therefore, f is convex.

(b)

(⇒) Suppose a set $A \subseteq \mathbb{C}$ be convex. We need to show $\sum_{k=1}^{n} t_k z_k \in A$. For $k = 2$, by the definition of convex set, the conclusion follows immediately.

Assume it is true for $k = n$, that is, for any $z_k \epsilon A, k = 1,2,\dots,n,$ and for any $t_k \geq 0, k = 1,2,\dots,n$ with $\sum_{k=1}^{n} t_k = 1,\quad \sum_{k=1}^{n} t_k\, z_k \,\epsilon A.$ We need to show for $k = n+1$, the statement is true. Let $z_k \epsilon A, k = 1,2,\dots,n+1,$ and $t_k \geq 0,\ k = 1,2,\dots,n+1$ with $\sum_{k=1}^{n+1} t_k = 1$. Let $t = \sum_{k=1}^{n} t_k$. If t=0, then it follows trivially. Assume that $t \neq 0$ and let $z = \sum_{k=1}^{n} \frac{t_k}{t} z_k$. Then, by the induction assumption, $z \epsilon A$. Since A is convex $tz + t_{n+1}z_{n+1} \epsilon A$. Therefore,

$$tz + t_{n+1}z_{n+1} = t\sum_{k=1}^{n} \frac{t_k}{t} z_k + t_{n+1}z_{n+1} = \sum_{k=1}^{n+1} t_k\, z_k \in A.$$

$(\Leftarrow)$ *suppose* $\sum_{k=1}^{n} t_k z_k$ *belongs to* A. We need to show A is convex.

Let $x_1, x_2 \in A$ with $0 < \alpha < 1$. Now since $\alpha + (1-\alpha) = 1$, where $t_1 = \alpha$ *and* $t_2 = 1 - \alpha$, by hypothesis we have $\alpha x_1 + (1-\alpha)x_2 \,\epsilon A$. Therefore, A is convex.

Proposition 3.1.5

A differentiable function f *on* $[a,b]$ *is convex iff* f' *is increasing.*

Proof

$(\Rightarrow)$ Suppose f is convex we need to show f' is increasing let $a \leq x < y \leq b$ and suppose $0 < t < 1$ since $0 < (1-t)x + ty - x = t(y-x)$, then the definition of convexity gives that

$$\frac{f((1-t)x+ty)-f(x)}{t(y-x)} \leq \frac{(1-t)f(x)+tf(y)-f(x)}{t(y-x)}$$

$$= \frac{f(x)-tf(x)+tf(y)-f(x)}{t(y-x)}$$

$$= \frac{t(f(y)-f(x))}{t(y-x)}$$

$$= \frac{f(y)-f(x)}{y-x}$$

This can be expressed as

$$\frac{f((1-t)x+ty)-f(x)}{(1-t)x+ty-x} \leq \frac{f(y)-f(x)}{y-x}$$

Now letting $t \to 0$ $then$ $(1-t)x+ty \to x$ this gives that

$$f'(x) \leq \frac{f(y)-f(x)}{y-x} \quad (3.1.5.1)$$

Similarly, using the fact that $0 > (1-t)x+ty-y = (1-t)(x-y)$

$$\frac{f((1-t)x+ty)-f(y)}{(1-t)(x-y)} \geq \frac{(1-t)f(x)+tf(y)-f(y)}{(1-t)(x-y)}$$

$$= \frac{f(x)-tf(x)+tf(y)-f(y)}{(1-t)(x-y)}$$

$$= \frac{f(x)(1-t)+f(y)(t-1)}{(1-t)(x-y)}$$

$$= \frac{(f(y)-f(x))(t-1)}{(1-t)(x-y)}$$

$$= \frac{f(y)-f(x)}{y-x}$$

Therefore $\frac{f((1-t)x+ty)-f(y)}{(1-t)x+ty-y} \geq \frac{f(y)-f(x)}{y-x}$

Now letting $t \to 1$ $then$ $(1-t)x+ty \to y$ gives that $f'(y) \geq \frac{f(y)-f(x)}{y-x}$ (3.1.5.2)

Now combining (3.1.5.1) and (3.1.5.2) we have $f'(x) \leq \frac{f(y)-f(x)}{y-x} \leq f'(y)$

Therefore f' is increasing.

($\Leftarrow$) Suppose that f' is increasing and that $x < u < y$ we want to show f is convex apply the Mean Value Theorem for differentiation to find r and s with $x < r < u < s < y$ such that

$$f'(r) = \frac{f(u)-f(x)}{u-x}$$

and

$$f'(s) = \frac{f(y) - f(u)}{y - u}$$

Since $f'(r) \leq f'(s)$ this gives that

$$\frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(u)}{y-u} \text{ Provided that } x < u < y$$

In particular by letting $u = (1-t)x + ty$ where $0 < t < 1$

$$\frac{f(u) - f(x)}{t(y-x)} \leq \frac{f(y) - f(u)}{(1-t)(y-x)}$$

$\Rightarrow (1-t)[f(u) - f(x)] \leq t[f(y) - f(u)]$

$\Rightarrow (1-t)[f\big((1-t)x + ty\big) - f(x)] \leq t[f(y) - f((1-t)x + ty]$

$\Rightarrow (1-t)f((1-t)x + ty - (1-t)f(x)) \leq tf(y) - tf((1-t)x + ty)$

$\Rightarrow (1-t)f\big((1-t)x + ty\big) + tf((1-t)x + ty) \leq tf(y) + (1-t)f(x)$

$\Rightarrow f((1-t)x + ty) \leq tf(y) + (1-t)f(x)$

Hence f is convex.

DEFINITION 3.1.2

A function $f: [a,b] \to \mathbb{R}$ such that $f(x) > 0$ for all x is said to be **logarithmically convex** if $\log f(x)$ is a convex function of x.

Remark 3.1.6

A logarithmically convex function is a convex function, but the converse is not always true.

Example 3.1.2

$f(x) = x^2$ is a convex function, but $\log f(x) = \log x^2 = 2\log|x|$ is not a convex function and thus $f(x) = x^2$ is not logarithmically convex. On the other hand, $f(x) = e^{x^2}$ is logarithmically convex as $\log e^{x^2} = x^2$ is a convex function.

Remark 3.1.7

If $f(x)$ has a second derivative in $[a, b]$, then a necessary and sufficient condition for it to be convex on that interval is that the second derivative $f''(x) \geq 0$ for all x in $[a, b]$ some examples of convex functions include x^p for $p = 1$ or even $p \geq 2$, $x \log x$ for $x > 0$, and $|x|$ for all x.

Before proving Theorem 3.1.10 note that to say that $\log M(x)$ is convex means that for $a \leq x < u < y \leq b$,

$(y-x)\log M(u) \leq (y-u)\log M(x) + (u-x)\log M(y)$ (3.1.6.1)

taking the exponential of both sides gives

$M(u)^{(y-x)} \leq M(x)^{(y-u)} M(y)^{(u-x)}$ whenever $a \leq x < u < y \leq b$ (3.1.6.2).

Also, since $\log M(x)$ is convex we have that $\log M(x)$ is bounded by

$\max\{\log M(a), \log M(b)\}$.That is, for $a \leq x \leq b$

$$M(x) \leq max\{M(a), M(b)\}$$

This gives the following

Corollary 3.1.8

Let $a < b$ and let G be the vertical strip $\{x + iy: a < x < b\}$.Suppose $f: G^{-} \to \mathbb{C}$ is continuous and f is not constant then $|f(z)| < sup\{|f(w)|: w \in \partial G\}$ *for all* z *in* G.

Lemma 3.1.9

Let $a < b$ *and let* G *be the vertical strip* $\{x + iy: a < x < b\}$. *Suppose* $f: G^{-} \to \mathbb{C}$ *is continuous and further suppose that* $|f(z)| \leq 1$ *for* z *in* ∂G *then* $|f(z)| \leq 1$ *for all* z *in* G.

Proof

For each $\epsilon > 0$ let $g_\epsilon(z) = [1 + \epsilon(z - a)]^{-1}$ for each z *in* G^{-} then for

$z = x + iy$ *in* G^{-}

$$|g_\epsilon(z)| \leq |Re[1+\epsilon(z-a)]|^{-1}$$

$$= [1+\epsilon(x-a)]^{-1} \leq 1$$

So for z in ∂G $|f(z)g_\epsilon(z)| \leq 1$. Also, since f is bounded by B in G.

$$|f(z)g_\epsilon(z)| \leq B|1+\epsilon(z-a)|^{-1} \quad (3.1.8.1)$$

$$\leq B\,[\epsilon|Im\,z|]^{-1}$$

So if $R = \left\{x+iy: a \leq x \leq b, |y| < \frac{B}{\epsilon}\right\}$ inequality (3.1.8.1) gives $|f(z)g_\epsilon(z)| \leq 1$ for z in ∂R. It follows from the Maximum Modulus Theorem that

$|f(z)g_\epsilon(z)| \leq 1$ *for* z *in* R. But if $|Im\,z| > \frac{B}{\epsilon}$ then (3.1.8.1) gives that $|f(z)g_\epsilon(z)| \leq 1$ Thus for all z in G.

$$|f(z)| \leq |1+\epsilon(z-a)|.$$

Letting ϵ approach zero

$$|f(z)| \leq 1 \text{ for all } z \text{ in } G.$$

An entire function (Integral function) is a function which is defined and analytic in the whole complex plane.

Theorem 3.1.10

Let $a < b$ *and let* G *be a vertical strip* $\{x+iy: a < x < b\}$. *Suppose* $f: G^- \to \mathbb{C}$ *is continuous and analytic in* G. *If we define* $M: [a,b] \to \mathbb{R}$ *by* $M(x) = sup\{|f(x+iy)|: -\infty < y < \infty\}$ (3.1.10.1) *and* $|f(z)| < B$ *for all* z *in* G, *then* $\log M(x)$ *is a convex function.*

Proof

To prove the theorem we need only establish $M(u)^{b-a} \leq M(a)^{b-u}M(b)^{u-a}$ *for* $a < u < b$. Recall that for a constant $A > 0$. $A^z = exp(z\log A)$ is an entire function of z with no zeros. Now define $g(z) = M(a)^{\frac{(b-z)}{(b-a)}} M(b)^{\frac{z-a}{b-a}}$ is an entire, never vanishes, and (because $|A^z| = A^{Rez}$) for $z = x + iy$

$$|g(z)| = M(a)^{\frac{(b-x)}{(b-a)}} M(b)^{\frac{(x-a)}{(b-a)}} \quad (3.1.10.2)$$

(It is assumed here that $M(a)$ *and* $M(b) \neq 0$. However, if either $M(a)$ *or* $M(b)$ is zero then $f \equiv 0$) since the expression on the right hand side of (3.1.10.2) is continuous for x in $[a, b]$ and never vanishes, $|g|^{-1}$ must be bounded in G^{-}. Also, $|g(a+iy)| = M(a)$ and $|g(b+iy)| = M(b)$ so that $|\frac{f(z)}{g(z)}| \leq 1$ for z in ∂G;and $\frac{f}{g}$ satisfies the hypothesis of Lemma (3.1.9).Thus $|f(z)| \leq |g(z)|, z \in G$.

Using (3.1.10.2) this gives for $a < u < b$

$$M(u) \leq M(a)^{\frac{(b-u)}{(b-a)}} M(b)^{\frac{(u-a)}{(b-a)}} \quad (3.1.10.3)$$

This is the desired conclusion.

Example 3.1.3

Show that if $f: (a, b) \to \mathbb{R}$ is convex then f is continuous Does this remains true if f is defined on the closed interval $[a, b]$?

Solution

Since f is convex on (a, b) and if $a < x < u < y < b$,

We have $\frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(y)-f(u)}{y-u}$ (3.1.3.1) which we want to show.

Let $a < x < u < y < b, u = \lambda x + (1-\lambda)y$ with $\lambda = \frac{y-u}{y-x} \in (0,1)$ since f *is convex on* (a, b) *we get* $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ *this implies*

$f(u) \leq \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y)$

$\Rightarrow (u-x)f(y) + (y-u)f(x) - (y-x)f(u) \geq 0$ (3.1.3.2)

$\Rightarrow uf(y) - xf(y) + yf(x) - uf(x) - yf(u) + xf(u) \geq 0$

From (3.1.3.2) we get (rearrange terms and add/subtract a term)

$$\Rightarrow uf(y) - uf(x) - xf(y) + xf(x) - yf(u) - yf(x) + xf(u) - xf(x) \geq 0$$

$$\Rightarrow (y - x)\big(f(y) - f(u)\big) - (y - u)(f(y) - f(x)) \geq 0$$

$\Rightarrow \frac{f(y)-f(x)}{y-x} \leq \frac{f(y)-f(u)}{y-u}$. Thus we have proved (3.1.3.1)

Given $x \in (a\,, b)$ choose $\delta > 0$ such that $[x - \delta\,, x + \delta] \subset (a\,, b)$

Claim

$$\frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{\delta}\, for\ all\ z \in (x - \delta, x + \delta) \quad (3.1.3.3)$$

$is\ equivalent\ to\ \frac{f(x)-f(x-\delta)}{\delta}(z - x) \leq f(z) - f(x) \leq \frac{f(x+\delta)-f(x)}{\delta}(z - x)$ Now taking the limit $z \to x$, we have that $\frac{f(x)-f(x-\delta)}{\delta}(z - x) \to 0\ and\ \frac{f(x+\delta)-f(x)}{\delta}(z - x) \to 0$.

Thus $f(z) - f(x) \to 0, that\ is\ |z - x| < \delta \Rightarrow |f(z) - f(x)| < \varepsilon$ which shows that f is continuous. Now it remains to show the claim is true. The proof uses (3.1.3.1) first, consider a point $z \in (x - \delta\,, x)$ and apply the second inequality in (3.1.3.1) gives $\frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(z)-f(x)}{z-x}$ which gives the first inequality in (3.1.3.3). Applying the outer inequality (3.1.3.1) into the three points $z < x < x + \delta$ gives $\frac{f(x)-f(z)}{x-z} \leq \frac{f(x+\delta)-f(x)}{\delta} \Rightarrow$ $\frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{\delta}$ which gives the second inequality in (3.1.3.3). . Now consider the case $x < z < x + \delta$. Then the first inequality in (3.1.3.3) applied to the three points $x - \delta, x, z$

$\frac{f(x)-f(x-\delta)}{x-x+\delta} \leq \frac{f(z)-f(x)}{z-x} \Rightarrow \frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(z)-f(x)}{z-x}$. Thus we have proved the claim.

The statement is not true if f is defined on the closed interval $[a, b]$. . Here is a counter example. Define $f\colon [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in [a, b) \\ 1, & x = b \end{cases}$$

Clearly, f is convex on $[a,b]$, but f is not continuous at $x = b$ (but the function is *continuous on* (a,b).

Example 3.1.4

Show that a function $f:[a,b] \to \mathbb{R}$ is convex if and only if any of the following equivalent conditions is satisfied:

(a) $a \le x < u < y \le b$ gives $\det\begin{pmatrix} f(u) & u & 1 \\ f(x) & x & 1 \\ f(y) & y & 1 \end{pmatrix} \ge 0$;

(b) $a \le x < u < y \le b$ gives $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(x)}{y-x}$;

(c) $a \le x < u < y \le b$ gives $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-u}$

Solution

(a) Suppose f is convex on$[a,b]$ and let $a \le x < u < y \le b$, $u = \lambda x + (1-\lambda)y$ with $\lambda = \frac{y-u}{y-x} \in (0,1)$

Since f is convex on $[a,b]$, we have $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

$$\Rightarrow \; f(u) \le \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y)$$

$$\Rightarrow \; (y-x)f(u) \le (y-u)f(x) + (u-x)f(y)$$

$$\Rightarrow (y-x)f(x) + (u-x)f(y) - (y-x)f(u) \ge 0$$

$$\Rightarrow yf(x) - uf(x) + uf(y) - xf(y) - yf(u) + xf(u) \ge 0$$

$$\Rightarrow yf(x) - xf(y) + u\big(f(y) - f(x)\big) + f(u)(x-y) \ge 0$$

Therefore

$$\det\begin{pmatrix} f(u) & u & 1 \\ f(x) & x & 1 \\ f(y) & y & 1 \end{pmatrix} \ge 0$$

(b) Suppose f is convex on $[a,b]$ and let $a \le x < u < y \le b$, and $u = \lambda x + (1-\lambda)y$ with $\lambda = \frac{y-u}{y-x} \in (0,1)$.

Since f is convex on $[a,b]$ we get $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$

$\Rightarrow f(u) \le \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y)$

$\Rightarrow (y-x)f(u) \le (y-u)f(x) + (u-x)f(y)$

$\Rightarrow yf(u) - xf(u) \le yf(x) - uf(x) + uf(y) - xf(y)$

$\Rightarrow yf(x) - uf(x) + uf(y) - xf(y) - yf(u) + xf(u) \ge 0$

$\Rightarrow yf(x) - yf(u) - uf(x) + uf(y) - xf(y) + xf(u) \ge 0$

$\Rightarrow uf(y) - uf(x) + yf(x) - yf(u) + xf(u) - xf(y) \ge 0$

$\Rightarrow uf(y) - uf(x) + yf(x) - xf(x) - yf(u) + xf(u) - xf(y) + xf(x) \ge 0$

$\Rightarrow u\big(f(y) - f(x)\big) + (y-x)f(x) + (x-y)f(u) + x(f(x) - f(y)) \ge 0$

$\Rightarrow (y-x)\big(f(x) - f(u)\big) + (u-x)(f(y) - f(x)) \ge 0$

$\Rightarrow (y-x)\big(f(x) - f(u)\big) \ge -(u-x)(f(y) - f(x))$

$\Rightarrow \frac{f(x)-f(u)}{-(u-x)} \le \frac{f(y)-f(x)}{y-x}$

$\Rightarrow \frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(x)}{y-x}$

(c) Suppose f is convex on $[a,b]$ and let $a \le x < u < y \le b$ and $u = \lambda x + (1-\lambda)y$ with

$\lambda = \frac{y-u}{y-x} \in (0,1)$.

Since f is convex on $[a,b]$ we have $f(\lambda x + (1-\lambda)y \le \lambda f(x) + (1-\lambda)f(y)$

$\Rightarrow f(u) \le \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y)$

$\Rightarrow (y-x)f(u) \leq (y-u)f(x) + (u-x)f(y)$

$\Rightarrow yf(u) - xf(u) \leq yf(x) - uf(x) + uf(y) - xf(y) - xf(y)$

$\Rightarrow yf(x) - uf(x) + uf(y) - xf(y) - yf(u) + xf(u) \geq 0$

$\Rightarrow yf(x) - yf(u) + uf(y) - uf(x) - xf(y) + xf(u) \geq 0$

$\Rightarrow yf(x) - yf(u) + uf(y) + uf(u) - uf(x) - xf(y) + xf(u) - uf(u) \geq 0$

$\Rightarrow yf(x) - uf(x) - yf(u) + uf(u) + uf(y) - xf(y) + xf(u) - uf(u) \geq 0$

$\Rightarrow (y-u)f(x) + f(u)(y-u) + (u-x)(f(y) - f(u) \geq 0$

$\Rightarrow (y-x)f(x) - f(u)(y-u) + (u-x)f(y) - (u-x)f(u) \geq 0$

$\Rightarrow (y-u)\big(f(x) - f(u)\big) + (u-x)(f(y) - f(u)) \geq 0$

$\Rightarrow (y-x)\big(f(x) - f(y)\big) \geq -(u-x)(f(y) - f(u))$

$\Rightarrow \frac{f(x)-f(u)}{-(u-x)} \leq \frac{f(y)-f(u)}{y-u}$

$\Rightarrow \frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(u)}{y-u}$

In the next section we will state and prove the Hadamard's Three Circles Theorem we can prove this fact using Theorem 3.1.10 and we also apply the concept of convex functions. That is

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2)$$

Here if we take $\lambda = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}$ and $1 - \lambda = \frac{\log r - \log r_1}{\log r_2 - \log r_1}$ hence the above inequality can be expressed as $\log M(r) \leq \lambda \log M(r_1) + (1-\lambda)\log M(r_2)$, therefore $\log M(r)$ is a convex function of $\log r$ that is, the graph of $\log M(r)$ (versus $\log r$) has the property that the part of the graph between $(\log c, \log M(c))$ *and* $(\log d, \log M(d))$ $r_1 \leq c \leq d \leq r_2$ lies below (coincides with) the line segment joining these two end points.

3.2 THEOREM (Hadamard's Three Circles Theorem)

Hadamard's theorem is concerned with the relation between the maximum absolute value of analytic function on three concentric circles. If we put

$$M(r) = \max_{|z|=r} |f(z)|$$

Then the theorem state that $\log M(r)$ is a convex function of $\log r$ for $r_1 < |z| < r_2$, if $f(z)$ is regular for $r_1 < |z| < r_2$. This is an immediate consequence of the fact that if $|f(z)| \leq A|z|^{\lambda}$ on two circles about the origin, then it is also true between the circles and this in turn is seen by applying the principle of maximum to $\frac{f(z)}{z^{\lambda}}$. The bound is attainable with in the ring only for $f(z) = \alpha z^{\lambda}$ with $|\alpha| = A$.

Hadamard's three circles theorem is analogue of theorem 3.1.10 for an annulus consider $(0; R_1, R_2) = A$ *where* $0 < R_1 < R_2 < \infty$. If the strip $\{x + iy: \log R_1 < x < \log R_2\}$ then the exponential function maps G onto A and ∂G onto ∂A.

Theorem 3.2.1 (Hadamard's Three Circles Theorem)

Let $0 < R_1 < R_2 < \infty$ *and suppose* f is analytic and not identically zero on $ann\,(0; R_1, R_2)$. If $R_1 < r < R_2$, define $M(r) = \max\ \{|f(re^{i\theta})|: 0 \leq \theta \leq 2\pi\}$.

Then for $R_1 < r_1 \leq r \leq r_2 < R_2$ and $r_1 \neq r_2$

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2)$$

[This inequality says quite plainly that $\log M(r)$ is a convex function of $\log r$]

Proof

Consider the $ann\,(0; R_1, R_2) = A$ where $0 < R_1 < R_2 < \infty$ and suppose G is a strip $\{x + iy: \log R_1 < x < \log R_2\}$ then the exponential function e^z *maps* G *onto* A. Let us define $g: G \to \mathbb{C}$ by $g(z) = f(e^z)$. Then g is analytic on G and thus, by theorem 3.1.10 if

$$N(x) = \sup\ \{|g(x + iy)|: -\infty < y < \infty\}$$

then $logN(x)$ is a convex function. Since $x = logr$, we have $N(x) = N(logr)$. Also we get

$$N(\log r) = sup\,\{|g(\log r + iy)|: -\infty < y < \infty\}$$

$$= sup\{|f(e^{\log r + iy})|: -\infty < y < \infty\}$$

$$= sup\{|f(re^{iy})|: -\infty < y < \infty\}$$

$$= sup\{|f(re^{i\theta})|: 0 \le \theta \le 2\pi\}$$

$$= M(r)$$

From equation (3.1.6.2) we have

$$N(u)^{(y-x)} \le N(x)^{y-u} N(y)^{u-x}$$

whenever $logR_1 \le x < u < y \le logR_2$. Therefore,

$$N(\log r)^{(\log r_2 - \log r_1)} \le N(\log r_1)^{(\log r_2 - \log r)} N(\log r_2)^{(\log r - \log r_1)},$$

whenever $R_1 < r_1 \le r \le r_2 < R_2$ and $r_1 \ne r_2$. Then

Since $M(r) = N(logr)$. it follows that

$$M(r)^{(\log r_2 - \log r_1)} \le M(r_1)^{(\log r_2 - \log r)} M(r_2)^{(\log r - \log r_1)}.$$

Now taking the logarithm of both sides gives

$$\log M(r)^{(\log r_2 - \log r_1)} \le \log M(r_1)^{(\log r_2 - \log r)} + \log M(r_2)^{(\log r - \log r_1)}$$

$$\Rightarrow (\log r_2 - \log r_1) \log M(r) \le (\log r_2 - \log r)\ \log M(r_1) + (\log r - \log r_1) \log M(r_2)$$

$$\Rightarrow \log M(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2)$$

This is the desired result.

3.3 PHRAGMEN LINDELOF THEOREM

The context is that the Maximum Modulus Principle in complex analysis does not apply to unbounded regions. That is, an analytic function on unbounded region may be bounded by 1 on the edges but be violently unbounded in the interior.

The simplest example is $f(z) = \exp(exp^z)$ for z real and going to $+\infty$ this function blows up. Indeed

$$|expexp^{(x+iy)}| = expRe\left(e^{x+iy}\right) = expRe(e^x(cosy + isiny))$$

$$= expRe(e^x cosy + ie^x siny)$$

$$= expe^x cosy$$

Thus, for fixed $y = Imz$ with $cosy > 0$ the function blows up as $x = Rez \to +\infty$. On the other hand, for $cosy = 0$ the function is bounded. Thus, on the strip $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, the function $\exp(exp^z)$ is bounded on the edges but blows up as $x \to +\infty$. This example suggests growth conditions under which a bound of 1 on the edges implies the same bound throughout the strip.

Theorem 3.3.1

Let f be analytic function on the horizontal strip $\left\{z: -\frac{\pi}{2} \le y \le \frac{\pi}{2} \text{ and } x \ge 0\right\}$ if $|f(z)| \le e^{e^{c\,Rez}}$ (for some constant $0 \le c \le 1$) then $|f(z)| \le 1$ on the edges of the half strip implies $|f(z)| \le 1$ in the interior as well).

Proof

The proof is a reduction of in the usual Maximum Modulus Theorem. Take any fixed D in the range $c < D < 1$ the function $F_\epsilon(z) = f(z)/e^{\epsilon e^{D-z}}$ $(for\ \epsilon > 0)$ is certainly bounded by 1 on the edges of the half strip and in the interior goes to 0 uniformly in y as $x \to +\infty$, for fixed $\epsilon > 0$. (the uniform decay in the interior is where the modification with D is used). Thus, on a rectangle $R_r = \left\{z: -\frac{\pi}{2} \le y \le \frac{\pi}{2}, and\ 0 \le x \le T\right\}$ for sufficiently large $T > 0$ depending upon z, the function F_z is bounded by 1 on the edge. The usual Maximum Modulus Principle implies that F_z is bounded by 1 throughout.

That is, for each fixed z_0 in the half strip, $|f(z_0)| \leq e^{\epsilon e^{DRez_0}}$ (for all $\epsilon > 0$) we can let $\epsilon \to 0^+$ giving $|f(z_0)| \leq 1$.

[The maximum Modulus Principle in complex analysis is that an analytic function f on a bounded region in $\mathbb{C}$ with $|f(z)| \leq 1$ on the edges is bounded by 1 in the interior as well]

Note that a curve $\gamma: [a, b] \to \mathbb{C}$ is said to be closed if $\gamma(a) = \gamma(b)$

If γ is closed rectifiable curve in G then γ is **homotopic** to zero ($\gamma \sim 0$) if γ is homotopic to a constant curve.

An open set G is **simply connected** if G is connected and every closed curve in G is homotopic to zero.

If G is an open connected set in $\mathbb{C}$ and $f: G \to \mathbb{C}$ is continuous function such that $z = expf(z)$ for all z in G then f is a branch of the logarithm.

The domain of the principal branch of the logarithm is simply connected

The next corollary is taken from John B. Conway page94-95 it helps us to prove Theorem3.3.3

Corollary 3.3.2

Let G be simply connected and let $f: G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any z in G. Then there is an analytic function $g: G \to \mathbb{C}$ such that $f(z) = exp^{g(z)}$. If $z_o \in G$ and $e^{w_o} = f(z_o)$ we may choose G such that $g(z_o) = w_o$.

THEOREM 3.3.3 (Phragmen Lindelöf Theorem)

Let G be a simply connected region and let f be analytic function on G. Suppose there is an analytic function $\varphi: G \to \mathbb{C}$ which never vanishes and is bounded on G. If M is a constant and $\partial_\infty G = A \cup B$ such that:

(a) For every a in A, $\lim_{z \to a} sup|f(z)| \leq M$

(b) For every b in B, and $\eta > 0$, $\lim_{z \to b} sup|f(z)|\,|\varphi(z)|^\eta \leq M$; then $|f(z)| \leq M \; for \; all \; z \; in \; G.$

Proof

Suppose $|\varphi(z)| \leq K$ for all z in G. Since G is simply connected there is an analytic branch of $\log \varphi(z)$ on G. Hence $g(z) = exp(\eta \log \varphi(z))$ an analytic branch of $\varphi(z)^\eta$ for $\eta > 0$ and $|g(z)| = |\varphi(z)|^\eta$. Now define F: $G \to \mathbb{C}$ by $F(z) = f(z)g(z)k^{-\eta}$; then F is analytic on G and $|F(z)| \leq |f(z)|$ (since $|\varphi(z)| \leq k$ for all z in G). Hence by conditions (a) and (b) on $\partial_\infty G$, F satisfies the hypothesis of Maximum Modulus Theorem ($Third\ version$) Thus $|F(z)| \leq \max(M, k^{-\eta}M)$ $for\ all\ z\ in\ G\ this\ gives$

$|f(z)| \leq |{}^{k}/_{\varphi(z)}|^\eta \max\ (M, k^{-\eta})$ $for\ all\ z\ in\ G\ and\ for\ all\ \eta > 0\ letting\ \eta \to 0^+$ gives that $|f(z)| \leq M$ for all z in G.

Corollary 3.3.4

Let $a \geq \frac{1}{2}$ and put

$$G = \left\{z: |argz| < \frac{\pi}{2a}\right\},$$

Suppose that f is analytic on G and there is a constant M such that $\lim_{z\to w} sup|f(z)| \leq M$ for all w in ∂G. If there are positive constants p and $b < a$ such that

$$|f(z)| \leq pexp(|z|^b)$$

$for\ all\ z\ with\ |z|\ sufficiently\ large\ ;then\ |f(z)| \leq M\ for\ all\ z\ in\ G.$

Proof

Suppose $b < c < a$ and put $\varphi(z) = exp(-z^c)$ for z in G. $If\ z = re^{i\theta}, |\theta| < \frac{\pi}{2a}$ then $Rez^c = r^c cosc\theta$. So for z in G.

$$|\varphi(z)| = \exp\,(-r^c cosc\theta)$$

$when\ z = re^{i\theta}$. Since $c < a,\ cosc\theta \geq \rho > 0$ for all z in G. This gives that φ is bounded on G. Also, if $\eta > 0$ and $z = re^{i\theta}$ is sufficiently large,

$$|f(z)||\varphi(z)^\eta| \leq pexp(r^b - \eta r^c cosc\theta)$$

$$\leq pexp(r^b - \eta r^c \rho)$$

But $r^b - \eta r^c \rho = r^c(r^{b-c} - \eta\rho)$. *Since* $b < c, r^{b-c} \to 0^+$ *as* $r \to \infty$ *so that*

$r^b - \eta r^c \rho \to -\infty$ *as* $r \to \infty$. Thus

$$\lim sup|f(z)||\varphi(z)|^\eta = 0$$

Hence, f and φ satisfy the hypothesis of the Phragmen-Lindelöf Theorem so that $|f(z)| \leq M$ for each z in G.

Remark 3.3.5

The size of the angle of the sector G is the only relevant fact in this corollary; its position is inconsequential. So if G is any sector of angle $\frac{\pi}{a}$ the conclusion remains valid.

Corollary 3.3.6

Let $a \geq \frac{1}{2}$,

$$G = \{z: |argz| < \frac{\pi}{2a}\},$$

and suppose that for every w *in* ∂G, $\lim_{z\to w} sup|f(z)| \leq M$. Moreover, assume that for every $\delta > 0$ there is a constant p (which may depend on δ) such that

$$|f(z)| \leq pexp(\delta|z|^a) \qquad (3.2.15.1)$$

for z *in* G *and* $|z|$ *is sufficiently large. Then* $|f(z)| \leq M$ *for all* z *in* G.

Proof

Let $\epsilon > 0$ be arbitrary. Define $F: G \to \mathbb{C}$ by $F(z) = f(z)exp(-\epsilon z^a)$ suppose $x > 0$ and δ *is* chosen with $0 < \delta < \epsilon$ then there is a constant p with $|F(z)| \leq pexp[(\delta - \epsilon)x^a]$ but then $|F(z)| \to 0$ *as* $x \to \infty$ in $\mathbb{R}$;so $M_1 = sup\{|F(x)|: 0 < x < \infty\} < \infty$.Define

$M_2 = max\{M_1, M\}$ and

$$H_+ = \{z \in G: 0 < argz < \frac{\pi}{2a}\}$$

$$H_- = \{z \in G: 0 > argz > \frac{-\pi}{2a}\}$$

Then $\lim_{z \to w} sup|f(z)| \leq M_2$ for all z in ∂H_+ and ∂H_- using the hypothesis (3.2.15.1), corollary(3.2.13) gives $|F(z)| \leq M_2$ for all z in $H_+$and H_- hence $|F(z)| \leq M_1$ for all z in G.

We claim $M_2 = M$. if $M_2 = M_1 > M$ then $|F|$ assume its maximum value in G at $some\ point\ x,\ 0 < x < \infty$ ($since\ |F(x)| \to 0\ as\ x \to \infty\ and \lim_{x \to 0} sup|f(x)| = \lim_{x \to 0} sup|F(x)| \leq M < M_1$). So F is constant by Maximum Modules Principle $M = M_1$ and $|F(z)| \leq M$ for all z in G; that is $|f(z)| \leq Mexp(\epsilon Rez^a)\ for\ all\ z\ in\ G$; as M is independent of ϵ, we can let $\epsilon \to 0$ and get $|f(z)| \leq M for\ all\ z\ in\ G$

Example 3.4

Let $G = \{ z: z \neq 0\ and\ |argz| < \frac{\pi}{2a}\}$ and let $f(z) = exp(z^a)$ for $z \in G$. Then $|f(z)| = exp(|z|)^a cosa\theta)$ where $\theta = argz$. So for z in ∂G, $|f(z)| = 1$; but $f(z)$ is unbounded in G.. In fact on any ray in G we have that $|f(z)| \to \infty$. This shows that the growth condition (3.2.15.1) is very delicate and cannot be improved.

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