ON THE LENGTH OF D-MODULE

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Abstract

This paper is concerned about on the length of D-modules,and the point of view that of modules over the Weyl Algebra.I have only tied to present the length of A_2-modules and its composition series by considering polynomial rings in two variables.

Notation

- K denotes a field of characteristic zero.
- $\mathbb{C}$ denotes a field of complex numbers.
- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{N}_0$ denotes the set of natural numbers with 0.
- $\mathbb{Z}$ is the set of integers.
- K[x,y]=k[x][y] denotes the ring of polynomials in two variables over a field k.
- $\mathbb{C}[x, y]$ is the ring of polynomials in two variables over $\mathbb{C}$.
- Differentiation (in the variables x,y) can be considered as a map

  $\partial x : K[x, y] \longrightarrow K[x, y]$.

  $\partial y : K[x, y] \longrightarrow K[x, y]$.
- $End_k(K[x, y])$ denotes the set of endomorphism from K[x,y] to K[x,y].
- $\ell(M)$ Denotes length of an R-module M.

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Introduction

Under the cryptic name of D-Module hides a module over a ring of differential operator. The major interest of such D-module is as an approach to the theory of linear partial differential operator equation.The first case of algebraic D-modules are modules over the Weyl algebra A_n over a field of characteristic 0.The history of the Weyl Algebra begins with the birth of quantum mechanics,in the year 1925.What Heisenberg originally were introduce quantum theoretical analogies of the Fourier series.

This paper is concerned about on the length of D-modules,and the point of view that of modules over the Weyl Algebra.I have only tied to present the length of A_2-modules and its composition series by considering polynomial rings in two variables.The work in this seminar report was partially supported by the idea of a S.C.Coutinho,from his book a primer of Algebra D-modules in preliminary part.

It the purpose of this paper to provide basic properties of Weyl algebra and introduce the number of important examples of modules over the Weyl algebra,and mainly focused on the composition series and its length.This project report is divided in to two parts.The first part is concerned with preliminary;definition and basic properties of Weyl algebra,the ideal structure of the Weyl algebra and module over Weyl algebra.We will discuss the idea structure of Weyl algebra and the degree of an operator like the degree of polynomial.Most importantly,it is shown that the Weyl algebra is a simple domain and only invertible elements of A_2 are non-zero constant.

The second part of this project is concerned with simple A_2,composition series and length of A_2 and collect a number of examples of modules over Weyl algebra.All examples we discuss here are the polynomial rings in two variables.It is also proved that in this paper every simple module is cyclic.

Chapter 1

PRELIMINARIES

1.1 The Weyl algebra and its basic properties

In this section we will define the Weyl Algebra and present its basic properties. The Weyl algebra is introduced as a ring of operators on a vector space of infinite dimensions.

1.1.1 Algebra

Definition 1.1.1. An algebra A over a field F is a non empty set A together with three operation. Addition(+),multiplication (.) and scalar multiplication. For which

i. A together with addition and scalar multiplication is a vector space over F.

ii. A is a ring under addition and multiplication.

iii. If $\alpha \in F$ and $a, b \in A$, then $\alpha(a.b) = (\alpha.a)b = a(\alpha.b)$.

Definition 1.1.2. Let R be a ring. If there exists $n \in N$ such that n a = 0 for all $a \in R$, then the smallest such positive integer is called the characteristics of R and denoted by char R. If no such positive integer exists, then we say R has characteristic zero (That is char R= 0).

Example 1.1.1. *1. Char Z = 0*

2. Char $Z_n = n$, where $n \in N$

Definition 1.1.3. Let R be a ring. A left module M over R (or a left R-module) is an additive abelian group, with a function $R \times M \longrightarrow M$ such that for all $r, s \in R$ and for all $a, b \in M$ it satisfies:

i. r (a + b) = r a + r b.

ii.(r s) a = r(s a).

iii. (r + s) a = r a + s a.

If R has identity element 1_R and

iv. $1_R.a = a$ for all $a \in M$

Example 1.1.2. *1. Let (A,+) be an abelian group and $(Z, +, \cdot)$ be the ring then define $f : Z \times A \longrightarrow A$ by $f(n, a) = na$. Hence A is Z-module.*

2. Let $(S, +, \cdot)$ be a ring and R be a subring of S .Then S is an R-module.

Define $f : R \times S \longrightarrow S$ by $f(r, s) = rs$ (ring multiplication).

3. Let $(R, +, \cdot)$ be a ring and I be an ideal of R. Then R/I is an R-module with the action defined by $f(r, a + I) = ra + I$.

Definition 1.1.4. Let A and B be R-modules.A mapping $f : A \longrightarrow B$ is called R-module homomorphism if satisfies

i $f(x + y) = f(x) + f(y)$

ii $f(ax) = af(x)$ for all $x, y \in A$, $a \in R$

Lemma 1.1.3. *let f be an R-module homomorphism of A in to B where A and B are R-module. Then*

i) f(0)=0

ii) f(-x)=-f(x) for all $x \in A$

iii) f(x-y)=f(x)-f(y) for all $x, y \in A$

Proof. (i) $f(0) = f(0+0) = f(0) + f(0)$ since f be an R-module homomorphism this implies $f(0) - f(0) = f(0) = 0$.

(ii) $f(-x) = f(0 + (-x)) = f(0) - f(x)$ since f be an R-module homomorphism then by (i) above we get $f(-x) = -f(x)$.

(iii) $f(x - y) = f(x + (-y)) = f(x) + f(-y) = f(x)f(y)$ since f be an R-module homomorphism and by (ii). □

Example 1.1.4.

(1.) let A and B are R-modules. Define $0 : A \longrightarrow B$ *by* $0(x) = 0$ *for all* $x \in A$. *Then 0 is an R-module homomorphism of A into B.*

(2.) let R be a commutative ring and A be R-module. Define $f_r : A \longrightarrow A$ *by* $f_r(x) = rx$ *for all* $x \in A$, $r \in R$. *Then* f_r *is an R-module homomorphism of A into A.*

Remark 1.1.1. If R is a division ring, then an R-module homomorphism is called a linear transformation.

i) f is an R-module monomorphism if f is injective.

ii) f is an R-module epimorphism if f is surjective.

iii)f is an R-module isomorphism if f is bijetive.

Theorem 1.1.5. *letf* $: A \longrightarrow B$ *be an R-module homomorphism. Then*

1)f is an R-module monomorphism if and only if kerf=0.

2) f is an R-module epimorphism if and only if Imf=B.

3) f is an R-module isomorphism if and only if f is bijective.

Proof. 1) since kerf $= \{x \in A : f(x) = 0\}$. Then let $x \in$ kerf then $f(x) = 0$

$\Longrightarrow f(x) = 0 = f(o)$ by lemma(1.1.3)

$\Longrightarrow x = 0$ since f is monomorphism by remark $0 \in kerf$

Therefore kerf=0.

Conversely suppose f(x)=f(y) for $x, y \in A \Longrightarrow f(x) - f(y) = 0 \Longrightarrow f(x - y) = 0$.

since f is R-module homomorphism

$\Longrightarrow x - y \in kerf \Longrightarrow x - y = 0$ by hypothesis $\Longrightarrow x = y$. Hence f is injective.

Therefore by remark f is R-module monomorphism.

2) Suppose f is an R-module epimorphism then f is surjective.

$\Longleftrightarrow$ Imf=B since Imf=$\{b \in B : f(a) = b$ for some $a \in x\}$.

3) Suppose f is an R-module isomorphism then by above remark f is bijective.

Conversely suppose f is bijective then by 1 and 2 f is monomorphism and epimorphism.

Hence f is isomorphism. □

Definition 1.1.5. Let R be a ring and A be an R-module. A non empty subset B of A is called a **submodule** of A if :

i) $x, y \in B$ implies that $x - y \in B$.

ii) $r \in R, x \in B$ implies that $rx \in B$.

Example 1.1.6. *1) Let A be an R-module and f :A $\longrightarrow$ B be an R-module homomorphism. Then*

a) Kerf is a submodule of A.

b) Imf is a submodule of B.

c) If C is any submodule of B then $f^{-1}(c) = \{a \in A : f(a) \in C\}$ is a submodule of A.

2) Let A be R-module and $x \in A$. Define Rx= $\{rx : r \in R\}$. Hence Rx is

submodule of R.

Theorem 1.1.7. *Let B be a submodule of an R-module A.Then*

i) A/B is an R-module.

ii) $\pi : A \longrightarrow A/B$ *given by* $\pi(a) = a + B$ *is an R-module homomorphism with kernel B.The map* π *is called the canonical epimorphism(or projection).*

Proof. i) A is an abelian group and B is subgroup of A.Then A/B is a Well defined abelian group .

A/B= $\{x + B : x \in A\}$. Define $f : RXA/B \longrightarrow A/B$ by f(r,a+B)=ra+B.

Well defined: Let a + B=b + B for some $a, b \in A$ then .We have to show that f(r,a + B) =f(r,b + B). Now a + B=b + B implies $a - b \in B$.

Thus $ra - b \in B$. Hence $ra - rb \in B$. It follows that ra +B=rb + B .

Hence f(r,a + B) =f(r,b + B). Thus f is well defined.

a) f(r,a + B +b + B)=f(r,a + b + B)=r(a + b)+B=ra+ rb+B

= ra+B+rb+B=f(r,a + B) +f(r,b+B).

b) f(r+s,a + B)=(r+s)a+B=ra+B+sa+B

=f(r,a + B)+ f(s,a + B).

c) f(rs,a + B)=(rs)a+B= rf(sa)+B=rf(s,a)+ B)

= r(f(s,a+B)).

d) f(1,a+B)=1a+B=a+B.

Therefore A/B is R-module.

ii) $\pi : A \longrightarrow A/B$ given by $\pi(a) = a + B$.

We have to show that it is an R-module homomorphism with kernel B.

$$\pi(a + b) = a + b + B = A + B + b + B = \pi(a) + (b).$$

$\pi(ra) = ra + B = r(a + B) = r(\pi(a))$.

Thus π is an R-module homomorphism.

Let $a + B \in A/B$ then $\pi(a) = a + B$ for some $a \in A$. Hence π onto.

Ker $\pi = \{a \in A : \pi(a) = B\} = \{a \in A : a+B = B\} = \{a \in A : a \in B\} = A \cap B = B$.

Thus Kerπ =B. □

Definition 1.1.6. The R-module A/B is called quotient of A by B.

Theorem 1.1.8. *Let B and C be submodules of a module A over a ring R.There is an R-module isomorphism, $B + C \cong B/B \cap C$.*

Definition 1.1.7. Let R be commutative ring with unity. An **R-algebra** (or algebra over R) is a ring A such that:

i. A is a left R-module with unity.

ii. r(ab) = (ra) b=a(rb) for all $r \in R$, for all $a, b \in A$.

Theorem 1.1.9. *For any vector spaceV_k, $Hom_k(V, V)$ is an algebra over K.*

Proof. Clearly$Hom_k(V, V)$ is a vector space.

For any $T_1, T_2 \in Hom_k(V, V)$,the composition $T_1T_2 : V \longrightarrow V$ is such that for any $v_1v_2 \in V$, $\alpha, \beta \in K$,

$$\begin{aligned}(T_1T_2)(\alpha v_1 + \beta v_2) &= T_1[T_2(\alpha v_1 + \beta v_2)] \\ &= T_1[\alpha T_2)(v_1) + \beta T_2(v_2)] \\ &= \alpha T_1T_2[(v_1)] + \beta T_1T_2[(v_2)] \\ &= \alpha[(T_1T_2)(v_1)] + \beta[(T_1T_2)(v_2)]\end{aligned}$$

Therefore T_1T_2 is a linear transformation on V. So that $T_1T_2 \in Hom_K(V, V)$. Now for $T_1, T_2, T_3 \in Hom_K(V, V), v \in V, and\ \alpha \in K$,

$$\begin{aligned}[T_1(T_2+T_3)](v) &= T_1[(T_2+T3)(v)] \\ &= T_1[(T_2(v)+T_3)(v)] \\ &= (T_1T_2)(v)+(T_1T_3)(v) \\ &= [(T_1T_2)+(T_1T3)](v)\end{aligned}$$

Therefore $T_1(T_2+T_3) = T_1T_2 + T_1T_3$, Similarly $(T_1+T_3)T_1 = T_2T_1 + T_3T_1$. Since the resultant composition is an associative operation among all mapping from $V \longrightarrow V$. The same holds for all members of $Hom_k(V,V)$.

Therefore $(Hom_k(V,V),+,o)$ is a ring. Now

$$\begin{aligned}[T_1(\alpha T_2)](v) &= T1[(\alpha T_2)(v)] \\ &= \alpha(T_1T_2)(v) \\ &= (\alpha T_1)T_2)(v). \\ HenceT_1(\alpha T_2) &= (\alpha T_1)T_2 \\ &= \alpha(T_1T_2).\end{aligned}$$

Therefore $Hom_k(V,V)$ is an algebra over K. □

Example 1.1.10. *1.Let $f : R \longrightarrow M$ is a ring homomorphism. Define $a.x = f(a).x$, for $a \in R$, $x \in M$, then M is an R-algebra.*

Proof. Let$a, b \in R$ and$x, y \in M$, then

$$\begin{aligned}(i)a(x+y) &= f(a)(x+y) \\ &= f(a)x + f(a)y \\ &= ax + ay\end{aligned}$$

$$Therefore \quad a(x+y) = ax + ay.$$

$$(ii)(a+b)x \quad = f(a+b)x$$
$$= (f(a)+f(b))x \quad (since \quad f \quad is \quad homomorphism).$$
$$= f(a)x + f(b)x$$
$$= ax + by$$

$Therefore \quad (a+b)x = ax + by$

$$(iii) \quad (ab)x \quad = f(ab)x$$
$$= (f(a)f(b))x$$
$$= f(a)(f(b))x$$
$$= a(bx$$

$Therefore \quad (ab)x = a(bx)$

(iv) $1.x = x$ for all $x \in M$. Hence (M, +) is a unitary (left) R-module.

(v) $\alpha(xy) = f(\alpha)(xy) = (f(\alpha)x)y = (\alpha x)y$. Therefore $\alpha(xy) = (\alpha x)y$.

$$And \quad \alpha(xy) \quad = f(\alpha)(xy)$$
$$= (f(\alpha)x)y$$
$$= (xf(\alpha))y \quad (since \quad M \quad is \quad acommutativering).$$
$$= (x\alpha)y$$
$$= (\alpha x)y)$$
$$= \alpha(xy)$$

Therefore $\alpha(xy) = \alpha(xy)$. Therefore $\alpha(xy) = (\alpha x)y = x(\alpha y)$. Therefore M is an R-algebra. □

2 . If K is a commutative ring with identity, then the polynomial ring K[x,y] is a K-algebra with respective to K-module structure given in the usual way.

Definition 1.1.8. A **subalgebra** of an algebra A is a subset S of an algebra that is also has the structure of an algebra of the same type when the algebraic operations are restricted to S.Or it is a subset of algebra, closed under all its operations and carrying the induced operations.

Definition 1.1.9. Let $End_k(K[x,y])$ be the set of endomorphisms of K[x,y] over K. The algebra operation in the endomorphism ring addition and composition of operators are defined as:

$$(i)\Psi_1+\Psi_2 : K[x,y] \longrightarrow K[x,y] by (\Psi_1+\Psi_2)(f) = \Psi_1(f)+\Psi_2(f) \quad (def \quad 1.1.9.1)$$

and

$$(ii)(\Psi_1 o \Psi_2)(f) = \Psi_1\Psi_2 : K[x,y] \longrightarrow K[x,y] by (\Psi_1\Psi_2(f) = \Psi_1(\Psi_2(f)) \quad (def \quad 1.1.9.2)$$

where$f \in K[x,y], \Psi, \Psi_2 \in End_k(K[x,y])$, + is addition of endomorphism and o is the composition of endomorphism.

Theorem 1.1.11. *$(End_k(K[x,y]),+,o)$ is a ring.*

Proof. 1. Let$f_1, f_2 \in K[x,y], \Psi_1, \Psi_2 \in End_k(K[x,y])$ and $\alpha \in K$.

$$\begin{aligned} Then \quad (\Psi_1+\Psi_2)(f_1+f_2) &= \Psi_1(f_1+f_2)+\Psi_2(f_1+f_2) \quad (by \quad def \quad 1.1.9.1) \\ &= \Psi_1(f_1)+\Psi_1(f_2)+\Psi_1(f_1)+\Psi_2(f_2) \\ &= \Psi_1(f_1)+\Psi_2(f_1)+\Psi_1(f_2)+\Psi_2(f_2). \\ &= (\Psi_1+\Psi_2)(f_1)+(\Psi_1+\Psi_2)(f_2) \quad (by \quad def \quad 1.1.9.1) \end{aligned}$$

Therefore $(\Psi_1 + \Psi_2)(f_1 + f_2) = (\Psi_1 + \Psi_2)(f_1) + (\Psi_1 + \Psi_2)(f_2)$.and

$$\begin{aligned}(\Psi_1 + \Psi_2)(\alpha f) &= (\Psi_1)(\alpha f) + \Psi_2(\alpha f) \quad (by \quad def \quad 1.1.9.1)\\ &= \alpha(\Psi_1)(f) + \alpha(\Psi_2)(f)\\ &= \alpha\Psi_1(f) + \Psi_2(f)\\ &= \alpha(\Psi_1 + \Psi_2)(f)\end{aligned}$$

Therefore $(\Psi_1 + \Psi_2)(\alpha f) = \alpha(\Psi_1 + \Psi_2)(f)$.

Therefore $(\Psi_1 + \Psi_2) \in End_k(K[x, y])$ And

$$\begin{aligned}\Psi_1\Psi_2(f_1 + f_2) &= \Psi_1(\Psi_2(f_1 + f2)) \quad (by \quad def \quad 1.1.9.2)\\ &= \Psi_1(\Psi_2(f_1) + \Psi_2(f_2)) \quad (by \quad def \quad 1.1.9.1)\\ &= \Psi_1(\Psi_2(f_1)) + \Psi_1(\Psi_2(f_2))\\ &= (\Psi_1\Psi_2)(f_1)) + (\Psi_1\Psi_2)(f_2)\end{aligned}$$

Therefore $\Psi_1\Psi_2(f_1 + f_2) = (\Psi_1\Psi_2)(f_1)) + (\Psi_1\Psi_2)(f_2)$.And

$$\begin{aligned}\Psi_1\Psi_2(\alpha(f_1 + f_2)) &= \Psi_1(\Psi_2(\alpha(f_1 + f2)))\\ &= \Psi_1(\alpha\Psi_2(f_1 + f_2 \quad (since \quad \Psi_2 \quad is \quad ring \quad homomorphism)\\ &= \alpha\Psi_1(\Psi_2(f_1 + f_2)) \quad (since \quad \Psi_1 \quad is \quad ring \quad homomorphism)\end{aligned}$$

Therefore, $\psi_1\psi_2 \in End_K(K[x, y])$.

2) For any $\psi_1, \psi_2, \psi_3 \in End_K(K[x,y])$ and for all $f \in K[x,y]$.

$$\begin{aligned}((\psi_1+\psi_2)+\psi_3)(f) &= ((\psi_1+\psi_2)(f))+\psi_3(f) \quad (by \quad def \quad 1.1.9).\\ &= (\psi_1(f)+\psi_2(f))+\psi_3(f) \quad (by \quad def \quad 1.1.9.1)\\ &= \psi_1(f)+(\psi_2(f))+\psi_3(f)) \quad (+ is \quad associative \quad in \quad K[x,y])\\ &= \psi_1(f)+((\psi_2+\psi_3)(f)) \quad (by \quad def \quad 1.1.9.1)\\ &= [\psi_1+(\psi_2+\psi_3)](f) \quad (by \quad def \quad 1.1.9.1)\end{aligned}$$

Therefore $(\psi_1+\psi_2)+\psi_3 = \psi_1+(\psi_2+\psi_3)$.

3) 0: K[x,y] $\longrightarrow$ K[x,y] defined by 0(f) = 0 (the zero operator in K[x,y]), is the zero element in $End_k(K[x,y])$.

So that 0 is an identity element for + on $End_K(K[x,y])$.

4) For any $\psi \in End_k(K[x,y])$ the map ψ: K[x,y] $\longrightarrow$ K[x,y] given by $(-\psi)(f) = -\psi(f)$, for all $f \in K[x,y]$.

Since $(\psi+(-\psi))(f) = \psi(f)+(-\psi)(f) = \psi(f)-\psi(f) = 0$ and $((-\psi)+(\psi))(f) = -\psi(f)+\psi(f) = 0$. Hence $-\psi$ is an additive inverse of ψ.

Therefore every element in End_k (K[x,y]) has an inverse.

5) For any $\psi_1, \psi_2 \in End_k(K[x,y])$ and $f \in K[x,y]$.

$$\begin{aligned}(\psi_1+\psi_2)(f) &= \psi_1(f)+\psi_2(f) \quad (by \quad def \quad 1.1.9.1)\\ &= \psi_2(f)+\psi_1(f) \quad (since + is \quad commutative \quad on \quad K[x,y])\\ &= (\psi_2+\psi_1)(f) \quad (by \quad def \quad 1.1.9.1)\end{aligned}$$

$\psi_1+\psi_2 = \psi_2+\psi_1$

Therefore + is commutative over $End_k(K[x,y])$.

Therefore $(End_k(K[x,y]),+)$ is abelian group.

6) Let $\psi,\psi_1,\psi_2 \in End_k(K[x,y])$ then

$$\begin{aligned}(\psi o(\psi_1+\psi_2))(f) &= \psi(\psi_1+\psi_2)(f)) \quad (by \quad def \quad 1.1.9.2)\\ &= \psi(\psi_1(f)+\psi_2)(f)) \quad (by \quad def \quad 1.1.9.1)\\ &= \psi(\psi_1(f))+\psi(\psi_2)(f))\\ &= (\psi o\psi_1)(f)+(\psi o\psi_2))(f)\end{aligned}$$

Therefore $(\psi o(\psi_1+\psi_2))(f)=\psi o\psi_1+\psi o\psi_2$. Similarly $(\psi_1+\psi_2)o\psi=\psi_1 o\psi+\psi_2 o\psi$. Therefore "o" is distributive over addition.

$$\begin{aligned}7)((\psi_1 o\psi_2)o\psi_3)(f) &= (\psi_1 o\psi_2)(\psi_3((f)) \quad (def \quad 1.1.9.2)\\ &= \psi_1(\psi_2(\psi_3(f))) \quad (def \quad 1.1.9.1)\\ &= \psi_1(\psi_2 o\psi_3(f))\\ &= (\psi_1 o(\psi_2\psi_3(f))\end{aligned}$$

Therefore $((\psi_1 o\psi_2)o\psi_3)(f)=(\psi_1 o(\psi_2\psi_3(f))$

Therefore o is associative on $End_k(K[x,y])$. Hence $(End_k(K[x,y]),+,o)$ is a ring. □

Definition 1.1.10. Let $\hat{x},\hat{y}$ be the operators of K[x,y] which are defined on a polynomial $f \in K[x,y]$.

$\hat{x}$: K[x,y] $\longrightarrow$ K[x,y] by the formula $\hat{x}(f)=xf$.

$\hat{y}$: K[x,y] $\longrightarrow K[x,y]$ by the formula $\hat{y}(f)=yf$.

Similarly ∂x : K[x,y] $\longrightarrow K[x,y]$ and ∂y: K[x,y] $\longrightarrow$ K[x,y] are operators of K[x,y] defined on the polynomial $f \in K[x,y]$ by the

Formula $\partial x(f)=\dfrac{\partial f}{\partial_x}$ and $\partial y(f)=\dfrac{\partial f}{\partial_y}$

Lemma 1.1.12. *The operators $\hat{x}, \hat{y}, \partial x$ and ∂y are linear operators of K[x,y].*

Proof. Let $f, f_1, f_2 \in K[x, y]$ and $\alpha, \in K$. Then:

1) *i.* $\hat{x}(f_1 + f_2) = x(f_1 + f_2) = xf_1 + xf_2 = \hat{x}(f_1) + \hat{x}(f_2)$.

$\hat{y}(f_1 + f_2) = y(f_1 + f_2) = yf_1 + yf_2 = \hat{y}(f_1) + \hat{y}(f_2)$.

ii. $\hat{x}(\alpha f) = x(\alpha f) = \alpha xf = \alpha \hat{x} f$

$\hat{y}(\alpha f) = y(\alpha f) = \alpha yf = \alpha \hat{y} f$

Therefore $\hat{x}, \hat{y}$ are linear operator.

2) *i.* $\partial x(f_1 + f_2) = \dfrac{\partial(f_1 + f_2)}{\partial_x} = \dfrac{\partial f_1}{\partial_x} + \dfrac{\partial f_2}{\partial_x} = \partial x f_1 + \partial x f_2$

$\partial y(f_1 + f_2) = \dfrac{\partial(f_1 + f_2)}{\partial_y} = \dfrac{\partial f_1}{\partial_y} + \dfrac{\partial f_2}{\partial_y} = \partial y f_1 + \partial y f_2$

ii. $\partial x(\alpha f) = \dfrac{\partial(\alpha f)}{\partial_x} = \alpha(\dfrac{\partial f}{\partial_x}) = \alpha(\partial x f)$

$\partial y(\alpha f) = \dfrac{\partial(\alpha f)}{\partial_y} = \alpha(\dfrac{\partial f}{\partial_y}) = \alpha(\partial y f)$

Therefore ∂x and ∂y are linear operator. □

Definition 1.1.11. Let $n \geq 1$ be an integer. The n^{th} **Weyl algebra** A_n is the K-sub algebra of $End_k(K[X])$ generated by the operators $\hat{x}_1, \hat{x}_2, \hat{x}_3 ... \hat{x}_n, \partial_1, \partial_2, \partial_3 ... \partial_n$. Take n=2 then.The 2^{nd} **weyl algebra** A_2 is the k-sub algebra of $End_k(K[x, y])$ generated by the operators $\hat{x}, \hat{y}, \partial x, \partial y$.

Note:

- The elements of A_2 are linear combinations over K of monomials in the generators $\hat{x}, \hat{y}, \partial x, \partial y$.
- $\partial^j(f)$ is the j^{th} derivative of f with respect to x,y where $f \in K[x, y]$.

Example 1.1.13. *1.* $x\partial_x + y\partial_y^2$ *is an element of* A_2.

2. $x^2\partial_x^3 + x\partial_y$ *is an element of* A_2

Lemma 1.1.14. *The Weyl algebra A_2 is not commutative .*

Proof. Consider the operator $\partial x\hat{x}$ and apply it to a polynomial $f \in K[x, y]$.

$$\begin{aligned} Now \quad \partial x\hat{x}(f) &= \partial x(\hat{x}.f) \quad (by \quad def 1.1.9.2) \\ &= \partial x(x.f) \quad (by \quad def \quad 1.1.10) \\ &= f\partial x(x) + x\partial x(f) \quad (Product \quad rule \quad of \quad differentiation) \\ &= f(\frac{\partial x}{\partial_x}) + x\partial(f) \\ &= f.1 + x\partial x(f) \quad (Since \quad (\frac{\partial x}{\partial_x}) = 1) \\ &= (1 + x\partial x)f \end{aligned}$$

Therefore $\partial x\hat{x}(f) = (1 + x\partial x)f \quad (*)$ (where 1 stands for identity operator).From $(*)$ we have $\partial x\hat{x} = 1 + \hat{x}.\partial x.$ similarly for $\partial y\hat{y}$.

Therefore the Weyl algebra A_2 is not commutative .

Remark 1.1.2. If $p, q \in A_2$, then their commutator is the operator [p, q] = p q q p.

Note:

- $[\partial x, \hat{x}] = \partial x\hat{x} - \hat{x}.\partial x. = 1$
- $[\partial y, \hat{y}] = \partial y\hat{y} - \hat{y}.\partial y. = 1$
- $[\partial x, \hat{y}] = \partial x\hat{y} - \hat{y}\partial x = 0$
- $[\partial y, \hat{x}] = \partial y\hat{x} - \hat{x}\partial y = 0$
- $[\hat{x}, \hat{y}] = \hat{x}\hat{y} - \hat{x}\hat{y} = 0$
- $[\partial x, \partial y] = \partial x\partial y - \partial y\partial x = 0$ □

1.1.2 Canonical form

In this section we construct a basis for Weyl algebra as K-vector space. This basis is known as the **canonical basis**. If an element of A_2 is written as a linear combination of this basis, then we say that it is in **canonical form**. Of course, to compare two elements in canonical form, it is enough to compare the coefficients of their linear combinations. It is easier to describe the canonical basis if we use a multi-index notation. The multi-index α is an element of $\mathbb{N}^2, \alpha = (\alpha_1, \alpha_2)$ and X = (x,y) then $X^\alpha = x^{\alpha_1} y^{\alpha_2}$ (the monomial).

Definition 1.1.12. The degree of X^α is the length $|\alpha|$ of the multi-index α, $|\alpha| = \alpha_1 + \alpha_2$.

Example 1.1.15. *The degree of* x^2y^5 *is 2 +5=7 .*

Note:

- A pair (α, β) of a multi-index in $\mathbb{N}^2$ is itself a multi-index in $\mathbb{N}^4$, so it makes a sense to talk of its length.
- For a multi-index $\beta \in \mathbb{N}^2$ we define $\beta! = \beta_1!\beta_2!$.

Convention:

- We will write x and y for both variable and the corresponding operator.
- For the generators of A_2 we write simply x,y and $\partial x, \partial y$.

Lemma 1.1.16. *Let* $\delta, \beta \in \mathbb{N}^2$ *and assume that* $|\delta| \leq |\beta|$ *then* $\partial^\beta(X^\delta) = \beta!$ *If* $\delta = \beta$ *and 0 if* $\delta < \beta$.

Proof. **Case I**. Suppose $\delta = \beta$.

$$\begin{aligned} Then \quad \partial^{\beta}(X^{\delta}) &= \partial^{\beta}(X^{\delta}) = (\partial_x^{\beta_1}\partial_y^{\beta_2})(x^{\beta_1}y^{\beta_2}) \\ &= \partial_x^{\beta_1}(\partial_y^{\beta_2}x^{\beta_1})y^{\beta_2} \\ &= (\partial_x^{\beta_1}x^{\beta_1})(\partial_y^{\beta_2}y^{\beta_2}) = \beta_1!\beta_2! = \beta! \end{aligned}$$

Therefore $\partial^{\beta}(X^{\delta}) = \beta_1!\beta_2! = \beta!$

Case II: Suppose $\delta \neq \beta$ this implies $|\delta| < |\beta|$, now assume $\delta_1 < \beta_1$.Then

$$\begin{aligned} \partial^{\beta}(X^{\delta}) &= (\partial_x^{\beta_1}x^{\beta_1})(\partial_y^{\beta_2}y^{\beta_2}) \\ &= 0.(\partial_y^{\beta_2}y^{\beta_2}) \quad since \quad \delta_1 < \beta_1 \\ &= 0 \end{aligned}$$

Hence $\partial^{\beta}(X^{\delta}) = 0$.Similarly for the case $\delta_2 < \beta_2$.Therefore $\partial^{\beta}(X^{\delta}) = 0$. □

Proposition 1.1.17. *The set* $B = \{X^{\alpha}\partial^{\beta} : \alpha, \beta \in \mathbb{N}^2, X^{\alpha} = x^{\alpha_1}.y^{\alpha_2}, \partial^{\beta} = \partial_x^{\beta_1}.\partial_y^{\beta_2}\}$ *is a basis of* A_2 *as a vector space over K.*

Proof. Apply the formula for the derivative of polynomials in terms of multi- indices.

i) Let $c_{\alpha\beta} \in K$ for $\alpha, \beta \in \mathbb{N}^2$ such that $\Sigma c_{\alpha\beta}X^{\alpha}\partial^{\beta} = 0$, Where 0 is the zero operator. It is enough to show for the case f is a monomial, since a polynomial is the linear combination of monomials. Let $f \in K[x, y]$ be a monomial of degree β. That is $f = X^{\beta} = x^{\beta_1}y^{\beta_2}$ where $\beta! = \beta_1!\beta_2!$ and X = (x, y).

Now, $\Sigma c_{\alpha\beta}X^{\alpha}\partial(f) = \Sigma c_{\alpha\beta}X^{\alpha}\partial(X^{\beta}) = 0(f) = 0$.

This implies $\beta!\Sigma c_{\alpha\beta}X^{\alpha} = 0$ (by Lemma 1.1.15)

This implies $\Sigma c_{\alpha\beta}X^{\alpha} = 0$ (since $\beta! \neq 0$). Since $X^{\alpha} \neq 0$, we have $c_{\alpha\beta} = 0$ for all $\alpha, \beta \in \mathbb{N}^2$.

Hence B is Linearly Independent.

ii) Let $D \in A_2$, then D $=\Sigma c_{\alpha\beta} X^\alpha \partial^\beta$ is a finite linear combination, $c_{\alpha\beta} \in K$. We need to show that if some $c_{\alpha\beta}(\in K) \neq 0$ then $D \neq 0$. But D is a linear operator of K[x,y].

Case I : Suppose D = 0. Then $c_{\alpha\beta} = 0 \ \forall \alpha, \beta \in N^2$. Since B is linearly independent. But $c_{\alpha\beta} \neq 0$. This implies $D \neq 0$.

Case II: Suppose $D \neq 0$. Then there exists non-zero $f \in K[x, y]$ such that $D(f) \neq 0$. (Otherwise D (f) = 0 for all $f \in K[x, y]$. This implies D = 0.) Choose δ be the index such that $c_{\alpha\delta} \neq 0$ for some index α, but $c_{\alpha\beta} = 0$ for all indices β such that $|\beta| < |\delta|$. Now D $(X^\delta) = \Sigma c_{\alpha\delta} X^\alpha \partial^\delta (X^\delta) = \delta! \Sigma c_{\alpha\delta} X^\alpha$ (by Lemma 1.1.15). This implies D $(X^\delta) \neq 0$, since at least one of the coefficients $c_{\alpha\delta} \neq 0$ by the choice of δ. Hence $D \neq 0$. This implies D $=\Sigma c_{\alpha\beta} X^\alpha \partial^\beta$ for some $c_{\alpha\delta} \neq 0$. This is a contradiction with our hypothesis.

Therefore B is a basis of A_2. □

Definition 1.1.13. An element D of A_2 is written in the linear combination of the basis $\{X^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^2, X^\alpha = x^{\alpha_1}.y^{\alpha_2}, \partial^\beta = \partial_x^{\beta_1}.\partial_y^{\beta_2}\}$, then we say that it is in **canonical form**.

Example 1.1.18. *Find the canonical form of* $\partial_y^2 x + \partial_x x$ *in* A_2.

Solution:

$$\begin{aligned} \partial_y^2 x + \partial_x x &= \partial_y \partial_y x + \partial_x x \\ &= \partial_y x \partial_y + (1 + x\partial_x) \quad (Since \partial_x x = 1 + x\partial_x) \\ &= (x\partial_y)\partial_y + 1 + x\partial_x \\ &= 1 + x\partial_x + x\partial_y^2 \end{aligned}$$

Hence $1 + x\partial_x + x\partial_y^2$ *is the canonical form of* $\partial_y^2 x + \partial_x x$ *in* A_2.

Proposition 1.1.19. *Let R be a K-algebra and $\lambda_1, \lambda_2 \in K$. Then the commutator of two elements of R satisfies*

$[a, \lambda_1 b_1 + \lambda_2 b_2] = \lambda_1[a, b_1] + \lambda_2[a, b_2]$ *for* $a, b \in R$.

Proof.

$$\begin{aligned}
[a, \lambda_1 b_1 + \lambda_2 b_2] &= a(\lambda_1 b_1 + \lambda_2 b_2) - (\lambda_1 b_1 + \lambda_2 b_2)a \\
&= a\lambda_1 b_1 + a\lambda_2 b_2 - \lambda_1 b_1 a - \lambda_2 b_2 a \\
&= \lambda_1 a b_1 + \lambda_2 a b_2 - \lambda_1 b_1 a - \lambda_2 b_2 a \\
&= \lambda_1 a b_1 - \lambda_1 b_1 a + \lambda_2 a b_2 - \lambda_2 b_2 a \\
&= \lambda_1(a b_1 - b_1 a) + \lambda_2(a b_2 - b_2 a) \\
&= \lambda_1[a, b_1] + \lambda_2[a, b_2]
\end{aligned}$$

Therefore $[a, \lambda_1 b_1 + \lambda_2 b_2] = \lambda_1[a, b_1] + \lambda_2[a, b_2]$ □

1.1.3 Ideal structure of the Weyl algebra

In this section we will discuss the ideal structure of the Weyl algebra, the degree of an operator and use it to show that the Weyl algebra is a domain whose only proper two sided ideal is zero.

1.1.3.1 The degree of an operator

The degree of an operator of A_2, to be introduced in this section, behaves, in many ways, like the degree of a polynomial. The differences are accounted for by the non-commutatively of A_2.

Definition 1.1.14. The degree of $P \in A_2$, P $= \Sigma c_{\alpha\beta} X^\alpha \partial^\beta$ is

$\deg(\mathrm{P}) = \max\{|\alpha| +, |\beta| \text{ where } c_{\alpha\beta} \neq 0\}$.

Note:

As a degree of zero polynomial we use the Convention that the zero operator has degree $-\infty$.

Example 1.1.20. *1. The degree of* $2x\partial_y + xy^3\partial_x\partial_y$ *is 6.*

2. The degree of $2x\partial_y + x^2y\partial_x\partial_y$ *is 5.*

3. The degree of $x^2y\partial_x^3 5y^3\partial_x^2\partial_y + \partial_x^2\partial y^2 1$ *is 6.*

Theorem 1.1.21. *For* $D, D' \in A_2$.

1. Deg (D + D') ≤ max $\{deg(D), deg(D')\}$

2. Deg (DD') = deg (D) + deg (D')

3. Deg [D, D'] ≤ deg (D) + deg (D') - 2.

Proof. 1.**Case I:** Either D or D' is zero. WLOG assume D is zero. Then D+D'= D'. This implies deg (D+D') = deg (D') = max $\{-\infty, deg(D')\} \leq max\{deg(D), deg(D')\}$. Since $-\infty \leq x$, for all $x \in \mathbb{C}$.

Therefore, deg (D + D') ≤ max $\{deg(D), deg(D')\}$.

Case II: Either D or D' is non zero constant. WLOG assume that D' is a non-zero constant.

(i) For D ≠ 0, deg (D+D') = deg(D)= max $\{deg(D), 0\}$ = max $\{deg(D), deg(D')\}$

Therefore deg (D+D') =max $\{deg(D), deg(D')\}$

(ii) For D = 0 ,deg (D+D') = deg(D') = max $\{deg(D), deg(D')\}$

Case III: Let $D, D' \in A_2$, we can write in canonical form. Then so is D+D'. Hence deg (D+D') ≤ max $\{deg(D), deg(D')\}$. Therefore deg (D+D') ≤ max $\{deg(D), deg(D')\}$

We prove 2 and 3 by induction on deg (D) + deg (D') = k. □

Example 1.1.22. *(1) Let D =*$y\partial_x$ *and D' =* $x^2\partial_x$,

Then deg (D) = 2 and deg (D') = 3.

i) $D + D' = y\partial_x + x^2\partial_x$, . *This implies Deg (D + D') = 3*

Hence deg (D + D') =max $\{deg(D), deg(D')\}$

$$\begin{aligned} ii) \quad DD' &= y\partial_x x^2\partial_x \\ &= y(\partial_x x)x\partial_x \\ &= y(1 + x\partial_x)x\partial_x \\ &= y + yx\partial_x x\partial_x \\ &= y + yx(1 + x\partial_x)\partial_x \\ &= y + yx\partial_x + yx^2\partial_x^2 \end{aligned}$$

Deg (DD') = 5 = 2 + 3. Therefore Deg (DD') = deg (D) + deg (D')

$$\begin{aligned} (iii) \quad [D, D'] &= DD' - D'D \\ &= y\partial_x x^2\partial_x - x^2\partial_x y\partial_x \\ &= y(\partial_x x)x\partial_x - x^2\partial_x y\partial_x \\ &= y(1 + x\partial_x)x\partial_x - x^2\partial_x y\partial_x \\ &= yx\partial_x + yx\partial_x x\partial_x - x^2 y\partial_x\partial_x \\ &= yx\partial_x + yx(1 + x\partial_x)\partial_x - x^2 y\partial_x^2 \\ &= yx\partial_x + yx\partial_x + yx^2\partial_x^2 - x^2 y\partial_x^2 \\ &= 2yx\partial_x \end{aligned}$$

Deg [D, D'] = $deg(2yx\partial_x)$ *= 3 = deg (D) + deg (D') 2.*

Corollary 1.1.23. *The algebra* A_2 *is a domain.*

Proof. Let $P, Q \in A_2$ with PQ = 0 (the zero operator). Then we want to show P = 0 or Q = 0. Suppose not! That is assume that $P \neq 0$ and $Q \neq 0$. Let deg (P) = n and deg (Q) = m for some $n, m \geq 0$. This implies deg (P) + deg (Q)=n + m ≥ 0. This implies deg (PQ)= $s \geq 0$ for some s=m + n. This implies $PQ \neq 0$. This contradicts to our hypothesis PQ = 0. Therefore P = 0 or Q = 0. Therefore A_2 is a domain. □

Corollary 1.1.24. *The only invertible elements of A_2 are non zero constants.*

Proof. Let $P(\neq 0) \in A_2$ has an inverse. Then there exists $P^{-1} \in A_2$ such that $PP^{-1} = P^{-1}P = 1$. This implies deg $(PP^{-1}) = deg(1) = 0$.
This implies $Deg(PP^{-1}) = deg(P) + deg(P^{-1}) = 0$. This implies
$deg(P) = deg(P^{-1}) = 0$. Since the degree of non- zero operators is always non negative. Therefore P is constant. □

Definition 1.1.15. A **simple ring** is a non-zero ring that has no two-sided ideal other than the zero ideal and itself.

Theorem 1.1.25. *The Weyl algebra A_2 is simple.*

Proof. Let I be a non-zero two sided ideal of A_2. And choose an element $P(\neq 0)$ of I which has the smallest degree in I.
Case I: If deg (P) = 0 (that is p is constant). This implies P is a unit in I.
(by cor. 1.2.24). Then there exists $P^{-1} \in I$ such that $P^{-1}P = PP^{-1} = 1 \in I$, since I is two sided ideal.
Now let $q \in A_2, 1 \in I$, then q.1 = 1.q =$q \in I$. This implies $I =< 1 >$. Hence $A_2 \in I$, but $I \in A_2$.Therefore $I = A_2$.
Case II: Suppose that (α, β) is a multi- index of length k. Let P=$\Sigma c_{\alpha\beta} X^{\alpha} \partial^{\beta}$, where $0 \neq c_{\alpha\beta} \in K$ for $\alpha, \beta \in N^2$. This implies deg (p)= $\alpha + \beta > 0$.

Now, $x \in A_2$, $P \in I$. This implies $xP, Px \in I$ (Since I is two sided ideal ofA_2). If $\beta \neq 0$ then $xP - Px \in I$, otherwise β = (0,0). Then xP Px= 0 $\in I$, this is the contradiction to the fact $I \neq 0$. But from Theorem 1.2.21 we have,

deg (xP Px) = deg [x, P] $\leq$ deg(x) + deg (P) 2 = 1 + $\alpha + \beta$ -2 = $\alpha + \beta$- 1. This contradicts to the choice of P which is p has the smallest degree in I. Therefore $\beta = 0$. Now since $\alpha + \beta > 0$ and $\beta = 0$ we have $\alpha \neq 0$. This implies $0 \neq xPPx \neq [x, p] \in I$. Moreover deg [x, p] $\leq$ deg(x) + deg (P) - 2 = 1+ $\alpha + \beta$ - 2 = $\alpha + \beta - 1$ which is also a contradiction to the choice of P which is P has the smallest degree in I. Therefore $\alpha = 0$. This implies I contain only the constant element(s). Hence A_2 does not contain the two sided proper ideals.

Therefore A_2 is simple. From this we conclude that the kernel of an endomorphism of A_2 is two sided ideal. □

Note: The only two-sided ideals in A_2 are (0) and A_2.

Corollary 1.1.26. *Every non zero endomorphism of A_2 is injective.*

Proof. Let $\psi : A_2 \longrightarrow A_2$ be an endomorphism. We know that the kerψ is sub ring of ψ. But we know that the only subring of A_2 other than itself is 0.

Therefore ker$\psi = 0$. Therefore ψ is injective. □

1.1.4 Modules over the Weyl Algebra

In this section we introduce the number of important examples of modules over the Weyl algebra. The prototype of all the examples we discuss here is the polynomial ring in two variables. The reader is expected to be familiar with the basic notions of module theory.

1.1.4.1 The polynomial ring

In section 1.1, the Weyl Algebra was constructed as a sub ring of an endomorphism ring. Writing $\mathbb{C}[x, y]$ for the polynomial ring $\mathbb{C}[x, y]$, where $\mathbb{C}$ is the field of complex numbers, we have that A_2 is a sub ring of $End_{\mathbb{C}}(\mathbb{C}[x, y])$ and one deduces from this that the polynomial ring is a leftA_2-module. Thus the action of $\hat{x}$ and $\hat{y}$ on $\mathbb{C}[x, y]$ is by straightforward multiplication while ∂x and ∂y act by differentiation with respect to x and y.

Definition 1.1.16. Let R be a ring. A left module M over R (or a left R-module) Let M be a K-vector space .To define an action of A_2 on M we begin by prescribing values of x,y, u, for every $u \in M$. Since the ideal of the relations is generated by a finite number of elements. Then to say M is A_2-module it suffices to show that:
$[\partial x, \hat{x}]u = [\partial y, \hat{y}]u = \delta u$
$[\hat{x}, \hat{y}]u = [\partial x, \partial y]u = 0$ for every $u \in M$ and $\hat{x}, \hat{y}$, ∂_x and ∂_y are operators of K[x,y]

Theorem 1.1.27. *$\mathbb{C}[x, y]$ is A_2-module with the action of x ,y,∂_x and ∂_y on $\mathbb{C}[x, y]$ by*
x: $\mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]$ which is given by $xf = x.f$
y: $\mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]$ which is given by $yf = y.f$
$\partial x : \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]$ which is given by $\partial_x(f) = \dfrac{\partial(f)}{\partial_x}$
and $\partial y : \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]$ which is given by $\partial_y(f) = \dfrac{\partial(f)}{\partial_y}$

Proof. We know that $(\mathbb{C}[x, y], +)$ is an abelian group and $(A_2, +, o)$ is a ring. Let $P, Q \in A_2$ and $f, g \in \mathbb{C}[x, y]$.
i) P (f + g) = P f + P g (by linearity of P)
ii) (PQ) f = P(Q (f)) = P (Q f) (by def 1.1.9.2)

iii) (P + Q) f = P(f) + Q (f) (by def 1.1.9.1)

iv) 1f = f, $\forall f \in \mathbb{C}[x, y]$, where 1 stands for identity operator of A_2. **Moreover:**

a) $[\partial x, x](f) = (\partial_x x - x\partial_x)(f) = 1f = f$

$[\partial_y, y](f) = (\partial_y y - y\partial_y)(f) = 1f = f$

b) $[\partial_x, \partial_y](f) = (\partial_x\partial_y - \partial_y\partial_x)$ (f) = 0(f) = 0

c) [x, y] (f) = (xy-yx)(f)= 0(f) = 0

Therefore $\mathbb{C}[x, y]$ is a left A_2- module. □

Chapter 2

The Length 0f A_2-module

In section we describe the definition of simple modules and collects examples of simple A_2- modules. It is true that every simple module is cyclic but the converse is not true. Also we will describe the general study of modules with finite composition series and the length of modules.

2.1 Simple A_2- Modules

In section we describe the definition of simple modules and collects examples of simple A_2- modules. It is true that every simple module is cyclic but the converse is not true.

Definition 2.1.1. let M be an A_2-module. M is said to be simple module over A_2 if it has no proper sub modules. That is it has no sub modules except 0 and M itself.

Theorem 2.1.1. $\mathbb{C}[x, y]$ *as* A_2- *module is simple.*

Proof. Let $f \in \mathbb{C}[x, y]$, $f \neq 0$. Then $A_2 f$ be a sub module of $\mathbb{C}[x, y]$. Let $f = \Sigma c_\alpha X^\alpha$,

$\alpha = (\alpha_1, \alpha_2)$, $X^\alpha = (x^{\alpha_1}.y^{\alpha_2})$,$|\alpha| = \alpha_1 + \alpha_2$ and $X = (x, y)$. Then
$\partial^\alpha(f) = \partial^\alpha(\Sigma c_\alpha X^\alpha) = \Sigma c_\alpha \alpha!$. This implies $\partial^\alpha(f) = \Sigma c_\alpha \alpha!$. Thus $\dfrac{\partial^\alpha(f)}{\Sigma c_\alpha \alpha!} = 1$. This implies $1 \in A_2 f$. Therefore $A_2 f = \mathbb{C}[x, y]$. And hence $\mathbb{C}[x, y]$ is simple as A_2- module.

□

Definition 2.1.2. A left R-module M is called cyclic if M can be generated by a single element.

Proposition 2.1.2. *Every simple A_2- module is cyclic.*

Proof. Let M be an A_2-module. For each $a \in M$ and $a \neq 0$, consider $A_2 a$ is sub module of M. Since M is simple $A_2 a = 0$ or $A_2 a = M$. But $A_2 a \neq 0$, Then $A_2 a = M$ for some a. Therefore, M is cyclic.

Note:

♦ Any simple A_2-module M is generated by any of its non zero elements.

That is M = A_2a, for $a \in M$ and $a \neq 0$.

♦ A cyclic module may not be simple. □

Example 2.1.3. *Let $\mathbb{C}[x, y]$ be the polynomial ring in x and y over the field of the set complex numbers $\mathbb{C}$. We define $\mathbb{C}[x, y]_x = \{\dfrac{p}{x^r} : p \in \mathbb{C}[x, y]$ and $r \in N.\}$. Here $\mathbb{C}[x, y]_x$ is cyclic but not simple. Because $\mathbb{C}[x, y]_x$ is generated by $(\frac{1}{x})$ and it has a chain of sub modules $0 \subset \mathbb{C}[x, y] \subset \mathbb{C}[x, y]_x$ and $\mathbb{C}[x, y] \neq \mathbb{C}[x, y]_x$.*

Proof. claim 1. $\mathbb{C}[x, y] \subset \mathbb{C}[x, y]x$.
Let $t \in \mathbb{C}[x, y]$, then $\dfrac{x^r t}{x^r} = \dfrac{q}{x^r} \in \mathbb{C}[x, y]_x$, (since $q = x^r\ t \in \mathbb{C}[x, y]$).
Therefore $\mathbb{C}[x, y] \subset \mathbb{C}[x, y]_x$.
claim 2. $\mathbb{C}[x, y]_x$ is generated by $(\frac{1}{x})$.

Let $t \in \mathbb{C}[x,y]_x$ and $t \neq 0$. Then $t = \frac{q}{x^r}$ where $q \in \mathbb{C}[x,y]$. Let $k = r-1$ and consider $D = \frac{q\partial_x^k}{k!(-1)^k} \in A_2$. Recall that: $\partial_x^n(\frac{1}{x}) = \frac{(-1)^n n!}{x^{n+1}}$

$$\begin{aligned} Now \quad D(\frac{1}{x}) &= \frac{q\partial_x^k}{k!(-1)^k}(\frac{1}{x}) \\ &= (\frac{q}{k!(-1)^k})(\partial_x^k(\frac{1}{x}) \\ &= (\frac{q}{k!(-1)^k})(\frac{(-1)^k(k!)}{x^{k+1}}) \\ &= \frac{q}{x^{k+1}} \\ &= \frac{q}{x^r} \qquad ....(since \quad k = r-1 \quad implies \quad r = k+1) \\ &= t \end{aligned}$$

Hence $D(\frac{1}{x}) = t$.

Since t is an arbitrary element of $\mathbb{C}[x,y]_x$ and $t \neq 0$, we have $\mathbb{C}[x,y]_x$ is generated by $(\frac{1}{x})$.

Therefore $\mathbb{C}[x,y]_x$ is cyclic but not simple.

□

Definition 2.1.3. Let f be a non- zero polynomial.

$K[x,y]_f = \{\frac{p}{f^r} : p \in K[x,y], r \in N_0\}$ is the set of rational functions of the form $(\frac{p}{f^r})$, where N_0 is the set of whole numbers.

Theorem 2.1.4. *Let $\mathbb{C}[x,y]$ be the field of complex numbers. Then $(\mathbb{C}[x,y]_f, +)$ is an abelian group.*

Proof. We know that the set of rational functions $\mathbb{C}(x,y)$ is an abelian group. Then it is enough to show that a sub set $\mathbb{C}[x,y]_f$ is a sub group of $\mathbb{C}(x,y)$.

Let $D_1, D_2 \in \mathbb{C}[x,y]_f$. Then $D_1 = \frac{p}{f^r}$ and $D_2 = \frac{h}{f^t}$ then:

$D_1 - D_2 = \frac{p}{f^r} - \frac{h}{f^t} = \frac{pf^t - hf^r}{f^r f^t} = \frac{pf^t - hf^r}{f^{r+t}} = \frac{q}{f^{r+t}} = \frac{q}{f^s} \in \mathbb{C}[x,y]_f \in$, where $q = pf^t - hf^r \in \mathbb{C}[x,y], s = r + t \in N$.

Hence $\mathbb{C}[x,y]_f$ is a sub group of $\mathbb{C}(x,y)$.

Therefore $(\mathbb{C}[x,y]_f, +)$ is an abelian group. □

Theorem 2.1.5. *Let f be a non zero and non constant polynomial, then* $\mathbb{C}[x,y]_f$ *is a left* A_2*- module with the action of x,y,*∂x *and* ∂y *by* $(\hat{x}, \frac{p}{f^r}) \longrightarrow \hat{x}\frac{p}{f^r} = x\frac{p}{f^r}$, $(\hat{y}, \frac{p}{f^r}) \longrightarrow \hat{y}\frac{p}{f^r} = y\frac{p}{f^r}$ *and*

$(\partial x, \frac{p}{f^r}) \longrightarrow \partial x(\frac{p}{f^r}) = \frac{f^r\partial_x p - p\partial_x f^r}{(f^r)^2} = \frac{f^r\partial_x p - pr f^{r-1}\partial_x f}{(f^r)^2}$,

$(\partial y, \frac{p}{f^r}) \longrightarrow \partial y(\frac{p}{f^r}) = \frac{f^r\partial_y p - p\partial_y f^r}{(f^r)^2} = \frac{f^r\partial_y p - pr f^{r-1}\partial_y f}{(f^r)^2}$

Proof. Since $(\mathbb{C}[x,y]_f, +)$ is an abelian group. (*by Theorem* 2.1.8)

Now let p, q, h, f $\in \mathbb{C}[x,y]$ and $\frac{p}{f^r}, \frac{q}{f^t}, \frac{h}{f^s} \in \mathbb{C}[x,y]_f$ and $\hat{x}, \hat{y}, \partial x, \partial y \in A_2$ where r, t,

s $\in N_0$

$$
\begin{aligned}
1.\hat{x}(\frac{p}{f^r}+\frac{q}{f^t}) &= x(\frac{p}{f^r}+\frac{q}{f^t}) \\
&= x\frac{(pf^t+qf^r)}{f^rf^t} \\
&= (\frac{xpf^t+xqf^r}{f^rf^t}) \\
&= \frac{x(pf^t)}{f^rf^t}+\frac{x(qf^r)}{f^rf^t} \\
&= (\frac{x(p)}{f^r}+(\frac{x(q)}{f^t} \\
&= \hat{x}(\frac{p}{f^r})+\hat{x}(\frac{p}{f^r}) \quad and \\
\partial x(\frac{p}{f^r}+\frac{q}{f^t}) &= \partial_x(\frac{p}{f^r}+\frac{q}{f^t})
\end{aligned}
$$

$$= \frac{f^rf^t[\partial_x(pf^t+qf^r)]-(pf^t+qf^r)(\partial_x(f^rf^t))}{(f^rf^t)^2}$$

$$= \frac{f^rf^t[\partial_x(pf^t)+\partial_x(qf^r)]-(pf^t+qf^r)(\partial_x(f^rf^t))}{(f^rf^t)^2}$$

$$= \frac{f^rf^t[\partial_x(pf^t)+\partial_x(qf^r)]}{(f^rf^t)^2}-\frac{(pf^t+qf^r)(\partial_x(f^rf^t))}{(f^rf^t)^2}$$

$$= \frac{f^rf^t[f^t\partial_xp+p\partial_xf^t+f^r\partial_xq+q\partial_xf^r]}{(f^rf^t)^2}-\frac{(pf^t+qf^r)(\partial_x(f^rf^t))}{(f^rf^t)^2}$$

$$= \frac{f^r(f^t)^2\partial_xp+f^rf^tp\partial_xf^t+(f^r)^2f^t\partial_xq+f^rf^tq\partial_xf^r}{(f^rf^t)^2}-\frac{(pf^t+qf^r)(\partial_x(f^rf^t))}{(f^rf^t)^2}$$

$$= \frac{p\partial_xf^t}{f^rf^t}+\frac{\partial_x(p)}{f^r}+\frac{q\partial_xf^r}{f^rf^t}+\frac{\partial_x(q)}{f^t}-\frac{(pf^t+qf^r)(\partial_x(f^rf^t))}{(f^rf^t)^2}$$

$$= \frac{p\partial_xf^t}{f^rf^t}+\frac{\partial_x(p)}{f^r}+\frac{q\partial_xf^r}{f^rf^t}+\frac{\partial_x(q)}{f^t}-\frac{p\partial_xf^t}{f^rf^t}-\frac{p\partial_xf^r}{(f^r)^2}-\frac{q\partial_xf^t}{(f^t)^2}-\frac{q\partial_xf^r}{f^rf^t}$$

$$= \frac{\partial_x(p)}{f^r}-\frac{p\partial_xf^r}{(f^r)^2}+\frac{\partial_x(q)}{f^t}-\frac{q\partial_xf^t}{(f^t)^2}$$

2. $(\hat{x}+\partial x)(\frac{p}{f^r}) = \frac{(x+\partial_x)p}{f^r} = \frac{x(p)}{f^r} + \frac{\partial_x p}{f^r} = \hat{x}(\frac{p}{f^r}) + \partial x(\frac{p}{f^r})$

3. $(\partial x\hat{x}) = \frac{\partial}{\partial_x}(\hat{x}\frac{p}{f^r}) = \frac{\partial}{\partial_x}(x\frac{p}{f^r}) = \partial x(\hat{x}\frac{p}{f^r})$ and

$(\hat{x}\partial x)(\frac{p}{f^r}) = x(\frac{\partial}{\partial_x}\frac{p}{f^r}) = \hat{x}(\partial x\frac{p}{f^r})$

4. $1(\frac{p}{f^r}) = 1\frac{p}{f^r} = \frac{1p}{f^r} = \frac{p}{f^r}$ .Similarly to $\hat{y}, \partial y$.

Moreover:

a) $[\partial x, \hat{x}](\frac{p}{f^r}) = (\partial x\hat{x} - \hat{x}\partial x)(\frac{p}{f^r}) = 1(\frac{p}{f^r})$

b) $[\partial x, \partial y](\frac{p}{f^r}) = (\partial x\partial y - \partial y\partial x)(\frac{p}{f^r}) = 0(\frac{p}{f^r}) = 0$

c) $[\hat{x}, \hat{y}](\frac{p}{f^r}) = (\hat{x}\hat{y} - \hat{y}\hat{x})(\frac{p}{f^r}) = 0(\frac{p}{f^r}) = 0$

Therefore $\mathbb{C}[x,y]_f$ is a left A_2-module. □

Example 2.1.6. *Let $f = x+y+1$ and let $p \in \mathbb{C}[x,y]$. Then $\mathbb{C}[x,y]_{x+y+1}$ is A_2-module.*

Proof. $\hat{x}(\frac{p}{f^r}) = x(\frac{p}{(x+y+1)^r}) = \frac{xp}{(x+y+1)^r} \in \mathbb{C}[x,y]_{x+y+1}$

$$\hat{y}(\frac{p}{f^r}) = y(\frac{p}{(x+y+1)^r}) = \frac{yp}{(x+y+1)^r} \in \mathbb{C}[x,y]_{x+y+1}$$

$$\partial x(\frac{p}{f^r}) = \partial x(\frac{p}{(x+y+1)^r}) = \frac{(x+y+1)^r\partial_x(p) - r(x+y+1)^{r-1}p}{((x+y+1)^r)^2}$$

$$= \frac{\partial_x(p) - r(x+y+1)^{-1}p}{(x+y+1)^r} \in \mathbb{C}[x,y]_{x+y+1}$$

$$\partial y(\frac{p}{f^r}) = \partial y(\frac{p}{(x+y+1)^r}) = \frac{(x+y+1)^r\partial_y(p) - r(x+y+1)^{r-1}p}{((x+y+1)^r)^2}$$

$$= \frac{\partial_y(p) - r(x+y+1)^{-1}p}{(x+y+1)^r} \in \mathbb{C}[x,y]_{x+y+1}$$

Moreover:

a) $[\partial x, \hat{x}](\frac{p}{(x+y+1)^r}) = (\partial x\hat{x} - \hat{x}\partial x)(\frac{p}{(x+y+1)^r}) = 1(\frac{p}{(x+y+1)^r})$

b) $[\partial x, \partial y](\frac{p}{(x+y+1)^r}) = (\partial x\partial y - \partial y\partial x)(\frac{p}{(x+y+1)^r}) = 0(\frac{p}{(x+y+1)^r}) = 0$

c) $[\hat{x}, \hat{y}](\frac{p}{(x+y+1)^r}) = (\hat{x}\hat{y} - \hat{y}\hat{x})(\frac{p}{(x+y+1)^r}) = 0(\frac{p}{(x+y+1)^r}) = 0$

Therefore $\mathbb{C}[x,y]_{x+y+1}$ is A_2- module.

Note:

$\mathbb{C}[x,y]_f$ is not simple as$A_2 - module$. We will see in the next section the reason why it is not simple. □

2.2 Composition series and Length of A_2- Modules

In this section we will see the length of A_2-modules by considering of polynomial rings in two variable. And then by introducing its quotient module, we will describe its properties. A module M which has a finite composition series is called a module of finite length. The length of a finite composition series of M is said to be the length of M. In this section only we have the case for A_2-moduleof finite length.

Let $f = xy(x+y)$ then $\mathbb{C}[x,y]_f = \mathbb{C}[x,y]_{xy(x+y)}$ is not simple. Because the chain $0 \subset \mathbb{C}[x,y] \subset \mathbb{C}[x,y]_x \subset \mathbb{C}[x,y]_{xy} \subset \mathbb{C}[x,y]_{xy(x+y)}$ is the sequence of sub modules of $\mathbb{C}[x,y]_f$ .In this section we will see the quotient modules $\mathbb{C}[x,y]_x/\mathbb{C}[x,y], \mathbb{C}[x,y]_{xy}/\mathbb{C}[x,y]_x$, $\mathbb{C}[x,y]_{x(x+y)}/\mathbb{C}[x,y]_{xy}$ are simple or not $as A_2$-module.

Definition 2.2.1. Let M be an A_2-module then a chain of sub modules of M is a sequence of sub modules with strict inclusions,

$0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_k = M$ such a chain is said to have length k (the number of links).

Example 2.2.1. *1. The zero modules are the only one with length 0.*

2. Modules with length 1 are precisely the simple module.

Definition 2.2.2. Let M be an A_2-module. A composition series of the module M is a sequence of sub modules with strict inclusions, $0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_k = M$ and M_i/M_{i-1} is simple $(i = 1, 2, ..., k)$

Theorem 2.2.2. (***Jordan-Holder Theorem***)

Any two composition series of a non zero module are equivalent in the sense that both have the same length and the same simple quotient and isomorphism .To be precise,let $0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_{r-1} \subset M_r = M$ *and*

$0 = N_0 \subset N_1 \subset N_2 \subset ... \subset N_{t-1} \subset N_t = M$ *be any two composition series for M.Then*

i) $r=t$

ii) $\forall$ *i,* *i=1...r* $\exists j$, *j=1...t such that* $M_i/M_{i-1} = N_j/N_{j-1}$ *and vice-versa.*

Proof. We proof the result by induction on the length of one of the composition series,say r of the first.If r=1,then M is simple and hence $N_{r-1} = (0)$,i.e.,t=1 and the result follows.suppose $r \geq 2$ and assume the induction hypothesis that the theorem is true for any module having some composition series of length at most r-1.We distinguish two possibilities.

case 1: $M_{r-1} = N_{t-1}$

Now we find that M_{r-1} has two composition series,namely,

$0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_{r-1}$ and

$0 = N_0 \subset N_1 \subset N_2 \subset ... \subset N_{r-1}$.

Since the first one is of length r-1,we get by induction that

1. r-1=t-1,i.e.,r=t and
2. $\forall$ i, i=1...r $\exists j$, j=1...t such that $M_i/M_{i-1} = N_j/N_{j-1}$ and vice-versa.

Since $M/M_{r-1} = M/N_{t-1}$,the result follows.

case 2: $M_{r-1} \neq N_{t-1}$

Let $M' = M_{r-1} + N_{t-1}$ which is a submodule of M containg the maxima submodules M_{r-1} and N_{t-1} and so $M' = M$.Furthermore,we have

($\bigstar\bigstar$) $\{M/M_{r-1} = (M_{r-1}+N_{t-1})/M_{r-1} \cong N_{t-1}/(M_{r-1}\cap N_{t-1}) = N_{t-1}/K = simple$

$\{M/N_{t-1} = (M_{r-1}+N_{t-1})/N_{t-1} \cong M_{t-1}/(M_{r-1}\cap N_{t-1}) = M_{r-1}/K = simple,$

where $K = (M_{r-1} \cap N_{t-1})$ which is a submodule of a module of finite length M and so K is also a module of finite length and hence has some composition series,say $0 = K_0 \subset K_1 \subset K_2 \subset ... \subset K_{r-1} \subset K_r = K$.Thus we get 4 composition series for $M = (M_{r-1} + N_{t-1}), namely$

$$\begin{aligned} 0 &= K_0 \subset K_1 \subset K_2 \subset ... \subset K_{r-1} \subset K_r \subset M_{r-1} \subset (M_{r-1} + N_{t-1}) = M, \\ 0 &= M_0 \subset M_1 \subset M_2 \subset ... \subset M_{r-1} \subset M_r \subset (M_{r-1} + N_{t-1}) = M, \\ 0 &= N_0 \subset N_1 \subset N_2 \subset ... \subset N_{t-1} \subset N_t \subset (M_{r-1} + N_{t-1}) = M \quad and \\ 0 &= K_0 \subset K_1 \subset K_2 \subset ... \subset K_{r-1} \subset K_r \subset N_{t-1} \subset (M_{r-1} + N_{t-1}) = M. \end{aligned}$$

Of these,the first two(resp.the last two) are equivalent by case 1 whereas the first and last are equivalent by ($\bigstar\bigstar$) above and hence we get that 2^{nd} and 3^{rd} are equivalent,as required. □

Definition 2.2.3. Let M be an A_2-module.The length of any of it's composition series (which is independent of the series) is called **the length of the module** and is denoted by $\ell(M)$. (If M has no composition series,we say $\ell(M) = \infty$).

Let $f = xy(x + y)$ then $\mathbb{C}[x, y]_f = \mathbb{C}[x, y]_{xy(x+y)}$ is not simple. Because the chain $0 \subset \mathbb{C}[x, y] \subset \mathbb{C}[x, y]_x \subset \mathbb{C}[x, y]_{xy} \subset \mathbb{C}[x, y]_{xy(x+y)}$ is the sequence of sub modules of $\mathbb{C}[x, y]_f$ with strict inclusion.We want to find the length of $\mathbb{C}[x, y]_f$

Proposition 2.2.3.

i. $\frac{1}{x}(mod(\mathbb{C}[x,y]))$ *is the generator of* $(\mathbb{C}[x,y]_x)/(\mathbb{C}[x,y])$.

ii. $(\mathbb{C}[x,y]_x)/(\mathbb{C}[x,y])$ *is simple as* A_2*-module.*

Proof. i) Let $t \in \mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ and $t \neq 0$. Then $t = \frac{q}{x^r} + \mathbb{C}[x,y]$ where $q \in \mathbb{C}[x,y]$, and $r \in N$.

Let $k = r-1$ and consider $D = \frac{q\partial_x^k}{k!(-1)^k} \in A_2$. Recall that: $\partial_x^n(\frac{1}{x}) = \frac{(-1)^n n!}{x^{n+1}}$

$$
\begin{aligned}
Now \quad D((\frac{1}{x}) + \mathbb{C}[x,y]) \quad &= \frac{q\partial_x^k}{k!(-1)^k}(\frac{1}{x} + \mathbb{C}[x,y]) \\
&= (\frac{q}{k!(-1)^k})(\partial_x^k(\frac{1}{x} + \mathbb{C}[x,y]) \\
&= (\frac{q}{k!(-1)^k})(\frac{(-1)^k(k!)}{x^{k+1}}) + \mathbb{C}[x,y] \\
&= \frac{q}{x^{k+1}} + \mathbb{C}[x,y] \\
&= \frac{q}{x^r} + \mathbb{C}[x,y] \qquad (since \quad k = r-1 \quad implies \quad r = k+1) \\
&= t
\end{aligned}
$$

Hence, $D((\frac{1}{x}) + \mathbb{C}[x,y]) = t$, since t is an arbitrary non-zero element of $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ Therefore $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ is generated by $(\frac{1}{x} + \mathbb{C}[x,y])$.

ii) Let $t \in \mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ and $t \neq 0$. Then $t = \frac{p}{x^r} + \mathbb{C}[x,y]$ where $p \in \mathbb{C}[x,y]$, and $r \in N$. then $t = \frac{\Sigma c_{\alpha\beta} x^\alpha y^\beta}{x^r} + \mathbb{C}[x,y]$ where $c_{\alpha\beta} \in \mathbb{C}, r \in N, p = \Sigma c_{\alpha\beta} x^\alpha y^\beta$ is finite and $c_{\alpha\beta} \neq 0$. It is enough to show for the case $s < r$, since if $s \geq r$ we can

write $\frac{p}{x^r} + \mathbb{C}[x,y] = q + \frac{h}{x^r} + \mathbb{C}[x,y]$ where $q, h \in \mathbb{C}[x,y]$ and $deg(h) < r$. Thus $t = \frac{p}{x^r} + \mathbb{C}[x,y] = \frac{h}{x^r} + \mathbb{C}[x,y]$.Assume $\alpha < r$, and let c_k be a non- zero coefficient of smallest power of x in p. Then $t = \frac{\Sigma_{i=k}^{k} c_i x^i y^t}{x^r} + \mathbb{C}[x,y]$ where $c_k \in \mathbb{C}, n \in N, y^t$ is finite sum of y and $c_n \neq 0$. Then
$t = (\frac{c_k}{x^r}x^k y^t + \frac{c_{k+1}}{x^r}x^{k+1}y^t + \frac{c_{k+2}}{x^r}x^{k+2}y^t + ... + \frac{c_n}{x^r}x^n y^t) + \mathbb{C}[x,y]$. consider

$$D = \frac{\partial_y^t x^{r-(k+1)}}{t! c_k} \in A_2$$

$$Now \quad D(t) = (\frac{\partial_y^t x^{r-(k+1)}}{t! c_k})((\frac{c_k}{x^r}x^k y^t + \frac{c_{k+1}}{x^r}x^{k+1}y^t + \frac{c_{k+2}}{x^r}x^{k+2}y^t + ... + \frac{c_n}{x^r}x^n y^t) + \mathbb{C}[x,y])$$

$$= \frac{\partial_y^t}{t!}(\frac{x^{r-(k+1)}}{c_k}(\frac{c_k}{x^r}x^k y^t + \frac{c_{k+1}}{x^r}x^{k+1}y^t + \frac{c_{k+2}}{x^r}x^{k+2}y^t + ... + \frac{c_n}{x^r}x^n y^t) + \mathbb{C}[x,y])$$

$$= \frac{\partial_y^t}{t!}((\frac{x^{r-(k+1)}}{c_k}\frac{c_k}{x^r}x^k y^t) + (\frac{x^{r-(k+1)}}{c_k}\frac{c_{k+1}}{x^r}x^{k+1}y^t) + (\frac{x^{r-(k+1)}}{c_k}\frac{c_{k+2}}{x^r}x^{k+2}y^t) + ...$$

$$+(\frac{x^{r-(k+1)}}{c_k}\frac{c_n}{x^r}x^n y^t) + \mathbb{C}[x,y])$$

$$= \frac{\partial_y^t}{t!}(\frac{y^t}{x} + \frac{c_{k+1}y^t}{c_k} + \frac{c_{k+2}xy^t}{c_k} + ... + \frac{c_n y^t}{c_k})$$

$$= \frac{\partial_y^t}{t!}(\frac{y^t}{x}) \qquad \text{since } (\frac{c_{k+1}}{c_k}y^t + \frac{c_{k+2}}{c_k}xy^t + ... + \frac{c_n}{c_k}x^n y^t) \in \mathbb{C}[x,y])$$

$$= \frac{\partial_y^t y^t}{t!x} + \mathbb{C}[x,y] = \frac{t!}{t!x}$$

$$= \frac{1}{x} + \mathbb{C}[x,y]$$

Hence $D(t) = \frac{1}{x} + \mathbb{C}[x,y]$. Since $\frac{1}{x} + \mathbb{C}[x,y]$ is a generator of $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ and t is

a non-zero arbitrary element of $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$. we have $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ is simple as A_2- module.Therefore $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ is simple as A_2- module. □

But $\mathbb{C}[x,y]_{xy}/\mathbb{C}[x,y]_x$ is not simple as A_2- module.Because $\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y$ is submodule between $\mathbb{C}[x,y]_x$ and $\mathbb{C}[x,y]_{xy}$

Definition 2.2.4. Let M be an A_2-module.If $0 = M_0 \subset M_1 \subset ... \subset M_r = M$ is a composition series of M,then the set $DF(M) := \{M_i/M_{i-1}\}_{i=1}^r$ of simple A_2-module is the set of decomposition factors of M.

We have the following proposition on the decomposition factors of A_2-modules.

Proposition 2.2.4. *Let M be an A_2-module. Let N a submodule of M. Consider the exact sequence of A_2-module $N \subset M \xrightarrow{\pi} M/N$. Then*

(a) $DF(M) = DF(N) \cup DF(M/N)$ and

(b) $\ell(M) = \ell(N) + \ell(M/N)$

Proof. Consider $M \xrightarrow{\pi} M/N = F_k \supset F_{k-1} \supset ... \supset F_1 \supset F_0 = 0$

Then $F_j = M_j/N$, where $M \supset M_j = \pi^{-1}(F_j)$ for j=0,1,,k.But

$M_j/M_{j-1} = \pi^{-1}(F_j)/^{-1}F_{j-1} \cong M_j/N/M_{j-1}/N = F_j/F_{j-1}$

Hence if F_j/F_{j-1} are simple, then M_j/M_{j-1} also are simple.

Suppose $0 = N_0 \subset N_1 \subset ...N_r = N$ is composition series of N.Then

$N_0 \subset N_1 \subset ... \subset N_r = N = M_0 \subset M_1 \subset ... \subset M_k = M$ is composition series of M.

Therefore $\{N_i/N_{i-1}\}_{i=0}^r \cup \{F_j/F_{j-1} \cong M_j/M_{j-1}\}_{j=0}^k$ is the set of decomposition factors of M.

$\ell(M) = r + k = \ell(N) + \ell(M/N)$. □

Proposition 2.2.5. *The length of* $\mathbb{C}[x,y]_{xy}$ *is 4*

Proof. consider the sequence $0 \subset \mathbb{C}[x,y] \subset \mathbb{C}[x,y]_x \subset \mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y \subset \mathbb{C}[x,y]_{xy}$.
The inclusion is strict and

i. $\mathbb{C}[x,y]/(0)$ is simple as A_2-module since $\mathbb{C}[x,y]/(0) \cong \mathbb{C}[x,y]$.

ii. $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$ is simple as A_2-module (by proposition 2.2.3).

iii. $(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y)/\mathbb{C}[x,y]_x \cong \mathbb{C}[x,y]_y/\mathbb{C}[x,y]$ (by theorem 1.1.8) and $\mathbb{C}[x,y]_y/\mathbb{C}[x,y]$ is simple as A_2-module (by proposition 2.2.3).

Hence $(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y)/\mathbb{C}[x,y]_x$ is simple as A_2-module.

iv. $\mathbb{C}[x,y]_{xy}/(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y)$ is a simple A_2 module generated by $\dfrac{1}{xy}$.

Therefore $0 \subset \mathbb{C}[x,y] \subset \mathbb{C}[x,y]_x \subset \mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y \subset \mathbb{C}[x,y]_{xy}$ is composition series of $\mathbb{C}[x,y]_{xy}$ (by definition 2.2.2).

Therefore the length of $\mathbb{C}[x,y]_{xy}$ is 4. □

Proposition 2.2.6. *The length of* $\mathbb{C}[x,y]_{xy(x+y)}$ *is 6.*

Proof. Consider the sequence

$0 \subset \mathbb{C}[x,y] \subset \mathbb{C}[x,y]_x \subset \mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y \subset \mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y} \subset \mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+y)}$.

$\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y}\big)/\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y\big) \cong \mathbb{C}[x,y]_{x+y}/\mathbb{C}[x,y]$

(by theorem 1.1.8) and $\mathbb{C}[x,y]_{x+y}/\mathbb{C}[x,y]$ is simple as A_2-module (by proposition 2.2.3)

so, $\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y}\big)/\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y\big)$ is simple as A_2-module

Since $\dfrac{1}{xy(x+y)} = \dfrac{1}{xy} - \dfrac{1}{x(x+y)}$ then $\dfrac{1}{xy(x+y)} \in \mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+y)}$.

$\mathbb{C}[x,y]_{xy(x+y)} = \mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+y)}$ thus

$\big(\mathbb{C}[x,y]_{xy(x+y)}\big)/\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y}\big) = \big(\mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+y)}\big)/\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y}\big)$.

$\mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+y)}/\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y} \cong \mathbb{C}[\frac{1}{xy}] \bigoplus \mathbb{C}[\frac{1}{x(x+y)}]$

since $\mathbb{C}[\frac{1}{xy}] \cong \bigoplus_{i,j<0} \mathbb{C}(\frac{1}{x^i y^j})$ and $\mathbb{C}[\frac{1}{x(x+y)}] \cong \bigoplus_{k,r<0} \mathbb{C}(\frac{1}{x^k(x+y)^r})$

Let M=$\bigoplus_{i,j<0} \mathbb{C}(\frac{1}{x^i y^j})$ and N=$\bigoplus_{k,r<0} \mathbb{C}(\frac{1}{x^k(x+y)^r})$, then $M \bigoplus N/M \cong N$.Thus $M \bigoplus N/M$ is simple.Hence $0 \subset N \subset M \bigoplus N$ is a composition series and $\ell(M \bigoplus N)$=2.

Therefore $\ell\big(\mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+y)}\big)/\big(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y}\big) = 2$ and

$\ell(\mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{x+y})$=4.

Therefore,The length of $\mathbb{C}[x,y]_{xy(x+y)}$ is 6 (by proposition 2.2.4). □

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