ON DISTRIBUTION THEORY AND SOBOLEV SPACE

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GRADUATE PROJECT SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER SCIENCE IN MATHEMATICS

Addis Ababa, Ethiopia
November 2012

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship, or any other similar title to me.

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november 2012 **BIZUNEH GIRMA**

Permission

This is to certify that this project is compiled by ***Mr. BIZUNEH GIRMA*** *in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.*

Addis Ababa ________________________________

november 2012 **Mengistu Goa Sangago(Ph.D)**

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Acknowledgement

First and for most I would like to thank almighty God for being on my side in my endeavors. And also would like to express my gratitude to my advisor Dr. Mengistu Goa. for providing me valuable comments, reference materials and correcting mistakes would also like to express my appreciation to my brother Ato Sebsibe Teferi for providing me valuable comments and reference materials. Lastly, I would like to extend many thanks to my brother Tariku welde, Samson Haile and Fanos Dechasa who have been helping me in all activities.

Abstract

This project contains two parts; theory of distribution and Sobolev spaces. In this project, we discussed theory of distribution and also the differentiations of distribution, multiplication by smooth function of distribution, direct product of distribution, convolution of distribution and the Fourier transform of function of slow growth. The theory of Sobolev spaces has been originated by Russians mathematician S.L.Sobolev around 1938. These spaces were not introduced for some theoretical purposes, but for the need of the theory of partial differential equations.

Introduction

A generalized function (distribution) is a generalization or extension of the classical notion of function. On the one hand, this generalization permits expressing in a mathematically proper form such idealized concept as the density of a material point the density of a point charge or dipole, the special density of a simple or double layer the density of an instantaneous point source, the magnitude of an instantaneous force applied to a point, and so forth.

On the other hand, the notation of generalized function can reflect the fact that, in reality, one cannot measure the value of physical quantity at a point. But can only measure the mean values with in sufficient small neighborhood of the point and proclaim the limit of the sequence of those mean values as the value of the physical quantity at the given point.

The theory of Sobolev space has been originated by the Russian Mathematician S.L.Sobolev around 1938. These spaces were not introduced for some theoretical purpose, but for the need of the theory of partial differential equations. Since elements of such spaces are special class of distributions. Sobolev spaces are an example of banach spaces or, sometimes, Hilbert spaces are interesting objects for themselves.

Notation and Symbols

- N: the set of natural numbers.
- N_0: the set of non-negative integers.
- R: the set of all real numbers.
- ℂ: the set of all complex numbers.
- N_o^n : N_0 x.....x N_0 (n-times): the set of multi- indices.
- R^n: n-dimensional Euclidean space and x= {$x_1, x_2, \ldots x_n$} be a variable element of R^n.
- Domain Ω: an open connected region
- A domain Ω in R^n: An open connected subset $\Omega \subseteq R^n$
- $\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$: The scalar product in R^n.

For an arbitrary non- empty set $\Omega \subset R^n$ we shall denote by:

- C (Ω): The set of all continuous functions in Ω.
- $C^1(\Omega)$:The set of all once continuously differentiable functions in Ω.
- $C^2(\Omega)$: The set of all twice continuously differentiable functions in Ω.
- $C^k(\Omega)$: The space of functions having all derivatives of order ≤ k continuous in Ω
- $C^o(\Omega) = C(\bar{\Omega})$: The set of all continuous function on $\bar{\Omega}$.
- $\subset\subset$: Strictly Compact
- $C^\infty(\Omega)$: The space of infinitely continuously differentiable functions on Ω.
- $C_o^\infty(\Omega)$: The space of functions in $C^\infty(\Omega)$ with compact support.
- $\mathcal{F}(f) = \hat{f}(f \in S(R^n))$:The Fourier transform of a function f.

For a measurable non-empty set $\Omega \subset R^n$ we shall denote by:

- Space: for $1 \le p < \infty$, $L_p(\Omega)$ denote the set of all measurable function defined on Ω such that, $\int_\Omega |f(x)|^p dx < \infty$,where the integrable is taken in the sense of lebesgue.
- L^p Space norm is defined by $\|f\|_{L_p(\Omega)} = (\int_\Omega |f|^p)^{\frac{1}{p}} < \infty$.
- $L^\infty(\Omega)$:the Banch space of function f measurable on Ω such that the norm
$$\|f\|_{L^\infty(\Omega)} = \underset{x \in \Omega}{ess\ sup} |f(x)| < \infty.$$
- A multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of non –negative integer numbers and
$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$
- Partial derivative will be denoted by:
- $\partial_j := \frac{\partial}{\partial x_j}, 1 \le j \le n$, and we write $\partial = \frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$

For $\alpha \in N_0^n$, $\alpha \neq 0$, *we shall write*:

$$D^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n} f}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \cdots \partial {x_n}^{\alpha_n}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial_n}\right)^{\alpha_n} f.$$

For Ω be a measurable set and $1 \leq p \leq \infty$.

Hoelder inequality: Let $1 \leq p \leq \infty$ and let q denote the conjugate

exponent defined by

$\frac{1}{p} + \frac{1}{q} = 1$, (q=∞ if p=1, q=1 if p=∞). If $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$, then

$fg \in L_1(\Omega)$ and $\|fg\|_{L_1(\Omega)} \leq \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}$.

Minkowski's inequality: If $f, g \in L_p(\Omega)$, then $f + g \in L_p(\Omega)$ and

$$\|f + g\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}.$$

Young inequality: For p, q, r ∈ [1 , ∞] : $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and $f \in L_p$, $g \in L_q$ we have

$(f * g) \in L_r$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$ By $(f * g)$ (x) $= \int f(y) g(x - y) dy$ we denote the convolution of f and g.

Fubini's Theorem :(changes of order of integration).

If a function $f(x, y)$ defined in R^{n+m}, $x \in R^n$ and $\in R^m$, is measurable, and there exists a repeated integral of the function $|f(x, y)|$ and

$$\int [\int |f(x, y)|\, dx] dy < \infty$$

then $f(x, y)$is integrable, and the integrals

$$\int f(x, y) dx, \quad \int f(x, y) dy$$

exist almost everywhere and are integrable, and the equations

$$\int [\int f(x, y) dy]\, dx = \int \int f(x, y) dx dy = \int [\int f(x, y) dx]\, dy.$$

CHAPTER ONE: DISTRIBUTION THEORY

1.1 DISTRIBUTIONS (GENERALIZED FUNCTIONS):

A generalized function (distribution) is a generalization or an extension of the classical notion of a function.

1.1.1 The Space of Basic Functions D(Ω)

On a material which has point of mass 1, assume that the point is on the origin of the coordinates, the mean density $f_\varepsilon\,(x)$ is given by

$$f_\varepsilon\,(x) = \begin{cases} \frac{3}{4\pi\varepsilon^3} & \text{if } |x| < \varepsilon \\ 0 & \text{if } |x| > \varepsilon \end{cases} \qquad \text{.................................1}$$

As $\varepsilon \to 0$, the mean density $f_{\varepsilon\,(x)}$ becomes a function

$$\boldsymbol{\delta}(\boldsymbol{x}) = \begin{cases} \infty, if \;\; x = 0 \\ 0, if \;\; x \neq 0 \end{cases} \qquad \text{..............................2}$$

The integral of the density over the entire space yield the total mass of substance,

that is,

$$\int \delta(x)\,dx = 1 \qquad \text{....................3}$$

But for the functional $\delta(x)$ defined by (2) $\int \delta(x)dx$ =0. This means that the function does not satisfy the requirement of (3).

Let us now find a somewhat different limit of the sequences of mean densities $f_\varepsilon\,(x)$, the so called weak limit.

It will readily be seen that for any continuous function $\varphi(x)$,

$$\lim_{\varepsilon\to 0} \int f_\varepsilon\;(x)\,\varphi(x)dx = \varphi(0) \text{------------------------------------4}$$

Formula (4) states that the weak limit of a sequence of function $f_\varepsilon\,(x)$, $\varepsilon \to 0$ is a functional $\varphi(0)$, that relates to every continuous function $\varphi(x)$, a number$\varphi(0)$, which is its value at the point x=0. It is this functional which is taken as the definition of the density $\delta(x)$, and this is the well known Dirac $\boldsymbol{\delta}$ function.

So $f_\varepsilon(x) \xrightarrow{weak} \delta(x)$ as $\varepsilon \to 0$

The value of the function δ on the function φ (the number $\varphi(0)$) will be denoted,

$$\delta(\varphi) = \langle \delta, \varphi \rangle = \varphi(0)$$

This is the value of the functional δ on the function φ.

It is view point that serves as the basis for defining an arbitrary generalized function as a continuous linear functional on collection of sufficiently "good" so called basic functions.

In this sub-section we introduce the important space of basic function D (Ω) for any open set $\Omega \subset R^n$. We use $C^n(\Omega)$ to denote the set of all functions $f(x)$ that are continuous in Ω together with all derivatives $D^\alpha f(x)$, $|\alpha| \leq n$; $C^\infty(\Omega)$ is the collection of all infinitely differentiable function in Ω.

We introduce the norm in $C^n(\Omega)$ for n<∞ via the formula

$$\|\varphi\|_{n,\infty,\Omega} = \sum_{|\alpha| \leq n} \|D^\alpha \varphi\|_{L_\infty(\Omega)}.$$

Then $C^n(\Omega)$ is a Banach space, since completeness follows from the fact that, on a compact subsets, the uniform limit of continuous function is continuous.

Note that if $m \geq n$, then $\|\varphi\|_{m,\infty,\Omega} \geq \|\varphi\|_{n,\infty,\Omega}$, so we have a nested sequence of norms.

Definition 1.1.1: Let f be a complex valued function defined on an open subset $\Omega \subset R^n$. We call support of f and denote it by supp$f = \overline{\{x \in \Omega / f(x) \neq 0\}}$

The support of f is the smallest relative closed set outside of which f is identically zero.

Definition 1.1.2: Let $\Omega \subset R^n$ be an open set. Included in the set of Test functions D(Ω) are all functions which have compact support and infinitely differentiable functions in Ω.

That is,

$$D(\Omega) = \{\varphi \in C^\infty(\Omega) / \ supp\varphi \ is \ compact\} = C_0^\infty(\Omega)$$

Convergence in D(Ω): A given sequence $\{\varphi_n\}$ in D(Ω) converges to φ in D(Ω) if

i. ∃ a compact set $k \subset\subset \Omega$ such that supp$\varphi_n \subset k$, ∀n.

ii. $\forall \alpha \in N_0^n$, $D^\alpha \varphi_n(x) \Rightarrow D^\alpha \varphi(x)$, $x \in \Omega$, as $n \to \infty$,

In this case we shall write $\varphi_n \to \varphi$, as $n \to \infty$ in D (Ω).

The linear set D (Ω) equipped with convergence is called the space of test/ basic function D(Ω).

Example 1.1.1: $\varphi(x) = \begin{cases} e^{-\left(\frac{1}{1-|x|^2}\right)}, & when\ |x| < 1 \\ 0, when & |x| \geq 1 \end{cases}$ is in **D.**

For $|x| < 1$

$$\frac{\partial \varphi}{\partial x_j} = \frac{-2x_j}{(1-|x|^2)^2}\ \varphi(x)\ , \qquad |x|^2 = x_1^2 + \cdots + x_n^2$$

Any partial derivative of higher order has the form $\frac{p(x)}{(1-|x|^2)^n} \cdot \varphi(x)$; p(x) is a polynomial

Now put t=$|x|$

Then $\forall x \in R^n \backslash \{0\}$, x $\longmapsto$ $\|x\|$=t. Then

$$\varphi(x) = \varphi(t) = \begin{cases} e^{-\left(\frac{1}{1-t^2}\right)}, & when\ t < 1 \\ 0 & , when\ t \geq 1 \end{cases}$$

Then $D^n \varphi(t) = \frac{p(t)}{(1-|x|^2)^n} \cdot \varphi(t)$ which is infinitely differentiable.

Also

$$\text{Supp}\ (\varphi) = \overline{\{x \in \Omega: \varphi(x) \neq 0\}}$$
$$= \{x \in \Omega:\ |x| \leq 1\}$$

Obviously, the supp (φ) is a compact set (since it is closed and bounded) and $\varphi \in C^\infty(\Omega)$.

1 .1 .2 The Space of Distributions D'.

Definition 1.1.3: A continuous linear functional on the space of test functions D is called a generalized function /distribution/.The set of all generalized function is denoted by $D' = D'(\mathrm{R}^n)$.

We will write the value of the functional (generalized function) f on the basic function φ as $\langle f, \varphi \rangle$.

We say that $f \in D'$ if it satisfies the following conditions.

- A distribution $f \in D'$, is a functional on the space of basic function D, that is, with each basic function φ there is associated a (complex valued) number $\langle f, \varphi\rangle$.
- A distribution $f \in D'$, is a linear functional on D, that is,
  If $\varphi, \psi \in D$ and $\lambda, \beta \in \mathbb{C}, then$
  $$\langle f, \lambda\varphi + \beta\psi \rangle = \lambda \langle f, \varphi\rangle + \beta \langle f, \psi\rangle.$$
- A distribution $f \in D'$ is a continuous functional on D, that is, if $\varphi_n \to \varphi$ in D(Ω)as n → ∞ ,then
  $$\langle f, \varphi_n\rangle \to \langle f, \varphi\rangle \text{ as } n \to \infty.$$

The generalized functions f and g specified in Ω are said to be equal in Ω if they are equal as functional on D(Ω) , that is, if for any φ in D(Ω) , $\langle f, \varphi\rangle = \langle g, \varphi\rangle$. We will write: $f = g$ in Ω or $f(x) = g(x), x \in \Omega$.

Note: The space D' is linear if we define the linear combination $\lambda f + \mu g$ of the generalized function f and g in D(Ω) as a functional acting via the formula:

$$\langle \lambda f + \mu g, \varphi\rangle = \lambda\langle f, \varphi\rangle + \mu \langle g, \varphi\rangle, \ \varphi \in D.$$

Proof: we what to show that
$\lambda f + \mu g$ *is linear and continuous function on* D, *i.e. belong to* D'.

Let $\varphi \in D$ and $\psi \in D$ and α, β are any complex numbers. Then,

$$\begin{aligned}\langle \lambda f + \mu g, \alpha\varphi + \beta\psi\rangle &= \lambda\langle f, \alpha\varphi + \beta\psi\rangle + \mu \langle g, \alpha\varphi + \beta\psi\rangle \\ &= \alpha[\lambda\langle f, \varphi\rangle + \mu\langle g, \varphi\rangle] + \beta[\lambda\langle f, \psi\rangle + \mu\langle g, \psi\rangle] \\ &= \alpha\langle \lambda f + \mu g, \varphi\rangle + \beta\langle \lambda f + \mu g, \psi\rangle\end{aligned}$$

Hence, $\lambda f + \mu g$ is linear

Its continuity follows from the continuity of the functional f and g. If $\varphi_k \longrightarrow \varphi$ as k→ ∞ in D, then

$$\langle \lambda f + \mu g, \varphi_k \rangle = \lambda\langle f, \varphi_k\rangle + \mu\langle g, \varphi_k\rangle \longrightarrow \lambda\langle f, \varphi\rangle + \mu\langle g, \varphi\rangle = \langle \lambda f + \mu g, \varphi \rangle \text{ as } k \to \infty$$

Hence, $\lambda f + \mu g$ is continuous.

Therefore, $\lambda f + \mu g$ is a linear and continuous on D. i.e. $\lambda f + \mu g \in D'$.

Convergence in D'**:** Let $\{f_n\}$ be a sequence of distributions. Then f_n is said to be convergent to the distribution f if and only if

$$\langle f_n, \varphi\rangle \xrightarrow{n\to\infty} \langle f, \varphi\rangle \text{ for all } \varphi \in D(\Omega)$$

In this case we shall write $f_n \to f$ as $n \to \infty$ in D'

This convergence is called weak convergence.

The linear set D' together with the convergence which it is equipped is called the space of generalized function.

Theorem: 1.1.2 Suppose that f:D$(\Omega) \longrightarrow \mathbb{C}$ is linear .Then

$f \in D'(\Omega)$ if and only if for every $k \subset\subset \Omega$, there are $n \geq 0$ and C>0 such that

$$|f(\varphi)| \leq C|\varphi|_{n,\infty,\Omega} \text{ for every } \varphi \in D_k.$$

Proof: Suppose that $f \in D'(\Omega)$, but suppose also that the conclusion is false. Then there is some $k \subset\subset \Omega$ such that for every $n \geq 0$ *and* $m \geq 0$,we have some $\varphi_{n,m} \in D_k$ such that

$$|f(\varphi_{n,m})| > m\|\varphi_{n,m}\|_{n,\infty,\Omega}.$$

Normalize by setting $\widehat{\varphi}_j = \varphi_j/(j\|\varphi_j\|_{j,\infty,\Omega}) \in D_k$.

Then $|f(\hat{\varphi}_j)| > 1$, *but* $\hat{\varphi}_j \longrightarrow 0$ in D(Ω) (since $\|\hat{\varphi}_j\|_{n,\infty,\Omega} \leq \|\hat{\varphi}_j\|_{j,\infty,\Omega}$=1/j for $j \geq n$),

Contradicting the hypothesis.

For the converse, suppose that $\varphi_j \to 0$ in D(Ω).

Then there is some $k \subset\subset \Omega$ such that supp $(\varphi_j) \subset k$ for all j, and , by hypothesis , some n and C such that $|f(\varphi_j)| \leq C\|\varphi_j\|_{n,\infty,\Omega} \to 0$.

That is f is continuous at 0.

Definition1.1.4: A measurable function $f: \Omega \longrightarrow C$ is said to be locally integrable if

$$\int_K |f(x)|\, dx < \infty \text{ for all compact } K \subset \Omega.$$

We denote by $L^1_{loc}(\Omega)$.

Regular Distributions

The simplest example of a generalized function is the functional generated by the function f(x) locally integrable in R^n:

$$\langle f, \varphi\rangle = \int_{\Omega} f(x)\, \varphi(x)dx, \quad \varphi \in D \text{.................}1$$

From the property of linearity of the integral follows the linearity of this functional:

$$\begin{aligned}\langle f, \lambda\varphi + \beta\psi\rangle &= \int f(x)[\lambda\varphi(x) + \beta\psi(x)]dx \\ &= \lambda \int f(x)\varphi(x)dx + \beta \int f(x)\psi(x)dx \\ &= \lambda\, \langle f, \varphi\rangle + \beta\langle f, \psi\rangle\end{aligned}$$

While from the theorem which concerns proceeding to the limit under the integral sign follows the continuity of this functional on D:

$$\langle f, \varphi_n\rangle = \int f(x)\varphi_n(x)dx \;\rightarrow\; \int f(x)\varphi(x)dx = \langle f, \varphi\rangle$$

as n $\rightarrow \infty$ if $\varphi_n \rightarrow \varphi$ as n $\rightarrow \infty$ in D.

Thus the functional defined in (1) defines a generalized function belonging to D'.

Definition1.1.5: For $T_f \in D'(\Omega)$, if there is $f \in L^1_{loc}(\Omega)$ via the formula:

$\langle T_f, \varphi\rangle = \int_{\Omega} f(x)\, \varphi(x)dx, \quad \varphi \in D$, are called regular distributions and all other generalized functions are called singular distributions.

Example 1.1.3: Let $x_0 \in \Omega_0$ be a fixed element. We consider the function

δ_{x_0}: D (Ω) $\rightarrow \mathbb{C}$ defined by $\varphi \mapsto \varphi(x_0)$.

We have for each compact K $\subset \Omega$

$$\left|\delta_{x_0}(\varphi)\right| = |\varphi(x_0)| \leq \|\varphi\|_{0,\infty,\Omega}$$

Implies by theorem1.1.2 that $\delta_{x_0}$is continuous. We call $\delta_{x_0}$the Dirac mass, distribution or delta function at x_0. In the case $x_0 = 0$ we will write simple δ instead of δ_0.

The distribution $\delta_{x_0}$is not a regular distribution .This we will prove now.

Let $\Omega_0 = \Omega\backslash\{x_0\}$.Let K $\subset \Omega_0$ be a compact set.

Then (by definition) we have that $D_K(\Omega_0)$ is the set of all

$\varphi: \Omega_0 \longrightarrow$ C, $\varphi \in C^\infty(\Omega_0)$, supp ($\varphi$)$\subseteq K \subset \Omega_0$.Now we consider $\varphi \in \delta_{x_0}|D_K(\Omega_0)$.

For this φ we have$\varphi: \Omega \longrightarrow$ C , $\varphi \in C^\infty(\Omega)$, supp ($\varphi$) $\subseteq K \subset \Omega_0$.

Therefore, we have $\varphi(x_0) = 0$. Since $x_0 \notin \Omega_0$, i.e.$x_0 \notin supp(\varphi)$.

This implies $\delta_{x_0}|D_K(\Omega_0)$=0, i.e. $\langle\delta_{x_0},\varphi\rangle = 0$ for all $\varphi \in D_K(\Omega_0)$.

Now we assume that there is an $f \in L^1_{loc}(\Omega)$ such that $\delta_{x_0} = f$.

$$\langle\delta_{x_0},\varphi\rangle = \langle f,\varphi\rangle = \int_\Omega f(x)\varphi(x)dx = \int_{\Omega_0} f(x)\varphi(x)dx = 0 \text{ for all } \varphi \in D_K(\Omega_0).$$

immediately

$$f(x) = 0 \text{ for almost all } x \in \Omega_0,$$

and therefore,

$$f(x) = 0 \text{ for almost all } x \in \Omega.$$

From this it follows δ_{x_0}=0, i.e.$\langle\delta_{x_0},\varphi\rangle = 0$ for all $\varphi \in D(\Omega)$.

But this is not true, because for a function $\varphi \in D(\Omega)$ for which $x_0 \in supp(\varphi)$ we get

$$\langle\delta_{x_0},\varphi\rangle = \varphi(x_0) \neq 0 \text{ , i.e } \delta_{x_0} \neq 0.$$

Definition 1.1.6: The support of a generalized function f is the completion of Ω_f to Ω ,

so that, Supp$f = \Omega \setminus \Omega_f$:supp$f$ is closed set in Ω. Where Ω_f is the largest open set in which f vanishes. Also, the support of f is the smallest closed subset of Ω out side of which the distribution f is zero.

Definition1.1.7: A distribution $f \in D'(\Omega)$ has a compact support if and only if there is a compact set K $\subseteq \Omega$ such that supp $\varphi \cap$ K = { } implies $\langle f,\varphi\rangle$= 0 $\forall\ \varphi \in$ D(Ω).

Lemma1.1.1: Let $\{f_k\}$ be a sequence of distributions.

If there is a linear mapping f: D $(R^n) \longrightarrow$ C such that

$$\langle f_k,\varphi\rangle \xrightarrow{k\to\infty} \langle f,\varphi\rangle \text{ for all } \varphi \in D(R^n),$$

then f is a distribution.

Proof: We have only to show that f is linear and continuous,

i.e. $f \in D'(R^n)$. From

$$\langle f,\varphi\rangle = \lim_{k\to\infty}\langle f_k,\varphi\rangle$$

We get $\langle f,\varphi_1+\varphi_2\rangle = \lim_{k\to\infty}\langle f_k,\varphi_1+\varphi_2\rangle$

$= \lim_{k\to\infty}(\langle f_k,\varphi_1\rangle + \langle f_k,\varphi_2\rangle)$

$= \lim_{k\to\infty}\langle f_k,\varphi_1\rangle + \lim_{k\to\infty}\langle f_k,\varphi_2\rangle$

$= \langle f,\varphi_1\rangle + \langle f,\varphi_2\rangle$ and

$$\langle f,\alpha\varphi\rangle = \lim_{k\to\infty}\langle f_k,\alpha\varphi\rangle$$
$$=\alpha\lim_{k\to\infty}\langle f_k,\varphi\rangle$$
$$=\alpha\langle f,\varphi\rangle$$

i.e. f is linear. For the proof of the continuity of f we consider a sequence (φ_v), where $\varphi_v \in D(R^n)$ and $\varphi_v \longrightarrow 0$ as $v \longrightarrow \infty$.We have to prove that $\langle f,\varphi_V\rangle \longrightarrow 0$ as $v \to \infty$, i.e. we have to prove that f is continuous at 0.

We assume that the last is not true. Then there is an $\alpha > 0$ such that

$$|\langle f,\varphi_v\rangle| \geq 2\alpha \text{ for all v} \in N.$$

Since$\langle f,\varphi\rangle = \lim_{k\to\infty}\langle f_k,\varphi\rangle$ for each $\varphi \in D(R^n)$ we get particularly $\langle f,\varphi_v\rangle = \lim_{k\to\infty}\langle f_k,\varphi_v\rangle$ and therefore

$$2\alpha \leq |\langle f,\varphi_v\rangle| = \left|\lim_{k\to\infty}\langle f_k,\varphi_v\rangle\right| = \lim_{k\to\infty}|\langle f_k,\varphi_v\rangle| \text{ for all v} \in N.$$

From this it follows that for each $\epsilon > 0$ and for each v $\in N$ there is an index k_v such that

$$2\alpha - \epsilon \leq \left|\langle f_{k_v},\varphi_v\rangle\right| \text{ for all v} \in N.$$

To make it easier we choose $\epsilon = \alpha$.Then we have that for each v $\in N$ there is an index k_v such that

$$\alpha \leq \left|\langle f_{k_v},\varphi_v\rangle\right| \text{ for all v} \in N. \text{ ------------------------*}$$

since $\varphi_v \longrightarrow 0$ as v $\to \infty$ in D(φ),then $\langle f,\varphi_v\rangle \to 0$ as v $\to \infty$ .Then we get that $\langle f_{k_v},\varphi_v\rangle \to 0$ as v $\to \infty$ for all k $\in N$, since $f_k \in D'(R^n)$.This implies that $\langle f_{k_v},\varphi_v\rangle \longrightarrow 0$ as v $\to \infty$.

But this is a contradiction to *.

Example 1.1.4: We consider Cauchy's principal value of

$$\int_R \frac{\varphi(x)}{x}dx, \quad \text{where } \varphi \in D(R).$$

Cauchy's principal value is defined as

$$\text{CPV}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}dx\right]$$

We define the mapping

$$\langle p,\varphi\rangle = \text{CPV}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx$$

and we show that p is a distribution. First we show that the mapping p_k defined by

$$\langle p_k,\varphi\rangle=\int_{|x|>k}\frac{\varphi(x)}{x}dx,\ \varphi\ \in D(R),$$ is a regular distribution for each k $\in N$.

For this we define

$$f_k(x)=\begin{cases}0, if\ \ |x|\le\frac{1}{k}\\ \frac{1}{x}, if\ \ |x|>\frac{1}{k}\end{cases},\quad k\in N.$$

Then f_k is locally integrable. To see this we consider an arbitrary compact set k $\subset R$. Let a=mink, b= maxk and c=max{|a|, |b|, $\frac{1}{k}$}. Then we have

$$\int_k |f_k(x)|\,dx\le\int_{-c}^{c}|f_k(x)|dx\ =\int_{-c}^{-\frac{1}{k}}|f_k(x)|dx+\int_{-\frac{1}{k}}^{\frac{1}{k}}|f_k(x)|dx+\int_{\frac{1}{k}}^{c}|f_k(x)|dx$$

$$=-\ln(-x)\Big|_{-c}^{\frac{-1}{k}}+lnx\Big|_{\frac{1}{k}}^{c}=-\left(ln\frac{1}{k}-lnc\right)+lnc-ln\frac{1}{k}$$

$$=2[lnc-\ln\tfrac{1}{k}]<\infty.$$

By definition 1.1.5 we get that

$$\langle T_{f_k},\varphi\rangle=\int_{-\infty}^{\infty}f_k(x)\varphi(x)dx=\int_{-\infty}^{\frac{-1}{k}}\frac{\varphi(x)}{x}dx+\int_{\frac{1}{k}}^{\infty}\frac{\varphi(x)}{x}dx$$

$$=\int_{|x|>\frac{1}{k}}\frac{\varphi(x)}{x}dx=\langle p_k,\varphi\rangle$$ is

a regular distribution.

Now we have

$$\lim_{k\to 0}\langle T_{f_k},\varphi\rangle=\lim_{k\to 0}\langle p_k,\varphi\rangle$$

$$=\lim_{k\to 0}\left[\int_{-\infty}^{-\frac{1}{k}}\frac{\varphi(x)}{x}dx+\int_{\frac{1}{k}}^{\infty}\frac{\varphi(x)}{x}dx\right]$$

$$=\mathrm{CPV}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx=\langle p,\varphi\rangle$$

i.e p is the (weak) limit of a sequence of (regular) distributions. Therefore, by lemma 1.1.1 we get that p is a distribution.

1.2 Differentiation, multiplication by smooth function, direct product and convolution of distribution.

Differentiation of distribution

Definition 1.2.1: Let $\Omega \subseteq R^n, f \in D^{'}(\Omega)$ and α be a multi − index . Then the mapping

$$D^{\alpha} : D^{'}(\Omega) \longrightarrow D^{'}(\Omega) \text{ defined by}$$

$$\langle D^{\alpha} f, \varphi\rangle = (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi\rangle, \quad \forall\varphi \in D(\Omega)$$

is called the derivative of f

Let us see whether $\boldsymbol{D^{\alpha} f} \in D^{'}(\Omega)$

Clearly $D^{\alpha} f$ is a functional since $f \in D'$

Linearity: Let $\varphi,\psi \in D$ *and* $\lambda, \beta \in \mathbb{C}$, then

$$\begin{aligned}\langle D^{\alpha} f, \lambda\varphi + \beta\psi\rangle &= (-1)^{|\alpha|} \langle f, D^{\alpha}(\lambda\varphi + \beta\psi)\rangle \\ &= (-1)^{|\alpha|} \langle f, \lambda D^{\alpha}\varphi + \beta D^{\alpha}\psi\rangle \\ &= \lambda(-1)^{|\alpha|} \langle f, D^{\alpha}\varphi\rangle + \beta(-1)^{|\alpha|} \langle f, D^{\alpha}\psi\rangle \\ &= \lambda \langle D^{\alpha} f, \varphi\rangle + \beta\langle D^{\alpha} f, \psi\rangle\end{aligned}$$

Hence $D^{\alpha} f$ is a linear functional on D.

Continuity: Let $\varphi_n \to \varphi$ as n → ∞ in D, then

$$\langle D^{\alpha} f, \varphi_n\rangle = (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi_n\rangle \to (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi\rangle = \langle D^{\alpha} f, \varphi\rangle.$$

This implies that $\langle D^{\alpha} f, \varphi_n\rangle \to \langle D^{\alpha} f, \varphi\rangle$ as n → ∞.

Hence $D^{\alpha} f$ is continuous.

$\therefore D^{\alpha} f \in D^{'}$.

Example 1.2.1: The Dirac measure δ on R^n defined by

$\langle\delta, \varphi\rangle = \varphi(0)$, for all $\varphi \in D(\Omega)$ is a distribution.

Let Ω = R and consider that Heaviside *function*

$$H(x) = \begin{cases} 1, if\ x > 0 \\ 0, if\ x < 0 \end{cases}$$

Then the derivative of H(x) in the sense of distribution is defined by:

$$\langle DH, \varphi\rangle = -\langle H, \varphi'\rangle = -\int_{-\infty}^{\infty} H(x)\, \varphi^{'}(x)dx$$

$$=-\int_{-\infty}^{0} H(x)\varphi'(x)dx - \int_{0}^{\infty} H(x)\varphi'(x)dx$$
$$= -\int_{-\infty}^{0} 0\varphi' dx - \int_{0}^{\infty} 1\varphi'(x)dx$$
$$= -\int_{0}^{\infty} \varphi'(x)dx$$
$$=-\lim_{b\to\infty}\int_{0}^{b} \varphi'(x)dx = -\lim_{b\to\infty}[\varphi(x)\Big|_0^b]$$
$$=-\lim_{b\to\infty}[\varphi(b)-\varphi(0)]$$
$$= \varphi(0)$$
$$= \langle\delta,\varphi\rangle$$

Therefore, $H'=\delta$ the Dirac measure.

Proposition1.2.1: If $f \in D'(\Omega)$ and α *and* β are multi-indices, then

$$D^{\alpha}(D^{\beta}f) = D^{\beta}(D^{\alpha}f) = D^{\alpha+\beta}f.$$

Proof: $\langle D^{\alpha+\beta}f,\varphi\rangle = (-1)^{|\alpha|+|\beta|}\langle f, D^{\alpha+\beta}\varphi\rangle$

$$=(-1)^{|\beta|}\langle D^{\alpha}f, D^{\beta}\varphi\rangle = \langle D^{\beta}(D^{\alpha}f),\varphi\rangle$$
$$=(-1)^{|\alpha|}\langle D^{\beta}f, D^{\alpha}\varphi\rangle = \langle D^{\alpha}(D^{\beta}f),\varphi\rangle$$

Hence $D^{\alpha+\beta}f = D^{\beta}(D^{\alpha}f)=D^{\alpha}(D^{\beta}f)$

Lemma 1.2.1:The operation of differentiation D^{α} is linear and continuous from D' into D', that is,

Linearity: let f, g $\in D'(\Omega)$ and λ , $\beta \in \mathbb{C}$, we have

$$\langle\lambda f + \beta g,\varphi\rangle = \lambda\langle f,\varphi\rangle + \beta\langle g,\varphi\rangle, \forall\ \varphi \in D(\Omega).$$

then $D^{\alpha}\langle\lambda f + \beta g,\varphi\rangle = \langle D^{\alpha}(\lambda f + \beta g),\varphi\rangle$

$$= (-1)^{|\alpha|}\langle\lambda f + \beta g, D^{\alpha}\varphi\rangle$$
$$=(-1)^{|\alpha|}\lambda\langle f, D^{\alpha}\varphi\rangle + (-1)^{|\alpha|}\beta\langle g, D^{\alpha}\varphi\rangle$$
$$= \lambda\langle D^{\alpha} f,\varphi\rangle + \beta\langle D^{\alpha} g,\varphi\rangle$$

Continuity: suppose $f_k \to 0$, k $\to\infty$ in $D'(\Omega)$. Then for all $\varphi \in D(\Omega)$, we have

$\langle D^{\alpha}f_k,\varphi\rangle = (-1)^{|\alpha|}\langle f_k, D^{\alpha}\varphi\rangle \to 0$, k $\to\infty$, and this implies that

$D^{\alpha}f_k \to 0$, as k$\to\infty$ in $D'(\Omega)$.

Lemma 1.2.2 (Leibniz Rule).Let$f \in C^{\infty}(\Omega)$, u$\in D^{'}(\Omega)$, and α a multi-index. Then

$$D^{\alpha}(fu)=\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}fD^{\beta}u \in D^{'}(\Omega),$$

Where

$$\binom{\alpha}{\beta}=\frac{\alpha!}{(\alpha-\beta)!\beta!},$$

$\alpha! = \alpha_1!\,\alpha_2!\cdots\alpha_n!$ and $\beta \leq \alpha$ means that β is a multi-index with $\beta_i \leq \alpha_i$ for i=1,… n.

If u $\in C^{\infty}(\Omega)$, this is just the product rule for differentiation.

Proof. By the previous proposition, we have the theorem if it is true for multi-indices that have a single non zero component, say the first component. We proceed by induction on n=$|\alpha|$.

The result holds for n=0, but we will need the result for n=1.Denote D^{α} by D_1^{α}.

When n=1, for any $\varphi \in D(\Omega)$

$$\begin{aligned}\langle D_1(uf),\varphi\rangle &= -\langle fu, D_1\varphi\rangle\\ &= -\langle u, fD_1\varphi\rangle\\ &= -\langle u, D_1(f\varphi)-D_1f\varphi\rangle\\ &= \langle D_1u, f\varphi\rangle+\langle u, D_1f\varphi\rangle\\ &= \langle fD_1u+D_1fu,\varphi\rangle,\end{aligned}$$

and the result holds.

Now assume the result for derivative up to order n-1. Then

$$D_1^n(fu)=D_1D_1^{n-1}(fu)$$

$$=D_1\sum_{j=0}^{n-1}\binom{n-1}{j}D_1^{n-1-j}fD_1^{j}u$$

$$=\sum_{j=0}^{n-1}\binom{n-1}{j}(D_1^{n-j}fD_1^{j}u+D_1^{n-1-j}fD_1^{j+1}u)$$

$$=\sum_{j=0}^{n-1}\binom{n-1}{j}D_1^{n-j}fD_1^{j}u+\sum_{j=0}^{n-1}\binom{n-1}{j-1}D_1^{n-j}fD_1^{j}u$$

$$=\sum_{j=0}^{n-1}\binom{n}{j} D_1^{n-j} f D_1^{j} u \,,$$

where the last equality follows from the combinatorial identify

$$\binom{n}{j}=\binom{n-1}{j}+\binom{n-1}{j-1},$$

and so the induction proceeds..

Theorem1.2.1: If $\langle f_k,\varphi\rangle \xrightarrow{k\to\infty} \langle f,\varphi\rangle$ for all $\varphi \in \mathrm{D}(\Omega)$, then

$\langle D^{\alpha} f_k,\varphi\rangle \xrightarrow{k\to\infty} \langle \mathrm{D}^{\alpha} f,\varphi\rangle$ for all $\varphi \in \mathrm{D}(\Omega)$, for all multi indices.

Proof: $\langle D^{\alpha} f_k,\varphi\rangle = (-1)^{|\alpha|}\langle f_k,\mathrm{D}^{\alpha}\varphi\rangle \xrightarrow{k\to\infty} (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi\rangle = \langle D^{\alpha} f,\varphi\rangle$ for all $\varphi \in D(\Omega)$.

Multiplication of a distribution by a function $f \in C^{\infty}(\Omega)$

Definition1.2.2; Let $\mathrm{T} \in D'(\Omega)$, $\varphi \in \mathrm{D}(\Omega)$ and $f \in C^{\infty}(\Omega)$.

Then $\langle f\cdot \mathrm{T},\varphi\rangle = \langle \mathrm{T}, f\cdot\varphi\rangle$ $\varphi \in D(\Omega)$ is called the prodact of f and T.

Example1.2.2: Let $\mathrm{T}=\delta$. Then

$$\begin{aligned}\langle f\cdot \mathrm{T},\varphi\rangle &= \langle f\cdot\delta,\varphi\rangle\\ &= \langle \delta, f\cdot\varphi\rangle\\ &= f(0)\varphi(0)\\ &= f(0)\delta(\varphi)\end{aligned}$$

Therefore $(f\cdot\delta) = f(0)\delta$

If f_1 and f_2 are two distributions, we cannot assign a distribution $\Pi(f_1, f_2)$ in the same way that it amount of the usual product for regular distribution and is at the same time continuous in both distribution. Let $f_k(x) = \sin(kx)$ in $D'(\Omega)$. Then

$$\pi(f_k,f_k) = \sin^2(kx)\ (\text{i.e. } \pi(f_k,f_k) = \pi(f_k)\,\pi(f_k) = f_k(x)\, f_k(x)).$$

Although $f_k \to 0$ in D' as $k\to\infty$

$$\pi(f_k,f_k) \to \frac{1}{2} \text{ as } k\to\infty$$

Solution:

For any test functionφ, we have

$$\langle f_k, \varphi\rangle = \int_R \sin(kx)\varphi(x)dx$$

$$= -\frac{1}{k}\int_R \cos(kx)\varphi'(x)dx, \text{ using integration by part}$$

So that $|\langle f_k, \varphi\rangle| \xrightarrow{k\to\infty} 0$, for all $\varphi \in D(\Omega)$

This implies that $f_k \to 0$ in $D'(\Omega)$.

Similarly

$$\langle \pi(f_k, f_k), \varphi\rangle = \int_R \sin^2(kx)\, \varphi(x)dx$$

$$= \int_R \frac{1-\cos(2kx)}{2}\varphi(x)dx$$

$$= \int_R \frac{\varphi(x)}{2}dx - \int_R \frac{\cos(2kx)}{2}\varphi(x)dx$$

$$= \frac{1}{2}\int_R \varphi(x)dx - \frac{1}{2}\int_R \cos(2kx)\varphi(x)dx$$

$$= \frac{1}{2}\int_R \varphi(x)dx - \frac{1}{4k}\int_K \sin(2kx)\varphi'(x)dx \text{ using integration by part.}$$

So that, $|\langle \pi(f_k, f_k), \varphi\rangle| \xrightarrow{k\to\infty} \frac{1}{2}\int_R \varphi(x)dx = \langle \frac{1}{2}, \varphi\rangle$

This implies that

$$\pi(f_k, f_k) \to \frac{1}{2} \text{ as } k \to \infty$$

Theorem1.2.2: Let $f \in D'(\Omega)$ and $\varphi \in D(\Omega)$. Then

$$\text{Supp}(f\varphi) \subset \text{supp} f \cap \text{supp}\varphi$$

Proof: Let $\Omega_{f\varphi}$ be the largest open set in Ω which contains both Ω_f and Ω_φ.

That is $\Omega_f \cup \Omega_\varphi \subset \Omega_{f\varphi}$.

$$\begin{aligned}\text{Supp} f\varphi = \Omega \backslash \Omega_{f\varphi} &\subset \Omega \backslash (\Omega_f \cup \Omega_\varphi) \\ &\subset (\Omega \backslash \Omega_\varphi) \cap (\Omega \backslash \Omega_\varphi) \\ &\subset \text{Supp} f \cap \text{Supp}\varphi.\end{aligned}$$

Direct Product of Distribution

Definition1.2.3: Let f(x) and g(y) be locally integrable functions in the open set $\Omega_1 \subset R^n$ and $\Omega_2 \subset R^m$, respectively. The function f(x) g(y) is also be locally integrable in R^{n+m}. It defines a (regular) generalized functions f(x) g(y) in $D'(\Omega_1 \text{ x } \Omega_2)$ operating on the basic functions $\varphi(x, y)$ in $D(\Omega_1 \text{x} \Omega_2)$ via the formula:

$$\begin{aligned}< f(x) \otimes g(y), \varphi >&= \int_{\Omega_1 \otimes \Omega_2} f(x) g(y) \varphi(x, y) dxdy \\ &= \int_{\Omega_1} f(x) \int_{\Omega_2} g(y)\, \varphi(x, y) dy\, dx = \left\langle f(x), \left\langle g(y), \varphi(x, y) \right\rangle \right\rangle \\ &= \int_{\Omega_2} g(y) \int_{\Omega_1} \text{f(x)}\ \varphi(x, y) dxdy \\ &= \left\langle g(y), \left\langle f(x), \varphi(x, y) \right\rangle \right\rangle\end{aligned}$$

That is $\langle f(x)g(y), \varphi \rangle = \langle f(x), \langle g(y), \varphi(x, y) \rangle \rangle$

$\langle g(y)f(x), \varphi \rangle = \langle g(y), \langle f(x), \varphi(x, y) \rangle \rangle$

The Convolution of distributions

Definition 1.2.4: Let f and g be locally integrable function in R^n .If the integral

$\int f(y) g(x - y) dy$ exist for almost all $x \in R^n$,

and if the function

$$(f * g)(x) = \int f(y) g(x - y) dy$$

is locally integrable in R^n, then $f * g$ is called the convolution of f and g.

The convolution $f * g$ defines a(regular)generalized function acting on the test functions $\varphi \in D(R^n)$ according to the rule:

$$\langle f * g, \varphi \rangle = \int (f * g)(x)\varphi(x)dx$$
$$= \int \varphi(x)[\int f(y)g(x-y)dy]dx$$
$$= \int f(y)[\int g(x-y)\varphi(x)dx]dy$$
$$= \int f(y) \int g(z)\varphi(z+y)dzdy, \text{ set: z=x-y}$$

(by virtue of fubini's theorem), that is,

$$\langle f * g, \varphi \rangle = \int \int f(x)g(y)\varphi(x+y)dxdy, \forall \varphi \in D(R^n).$$

Condition for the existence of a convolution: - Let f be an arbitrary and g a generalized function with compact support. Then the convolution $f * g$ exists in $D'$and appears in the form: $\langle f * g, \varphi \rangle = \langle f(x) \otimes g(y),\ \eta(y)\varphi(x+y) \rangle, \varphi \in D$----------------------- (1)

When η is any test function equal to 1 in the neighborhood of the support of g. For this the convolution is continuous with respect to f and g separately:

If $f_k \to f as k \to \infty in D^{'}, then\ f_k * g \to f * g\ as\ k \to \infty\ in\ D'$

If $g_k \to g\ as\ k \to \infty\ and\ for\ acertain\ R$, supp$g_k\, c\ u_R$, then

$$f * g_k \to f * g\ as\ k \to \infty\ in\ D'$$

There are some cases, where the convolution $f * g$ exists.

We want to give two examples now.

Example1.2.4. Let $\in L^P(R^n)$, $g \in L^q(R^n)$, $\frac{1}{p} + \frac{1}{q} \geq 1$.

Then the convolution exist and $f * g \in L^r(R^n)$, where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

To prove it we choose$\alpha \geq 0, \geq 0$, $s \geq 1, t \geq 1$ such that

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1,\ \alpha r = \text{p} = (1-\alpha)\text{ s}, \qquad \beta r = \text{q} = (1-\beta)\text{ t}$$

Then

$$\text{P} + \frac{pr}{s} = \text{r} = \text{q} + \frac{qr}{t}\ .$$

The last comes from $\text{P} + \frac{pr}{s} = \text{pr}\ (\frac{1}{r} + \frac{1}{s}) = pr(1 - \frac{1}{t}) = \text{pr}\frac{1}{p} = \text{r}$. Then we get (using the Holder inequality and Theorem of Fubini)

$$\|f * g\|_{L^r}^r = \int_{R^n} | \int_{R^n} f(y)g(x-y)dy|^r \, dx$$

$$\leq \int_{R^n} [\int_{R^n} |f(y)|^\alpha \cdot |g(x-y)|^\beta \cdot |f(y)|^{1-\alpha} \cdot |g(x-y)|^{1-\beta} \text{dy}]\text{dx}$$

$$\leq \int_{R^n} \int_{R^n} |f(y)|^{\alpha r} \cdot |g(x-y)|^{\beta t} \text{dy} \, [\int_{R^n} |f(y)|^{(1-\alpha)s} \, dy]^{\frac{r}{s}} \cdot$$

$$[\int_{R^n} |g(x-y)|^{(1-\beta)t} \, dy]^{\frac{r}{t}} \text{dx}$$

$$\leq \|f\|_{L^p}^r \|g\|_{L^q}^r$$

Example1.2.5. Let $f \in L_{loc}^1(R^n)$ and g $\in L_{loc}^1(R^n)$, where

Supp f $\subseteq$A, supp g $\subseteq$B, and Let the set

D_R= {(x, y) ∈ A × B : ||x+y|| ≤ R} ⊆ $R^n \times R^n = R^{2n}$ be bounded for any R > 0.

Then $\int_{R^n} f(y)$g(x−y)dy exist for almost all x ∈ R^n, i.e $f * g$ exists.

This can be proved as follows (using Theorem of Fubini). We have for each R>0

$$\int_{\|x\| \leq R} |(f * g)(\text{x})|\text{dx} = \int_{\|x\| \leq R} | \int_{R^n} f(y)\text{g(x−y)dy} |\text{dx}$$

$$\leq \int_{\|x\| \leq R} \int_{R^n} |f(y)| \cdot |\text{g}(x-y)| \text{dydx}$$

$$\leq \int_{\|\xi + y\| \leq R} |f(y)| \cdot |\text{g}(\xi)| \text{dy} d\xi$$

$$\leq \int_{D_R} |f(y)| \cdot |\text{g}(\xi)| \text{dy} d\xi < \infty.$$

The last we get since D_R is bounded.

So we have that for each compact set K⊆ R^n there is an R>0

Such that K ⊆ {x ∈ R^n: ||x|| ≤ R}. So we have

$$\int_K | (f * g)(x)|\text{dx} \leq \int_{\|x\| \leq R} | (f * g)(x)|\text{dx} < \infty ,$$

i.e. $f * g$is locally integrable .

Obviously, if there is a set A' (or a set B')such that

supp f ⊆ A' ⋐ A (or supp g ⊆ B' ⋐ B) such that f (or g) is zero outside of

A' (or B'), then we have that

D_R= {(x,y)∈ $A' \times B$/ ||x+y|| ≤ R} ⊆ $R^n \times R^n = R^{2n}$

or

D_R= {(x,y)∈ $A \times B'$/ ||x+y|| ≤ R} ⊆ $R^n \times R^n = R^{2n}$) is bounded.

Theorem 1.2.4 Let $f\ ,g \in D^{'}(R^n)$, let $f * g$ exist and α be a multi index.

Then $(D^\alpha f) * g$ $\ and\ f * D^\alpha g\ exist\ and$

$$D^\alpha(f * g)\ =(\ D^\alpha f) * g\ =f * (D^\alpha g\ )$$

Proof $\langle\frac{\partial}{\partial xj}(f * g), \varphi\rangle$= -$\langle f * g\ ,\frac{\partial\varphi}{\partial xj}\rangle$

$$= -\ \langle f(x) \otimes g(y),\ \frac{\partial\varphi\,(x+y)}{\partial x_j}\rangle$$

$$= -\ \langle g(y), \langle f(x), \frac{\partial\varphi\,(x+y)}{\partial x_j}\rangle\rangle$$

$$=\ -\langle g(y), -\ \langle\frac{\partial}{\partial x_j} f, \varphi(x+y)\rangle\rangle$$

$$=\ \langle\frac{\partial f}{\partial x_j} * g\ ,\varphi\ \rangle$$

Similarly, $\frac{\partial}{\partial xj}\,(f * g) = \frac{\partial f}{\partial xj} * g = f * \frac{\partial g}{\partial x_j}$

So that by induction for any multi-index $\alpha \in N_0^n$.

$$D^\alpha\ (f * g) = D^\alpha f\ * g = f * D^\alpha g.$$

Lemma1.2.4: The convolution of distribution f in D′ with δ exist and is equal to f.

i.e. $f*\delta= \delta*f=f$

Proof. In fact, by virtue of (1) for all $\varphi \in D$ we have

$$\langle f * \delta, \varphi\ \rangle = \langle f(x)\otimes\delta(y), \eta(y)\ \varphi(x+y)\rangle$$

$$=\langle f(x), \langle\delta(y), \eta(y)\ \varphi(x+y)\rangle$$

$=\langle f, \varphi\rangle$, since $\langle\delta(y), \eta(y)\ \varphi(x+y)\rangle = \varphi(x)$

Similarly,$\langle\delta*f,\varphi\rangle = \langle\delta(x) \otimes f\ (y), \eta(x)\ \varphi(x+y)\rangle$

$$= \langle f(y)\otimes\delta(x), \eta(x)\ \ \varphi(x+y)\rangle$$

$$= \langle f(y), \langle\delta(x), \eta(x)\ \ \varphi(x+y)\rangle$$

$=\langle f,\varphi\rangle$, since $\langle\delta(x), \eta(x)\ \varphi(x+y)\rangle = \varphi(y)$

Hence, $f*\delta= \delta*f=f$

Theorem 1.2.5: Let $f, g\ \in\ D^{'}(R^n)$ and suppose that at least one has compact support. We have

$$\mathrm{Supp}(f*g) \subset \mathrm{Supp}f\ + \mathrm{Supp}g$$

Proof: Let A=Supp$f$ and B= Supp$g$. Since A and B are closed and at least one is compact, the set

$$A + B = \{x + y : x \in A, y \in B\}$$

is closed. Let us show that $f * g$ is equal to zero on the open set $\Omega = (A + B)^c$.

Indeed, if $\varphi \in D(\Omega)$, the support of $\varphi(\xi + \eta)$ is contained in the open set

$$\{(\xi, \eta) \in R^n \times R^n : \xi + \eta \in \Omega\}$$

On the other hand, the support of $f \otimes g$ is in A × B.

Since $(\xi, \eta) \in A \times B$ implies $\xi + \eta \in A + B$, the support of $f \otimes g$ does not intersect the support of $\varphi(\xi + \eta)$; Consequently,

$$\langle f * g, \varphi \rangle = \langle f_\xi \otimes g_\eta, \varphi(\xi + \eta) \rangle \text{ for all } \varphi \in D(\Omega).$$

Therefore, the support of the convolution $f * g$ is contained in the sum of the supports of f *and* g

i.e. $\text{Supp}(f * g) \subset \text{Supp}f + \text{Supp}g$

1.3. Tempered Distribution (generalized functions of slow growth)

1.3.1. The Space of basic (rapidly decreasing) functions

Definition1.3.1: Let $f : R^n \to \mathbb{C}$ *and* $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. The fucnction f is said to be rapidly decreasing if and only if

$$\lim_{|x| \to \infty} x^\alpha f(x) = 0 \text{ } for \ all \ \alpha \in N_0{}^n$$

The space S $(R^n) = \{f \in C^\infty(\Omega) \,/ D^\beta f$ *is* rapidly decreasing for all $\beta \in N_0^n\}$ is called the space of rapidly decreasing functions.

Therefore , the space S= S(R^n) of all functions infinitely differentiable in R^n that decrease together with all their derivatives, as $|x| \to \infty$, faster than any power of $|x|^{-1}$ is called the space of rapidly decreasing functions. For instance, $\varphi(x) = e^{-|x|^2} \in S$.

Convergence in S: The sequence of function $\varphi_{1,} \varphi_2$, --------- belonging to S converges to a function φ in S, denoted by $\varphi_k \to \varphi$ as k → ∞ in S. if for all α and β

$$x^\beta D^\alpha \varphi_k(x) \Rightarrow x^\beta D^\alpha \varphi(x), \text{ as } k \to \infty$$

1.3.2 The Space of Tempered Distributions S'

Definition1.3.2: A generalized function of slow growth is any continuous linear functional on the space S of test functions. We denote by S'= S'(R^n) the set of all generalized functions of slow growth.

Convergence in S': A sequence of generalized functionsf_1,f_2,------- taken from S' converges to the generalized function f ∈S', denoted by $f_k \to f$, as k → ∞ in S', if for any $\varphi \in S$, $\langle f_k, \varphi\rangle \to \langle f, \varphi\rangle$, k → ∞. This convergence is called a weak convergence of a sequence of functional.

The linear set **S'(Ω)** equipped with convergence is termed the spaces of generalized functions of slow growths S'

The space S'(R^n) is a sub space of D'(R^n), that is, let$f \in S'$, the$\langle f, \varphi \rangle$ is defined for all $\varphi \in S$ but D ⊂ S.

Let φ ∈ D, which implies φ ∈ S the $\langle f, \varphi\rangle$ is defined. Since φ is arbitrary in D, ⟨f, φ⟩ is defined for all φ ∈ D.

$\therefore f \in D'$, hence S'⊂D'

Convergence S' implies convergence D', that is,

Letf_1, f_2, ------ be a sequence function in S' converges to$f \in S'$.

i.e. $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ as n → ∞ for all $\varphi \in S$

As D⊂ S, $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ as n → ∞ for all $\varphi \in D$.

Then it follows that convergence in D'.

Example1.3.2 a) We say that a continuous function f is slowly increasing at infinity if there exists an integer k ≥ 0 such that $(1+r^2)^{-k/2}$ f(x) is bounded in R^n

Every continuous function f slowly increasing at infinity defines a tempered distribution. In fact, set

$$\langle f, \varphi\rangle = \int_{R^n} f(x)\, \varphi(x) dx,\ \forall \varphi \in S$$

We have $|\langle f, \varphi\rangle| \le \int_{R^n} |f(x)\varphi(x)|\, dx = \int_{R^n} \left|(1+r^2)^{-k/2} f(x)\, (1+r^2)^{k/2}\varphi(x)\right| dx$

$$\le C \int_{R^n} \left|(1+r^2)^{k/2}\varphi(x)\right| dx$$

Where $\left|(1+r^2)^{-k/2} f(x)\right| \le C$ (constant)

By observing that, if a sequence $\{\varphi_j\}$ converges to zero in S, then for every $k \geq 0$, $((1+r^2)^{k/2}\varphi_j)$ converges to zero in L_1, we conclude that f defines an element of S'.

b) Any derivative (in the sense of distributions) of a continuous function f slowly increasing at infinity defines a tempered distribution.

Let $T = \partial^\alpha f$ and define

$$\langle T, \varphi\rangle = (-1)^{|\alpha|} \int_{R^n} f(x)\, \partial^\alpha \varphi(x)\mathrm{dx},\ \forall \varphi \in S$$

We have $|\langle T, \varphi\rangle| \leq \int_{R^n} |f(x)||\partial^\alpha \varphi(x)|\,\mathrm{dx}$

$$= \int_{R^n} \left|(1+r^2)^{-k/2} f(x)(1+r^2)^{k/2}\partial^\alpha \varphi(x)\right|\mathrm{dx}$$

$$\leq C\int_{R^n} \left|(1+r^2)^{k/2}\partial^\alpha \varphi(x)\right|\mathrm{dx}$$

Observe that, if a sequence $\{\varphi_j\}$ converges to zero in S, then for every $k \geq 0$ and every $\alpha \in N_o^n$, the sequence $((1+r^2)^{-k/2}\, \partial^\alpha \varphi_j)$ converges to zero in L_1. Hence $T = \partial^\alpha f$ defines a tempered distribution.

c) Let $f: R \to R$ be given by $f(x) = e^x \cos(e^x)$.

Then $f(x) = (\sin(e^x))'$. We set $g(x) = \sin(e^x)$.

Then we have $f(x) = g'(x)$. The function f is not of slow growing but it generates a tempered distribution by

$$\langle f, \varphi\rangle = \int_{R^n} e^x \cos(e^x)\, \varphi(x)dx \text{ for all } \varphi \in S(R^n)$$

$$= \int_{R^n} (\sin(e^x))'\, \varphi(x)$$

$$= -\int_{R^n} \sin(e^x)\, \varphi'(x)dx$$

Since g is a function of slow growth, i.e.

$|\sin(e^x)| \leq 1 \leq 1 + |x|$ for all $x \in R^n$.

We get that g is a tempered distribution. This implies that f is a tempered distribution.

1.4. Fourier transform of distribution function of slow growth.

1.4.1. The Fourier transform of test functions belonging to S.

Definitions 1.4.1: The Fourier transform of a function $f \in S(R^n)$ is defined by the integral.

$$(\mathcal{F}\mathrm{f})\,(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} f(x)\, e^{-i\langle x,\xi\rangle}\mathrm{dx}$$

Where $\langle x, y\rangle = x\xi = x_1\xi_1 + \cdots + x_n\xi_n$.

Definition1.4.2: The inverse Fourier transform of a function $f \in S(R^n)$ is defined by the integral

$$(\mathcal{F}^{-'} f)(\xi) = \check{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} f(x)\, e^{i\langle x,\xi\rangle} dx$$

Lemma1.4.1: Let $f \in S(R^n)$ and α be a multi-index.

Then

i) $D^\alpha(\mathcal{F}f)(\xi) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha f)(\xi)$

ii) $\mathcal{F}(D^\alpha f)(\xi) = (i)^{|\alpha|}\, \xi^\alpha(\mathcal{F}f)(\xi)$

Proof. i) $D^\alpha(\mathcal{F}f)(\xi) = \frac{\partial^\alpha}{\partial \xi^\alpha}[(2\pi)^{-n/2} \int_{R^n} f(x)\, e^{-i\langle x,\xi\rangle} dx]$

$$= (2\pi)^{-n/2} \int_{R^n} f(x) \frac{\partial^\alpha}{\partial \xi^\alpha} e^{-i\langle x,\xi\rangle} dx$$

$$= (2\pi)^{-n/2} \int_{R^n} f(x)(-ix_1)^{\alpha_1}(-ix_2)^{\alpha_2} \cdot \ldots \cdot (-ix_n)^{\alpha_n} e^{-i\langle x,\xi\rangle} dx$$

$$= (-i)^{\alpha_1+\ldots+\alpha_n} (2\pi)^{-n/2} \int_{R^n} f(x) x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} e^{-i\langle x,\xi\rangle} dx$$

$$= (-i)^{|\alpha|}(2\pi)^{-n/2} \int_{R^n} f(x) x^\alpha e^{-i\langle x,\xi\rangle} dx$$

$$= (-i)^{|\alpha|}\mathcal{F}(x^\alpha f)(\xi)$$

ii) $\mathcal{F}(D^\alpha f)(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x,\xi\rangle} D^\alpha f(x) dx$

$$= (-1)^{|\alpha|}(2\pi)^{-n/2} \int_{R^n} f(x) \frac{\partial^\alpha}{\partial x^\alpha} e^{-i\langle x,\xi\rangle} dx$$

$$= (-1)^{|\alpha|}(2\pi)^{-n/2} \int_{R^n} f(x)(-i\xi_1)^{\alpha_1} \cdot \ldots \cdot (-i\xi_n)^{\alpha_n} e^{-i\langle x,\xi\rangle} dx$$

$$= (-1)^{|\alpha|}(2\pi)^{-n/2} \int_{R^n} f(x)(-i)^{|\alpha|}\xi^\alpha e^{-i\langle x,\xi\rangle} dx$$

$$= (-1)^{|\alpha|}(-i)^{|\alpha|}\xi^\alpha(2\pi)^{-n/2} \int_{R^n} f(x) e^{-i\langle x,\xi\rangle} dx$$

$$= i^{|\alpha|}\xi^\alpha[\mathcal{F}f](\xi).$$

We note that the space of test functions D is not mapped in to itself by the Fourier transform since the Fourier transform of a function with compact support is an analytic function, and consequently is either not of compact support or zero.

Theorem1.4.1: The Fourier transforms

$$\mathcal{F}: L_1(R^n) \longrightarrow L_\infty(R^n)$$

is a bounded linear operator, and

$$\|\hat{f}\|_{L_\infty(R^n)} \le (2\pi)^{\frac{-n}{2}} \|f\|_{L_1(R^n)}.$$

Proof: From $(\mathcal{F}f)\ (\xi) = (2\pi)^{-n/2} \int_{R^n} f(x)\, e^{-ix\xi}\,dx$ it follows

$$\|\mathcal{F}f\|_\infty = \sup_{\xi\in R^n} |\ \mathcal{F}f(\xi)|$$

$$= (2\pi)^{-n/2} \sup_{\xi\in R^n} |\int_{R^n} f(x)\, e^{-ix\xi}\,dx|$$

$$\leq (2\pi)^{-n/2} \sup_{\xi\in R^n} |e^{-ix\xi}| \cdot \int_{R^n} |f(x)|dx$$

$$= (2\pi)^{-n/2} ||f||_{L^1(\Omega)}\ .$$

This implies that $\mathcal{F}$ is bounded and there for continuous and it is a mapping from $L^1(\Omega)$ in to $L^\infty(\Omega)$. Furthermore, it follows

$$||\ \mathcal{F}|| = \sup_{|f|\leq 1} ||\ \mathcal{F}f|\,| \leq (2\pi)^{-n/2}.$$

From the general theory of the Fourier Transform it follows that the function $f(x)$ is expressed in terms of its Fourier Transform $\mathcal{F}[f](\xi)$ with the aid of the inverse Fourier Transform$\mathcal{F}^{-1}$:

$$f\ =\ \mathcal{F}^{-1}[\mathcal{F}[f]](\xi) = \mathcal{F}[\mathcal{F}^{-1}[f]](\xi) \quad \text{----------------------------1}$$

Where

$$\mathcal{F}^{-1}[f](\xi) = (2\pi)^{-n/2} \int_{R^n} f(x)\, e^{i\langle x,\xi\rangle}dx$$

$$= \mathcal{F}[f](-\xi)$$

$$= (2\pi)^{-n/2} \int_{R^n} f(-x)\, e^{-i\langle x,\xi\rangle}dx$$

$$= \mathcal{F}[f(-x)](\xi) \qquad \text{--------------------------------2}$$

Theorem1.4.2:The Fourier Transform of a convolution.

If f and g belong to S(R^n). We have the Fourier Transform of $\mathcal{F}(f * g)$ is equal to $(2\pi)^{n/2}\mathcal{F}(f)\mathcal{F}(g)$.

That is, $\mathcal{F}(f*g) = (2\pi)^{n/2}\hat{f}(\xi)\hat{g}(\xi)$.

Proof:by Fubini's theorem we get

$$\mathcal{F}(f\ *\ g)(\ \xi) = (2\pi)^{\frac{-n}{2}}(\int_{R^n} e^{-i\langle x,\xi\rangle}(f*g)(x)dx$$

$$= (2\pi)^{\frac{-n}{2}} \int_{R^n} e^{-i\langle x,\xi\rangle} \left[\int_{R^n} f(y)g(x-y)dy\right]dx$$

$$= \int_{R^n}\int_{R^n} (2\pi)^{\frac{-n}{2}} e^{-i\langle y,\xi\rangle}(e)^{-i\langle x-y,\xi\rangle}\, f(y)g(x-y)dxdy$$

$$= (2\pi)^{n/2}(2\pi)^{-n/2} \int_{R^n} e^{-i\langle y,\xi\rangle}[(2\pi)^{-n/2} \int_{R^n} e^{-i\langle x-y,\xi\rangle}\, g(x-y)dx]f(y)dy$$

$=(2\pi)^{n/2}\mathcal{F}(f)\mathcal{F}(g)$

$=(2\pi)^{n/2}\hat{f}(\xi)\hat{g}(\xi)$.

Example1.4.1: Let $\gamma(x)=e^{\frac{-x^2}{2}}$, x $\in R^n$, and $a \neq 0$.Then

$$(\mathcal{F}\gamma_a)(\xi)=\frac{1}{a^n}(\mathcal{F}\gamma)\left(\frac{\xi}{a}\right),\quad \text{we set } \gamma_a(\mathrm{x})=\gamma(ax)\text{, for } a\in R.$$

Proof: We have (using the substitution y=ax, i.e. x= $\frac{y}{a}$ and dx =$\frac{1}{a^n}$dy)

$$(\mathcal{F}\gamma_a)(\xi)\ =(\mathcal{F}\gamma)(a\xi)$$

$$=(2\pi)^{\frac{-n}{2}}\int_{R^n}e^{\frac{-(ax)^2}{2}}e^{-i\langle \mathrm{x},\xi\rangle}\mathrm{dx}$$

$$=(2\pi)^{\frac{-n}{2}}\int_{R^n}e^{\frac{-(y)^2}{2}}e^{-iy\frac{\xi}{a}}\cdot\frac{1}{a^n}\mathrm{dy}$$

$$=\frac{1}{a^n}(2\pi)^{\frac{-n}{2}}\int_{R^n}e^{\frac{-(y)^2}{2}}e^{-iy\frac{\xi}{a}}\mathrm{dy}$$

$$=\frac{1}{a^n}(\mathcal{F}\gamma)\left(\frac{\xi}{a}\right)$$

1.4.2 The Fourier transform of distribution belonging to S'.

Definition1.4.2:Let f(x) be an integrable function on R^n. Then its Fourier transform defined as $(\mathcal{F}\mathrm{f})(\xi)=(2\pi)^{-n/2}\int_{R^n}f(\mathrm{x})e^{-i\langle \mathrm{x},\xi\rangle}$ dx for all $f \in S'$, $|(\mathcal{F}\mathrm{f})(\xi)| \leq \int|f(x)|\mathrm{dx} < \infty$ is a continuous function bounded in R^n and , consequently, defines a generalized function from S',

$$\langle\mathcal{F}f(\xi),\varphi\rangle=\langle\hat{f},\varphi\rangle=\int\hat{f}(\xi)\,\varphi(\xi)\,d\xi$$

$$=\int\left[(2\pi)^{-n/2}\int f(x)e^{-i\langle \mathrm{x},\xi\rangle}\,dx\right]\varphi(\xi)\mathrm{d}\,\xi$$

$$=\int f(x)\,[(2\pi)^{-n/2}\int\varphi(\xi)\,e^{-i\langle \mathrm{x},\xi\rangle}d\,\xi]dx$$

$$=\int f(x)(\mathcal{F}\varphi)(x)dx$$

$$=\langle f,\mathcal{F}\,\varphi\rangle$$

$$=\langle f,\hat{\varphi}\rangle\qquad \varphi\in \mathrm{S}\qquad \text{--------------------------3}$$

Example1.4.2: If f=δ, then

$$\langle\mathcal{F}\delta,\varphi\ \rangle=\langle\delta,\hat{\varphi}\rangle$$

$$=(\hat{\varphi})(0)$$

$$=(2\pi)^{-n/2}\int_{R^n}\varphi(\mathrm{x})e^{-i\langle \mathrm{x},0\rangle}\mathrm{dx}$$

$=(2\pi)^{-n/2}\int_{R^n}\varphi(\mathrm{x})\mathrm{dx}$

$=(2\pi)^{-n/2}\langle 1,\varphi\rangle$

$=\langle(2\pi)^{-n/2},\varphi\rangle,\quad \varphi\in\mathrm{S}$

$\therefore\ \mathcal{F}(\delta)=(2\pi)^{-n/2}$

We shall prove that the operation $\mathcal{F}^{-1}$ is the inverse operation to the Fourier transform $\mathcal{F}$, that is,

$\mathcal{F}^{-1}[\mathcal{F}[f]](\xi)=f\ ,\quad \mathcal{F}[\mathcal{F}^{-1}[f]](\xi)\ =f\ ,\qquad f\in S'$ ----------------------4

In fact, by virtue of equation (1)−(3), for all $\varphi\in S$ we obtain the equations

$$\begin{aligned}\langle\mathcal{F}^{-1}\big[\mathcal{F}[f]\big](\xi),\varphi\rangle &= \langle\mathcal{F}[\mathcal{F}[f](-\xi)],\varphi\rangle\\ &=\langle\mathcal{F}[f](-\xi),\mathcal{F}[\varphi]\rangle\\ &=\langle\mathcal{F}[f],\mathcal{F}[\varphi](-\xi)\rangle\\ &=\langle\mathcal{F}[f],\mathcal{F}^{-1}[\varphi]\rangle\\ &=\langle f,\mathcal{F}[\mathcal{F}^{-1}[\varphi]]\rangle=\langle f,\varphi\rangle\\ &=\langle f,\mathcal{F}^{-1}[\mathcal{F}[\varphi]\rangle\\ &=\langle\mathcal{F}^{-1}[f],\mathcal{F}[\varphi]\rangle\\ &=\langle\mathcal{F}[\mathcal{F}^{-1}[f]],\varphi\rangle\end{aligned}$$

From which equation (4) follow.

Properties of the Fourier transform

i)Differentiating the Fourier transform

If $f\in S'$, then $\mathrm{D}^{\alpha}\mathcal{F}[\mathrm{f}]=\mathcal{F}[(\mathrm{ix})^{\alpha}\mathrm{f}]$

i.e. Let $\varphi\in\mathrm{S}$, then $\langle\mathrm{D}^{\alpha}\mathcal{F}[\mathrm{f}],\varphi\rangle=(-1)^{|\alpha|}\langle\mathcal{F}[\mathrm{f}],\mathrm{D}^{\alpha}\varphi\rangle$

$$\begin{aligned}&=(-1)^{|\alpha|}\langle\mathrm{f},\mathcal{F}[\mathrm{D}^{\alpha}\varphi]\rangle\\ &=(-1)^{|\alpha|}\langle\mathrm{f},(-\mathrm{ix})^{\alpha}\mathcal{F}[\varphi]\rangle\\ &=(-1)^{|\alpha|}\langle(-\mathrm{ix})^{\alpha}\mathrm{f},\mathcal{F}[\varphi]\rangle\\ &=\langle(\mathrm{ix})^{\alpha}\mathrm{f},\mathcal{F}\varphi\rangle\\ &=\langle\mathcal{F}(\mathrm{ix})^{\alpha}\mathrm{f},\varphi\rangle\end{aligned}$$

$\therefore\ \mathrm{D}^{\alpha}[\mathrm{f}](\xi)=\mathcal{F}[(\mathrm{ix})^{\alpha}\mathrm{f}]\ (\xi)$

ii)The Fourier transform of a derivative

If $f \in S'$ then $\mathcal{F}[D^{\alpha}f] = (i\xi)^{\alpha}[\mathcal{F}f]$

i.e. Let $\varphi \in S$,then $\langle \mathcal{F}[D^{\alpha} f, \varphi]\rangle = \langle D^{\alpha} f, \mathcal{F}[\varphi]\rangle$

$= (-1)^{|\alpha|}\langle f, D^{\alpha}[\mathcal{F}\varphi]\rangle$

$=(-1)^{|\alpha|}\langle f, \mathcal{F}(-i\xi)^{\alpha}\varphi\rangle$

$=(-1)^{\alpha}\langle \mathcal{F}[f], (-i\xi)^{\alpha}\varphi\rangle$

$=(-1)^{|\alpha|}\langle (-i\xi)^{\alpha}\mathcal{F}[f], \varphi\rangle$

$=\langle (i\xi)^{\alpha}\mathcal{F}[f], \varphi\rangle$

$\therefore \mathcal{F}[\, D^{\alpha}f]\,(\xi)=(i\xi)^{\alpha}[\mathcal{F}f](\,\xi)$

CHAPTER –TWO: SOBOLEV SPACE

2.1. Weak Derivative

Definition2.1.1: Let $f \in L^1_{loc}(\Omega)$ be a locally integrable function on the open set $\Omega \subseteq R^n$. Given a multi-index$=(\alpha_1, \alpha_2, \dots \alpha_n)$, if there exists a locally integrable function $g \in L^1_{loc}(\Omega)$ such that

$\int f D^\alpha \varphi dx = (-1)^{|\alpha|} \int D^\alpha f \varphi dx = (-1)^{|\alpha|} \int g\varphi dx$ for all test functions, then we say that g is the weak α-th derivative of f , and write $g = D^\alpha f$.

Example2.1.1: For n=1, Ω=R, $|x|'_w$=sgnx

Proof: for all $\varphi \in$ D (Ω)

$$\int_{-\infty}^{\infty} |x| \varphi'(x) dx = -\int_{-\infty}^{0} x\varphi'(x) dx + \int_{0}^{\infty} x\varphi'(x) dx$$

$$= -x\varphi(x) \Big|_{-\infty}^{0} + x\varphi(x) \Big|_{o}^{\infty} + \int_{-\infty}^{0} \varphi(x) dx - \int_{0}^{\infty} \varphi(x)\, dx$$

$$= \int_{-\infty}^{0} \varphi(x) dx - \int_{0}^{\infty} \varphi(x) dx \text{..........................} 1$$

$$(-1) \int_{-\infty}^{\infty} \operatorname{sgn} x\varphi(x) dx = (-1)\left[-\int_{-\infty}^{0} \varphi(x) dx + \int_{0}^{\infty} \varphi(x) dx \right]$$

$$= \int_{-\infty}^{0} \varphi(x) dx - \int_{0}^{\infty} \varphi(x) dx \text{...........................} (2)$$

Hence from (1) and (2) we can see that $|x|'_W$ =sgnx.

Lemma 2.1.1 (Uniqueness of weak derivative)

Assume $f \in L^1_{loc}(\Omega)$ and let $h, g \in L^1_{loc}(\Omega)$ be weak α-th derivative of f , so that

$$\int f D^\alpha \varphi dx=(-1)^{|\alpha|} \int g\varphi\, dx=(-1)^{|\alpha|} \int h\varphi dx$$

for all test function $\varphi \in D(\Omega)$. Then $g(x)=h(x)$ for $a.e.\ x \in \Omega$.

Proof. By the assumptions, the function $(g-h) \in L'_{loc}(\Omega)$ satisfies

$$\int (g-h)\varphi dx=0 \quad \forall\varphi \in D(\Omega)$$

$g(x)$ −h(x) for a.e

$$g = h \ \ a.e$$

Lemma 2.1.2: Assume that $f \in L^1_{loc}(\Omega)$ has weak derivative $D^\alpha f$ for every $|\alpha| \le k$.

Then, for every pair of multi-indices α, β with $|\alpha|+|\beta| \le k$ one has

$$D^\alpha(D^\beta f)=D^\beta(D^\alpha f)=D^{\alpha+\beta} f.$$

Proof: Consider any test function $\varphi \in D(\Omega)$ .Using the fact that $D^\beta \varphi \in D(\Omega)$ is a test function as well, we obtain

$$\int_\Omega D^\alpha f D^\beta \varphi dx=(-1)^{|\alpha|} \int_\Omega f(D^{\alpha+\beta}\varphi)\mathrm{dx}$$

$$=(-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_\Omega (D^{\alpha+\beta} f)\, \varphi dx$$

$$=(-1)^{|\beta|} \int_\Omega (D^{\alpha+\beta} f)\, \varphi dx.$$

By definition, this means that $D^{\alpha+\beta} f = D^\beta(D^\alpha f)$. Exchanging the roles of the multi-indices α *and* βin the previous computation one obtains $D^{\alpha+\beta} f = D^\alpha(D^\beta f)$.

Lemma 2.1.3 :(Convergence of weak derivative).

Consider a sequence of functions $f_n \in L^1_{loc}(\Omega)$. For a fixed multi-index α , assume that each f_n admits the weak derivative $g_n = D^\alpha f_n$

If $f_n \longrightarrow f$ and $g_n \longrightarrow g$ in $L^1_{loc}(\Omega)$, then $g = D^\alpha f$.

Proof. For every test function $\varphi \in D(\Omega)$, a direct computation yeildes

$$\int_\Omega g\varphi dx = \lim_{n\to\infty} \int_\Omega g_n \varphi\, dx = \lim_{n\to\infty} \int_\Omega D^\alpha f_n \varphi\, dx$$

$$= \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega f_n D^\alpha \varphi\, dx$$

$$= (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx.$$

$$= \int_\Omega D^\alpha f \varphi \mathrm{dx}$$

By definition, this means that g is the α-th weak derivative of f.

Theorem2.1.1: If $D^\alpha u = v$ and D^β v=w, then $D^{\alpha+\beta}$ u=w in weak sense.

Proof: Let $\varphi \in D(\Omega)$ $\phi = D^\beta \varphi$. Then

$$\int_\Omega u D^{\alpha+\beta} \varphi dx = (-1)^{|\alpha|} \int_\Omega v\, \phi dx$$

$$= (-1)^{|\alpha|} \int_\Omega v\, D^\beta \varphi dx$$

$$= (-1)^{|\alpha+\beta|} \int_\Omega D^\beta v \varphi\, dx$$

$$= (-1)^{|\alpha+\beta|} \int_\Omega w \varphi\, dx$$

Another definition of the Weak Derivative

Definition 2.1.2: Let $\Omega \subset R^n$ be an open set, $\alpha \in N_o^n, \alpha \neq 0$ and $f, g \in L^1_{loc}(\Omega)$. The function g is a weak derivative of the function f on Ω (i.e. $g = D^\alpha f$) if there exist $\psi_k \in C^\infty(\Omega), k \in N$,

$\psi_k \longrightarrow f,\ D^\alpha \psi_k \longrightarrow g\ in\ L^1_{loc}(\Omega)$ as k⟶ ∞.

Theorem 2.1.2: Definition (1) and (2) are equivalent

Proof: (1) ⇒(2)

Suppose; $\psi_k \longrightarrow f, D^\alpha \psi_k \longrightarrow g\ in\ L_{loc}{}^1(\Omega)$ as k⟶ ∞.

$$\psi_k \longrightarrow f \Longrightarrow D^\alpha \psi_k \longrightarrow D^\alpha f\ and\ D^\alpha \psi_k \longrightarrow g$$

Let $\lim_{k\to\infty} \int_\Omega D^\alpha\, \psi_k \varphi dx = \lim_{k\to\infty} (-1)^{|\alpha|} \int_\Omega \psi_k\, D^\alpha \varphi dx$

$$= (-1)^{|\alpha|} \int_\Omega f\, D^\alpha \varphi dx$$

$$= \int_\Omega D^\alpha f\, \varphi dx \text{---------------------------} 1$$

And

$$\lim_{k\to\infty} \int_\Omega D^\alpha\, \psi_k \varphi dx = \int_\Omega g\, \varphi dx \text{ ------------------------} 2$$

By combining of equation (1) and (2), we will get

$$\int_\Omega D^\alpha f \varphi dx = \int_\Omega g \varphi dx$$

But $\int_\Omega g \varphi dx = \int_\Omega D^\alpha f \varphi dx$ by definition (1).

This implies that g is a weak derivative of f on Ω.

(2) ⇒ (1):

$$\text{Let} \int_\Omega D^\alpha f \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx, \forall \varphi \in C_o^\infty(\Omega)$$

Since $\psi_k \to f, D^\alpha \psi_k \to g = D^\alpha f$ *and* $g = D_w{}^\alpha f$ (by definition 2)

then we get $\int_\Omega D^\alpha f \varphi dx = \int_\Omega g\varphi dx = \int_\Omega D_w^\alpha f\varphi dx$

Hence the proof.

Lemma2.1.4 :(Weak differentiation under the integral sign).

Let $\Omega \subset R^n$ be an open set, $A \subset R^n$ a measurable set and let $\alpha \in N_0^n, \alpha \neq 0$.

Suppose that the function f is defined on $\Omega \times A$, for almost every $y \in A f(.,y) \in L_{loc}^1(\Omega)$ and there exist a weak derivative $D_w^\alpha f(.,y)$ on Ω.

Moreover, suppose that $f, D_w^\alpha f \in L_1(k\times A)$ for each compact $k \subset \Omega$. Then on Ω

$$D_w^\alpha\left(\int_A f(x,y)dy\right)=\int_A (D_w^\alpha f)(x,y)dy$$

Proof: For all $\varphi \in D(\Omega)$the function f(x,y)$(D^\alpha\varphi)$(x) and $(D_w^\alpha f)(x,y)\varphi(x)$ belongs to $L_1(\Omega \times A)$.

Since $\int_{\Omega\times A} |f(x,y)(D^\alpha \varphi)(x)|dxdy \leq M\int_{supp\varphi\ \times A} |f|dxdy < \infty$

Where $M=\max_{x\in\Omega} |(D^\alpha\varphi)(x)|$.

Therefore, starting from definition (weak), we can use Fubini's theorem twice to change the order of integration and deduce that for all $\varphi \in D(\Omega)$

$$\int_\Omega\left[\int_A (D_w^\alpha f)(x,y)dy\right]\varphi(x)dx = \int_A\left[\int_\Omega (D_w^\alpha f)(x,y)\varphi(x)dx\right]dy$$

$$=(-1)^{|\alpha|}\int_A\left[\int_\Omega f(x,y)(D^\alpha \varphi)(x)dx\right]dy$$

$$=(-1)^{|\alpha|}\int_\Omega\left[\int_A f(x,y)dy\right](D^\alpha \varphi)(x)dx$$

$$=(-1)^{|\alpha|}\int_\Omega (D^\alpha \varphi)(x)\left[\int_A f(x,y)dy\right]dx$$

(By Fubini's theorem)

$$= \int_\Omega \varphi(x) D_w^\alpha\left[\int_A f(x,y)dy\right]dx \text{..............................} 1$$

But $\int_\Omega\left[\int_A (D_w^\alpha f)(x,y)dy\right]\varphi(x)dx = \int_\Omega \varphi(x)\left[\int_A (D_w^\alpha f)(x,y)dy\right]dx$ 2

By combining (1) and (2) we will get

$$\int_A (D_W^\alpha f(x,y)dy = D_w^\alpha\left[\int_A f(x,y)dy\right].$$

2.2. Sobolev Spaces

Definition2.2.1: Let $\Omega \subset R^n$ be an open set, $1 \leq P \leq \infty$, and m $\geq$ 0 be an integer.

The Sobolev space $W_p^m(\Omega)$ of order m is defined by

$$W_p^m(\Omega) = \{f \in L_p(\Omega) : D^\alpha f \in L_p(\Omega), \text{for all multi-indices } \alpha \text{ such that } |\alpha| \leq m\}.$$

Definition2.2.2: For $f \in W_p^m(\Omega)$, the $W_p^m(\Omega)$ norm is

$$\|f\|_{W_p^m(\Omega)} = \{\sum_{|\alpha|\leq m} \|D^\alpha f\|_{L_p(\Omega)}^p\}^{\frac{1}{p}} \quad \text{if } p < \infty \text{ --------------------1}$$

and

$$\|f\|_{W_p^m(\Omega)} = \max_{|\alpha|\leq m} \|D^\alpha f\|_{L_\infty(\Omega)} \quad \text{if } p = \infty. \text{ --------------------2}$$

Theorem2.2.1: The space $W_p^m(\Omega)$ is a Banach space.

Proof: 1.Let $f_1, f_2 \in W_p^m(\Omega)$.For $|\alpha| \leq m$,call $D^\alpha f_1, D^\alpha f_2$ their weak derivatives.

Then, for any $\lambda, \mu \in \mathbb{R}$, the linear combination $\lambda f_1 + \mu f_2$ is a locally integrable function. Its weak derivatives are

$$D^\alpha(\lambda f_1 + \mu f_2) = \lambda D^\alpha f_1 + \mu D^\alpha f_2$$

Therefore, $D^\alpha(\lambda f_1 + \mu f_2) \in L_p(\Omega)$ for every $|\alpha| \leq m$.

This proves that $W_p^m(\Omega)$ is a vector space.

2. Next, we check that (1) (2) are a norm.

Indeed, for $\lambda \in \mathbb{R}$ and $f \in W_p^m(\Omega)$ one has

$$\|\lambda f\|_{W_p^m(\Omega)} = |\lambda| \|f\|_{W_p^m(\Omega)}$$

$$\|f\|_{W_p^m(\Omega)} \geq \|f\|_{L_p} \geq 0,$$

with equality holding if and only if $f = 0$.

Moreover, if $f_1, f_2 \in W_p^m(\Omega)$, then for $1 \leq p < \infty$ Minkowski's inequality yields

$$\|f_1 + f_2\|_{W_p^m} = (\sum_{|\alpha|\leq m} \|D^\alpha f_1 + D^\alpha f_2\|_{L_p}^p)^{\frac{1}{p}}$$

$$\leq (\sum_{|\alpha|\leq m} (\|D^\alpha f_1\|_{L_p} + \|D^\alpha f_2\|_{L_p})^p)^{\frac{1}{p}}$$

$$\leq (\sum_{|\alpha|\leq m} \|D^\alpha f_1\|^p)^{\frac{1}{p}} + (\sum_{|\alpha|\leq m} \|D^\alpha f_2\|^p)^{\frac{1}{p}}$$

$$= \|f_1\|_{W_p^m} + \|f_2\|_{W_p^m}.$$

In the case $p=\infty$, the above computation is replaced by

$$\|f_1 + f_2\|_{W_p^m} = \sum_{|\alpha|\leq m} \|D^\alpha f_1 + D^\alpha f_2\|_{L_\infty}$$
$$\leq \sum_{|\alpha|\leq m} (\|D^\alpha f_1\|_{L_\infty} + \|D^\alpha f_2\|_{L_\infty})$$
$$= \|f_1\|_{W_\infty^m} + \|f_2\|_{W_\infty^m}.$$

3. To conclude the proof, we need to show that the space $W_p^m(\Omega)$ is complete, hence is a Banach space.

Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $W_p^m(R^n)$.

Then $\{D^\alpha f_j\}_{j=1}^{\infty}$ are Cauchy sequence in $L_P(R^n)$ for $|\alpha| \leq k$. Hence there are $f^\alpha \in L_P(R^n)$ with for $|\alpha| \leq k$. Hence there are $f^\alpha \in L_P(R^n)$ with

$$D^\alpha f_j \longrightarrow f^\alpha \text{ in } L_p(R^n), |\alpha| \leq k, \text{and } f^0 = f.$$

It follows from

$$\int_{R^n} D^\alpha f_j(x)\varphi(x)dx = (-1)^{|\alpha|} \int_{R^n} f_j(x)\, D^\alpha \varphi(x)dx, \quad \varphi \in S(R^n),$$

And holder's inequality applied to $D^\alpha f_j \longrightarrow f^\alpha, f_j \longrightarrow f \in L_P(R^n)$ and $\varphi, D^\alpha\varphi \in (R^n)$ that

$$\int_{R^n} f^\alpha(x)\varphi(x)dx = (-1)^{|\alpha|} \int_{R^n} f(x) D^\alpha \varphi(x)dx, \varphi \in S(R^n),$$

Then $f^\alpha = D^\alpha f$, $|\alpha| \leq k$, and $f \in W_P^m(R^n)$ with

$$f_j \longrightarrow f \text{ in } W_P^m(R^n) \text{ for } j \longrightarrow \infty$$

Consequently, $W_P^m(R^n)$ is a Banach space.

Definition2.2.3: We denote the mth order Sobolev space in $L_2(\Omega)$ by

$$H^m(\Omega) = W_2^m(\Omega).$$

And the Sobolev space $W_2^m(\Omega)$ equipped with the scalar product

$$(f,g)_{W_2^m(\Omega)} = \int_\Omega \sum_{|\alpha|\leq m} \partial^\alpha f(x) \overline{\partial^\alpha g(x)} dx$$

becomes Hilbert spaces.

Definition 2.2.4: The closure of $C_0^\infty(\Omega)$ in the norm of $W_p^m(\Omega)$ is denoted by $w_p^m(\Omega)$.

So, w_p^m is a sub space in the space $W_p^m(\Omega)$.

Proposition 2.2.2: Let $f \in W_p{}^m(\Omega)$ and $g \in w_p^m$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_\Omega \partial^\alpha f g dx = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha g dx, |\alpha| \leq m \text{ ----------------------1}$$

Proof: Let $g_n \in C_0^{\infty}(\Omega)$ *and* $g_n \to g$ *as* $n \to \infty$ *in* $w_p^m(\Omega)$.

By definition of the weak derivative $\partial^{\alpha} f$, we have

$$\int_{\Omega} \partial^{\alpha} fg dx = (-1)^{|\alpha|} \int_{\Omega} f \partial^{\alpha} g_n dx \text{..............................} (2)$$

Let us show that

$$\int_{\Omega} \partial^{\alpha} f g_n dx \xrightarrow{n \to \infty} \int_{\Omega} \partial^{\alpha} f g dx$$

$$\int_{\Omega} f \, \partial^{\alpha} g_n dx \xrightarrow{n \to \infty} \int_{\Omega} f \, \partial^{\alpha} g dx$$

We have

$$\left| \int_{\Omega} \partial^{\alpha} f (g_n - g) dx \right| \leq \left(\int_{\Omega} |\partial^{\alpha} f|^p \right)^{1/p} \left(\int_{\Omega} |g_n - g|^{q'} \right)^{1/q'}$$

$$\leq \| f \|_{W_p^m(\Omega)} \| g_n - g \|_{W_p^m(\Omega)}$$

$$\to 0 \text{ as } n \to \infty$$

$$\left| \int_{\Omega} f (\partial^{\alpha} g_n - \partial^{\alpha} g) dx \right| \leq \left(\int_{\Omega} |f|^p \right)^{1/p} \left(\int_{\Omega} |\partial^{\alpha} g_n - \partial^{\alpha} g|^{q'} dx \right)^{1/q'}$$

$$\leq \| f \|_{W_p^m(\Omega)} \| g_n - g \|_{W_p^m(\Omega)}$$

$$\to 0 \text{ as } n \to \infty$$

Tending to the limit (2) as $n \to \infty$ we obtain (1)

Conclusion

In classical sense it is not possible to handle discontinuous function at the point of discontinuity, but in distributional sense (which is an extension of the classical one) it is handled as if it is continuous and differentiation is computed. We note that the space of test functions D is not mapped in to itself by the Fourier transform since the Fourier transform of a function with compact support is an analytic function, and consequently is either not of compact support or zero. A sobolev space is vector space of functions equipped with a norm that is a combination of L^p-norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space.

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