ENTANGLEMENT AND SQUEEZING PROPERTIES OF TWO-MODE CAVITY LIGHT FROM THREE-LEVEL ATOM AND SUBHARMONIC GENERATING SYSTEM

A Thesis Submitted to the
Graduate programs
Addis Ababa University

In Partial Fulfilment of
the Requirement for the Degree
of Master of Sciences in Physics (Quantum optics)

by
Muhammed Umer

Addis Ababa, Ethiopia
31 August 2020

ADDIS ABABA UNIVERSITY
GRADUATE PROGRAMS

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by
Muhammed Umer

Department of Physics
College of Natural and Computational Sciences

Approved by the Examination Commitee
External Examiner: ____
Internal Examiner: ____
Advisor: Dr. Deribe Hirpo ____________________________________
Chairman: ____

Abstract

The aim of this thesis is to study the quantum properties of light produced by a non-degenerate three-level atom driven by strong coherent light in a cavity containing a parametric amplifier and coupled to a two-mode vacuum reservoir. By using the master equation of the system, the stochastic differential equations of the expectation value of the operators are obtained. Further manipulating of the quantum Langevin equations, the decoupled stochastic differential equations between cavity mode and atomic operators are established. On the other hand the correlation properties of the noise operators are obtained by using the quantum Langevin equations. For cavity mode operators, the steady state solutions of the decoupled differential equations are used to calculate the photon statistics of single mode as well as the two mode cavity light beams. The quadrature variances of the single modes as well as the two-mode are also obtained from the steady state solutions of the operators. The single modes are in chaotic state, where as the two-mode can be in squeezed state for some combinations of the considered parameters. The quadrature squeezing occurs in both minus quadrature and plus quadrature for different values of Ω, and μ. The maximum squeezing occurs in the plus quadrature with a quadrature squeezing of 60%. Moreover, the entanglement of the cavity radiation is analyzed using different criteria. It is found that the presence of parametric amplifier and coherent light enhance entanglement of the cavity light. The Duan et. al. and logarthmic negativity entanglement criteria indicate that the quantum optical system generates entangled photons.

Acknowledgments

This thesis has benefited from the advice and help of my advisor Dr. Deribe Hirpo, and a valuable support was provided from Dr. Fesseha Kassahun's material. According to this I would like to thank both of them and other instructors who supported me in the first year courses. I acknowledge the support and encouragements of my friends. Finally I would like to acknowledge my parents, with out them I might not have had the energy to engage in this MSc program.

Contents

List of Figures

Chapter 1

Introduction

The quantum mechanical discription of light is studied in quantum optics. Quantum optics starts from the quantization of electric and magnetic fields, which is different from classical physical optics.
There are some physical phenomena, which cannot be described in classical theory. From these squeezed light and photon anti-bunching can be listed. Squeezed state is minimum uncertainty state of light in the complex amplitude. The product of state of the fluctuations, disperssions, in the amplitude and the phase always give the minimum limit. Forexample for a single mode light beam the minimum limit is one. If the amplitude is measured precisely the phase fluctuates according to the limit or vice-versa. The coherent light has equal fluctuations in the amplitude and the phase, for a single mode it is one. The other physical phenomena is photon anti-bunching. Photon anti-bunching is detecting two photons in a different time. For example in a two level atom, after the atom emits a photon, it is in the lower level. The atom to emit another photon, it has to be in excited state again. It takes some time for the atom to absorb a photon and make a transition to the upper level from which another photon is emitted. Hence it needs additional time. This phenomenon is photon anti-bunching, which is non-classical property of light[1][2].

Quantum entanglement is a label for the observed physical phenomenon that occurs when a pair or group of particles is generated, interact, or share spatial proximity in a way such that the quantum state of each particle of the pair or group cannot be described independently of the state of the others, even when the particles are separated by a large distance [5].
Measurements of physical properties such as position, momentum, spin, and polarization performed on entangled particles are found to be perfectly correlated. For example, if a pair of entangled particles is generated such that their total spin is known to be zero, and one particle is found to have clockwise spin on a first axis, then the spin of the other particle, measured on the same axis, will be found to be counterclockwise. However, this behavior gives rise to seemingly paradoxical effects: any measurement of a property of a particle results in an irreversible wave function collapse of that particle and will change the original quantum state. In the case of entangled particles, such a measurement will affect the entangled system as a whole [5].
Quantum entanglement attracts the attention of scientists since it has a lot of applications. From these quantum cryptography [6], quantum computation and

communication [7], quantum dense coding [8], quantum teleportation [9], entanglement swapping [10], sensitive measurements [11], and quantum telecloning [12] can be listed [13].
Inequalities derived in classical electromagnetic theory may be violated according to quantum mechanics, especially in entangled states. In particular, Cauchy–Schwarz and Bell inequalities have been used to test the classical electromagnetic theory against quantum theory. Violation of Cauchy–Schwarz inequality has been studied in systems such as two-photon laser, parametric amplifier, and resonance fluorescence [14].
There are some cheking criterias for entanglement in addition to Cauchy-Schwarz and Bell inequalities. These are Duan et. al. criteria [15], Hillery-Zubairy criterion [16], violation of Uncertainity inequality [17] and soon based on the system, which is tested.

To understand the forgoing research it is better to know about its components. These are non-degenerate parametric down conversion (parametric amplifier), non-degenerate three-level atom in a cavity, the coherent light and two-mode vacuum reservoir.
Non-degenerate parametric amplifier is a phenomenon in the nonlinear optics, where a pump mode of frequency ω splits in to two modes at frequency ω_a and ω_b. These frequency sum to the pump frequency, $\omega = \omega_a + \omega_b$. It is conventional to designate one mode as the Signal and the other as the Idler as shown in Fig. 1.1. This process may occur in a medium with a second-order nonlinear susceptibility χ^2 [3].
The non-degenerate parametric amplifier exhibits quantum mechanical correlations which violate certain classical inequalities. These quantum correlations may be further exploited to give squeezing [3]. Any input state, whether it is entangled or not, for the parametric amplifier will lead to an entangled output state [4].

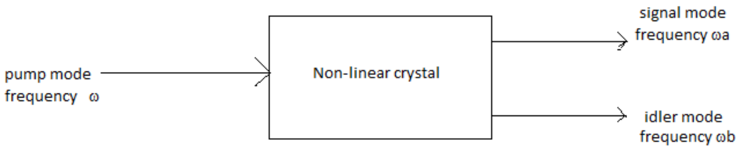

Figure 1.1: Parametric amplifier

Non-degenerate three-level atom is an atom, which is its valence electron has three possible energy levels. The energy gap between the first level and the second level is different from the energy gap between the second level and the third level as shown in Fig. 1.2. Two-mode vacuum reservoir is a vacuum reservoir which damped only two-mode lights in the cavity.

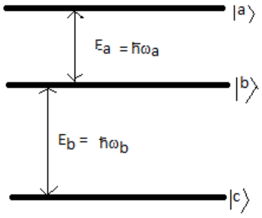

Figure 1.2: Non-degenerate three level atom

The aim of this research is to study the squeezing and entanglement property of the cavity light produced by parametric amplifier and non-degenerate three-level atom coupled to strong coherent light and the cavity is coupled to two-mode vacuum reservoir. In chapter 2 employing the master equation of the system under consideration, the time evolution of the cavity mode and atomic operators are obtained. From the time evolution of atomic operators the quantum Langevin equations are obtained. These equations are used to obtain the correlation properties of the noise operators and to decouple the time evolution equations. The decoupled equations will lead us to the steady state solution of the cavity mode as well as the atomic operators. In chapter 3 the steady state solutions of the evolution equations are used to analyze photon numbers and photon number fluctuations. Quadrature variances are obtained on chapter 4. Based on these results the entanglement property of photons is tested in chapter 5.

Chapter 2

Equations of Evolution

In all physical processes there is an associated loss mechanism [3]. In this research, a non-degenerate three level atom in a cavity is interacting with parametric amplifier and driving coherent light. There is also an interaction between the cavity and a two-mode vacuum reservoir. This makes the system to be damped. The time evolution of the damping process and the internal interaction is governed by the Master equation.

In this chapter employing the master equation the time evolution of the cavity mode and atomic operators are obtained. From these equations the steady state solutions will be gained.

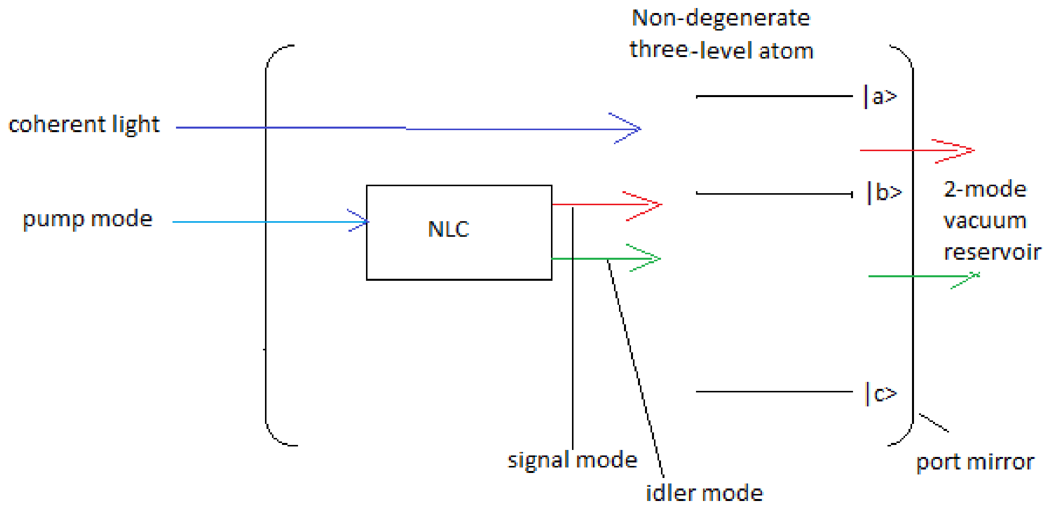

Figure 2.1: Non-degenerate three-level atom in a cavity containing a parametric amplifier and coupled to a two-mode vacuum reservoir.

Before obtaining the master equation for our system, we first write the Hamiltonians describing various interactions.

In this system there are three cavity interactions. One of these is the driving coherent light interacts with the three-level atom. In this interaction the coherent light couples the top and bottom levels of the atom. The Hamiltonian describing the coupling of the top and bottom levels of the three-level atom by driving coherent light is given as [18]:

$$\hat{H}_1 = i\lambda(\hat{\sigma}_c^\dagger \hat{c} - \hat{c}^\dagger \hat{\sigma}_c), \tag{2.1}$$

where

$$\hat{\sigma}_c = |c\rangle\langle a|, \tag{2.2}$$

is a lowering atomic operator, λ is the coupling constant, $\hat{c}$ is annihilation operator for the driving coherent light, and a strong driving coherent light can be treated classically. Thus we can write $\hat{c} = \hat{c}^\dagger = \beta$ and re-express Eq.(2.1) as

$$\hat{H}_1 = i\frac{\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c), \tag{2.3}$$

where

$$\Omega = 2\lambda\beta, \tag{2.4}$$

is a positive real c-number proportional to the amplitude of the coupling coherent light.
The interaction of non-degenerate three-level atom with a two-mode cavity light is expressible as [19]:

$$\hat{H}_2 = ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{b} - \hat{b}^\dagger \hat{\sigma}_b), \tag{2.5}$$

in which

$$\begin{aligned} \hat{\sigma}_a &= |b\rangle\langle a|, & (2.6) \\ \hat{\sigma}_b &= |c\rangle\langle b|, & (2.7) \end{aligned}$$

are atomic operators, g is the coupling constant, $\hat{a}(\hat{b})$ is annihilation operator for cavity light mode a (mode b) respectively.
The pump mode drives the non-linear crystal, Parametric amplifier. The light emerging from the non-linear crystal does not couple the top and bottom levels. The parametric interaction is described by [17]:

$$\hat{H}_3 = i\mu(\hat{a}\hat{b} - \hat{a}^\dagger \hat{b}^\dagger), \tag{2.8}$$

where μ is the coupling constant, $\hat{a}$ is annihilation operator for the signal mode and $\hat{b}$ is for the Idler mode.

The Master equation for the light produced by the system under consideration is [2]:

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \frac{k_1}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{k_2}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}), \tag{2.9}$$

where

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3, \tag{2.10}$$

is the total Hamiltonian of the system, k_1 and k_2 is the damping constant for cavity mode a and cavity mode b respectively.
Employing the Master equation along with the relation [2]

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{2.11}$$

the time evolution of both atomic operators and cavity mode operators can be obtained. Before calculating the time evolution of each operator, there are general

relations which are used many times in the next section. These are:
for any operator $\hat{A}$

$$Tr([\hat{H},\hat{\rho}]\hat{A}) = Tr(\hat{H}\hat{\rho}\hat{A} - \hat{\rho}\hat{H}\hat{A}), \tag{2.12}$$

and utilizing the cyclic property of the trace operator we see that

$$\begin{aligned} Tr([\hat{H},\hat{\rho}]\hat{A}) &= Tr(\hat{\rho}\hat{A}\hat{H} - \hat{\rho}\hat{H}\hat{A}) \\ &= Tr(\hat{\rho}(\hat{A}\hat{H} - \hat{H}\hat{A})) \\ &= Tr(\hat{\rho}[\hat{A},\hat{H}]) \\ &= \langle[\hat{A},\hat{H}]\rangle. \end{aligned} \tag{2.13}$$

And for the operator $\hat{a}$ which commutes with any operator $\hat{b}$

$$\begin{aligned} Tr([2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}]\hat{a}) &= Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}) \\ &= Tr(2\hat{\rho}\hat{b}^\dagger\hat{a}\hat{b} - \hat{\rho}\hat{a}\hat{b}^\dagger\hat{b} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}) \\ &= Tr(2\hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}) \\ &= Tr(2\hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - 2\hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}) \\ &= 0, \end{aligned} \tag{2.14}$$

And also

$$Tr([2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}]\hat{b}) = 0. \tag{2.15}$$

By using these frequently used relations, the time evolution of operators can be calculated.

2.1 Time evolution of cavity mode operators

The time evolution of the annihilation operator for mode a can be calculated employing the master equation in the relation

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a}).$$

We can then write

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\{-i[\hat{H},\hat{\rho}]\hat{a} + \frac{k_1}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})\hat{a} + \frac{k_2}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b})\hat{a}\}$$

By using Eqs. (2.13), (2.14), and (2.15) and applying the trace cyclic property, the above equation becomes

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}\rangle &= -i\langle[\hat{a},\hat{H}]\rangle + \frac{k_1}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}) \\ &= -i\langle[\hat{a},\hat{H}]\rangle + \frac{k_1}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &= -i\langle[\hat{a},\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}[\hat{a}^\dagger\hat{a}+1]\hat{a}) \\ &= -i\langle[\hat{a},\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}) \\ &= -i\langle[\hat{a},\hat{H}]\rangle - \frac{k_1}{2}\langle\hat{a}\rangle. \end{aligned} \tag{2.16}$$

The commutation of the cavity mode operator with the Hamiltonian of the system can be written as

$$\begin{aligned} -i\langle[\hat{a},\hat{H}]\rangle &= -i\langle[\hat{a},\hat{H}_1+\hat{H}_2+\hat{H}_3]\rangle \\ &= -i\langle[\hat{a},\hat{H}_1]+[\hat{a},\hat{H}_2]+[\hat{a},\hat{H}_3]\rangle \end{aligned}$$

Since H_1 is purely a combination of atomic operators, it commutes with the cavity mode operators and thus

$$-i\langle[\hat{a},\hat{H}]\rangle = -i\langle[\hat{a},\hat{H}_2]+[\hat{a},\hat{H}_3]\rangle. \tag{2.17}$$

Making use of the expression for Hamiltonian described by Eqs.(2.5) and (2.8), we can put Eq.(2.17) as

$$-i\langle[\hat{a},\hat{H}]\rangle = -i\langle[\hat{a},ig(\hat{\sigma}_a^{\dagger}\hat{a}-\hat{a}^{\dagger}\hat{\sigma}_a+\hat{b}\hat{\sigma}_b^{\dagger}-\hat{b}^{\dagger}\hat{\sigma}_b)]+[\hat{a},i\mu(\hat{a}\hat{b}-\hat{a}^{\dagger}\hat{b}^{\dagger})]\rangle. \tag{2.18}$$

Employing the commutation relation

$$[\hat{A}\hat{B},\hat{C}\hat{D}] = \hat{C}\hat{A}[\hat{B},\hat{D}]+\hat{C}[\hat{A},\hat{D}]\hat{B}+\hat{A}[\hat{B},\hat{C}]\hat{D}+[\hat{A},\hat{C}]\hat{B}\hat{D} \tag{2.19}$$

and noting that cavity mode operators $\hat{a}$ and $\hat{b}$ commute with each other and atomic operators, one observes that

$$\begin{aligned} -i\langle[\hat{a},\hat{H}]\rangle &= -g\langle\hat{\sigma}_a[\hat{a},\hat{a}^{\dagger}]\rangle-\mu\langle\hat{b}^{\dagger}[\hat{a},\hat{a}^{\dagger}]\rangle \\ &= -g\langle\hat{\sigma}_a\rangle-\mu\langle\hat{b}^{\dagger}\rangle. \end{aligned} \tag{2.20}$$

Combination of Eq.(2.20) with the expression in Eq.(2.16) gives

$$\frac{d}{dt}\langle\hat{a}\rangle = -g\langle\hat{\sigma}_a\rangle-\mu\langle\hat{b}^{\dagger}\rangle-\frac{k_1}{2}\langle\hat{a}\rangle. \tag{2.21}$$

The time evolution of the second moment of annihilation operator for mode a can be calculated as

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}^2\rangle &= Tr(\frac{d\hat{\rho}}{dt}\hat{a}^2) \\ &= Tr\{-\imath[\hat{H},\hat{\rho}]\hat{a}^2+\frac{k_1}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a})\hat{a}^2 \\ &+ \frac{k_2}{2}(2\hat{b}\hat{\rho}\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{b}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\hat{b})\hat{a}^2\}. \end{aligned}$$

Employing Eqs. (2.13), (2.14), and (2.15), the previous equation can be rewritten as

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}^2\rangle &= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^3) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}[\hat{a}^\dagger\hat{a}+1]\hat{a}) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^2) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}(\hat{a}^\dagger\hat{a}+1)\hat{a}^2 - \hat{\rho}\hat{a}^2) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^2) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle + \frac{k_1}{2}Tr(-2\hat{\rho}\hat{a}^2) \\
&= -i\langle[\hat{a}^2,\hat{H}]\rangle - k_1\langle\hat{a}^2\rangle. \qquad (2.22)
\end{aligned}$$

And one can easily notice that

$$-i\langle[\hat{a}^2,\hat{H}]\rangle = -i\langle[\hat{a}^2,\hat{H}_2] + [\hat{a}^2,\hat{H}_3]\rangle. \qquad (2.23)$$

Inserting the expression of Hamiltonian and applying Eq.(2.19) in the previous commutation, we find that

$$-i\langle[\hat{a}^2,\hat{H}]\rangle = -2g\langle\hat{\sigma}_a\hat{a}\rangle - 2\mu\langle\hat{b}^\dagger\hat{a}\rangle. \qquad (2.24)$$

Combination of Eq.(2.24) with the expression in Eq.(2.22) gives

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -2g\langle\hat{\sigma}_a\hat{a}\rangle - 2\mu\langle\hat{b}^\dagger\hat{a}\rangle - k_1\langle\hat{a}^2\rangle. \qquad (2.25)$$

In a similar procedure, one readily obtains

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle &= -g(\langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\sigma_a^\dagger\hat{a}\rangle) - \mu(\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle) - k_1\langle\hat{a}^\dagger\hat{a}\rangle, \qquad (2.26) \\
\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle &= -g(\langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a^\dagger\hat{a}\rangle) - \mu(\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle) - k_1\langle\hat{a}\hat{a}^\dagger\rangle + k_1, \qquad (2.27)
\end{aligned}$$

and for mode b operators

$$\begin{aligned}
\frac{d}{dt}\langle\hat{b}\rangle &= -g\langle\hat{\sigma}_b\rangle - \mu\langle\hat{a}^\dagger\rangle - \frac{k_2}{2}\langle\hat{b}\rangle, \qquad (2.28) \\
\frac{d}{dt}\langle\hat{b}^2\rangle &= -2g\langle\hat{\sigma}_b\hat{b}\rangle - 2\mu\langle\hat{a}^\dagger\hat{b}\rangle - k_2\langle\hat{b}^2\rangle, \qquad (2.29) \\
\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle &= -g(\langle\hat{b}^\dagger\hat{\sigma}_b\rangle + \langle\sigma_b^\dagger\hat{b}\rangle) - \mu(\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \langle\hat{b}\hat{a}\rangle) - k_2\langle\hat{b}^\dagger\hat{b}\rangle, \qquad (2.30) \\
\frac{d}{dt}\langle\hat{b}\hat{b}^\dagger\rangle &= -g(\langle\hat{b}^\dagger\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\hat{b}\rangle) - \mu(\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle) - k_2\langle\hat{b}\hat{b}^\dagger\rangle + k_2, \qquad (2.31)
\end{aligned}$$

And for inter-modal operators

$$
\begin{aligned}
\tfrac{d}{dt}\langle\hat{a}\hat{b}\rangle = &\quad -g(\langle\hat{\sigma}_a\hat{b}\rangle+\langle\hat{\sigma}_b\hat{a}\rangle)-\mu(\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{b}^\dagger\hat{b}\rangle+1)-\frac{(k_2+k_1)}{2}\langle\hat{a}\hat{b}\rangle, && (2.32)\\
\tfrac{d}{dt}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle = &\quad -g(\langle\hat{\sigma}_a^\dagger\hat{b}^\dagger\rangle+\langle\hat{\sigma}_b^\dagger\hat{a}^\dagger\rangle)-\mu(\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{b}^\dagger\hat{b}\rangle+1)-\frac{(k_2+k_1)}{2}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle && (2.33)\\
\tfrac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = &\quad -g(\langle\hat{\sigma}_a^\dagger\hat{b}\rangle+\langle\hat{a}^\dagger\hat{\sigma}_b\rangle)-\mu(\langle\hat{b}^2\rangle+\langle\hat{a}^{\dagger 2}\rangle)-\frac{(k_2+k_1)}{2}\langle\hat{a}^\dagger\hat{b}\rangle, && (2.34)\\
\tfrac{d}{dt}\langle\hat{b}^\dagger\hat{a}\rangle = &\quad -g(\langle\hat{b}^\dagger\hat{\sigma}_a\rangle+\langle\hat{\sigma}_b^\dagger\hat{a}\rangle)-\mu(\langle\hat{b}^{\dagger 2}\rangle+\langle\hat{a}^2\rangle)-\frac{(k_2+k_1)}{2}\langle\hat{b}^\dagger\hat{a}\rangle. && (2.35)
\end{aligned}
$$

2.2 Time evolution of atomic operators

Before calculating the time evolution of the atomic operators, we first simplify frequently used relations.

$$
\begin{aligned}
[\hat{H},\hat{\rho}] &= i\frac{\Omega}{2}(\hat{\sigma}_c^\dagger-\hat{\sigma}_c)\hat{\rho}-i\frac{\Omega}{2}\hat{\rho}(\hat{\sigma}_c^\dagger-\hat{\sigma}_c)+ig(\hat{\sigma}_a^\dagger\hat{a}-\hat{a}^\dagger\hat{\sigma}_a\\
&+ \hat{\sigma}_b^\dagger\hat{b}-\hat{b}^\dagger\hat{\sigma}_b)\hat{\rho}-ig\hat{\rho}(\hat{\sigma}_a^\dagger\hat{a}-\hat{a}^\dagger\hat{\sigma}_a+\hat{\sigma}_b^\dagger\hat{b}-\hat{b}^\dagger\hat{\sigma}_b)\\
&+ i\mu(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger)\hat{\rho}-i\mu\hat{\rho}(\hat{a}\hat{b}-\hat{a}^\dagger\hat{b}^\dagger)\\
&= i\frac{\Omega}{2}(\hat{\sigma}_c^\dagger\hat{\rho}-\hat{\sigma}_c\hat{\rho}-\hat{\rho}\hat{\sigma}_c^\dagger+\hat{\rho}\hat{\sigma}_c)+ig(\hat{\sigma}_a^\dagger\hat{a}\hat{\rho}-\hat{a}^\dagger\hat{\sigma}_a\hat{\rho}\\
&+ \hat{\sigma}_b^\dagger\hat{b}\hat{\rho}-\hat{b}^\dagger\hat{\sigma}_b\hat{\rho}-\hat{\rho}\hat{\sigma}_a^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a-\hat{\rho}\hat{\sigma}_b^\dagger\hat{b}+\hat{\rho}\hat{b}^\dagger\hat{\sigma}_b)\\
&+ i\mu(\hat{a}\hat{b}\hat{\rho}-\hat{a}^\dagger\hat{b}^\dagger\hat{\rho}-\hat{\rho}\hat{a}\hat{b}+\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger). \qquad (2.36)
\end{aligned}
$$

All the atomic operators commute with terms in the master equation containing k_1 and k_2. Thus the trace of the product of atomic operators and kappa terms is zero. This implies the time evolution of atomic operators will not contain kappas. Based on this the time evolution of atomic operators can be obtained with the use of the master equation in the relation

$$
\begin{aligned}
\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= Tr(\frac{d\hat{\rho}}{dt}\hat{\sigma}_a)\\
&= Tr\{[-i[\hat{H},\hat{\rho}]+\frac{k_1}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^\dagger\hat{a})\\
&+ \frac{k_2}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger-\hat{b}^\dagger\hat{b}\hat{\rho}-\hat{\rho}\hat{b}^\dagger\hat{b})|b\rangle\langle a|\}\\
&= Tr(-i[\hat{H},\hat{\rho}]|b\rangle\langle a|)\\
&= -i\langle[\hat{\sigma}_a,\hat{H}]\rangle. \qquad (2.37)
\end{aligned}
$$

Substituting Eq.(2.36) in to Eq.(2.37) yields

$$
\begin{aligned}
\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= Tr\{\frac{\Omega}{2}((\langle a|\hat{\sigma}_c^\dagger\hat{\rho}|b\rangle-\langle a|\hat{\sigma}_c\hat{\rho}|b\rangle-\langle a|\hat{\rho}\hat{\sigma}_c^\dagger|b\rangle+\langle a|\hat{\rho}\hat{\sigma}_c|b\rangle)\\
&+ g(\langle a|\hat{\sigma}_a^\dagger\hat{a}\hat{\rho}|b\rangle-\langle a|\hat{a}^\dagger\hat{\sigma}_a\hat{\rho}|b\rangle+\langle a|\hat{\sigma}_b^\dagger\hat{b}\hat{\rho}|b\rangle-\langle a|\hat{b}^\dagger\hat{\sigma}_b\hat{\rho}|b\rangle\\
&- \langle a|\hat{\rho}\hat{\sigma}_a^\dagger\hat{a}|b\rangle+\langle a|\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a|b\rangle-\langle a|\hat{\rho}\hat{\sigma}_b^\dagger\hat{b}|b\rangle+\langle a|\hat{\rho}\hat{b}^\dagger\hat{\sigma}_b|b\rangle)\\
&+ \mu(\hat{\rho}\hat{a}\hat{b}|b\rangle\langle a|-\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger|b\rangle\langle a|-\hat{\rho}\hat{a}\hat{b}|b\rangle\langle a|+\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger|b\rangle\langle a|)\}. \qquad (2.38)
\end{aligned}
$$

Substituting the expression of atomic operators in Eq.(2.38), one can get

$$\begin{aligned}
\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= Tr\{\frac{\Omega}{2}(\langle a|a\rangle\langle c|\hat{\rho}|b\rangle - \langle a|c\rangle\langle a|\hat{\rho}|b\rangle - \langle a|\hat{\rho}|a\rangle\langle c|b\rangle + \langle a|\hat{\rho}|c\rangle\langle a|b\rangle) \\
&+ g(\langle a|a\rangle\langle b|\hat{a}\hat{\rho}|b\rangle - \langle a|\hat{a}^\dagger|b\rangle\langle a|\hat{\rho}|b\rangle + \langle a|b\rangle\langle c|\hat{b}\hat{\rho}|b\rangle - \langle a|\hat{b}^\dagger|c\rangle\langle b|\hat{\rho}|b\rangle \\
&- \langle a|\hat{\rho}|a\rangle\langle b|\hat{a}|b\rangle + \langle a|\hat{\rho}\hat{a}^\dagger|b\rangle\langle a|b\rangle - \langle a|\hat{\rho}|b\rangle\langle c|\hat{b}|b\rangle + \langle a|\hat{\rho}\hat{b}^\dagger|c\rangle\langle b|b\rangle)\}.
\end{aligned}$$

Since the atomic states are orthonormal to each other, and the cavity mode operators donot act on the atomic states the above equation becomes

$$\begin{aligned}
\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= Tr\{\frac{\Omega}{2}\langle c|\hat{\rho}|b\rangle + g(\langle b|\hat{a}\hat{\rho}|b\rangle - \langle a|\hat{\rho}|a\rangle\hat{a} + \langle a|\hat{\rho}\hat{b}^\dagger|c\rangle\} \\
&= Tr\{\frac{\Omega}{2}\hat{\rho}|b\rangle\langle c| + g(\hat{\rho}|b\rangle\langle b|\hat{a} - \hat{\rho}|a\rangle\langle a|\hat{a} + \hat{\rho}\hat{b}^\dagger|c\rangle\langle a|\} \\
&= \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle + g\{\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle\}. \qquad (2.39)
\end{aligned}$$

Following the same procedure, it can be established that

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = \frac{-\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle - g\{\langle\hat{a}^\dagger\hat{\sigma}_c\rangle - \langle\hat{\eta}_c\hat{b}\rangle + \langle\hat{\eta}_b\hat{b}\rangle\}, \qquad (2.40)$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = \frac{\Omega}{2}\{\langle\eta_c\rangle - \langle\eta_a\rangle\} + g\{\langle\hat{\sigma}_b\hat{a}\rangle - \langle\hat{\sigma}_a\hat{b}\rangle\}, \qquad (2.41)$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = \frac{\Omega}{2}\{\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle\} + g\{\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle\}, \qquad (2.42)$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = g\{\langle\hat{\sigma}_b^\dagger\hat{b}\rangle - \langle\hat{\sigma}_a^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b\rangle\}, \qquad (2.43)$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -\frac{\Omega}{2}\{\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle\} - g\{\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b\rangle\}. \qquad (2.44)$$

2.3 Quantum Langevin equations

Since the cavity modes are non-degenerate, there are two quantum Langevin equations for the cavity mode operators. These equations tell the time evolution of the annihilation operators of the system. The quantum Langevin equations of this system are proposed from Eq.(2.21) and Eq.(2.28).
The quantum Langevin equation for mode a can thus be written as [2]

$$\frac{d}{dt}\hat{a}(t) = -g\hat{\sigma}_a - \mu\hat{b}^\dagger - \frac{k_1}{2}\hat{a}(t) + \hat{F}_1(t), \qquad (2.45)$$

And for mode b

$$\frac{d}{dt}\hat{b}(t) = -g\hat{\sigma}_b - \mu\hat{a}^\dagger - \frac{k_2}{2}\hat{b}(t) + \hat{F}_2(t), \qquad (2.46)$$

where $\hat{F}_1(t)$ and $\hat{F}_2(t)$ are noise operators for mode a and mode b respectively. By comparing Eq.(2.21) and the expectation value of Eq.(2.45) it can be easily deduced that

$$\langle\hat{F}_1(t)\rangle = 0, \qquad (2.47)$$

and also from Eq.(2.28) and the expectation value of Eq.(2.46)

$$\langle\hat{F}_2(t)\rangle = 0. \qquad (2.48)$$

These quantum Langevin equations are first order differential equations. First order differential equation with the form

$$\frac{dy}{dx}+p(x)y=q(x) \tag{2.49}$$

has a solution of the form [21]:

$$y(x)=y(0)e^{-\int p(x)dx}+\int e^{-\int p(x)dx}q(x)dx. \tag{2.50}$$

Accordingly, the solution of the quantum Langevin equations specified by Eqs. (2.45) and (2.46) can be expressed as [2]:

$$\begin{aligned}
\hat{a}(t) &= \hat{a}(0)e^{-k_1t/2}-\int_0^t e^{-k_1(t-t')/2}[i[\hat{a}(t'),\hat{H}(t')]-\hat{F}_1(t')]dt', && (2.51)\\
\hat{a}^\dagger(t) &= \hat{a}^\dagger(0)e^{-k_1t/2}-\int_0^t e^{-k_1(t-t')/2}[i[\hat{a}^\dagger(t'),\hat{H}(t')]-\hat{F}_1^\dagger(t')]dt', && (2.52)\\
\hat{b}(t) &= \hat{b}(0)e^{-k_2t/2}-\int_0^t e^{-k_2(t-t')/2}[i[\hat{b}(t'),\hat{H}(t')]-\hat{F}_2(t')]dt', && (2.53)\\
\hat{b}^\dagger(t) &= \hat{b}^\dagger(0)e^{-k_2t/2}-\int_0^t e^{-k_2(t-t')/2}[i[\hat{b}^\dagger(t'),\hat{H}(t')]-\hat{F}_2^\dagger(t')]dt'. && (2.54)
\end{aligned}$$

The solution of quantum Langevin equations are helpfull to calculate the correlation properties of the noise operators.

2.3.1 Correlation properties of cavity mode noise operators

With the aid of the quantum Langevin equation for cavity mode a and the relation [2]

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle=\langle\frac{d\hat{a}(t)}{dt}\hat{a}(t)\rangle+\langle\hat{a}(t)\frac{d\hat{a}(t)}{dt}\rangle, \tag{2.55}$$

we observe that

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle &= \langle(-i[\hat{a}(t),\hat{H}(t)]-\frac{k_1}{2}\hat{a}(t)+\hat{F}_1(t))\hat{a}(t)\rangle\\
&+ \langle\hat{a}(t)(-i[\hat{a}(t),\hat{H}(t)]-\frac{k_1}{2}\hat{a}(t)+\hat{F}_1(t))\rangle\\
&= \langle(-i[\hat{a}(t),\hat{H}(t)]\hat{a}(t)-\frac{k_1}{2}\hat{a}(t)\hat{a}(t)+\hat{F}_1(t))\hat{a}(t)\rangle\\
&+ \langle(-i\hat{a}(t)[\hat{a}(t),\hat{H}(t)]-\frac{k_1}{2}\hat{a}(t)\hat{a}(t)+\hat{a}(t)\hat{F}_1(t))\rangle\\
&= -i\langle[\hat{a}(t),\hat{H}(t)]\hat{a}(t)\rangle-i\langle\hat{a}(t)[\hat{a}(t),\hat{H}(t)]\rangle\\
&- k_1\langle\hat{a}^2(t)\rangle+\langle\hat{F}_1(t))\hat{a}(t)\rangle+\langle\hat{a}(t)\hat{F}_1(t))\rangle\\
&= -i\langle[\hat{a}^2(t),\hat{H}(t)]\rangle-k_1\langle\hat{a}^2(t)\rangle+\langle\hat{F}_1(t)\hat{a}(t)\rangle\\
&+ \langle\hat{a}(t)\hat{F}_1(t)\rangle, && (2.56)
\end{aligned}$$

or

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle=-i\langle[\hat{a}^2(t),\hat{H}(t)]\rangle-k_1\langle\hat{a}^2(t)\rangle+\langle\hat{F}_1(t)\hat{a}(t)\rangle+\langle\hat{a}(t)\hat{F}_1(t)\rangle. \tag{2.57}$$

Comparing Eq.(2.57) and Eq.(2.25), we realize that

$$\langle \hat{F}_1(t)\hat{a}(t)\rangle + \langle \hat{a}(t)\hat{F}_1(t)\rangle = 0,$$

which with the use of the solution of the quantum Langevin equation for mode a becomes

$$\begin{aligned}\langle \hat{F}_1(t)\hat{a}(0)\rangle e^{-k_1 t/2} - \int_0^t e^{-k_1(t-t')/2}[i\langle \hat{F}_1(t)[\hat{a}(t'),\hat{H}(t')]\rangle - \langle \hat{F}_1(t)\hat{F}_1(t')\rangle]dt' \\ +\langle \hat{a}(0)\hat{F}(t)\rangle e^{-k_1 t/2} - \int_0^t e^{-k_1(t-t')/2}[i\langle [\hat{a}(t'),\hat{H}(t')]\hat{F}_1(t)\rangle - \langle \hat{F}_1(t')\hat{F}(t)\rangle]dt' \\ = 0. \quad (2.58)\end{aligned}$$

As noise operator at some time t does not affect system operator at an earlier time,[2]

$$\begin{aligned}\langle \hat{F}_1(t)\hat{a}(0)\rangle &= \langle \hat{F}(t)\rangle\langle \hat{a}(0)\rangle = 0, && (2.59)\\ \langle \hat{a}(0)\hat{F}_1(t)\rangle &= \langle \hat{a}(0)\rangle\langle \hat{F}(t)\rangle = 0, && (2.60)\\ \langle [\hat{a}(t'),\hat{H}(t')]\hat{F}_1\rangle &= \langle [\hat{a}(t'),\hat{H}(t')]\rangle\langle \hat{F}_1\rangle = 0, && (2.61)\\ \langle \hat{F}_1[\hat{a}(t'),\hat{H}(t')]\rangle &= \langle \hat{F}_1\rangle\langle [\hat{a}(t'),\hat{H}(t')]\rangle = 0. && (2.62)\end{aligned}$$

With the aid of these result, we can put Eq. (2.58) as

$$\int_0^t e^{-k_1(t-t')/2}\langle \hat{F}_1(t)\hat{F}_1(t')\rangle dt' + \int_0^t e^{-k_1(t-t')/2}\langle \hat{F}_1(t')\hat{F}_1(t)\rangle dt' = 0 \quad (2.63)$$

and assuming

$$\langle \hat{F}_1(t)\hat{F}_1(t')\rangle = \langle \hat{F}_1(t')\hat{F}_1(t)\rangle, \quad (2.64)$$

we notice that

$$\int_0^t e^{-k_1(t-t')/2}\langle \hat{F}_1(t)\hat{F}_1(t')\rangle dt' = 0 \quad (2.65)$$

or

$$\langle \hat{F}_1(t)\hat{F}_1(t')\rangle = 0. \quad (2.66)$$

In a similar pattern, it can be verified that

$$\begin{aligned}\langle \hat{F}_1(t')\hat{F}_1^\dagger(t)\rangle &= k_1\delta(t-t'), && (2.67)\\ \langle \hat{F}_1^\dagger(t')\hat{F}_1(t)\rangle &= 0, && (2.68)\\ \langle \hat{F}_2(t)\hat{F}_2(t')\rangle &= 0, && (2.69)\\ \langle \hat{F}_2(t')\hat{F}_2^\dagger(t)\rangle &= k_2\delta(t-t'), && (2.70)\\ \langle \hat{F}_2^\dagger(t')\hat{F}_2(t)\rangle &= 0. && (2.71)\end{aligned}$$

The quantum Langevin equations are helpfull to obtain the correlation properties, while they are not helpfull to study the analysis of the system. Therefore the analysis of the system is done using the evolutions of the expectation values. But they are coupled equations between the atomic operators and the cavity mode operators. Consequently it can be made easy by decoupleing the product of atomic and cavity mode operators.

2.4 The decoupled differential equations

For the sake of decoupling of cavity mode operators and atomic operators we employ the large time approximation. In this approximation, the time differentiation of cavity mode operators are expected to be close to zero.
From this perspective, Eq.(2.45) and Eq.(2.46) become

$$\hat{a} = \frac{-2g}{k_1}\hat{\sigma}_a - \frac{2\mu}{k_1}\hat{b}^\dagger + \frac{2}{k_1}\hat{F}_1, \tag{2.72}$$

$$\hat{a}^\dagger = \frac{-2g}{k_1}\hat{\sigma}_a^\dagger - \frac{2\mu}{k_1}\hat{b} + \frac{2}{k_1}\hat{F}_1^\dagger, \tag{2.73}$$

$$\hat{b} = \frac{-2g}{k_2}\hat{\sigma}_b - \frac{2\mu}{k_2}\hat{a}^\dagger + \frac{2}{k_2}\hat{F}_2, \tag{2.74}$$

and

$$\hat{b}^\dagger = \frac{-2g}{k_2}\hat{\sigma}_b^\dagger - \frac{2\mu}{k_2}\hat{a} + \frac{2}{k_2}\hat{F}_2^\dagger. \tag{2.75}$$

Substituting Eq.(2.75) in to Eq.(2.72), and Eq.(2.73) in to Eq.(2.74), one can arrive at

$$\hat{a} = \frac{4\mu g\Gamma_1}{k_1}[\hat{\sigma}_b^\dagger - \frac{k_2}{2\mu}\hat{\sigma}_a] - \frac{4\mu\Gamma_1}{k_1}\hat{F}_2^\dagger + 2\Gamma_2\hat{F}_1, \tag{2.76}$$

$$\hat{a}^\dagger = \frac{4\mu g\Gamma_1}{k_1}[\hat{\sigma}_b - \frac{k_2}{2\mu}\hat{\sigma}_a^\dagger] - \frac{4\mu\Gamma_1}{k_1}\hat{F}_2 + 2\Gamma_2\hat{F}_1^\dagger, \tag{2.77}$$

$$\hat{b} = \frac{4\mu g\Gamma_2}{k_2}[\hat{\sigma}_a^\dagger - \frac{k_1}{2\mu}\hat{\sigma}_b] - \frac{4\mu\Gamma_2}{k_2}\hat{F}_1^\dagger + 2\Gamma_1\hat{F}_2, \tag{2.78}$$

and

$$\hat{b}^\dagger = \frac{4\mu g\Gamma_2}{k_2}[\hat{\sigma}_a - \frac{k_1}{2\mu}\hat{\sigma}_b^\dagger] - \frac{4\mu\Gamma_2}{k_2}\hat{F}_1 + 2\Gamma_1\hat{F}_2^\dagger, \tag{2.79}$$

where

$$\gamma_{c1} = \frac{4g^2}{k_1}, \gamma_{c2} = \frac{4g^2}{k_2}, \Gamma_1 = \frac{k_1}{k_1k_2 - 4\mu^2}, \Gamma_2 = \frac{k_2}{k_1k_2 - 4\mu^2}.$$

For $k_1 = k_2 = k$

$$\gamma_{c1} = \gamma_{c2} = \gamma_c = \frac{4g^2}{k}, \tag{2.80}$$

$$\Gamma_1 = \Gamma_2 = \Gamma = \frac{k}{k^2 - 4\mu^2}. \tag{2.81}$$

Eq.(2.76) to Eq.(2.79) help to decouple the product of cavity mode and atomic operators appearing in their equation of evolution. We thus use these equations in the time evolution relations specified by Eq.(2.25) to Eq.(2.27), Eq.(2.29) to Eq.(2.35), and Eq.(2.39) to Eq.(2.44).
Utilizing Eq.(2.76) in Eq.(2.25), we have

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}^2\rangle &= -2g\langle(\frac{4\mu g\Gamma_1}{k_1}[\hat{\sigma}_b^\dagger - \frac{k_2}{2\mu}\hat{\sigma}_a] - \frac{4\mu\Gamma_1}{k_1}\hat{F}_2^\dagger + 2\Gamma_2\hat{F}_1)\hat{\sigma}_a\rangle - 2\mu\langle\hat{b}^\dagger\hat{a}\rangle - k_1\langle\hat{a}^2\rangle \\
&= -2g(-2g\Gamma_2\langle\hat{\sigma}_a\hat{\sigma}_a\rangle + \frac{4\mu g\Gamma_1}{k_1}\langle\hat{\sigma}_b^\dagger\hat{\sigma}_a\rangle - \frac{4\mu\Gamma_2}{k_2}\langle\hat{F}_1\hat{\sigma}_a\rangle + 2\Gamma_1\langle\hat{F}_2^\dagger\hat{\sigma}_a\rangle) \\
&- 2\mu\langle\hat{b}^\dagger\hat{a}\rangle - k_1\langle\hat{a}^2\rangle.
\end{aligned}$$

Since the correlation of noise operator and atomic operator is very small, one can assume

$$\langle \hat{F}_2^\dagger(t)\hat{\sigma}_a(t)\rangle = \langle \hat{F}_2^\dagger(t)\rangle\langle \hat{\sigma}_a(t)\rangle = 0, \tag{2.82}$$

$$\langle \hat{F}_1(t)\hat{\sigma}_a(t)\rangle = \langle \hat{F}_1(t)\rangle\langle \hat{\sigma}_a(t)\rangle = 0, \tag{2.83}$$

and considering the orthonormalization of the atomic states, it follows that

$$\hat{\sigma}_a\hat{\sigma}_a = \hat{\sigma}_b^\dagger\hat{\sigma}_a = 0. \tag{2.84}$$

With the aid of these relations, the time evolution of the second moment of cavity mode operator a becomes

$$\frac{d}{dt}\langle \hat{a}^2\rangle = -2\mu\langle \hat{b}^\dagger\hat{a}\rangle - k_1\langle \hat{a}^2\rangle. \tag{2.85}$$

By following the same methods, the other decoupled equations of evolutions based on the substitiution become

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}\hat{a}^\dagger\rangle &= -\gamma_{c1}\Gamma_1\mu\{\frac{-k_2}{\mu}\langle \hat{\eta}_a\rangle + \langle \hat{\sigma}_c\rangle + \langle \hat{\sigma}_c^\dagger\rangle\}\\ &- \mu(\langle \hat{a}^\dagger\hat{b}^\dagger\rangle + \langle \hat{a}\hat{b}\rangle) - k_1\langle \hat{a}\hat{a}^\dagger\rangle + k_1,\end{aligned} \tag{2.86}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^\dagger\hat{a}\rangle &= -\gamma_{c1}\Gamma_1\mu\{\frac{-k_2}{\mu}\langle \hat{\eta}_a\rangle + \langle \hat{\sigma}_c\rangle + \langle \hat{\sigma}_c^\dagger\rangle\}\\ &- \mu(\langle \hat{a}^\dagger\hat{b}^\dagger\rangle + \langle \hat{a}\hat{b}\rangle) - k_1\langle \hat{a}^\dagger\hat{a}\rangle,\end{aligned} \tag{2.87}$$

$$\frac{d}{dt}\langle \hat{b}^2\rangle = -2\mu\langle \hat{a}^\dagger\hat{b}\rangle - k_2\langle \hat{b}^2\rangle, \tag{2.88}$$

$$\frac{d}{dt}\langle \hat{b}\hat{b}^\dagger\rangle = \gamma_{c2}\Gamma_2 k_1\langle \hat{\eta}_b\rangle - \mu(\langle \hat{b}^\dagger\hat{a}^\dagger\rangle + \langle \hat{b}\hat{a}\rangle) - k_2\langle \hat{b}\hat{b}^\dagger\rangle + k_2, \tag{2.89}$$

$$\frac{d}{dt}\langle \hat{b}^\dagger\hat{b}\rangle = \gamma_{c2}\Gamma_2 k_1\langle \hat{\eta}_b\rangle - \mu(\langle \hat{b}^\dagger\hat{a}^\dagger\rangle + \langle \hat{b}\hat{a}\rangle) - k_2\langle \hat{b}^\dagger\hat{b}\rangle, \tag{2.90}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}\hat{b}\rangle &= \gamma_{c2}\mu\Gamma_2(\frac{-k_1}{2\mu}\langle \hat{\sigma}_c\rangle + \langle \hat{\eta}_a\rangle) - \gamma_{c1}\mu\Gamma_1\langle \hat{\eta}_b\rangle\\ &- \mu(\langle \hat{a}^\dagger\hat{a}\rangle + \langle \hat{b}^\dagger\hat{b}\rangle + 1) - \frac{(k_2+k_1)}{2}\langle \hat{a}\hat{b}\rangle,\end{aligned} \tag{2.91}$$

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^\dagger\hat{b}^\dagger\rangle &= \gamma_{c2}\mu\Gamma_2(\frac{-k_1}{2\mu}\langle \hat{\sigma}_c^\dagger\rangle + \langle \hat{\eta}_a\rangle) - \gamma_{c1}\mu\Gamma_1\langle \hat{\eta}_b\rangle)\\ &- \mu(\langle \hat{a}^\dagger\hat{a}\rangle + \langle \hat{b}^\dagger\hat{b}\rangle + 1) - \frac{(k_2+k_1)}{2}\langle \hat{a}^\dagger\hat{b}^\dagger\rangle,\end{aligned} \tag{2.92}$$

$$\frac{d}{dt}\langle \hat{a}\hat{b}^\dagger\rangle = \frac{d}{dt}\langle \hat{b}^\dagger\hat{a}\rangle = -\mu(\langle \hat{b}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle) - \frac{(k_2+k_1)}{2}\langle \hat{b}^\dagger\hat{a}\rangle, \tag{2.93}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{b}\rangle = \frac{d}{dt}\langle \hat{b}\hat{a}^\dagger\rangle = -\mu(\langle \hat{b}^2\rangle + \langle \hat{a}^{\dagger 2}\rangle) - \frac{(k_2+k_1)}{2}\langle \hat{a}^\dagger\hat{b}\rangle. \tag{2.94}$$

The equations of evolution for the expectation value of atomic operators can also be written as

$$\begin{aligned}
\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= \{\frac{\Omega}{2}+\gamma_{c1}\mu\Gamma_1\}\langle\hat{\sigma}_b^\dagger\rangle-\{\frac{\gamma_{c1}\Gamma_1k_2}{2}+\frac{\gamma_{c2}\Gamma_2k_1}{2}\}\langle\hat{\sigma}_a\rangle, & (2.95)\\
\frac{d}{dt}\langle\hat{\sigma}_b\rangle &= -\frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle-\frac{\gamma_{c2}\Gamma_2k_1}{2}\langle\hat{\sigma}_b\rangle, & (2.96)\\
\frac{d}{dt}\langle\hat{\sigma}_c\rangle &= (\frac{\Omega}{2}+\gamma_{c1}\mu\Gamma_1)\langle\hat{\eta}_c\rangle-\frac{\Omega}{2}\langle\hat{\eta}_a\rangle-\frac{\gamma_{c1}\Gamma_1k_2}{2}\langle\hat{\sigma}_c\rangle-\gamma_{c2}\Gamma_2\mu\langle\hat{\eta}_b\rangle), & (2.97)\\
\frac{d}{dt}\langle\hat{\eta}_a\rangle &= (\frac{\Omega}{2}+\gamma_{c1}\Gamma_1\mu)(\langle\hat{\sigma}_c^\dagger\rangle+\langle\hat{\sigma}_c\rangle)-\gamma_{c1}\Gamma_1k_2\langle\hat{\eta}_a\rangle, & (2.98)\\
\frac{d}{dt}\langle\hat{\eta}_b\rangle &= \gamma_{c1}\Gamma_1k_2\langle\hat{\eta}_a\rangle-\gamma_{c2}\Gamma_2k_1\langle\hat{\eta}_b\rangle-\gamma_{c1}\mu\Gamma_1(\langle\hat{\sigma}_c^\dagger\rangle+\langle\hat{\sigma}_c\rangle), & (2.99)\\
\frac{d}{dt}\langle\hat{\eta}_c\rangle &= -\frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle+\langle\hat{\sigma}_c\rangle)+\gamma_{c2}\Gamma_2k_1\langle\hat{\eta}_b\rangle. & (2.100)
\end{aligned}$$

2.4.1 The steady state solutions of the decoupled differential equations

The steady state solution of Eq.(2.21) is expressible as

$$\langle\hat{a}\rangle=\frac{-2g}{k_1}\langle\hat{\sigma}_a\rangle-\frac{2\mu}{k_1}\langle\hat{b}^\dagger\rangle. \tag{2.101}$$

Taking the adjoint of the whole equation gives

$$\langle\hat{a}^\dagger\rangle=\frac{-2g}{k_1}\langle\hat{\sigma}_a^\dagger\rangle-\frac{2\mu}{k_1}\langle\hat{b}\rangle. \tag{2.102}$$

By following the same method one readily obtains

$$\langle\hat{b}\rangle=\frac{-2g}{k_2}\langle\hat{\sigma}_b\rangle-\frac{2\mu}{k_2}\langle\hat{a}^\dagger\rangle, \tag{2.103}$$

$$\langle\hat{b}^\dagger\rangle=\frac{-2g}{k_2}\langle\hat{\sigma}_b^\dagger\rangle-\frac{2\mu}{k_2}\langle\hat{a}\rangle. \tag{2.104}$$

Substituting Eq.(2.104) in to Eq.(2.101) gives

$$\langle\hat{a}\rangle=\frac{-2gk_2}{k_1k_2-4\mu^2}\langle\hat{\sigma}_a\rangle+\frac{4g\mu}{k_1k_2-4\mu^2}\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.105}$$

and also substituting Eq.(2.102) to Eq.(2.103) gives

$$\langle\hat{b}\rangle=\frac{-2gk_1}{k_1k_2-4\mu^2}\langle\hat{\sigma}_b\rangle+\frac{4g\mu}{k_1k_2-4\mu^2}\langle\hat{\sigma}_a^\dagger\rangle. \tag{2.106}$$

The steady state solution of $\langle\hat{\sigma}_b\rangle$ and $\langle\hat{\sigma}_a\rangle$ can be obtained by setting $\frac{d}{dt}\langle\hat{\sigma}_a\rangle=0$ and $\frac{d}{dt}\langle\hat{\sigma}_b\rangle=0$. It then follows from Eqs. (2.95) and (2.96) that

$$\begin{aligned}
\langle\hat{\sigma}_a\rangle &= \frac{\Omega(k_1k_2-4\mu^2)+8g^2\mu}{4g^2(k_1+k_2)}\langle\hat{\sigma}_b^\dagger\rangle, & (2.107)\\
\langle\hat{\sigma}_a^\dagger\rangle &= -\frac{4g^2k_1}{\Omega(k_1k_2-4\mu^2)}\langle\hat{\sigma}_b\rangle, & (2.108)\\
\langle\hat{\sigma}_a\rangle &= -\frac{4g^2k_1}{\Omega(k_1k_2-4\mu^2)}\langle\hat{\sigma}_b^\dagger\rangle. & (2.109)
\end{aligned}$$

To satisfy Eq.(2.109) and Eq.(2.107), $\langle\hat{\sigma}_a\rangle = \langle\hat{\sigma}_b\rangle = \langle\hat{\sigma}_a^\dagger\rangle = \langle\hat{\sigma}_b^\dagger\rangle$=0.
Substituting this result in Eqs.(2.105) and (2.106) yields

$$\langle\hat{a}\rangle = \langle\hat{b}\rangle = \langle\hat{a}^\dagger\rangle = \langle\hat{b}^\dagger\rangle = 0. \tag{2.110}$$

Consiquently, $\hat{a}$ and $\hat{b}$ are Gaussian variables, with zero mean value. For this system the steady state solution can help to describe its analysis. Therefore the whole steady state solutions of the decoupled equations of evolution have to be found. Whereas the steady state equations are simultaneous, it is better to start from the atomic operators.
from Eq.(2.97) to Eq.(2.100)

$$\begin{aligned}
0 &= (\frac{\Omega}{2} + \gamma_{c1}\mu\Gamma_1)\langle\hat{\eta}_c\rangle - \frac{\Omega}{2}\langle\hat{\eta}_a\rangle - \frac{\gamma_{c1}\Gamma_1 k_2}{2}\langle\hat{\sigma}_c\rangle - \gamma_{c2}\Gamma_2\mu\langle\hat{\eta}_b\rangle),\\
0 &= (\frac{\Omega}{2} + \gamma_{c1}\Gamma_1\mu)(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) - \gamma_{c1}\Gamma_1 k_2\langle\hat{\eta}_a\rangle,\\
0 &= \gamma_{c1}\Gamma_1 k_2\langle\hat{\eta}_a\rangle - \gamma_{c2}\Gamma_2 k_1\langle\hat{\eta}_b\rangle - \gamma_{c1}\mu\Gamma_1(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle),\\
0 &= -\frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) + \gamma_{c2}\Gamma_2 k_1\langle\hat{\eta}_b\rangle. \qquad (2.111)
\end{aligned}$$

And from the normalization of the atomic state, one can write

$$\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle = 1. \tag{2.112}$$

From the globalization principle of normalization, one can obtain

$$\langle\hat{\eta}_a\rangle_{ss} + \langle\hat{\eta}_b\rangle_{ss} + \langle\hat{\eta}_c\rangle_{ss} = 1. \tag{2.113}$$

There are five equations, which are obtained from the linear combination of only atomic operators.
In view of Eq. (2.111), we see that

$$\begin{aligned}
\langle\hat{\sigma}_c\rangle &= \frac{2}{\gamma_{c1}\Gamma_1 k_2}\{\frac{\Omega}{2}(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle) + \gamma_{c1}\Gamma_1\mu\langle\hat{\eta}_c\rangle - \gamma_{c2}\Gamma_2\mu\langle\hat{\eta}_b\rangle\}, \qquad (2.114)\\
\langle\hat{\eta}_a\rangle &= \frac{(\Omega/2 + \gamma_{c1}\Gamma_1\mu)}{\gamma_{c1}\Gamma_1 k_2}(\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle), \qquad (2.115)\\
\langle\hat{\eta}_b\rangle &= \frac{-\mu}{k_1}(\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle) + \frac{k_2}{k_1}\langle\hat{\eta}_a\rangle, \qquad (2.116)\\
\langle\hat{\eta}_b\rangle &= \frac{\Omega}{2\gamma_{c2}\Gamma_2 k_1}(\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle). \qquad (2.117)
\end{aligned}$$

We note from Eq. (2.114) that $\langle\hat{\sigma}_c\rangle = \langle\hat{\sigma}_c^\dagger\rangle$. We can therefore write

$$\langle\hat{\sigma}_c\rangle + \langle\hat{\sigma}_c^\dagger\rangle = 2\langle\hat{\sigma}_c\rangle.$$

On the basis of this result the atomic operators can be expressed interms of $\langle\hat{\eta}_a\rangle$ as

$$\begin{aligned}
\langle\hat{\eta}_b\rangle &= \frac{\Omega k_2}{k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)}\langle\hat{\eta}_a\rangle, \qquad (2.118)\\
\langle\hat{\sigma}_c\rangle &= \frac{\gamma_{c2}\Gamma_2 k_2}{\Omega + 2\gamma_{c1}\Gamma_1\mu}\langle\hat{\eta}_a\rangle, \qquad (2.119)\\
\langle\hat{\eta}_c\rangle &= \frac{1}{k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)^2}\{\gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2 + \Omega k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)\\
&+ 2\Omega k_2\gamma_{c1}\Gamma_1\mu\}\langle\hat{\eta}_a\rangle. \qquad (2.120)
\end{aligned}$$

Substituting Eqs.(2.118) and (2.120) in to Eq.(2.113) gives

$$\begin{aligned} \langle\hat{\eta}_a\rangle\{1 + \frac{\Omega k_2}{k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)} + \frac{1}{k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)^2}\{\gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2 \\ +\Omega k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu) + 2\Omega k_2\gamma_{c1}\Gamma_1\mu\}\} = 1, \end{aligned} \tag{2.121}$$

So that solving for $\langle\hat{\eta}_a\rangle$ results in

$$\langle\hat{\eta}_a\rangle = \frac{k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)^2}{2k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)(\Omega + \gamma_{c1}\Gamma_1\mu) + \Omega k_2(\Omega + 4\gamma_{c1}\Gamma_1\mu) + \gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2}. \tag{2.122}$$

Substituting Eq.(2.122) in to Eq.(2.118) and Eq.(2.120), one readily obtains

$$\langle\hat{\eta}_b\rangle = \frac{\Omega k_2(\Omega + 2\gamma_{c1}\Gamma_1\mu)}{2k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)(\Omega + \gamma_{c1}\Gamma_1\mu) + \Omega k_2(\Omega + 4\gamma_{c1}\Gamma_1\mu) + \gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2}, \tag{2.123}$$

$$\langle\hat{\eta}_c\rangle = \frac{\Omega k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu) + \gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2 + 2\Omega k_2\gamma_{c1}\Gamma_1\mu}{2k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)(\Omega + \gamma_{c1}\Gamma_1\mu) + \Omega k_2(\Omega + 4\gamma_{c1}\Gamma_1\mu) + \gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2}, \tag{2.124}$$

$$\langle\hat{\sigma}_c\rangle = \frac{\gamma_{c2}\Gamma_2 k_2 k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)}{2k_1(\Omega + 2\gamma_{c1}\Gamma_1\mu)(\Omega + \gamma_{c1}\Gamma_1\mu) + \Omega k_2(\Omega + 4\gamma_{c1}\Gamma_1\mu) + \gamma_{c2}\Gamma_2\gamma_{c1}\Gamma_1 k_1 k_2^2}. \tag{2.125}$$

Based on the condition of Eqs.(2.80) and (2.81), Eqs. (2.126) to(2.129) become

$$\langle\hat{\eta}_a\rangle = \frac{[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]^2}{[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]^2 + \gamma_c^2 k^4 + \Omega(k^2 - 4\mu^2)[2\Omega(k^2 - 4\mu^2) + 6\gamma_c\mu k]}, \tag{2.126}$$

$$\langle\hat{\eta}_b\rangle = \frac{\Omega(k^2 - 4\mu^2)[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]}{[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]^2 + \gamma_c^2 k^4 + \Omega(k^2 - 4\mu^2)[2\Omega(k^2 - 4\mu^2) + 6\gamma_c\mu k]}, \tag{2.127}$$

$$\langle\hat{\eta}_c\rangle = \frac{\gamma_c^2 k^4 + \Omega(k^2 - 4\mu^2)[\Omega(k^2 - 4\mu^2) + 4\gamma_c\mu k]}{[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]^2 + \gamma_c^2 k^4 + \Omega(k^2 - 4\mu^2)[2\Omega(k^2 - 4\mu^2) + 6\gamma_c\mu k]}, \tag{2.128}$$

$$\langle\hat{\sigma}_c\rangle = \frac{\gamma_c k^2[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]}{[\Omega(k^2 - 4\mu^2) + 2\gamma_c\mu k]^2 + \gamma_c^2 k^4 + \Omega(k^2 - 4\mu^2)[2\Omega(k^2 - 4\mu^2) + 6\gamma_c\mu k]}. \tag{2.129}$$

Furthermore, the steady state solutions of cavity mode operators can be expressed as

$$\langle\hat{a}^2\rangle = \frac{-2\mu}{k_1}\langle\hat{a}\hat{b}^\dagger\rangle, \tag{2.130}$$

$$\langle\hat{b}^2\rangle = \frac{-2\mu}{k_2}\langle\hat{b}\hat{a}^\dagger\rangle, \tag{2.131}$$

$$\langle\hat{a}^\dagger\hat{b}\rangle = \frac{-2\mu}{k_1 + k_2}(\langle\hat{a}^{\dagger 2}\rangle + \langle\hat{b}^2\rangle), \tag{2.132}$$

$$\langle\hat{a}\hat{b}^\dagger\rangle = \frac{-2\mu}{k_1 + k_2}(\langle\hat{a}^2\rangle_{ss} + \langle\hat{b}^{\dagger 2}\rangle). \tag{2.133}$$

Based on Eq.(2.130) to Eq.(2.133), it is evident that

$$\langle\hat{a}^2\rangle = \langle\hat{a}^{\dagger 2}\rangle = 0, \tag{2.134}$$

$$\langle\hat{b}^2\rangle = \langle\hat{b}^{\dagger 2}\rangle = 0, \tag{2.135}$$

$$\langle\hat{a}^\dagger\hat{b}\rangle = \langle\hat{b}^\dagger\hat{a}\rangle = 0. \tag{2.136}$$

Moreover, the steady state form of the expectation value of product of cavity mode operators can be expressed as

$$\begin{aligned}
\langle\hat{a}\hat{a}^\dagger\rangle &= \frac{-\gamma_{c1}\Gamma_1\mu}{k_1}(\langle\hat{\sigma}_c\rangle+\langle\hat{\sigma}_c^\dagger\rangle)+\frac{\gamma_{c1}\Gamma_1 k_2}{k_1}\langle\hat{\eta}_a\rangle \\
&- \frac{\mu}{k_1}(\langle\hat{a}\hat{b}\rangle+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle)+1 \qquad (2.137) \\
\langle\hat{a}^\dagger\hat{a}\rangle &= \gamma_{c1}\Gamma_1(\frac{k_2}{k_1}\langle\hat{\eta}_a\rangle-\frac{2\mu\gamma_{c2}\Gamma_2}{\Omega}\langle\hat{\eta}_b\rangle)-\frac{\mu}{k_1}(\langle\hat{a}\hat{b}\rangle+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle), \qquad (2.138) \\
\langle\hat{b}^\dagger\hat{b}\rangle &= \frac{\gamma_{c2}\Gamma_2 k_1}{k_2}\langle\hat{\eta}_b\rangle-\frac{\mu}{k_2}(\langle\hat{a}\hat{b}\rangle+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle), \qquad (2.139) \\
\langle\hat{a}\hat{b}\rangle &= \langle\hat{b}\hat{a}\rangle=\frac{-2\gamma_{c2}\Gamma_2\mu}{k_1+k_2}\langle\hat{\eta}_a\rangle-\frac{2\gamma_{c1}\Gamma_1\mu}{k_1+k_2}\langle\hat{\eta}_b\rangle+\frac{\gamma_{c2}\Gamma_2 k_1}{k_1+k_2}\langle\hat{\sigma}_c\rangle \\
&- \frac{2\mu}{k_1+k_2}(\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{b}^\dagger\hat{b}\rangle+1), \qquad (2.140) \\
\langle\hat{a}^\dagger\hat{b}^\dagger\rangle &= \langle\hat{b}^\dagger\hat{a}^\dagger\rangle=\frac{-2\gamma_{c2}\Gamma_2\mu}{k_1+k_2}\langle\hat{\eta}_a\rangle-\frac{2\gamma_{c1}\Gamma_1\mu}{k_1+k_2}\langle\hat{\eta}_b\rangle+\frac{\gamma_{c2}\Gamma_2 k_1}{k_1+k_2}\langle\hat{\sigma}_c^\dagger\rangle \\
&- \frac{2\mu}{k_1+k_2}(\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{b}^\dagger\hat{b}\rangle+1). \qquad (2.141)
\end{aligned}$$

Based on Eqs.(2.140), Eq.(2.141), and (2.119), we notice that

$$\begin{aligned}
\langle\hat{a}\hat{b}\rangle+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle &= \frac{-4\gamma_{c2}\Gamma_2\mu}{k_1+k_2}\langle\hat{\eta}_a\rangle+\frac{2}{k_1+k_2}[\frac{(\gamma_{c2}\Gamma_2 k_1)^2}{\Omega}-2\gamma_{c1}\Gamma_1\mu]\langle\hat{\eta}_b\rangle \\
&- \frac{4\mu}{k_1+k_2}(\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{b}^\dagger\hat{b}\rangle+1). \qquad (2.142)
\end{aligned}$$

Upon combining Eqs. (2.142) and (2.137), we find that

$$\begin{aligned}
\langle\hat{a}^\dagger\hat{a}\rangle &= \gamma_{c1}\Gamma_1(\frac{k_2}{k_1}\langle\hat{\eta}_a\rangle-\frac{2\mu\gamma_{c2}\Gamma_2}{\Omega}\langle\hat{\eta}_b\rangle)+\frac{4\mu^2}{k_1(k_1+k_2)}(\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{b}^\dagger\hat{b}\rangle+1) \\
&- \frac{\mu}{k_1}\{\frac{-4\gamma_{c2}\Gamma_2\mu}{k_1+k_2}\langle\hat{\eta}_a\rangle+\frac{2}{k_1+k_2}[\frac{(\gamma_{c2}\Gamma_2 k_1)^2}{\Omega}-2\gamma_{c1}\Gamma_1\mu]\langle\hat{\eta}_b\rangle\} \\
&= \frac{[k_2(k_1+k_2)\gamma_{c1}\Gamma_1+4\mu^2\gamma_{c2}\Gamma_2]}{k_1(k_1+k_2)-4\mu^2}\langle\hat{\eta}_a\rangle+\frac{4\mu^2}{k_1(k_1+k_2)-4\mu^2}(\langle\hat{b}^\dagger\hat{b}\rangle+1) \\
&+ \frac{1}{k_1(k_1+k_2)-4\mu^2}\{4\mu^2\gamma_{c1}\Gamma_1-\frac{2\mu(\gamma_{c2}\Gamma_2 k_1)^2}{\Omega} \\
&- \frac{2k_1(k_1+k_2)\mu\gamma_{c1}\Gamma_1\gamma_{c2}\Gamma_2}{\Omega}\langle\hat{\eta}_b\rangle\}. \qquad (2.143)
\end{aligned}$$

Moreover, with Eq. (2.142) substituted into Eq. (2.139), the steady state mean photon number for mode b is expressible as

$$\begin{aligned}
\langle\hat{b}^\dagger\hat{b}\rangle &= \frac{1}{k_2(k_1+k_2)-4\mu^2}\{k_1(k_1+k_2)\gamma_{c2}\Gamma_2+4\mu^2\gamma_{c1}\Gamma_1-\frac{2\mu(\gamma_{c2}\Gamma_2 k_1)^2}{\Omega}\}\langle\hat{\eta}_b\rangle \\
&+ \frac{4\mu^2\gamma_{c2}\Gamma_2}{k_2(k_1+k_2)-4\mu^2}\langle\hat{\eta}_a\rangle+\frac{4\mu^2}{k_2(k_1+k_2)-4\mu^2}(\langle\hat{a}^\dagger\hat{a}\rangle+1). \qquad (2.144)
\end{aligned}$$

For the analysis of this system not all combination of the parameters give the allowed values of $\langle\hat{\eta}_a\rangle, \langle\hat{\eta}_b\rangle, \langle\hat{\eta}_c\rangle$ and the mean photon numbers. For example as we see from Fig. 2.3, the combination of $\Omega = 0.001, k = 0.8, \gamma_c = 0.4$ with $\mu > 0.4$ is not allowed in probability principle, as $\langle \hat{\eta}_b \rangle$ is negative. And some other values make the mean photon numbers negative. Therefore the steady state solutions can only be studied for certain independent variables. The computer program which generates these allowed variables value in python programing language is given in Appendix A and the result is shown graphically in Fig. 2.4.

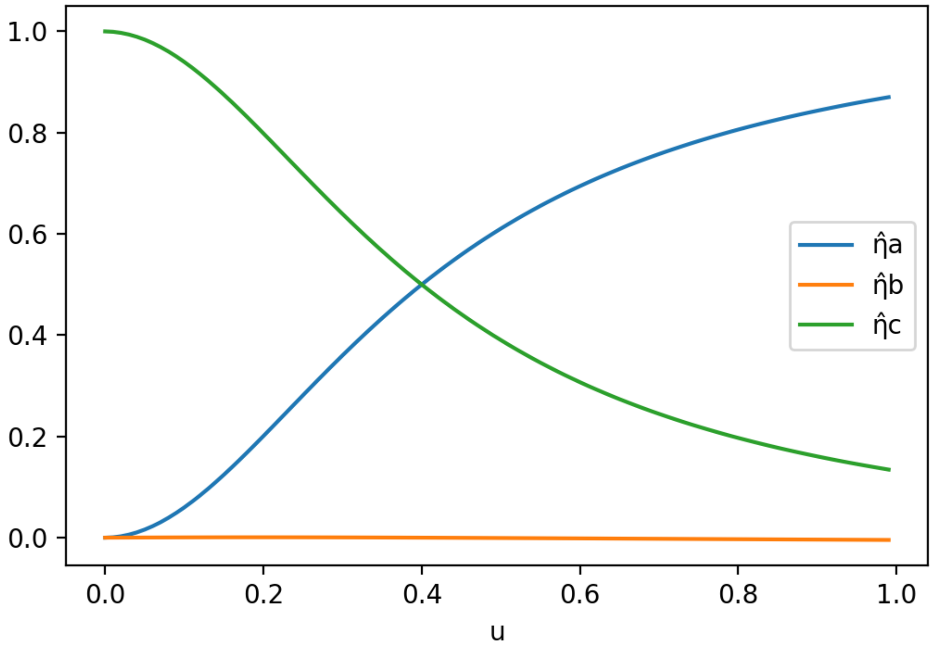

Figure 2.2: Plot of electron states probability vs μ [Eqs.2.126 - 2.128] for $\Omega = 0.001, \gamma_c = 0.4$ and $k = 0.8$.

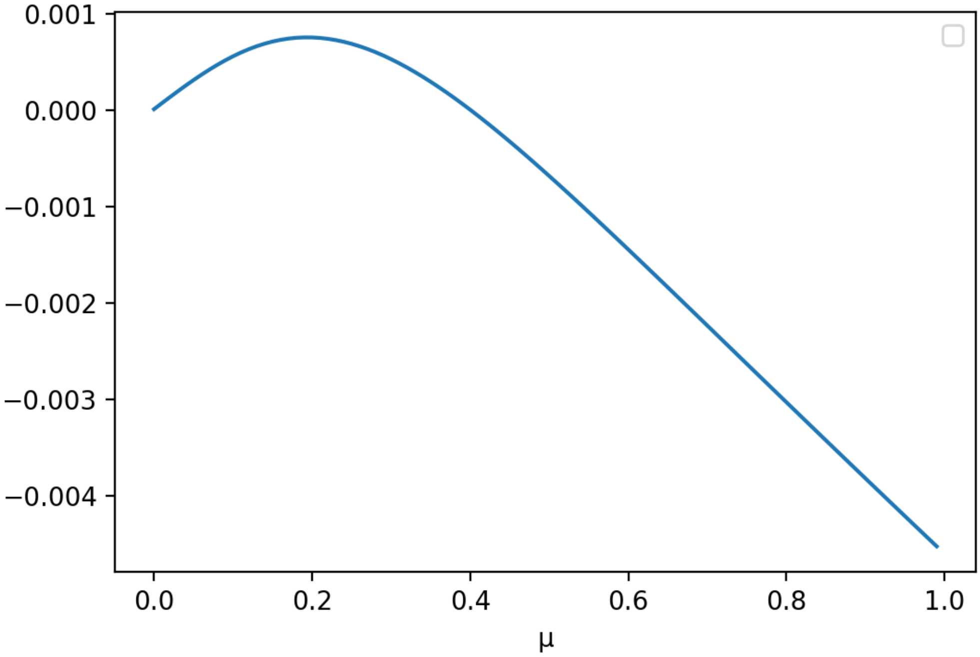

Figure 2.3: Plot of $\langle\hat{\eta}_b\rangle_{ss}$ vs μ [Eq. 2.127] for $\Omega = 0.001, \gamma_c = 0.4$ and $k = 0.8$.

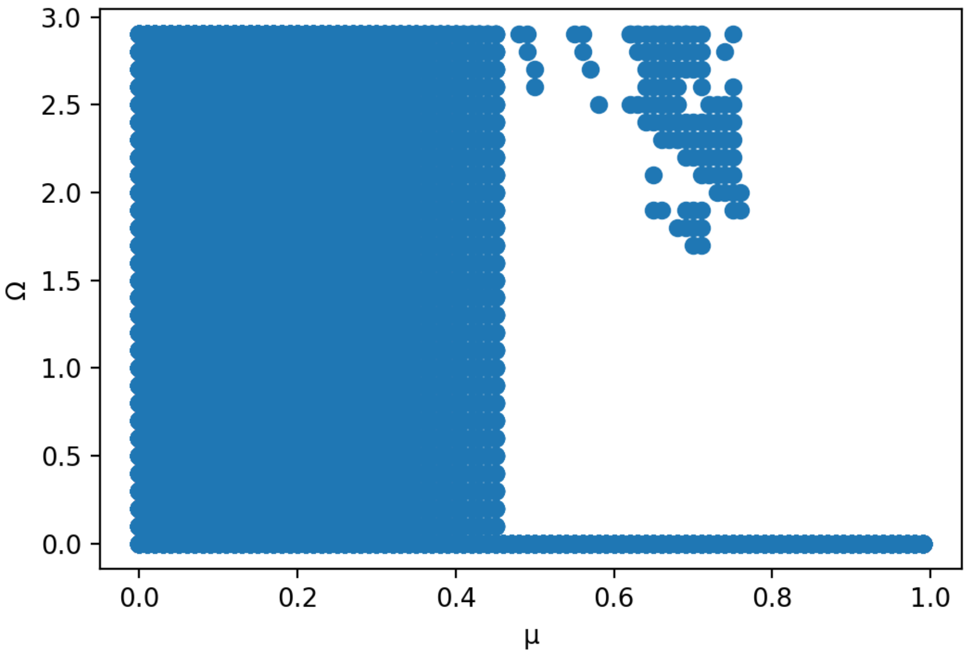

Figure 2.4: Plot of combination of Ω vs μ, 'blue' colored is for allowed values of Ω and μ.

Chapter 3

Photon Number Fluctuations

In this chapter mean photon number for each mode individually and in together will be analyzed. The photon number disperssion is also going to be obtained. Starting from this point the analysis is done for $k_1 = k_2$.

3.1 Mean photon number

Employing Eqs. (2.143) and (2.144) and setting $k_1 = k_2$, the mean photon number for modes a and b can be written at steady state as

$$\begin{aligned}\overline{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle &= \frac{\gamma_c}{(k^2-4\mu^2)^2}\{k^3\langle\hat{\eta}_a\rangle + [4\mu^2 k + \frac{\gamma_c \mu k^2(4\mu^2-3k^2)}{\Omega(k^2-4\mu^2)}]\langle\hat{\eta}_b\rangle\} \\ &+ \frac{2\mu^2}{k^2-4\mu^2}, \qquad (3.1)\\ \overline{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle &= \frac{\gamma_c}{(k^2-2\mu^2)(k^2-4\mu^2)}\langle\hat{\eta}_b\rangle\{k^3(\frac{k^2+2\mu^2}{k^2} - \frac{\mu\gamma_c k}{\Omega(k^2-4\mu^2)}) \\ &+ \frac{8\mu^4 k}{k^2-4\mu^2} + \frac{2\gamma_c\mu^3 k^2(4\mu^2-3k^2)}{\Omega(k^2-4\mu^2)^2}\} + \frac{4\gamma_c\mu^2 k}{(k^2-4\mu^2)^2}\langle\hat{\eta}_a\rangle \\ &+ \frac{2\mu^2}{k^2-4\mu^2}. \qquad (3.2)\end{aligned}$$

In these two equations the last term in the right side is the parametric amplifier contribution to the system and the terms in the wave brakets are due to interaction of the atom with cavity modes. The mean photon number for two-mode cavity light can be evaluated employing the annihilation operator

$$\hat{c} = \hat{a} + \hat{b}, \qquad (3.3)$$

and its adjoint

$$\hat{c}^\dagger = \hat{a}^\dagger + \hat{b}^\dagger, \qquad (3.4)$$

as

$$\begin{aligned}\langle \hat{c}^\dagger \hat{c} \rangle &= \langle (\hat{a}^\dagger + \hat{b}^\dagger)(\hat{a} + \hat{b}) \rangle \\ &= \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle + \langle \hat{b}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle, \qquad (3.5)\end{aligned}$$

so that taking into account Eq. (2.136), one finds

$$\langle \hat{c}^\dagger \hat{c} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle, \qquad (3.6)$$

or

$$\overline{n}_c = \overline{n}_a + \overline{n}_b, \tag{3.7}$$

where $\overline{n}_c$ is mean photon number for two modes, $\overline{n}_a$ and $\overline{n}_b$ are mean photon number for mode a and mode b respectively.
Substitiuting Eqs.(3.1) and (3.2) in to Eq.(3.7), one can show that

$$\overline{n}_c = \frac{\gamma_c k}{(k^2-4\mu^2)^2}\{(k^2+4\mu^2)\langle\hat{\eta}_a\rangle+[4\mu^2+k^2-\frac{4\gamma_c\mu k^3}{\Omega(k^2-4\mu^2)}]\langle\hat{\eta}_b\rangle\}+\frac{4\mu^2}{k^2-4\mu^2}. \tag{3.8}$$

In Eq.(3.8) the last term in the right side is the parameteric amplifier contribution. In the absence of stimulated emission, $\gamma_c = 0$, the mean photon number reduced to

$$\overline{n}_a = \overline{n}_b = \frac{2\mu^2}{k^2-2\mu^2}, \tag{3.9}$$

$$\overline{n}_c = \frac{4\mu^2}{k^2-2\mu^2}. \tag{3.10}$$

In the absence of the parametric amplifier, $\mu = 0$, the mean photon number can be expressed as

$$\overline{n}_a = \overline{n}_b = \frac{\gamma_c\Omega^2}{k(3\Omega^2+\gamma_c^2)}, \tag{3.11}$$

$$\overline{n}_c = \frac{2\gamma_c\Omega^2}{k(3\Omega^2+\gamma_c^2)}. \tag{3.12}$$

As we see Figs. 3.1, 3.2, and 3.3, the mean photon numbers are increased when Ω, γ_c, and μ are increased in the allowed regions.

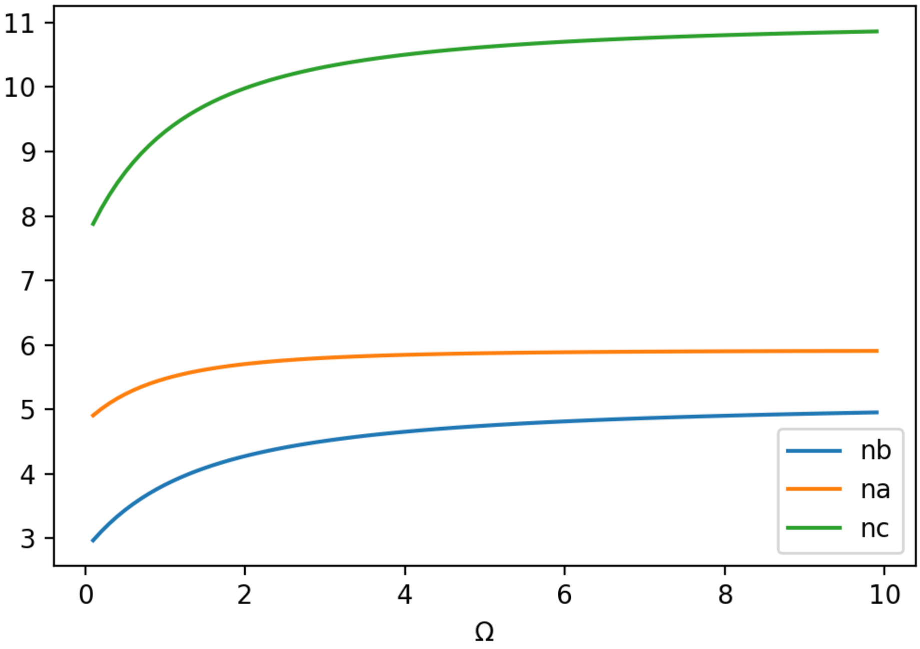

Figure 3.1: Plot of mean photon number vs Ω [Eqs. 3.1, 3.2, and, 3.8] for $\mu = 0.3, \gamma_c = 1.1$ and $k = 0.8$.

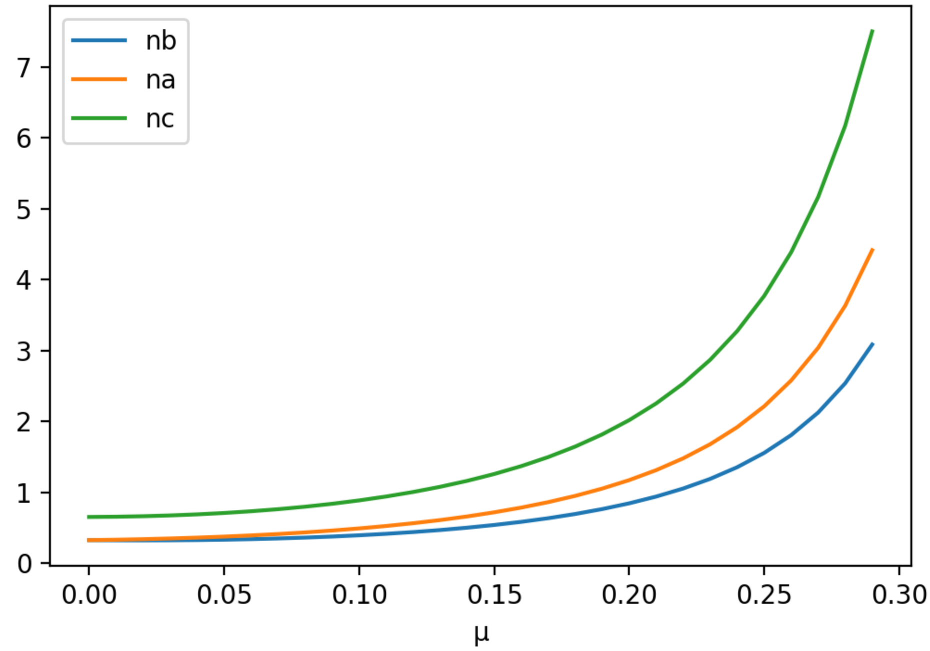

Figure 3.2: Plot of mean photon number vs μ [Eqs. 3.1, 3.2, and, 3.8] for $\Omega = 1, \gamma_c = 1.1$ and $k = 0.8$.

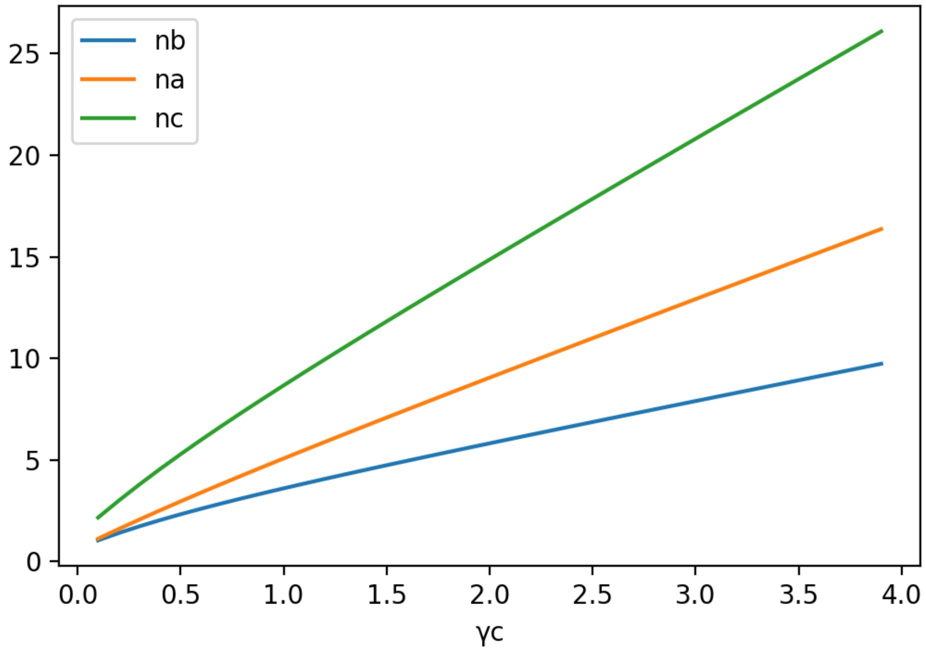

Figure 3.3: Plot of mean photon number vs γ_c [Eqs. 3.1, 3.2, and, 3.8] for $\mu = 0.3, \Omega = 1$ and $k = 0.8$.

3.2 Variance of the photon number

For mode a the variance at steady state can be calculated as

$$\begin{aligned}(\triangle\, n_a)^2 &= \langle \hat{n_a}^2\rangle - \langle \hat{n}_a\rangle^2 \\ &= \langle \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle - \langle \hat{a}^\dagger\hat{a}\rangle^2. \end{aligned} \tag{3.13}$$

Since $\hat{a}$ is gaussian variable with vanishing mean, the variance can be expressed as [14]:

$$\begin{aligned}(\triangle\, n_a)^2 &= \langle \hat{a}^\dagger\hat{a}\rangle^2 + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle + \langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{a}\hat{a}^\dagger\rangle - \langle \hat{a}^\dagger\hat{a}\rangle^2 \\ &= \langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle + \langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{a}\hat{a}^\dagger\rangle. \end{aligned} \tag{3.14}$$

According to Eq.(2.134), the photon number variance for mode a reduces to

$$\begin{aligned}(\triangle\, n_a)^2 &= \langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{a}\hat{a}^\dagger\rangle \\ &= \langle \hat{a}^\dagger\hat{a}\rangle(\langle \hat{a}^\dagger\hat{a}\rangle + 1) \\ &= \overline{n}_a^2 + \overline{n}_a, \end{aligned} \tag{3.15}$$

which indicate that cavity light mode a is in a chaotic state, since the photon number variance is greater than the mean photon number.

For mode b the variance at steady state can be calculated as

$$\begin{aligned}(\triangle\, \hat{n}_b)^2 &= \langle \hat{n}_b^2\rangle - \langle \hat{n}_b\rangle^2 \\ &= \langle \hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\rangle - \langle \hat{b}^\dagger\hat{b}\rangle^2. \end{aligned} \tag{3.16}$$

Since $\hat{b}$ is gaussian variable with vanishing mean, the variance can be expressed as

$$\begin{aligned}(\triangle\, n_b)^2 &= \langle\hat{b}^\dagger\hat{b}\rangle^2 + \langle\hat{b}^{\dagger 2}\rangle\langle\hat{b}^2\rangle + \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{b}\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{b}\rangle^2 \\ &= \langle\hat{b}^{\dagger 2}\rangle\langle\hat{b}^2\rangle + \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{b}\hat{b}^\dagger\rangle.\end{aligned}$$

According to Eq.(2.134), one can obtain

$$\begin{aligned}(\triangle\, n_b)^2 &= \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{b}\hat{b}^\dagger\rangle \\ &= \langle\hat{b}^\dagger\hat{b}\rangle(\langle\hat{b}^\dagger\hat{b}\rangle + 1) \\ &= \overline{n}_b^2 + \overline{n}_b. \qquad (3.17)\end{aligned}$$

As expected mode b is also a chaotic light.
The two-mode photon number variance is calculated by following the same methods as mode a and mode b. The two-mode annihilation operator, $\hat{c}$, is a gaussian variable as it is sum of gaussian variables. The two-mode photon number variance can therefore be expressed as

$$\begin{aligned}(\triangle\, n_c)^2 &= \langle\hat{c}^\dagger\hat{c}\rangle^2 + \langle\hat{c}^{\dagger 2}\rangle\langle\hat{c}^2\rangle + \langle\hat{c}^\dagger\hat{c}\rangle\langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2 \\ &= \langle(\hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{b}^\dagger + \hat{b}^{\dagger 2})\rangle\langle(\hat{a}^2 + 2\hat{a}\hat{b} + \hat{b}^2)\rangle + \langle\hat{c}^\dagger\hat{c}\rangle\langle\hat{c}\hat{c}^\dagger\rangle \\ &= 4\langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{a}\hat{b}\rangle + \langle\hat{c}^\dagger\hat{c}\rangle\langle\hat{c}\hat{c}^\dagger\rangle \\ &= 4\langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{a}\hat{b}\rangle + \overline{n}_c^2 + 2\overline{n}_c. \qquad (3.18)\end{aligned}$$

According to Eq.(2.140), Eq.(2.141), and the fact that $\hat{\sigma}_c = \hat{\sigma}_c^\dagger$, Eq.(3.14) becomes

$$(\triangle\, n_c)^2 = 4\langle\hat{a}\hat{b}\rangle^2 + \overline{n}_c^2 + 2\overline{n}_c. \qquad (3.19)$$

Based on Eq.(2.140) and Eq.(3.6), the previous equation can be written as

$$\begin{aligned}(\triangle\, n_c)^2 &= 4\{\frac{-\gamma_c\mu}{k^2 - 4\mu^2}\langle\hat{\eta}_a\rangle + (\frac{\gamma_c^2 k^3}{2\Omega(k^2 - 4\mu^2)} - \frac{\gamma_c\mu}{k^2 - 4\mu^2})\langle\hat{\eta}_b\rangle \\ &- \frac{\mu}{k}(\overline{n}_c + 1)\}^2 + (\overline{n}_c + 1)^2 - 1. \qquad (3.20)\end{aligned}$$

Chapter 4

Quadrature Fluctuation

In this chapter quadrature variances for single modes as well as two-mode light beam will be calculated. Based on these results the property of the light beams will be explained.
The plus and minus quadratures are the real and complex amplitudes of the field. Based on this they can be expressed mathematically, for mode a, as

$$\begin{aligned} \hat{a}_+ &= \hat{a}^\dagger + \hat{a}, \qquad (4.1) \\ \hat{a}_- &= i(\hat{a}^\dagger - \hat{a}). \qquad (4.2) \end{aligned}$$

Starting from this definitions, the variances can be calculated as

$$\begin{aligned} (\triangle\, a_+)^2 &= \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2 \\ &= \langle (\hat{a}^\dagger + \hat{a})(\hat{a}^\dagger + \hat{a}) \rangle - (\langle \hat{a}^\dagger \rangle + \langle \hat{a} \rangle)^2 \\ &= \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^2 \rangle, \qquad (4.3) \end{aligned}$$

so that using the result in Eq. (2.134), one obtains

$$(\triangle\, a_+)^2 = 2\overline{n}_a + 1. \qquad (4.4)$$

and also for minus quadrature, the variance can be calculated as

$$\begin{aligned} (\triangle\, a_-)^2 &= \langle \hat{a}_-^2 \rangle - \langle \hat{a}_- \rangle^2 \\ &= \langle i^2 (\hat{a}^\dagger - \hat{a})(\hat{a}^\dagger - \hat{a}) \rangle - (i^2 \langle \hat{a}^\dagger \rangle - \langle \hat{a} \rangle)^2 \\ &= -(\langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^2 \rangle) \\ &= 2\overline{n}_a + 1. \qquad (4.5) \end{aligned}$$

The commutation relation between the plus and minus quadratures for single mode system can be calculated as

$$\begin{aligned} [\hat{a}_+, \hat{a}_-] &= [\hat{a}^\dagger + \hat{a}, i(\hat{a}^\dagger - \hat{a})] \\ &= i[\hat{a}^\dagger, -\hat{a}] + i[\hat{a}, \hat{a}^\dagger] \\ &= 2i[\hat{a}, \hat{a}^\dagger] = 2i. \qquad (4.6) \end{aligned}$$

As the commutation relation is different from zero, there is uncertainty relation between the plus and minus quadratures. The uncertainty relation can be expressed as

$$\triangle\, a_+ \;\triangle\, a_- \geq \frac{1}{2} |\langle [\hat{a}_+, \hat{a}_-] \rangle| \qquad (4.7)$$

$$\triangle\, a_+ \triangle\, a_- \geq \frac{1}{2}|\langle(2i)\rangle|$$
$$\triangle\, a_+ \triangle\, a_- \geq \frac{1}{2}|2i|$$
$$\triangle\, a_+ \triangle\, a_- \geq 1. \tag{4.8}$$

For the system under consideration the product of the standard deviations gives

$$\triangle\, a_+ \triangle\, a_- = 2\overline{n}_a + 1. \tag{4.9}$$

Since the field is chaotic, the uncertainty limit is greater than the minimum relation.
In a similar procedure, the quadrature variances for mode b are expressible as

$$(\triangle\, b_+)^2 = 2\overline{n}_b + 1, \tag{4.10}$$

$$(\triangle\, b_-)^2 = 2\overline{n}_b + 1. \tag{4.11}$$

We realize that the product of the standard deviations is

$$\triangle\, b_+ \triangle\, b_- = 2\overline{n}_b + 1. \tag{4.12}$$

We next proceed to evaluate the two-mode quadrature variance of cavity light. The variance of quadratures for the two-mode light can be calculated using the two-mode plus and minus quadrature operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{4.13}$$
$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{4.14}$$

The plus quadrature variance is defined as

$$\begin{aligned}(\triangle\, c_+)^2 &= \langle\hat{c}_+^2\rangle - \langle\hat{c}_+\rangle^2\\ &= \langle(\hat{c}^\dagger + \hat{c})(\hat{c}^\dagger + \hat{c})\rangle - (\langle\hat{c}^\dagger\rangle + \langle\hat{c}\rangle)^2\\ &= \langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^2\rangle\\ &= 2\langle\hat{a}\hat{b}\rangle + 2\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + 2\overline{n}_c + 2.\end{aligned} \tag{4.15}$$

In view of Eq.(2.144), and for $k_1 = k_2$, the two-mode plus quadrature variance is expressible as

$$(\triangle\, c_+)^2 = -\frac{4\gamma_c\mu}{k^2 - 4\mu^2}(\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle) + \frac{2\gamma_c^2 k^3}{\Omega(k^2 - 4\mu^2)^2}\langle\hat{\eta}_b\rangle - \frac{4\mu}{k} + 2(1 - \frac{2\mu}{k})\overline{n}_c + 2. \tag{4.16}$$

Finally substituting Eq.(3.6) in Eq.(4.16), we find

$$(\triangle\, c_+)^2 = \frac{2\gamma_c k}{(k + 2\mu)^2}[\langle\hat{\eta}_a\rangle + (1 + \frac{\gamma_c k^2}{\Omega(k^2 - 4\mu^2)})\langle\hat{\eta}_b\rangle] - \frac{4\mu}{k + 2\mu} + 2. \tag{4.17}$$

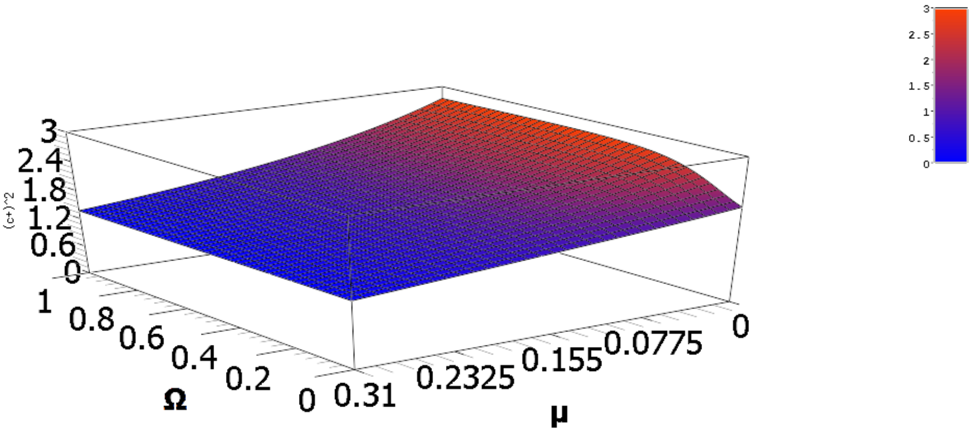

Figure 4.1: 3D Plot of the variance of plus-quadrature [Eq. 4.17] for $\gamma_c = 0.4$ and $k = 0.8$.

Employing the two-mode minus quadrature operator and following a similar method, the corresponding quadrature variance can be expressed as

$$(\triangle\, \hat{c}_{-})^2 = \frac{2\gamma_c k}{(k-2\mu)^2}[\langle\hat{\eta}_a\rangle + (1 - \frac{\gamma_c k^2}{\Omega(k^2-4\mu^2)})\langle\hat{\eta}_b\rangle] + \frac{4\mu}{k-2\mu} + 2. \tag{4.18}$$

In view of Figs. 4.1 and 4.2, the value of the plus and minus quadratures are around 2 for small values of Ω and μ. Consequently the two-mode light can be in squeezed state for those values of Ω and μ combinations. The computer simulation which shows these combinations, in which squeezed light is obtained, is given in Appendix B and the result is shown graphically in Fig. 4.3.

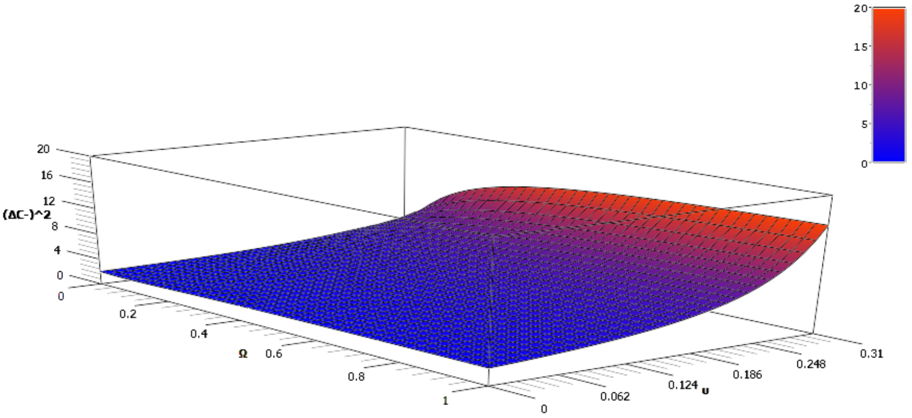

Figure 4.2: 3D Plot of the variance of minus-quadrature [Eq. 4.18] for $\gamma_c = 0.4$ and $k = 0.8$.

According to Fig. 4.3, for small values of Ω and μ the two-mode light is in squeezed state and also for a few other values. When we see Fig. 4.4, we get the squeezed quadrature is the minus quadrature for $\mu < 0.00025$ and the plus quadrature for $\mu > 0.00025$. This shows the squeezed quadrature vary when the given parameters varies. The squeezing increases when Ω and γ_c approaches to zero for small values of squeezed parameter values as shown in Fig. 4.6 and 4.7. When μ approaches to the largest value, the squeezing also increases which is shown in Fig. 4.5.

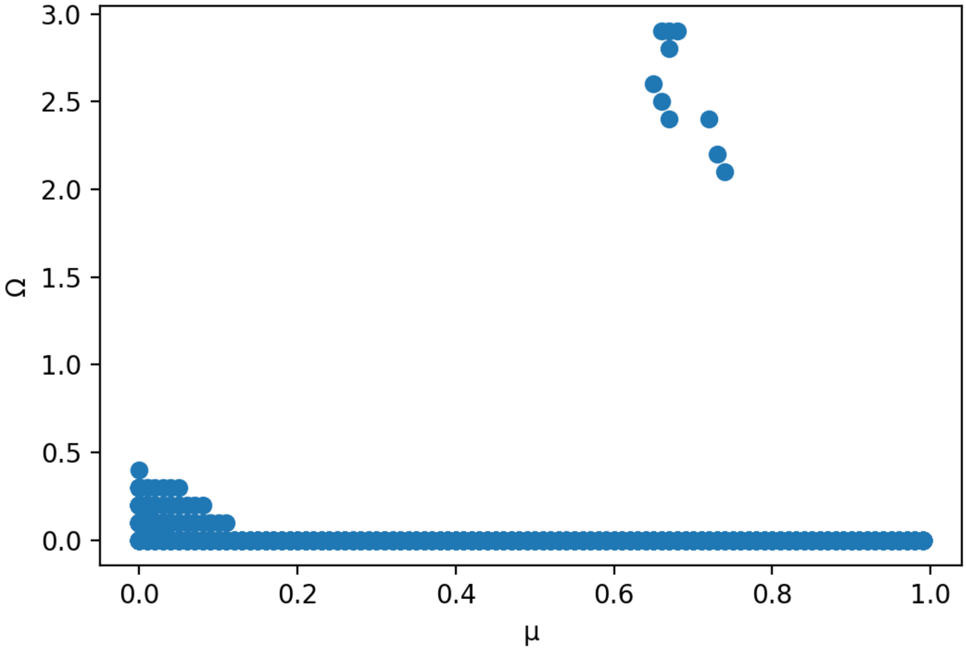

Figure 4.3: Plot of combination of Ω vs μ,(blue colored) which gives squeezed state

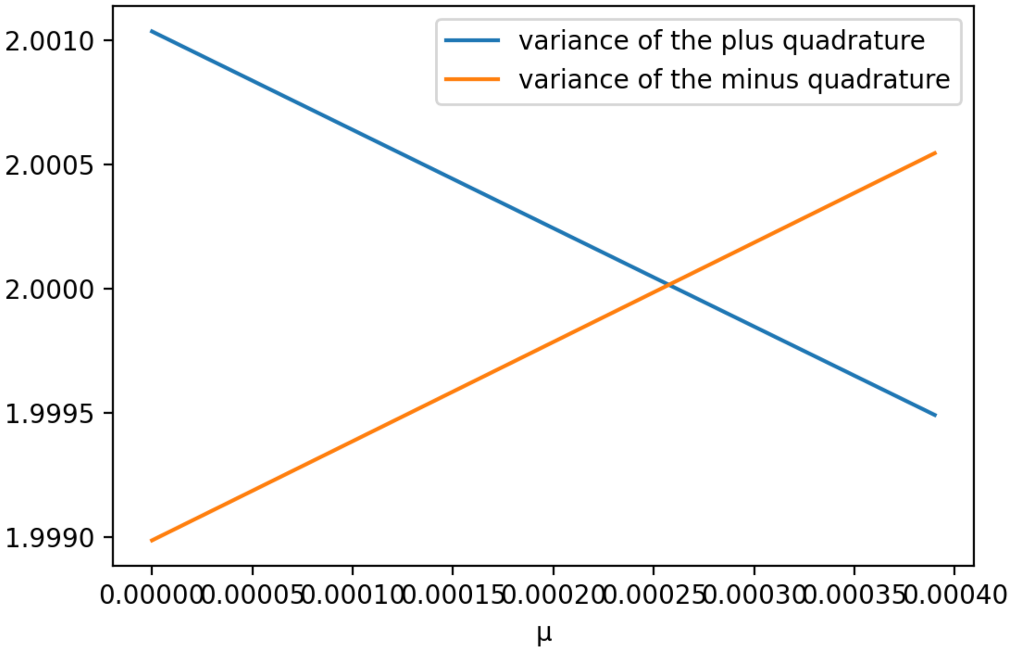

Figure 4.4: Plot of variance of minus quadrature and plus quadrature vs μ [Eqs. 4.17 and 4.18] for $\Omega = 0.001, \gamma_c = 0.4$ and $k = 0.8$.

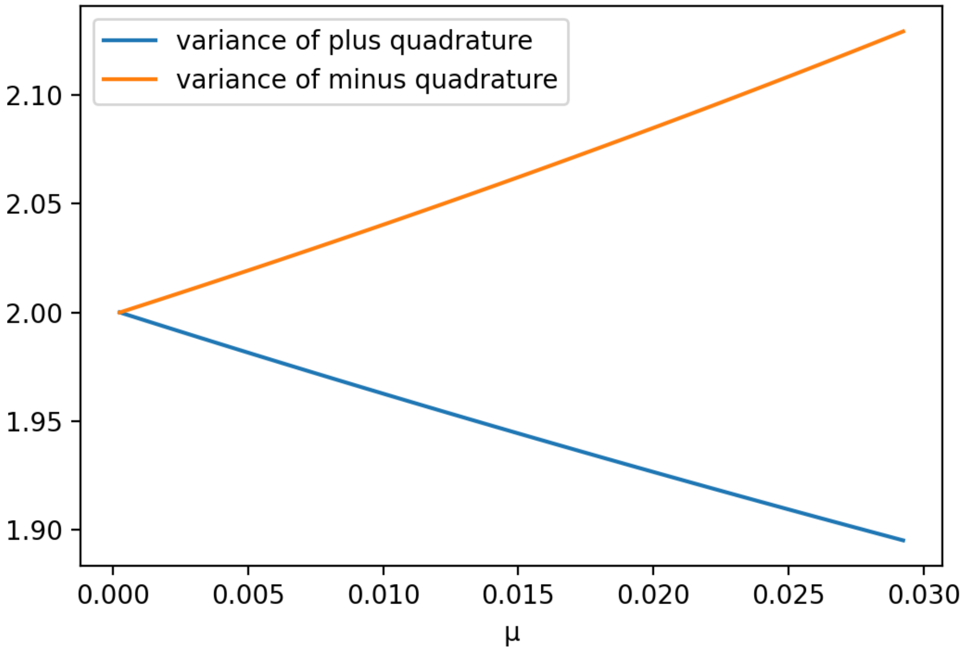

Figure 4.5: Plot of variance of minus quadrature and plus quadrature vs μ [Eqs. 4.17 and 4.18] for $\Omega = 0.001, \gamma_c = 0.4$ and $k = 0.8$.

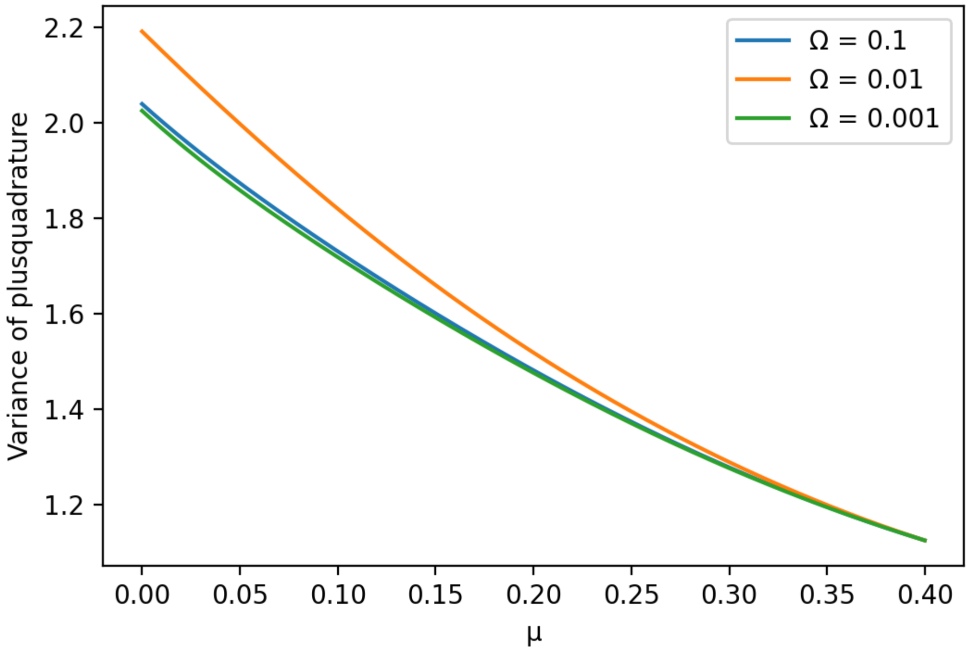

Figure 4.6: Plot of variance of plus quadrature vs μ [Eq. 4.17] for different values of Ω and $\gamma_c = 0.4$ and $k = 0.8$.

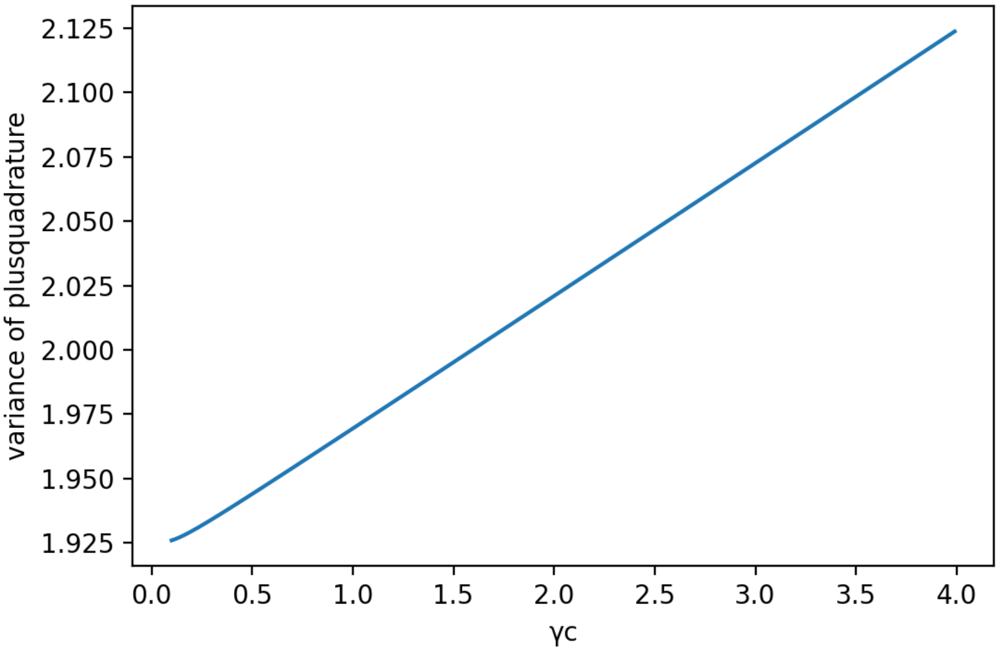

Figure 4.7: Plot of variance of plus quadrature vs γ_c [Eq. 4.17] for $\Omega = 0.01$ and $\mu = 0.02$ and $k = 0.8$.

4.1 Quadrature squeezing

The quadrature squeezing S relative to the vacuum quadrature variance can be expressed as [22]:

$$S_{\pm} = \frac{(\triangle\, \hat{c}_{\pm})_v^2 - (\triangle\, \hat{c}_{\pm})^2}{(\triangle\, \hat{c}_{\pm})_v^2}, \tag{4.19}$$

where $(\triangle\, \hat{c}_{\pm})_v^2$ is the two-mode quadrature variance of the vacuum state.
The vacuum state variances can be obtained by excluding the effect of the parametric amplifier, $\mu = 0$, and neglecting the coupling constat $\gamma_c = 0$ and its value is 2. In the maximum squeezed state the plus quadrature value is 0.8.
We notice from Fig 4.8 that a maximum squeezing of 60% occurs for $\mu = 0.761, \Omega = 1.87, k = 0.9, \gamma_c = 1.1$. The plus quadrature can be squeezed more and more if k approaches to 1, but it does not pass 60%.
The quadrature squeezing can be calculated as

$$(\triangle\, \hat{c}_{+})_v^2 = (\triangle\, \hat{c}_{-})_v^2 = 2 \tag{4.20}$$

$$\begin{aligned} S_{+} &= \frac{(\triangle\, \hat{c}_{+})_v^2 - (\triangle\, \hat{c}_{+})^2}{(\triangle\, \hat{c}_{+})_v^2} \\ &= \frac{2 - 0.8}{2} \\ S_{+} &= 0.6 = 60\%. \end{aligned} \tag{4.21}$$

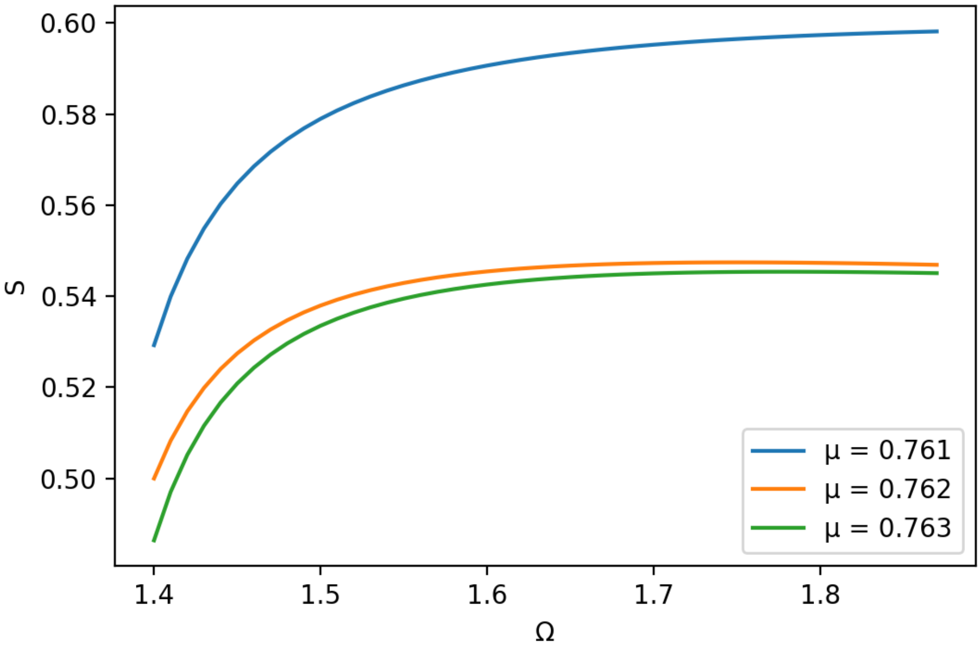

Figure 4.8: Plot of variance of plus quadrature vs Ω [Eq. 4.19] for different values of μ, $\gamma_c = 1.1$ and $k = 0.9$.

Chapter 5

Photon Entanglement

In this cahpter by using the previous results, we will measure the entanglement property of the two-mode light beam and its strength. Moreover we will see how the two-mode light violates the classical inequalties. Finally we will measure the entanglement by using the logarithmic negativity.
There are some checking criteria, which can be used independently, of entanglement between states.

5.1 Violation of Cauchy-Schwarz inequality

Cauchy-Schwarz inequality is classical inequality and it is violated for nonclassical effects, such as squeezing and photon-antibunching. For two-mode light, the quantum analogue of Cauchy–Schwarz inequality is given as [14]:

$$[g_{ab}^{(2)}]^2 \leq g_{aa}^{(2)} g_{bb}^{(2)} \tag{5.1}$$

where ${g_{aa}}^{(2)}, {g_{ab}}^{(2)}, {g_{bb}}^{(2)}$ is the second-order correlation of mode a, mode a and mode b, and mode b. These can be defined as

$$g_{ab}^{(2)} = \frac{\langle \hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}, g_{aa}^{(2)} = \frac{\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}, g_{bb}^{(2)} = \frac{\langle \hat{b}^{\dagger 2} \hat{b}^2 \rangle}{\langle \hat{b}^\dagger \hat{b} \rangle^2}. \tag{5.2}$$

For gaussian variables $\alpha_1, \alpha_2, \alpha_3$, and α_4, the expectation value of their product is expressed as [13]:

$$\langle \alpha_1 \alpha_2 \alpha_3 \alpha_4 \rangle = \langle \alpha_1 \alpha_2 \rangle \langle \alpha_3 \alpha_4 \rangle + \langle \alpha_1 \alpha_3 \rangle \langle \alpha_2 \alpha_4 \rangle + \langle \alpha_1 \alpha_4 \rangle \langle \alpha_2 \alpha_3 \rangle. \tag{5.3}$$

By using Eq.(5.3) in to Eqs.(5.2) and (5.1), Cauchy-Schwarz inequality becomes

$$|\langle \hat{a} \hat{b} \rangle|^2 \leq \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle. \tag{5.4}$$

For the given system

$$|\langle \hat{a} \hat{b} \rangle|^2 = \langle \hat{a} \hat{b} \rangle^2 \leq \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle \tag{5.5}$$

And this relation violated if

$$\langle \hat{a} \hat{b} \rangle^2 > \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle. \tag{5.6}$$

For all values of blue shaded parameters in Fig. 4.3, the Cauchy-Schwarz inequality is violated. Therefore those given values satisfy the neccessary conditon for entanglement but not sufficient. As an example

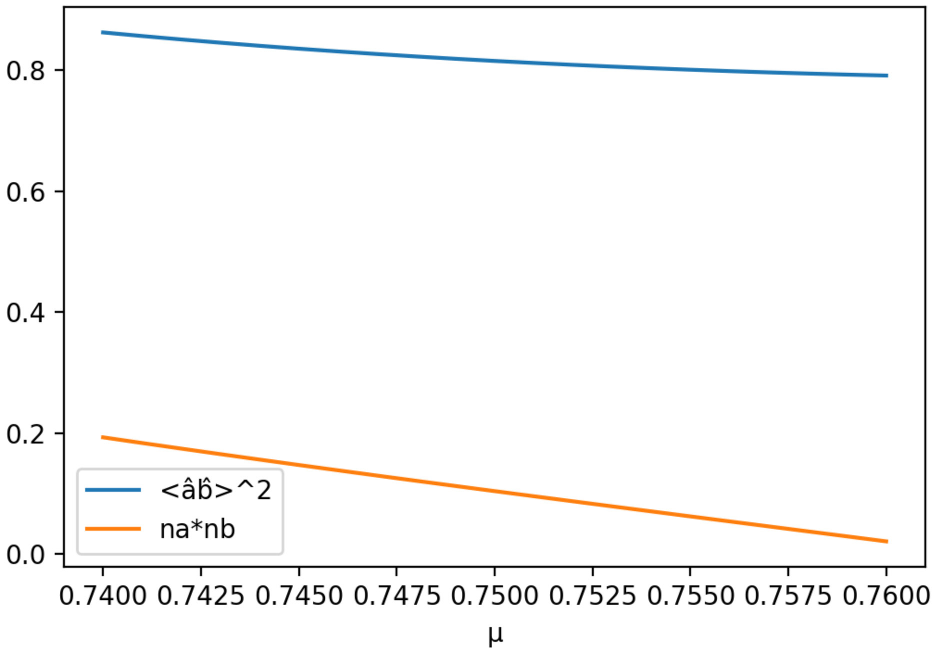

Figure 5.1: Plot of $\langle \hat{a}\hat{b}\rangle_{ss}^2$ vs μ and $\langle \hat{a}^\dagger \hat{a}\rangle_{ss}\langle \hat{b}^\dagger \hat{b}\rangle_{ss}$ vs μ [Eq, 5.6] for $\Omega = 1.87, \gamma_c = 1.1$ and $k = 0.9$.

5.2 Duan et. al. criteria

Duan et. al. criteria is sufficient and neccessary criteria for the entanglement of the two-mode light and can be stated as [23]:

$$(\triangle\ x)^2 + (\triangle\ y)^2 < 4, \tag{5.7}$$

$$x = \hat{a}_+ + \hat{b}_+ = \hat{c}_+, \tag{5.8}$$

$$y = \hat{a}_- - \hat{b}_-. \tag{5.9}$$

$$\begin{aligned}(\triangle\ x)^2 &= \langle x^2\rangle - \langle x\rangle^2 \\ &= \langle (\hat{a}_+ + \hat{b}_+)^2\rangle - \langle \hat{a}_+ + \hat{b}_+\rangle^2\end{aligned}$$

Substituting $\hat{a}_+ = \hat{a}^\dagger + \hat{a}$ and $\hat{b}_+ = \hat{b}^\dagger + \hat{b}$ in the above equation gives

$$(\triangle\ x)^2 = \langle (\hat{a}^\dagger + \hat{a} + \hat{b}^\dagger + \hat{b})(\hat{a}^\dagger + \hat{a} + \hat{b}^\dagger + \hat{b})\rangle - (\langle \hat{a}^\dagger + \hat{a} + \hat{b}^\dagger + \hat{b}\rangle)^2$$

From Eq.(2.134) to Eq.(2.136) and after a simple calculation

$$\begin{aligned}(\triangle\ x)^2 &= 2(\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}^\dagger \hat{b}^\dagger\rangle + \langle \hat{a}\hat{b}\rangle + \langle \hat{b}^\dagger \hat{b}\rangle + 1) \\ &= 2\overline{n_c} + 2\langle \hat{a}^\dagger \hat{b}^\dagger\rangle + 2\langle \hat{a}\hat{b}\rangle + 1.\end{aligned} \tag{5.10}$$

Comparing Eq.(4.16), Eq.(5.9), and Eq.(5.10)

$$(\triangle\ x)^2 = (\triangle\ \hat{c}_+)^2 = (\triangle\ y)^2. \tag{5.11}$$

Therefore the entanglement criteria becomes

$$(\triangle\ \hat{c}_+)^2 < 2. \tag{5.12}$$

According to Fig. 5.2, $2(\triangle\ \hat{c}_{+})^2$ is less than 4. Consequently if the two-mode light is squeezed then it is entangled. To show this, for $\Omega = (1.82 - 1.87), \gamma_c = 1.1$ and $k = 0.9$. $\mu = 0.761$ the result is shown in Fig. 5.2.

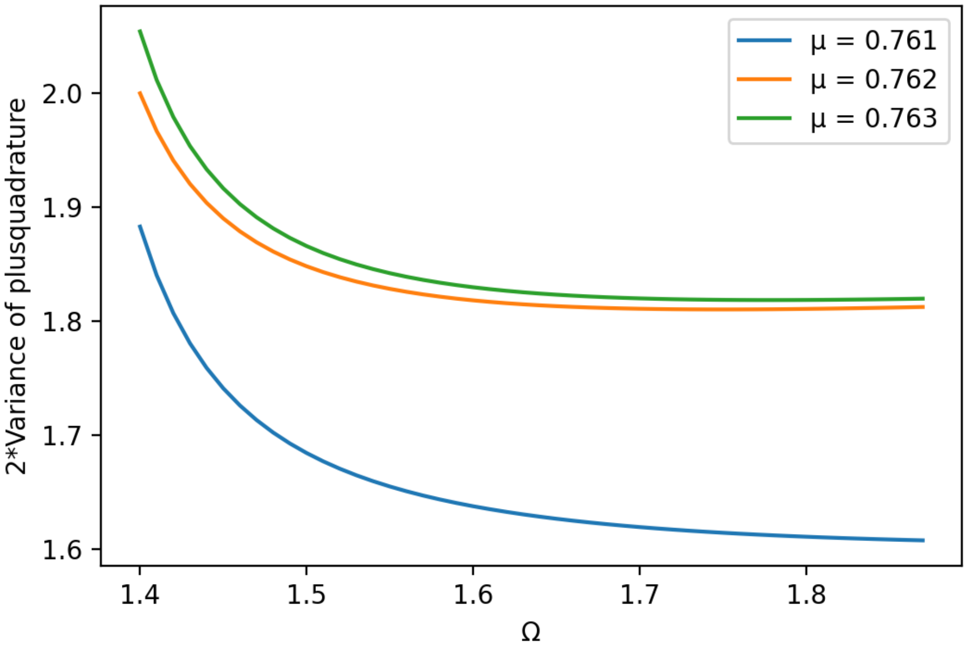

Figure 5.2: Plot of $(\triangle\ x)^2 + (\triangle\ y)^2$ vs Ω [Eq. 5.7] for $\mu = 0.761$, $\gamma_c = 1.1$ and $k = 0.9$.

5.3 Logarithmic negativity criteria

Logarithmic negativity criteria is another widely used entanglement measure, which is used for a two-mode continues variables based on the negativity of the partial transpose.

The logarithmic negativity for a two-mode cavity is defined as [24]:

$$E_N = max[0, -log_2 Vs] \tag{5.13}$$

In which V_s represents the smallest eigenvalue of the symplestic matrix. According to this criterion, the entanglement is achieved when E_N is positive within the region of the smallest eigenvalue of the covariance matrix $V_s < 1$.

The smallest possible eigenvalue of the covariance matrix is defined as

$$V_s = \{\frac{\sigma + (\sigma^2 - 4det\Gamma)^{\frac{1}{2}}}{2}\}^{\frac{1}{2}} \tag{5.14}$$

where the invariant matrix σ and covariant matrix Γ are defined as

$$\sigma = detA_1 + detA_2 - 2detA_{12} \tag{5.15}$$

$$\Gamma = \begin{pmatrix} A_1 & A_{12} \\ A_{12}^T & A_2 \end{pmatrix} \tag{5.16}$$

where A_1 and A_2 are the covariance matrices describing each mode separately while A_{12} is the inter-modal correlation that leads to the observed non-classical features detected by this criteria.

The elements of the matrix in Eq.(5.16) are given by

$$\Gamma_{ij} = \frac{1}{2}[\langle \hat{X}_i \hat{X}_j \rangle + \langle \hat{X}_j \hat{X}_i \rangle] - \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle \tag{5.17}$$

where $i, j = 1, 2, 3, 4$ and $\hat{X}_1 = \hat{a}_+, \hat{X}_2 = \hat{a}_-, \hat{X}_3 = \hat{b}_+, \hat{X}_4 = \hat{b}_-$ are the quadrature operators corresponding to each modes of the cavity light.

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{pmatrix} \tag{5.18}$$

$$A_1 = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} 2n_a + 1 & 0 \\ 0 & 2n_a + 1 \end{pmatrix} \tag{5.19}$$

$$A_{12} = \begin{pmatrix} \Gamma_{13} & \Gamma_{14} \\ \Gamma_{23} & \Gamma_{24} \end{pmatrix} = \begin{pmatrix} 2\langle \hat{a}\hat{b} \rangle & 0 \\ 0 & -2\langle \hat{a}\hat{b} \rangle \end{pmatrix} \tag{5.20}$$

$$A_{12}^T = \begin{pmatrix} \Gamma_{31} & \Gamma_{32} \\ \Gamma_{41} & \Gamma_{42} \end{pmatrix} = \begin{pmatrix} 2\langle \hat{a}\hat{b} \rangle & 0 \\ 0 & -2\langle \hat{a}\hat{b} \rangle \end{pmatrix} \tag{5.21}$$

$$A_2 = \begin{pmatrix} \Gamma_{33} & \Gamma_{34} \\ \Gamma_{43} & \Gamma_{44} \end{pmatrix} = \begin{pmatrix} 2n_b + 1 & 0 \\ 0 & 2n_b + 1 \end{pmatrix} \tag{5.22}$$

from which follows

$$det\Gamma = [(2n_a + 1)(2n_b + 1) - 4\langle\hat{a}\hat{b}\rangle^2]^2 \tag{5.23}$$

$$\sigma^2 = [(2n_a + 1)^2 + (2n_b + 1)^2 + 8\langle\hat{a}\hat{b}\rangle^2]^2 \tag{5.24}$$

The dependency of the covariant matrix on μ, Ω, γ_c is shown in Figs. 5.3, 5.4, and 5.5. The computer program which generates parameter values for $Vs < 1$ is expressed in Appendix C. The logarithmic negativity predicts as the other criterias predict eventhough the strength is different. According to logarithmic negativity criteria the maximum entangled state is 16%.

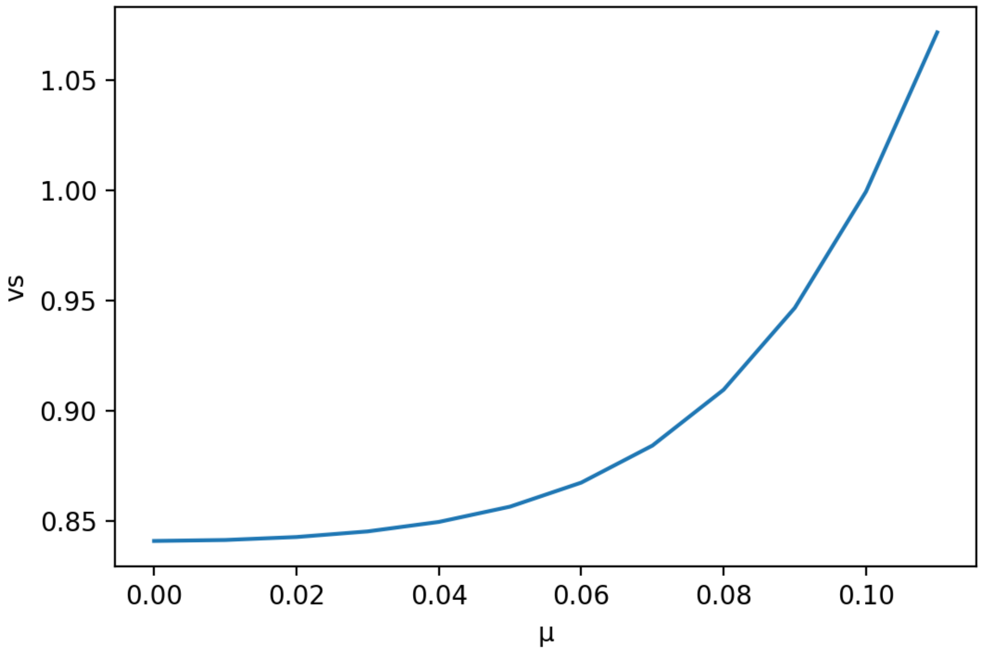

Figure 5.3: Plot of V_s vs μ [Eq. 5.14] for $\Omega = 0.001$, $\gamma_c = 0.4$ and $k = 0.8$.

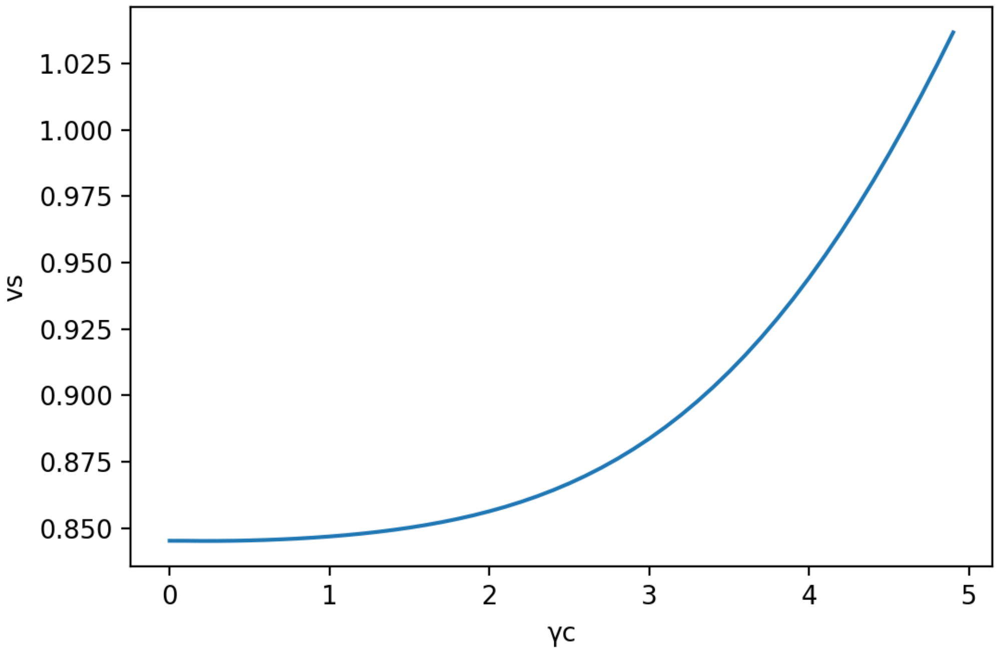

Figure 5.4: Plot of V_s vs γ_c [Eq. 5.14] for $\Omega = 0.001$, $\mu = 0.03$ and $k = 0.8$.

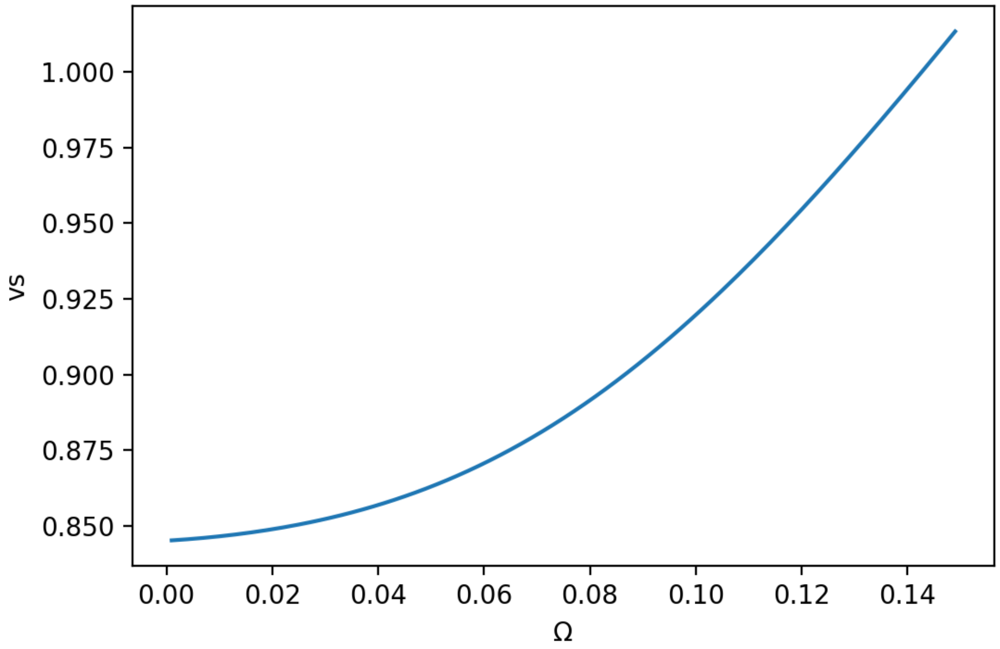

Figure 5.5: Plot of V_s vs Ω [Eq. 5.14] for $\mu = 0.03, \gamma_c = 0.4$ and $k = 0.8$.

Chapter 6

Conclusion

We have studied a light produced by a non-degenerate three-level atom in a cavity containing a parametric amplifier and driven by a strong coherent light coupled to a two-mode vacuum reservoir. The time evolution of the atomic and cavity mode operators is calculated by using the time evolution of the reduced density operator. The steady state solution of the operators are obtained. But this method is helpfull only for some range of parameter values. Forexample $\mu = [0, 0.31]$ and $\Omega = (0, 1]$. Each mode individually is in a chaotic state in any parameter but the two-mode in together is in squeezed state for some ranges of values, from these $\gamma_c = [0, 4], k = 0.8, \Omega = (0, 0.03]$ and $\mu = [0, 0.03]$. The squeezing in the minus quadrature is happened with a maximum quadrature squeezing of 0.19% relative to the vacuum variance. Moreover The squeezing in the plus quadrature is happened with a maximum quadrature squeezing of 60% relative to the vacuum variance. The maximum squeezing of the plus quadrature for the system occurs for $\Omega = 1.87, \mu = 0.761, k = 0.9$ and $\gamma_c = 1.1$. We can conclude from this, two modes in a chaotic state can be in squeezed state in together. Even if the 2-mode light is in entangled state for different values, the maximum entanglement entropy is 16%. A better result will be obtained, if the time evolution of the operators are calculatd analytically or by numerical analysis rather than Large-time approximation.

Bibliography

[1]. F.Diedrich, H.Walther, Nonclassical radiation of a single stored ion, Phys.Rev.lett.58,203(1987)

[2]. F. Kassahun,*Refined Quantum Analysis of Light*,(Create Space Independent Publishing Platform)(2014).

[3]. D.F.Walls, G.J.Milburn, *Quantum optics*, Springer, Berlin, 2008.

[4]. M.Hillery, M.Suhail, Entanglement conditions for two-mode states: Applications, Phys.Rev.lett **A74**, 032333(2006).

[5]. www.wikipedia.com-Quantum-entanglement.

[6]. T. Jennewein, C. Simon, G. Weihs, H. Weinfurter, A. Zeilinger, Quantum cryptography with entangled photons, Phys. Rev.Lett. **84**, 4729 (2000).

[7]. C.H. Bennett and D.P. DiVincenzo, Quantum information and computation ,Nature **404**, 247 (2000).

[8]. S. Barzanjeh, S. Pirandola, and C. Weedbrook, continuos variable dense coding by optomechanical cavities, Phys. Rev. **A 88**, 042331(2013).

[9]. N. Ganguly, S. Adhikari, A.S. Majumdar, and J. Chatterjee, Entanglement witness for Quantum Teleportation, Phys. Rev. Lett **107**, 270501(2011).

[10]. C. Branciard, N. Brunner, H. Buhrman, R. Cleve, N. Gisin, S. Portmann, D. Rosset, and M.Szegedy, Entangled states cannot be simulated in generalized Bell experiment, Phys. Rev. Lett. **109**, 100401 (2012).

[11]. T. Kitagawa, A. Aspect, M. Greiner, and E. Demler, phase-sensitive measurements for order parameters, Phys. Rev. Lett.**106**, 115302(2011).

[12]. S. Koike, H. Takahashi, H. Yonezawa, N. Takei, S.L. Braunstein, T. Aoki, A. Furusawa, Demonstration of quantum teleclonning of optical coherent states, Phys. Rev.lett. **96**, 060504 (2006).

[13]. C.Gashu, E.Mosisa, T.Abebe, Entanglement Quantification of Correlated photons Generated by Three-Level Laser with Parametric Amplifier and Coupled to a Two-Mode Vacuum Reservoir, Hindawi Advances in Mathematical physics, **Volume 2020**,ID 9027480.

[14]. E.Alebachew, VIOLATION OF CLASSICAL INEQUALITIES AND EPR CORRELATIONS IN A TWO-MODE THREE-LEVEL ATOMIC SYSTEM, International Journal of Quantum Information **vol.6**, No.4(2008)885-898.

[15]. L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Inseparability Criterion for Continuous Variable Systems, Phys. Rev. Lett. **84**, 2722 (2000).

[16]. M. Hillery and M.S. Zubairy, Entanglement conditions for two-mode states, Phys. Rev. Lett. **96**, 050503 (2006).

[17]. H.Nha, M.S.Zubairy, Uncertainity relation and Inseparability criterion, Phys.Rev.Lett.**101**. 130402(2008).

[18]. E.Alebachew, K.Fesseha, Interaction of a two-level atom with squeezed

light, quant-ph/0610123 **v2**, (2006)42.50 Dv.

[19]. T.Abebe, THE QUANTUM ANALYSIS OF A NONDEGENERATE THREE-LEVEL LASER WITH SPONTANEOUS EMISSION AND NOISELESS VACUUM RESERVOIR, Bulg.J.phys.45(2018).

[20]. B.Arfken, H.J.Weber, F.E.Harris, *Mathematical Methods of physicists*, Academic press, Oxford, 2013.

[21]. E.Alebachew, K.Fesseha, Coherently driven three-level laser with parametric amplifier, quant-ph/0506178 **v2**, (2005)42.50 Dv.

[22]. M.Al-Amri, S.Qamar, Entanglement in correlated spontaneous emission lasers, Springer Science+Business Media, LLC2009

[23]. K.Dechoum, P.D.Drummond, S.Chaturvedi, M.D.Redi, Critical fluctuations and entanglement in the nondegenerate parametric oscillator ,phys.rev.**A70**, 053807(2004).

[24]. S. Tesfa, Role of phase fluctuation and dephasing in the enhancing continuous variable entanglement of a two-photon coherent beat laser, Phys. Scr. **84**, 045403 (2011).

Appendix A

A python computer program to generate allowed values of Ω, μ, γ_c, and k according to the corresponding system is given as

```
import numpy as np

$i = np.arange(0.1, 1, 0.1)$

for k in i:

    $b = 4/k$
    $j = np.arange(0.1, b, 1)$
    for $\gamma_c$ in j:
        $x = np.arange(0,$any number$, 0.01)$
        for $\Omega$ in x:
            $n = np.arange(0, 1, 0.001)$
            for $\mu$ in n:
                expression of $\langle\hat{\eta}_a\rangle$
                expression of $\langle\hat{\eta}_b\rangle$
                expression of $\langle\hat{\eta}_c\rangle$
                if $\langle\hat{\eta}_a\rangle < 0$ :
                    continue
                if $\langle\hat{\eta}_b\rangle < 0$ :
                    continue
                if $\langle\hat{\eta}_c\rangle < 0$ :
                    continue
                expression of $\langle\hat{a}^\dagger\hat{a}\rangle$
                expression of $\langle\hat{b}^\dagger\hat{b}\rangle$
                if $\langle\hat{a}^\dagger\hat{a}\rangle < 0$:
                    continue
                if $\langle\hat{b}^\dagger\hat{b}\rangle < 0$ :
                    continue
                print $(\Omega, \mu, \gamma_c, k)$
```

Appendix B

A python computer program which generates the combination of $(\triangle\ c_+)^2, (\triangle\ c_-)^2, \Omega, \mu, \gamma_c$, and k for the uncertainity < 4.01 is expressed as

```
import numpy as np
$i = np.arange(0.1, 1, 0.1)$
for k in i:
    $b = 4/k$
    $j = np.arange(0.1, b, 1)$
    for $\gamma_c$ in j:
        $x = np.arange(0,$any number$, 0.01)$
        for $\Omega$ in x:
            $n = np.arange(0, 1, 0.001)$
            for $\mu$ in n:
                expression of $\langle\hat{\eta}_a\rangle$
                expression of $\langle\hat{\eta}_b\rangle$
                expression of $\langle\hat{\eta}_c\rangle$
                if $\langle\hat{\eta}_a\rangle < 0 :$
                    continue
                if $\langle\hat{\eta}_b\rangle < 0 :$
                    continue
                if $\langle\hat{\eta}_c\rangle < 0 :$
                    continue
                expression of $\langle\hat{a}^\dagger\hat{a}\rangle$
                expression of $\langle\hat{b}^\dagger\hat{b}\rangle$
                if $\langle\hat{a}^\dagger\hat{a}\rangle < 0$:
                    continue
                if $\langle\hat{b}^\dagger\hat{b}\rangle < 0 :$
                    continue
                expression of $(\triangle\ c_+)^2$
                expression of $(\triangle\ c_-)^2$
                expression of $(\triangle\ c_+)^2 * (\triangle\ c_-)^2$
                if expression of $(\triangle\ c_+)^2 * (\triangle\ c_-)^2 > 4.01$
                    continue
                print $((\triangle\ c_+)^2, (\triangle\ c_-)^2, \Omega, \mu, \gamma_c, k)$
```

Appendix C A python computer program which generates the values of parameters for entangled states, $Vs < 1$.

```
import numpy as np
$i = np.arange(0.1, 1, 0.1)$
for k in i:
    $b = 4/k$
    $j = np.arange(0.1, b, 1)$
    for $\gamma_c$ in j:
        $x = np.arange(0,$any number$, 0.01)$
        for $\Omega$ in x:
            $n = np.arange(0, 1, 0.001)$
            for $\mu$ in n:
                expression of $\langle \hat{\eta}_a \rangle$
                expression of $\langle \hat{\eta}_b \rangle$
                expression of $\langle \hat{\eta}_c \rangle$
                if $\langle \hat{\eta}_a \rangle < 0 :$
                    continue
                if $\langle \hat{\eta}_b \rangle < 0 :$
                    continue
                if $\langle \hat{\eta}_c \rangle < 0 :$
                    continue
                expression of $\langle \hat{a}^\dagger \hat{a} \rangle$
                expression of $\langle \hat{b}^\dagger \hat{b} \rangle$
                expression of $\langle \hat{a}\hat{b} \rangle^2$
                if $\langle \hat{a}^\dagger \hat{a} \rangle < 0$:
                    continue
                if $\langle \hat{b}^\dagger \hat{b} \rangle < 0 :$
                    continue
                expression of $Vs$
                if expression of $Vs > 1$
                    continue
                print $(Vs, \Omega, \mu, \gamma_c, k)$
```