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ON

ANALYSIS OF DIRECT SEGREGATED BOUNDARY- DOMAIN INTEGRAL EQUATIONS FOR VARIABLE-COEFFICIENT MIXED BVPs IN EXTERIOR DOMAINS

(SUBMITED IN PARTIAL FULFILMENT OF THE M.Sc. DEGREE IN MATHEMATICS)

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PERMISSION

This is to certify that this project is compiled by Shiferaw Geremew in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. Tsegaye Gedif

Signature: - _________

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Abstract

Direct segregated systems of boundary-domain integral equations are formulated from the mixed (Dirichlet-Neumann) boundary value problems for a scalar second order divergent elliptic partial differential equation with a variable coefficient in an exterior three-dimensional domain.

The boundary-domain integral equation system equivalence to the original boundary value problems and the Fredholm properties and invertibility of the corresponding boundary-domain integral operators are analyzed in weighted Sobolev spaces suitable for infinite domains. This analysis is based on the corresponding properties of the BVPs in Weighted Sobolev spaces that are proved as well.

Key words:

Partial Differential Equation; Variable coefficient; Mixed problem; Parametrix; Levi function;

Boundary-domain integral equations: Unbounded domain; Weighted Sobolev spaces.

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Chapter I

Preliminaries

1.1. Introduction

Partial Differential Equations (PDEs) with variable coefficients often arise in Mathematical modeling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced in homogeneity) in solid mechanics, electro magnectics thermo-conductivity; fluid flows through porous media and other areas of Medias of Physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, Preventing reduction of Boundary Value Problems (BVPs) for such PDEs to explicit boundary integral equations to be effectively solved numerically.

Nevertheless, for a rather wide class of variable-coefficients PDEs it is possible to use instead an explicit parametrix (Levi function) associated with a fundamental solution of corresponding frozen-coefficient PDEs and reduce BVPs for such PDEs in interior domains to systems of Boundary-Domain Integral Equations (BDIEs) for further numerical solution.

Our main goal here is to show that the mixed problems with variable coefficients in exterior domains can be reduced to some systems of BDIEs and investigate equivalence of the reduction and invertibility of the corresponding boundary-domain integral operators in the Weighted Sobolev spaces (that are more suitable for exterior domains than the standard Sobolev spaces).

To do this, we extend to exterior domains and Weighted spaces the methods developed for exterior domains and standard Sobolev (Bessel potential) spaces.

The analysis of the BDIEs is not only an intersecting and challenging Mathematical problems on its own right but is also useful for the BDIE discretisation and numerical solution to obtain by this way a numerical solution of the associated BVP.

Although the BDIE numerical applications are beyond the scope of this paper, this paper is arranged as follows. Chapter I describes some Prelimnaries. Chapter II is about Boundary Value Problems, Parametrix and Parametrix-Based Potentials and Integral equations. In Chapter III Segregated Boundary-Domain Integral Equation sytems (BDIEs)for a Mixed Problems, Equivalence and Uniqueness and Theorems and Boundary-Domain Integral operators (BDIO) Fredhmolm properties and Invetibilty are analyzed.

1.2 .Basic notation and spaces

Let $\Omega = \Omega^{+}$ be some unbounded (exterior) open three-dimensional region of $\mathbb{R}^3$ of such that $\Omega^{-} = \mathbb{R}^3 \setminus \overline{\Omega}$ is a bounded open domain. For simplicity, we assume that the boundary $\partial\Omega = \partial\Omega^{-}$ is simply connected, complete, infinitely smooth surface.

We consider below some boundary-domain integral equation systems associated with a mixed BVP for the following scalar elliptic differential equation.

$$Au(x) := A(x,\partial x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial xi}(a(x)\frac{\partial u(x)}{\partial xi}) = \text{f(x)} \quad \text{x} \in \Omega \qquad 1.1$$

Where u is the unknown function, while $a(x)$ >0 and f are given functions in Ω.

In what follows, $H^{s}(\Omega) = H_2^{s}(\Omega)$. $H^{s}(\partial\Omega) = H_2^{s}(\partial\Omega)$
Denote the Bessel potential spaces (coinciding with the Sobolev Slobodeskii spaces of $s \geq 0$)

$$H_2^{s}(\Omega) = \{g : g \in H^{s}(R^3),\ \sup p\ \ g \subset \partial\Omega\}$$

We also denote

$$\widetilde{H}^{s}(S_1) = \{g : g \in H^{s}(S), \sup p(g) \subset \overline{S_1}\}$$

$H^{s}(S_1) = \{r_{s_1} g : g \in H^{s}(S)\}$,where S_1 is a proper sub-manifold of a closed surface s and r_{s_1} is the restriction operator on S_1.

To make boundary value problems for(1.1) in infinite domains uniquely solvable. We will use Weighted Sobolev Spaces.

Let $\rho(x) = (1+|x|^2)^{1/2}$ be the Weight function,

$L_2(\rho^{-1},\Omega) := \{g : \rho^{-1} g \in L_2(\Omega)\}$ Be the Weighted Lebesgue space and $H^1(\Omega)$ be the Weighted Sobolev (Beppo-Levi) space.

$$H^1(\Omega) = \{g \in L_2(\rho^{-1};\Omega) : \nabla g \in L_2(\Omega)\},$$

$$\|g\|^2_{H^1(\Omega)} := \|\rho^{-1} g\|^2_{L_2(\Omega)} + \|\nabla g\|^2_{L_2(\Omega)} \qquad 1.2$$

For $u\epsilon H^1(\Omega)$ it follows that $u\epsilon H^1(\Omega^\pm)$, then $\gamma^\pm u\epsilon H^{\frac{1}{2}}(\partial\Omega)$ where $\gamma^\pm = \gamma^\pm_{\partial\Omega}$are the trace operators on $\partial\Omega$ from$\Omega^\pm$. We use γu for the traces $\gamma^\pm u$,if$\gamma^+ u = \gamma^- u$. We will also use notations $u^\pm$ for the traces$\gamma^\pm u$.

Assume that there are constants a_0,a_1 such that

$$a \in L_\infty(R^3) \; and \;\; 0 < a_0 < a(x) < a_1 < \infty \;\; for \;\; a.e. \;\; x \in (R^3). \qquad 1.3$$

For $u\epsilon H^1(\Omega)$ the co-normal derivative operators $a\partial_n u$ on $\partial\Omega$ may not exist in the classical (trace) sense.

However, if for a linear operator A, we introduce the space

$$H^{1,0}(\Omega; A) = \{ \; g \in H^1(\Omega) : Ag \in L_2(\rho;\Omega)\}$$

$$\|g\|^2_{H^{1,0}(\Omega,A)} := \|g\|^2_{H^1(\Omega)} + \|\rho Ag\|^2_{L_2(\Omega)}.$$

If $u\epsilon H^{1,0}(\Omega, A)$ one can correctly define the canonical co-normal derivative $T^\pm u\epsilon H^{\frac{-1}{2}}(\partial\Omega)$ as

$$\left\langle T^\pm u, w\right\rangle_{\partial\Omega} := \int_\Omega \left[(\gamma^\pm_{-1} w)Au + E(u, \gamma^\pm_{-1} w)\right]dx \qquad \forall w \in H^{\frac{1}{2}}(\partial\Omega) \qquad 1.4$$

Where $\gamma^\pm_{-1}: H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is the bounded right inverse to the trace operator

$\gamma^\pm: H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$.

The operator T^+: $u, v\epsilon H^{1,0}(\Omega; A)$ is continuous and gives the continuous extension on $H^{1,0}(\Omega; A)$ of the classical co-normal derivative operator $a\partial_n$, where$\partial_n = n.\nabla$ and $n = n^+$ is normal vector on $\partial\Omega$ directed out ward the exterior domain Ω. When$a \equiv 1$ we employ for T^+ the notation T^+_Δ, which is continuous on $H^{1,0}(\Omega; A)$ of classical co-normal derivative∂_n.

For $u, v\epsilon H^{1,0}(\Omega; A)$ the first Green identity holds in the form

$$\left\langle T^+ u, \gamma^+ v\right\rangle_{\partial\Omega} := \int_\Omega \left[vAu + E(u,v)\right]dx \qquad \forall v \in H^1(\Omega). \qquad 1.5$$

$$E(u,v) := \sum_{i=1}^{3} a(x)\frac{\partial u(x)}{\partial xi}\frac{\partial v(x)}{\partial xi} := a(x)\nabla u(x)\nabla v(x) \qquad 1.6$$

For any functions $u, v\epsilon H^{1,0}(\Omega; A)$ we have the second Green identity.

$$\int_\Omega [vAu - uAv]dx = \left\langle T^+ u, \gamma^+ v\right\rangle_{\partial\Omega} - \left\langle T^+ v, \gamma^+ u\right\rangle_{\partial\Omega}. \qquad 1.7$$

Chapter II

Boundary Value Problems, Parametrix and Parametrix-Based Potentials and Integral Equations

2.1. Boundary value problems

The mixed boundary value problem in an exterior domain Ω is defined as follows,

Find a function $u \in H^{1,0}(\Omega; A)$ satisfying the conditions

$$\mathrm{A}u = f \text{ in } \Omega \qquad 2.1$$

$$\gamma^{+}u = \varphi_{0} \text{ On } \partial_{D}\Omega \qquad 2.2$$

$$T^{+}u = \psi_{0} \text{ On } \partial_{N}\Omega \qquad 2.3$$

Where $\varphi_{o} \in H^{\frac{1}{2}}(\partial_{D}\Omega),\ \psi_{o} \in H^{\frac{-1}{2}}(\partial_{N}\Omega),\quad f \in L_{2}(\rho;\Omega)$ 2.4

Here $\partial\Omega = \overline{\partial_{D}\Omega} \bigcup \overline{\partial_{N}\Omega}$, while $\partial_{D}\Omega \neq \emptyset$ and $\partial_{N}\Omega \neq \emptyset$ are non intersecting

Simply connected sub-manifolds of $\partial\Omega$ with an infinitely smooth boundary curve

l: = $\overline{\partial_{D}\Omega} \bigcap \overline{\partial_{N}\Omega} \in \mathrm{C}^{\infty}$

If $\partial_{N}\Omega = \emptyset$, i.e. $\partial_{D}\Omega = \partial\Omega$, then we arrive at the Dirichlet problem for

$u \in H^{1,0}(\Omega; A)$,

$$\mathrm{A}u = f \text{ in } \Omega \qquad 2.5$$

$$\gamma^{+}u = \varphi_{0} \text{ On } \partial\Omega \qquad 2.6$$

Where $\varphi_{o} \in H^{\frac{1}{2}}(\partial\Omega),\ f \in L_{2}(\rho;\Omega)$

If $\partial_{D}\Omega = \emptyset$ i.e. $\partial_{N}\Omega = \partial\Omega$ in (2.1)-(2.4), then we arrive at the Neumann Problem

for $u \in H^{1,0}(\Omega; A)$,

$$\mathrm{A}u = f \text{ in } \Omega \qquad 2.7$$

$$T^{+}u = \psi_{0} \text{ On } \partial\Omega \qquad 2.8$$

$where\ \psi_o \in H^{\frac{-1}{2}}(\partial\Omega)$, $\quad f \in L_2(\rho;\Omega)$,

Let us denote by

$$\mathrm{A}_M : H^{1,0}(\Omega;A) \to L_2(\rho;\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{-1}{2}}(\partial_N\Omega)$$

$$\mathrm{A}_D : H^{1,0}(\Omega;A)) \to L_2(\rho;\Omega) \times H^{\frac{1}{2}}(\partial\Omega),$$

$$\mathrm{A}_N : H^{1,0}(\Omega;A) \to L_2(\rho;\Omega) \times H^{\frac{-1}{2}}(\partial\Omega)$$

The left hand side operators of, respectively, the mixed BVP (2.1)-(2.3),the Dirichlet BVP (2.5)-(2.6) and the Neumann BVP (2.7)-(2.8), which is evidently continuous.

Theorem 2.1

Under conditions (1.3) the mixed, Dirichlet and Neumann homogeneous problems are uniquely Solvable in $\mathrm{H}^{1,0}(\Omega;\mathrm{A})$ and the corresponding inverse operators

$$\mathrm{A}_M^{-1} : L_2(\rho;\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{\frac{-1}{2}}(\partial_N\Omega) \to H^{1,0}(\Omega;A)$$

$$\mathrm{A}_D^{-1} : L_2(\rho;\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A)$$

$A_N^{-1} : L_2(\rho;\Omega) \times H^{\frac{-1}{2}}(\partial\,\Omega) \to H^{1,0}(\Omega;A))$ are continuous.

2.2. Parametrix and parametrix-based potentials

It is well known that the function

$$P(x,y) = \frac{-1}{4\pi a(y)|x-y|} \qquad x,y \in \mathbb{R}^3 \qquad 2.9$$

Is the parametrix (Levi-function) for the operator $A(x,\partial x)$
i.e. $A(x,\partial x)\ P(x,y) = \delta(x-y) + R(x,y)$

Where $$R(x,y) = \sum_{i=1}^{3} \frac{xi-yi}{4\pi a(y)|x-y|^3} \frac{\partial a(x)}{\partial xi} \qquad x,y \in \mathbb{R}^3 \qquad 2.10$$

Proof:

To show P(x, y) is the parametrix (Levi-function) for the operator $A(x,\partial x)$

We have to show:

$$A(x,\partial x)\ P(x,y) = \delta(x-y) + R(x,y)$$

$$\Leftrightarrow\ A(x,\partial x)\ P(x,y) := \sum_{i=1}^{3}\frac{\partial}{\partial xi}\left(a(x)\frac{\partial p(x,y)}{\partial xi}\right)$$

$$:= \sum_{i=1}^{3}[a(x)\frac{\partial}{\partial xi}\left(\frac{\partial p(x,y)}{\partial xi}\right) + \frac{\partial a(x)}{\partial xi}\frac{\partial p(x,y)}{\partial xi}]$$

$$:= \sum_{i=1}^{3}[a(x)\frac{\partial}{\partial xi}\left(\frac{\partial}{\partial xi}\left(\frac{p_{\Delta}(x,y)}{a(y)}\right)\right) + \frac{\partial a(x)}{\partial xi}\frac{\partial}{\partial xi}\left(\frac{p_{\Delta}(x,y)}{a(y)}\right)]$$

$$:= \sum_{i=1}^{3}[a(x)\frac{\partial^2}{\partial xi^2}\left(\frac{p_{\Delta}(x,y)}{a(y)}\right)] + \sum_{i=1}^{3}[\frac{\partial}{\partial xi}\left(\frac{p_{\Delta}(x,y)}{a(y)}\right)\frac{\partial a(x)}{\partial xi}]$$

$$:= \sum_{i=1}^{3}[a(x)\frac{\partial^2}{\partial xi^2}\frac{-1}{4\pi a(y)|x-y|}] + \sum_{i=1}^{3}\ \frac{xi-yi}{4\pi a(y)|x-y|^3}\frac{\partial a(x)}{\partial xi}]$$

$$:= \delta(x-y) + R(x,y)$$ (Since $\frac{-1}{4\pi|x-y|}$ is the fundamental solution of the Laplacian operator Δ).

$$\therefore\ A(x,\partial x)\ P(x,y) = \delta(x-y) + R(x,y) \qquad x,y\in\mathbb{R}^3 \qquad 2.11$$

To obtain boundary-domain in integral equations, we will consider the coefficient a such that

$$a\in C^1(R^3)\ \text{ And }\ \rho\nabla a\in L_\infty(R^3) \qquad 2.12$$

Remark 2.1: *If a satisfies conditions* 1.3 *and the second condition in* 2.12 . *Then*

$\|ga\|H^1(\Omega)\le C_1\|g\|H^1(\Omega)$, $\left\|\frac{g}{a}\right\|H^1(\Omega)\le C_2\|g\|H^1(\Omega)$

Where constants C_1 and C_2 are independent of $g\epsilon H^1(\Omega)$.

i.e. a and $1/a$ are multipliers in space $H^1(\Omega)$.

For any fixed $y\epsilon\Omega$ and any ball $\Omega\backslash B_{\in}y$ centered at y with sufficiently small radius $\in\rightarrow 0$, we have $R(.,y)\epsilon L_2((\rho;\Omega\backslash B_{\in}y))$ and thus $p(y,.)\epsilon H^{1,0}(\rho;\Omega\backslash B_{\in}y)$ by 2.11.

For any $u\in H^{1,0}(\Omega;A)$.Here

$$Pg(y):=\int_{\Omega}P(x,y)g(x)dx, \qquad Rg(y):=\int_{\Omega}R(x,y)g(x)dx \qquad y\in\mathbb{R}^3. \qquad 2.13$$

are respectively, the parametrix-based volume Newton type and remainder potentials. While

$$Vg(y):=-\int_{\partial\Omega}p(x,y)g(x)ds(x) \quad ,Wg(y):=-\int_{\partial\Omega}\left[T(x,n(x),\partial x,)P(x,y)\right]g(x)ds(x) \quad y\in\mathbb{R}^3/\partial\Omega \quad 2.14$$

are the parametrix-based surface single layer and double layer potentials. The Newton type and the remainder potentials operators given by (2.13), for $\Omega=\mathbb{R}^3$will be denoted as P and R respectively.

Applying the second Green identity (1.7) with $v\ =\ p(y,.)$ and assuming u is the solution of operator A, We get the third Green identity.i.e:

$$\int_{\Omega}(vAu-uAv)dx=\left\langle T^+u,\gamma^+v\right\rangle_{\partial\Omega}-\left\langle T^+v,\gamma^+u\right\rangle_{\partial\Omega}$$

$$\Leftrightarrow\int_{\partial\Omega}T^+u(x)\gamma^+v(x)dS(x)-\int_{\partial\Omega}T^+v(x)\gamma^+u(x)dS(x)=\int_{\Omega}v(x)Au(x)d\Omega(x)-\int_{\Omega}u(x)Av(x)d\Omega(x)$$

$$\Leftrightarrow\int_{\partial\Omega}T^+u(x)\gamma^+P(x,y)dS(x)-\int_{\partial\Omega}T^+P(x,y)\gamma^+u(x)dS(x)=\int_{\Omega}P(x,y)Au(x)d\Omega(x)-\int_{\Omega}u(x)AP(x,y)d\Omega(x)$$

$$\Leftrightarrow\int_{\partial\Omega}T^+u(x)P(x,y)dS(x)-\int_{\partial\Omega}T^+P(x,y)\gamma^+u(x)dS(x)=\int_{\Omega}P(x,y)Au(x)d\Omega(x)-\int_{\Omega}u(x)\left[\delta(x-y)+R(x,y)\right]d\Omega(x)$$

$$\Leftrightarrow\int_{\Omega}u(x)\delta(x-y)d\Omega(x)+\int_{\Omega}R(x,y)u(x)d\Omega(x)+\int_{\partial\Omega}P(x,y)T^+u(x)dS(x)-\int_{\partial\Omega}T^+P(x,y)\gamma^+u(x)dS(x)$$
$$=\int_{\Omega}P(x,y)Au(x)d\Omega(x)$$

$$\Leftrightarrow u(y)+\int_{\Omega}R(x,y)u(x)d\Omega(x)+\int_{\partial\Omega}P(x,y)T^{+}u(x)dS(x)-\int_{\partial\Omega}T^{+}P(x,y)\gamma^{+}u(x)dS(x)=\int_{\Omega}P(x,y)Au(x)d\Omega(x)$$

Using equations 2.9,2.11,2.13 and 2.14 one operator third Green identity can be written as

$$u+Ru-VT^{+}u+W\gamma^{+}u=pAu \quad in \ \ \Omega \qquad 2.15$$

From definition (2.13) - (2.14) one can obtain representations of the parametrix-based potential operators in terms of their counter parts for $a=1$ (i.e. associated with the Laplace operator Δ)

$$Pg=\frac{1}{a}P_{\Delta}g \ \ , \qquad Rg=-\frac{1}{a}\sum_{i=1}^{3}\partial j\left[p_{\Delta}\left(g\partial ja\right)\right] \qquad 2.16$$

$$Vg=\frac{1}{a}V_{\Delta}g \ \ , \qquad Wg=\frac{1}{a}W_{\Delta}(ag) \qquad 2.17$$

Proof:

a) $Pg(y):=\int_{\Omega}P(x,y)g(x)dx$

$$:=\int_{\Omega}\frac{-1}{4\pi a(y)|x-y|}g(x)dx$$

$$:=\frac{1}{a(y)}\int_{\Omega}\frac{-1}{4\pi|x-y|}g(x)dx$$

$$:=\frac{1}{a(y)}\int_{\Omega}P_{\Delta}(x,y)g(x)dx \qquad where \quad P_{\Delta}(x,y)=\frac{-1}{4\pi|x-y|}$$

$$:=\frac{1}{a(y)}P_{\Delta}g(y)$$

$\therefore \ Pg=\frac{1}{a}P_{\Delta}g$.

b) $Rg(y):=\int_{\Omega}R(x,y)g(x)dx$

$$:=\int_{\Omega}\left[\sum_{i=1}^{3}\frac{xi-yi}{4\pi a(y)|x-y|^{3}}\frac{\partial a(x)}{\partial xi}\right]g(x)dx$$

$$:=\int_{\Omega}\left[\sum_{i=1}^{3}\frac{xi-yi}{4\pi a(y)|x-y|^{3}}g(x)\frac{\partial a(x)}{\partial xi}\right]dx$$

$$:= \int_{\Omega} \left[\sum_{i=1}^{3} \frac{-1}{a(y)} \frac{\partial}{\partial yi} \left(\frac{-1}{4\pi |x-y|} \right) g(x) \frac{\partial a(x)}{\partial xi} \right] dx$$

$$:= \int_{\Omega} \left[\sum_{i=1}^{3} \frac{-1}{a(y)} \frac{\partial}{\partial yi} P_{\Delta}(x,y) g(x) \frac{\partial a(x)}{\partial xi} \right] dx$$

$$:= \sum_{i=1}^{3} \frac{-1}{a(y)} \int_{\Omega} \frac{\partial}{\partial yi} \left(\frac{-1}{4\pi |x-y|} \right) g(x) \frac{\partial a(x)}{\partial xi}\, dx$$

$$:= \frac{-1}{a(y)} \int_{\Omega} \left[\sum_{i=1}^{3} \frac{\partial}{\partial yi} (P_{\Delta}(x,y)) g(x) \frac{\partial a(x)}{\partial xi} \right] dx$$

$$:= \frac{-1}{a(y)} \sum_{j=1}^{3} \partial j \left[p_{\Delta} \left(g(y) \partial j a(y) \right) \right]$$

$$\therefore\ Rg = -\frac{1}{a} \sum_{i=1}^{3} \partial j \left[p_{\Delta} \left(g \partial j a \right) \right]$$

where $p_{\Delta} h(y) = \frac{-1}{4\pi} \int \frac{h(x)}{|x-y|} dx$

c) $Vg(y) := -\int_{\partial\Omega} p(x,y) g(x) ds(x)$

$$:= -\int_{\partial\Omega} \frac{-1}{4\pi a(y) |x-y|} g(x) ds(x)$$

$$:= -\frac{1}{a(y)} \int_{\partial\Omega} \frac{-1}{4\pi |x-y|} g(x) ds(x)$$

$$:= -\frac{1}{a(y)} \int_{\partial\Omega} p_{\Delta}(x,y) g(x) ds(x)$$

$$= \frac{1}{a(y)} V_{\Delta} g(y)$$

$$\therefore\ Vg = \frac{1}{a} V_{\Delta} g$$

d) $Wg(y) := -\int_{\partial\Omega} \left[T\left(x, n(x), \partial x\right) P(x,y) \right] g(x) ds(x)$

$$:= -\int_{\partial\Omega}\left[\sum_{i=1}^{3} a(x)ni(x)\frac{\partial}{\partial xi}P(x,y)\right]g(x)ds(x)$$

$$:= -\int_{\partial\Omega}\left[\sum_{i=1}^{3} ni(x)\frac{\partial}{\partial xi}\frac{P_{\Delta}(x,y)}{a(y)}\right](ag)(x)ds(x)$$

$$:= -\int_{\partial\Omega}\left[\sum_{i=1}^{3} ni(x)\frac{\partial}{\partial xi}\frac{P_{\Delta}(x,y)}{a(y)}\right](ag)(x)ds(x)$$

$$:= -\frac{-1}{a(y)}\int_{\partial\Omega}\left[\sum_{i=1}^{3} ni(x)\frac{\partial}{\partial xi}P_{\Delta}(x,y)\right](ag)(x)ds(x)$$

$$= \frac{1}{a(y)}W_{\Delta}(ag)(y)$$

$$\therefore\ Wg = \frac{1}{a}W_{\Delta}(ag)$$

In addition to conditions (1.3) and (2.12) on the coefficient a, we will sometimes also need the condition

$$\rho^{2}\Delta a \in L_{\infty}(R^{3}) \qquad 2.18$$

Theorem 2.2

The following operators are continuous under the second condition in (2.12)

$$P : H^{-1}(R^{3}) \to H^{1}(R^{3}) \qquad 2.19$$

$$p : \tilde{H}^{-1}(\Omega) \to H^{1}(R^{3}) \qquad 2.20$$

$$R : L_{2}(\rho^{-1}; R^{3}) \to H^{1}(\Omega) \qquad 2.21$$

$$V : H^{\frac{-1}{2}}(\partial\Omega) \to H^{1}(\Omega) \qquad 2.22$$

$$W : H^{\frac{1}{2}}(\partial\Omega) \to H^{1}(\Omega) \qquad 2.23$$

While the following operators are continuous under the second condition in (2.12) and condition (2.18).

$$p : L_2(\rho;\Omega) \to H^{1,0}(R^3;A) \qquad 2.24$$

$$R : H^1(\Omega) \to H^{1,0}(\Omega;A) \qquad 2.25$$

$$V : H^{\frac{-1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A) \qquad 2.26$$

$$W : H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A) \qquad 2.27$$

Proof:

Let $\phi \in D(R^3) \subset H^{-1}(R^3)$ *. Then the Newton potential* $P_\Delta\phi = \frac{-1}{4\pi}\int_{R^3}\frac{\phi(x)}{|x-y|}dx$

Evidently belongs to $H^1(R^3)$ *and solves the poisson equation* $\Delta v = \phi$ *in* R^3 *. On the other hand , the Laplace operator form* $H^1(R^3)$ *to* $H^{-1}(R^3)$ *possesses a continuous inverse operator* $\Delta^{-1} : H^{-1}(R^3) \to H^1(R^3)$

.

Thus $P_\Delta\phi = \Delta^{-1}\phi$ *which due to the density of* $D(R^3)$ *in* $H^{-1}(R^3)$ *gives a continuous*

,

exetension of P_Δ *to the operator* $H^{-1}(R^3) \to H^1(R^3)$. Then the first relation in (2.16) and implies (2.19) under condition (2.12) and thus (2.20) immediately follows.

To prove (2.24), let us denote by $\tilde{g}$ the extension of a function $g \in L_2(\rho;\Omega)$ by zero outside Ω.

Evidently $\tilde{g} \in L_2(\rho;R^3) \subset H^{-1}(R^3)$ *and* $P_\Delta g = P_\Delta\tilde{g} \in H^1\left(R^3\right)$

Taking into account that $APg = g - \sum_{j=1}^{3}\partial j\left(\frac{\partial ja}{a}p_\Delta g\right)$ conditions (2.12) and (2.18) imply (2.24)

Let us prove the continuity of the operators (2.22) and (2.26)

For $\phi \in C^\infty(\partial\Omega)$, let us consider the single layer potential for the Laplace operator

$$V_\Delta\phi = \frac{1}{4\pi}\int_{\partial\Omega}\frac{1}{|x-y|}\phi(x)d\Gamma(x)$$

$Which\ \ evidently\ belongs\ \ to\ \ H^{1}(\Omega;\Delta)\ \ and$ solves the Dirichlet problem $\Delta V = 0\ \ in\ \ \Omega$ $\gamma^{+}v = w\ \ \ on\ \partial\Omega\ \ \ \ for\ \ v \in H^{1}(\Omega;\Delta)$, $where\ w = \gamma V_{\Delta}\phi$. By theorem (2.1), problem (2.28) is uniquely solvable and its solution is delivered by a continuous operator

$Q : H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;\Delta)$;

$i.e.\, V_{\Delta}\phi = Q\gamma V_{\Delta}\phi$.

Taking into account the continuity of the operator $\gamma V_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ and the density of $C^{\infty}(\partial\Omega)\ \ in\ H^{\frac{=1}{2}}(\partial\Omega)$. We arrive at the continuity of $V_{\Delta} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{1}(\Omega;\Delta)$.

Then the first relation in (2.17) imply the continuity of (2.22) and of (2.26) under conditions (2.12), (2.18), the continuity of (2.23) and (2.16) is proved by similar argument.

To prove the continuity of (2.21), let us consider $g \in L_{2}(\rho^{-1}; R^{3})$ due to the condition (2.12),

We have $g\partial ja \in L_{2}(R^{3})$. The second relation in (2.17) gives

$$Rg(y) = \frac{1}{4\pi a(y)} \sum_{j=1}^{3} \int_{R^{3}} \left[\partial yj \frac{1}{|x-y|} \right] g(x)\partial ja(x)dx$$

$$= \frac{1}{4\pi a(y)} \sum_{j=1}^{3} \int_{R^{3}} \left[\partial yj \frac{1}{|x-y|} \right] g(x)\partial ja(x)dx$$

$$= -\sum_{j=1}^{3} P[\partial j(g\partial ja)](y) \qquad 2.28$$

To justify the Gauss divergence theorem employed in (2.28), one can introduce a sequence of functions from $D(R^{3})$ converging to $g\partial ja\ \ in\ L_{2}(R^{3})$, which gradients with then converge to the gradient of $g\partial ja$ in $H^{-1}(R^{3})$ and thus $H^{-1}(R^{3})$. Then the continuity of (2.19) implies continuity of (2.21).

Let us prove continuity of (2.25). Since $H^{1}(R^{3})$ is continuously embedded in $C^{1}(\rho^{-1};\Omega)$,then the continuity of the operator $R : H^{1}(\Omega) \to H^{1}(\Omega)$ is implied by (2.21). Form any $g \in H^{1}(\Omega)$

We have

$$ARg = \sum_{k=1}^{3} \partial k\left(a\partial kR\right) = \nabla a.\nabla Rg + a\Delta Rg$$

$$= \nabla a.\nabla Rg + a^{2}\left[\Delta\left(\frac{1}{a}\right)\right]Rg + 2a\nabla\left(\frac{1}{a}\right).\nabla\left(aRg\right) + \Delta\left(aRg\right) \tag{2.29}$$

By the second relation in equation (2.17)

$$\Delta\left(aRg\right) = -\sum_{j-1}^{3} \partial j\Delta P_{\Delta}\left(g\partial ja\right) = -\nabla g.\nabla a - g\Delta a$$

Then (2.29) along with the conditions (2.12) and (2.18) imply the continuity of the operator $AR : H^{1}(\Omega) \to L_{2}(\rho^{-1};\Omega)$ and thus the operator (2.25).

Let us introduce the following boundary integral (pseudo differential) operators of the direct values and co-normal derivatives of the single and double layer potentials:

$$vg(y) := -\int_{\partial\Omega} P(x,y)g(x)ds(x) \tag{2.30}$$

$$wg(y) := -\int_{\partial\Omega} \left[T\left(x,n(x),\partial x\right)P(x,y)\right]g(x)ds(x) \tag{2.31}$$

$$w^{'}g(y) := -\int_{\partial\Omega} \left[T\left(y,n(y),\partial y\right)P(x,y)\right]g(x)ds(x) \tag{2.32}$$

$$L^{\pm}g(y) := -\left[T^{\pm}\left(y,n(y),\partial y\right)Wg(y)\right] := T^{\pm}Wg(y) \tag{2.33}$$

$Where\ y \in \partial\Omega$.

They can be also presented in terms of their counter parts for $a = 1$

i.e. associated with the Laplace Δ .

$$vg = \frac{1}{a}v_{\Delta}g \qquad wg = \frac{1}{a}w_{\Delta}(ag) \tag{2.34}$$

$$w^{'}g = w^{'}{}_{\Delta}g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]v_{\Delta}g\ , \tag{2.35}$$

$$L^{\pm}g = L^{\pm}{}_{\Delta}(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]W^{\pm}{}_{\Delta}(ag) \tag{2.36}$$

- $$vg(y) := -\int_{\partial\Omega} P(x,y)g(x)ds(x)$$

$$:= -\int_{\partial\Omega} \frac{-1}{4\pi a(y)|x-y|} g(x)ds(x)$$

$$:= \frac{-1}{a(y)} \int_{\partial\Omega} \frac{-1}{4\pi|x-y|} g(x)ds(x)$$

$$:= \frac{-1}{a(y)} \int_{\partial\Omega} P_{\Delta}(x,y)g(x)ds(x)$$

$$= \frac{1}{a(y)} v_{\Delta} g(y)$$

$$\therefore\ vg = \frac{1}{a} v_{\Delta} g$$

- $$wg(y) := -\int_{\partial\Omega} \left[T\left(x, n(x), \partial x\right)P(x,y)\right] g(x)ds(x)$$

$$:= -\int_{\partial\Omega} \left[\sum_{i=1}^{3} a(x)ni(x)\frac{\partial}{\partial xi} P(x,y)\right] g(x)ds(x)$$

$$:= -\int_{\partial\Omega} \left[\sum_{i=1}^{3} ni(x)\frac{\partial}{\partial xi} \frac{P_{\Delta}(x,y)}{a(y)}\right] (ag)(x)ds(x)$$

$$:= \frac{-1}{a(y)} \int_{\partial\Omega} \left[\sum_{i=1}^{3} ni(x)\frac{\partial}{\partial xi} P_{\Delta}(x,y)\right] (ag)(x)ds(x)$$

$$:= \frac{1}{a(y)} w_{\Delta}(ag)(y)$$

$$\therefore\ wg = \frac{1}{a} w_{\Delta}(ag)$$

- $$w^{'} g(y) := -\int_{\partial\Omega} \left[T\left(y, n(y), \partial y\right)P(x,y)\right] g(x)ds(x)$$

$$:= -\int_{\partial\Omega} \left[\sum_{i=1}^{3} a(y)ni(y)\frac{\partial}{\partial yi} P(x,y)\right] g(x)ds(x)$$

$$:= -\int_{\partial\Omega} \left[\sum_{i=1}^{3} a(y)ni(y)\frac{\partial}{\partial yi} \left(\frac{P_{\Delta}(x,y)}{a(y)}\right)\right] g(x)ds(x)$$

$$:= -\int_{\partial\Omega}\left[\sum_{i=1}^{3} a(y)ni(y)\{\frac{1}{[a(y)]^2}\left(a(y)\frac{\partial}{\partial yi}P_\Delta(x,y) - P_\Delta(x,y)\frac{\partial}{\partial yi}a(y)\}\right)\right]g(x)ds(x)$$

$$:= -\int_{\partial\Omega}\left[\sum_{i=1}^{3}\frac{1}{[a(y)]^2}\{\left([a(y)]^2 ni(y)\frac{\partial}{\partial yi}P_\Delta(x,y) - a(y)ni(y)P_\Delta(x,y)\frac{\partial}{\partial yi}a(y)\}\right)\right]g(x)ds(x)$$

$$:= -\int_{\partial\Omega}\left[\sum_{i=1}^{3}\left(ni(y)\frac{\partial}{\partial yi}P_\Delta(x,y)\right)\right]g(x)ds(x)$$

$$+\int_{\partial\Omega}\left[\sum_{i=1}^{3}\frac{1}{a(y)}\left(ni(y)P_\Delta(x,y)\frac{\partial}{\partial yi}a(y)\right)\right]g(x)ds(x)$$

$$= w'_\Delta g(y) + + \sum_{i=1}^{3}\frac{ni(y)}{a(y)}\frac{\partial a(y)}{\partial yi}\int_{\partial\Omega}P_\Delta(x,y)g(x)ds(x)$$

$$= w'_\Delta g(y) + \left[a(y)\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right]v_\Delta g(y)$$

$$\therefore\ w'g = w'_\Delta g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]v_\Delta g$$

- $L^{\pm}g(y) := -\left[T^{\pm}(y,n(y),\partial y)Wg(y)\right]$

$$:= \sum_{i=1}^{3} a(y)ni(y)\gamma^{\pm}\left[\frac{\partial}{\partial yi}Wg(y)\right]$$

$$:= a(y)\gamma^{\pm}\left[\frac{\partial}{\partial n(y)}Wg(y)\right]$$

$$:= a(y)\gamma^{\pm}\left[\frac{\partial}{\partial n(y)}\frac{W_\Delta(ag)(y)}{a(y)}\right]$$

$$:= a(y)\gamma^{\pm}\left[\frac{1}{a(y)}\frac{\partial}{\partial n(y)}W_\Delta(ag)(y) + W_\Delta(ag)(y)\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right]$$

$$:= a(y).\frac{1}{a(y)}\gamma^{\pm}\left[\frac{\partial}{\partial n(y)}W_\Delta(ag)(y)\right] + a(y)\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)W^{\pm}_\Delta(ag)(y)$$

$$:= \gamma^{\pm}\left[\frac{\partial}{\partial n(y)}W_\Delta(ag)(y)\right] + \left[a(y)\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right]W^{\pm}_\Delta(ag)(y)$$

$$= L^{\pm}{}_{\Delta}(ag)(y) + \left[a(y)(\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right] W^{\pm}{}_{\Delta}(ag)(y)$$

$$\therefore\ L^{\pm} g = L^{\pm}{}_{\Delta}(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right] W^{\pm}{}_{\Delta}(ag)$$

Where as well the subscript Δ means that the corresponding surface potentials are based on the harmonic fundamental solution $P_{\Delta}(x,y) = -\left(4\pi|x-y|\right)^{-1}$

Theorem 2.3:

Jump relations; *Let* $g_1 \in H^{\frac{-1}{2}}(\partial\Omega),\ and\ \ g_2 \in H^{\frac{1}{2}}(\partial\Omega)\ \ and\ a \in C^1(R^3)$. Then:

$$\gamma^{\pm} V g_1(y) = w g_1(y) \qquad 2.37$$

$$\gamma^{\pm} W g_2(y) = \mp\frac{1}{2} g_2(y) + w g_2(y) \qquad 2.38$$

$$T^{\pm} V g_1(y) = \pm\frac{1}{2} g_1(y) + w' g_1(y) \qquad 2.39$$

Where $y \in \partial\Omega$.

Taking trace and co-normal derivative of the third Green identity (2.15),

i.e. from $u + Ru - VT^{+}u + W\gamma^{+}u = pAu$

Take the trace of equation (2.15) ,we obtain

$$\Leftrightarrow\quad \gamma^{+}u + \gamma^{+}Ru - \gamma^{+}VT^{+}u + \gamma^{+}W\gamma^{+}u = \gamma^{+}pAu$$

$$\Leftrightarrow \gamma^{+}u + \gamma^{+}Ru - vT^{+}u - \frac{1}{2}\gamma^{+}u + w\gamma^{+}u = \gamma^{+}pAu$$

$$\Leftrightarrow \frac{1}{2}\gamma^{+}u + \gamma^{+}Ru - vT^{+}u + w\gamma^{+}u = \gamma^{+}pAu \qquad on\quad \partial\Omega$$

Again by taking the co-normal derivative of (2.15)

i.e. $T^{+}u + T^{+}Ru - T^{+}VT^{+}u + W\gamma^{+}u = T^{+}pAu$

$$\Leftrightarrow T^{+}u + T^{+}Ru - \frac{1}{2}T^{+}u - w'T^{+}u + L^{+}\gamma^{+}u = T^{+}pAu$$

$$\Leftrightarrow \frac{1}{2}T^{+}u + T^{+}Ru - w'T^{+}u + L^{+}\gamma^{+}u = T^{+}pAu \qquad on \quad \partial\Omega$$

Therefore;

$$u + Ru - VT^{+}u + W\gamma^{+}u = Pf \qquad in \qquad \Omega \tag{2.40}$$

$$\Leftrightarrow \frac{1}{2}\gamma^{+}u + \gamma^{+}Ru - vT^{+}u + w\gamma^{+}u = \gamma^{+}pAu \qquad on \ \partial\Omega \tag{2.41}$$

$$\Leftrightarrow \frac{1}{2}\gamma^{+}u + \gamma^{+}Ru - vT^{+}u + w\gamma^{+}u = \gamma^{+}pAu \qquad on \qquad \partial\Omega \tag{2.42}$$

For arbitrary functions u, f, Ψ, Φ, let us consider a more general “indirect” integral relation, associated with (2.15).

$$u + Ru - V\Psi + W\Phi = pf \qquad in\ \Omega \tag{2.43}$$

Lemma 2.1

Let $u \in H^{1}(\Omega), f \in L_{2}(\rho;\Omega), \Psi \in H^{\frac{-1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy (2.43) and conditions (2.12),(2.18) hold. Then u belongs to $H^{1,0}(\Omega;A)$ u is a solutions of the equation

$$Au = f \qquad \text{In} \qquad \Omega \tag{2.44}$$

$$While \qquad V(\Psi - T^{+}u) - W(\Phi - \gamma^{+}u) = 0 \quad in\ \Omega \tag{2.45}$$

Proof:

First of all, let us prove that $u\epsilon H^{1,0}(\Omega;A)$

Since $Au = \Delta(au) - \sum_{i=1}^{3}\frac{\partial}{\partial xi}(u(x)\frac{\partial a(x)}{\partial xi})$

And the last term is belongs to $L_2(\rho;\Omega)$. We need only to show that $\Delta(au)\epsilon L_2(\rho;\Omega)$,

Rewriting (2.43) as a function of u .

$$u = pf - Ru + V\Psi - W\Phi ,$$

Then multiply by $a(y)$

$$au = apf - aRu + aV\Psi - aW\Phi$$

$$= p_{\Delta} f - aRu + V_{\Delta}\Psi - W_{\Delta}(a\Phi)$$

Then take the Laplacian of the last equation

$$\Delta(au) = \Delta(p_{\Delta} f) - \Delta(aRu) + \Delta(V_{\Delta}\Psi) - \Delta(W_{\Delta}(a\Phi))$$

The last two terms in the right hand side are harmonic and also $Ru\epsilon H^2(\Omega)$ for $u\epsilon H^1(\Omega)$

$$\Delta(p_{\Delta} f) = f \in L_2(\rho;\Omega)$$

Implies $\Delta(au)\epsilon L_2(\rho;\Omega)$

Hence Au$\epsilon L_2(\rho;\Omega)$, u$\epsilon L_2(\rho;\Omega)$, Subtracting (2.43) from third Green identity (2.15)

$$i.e. \; - \begin{cases} u + Ru - VT^{+}u + W\gamma^{+}u = PAu \quad in \\ u + Ru - V\Psi + W\Phi = Pf \quad in \quad \Omega \end{cases} \Omega$$

$$-V(T^{+}u - \Psi) - W(\gamma^{+}u - \Phi) = P(Au - f) \quad in \quad \Omega$$

$$\Leftrightarrow -V\Psi^{*} - W\Phi^{*} = P(Au - f) \qquad in \quad \Omega \qquad 2.46$$

$$Where\; \Psi^{*} := T^{+}u - \Psi \,, \quad \Phi^{*} := \gamma^{+}u - \Phi$$

Multiplying equality (2.46) by $a(y)$ we get

$$-a(y)\, V\Psi^{*} - a(y)W\Phi^{*} = a(y)P(Au - f)$$

$$\Leftrightarrow -V_{\Delta}\Psi^{*} - W_{\Delta}\Phi^{*} = P_{\Delta}(Au - f) \quad in\; \Omega$$

Applying the Laplace operator Δ to the last equation and taking in to consideration that the both functions in the left hand-side are harmonic surface potentials, while the right-hand side functions is the classical Newtonian volume potential, We arrive at equation (2.44),substituting (2.44) back in to (2.46) leads to (2.45).

Lemma 2.2:

Let conditions (2.12), (2.18) hold

i) $If \;\; \Psi^{*} \in H^{\frac{-1}{2}}(\partial\Omega) \,, and \;\; V\Psi^{*} = 0 \;\; in \;\; \Omega \,, then \;\; \Psi^{*} = 0$

ii) $If \;\; \Phi^{*} \in H^{\frac{1}{2}}(\partial\Omega) \,, and \;\; W\Phi^{*} = 0 \;\; in \;\; \Omega \,, then \;\; \Phi^{*}(x) = c/a(x)$

$Where \;\; C \;\; is \;\; constant$.

iii) $Let \quad \partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega}\ , where \ \ \partial_D\Omega \ \ and \ \partial_N\Omega$ are non-empty non -intersecting, simply connected sub-manifolds of $\partial\Omega$ with infinitely smooth boundaries. If $\Psi^* \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega), \Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, $then$
$V\Psi^*(y) - W\Phi^*(y) = 0 \ \ in \ \Omega$, $and \ \Psi^* = 0 \ and \ \Phi^* = 0 \ on \ \partial\Omega$.

Proof:

The proofs of item (i) and (ii) coincide with the proofs of their counter parts for interior domains in [2, lemma 4.2].

To prove item (ii), we first remark that $\Phi_\Delta = C$ satisfies the equation $W_\Delta\Phi_\Delta = 0$ in the exterior domain $\Omega \ \ for \ \ any \ \ C = const.$ (This follows from the first Green identity (1.5) for the interior domain Ω^- employed for $v(x) = C$, $A = \Delta$, $u = -1/(4\pi|x-y|)$ and for any $y \in \Omega$). Let us check that there is no other solution of the equation in Ω in $H^{\frac{1}{2}}(\partial\Omega)$. By the Lyapunov-Tauber theorem, $T_\Delta^+ W_\Delta\Phi_\Delta = T_\Delta^- W_\Delta\Phi_\Delta = 0 \ \ on \ \partial\Omega$, $Which \ \ implies \ W_\Delta\Phi_\Delta = const.$ in the interior domain Ω^- due to the uniqueness up to a constant of the solution of the Neumann problem in $H^1(\Omega^-)$. Then the jump properties of the double layer potential give. Applying the second relation of (2.12) finalizes the proof of item (ii)

Chapter III

Segregated BDIEs for the Mixed Problems, Equivalence and Uniqueness Theorems and BDIO Fredholm Properties and Invertibility

3.1. Segregated BDIEs for the Mixed Problem

Let us fix an extension $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ of the given function φ_0 in the Dirichlet boundary condition (2.2) from $\partial_D\Omega$ to the whole of $\partial\Omega$ and an extension $\Psi_0 \in H^{\frac{-1}{2}}(\partial\Omega)$ of the given function ψ_0 in the Neumann boundary condition (2.3) from $\partial_N\Omega$ to the whole $\partial\Omega$.

We will explore different possibilities of the reducing BVP (2.1)-(2.3) to a system Boundary-Domain Integral Equation (BDIEs) and in all of them we represent in (2.40), (2.41), and (2.42) the trace of the function u and its co-normal derivative as $\gamma^+ u = \Phi_0 + \varphi$, $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$; $T^+ u = \Psi_0 + \psi$, $\psi \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$. And will regard the new unknown functions φ and ψ as formally segregated of u .

Thus we will look for the triplet:

$$U := (u, \psi, \varphi)^T \in H := H^{1,0}(\Omega; A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$$

$$\subset X := H^1(\Omega) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$$

Formulation of BDIE system (M11)

Using equation (2.40) in Ω we get

$$u + Ru - V(\Psi_0 + \psi) + W(\Phi_0 + \varphi) = Pf \qquad in \quad \Omega$$

$$u + Ru - V\psi + W\varphi = Pf + V\Psi_0 - W\Phi_0$$

$$u + Ru - V\psi + W\varphi = F_0 \qquad in \quad \Omega \qquad\qquad 3.1$$

Where $F_0 = Pf + V\Psi_0 - W\Phi_0$

Taking the restriction (2.41) on $\partial_D\Omega$ we obtain

$$r_{\partial_D\Omega}[\frac{1}{2}\gamma^+ u + \gamma^+ Ru - vT^+ u + w\gamma^+ u] = r_{\partial_D\Omega}[\gamma^+ Pf] \quad on\ \partial_D\Omega$$

$$\Leftrightarrow r_{\partial_D\Omega}[\frac{1}{2}\gamma^+ u + r_{\partial_D\Omega}\gamma^+ Ru - r_{\partial_D\Omega}vT^+ u + r_{\partial_D\Omega}w\gamma^+ u] = r_{\partial_D\Omega}[\gamma^+ Pf]$$

$$\Leftrightarrow\ r_{\partial_D\Omega}\frac{1}{2}\gamma^{+}u+r_{\partial_D\Omega}\gamma^{+}Ru-r_{\partial_D\Omega}vT^{+}u+r_{\partial_D\Omega}w\gamma^{+}u]=r_{\partial_D\Omega}[\gamma^{+}Pf]$$

$$\Leftrightarrow\ r_{\partial_D\Omega}\frac{1}{2}(\Phi_0+\varphi)+r_{\partial_D\Omega}\gamma^{+}Ru-r_{\partial_D\Omega}v(\Psi_0+\psi)+r_{\partial_D\Omega}w(\Phi_0+\varphi)]=r_{\partial_D\Omega}[\gamma^{+}Pf]$$

$$\Leftrightarrow\ \frac{1}{2}\varphi_0+r_{\partial_D\Omega}\gamma^{+}Ru-r_{\partial_D\Omega}v\psi+r_{\partial_D\Omega}w\varphi=r_{\partial_D\Omega}[\gamma^{+}Pf]+r_{\partial_D\Omega}V\Psi_0-r_{\partial_D\Omega}W\Phi_0$$

$$\Leftrightarrow\ r_{\partial_D\Omega}\gamma^{+}Ru-r_{\partial_D\Omega}v\psi+r_{\partial_D\Omega}w\varphi=r_{\partial_D\Omega}[\gamma^{+}Pf]+r_{\partial_D\Omega}V\Psi_0-r_{\partial_D\Omega}W\Phi_0-\frac{1}{2}\varphi_0$$

Where $F_0=Pf+V\Psi_0-W\Phi_0$

$$\Leftrightarrow\ \gamma^{+}F_0=\gamma^{+}[Pf+V\Psi_0-W\Phi_0]$$

$$\Leftrightarrow\ \gamma^{+}F_0=\gamma^{+}Pf+\gamma^{+}V\Psi_0-\gamma^{+}W\Phi_0$$

$$\Leftrightarrow\ \gamma^{+}F_0=\gamma^{+}Pf+v\Psi_0+\frac{1}{2}\Phi_0-w\Phi_0$$

$$\Leftrightarrow\ \gamma^{+}F_0-\Phi_0=\gamma^{+}Pf+v\Psi_0-w\Phi_0-\frac{1}{2}\Phi_0$$

$$\Leftrightarrow\ r_{\partial_{D\Omega}}[\gamma^{+}F_0-\Phi_0]=r_{\partial_{D\Omega}}[\gamma^{+}Pf+v\Psi_0-w\Phi_0-\frac{1}{2}\Phi_0]$$

$$\Leftrightarrow\ r_{\partial_{D\Omega}}\gamma^{+}F_0-\varphi_0=r_{\partial_{D\Omega}}\gamma^{+}Pf+r_{\partial_D\Omega}v\Psi_0-r_{\partial_D\Omega}w\Phi_0-\frac{1}{2}\varphi_0$$

$$r_{\partial_D\Omega}\gamma^{+}Ru-r_{\partial_D\Omega}v\psi+r_{\partial_D\Omega}w\varphi=r_{\partial_D\Omega}\gamma^{+}F_0-\varphi_0 \qquad on\ \partial_D\Omega \qquad 3.2$$

Taking the restriction of (2.42) *on* $\partial_N\Omega$ we get

$$r_{\partial_n\Omega}[\frac{1}{2}T^{+}u+T^{+}Ru-w'T^{+}u+L^{+}\gamma^{+}u=r_{\partial_N\Omega}[T^{+}Pf]$$

$$\Leftrightarrow\ r_{\partial_n\Omega}[\frac{1}{2}(\Psi_0+\psi)+T^{+}Ru-w'(\Psi_0+\psi)+L^{+}(\Phi_0+\varphi)]=r_{\partial_N\Omega}[T^{+}Pf]$$

$$\Leftrightarrow\ r_{\partial_N\Omega}T^{+}Ru-r_{\partial_N\Omega}w'\psi+r_{\partial_N\Omega}L^{+}\varphi=r_{\partial_N\Omega}[T^{+}Pf]+r_{\partial_N\Omega}w'\Psi_0-L^{+}\Phi_0-\frac{1}{2}\Psi_0$$

Where

$$F_0 = Pf + V\Psi_0 - W\Phi_0$$

$$\Leftrightarrow T^+F_0 = T^+[Pf + V\Psi_0 - W\Phi_0]$$

$$\Leftrightarrow T^+F_0 = T^+Pf + T^+V\Psi_0 - T^+W\Phi_0$$

$$\Leftrightarrow T^+F_0 = T^+Pf + \frac{1}{2}\Psi_0 + w'\Psi_0 - L^+\Phi_0$$

$$\Leftrightarrow T^+F_0 - \Psi_0 = T^+Pf + w'\Psi_0 - L^+\Phi_0 - \frac{1}{2}\Psi_0$$

$$\Leftrightarrow r_{\partial_N\Omega}[T^+F_0 - \Psi_0] = r_{\partial_N\Omega}[T^+Pf + w'\Psi_0 - L^+\Phi_0 - \frac{1}{2}\Psi_0]$$

$$\Leftrightarrow r_{\partial_N\Omega}T^+F_0 - \psi_0 = r_{\partial_N\Omega}T^+Pf + r_{\partial_N\Omega}w'\Psi_0 - r_{\partial_N\Omega}L^+\Phi_0 - \frac{1}{2}\psi_0$$

$$r_{\partial_N\Omega}T^+Ru - r_{\partial_N\Omega}w'\psi + r_{\partial_N\Omega}L^+\varphi = r_{\partial_N\Omega}T^+F_0 - \psi_0 \quad on \ \partial_N\Omega \qquad 3.3$$

Hence the above BDIEs (3.1)-(3.3) to gather

$$u + Ru - V\psi + W\varphi = F_0 \quad in \quad \Omega$$

$$r_{\partial_D\Omega}\gamma^+Ru - r_{\partial_D\Omega}v\psi + r_{\partial_D\Omega}w\varphi = r_{\partial_D\Omega}\gamma^+F_0 - \varphi_0 \quad on \quad \partial_D\Omega$$

$$r_{\partial_N\Omega}T^+Ru - r_{\partial_N\Omega}w'\psi + r_{\partial_N\Omega}L^+\varphi = r_{\partial_N\Omega}T^+F_0 - \psi_0 \quad on \ \partial_N\Omega$$

System (3.1)-(3.3) can be written as:

$$M^{11}U = F^{11}$$

Where $F_0 = Pf + V\Psi_0 - W\Phi_0$

And $F^{11} := \left[F_0, r_{\partial_D\Omega}\gamma^+F_0 - \varphi_0, r_{\partial_N\Omega}T^+F_0 - \psi_0\right]^T$ (where the superscript T-denotes transposition).

We denote the matrix operator of the left hand side of the system (M^{11}) as

$$M^{11} = \begin{bmatrix} I+R & -V & W \\ r_{\partial_D\Omega}\gamma^+R & -r_{\partial_D\Omega}v & r_{\partial_D\Omega}w \\ r_{\partial_{N\Omega}T^+\ R} & -r_{\partial_N\Omega}w' & r_{\partial_n\Omega}L^+ \end{bmatrix}$$

The notation M^{11} and the corresponding superscripts mean that the system includes the integral operators of the first kind both on the Dirichlet and Neumann parts of the boundary.

Formulation of BDIE system (M12)

If we use equation (2.40) in Ω and equation (2.41) on the whole of $\partial\Omega$, we arrive at the BDIE system (M12)

Using equation (2.40) in Ω we get

$$u + Ru - V(\Psi_0 + \psi) + W(\Phi_0 + \varphi) = Pf \quad in \quad \Omega$$

$$u + Ru - V\psi + W\varphi = Pf + V\Psi_0 - W\Phi_0$$

$$u + Ru - V\psi + W\varphi = F_0 \quad in \quad \Omega \qquad 3.4$$

Where $F_0 = Pf + V\Psi_0 - W\Phi_0$

Taking the restriction of equation (2.41) on $\partial\Omega$ we get

$$r_{\partial\Omega}[\frac{1}{2}\gamma^+u + \gamma^+Ru - vT^+u + w\gamma^+u] = r_{\partial_D\Omega}[\gamma^+Pf]$$

$$\Leftrightarrow \frac{1}{2}\gamma^+u + \gamma^+Ru - vT^+u + w\gamma^+u = \gamma^+Pf \qquad on \ \partial\Omega$$

$$\Leftrightarrow \frac{1}{2}(\Phi_0 + \varphi) + \gamma^+Ru - v(\Psi_0 + \psi) + w(\Phi_0 + \varphi) = \gamma^+Pf \quad on \ \partial\Omega$$

$$\Leftrightarrow \frac{1}{2}\varphi + \gamma^+Ru - v\psi + w\varphi = \gamma^+Pf + v\Psi_0 - w\Phi_0 - \frac{1}{2}\Phi_0$$

Where

$$F_0 = Pf + V\Psi_0 - W\Phi_0$$

$$\Leftrightarrow F_0 = Pf + V\Psi_0 - W\Phi_0$$

$$\Leftrightarrow T^+F_0 = T^+[Pf + V\Psi_0 - W\Phi_0]$$

$$\Leftrightarrow \gamma^+F_0 = \gamma^+Pf + v\Psi_0 + \frac{1}{2}\Phi_0 - w\Phi_0$$

$$\Leftrightarrow \gamma^{+}F_0 - \Phi_0 = \gamma^{+}Pf + v\Psi_0 - w\Phi_0 - \frac{1}{2}\Phi_0$$

$$\frac{1}{2}\varphi + \gamma^{+}Ru - v\psi + w\varphi = \gamma^{+}F_0 - \Phi_0 \qquad on \quad \partial\Omega \qquad 3.5$$

The above BDIEs (3.4)-(3.5) to gather

$$u + Ru - V\psi + W\varphi = F_0 \quad in \;\; \Omega$$

$$\frac{1}{2}\varphi + \gamma^{+}Ru - v\psi + w\varphi = \gamma^{+}F_0 - \Phi_0 \quad on\ \partial\Omega$$

The system (3.4)-(3.5) can be written as

$$M^{12}U = F^{12}$$

Where $F_0 = Pf + V\Psi_0 - W\Phi_0$

We denote the matrix operator of the left hand side of the systems (M12)

$$M^{12} := \begin{bmatrix} I+R & -V & W \\ \gamma^{+}R & -v & \frac{1}{2}I + w \end{bmatrix}$$

And $F^{11} := \left[F_0, \gamma^{+}F_0 - \Phi_0\right]^{T}$.

Formulation of BDIE system (M21)

Using equation (2.40) in Ω we get

$$u + Ru - V(\Psi_0 + \psi) + W(\Phi_0 + \varphi) = Pf \;\; in \quad \Omega$$

$$u + Ru - V\psi + W\varphi = Pf + V\Psi_0 - W\Phi_0 \quad in\ \Omega$$

$$u + Ru - V\psi + W\varphi = F_0 \qquad in \quad \Omega \qquad 3.6$$

Where $\qquad F_0 = Pf + V\Psi_0 - W\Phi_0$

Use equation (2.42) on the whole $\partial\Omega$, we get

$$r_{\partial_n\Omega}[\frac{1}{2}T^{+}u + T^{+}Ru - w'T^{+}u + L^{+}\gamma^{+}u] = r_{\partial_N\Omega}[T^{+}Pf] \quad on\ \partial\Omega$$

$$\Leftrightarrow \frac{1}{2}T^{+}u + T^{+}Ru - w'T^{+}u + L^{+}\gamma^{+}u = T^{+}Pf \quad on\ \partial\Omega$$

$$\Leftrightarrow \frac{1}{2}(\Psi_0 + \psi) + T^{+}Ru - w'(\Psi_0 + \psi) + L^{+}(\Phi_0 + \varphi) = T^{+}Pf \quad on\ \partial\Omega$$

$$\Leftrightarrow \frac{1}{2}\psi + T^{+}Ru - w'\psi + L^{+}\varphi = T^{+}Pf + w'\Psi_0 - L^{+}\Phi_0 - \frac{1}{2}\Psi_0$$

Where $F_0 = Pf + V\Psi_0 - W\Phi_0$

$$\Leftrightarrow T^{+}F_0 = T^{+}[Pf + V\Psi_0 - W\Phi_0]$$

$$\Leftrightarrow T^{+}F_0 - \Psi_0 = T^{+}Pf + w'\Psi_0 - L^{+}\Phi_0 - \frac{1}{2}\Psi_0$$

$$\frac{1}{2}\psi + T^{+}Ru - w'\psi + L^{+}\varphi = T^{+}F_0 - \Psi_0 \quad on\ \partial\Omega \qquad 3.7$$

The above BDIEs (3.6)-(3.7) to gather

$$u + Ru - V\psi + W\varphi = F_0 \ in\ \ \Omega$$

$$\frac{1}{2}\psi + T^{+}Ru - w'\psi + L^{+}\varphi = T^{+}F_0 - \Psi_0 \ on\ \partial\Omega$$

The system (3.6)-(3.7) can be written in the form

$$M^{21}U = F^{21}$$

Where $U := (u, \psi, \varphi)^T \in H^{1,0}(\Omega; A) \times \tilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$

We denote the matrix operator of the left hand side of the systems (M21) as

$$M^{21} = \begin{bmatrix} I + R & -V & W \\ & & \\ T^{+}R & \frac{1}{2}I - w' & L^{+} \end{bmatrix} \qquad 3.8$$

And $F^{21} := [F_0, T^{+}F_0 - \Psi_0]^T$

Formulation of BDIE system (M22)

Using equation (2.40) in Ω we get

$$u + Ru - V(\Psi_0 + \psi) + W(\Phi_0 + \varphi) = Pf \quad in \quad \Omega$$

$$\Leftrightarrow u + Ru - V\psi + W\varphi = Pf + V\Psi_0 - W\Phi_0 \quad in\ \Omega$$

$$\Leftrightarrow u + Ru - V\psi + W\varphi = F_0 \qquad in \quad \Omega \qquad 3.9$$

Where $F_0 = Pf + V\Psi_0 - W\Phi_0$

Taking the restriction (2.42) on $\partial_D\Omega$ we get

$$r_{\partial_D\Omega}[\frac{1}{2}T^+u + T^+Ru - w^{'}T^+u + L^+\gamma^+u = r_{\partial_N\Omega}[T^+Pf] \quad on \ \ \partial_D\Omega$$

$$\Leftrightarrow r_{\partial_D\Omega}[\frac{1}{2}(\Psi_0 + \psi) + T^+Ru - w^{'}(\Psi_0 + \psi) + L^+(\Phi_0 + \varphi)] = r_{\partial_{DN}\Omega}[T^+Pf] \quad on\ \partial_D\Omega$$

$$\Leftrightarrow \frac{1}{2}\Psi_0 + \frac{1}{2}\psi + r_{\partial_D\Omega}T^+Ru - r_{\partial_D\Omega}w^{'}\psi + r_{\partial_D\Omega}L^+\varphi = r_{\partial_{DN}\Omega}[T^+Pf] + r_{\partial_D\Omega}w^{'}\Psi_0 - L^+\Phi_0$$

$$\Leftrightarrow \frac{1}{2}\psi + r_{\partial_D\Omega}T^+Ru - r_{\partial_D\Omega}w^{'}\psi + r_{\partial_D\Omega}L^+\varphi = r_{\partial_{DN}\Omega}[T^+Pf] + r_{\partial_D\Omega}w^{'}\Psi_0 - r_{\partial_D\Omega}L^+\Phi_0 - \frac{1}{2}\Psi_0$$

Where

$$F_0 = Pf + V\Psi_0 - W\Phi_0$$

$$\Leftrightarrow \ T^+F_0 = T^+[Pf + V\Psi_0 - W\Phi_0]$$

$$\Leftrightarrow T^+F_0 = T^+Pf + T^+V\Psi_0 - T^+W\Phi_0$$

$$\Leftrightarrow T^+F_0 = T^+Pf + \frac{1}{2}\Psi_0 + w^{'}\Psi_0 - L^+\Phi_0$$

$$\Leftrightarrow T^+F_0 - \Psi_0 = T^+Pf + w^{'}\Psi_0 - L^+\Phi_0 - \frac{1}{2}\Psi_0$$

$$\Leftrightarrow r_{\partial_D\Omega}[T^+F_0 - \Psi_0] = r_{\partial_D\Omega}[T^+Pf + w^{'}\Psi_0 - L^+\Phi_0 - \frac{1}{2}\Psi_0]$$

$$\Leftrightarrow r_{\partial_D\Omega}T^+F_0 - \Psi_0 = r_{\partial_D\Omega}T^+Pf + r_{\partial_D\Omega}w^{'}\Psi_0 - r_{\partial_D\Omega}L^+\Phi_0 - \frac{1}{2}\Psi_0$$

$$\frac{1}{2}\psi + r_{\partial_D\Omega}T^+Ru - r_{\partial_D\Omega}w^{'}\psi + r_{\partial_D\Omega}L^+\varphi = r_{\partial_D\Omega}T^+F_0 - \Psi_0 \ \ on\ \partial_D\Omega \qquad 3.10$$

- Again by taking the restriction (2.41) on $\partial_N\Omega$

$$r_{\partial_N\Omega}[\frac{1}{2}\gamma^+u + \gamma^+Ru - vT^+u + w\gamma^+u] = r_{\partial_N\Omega}[\gamma^+Pf]$$

$$\Leftrightarrow r_{\partial_N\Omega}\frac{1}{2}\gamma^+u + r_{\partial_D\Omega}\gamma^+Ru - r_{\partial_D\Omega}vT^+u + r_{\partial_D\Omega}w\gamma^+u] = r_{\partial_{ND}\Omega}[\gamma^+Pf]$$

$$\Leftrightarrow\ r_{\partial_N\Omega}\frac{1}{2}(\Phi_0+\varphi)+r_{\partial_N\Omega}\gamma^+Ru-r_{\partial_N\Omega}v(\Psi_0+\psi)+r_{\partial_N\Omega}w(\Phi_0+\varphi)]=r_{\partial_N\Omega}[\gamma^+Pf]$$

$$\Leftrightarrow\ \frac{1}{2}\Phi_0+\frac{1}{2}\varphi+r_{\partial_N\Omega}\gamma^+Ru-r_{\partial_N\Omega}v\psi+r_{\partial_N\Omega}w\varphi=r_{\partial_N\Omega}[\gamma^+Pf]+r_{\partial_N\Omega}v\Psi_0-r_{\partial_N\Omega}w\Phi_0$$

$$\Leftrightarrow\ \frac{1}{2}\varphi+r_{\partial_N\Omega}\gamma^+Ru-r_{\partial_N\Omega}v\psi+r_{\partial_N\Omega}w\varphi=r_{\partial_N\Omega}[\gamma^+Pf]+r_{\partial_N\Omega}v\Psi_0-r_{\partial_N\Omega}w\Phi_0-\frac{1}{2}\Phi_0$$

Where

$$F_0=Pf+V\Psi_0-W\Phi_0$$

$$\Leftrightarrow\ \gamma^+F_0=\gamma^+Pf+\gamma^+V\Psi_0-\gamma^+W\Phi_0$$

$$\Leftrightarrow\ \gamma^+F_0=\gamma^+Pf+v\Psi_0+\frac{1}{2}\Phi_0-w\Phi_0$$

$$\Leftrightarrow\ \gamma^+F_0-\Phi_0=\gamma^+Pf+v\Psi_0-w\Phi_0-\frac{1}{2}\Phi_0$$

$$\Leftrightarrow\ r_{\partial_{N\Omega}}[\gamma^+F_0-\Phi_0]=r_{\partial_{N\Omega}}[\gamma^+Pf+v\Psi_0-w\Phi_0-\frac{1}{2}\Phi_0]$$

$$\Leftrightarrow\ r_{\partial_{N\Omega}}\gamma^+F_0-\Phi_0=r_{\partial_{N\Omega}}\gamma^+Pf+r_{\partial_N\Omega}v\Psi_0-r_{\partial_N\Omega}w\Phi_0-\frac{1}{2}\Phi_0$$

$$\frac{1}{2}\varphi+r_{\partial_N\Omega}\gamma^+Ru-r_{\partial_N\Omega}v\psi+r_{\partial_N\Omega}w\varphi=r_{\partial_N\Omega}\gamma^+F_0-\Phi_0\quad on\quad \partial_N\Omega \qquad 3.11$$

The above BDIEs (3.9)-(3.11) to gather

$$u+Ru-V\psi+W\varphi=F_0\quad in\quad \Omega$$

$$\frac{1}{2}\psi+r_{\partial_D\Omega}T^+Ru-r_{\partial_D\Omega}w'\psi+r_{\partial_D\Omega}L^+\varphi=r_{\partial_d\Omega}T^+F_0-\Psi_0\ on\ \ \partial_D\Omega$$

$$\frac{1}{2}\varphi+r_{\partial_N\Omega}\gamma^+Ru-r_{\partial_N\Omega}v\psi+r_{\partial_N\Omega}w\varphi=r_{\partial_N\Omega}\gamma^+F_0-\Phi_0\quad on\ \ \partial_N\Omega$$

The system (3.9)-(3.11) can be written in the form

$$M^{22}U=F^{22}$$

Where $U := (u,\psi,\varphi)^T \in H^{1,0}(\Omega;A)\times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)\times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$

We denote the matrix operator of the left hand side of the systems (M22) as

$$M^{22} := \begin{bmatrix} I+R & -V & W \\ r_{\partial_D\Omega}T^+R & -r_{\partial_D\Omega}(\frac{1}{2}I-w') & r_{\partial_D\Omega}L^+ \\ r_{\partial_N\Omega}\gamma^+R & -r_{\partial_N\Omega}v & r_{\partial_N\Omega}(\frac{1}{2}I+w) \end{bmatrix}$$

And $F^{22} := \left[F_0, r_{\partial_D\Omega}T^+F_0-\Psi_0, r_{\partial_N\Omega}\gamma^+F_0-\Phi_0\right]^T$

Remark 3.1

Note that the second relation in (2.16) means that if $a = const$ outside a bounded Sub-domain $\Omega' \subset \Omega$, then the operator R acts only on the restriction $\gamma_{\Omega''}u$.

This implies that all the BDIE systems reduce in this case to the BDIEs over Ω' *and* $\partial\Omega$, that are supplemented with the integral representation for u in $\Omega\setminus\Omega'$ given by the first equations of the systems.

The systems (M11), (M12), (M21) and (M22) can be written as

$$M^{\alpha\beta}U = F^{\alpha\beta},$$

Where $F^{\alpha\beta}$ denote their right hand sides and $\alpha,\beta = 1,2.$ If conditions (2.12) and (2.18) hold, then due to the mapping properties of the potentials, $F^{\alpha\beta} \subset X^{\alpha\beta}$ .while the operators $M^{\alpha\beta} : H \to F^{\alpha\beta}$ and $M^{\alpha\beta} : X \to Y^{\alpha\beta}$ are continuous for any $\alpha,\beta = 1,2.$ Here we denoted.

$$F^{11} := H^{1,0}(\Omega;A)xH^{\frac{1}{2}}(\partial_D\Omega)xH^{\frac{-1}{2}}(\partial_N\Omega),$$

$$F^{12} := H^{1,0}(\Omega;A)xH^{\frac{1}{2}}(\partial\Omega),$$

$$F^{21} := H^{1,0}(\Omega;A)xH^{\frac{-1}{2}}(\partial\Omega),$$

$$F^{22} := H^{1,0}(\Omega;A)xH^{\frac{-1}{2}}(\partial_D\Omega)xH^{\frac{1}{2}}(\partial_N\Omega),$$

$$Y^{11} := H^{1,0}(\Omega;A)xH^{\frac{1}{2}}(\partial_D\Omega)xH^{\frac{-1}{2}}(\partial_N\Omega),$$

$$Y^{12} := H^{1}(\Omega)xH^{\frac{1}{2}}(\partial\Omega),$$

$$Y^{21} := H^{1}(\Omega)xH^{-\frac{1}{2}}(\partial\Omega),$$

$$Y^{22} := H^{1,0}(\Omega;A)xH^{\frac{-1}{2}}(\partial_D\Omega)xH^{\frac{1}{2}}(\partial_N\Omega),$$

3.2. Equivalence and Uniqueness Theorems

Let us first prove the equivalence theorems.

Theorem 3.1

Let $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega), \psi_0 \in H^{\frac{-1}{2}}(\partial_N\Omega), f \in L_2(\rho;\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extension of φ_0 and ψ_0, respectively and conditions (2.12), (2.18) hold.

(i) If a function $u \in H^{1,0}(\Omega;A)$ solves the BVP (2.1)-(2.3); then the triplet (u,ψ,φ) ,where $\psi = T^{+}u - \Psi_0 \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega),\ \ \varphi = \gamma^{+}u - \Phi_0 \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),$ 3.12
Solves the BDIE systems (3.1)-(3.3) .

(ii) If a triplet $(u,\psi,\varphi) \in H^{1,0}(\Omega;A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, solves BDIE systems (3.1)- (3.3), then this solution is unique and u solves the BVP (2.1)-(2.3) and relations (3.12).

Proof:

(i) Let $u \in H^{1,0}(\Omega;A)$ be the solution to BVP (2.1)-(2.3). Its unique due to the homogeneous of the BVP (2.1)-(2.3), i.e. with $f = 0, \varphi_0 = 0, \psi_0 = 0$, has only the trivial solution.

Set $\psi = T^{+}u - \Psi_0 \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$, $\varphi = \gamma^{+}u - \Phi_0 \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then immediately follows from (2.40)-(2.42). This completes the proof (i).

(ii) Let now a triple $(u,\psi,\varphi) \in H^{1,0}(\Omega;A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves the BDIEs (3.1)-(3.3).

Taking the trace of (3.1) on $\partial_D\Omega$ using (2.37)-(2.38) and subtracting (3.2) from it,

$$-\begin{cases} r_{\partial_D\Omega}\gamma^+u + r_{\partial_D\Omega}\gamma^+Ru - r_{\partial_D\Omega}v\psi + r_{\partial_D\Omega}w\varphi = r_{\partial_D\Omega}\gamma^+F_0 \\ r_{\partial_D\Omega}\gamma^+Ru - r_{\partial_D\Omega}v\psi + r_{\partial_D\Omega}w\varphi = r_{\partial_D\Omega}\gamma^+F_0 - \varphi_0 \end{cases}$$

We obtain

$$r_{\partial_D\Omega}u = \varphi_0 \qquad on \quad \partial_D\Omega \qquad 3.13$$

i.e. u Satisfies the Dirichlet condition (2.2).

Taking the co-normal derivate of equation (3.1) on $\partial_N\Omega$ using (2.33), (4.39) and subtracting equation (3.3) from it,

$$-\begin{cases} r_{\partial_N\Omega}T^+u + r_{\partial_N\Omega}T^+Ru - r_{\partial_N\Omega}w'\psi + r_{\partial_N\Omega}L^+\varphi = r_{\partial_N\Omega}T^+F_0 \\ r_{\partial_N\Omega}T^+Ru - r_{\partial_N\Omega}w'\psi + r_{\partial_N\Omega}L^+\varphi = r_{\partial_N\Omega}T^+F_0 - \psi_0 \end{cases}$$

We obtain,

$$r_{\partial_N\Omega}T^+u = \psi_0 \qquad on \qquad \partial_N\Omega \qquad 3.14$$

i.e., u satisfies the Neumann condition (2.3). Taking in to account $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D\Omega$

And $\psi = 0, \Psi_0 = \psi_0$ on $\partial_N\Omega$, equation (3.13) and (3.14) imply that the first equation (3.12) is satisfied on $\partial_N\Omega$ and the second (3.12) is satisfied on $\partial_D\Omega$.

Equation (3.1) and lemma 2.1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is the solution of PDE (3.1) and

$$V\Psi^* - W\Phi^* = 0 \qquad in \ \ \Omega,$$

$where\ \Psi^* = \Psi_0 + \psi - T^+u \quad and\ \Phi^* = \Phi_0 + \varphi - \gamma^+u$. Due to the first equation (6.1) on $\partial_N\Omega$ and the second condition (6.1) on $\partial_D\Omega$, Already proved, $\Psi^* \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial_D\Omega)$,. Lemma 2.2 (iii) implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (3.12).

Uniqueness of the to BDIE system (2.1)-(2.3) follows from (3.12) along with remark (3.1) and theorem (2.1)

Theorem 3.2:

Let $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega), \psi_0 \in H^{\frac{-1}{2}}(\partial_N\Omega), f \in L_2(\rho;\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extension of φ_0 and ψ_0, respectively and conditions (2.12), (2.18) hold.

(i) If a function $u \in H^{1,0}(\Omega;A)$ solves the BVP (2.1)-(2.3); then the triplet (u,ψ,φ),

where $\psi = T^+u - \Psi_0 \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega),, \quad \varphi = \gamma^+u - \Phi_0 \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),$
3.15 Solves the BDIE systems (3.4)-(3.5).

(ii) If a triplet $(u,\psi,\varphi) \in H^{1,0}(\Omega;A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, solves BDIE systems (3.4)- (3.5), then this solution is unique and u solves the BVP (2.1)-(2.3) and relations (3.15) hold.

Proof:

(i) Let $u \in H^{1,0}(\Omega;A)$ be the solution to BVP (2.1)-(2.3). Its unique, due to the homogeneous of the BVP (2.1)-(2.3), i.e. with $f = 0, \varphi_0 = 0, \psi_0 = 0$, has only the trivial solution.

Set $\psi = T^+u - \Psi_0 \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$, $\varphi = \gamma^+u - \Phi_0 \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$,. Then immediately follows from (2.40)-(2.42). This completes the proof (i).

(iii) Let now a triple $(u,\psi,\varphi) \in H^{1,0}(\Omega;A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves the BDIEs (3.4)-(3.5).
Using the properties of the single and double layer potentials take the trace of (3.4) on $\partial\Omega$ and subtract it from equation (3.5)

$$-\begin{cases} \frac{1}{2}\varphi + \gamma^+Ru - v\psi + w\varphi = \gamma^+F_0 - \Phi_0 \\ \gamma^+u + \gamma^+Ru - v\psi - \frac{1}{2}\varphi + w\varphi = \gamma^+F_0 \end{cases}$$

We obtain

$$\gamma^+u = \Phi_0 + \varphi \quad on \ \partial\Omega \qquad 3.16$$

This means that the second equation in (3.15) holds. Since $\varphi(y)=0 \quad on\ \partial_D\Omega\ \ and\ \Phi_0=\varphi_0\ on$ $\partial_D\Omega$, we see that the Dirichlet condition (2.1) is satisfied.

Equation (3.4) and lemma 2.1 with $\Psi=\psi+\Psi_0$, $\Phi=\varphi+\Phi_0$ imply that u is the solution of equation (2.1) and

$$V(\Psi_0+\psi-T^+u)-W(\Phi_0+\varphi-\gamma^+u)=0 \quad in\ \Omega \qquad 3.17$$

Due to (3.16) the second equation in (3.17) vanishes, and by lemma 2.2 (i) WE obtain

$$\Psi_0+\psi-T^+u=0\ \ on\ \ \partial\Omega \qquad 3.18$$

This means that the first equation (3.15) is satisfied as well. Since $\psi=0\ on\ \partial_N\Omega\ \ and\ \Psi_0=\psi_0$ *on*

$\partial_N\Omega$, equation (3.18) implies that u satisfies the Neumann boundary condition (2.3).

Unique solvability of the BDIEs (3.4)-(3.5) then follows from remark (5.5).The unique solvability of the BVP (2.1)-(2.3) (from theorem 2.1), and (.15).

Theorem 3.3

Let $\varphi_0\in H^{\frac{1}{2}}(\partial_D\Omega),\psi_0\in H^{\frac{-1}{2}}(\partial_N\Omega),\ f\in L_2(\rho;\Omega)$ and let $\Phi_0\in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0\in H^{-\frac{1}{2}}(\partial\Omega)$ be some extension of φ_0 and ψ_0, respectively and conditions (2.12), (2.18) hold.

(i) If a function $u\in H^{1,0}(\Omega;A)$ solves the BVP (2.1)-(2.3); then the triplet (u,ψ,φ) ,where $\psi=T^+u-\Psi_0\in\widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega), \quad \varphi=\gamma^+u-\Phi_0\in\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),$
3.19
Solves the BDIE systems (3.9)-(3.11).

(ii) If a triplet $(u,\psi,\varphi)\in H^{1,0}(\Omega;A)\times\widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, solves BDIE systems (3.9)- (3.11), then this solution is unique and u solves the BVP (2.1)-(2.3) and relations (3.19).

Proof:

(i) Let $u\in H^{1,0}(\Omega;A)$ be the solution to BVP (3.1)-(3.3). It's unique, due to the homogeneous of the BVP (3.1)-(3.3), i.e. with $f=0,\varphi_0=0,\psi_0=0$, has only the trivial solution.

Set $\psi = T^{+}u - \Psi_0 \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega),$, $\varphi = \gamma^{+}u - \Phi_0 \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega),$. Then immediately follows from (2.40)-(2.42). This completes the proof (i).

(ii) Let now a triplet $(u,\psi,\varphi) \in H^{1,0}(\Omega; A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, solves BDIE systems (3.9)- (3.11),

Take the co-normal derivative of equation (3.9) on $\partial_D\Omega$ and subtract it from equation (3.10)

$$-\begin{cases} \frac{1}{2}\psi + r_{\partial_D\Omega}T^{+}Ru - r_{\partial_D\Omega}w'\psi + r_{\partial_D\Omega}L^{+}\varphi = r_{\partial_D\Omega}T^{+}F_0 - r_{\partial_D\Omega}\Psi_0 \\ r_{\partial_D\Omega}T^{+}u + r_{\partial_D\Omega}T^{+}Ru - \frac{1}{2}\psi - r_{\partial_D\Omega}w'\psi + r_{\partial_D\Omega}L^{+}\varphi = r_{\partial_D\Omega}F_0 \end{cases}$$

We obtain

$$\psi = r_{\partial_D\Omega}T^{+}u - r_{\partial_D\Omega}\Psi_0 \quad on \quad \partial_D\Omega \qquad 3.20$$

Further, take equation (3.9) on $\partial_N\Omega$ and subtract it from equation (3.11).

$$-\begin{cases} \frac{1}{2}\varphi + r_{\partial_N\Omega}\gamma^{+}Ru - r_{\partial_N\Omega}v\psi + r_{\partial_N\Omega}w\varphi = r_{\partial_N\Omega}\gamma^{+}F_0 - r_{\partial_N\Omega}\Phi_0 \\ r_{\partial_N\Omega}\gamma^{+}u + r_{\partial_N\Omega}\gamma^{+}Ru - r_{\partial_D\Omega}v\psi + r_{\partial_N\Omega}w\varphi - \frac{1}{2} = r_{\partial_N\Omega}F_0 \end{cases}$$

We obtain

$$\varphi = r_{\partial_N\Omega}\gamma^{+}u - r_{\partial_N\Omega}\Phi_0 \quad on \quad \partial_N\Omega \qquad 3.21$$

Equation (3.20) and (3.21) imply that the first equation in (3.19) is satisfied on $\partial_D\Omega$ and the second equation (3.19) is satisfied on $\partial_N\Omega$.

Equation (3.9) and Lemma 2.1 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is the solution of equation (3.1) *and* $V\Psi^{*}(y) - W\Phi^{*}(y) = 0 \qquad y \in \Omega$, *where* $\Psi^{*} = \Psi_0 + \psi - T^{+}u$ *and* $\Phi^{*} = \Phi_0 + \varphi - \gamma^{+}u$.

Due to (3.20) and (3.21), we have $\Psi^{*} \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$, $\Phi^{*} \in \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Lemma 2.2 (iii) implies $\Psi^{*} = \Phi^{*} = 0$ which completes the proof of conditions (3.19) the whole boundary $\partial\Omega$.

Taking in to account that $\varphi = 0 \quad on\ \partial_D\Omega \quad \Phi_0 = \varphi_0$ on $\partial_D\Omega$, and $\psi = 0 \quad on \quad \partial_N\Omega$ and

$\Psi_0 = \psi_0 \quad on\ \partial_N\Omega$, equations (3.19) imply the boundary conditions (2.2) and (2.3).

Unique solvability of (3.9)-(3.11) then follows from remark 3.1, the unique solvability of BVP (2.1)-(2.3) from (3.19).

The situation with uniqueness and equivalence for the (M21) differs from the one from the other systems.

Theorem 3.4:

Let $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega), \psi_0 \in H^{\frac{-1}{2}}(\partial_N\Omega), f \in L_2(\rho;\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extension of φ_0 and ψ_0, respectively and conditions (2.12), (2.18) hold

(i) Homogeneous BDIE system (M21)admits only one linearly independent solution $(u^0, \psi^0, \varphi^0) \in H^{1,0}(\Omega; A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, $where\ \ u^0$ is the solution of the mixed BVP

$$Au^0 = 0 \qquad in\ \Omega \qquad 3.22$$

$$r_{\partial_D\Omega}\gamma^+ u^0 = \frac{1}{a(x)} \qquad on \quad \partial_D\Omega \qquad 3.23$$

$$r_{\partial_N\Omega}T^+ u^0 = 0 \qquad on \qquad \partial_N\Omega \qquad 3.24$$

While

$$\psi_0 = T^+u^0, \varphi_0 = \gamma^+u^0 - {1}/{a(x)} \qquad on \qquad \partial\Omega \qquad 3.25$$

(ii) The non homogeneous (M21)is solvable , and any its solution $(u, \psi, \varphi) \in H^{1,0}(\Omega; A) \times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ Can be represented as

$$u = \widetilde{u} + Cu^0 \quad in \quad \Omega \qquad 3.26$$

$where \quad \widetilde{u} \quad solves\ the \quad BVP\ (2.1-(2.3)\ and\ C \quad is\ a\ cons\tan t,\ while$

$$\psi = T^+\widetilde{u} - \Psi_0 + C\psi_0, \qquad \varphi = \gamma^+\widetilde{u} - \Phi_0 + C\varphi_0 \ \text{on} \quad \partial\Omega \qquad 3.27$$

Proof:

Problem (3.22)-(3.24) is uniquely solvable in $H^{1,0}(\Omega;A)$ by theorem (2.1). Consequently the third Green identity (2.13) is applicable to u^0 leading to

$$u^0 + Ru^0 - V\psi^0 + W\varphi^0 = 0 \qquad in\ \ \Omega \qquad 3.28$$

With notations (3.25), if we take into account that $W(1/a(x)) = 0\ \ in\ \Omega$ due to the second relation in (2.17) and the equality $W_\Delta 1 = 0\ \ in\ \Omega$ (proof of lemma 2.2(ii)). Taking the co-normal derivative (3.28)

$$i.e.\ \ T^+[u^0 + TRu^0 - V\psi^0 + W\varphi^0 = 0] \quad on\ \ \partial\Omega$$

$$\Leftrightarrow T^+u^0 + T^+Ru^0 - \frac{1}{2}\psi^0 - w'\psi^0 + L^+\varphi^0 = 0\ \ on\ \partial\Omega$$

And substituting the first equation of (3.25) again we arrive at

$$\Leftrightarrow \quad \frac{1}{2}\psi^0 + T^+Ru^0 - w'\psi^0 + L^+\varphi^0 = 0\ on\ \partial\Omega \qquad 3.29$$

Equation (3.28)-(3.29) mean that the triplet (u^0,ψ^0,φ^0) solves the homogeneous BDIE system (M21).

To prove item (ii) and check that there exists only one linearly independent solution of the homogeneous BDIE system (M21), we proceed as follows. First we remark that the solvability of the non-homogeneous system (M21) follows from the solvability of the BVP (2.1)-(3.3) in $H^{1,0}(\Omega;A)$ and the deduction of the system (M21).

Let now a triplet $(u,\psi,\varphi)^T \in H^{1,0}(\Omega;A)\times \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)\times \widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, solves (generally non-homogeneous) BDIE systems (M21). Take the co-normal derivative of equation (3.6) on

And subtract it from equation (3.7).

$$-\begin{cases} \frac{1}{2}\psi + T^+Ru - w'\psi + L^+\varphi = T^+F_0 - \Psi_0 \\ \gamma^+u + T^+Ru - w'\psi - \frac{1}{2}\psi + L^+\varphi = T^+F_0 \end{cases}$$

We obtain

$$\Psi_0 + \psi - T^+ u = 0 \quad on \;\; \partial\Omega \qquad 3.30$$

Taking into account $\psi = 0 \; on \; \partial_N\Omega$ and $\Psi_0 = \psi_0 \;\; on \;\; \partial_N\Omega$, this implies that u satisfies the Neumann condition (2.3).And equation (3.6) and Lemma 2.2 with $\Psi = \psi + \Psi_0$, $\Phi = \varphi + \Phi_0$ imply that u is a solution of equation (2.1) and

$$V(\Psi_0 + \psi - T^+ u) - W(\Phi_0 + \varphi - \gamma^+ u) = 0 \quad in \;\; \Omega \qquad 3.31$$

Due to (3.30) the first term vanishes in (3.31), and by Lemma 2.2 (ii) we obtain

$$\Phi_0 + \varphi - \gamma^+ u = -C/a(x) \qquad on \;\; \partial\Omega \qquad 3.32$$

Where C is a constant.Taking into account $\varphi = 0 \;\; on \;\; \partial_D\Omega \;\; and \; \Phi_0 = \varphi_0 \;\; on \;\; \partial_D\Omega$, we conclude that u satisfies the Dirichlet condition (2.2).

$$\gamma^+ u = \varphi^0 + C/a(x) \; on \; \partial\Omega \qquad 3.33$$

instead of (2.1). Introducing notation $\tilde{u}$ by (3.26) in (3.30) prove the claim of item (ii).

The case $\varphi_0 = 0, \Phi_0 = 0, \psi_0 = 0, \Psi_0 = 0, f = 0$ leading to the homogeneous BVP system (M21) also implies the $\tilde{u}$ for this case satisfies homogeneous BVP (2.1)-(2.3) and thus $\tilde{u} = 0$ in (3.17) and (3.18) meaning that the triplet (u^0, ψ^0, φ^0) is the only linearly independent solution of the homogeneous BDIE system (M21).

This completes the proof of item (ii) and of the whole theorem.

3.3. BDIO Fredholm Properties and Invertibility

We will consider in this section the Fredholm properties and invertibility of the boundary-domain integral operators (BDIOs), starting from $M^{\alpha\beta} : H \to F^{\alpha\beta}$ and then, under more restrictive conditions on the coefficient a, of the operators $M^{\alpha\beta} : X \to F^{\alpha\beta}$, $\alpha, \beta = 1,2$.

3.3.1 Properties of operators $M^{\alpha\beta} : H \to F^{\alpha\beta}$

In this section, we will analyze the operator invertibility (or the Fredholm property when there is no invertibility) by proving first that the arbitrary right hand side functions from the corresponding spaces can be represented in terms of the parametrix-based potentials and using then the equivalence theorems.

To start with, Let us prove the following for exterior domain.

Lemma 3.1

For any function $F_{\bullet} \in H^{1,0}(\Omega; A)$, there exists a unique couple $(f_*, \Psi_*) = CF_* \in L_2(\Omega)xH^{\frac{-1}{2}}(\partial\Omega)$ such that

$$F_*(y) = pf_*(y) + V\Psi_*(y), \qquad y \in \Omega \tag*{3.34}$$

Where $C : H^{1,0}(\Omega; A) \to L_2(\rho; \Omega)xH^{\frac{-1}{2}}(\partial\Omega)$ is linear bounded operator.

Proof:

Suppose first there exist some functions $f_*(y)$ and $\Psi_*(y)$ satisfying (3.34) and find their expressions in terms of $F_*(y)$. Taking into account relations (2.17) and (2.18) for the volume and single layer potentials, can be rewritten as

$$a(y)F_*(y) = p_\Delta f_*(y) + V_\Delta \Psi_*(y), \qquad y \in \Omega \tag*{3.35}$$

Applying the Laplace operator to (3.35) we obtain that

$$f_* = \Delta(aF_*) \quad in \;\; \Omega \tag*{3.36}$$

Then (3.35) can be rewritten as

$$V_\Delta \Psi_*(y) = Q(y),\; y \in \Omega \tag*{3.37}$$

Where

$$Q(y := a(y)F_*(y) - p_\Delta[\Delta(aF_*)](y), y \in \Omega \tag*{3.38}$$

The trace of (3.37) on the boundary gives

$$v_\Delta \Psi_*(y) = \gamma^+ Q(y), \qquad y \in \partial\Omega \tag*{3.39}$$

Where $v_\Delta := v/_{a=1}$ is the direct value on $\partial\Omega$ of the single layer operator V_Δ associated with the Laplace operator. Since $v_\Delta : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$, $s \in R,$ is isomorphism. We obtain the following expression for Ψ_*

$$\Psi_*(y) = v_\Delta^{-1}\gamma^+ Q(y),\; y \in \partial\Omega \tag*{3.40}$$

Relations (3.35) and (3.40) imply uniqueness of the couple f_*, Ψ_*.

Now we have to prove that $f_*(y), \Psi_*(y)$ give by (3.36) and (3.40) satisfy (3.34). Indeed, the potential $V_\Delta \Psi_*(y)$ with $\Psi_*(y)$ given by (3.40) is a harmonic function, and one can check that Q given by (3.38) is also harmonic in Ω .Then condition (3.39) implies that $V_\Delta \Psi_*(y)$ and $Q(y)$ coincide in the Ω holds true, which implies (3.34). Thus (3.36)-(3.40) and (3.38)give

$$(f_*, \Psi_*) = CF_* := \left(\Delta(aF_*), v_\Delta^{-1}\gamma^+\left[aF_* - p_\Delta \Delta(aF_*)\right]\right),$$

And thus by remark 2.1 $C : H^{1,0}(\Omega; A) \to L_2(\rho; \Omega) x H^{\frac{-1}{2}}(\partial\Omega)$ is bounded operator.

Corollary 3.1

A couple $(F_0, F_1) \in H^{1,0}(\Omega; A) x H^{\frac{1}{2}}(\partial\Omega)$ can be uniquely represented as

$$F_0 = pf_* + V\Psi_* - W\Phi_* \; in \; \Omega \qquad 3.41$$

$$F_1 = \gamma^+ F_0 - \Phi_* \;\; on \; \partial\Omega \qquad 3.42$$

For some $(f_*, \Psi_*, \Phi_*) = C_*(F_0, F_1)^T$, where

$C_* : H^{1,0}(\Omega; A) x H^{\frac{1}{2}}(\partial\Omega) \to L_2(\rho; \Omega) x H^{\frac{-1}{2}}(\partial\Omega) x H^{\frac{-1}{2}}(\partial\Omega) x H^{\frac{1}{2}}(\partial\Omega)$ is a linear bounded operator.

Proof:

Taking $\Phi_* = \gamma^+ F_0 - F_1$ and applying Lemma 3.1 to $F_* = F_0 + W\Phi_*$, We prove existence of representation (3.41)-(3.42). To prove its uniqueness, we consider its homogeneous case, i.e., with $F_0 = 0, F_1 = 0$. Then (3.42) implies $\Phi_* = 0$ and thus by (3.41) and Lemma 3.1 we also obtain $\Psi_* = 0, f_* = 0$.

Using essentially the same reasoning as in Lemma 3.1 and corollary 3.1, one can prove the following statement that is similar to its counterpart for bounded domains.

Lemma 3.2

Let $\partial\Omega = \overline{\partial_D \Omega} \cup \overline{\partial_N \Omega}$, where $\partial_D \Omega$ and $\partial_N \Omega$ are nonempty nonintersecting simply connected sub manifolds of $\partial\Omega$ with infinitely smooth boundaries. For an arbitrary triplet

$$F = (F_0, F_1, F_2)^T \in H^{1,0}(\Omega; A) x H^{\frac{-1}{2}}(\partial_N \Omega) x H^{\frac{1}{2}}(\partial_D \Omega)$$

There exists a unique triplet

$$(f_*, \Psi_*, \Phi_*)^T = C_{\partial_N\Omega,\partial_D\Omega} F \in L_2(\rho;\Omega) x H^{\frac{-1}{2}}(\partial\Omega) x H^{\frac{1}{2}}(\partial\Omega) \qquad 3.43$$

Such that

$$F_0 = pf_* + V\Psi_* - W\Phi_* \;\; in \;\; \Omega^+ \qquad 3.44$$

$$F_1 = r_{\partial_N\Omega} T^+ F_0 - r_{\partial_N\Omega}\Psi_* \quad on \; \partial_N\Omega \qquad 3.45$$

$$F_2 = r_{\partial_D\Omega}\gamma^+ F_0 - r_{\partial_D\Omega}\Phi_* \;\; on \; \partial_D\Omega \qquad 3.46$$

Where

$C_{\partial_N\Omega,\partial_D\Omega} : H^{1,0}(\Omega;A) x H^{\frac{-1}{2}}(\partial\Omega) x H^{\frac{1}{2}}(\partial\Omega) \to L_2(\rho;\Omega) x H^{\frac{-1}{2}}(\partial\Omega) x H^{\frac{1}{2}}(\partial\Omega)$ is linear bounded operator.

Theorem 3.5

If conditions (2.12) and (2.18) hold, then the operators

$$M^{11} : H \to F^{11},\; M^{12} : H \to F^{12},\; M^{22} : H \to F^{22}, \qquad 3.47$$

are continuous and continuously invertible.

Proof:

Continuity of operators (3.47) follows from the volume and boundary potential mapping properties, Theorem 2.1.

Let us prove continuous invertibility of operator $M^{11} : H \to F^{11}$, By Lemma 3.2, any right hand side $F^{11} = (F_0, F_D, F_N) \in F^{11}$ of the equation $M^{11}U = F^{11}$ can be uniquely represented in form (3.44)-(3.46) , where $(f_*, \Psi_*, \Phi_*)^T = C_{\partial_N\Omega,\partial_D\Omega} F^{11}$ and the operator

$C_{\partial_N\Omega,\partial_D\Omega} : F^{11} = H^{1,0}(\Omega;A) x H^{\frac{1}{2}}(\partial_D\Omega) x H^{\frac{-1}{2}}(\partial_N\Omega) \to L_2(\rho;\Omega) x H^{\frac{-1}{2}}(\partial\Omega) x H^{\frac{1}{2}}(\partial\Omega)$ is continuous.

Then the equivalence Theorem 3.1 for the system (M11) and invertibility theorem 3.1 for the mixed problem imply that the equation $M^{11}U = F^{11}$ has a solution $U = (u,\psi,\varphi)^T = (M^{11})^{-1} F^{11}$, where the operator $(M^{11})^{-1} : F^{11} \to H$ is given by

$$u = A_M^{-1}\left[f_*, r_{\partial_D\Omega}\Phi_*, r_{\partial_N\Omega}\Psi_*\right]^T, \psi = T^+u - \Psi_*,\ \varphi = \gamma^+u - \Phi_*, \qquad 3.48$$

Where $(f_*, \Psi_*, \Phi_*)^T = C_{\partial_N\Omega, \partial_D\Omega}F^{11}$, and is evidently continuous. Thus the operators $(M^{11})^{-1}$ is the right inverse of the operator $M^{11} : H \rightarrow F^{11}$, but due to the injectivity of the latter implied by the equivalence theorem 3.1, the operator $(M^{11})^{-1}$ is in fact the two-side inverse.

Continuous invertibility of the operator $M^{22} : H \rightarrow F^{22}$, is proved similarly.

Let us prove the continuous invertibility of $M^{12} : H \rightarrow F^{12}$. By corollary 3.1, any right hand side $F^{12} = (F_0, F_1) \in F^{12}$ of equation $M^{12}U = F^{12}$ can be uniquely represented in form (3.41)-(3.42) for some $(f_*, \Psi_*, \Phi_*)^T = C_*F^{12}$, where the operator

$C_* : F^{12} = H^{1,0}(\Omega; A)xH^{\frac{1}{2}}(\partial\Omega) \rightarrow L_2(\rho;\Omega)xH^{\frac{-1}{2}}(\partial\Omega)xH^{\frac{-1}{2}}(\partial\Omega)xH^{\frac{1}{2}}(\partial\Omega)$ is continuous.

Then the equivalence Theorem 3.2 for the system (M12) and invertibility Theorem 3.1 for the mixed problem imply that the equation $M^{12}U = F^{12}$ has a solution $U = (u, \psi, \varphi)^T = (M^{12})^{-1}F^{12}$, where the operator $(M^{12})^{-1} : F^{12} \rightarrow H$ is given by expressions (3.48), where $(f_*, \Psi_*, \Phi_*)^T = C_*F^{12}$, and is evidently continuous. Thus the operator $(M^{12})^{-1}$ is the right inverse operator $M^{12} : H \rightarrow F^{12}$, but due to the injectivity of the latter implied by the equivalence theorem 3.2, the operator $(M^{12})^{-1}$ is in fact the two-side inverse.

Let us prove an assertion implied by Theorem 7.1 for the operator $M^{22} : H \rightarrow F^{22}$ for the particular case $a = 1 \text{ in } \Omega$, i.e., essentially for the purely boundary integral equation. We will need it to prove invertibility of the operator $M^{22} : X \rightarrow Y^{22}$ for variable a in section 3.3.2.

If $a = 1 \text{ in } \Omega$, then (2.1) becomes the classical Laplace equation, the remainder operator R vanishes, and the BDIE system (M22) splits into the system of two Boundary Integral Equations (BDIEs),

$$r_{\partial_D\Omega}\left(\frac{1}{2}\psi - w'_\Delta\psi + L_\Delta^+\varphi\right) = r_{\partial_D\Omega}T^+F_0 - r_{\partial_D\Omega}\Psi_0 \ \ on\ \partial_D\Omega, \qquad 3.49$$

$$r_{\partial_N\Omega}\left(\frac{1}{2}\varphi - v_\Delta\psi + w_\Delta\varphi\right) = r_{\partial_N\Omega}\gamma^+F_0 - r_{\partial_N\Omega}\Phi_0 \ \ on\ \ \partial_N\Omega \qquad 3.50$$

And the representation formula for u in terms of φ and ψ,

$$u = F_0 + V_\Delta\psi - W_\Delta\varphi \ \ in\ \Omega.$$

System (3.49)-(3.50) can be rewritten in the form

$$\hat{M}{}_{\Delta}^{22}\,\hat{U} = \hat{F}{}_{\Delta}^{22}, \qquad 3.51$$

Where $\hat{U}_{\Delta}^{T} := (\psi,\varphi) \in \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega)$,

$$\hat{M}{}_{\Delta}^{22} := \begin{bmatrix} r_{\partial_D\Omega}\left(\frac{1}{2}I - w_{\Delta}^{'}\right) & r_{\partial_D\Omega}L_{\Delta}^{+} \\ -r_{\partial_N\Omega}v_{\Delta} & r_{\partial_N\Omega}\left(\frac{1}{2}I + w_{\Delta}\right)\end{bmatrix}, \qquad 3.52$$

$$\hat{F}{}_{\Delta}^{22} := \begin{bmatrix} r_{\partial_D\Omega}T^{+}F_0 - r_{\partial_D\Omega}\Psi_0 \\ r_{\partial_N\Omega}\gamma^{+}F_0 - r_{\partial_N\Omega}\Phi_0 \end{bmatrix} \in H^{\frac{-1}{2}}(\partial_D\Omega)xH^{\frac{1}{2}}(\partial_N\Omega)$$

Moreover, the operator $\hat{M}_{\Delta}^{22} : \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{\frac{-1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega)$ is bounded and by Theorem 3.3 (employed for $a=1$) is also injective.

Theorem 3.6

The operator $\hat{M}_{\Delta}^{22} : \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)\times\widetilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{\frac{-1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega)$ is continuous invertible.

Proof:

A solution of system (3.51) with an arbitrary $\left(\hat{F}{}_{\Delta}^{22}\right)^{T} = \left(F_{1\Delta}^{22}, F_{2\Delta}^{22}\right) \in H^{\frac{-1}{2}}(\partial_D\Omega)xH^{\frac{1}{2}}(\partial_N\Omega)$ is delivered by the couple (ψ,φ) satisfying the extended system $M_{\Delta}^{22}U = F_{\Delta}^{0}$, where $U = (u,\psi,\varphi)^{T}, F_{\Delta}^{0} = (0, F_{1\Delta}^{22}, F_{2\Delta}^{22})^{T}$, and

$$M_{\Delta}^{22} := \begin{bmatrix} I & -V_{\Delta} & W_{\Delta} \\ 0 & r_{\partial_D\Omega}(\frac{1}{2}I - w_{\Delta}^{'}) & r_{\partial_D\Omega}L_{\Delta}^{+} \\ 0 & -r_{\partial_N\Omega}v_{\Delta} & r_{\partial_N\Omega}(\frac{1}{2}I + w_{\Delta}) \end{bmatrix}.$$

The operator $M_{\Delta}^{22} : H \to F^{22}$ has a continuous inverse due to Theorem 3.5 for $a=1$.

Consequently, the operator $\hat{M}{}_{\Delta}^{22}$ has a right bounded inverse, which is also a two-side inverse due to injectivity of the operator $\hat{M}{}_{\Delta}^{22}$. To analyze properties of the operator M^{21}, we will need the following assertion that appeared to be quite different from its counterpart for interior domains proved in [2].

Lemma 3.3

If the conditions (2.12) and (2.18) hold, then a function $F_* \in H^{1,0}(\Omega; A)$ can represented as

$$F_*(y) = pf_*(y) - W\Phi_*(y), \qquad y \in \Omega \qquad 3.53$$

For some $(f_\bullet, \Phi_\bullet) \in L_2(\rho;\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ if and only if

$$\int_{\partial\Omega} T_\Delta^+(aF_*)dS = 0 \qquad 3.54$$

Proof:

Suppose first there exists some functions $f_*(y)$ and $\Phi_*(y)$ satisfying (3.53).

Taking into account relations (2.16) and (2.17) for the Newtonian-type and double layer potentials, ansatz (3.53) can be rewritten as

$$a(y)F_*(y) = p_\Delta f_*(y) - W_\Delta[a\Phi_*](y), \qquad y \in \Omega \qquad 3.55$$

Applying the Laplace operator to (3.55) we obtain that

$$f_* = \Delta(aF_*) \qquad in \ \ \Omega. \qquad 3.56$$

Then (3.55) can be rewritten as

$$W_\Delta[a\Phi_*](y) = Q(y), \quad y \in \Omega \qquad 3.57$$

Where

$$Q(y) := p_\Delta[\Delta(aF_*)](y) - a(y)F_*(y), \qquad y \in \Omega \qquad 3.58$$

The trace of (3.57) on the boundary gives

$$\left[\frac{-1}{2}I + w_\Delta\right](a\Phi_*) \qquad on \qquad \partial\Omega \qquad 3.59$$

Equation (3.59) admits a solution $a\Phi_* \in H^{\frac{1}{2}}(\partial\Omega)$ if and only if the right hand side $\gamma^+ Q \in H^{\frac{1}{2}}(\partial\Omega)$

Satisfies the condition

$$\int_{\partial\Omega} \gamma^+ Q(x) T_\Delta^+ v(x) dS_x = 0, \qquad 3.60$$

Where $v \in H^1(\Omega)$ solves the Dirichlet problem $\Delta v = 0$ in Ω, $\gamma^+ v = 1 \;\; on \;\; \partial\Omega$.

Employing the second Green identity (1.7) associated with the operator Δ and substituting there (3.58), we have

$$\int_{\partial\Omega} T_\Delta^+ \{ p_\Delta[\Delta(aF_*)] - aF_* \} dS = 0\,. \qquad 3.61$$

We have $T_\Delta^+ p_\Delta[\Delta(aF_*)] = T_\Delta^- p_\Delta[\Delta(aF_*)]$ on $\partial\Omega$ since $p_\Delta[\Delta(aF_*)] \in H^{1,0}(R^3;\Delta) \subset H^2_{loc}(R^3)$ by Theorem 2.2. Keeping in mind that $p_\Delta[\Delta(aF_*)]$ is a harmonic function in bounded Ω^-, we obtain

$$\int_{\partial\Omega} T_\Delta^+ p_\Delta[\Delta(aF_*)] dS = 0\,,$$

Which reduces (3.61)-(3.54).

Let now (3.54) be satisfied. We have to prove that there exists a representation (3.53). First of all, let us note that if $F_* \in H^{1,0}(\Omega; A),$ then conditions (2.12) and (2.18) imply $aF_* \in H^{1,0}(\Omega;\Delta),$ and the co-normal derivative $T_\Delta^+(aF_*)$ is well defined on $\partial\Omega$. Then (3.54) implies (3.61).

Let $a\Phi_* \in H^{\frac{1}{2}}(\partial\Omega)$ be a solution of (3.59) with Q given by (3.58), while $f_* \in L_2(\rho;\Omega)$ be given by (3.56). Then the potential $W_\Delta[a\Phi_*] \in H^1(\Omega)$ is a harmonic function, and one can check that $Q \in H^1(\Omega)$ is also harmonic. Since (3.59) implies that they coincide on the boundary, the two harmonic functions should coincide also in the domain, which implies (3.53).

Lemma 3.3 implies the following corollary

Corollary 3.2

If conditions (2.12) and (2.18) hold, then a couple $(F_0, F_1) \in H^{1,0}(\Omega; A) x H^{\frac{-1}{2}}(\partial\Omega)$ can be represented as

$$F_0(y) = pf_*(y) + V\Psi_*(y) - W\Phi_*(y), \qquad y \in \Omega, \qquad 3.62$$

$$F_1(y) = T^+F_0(y) - \Psi_*(y), \quad y \in \partial\Omega \qquad 3.63$$

For some $(f_*, \Psi_*, \Phi_*) \in L_2(\rho;\Omega)xH^{\frac{-1}{2}}(\partial\Omega)xH^{\frac{1}{2}}(\partial\Omega)$ if and only if

$$f(F_0, F_1) = \int_{\partial\Omega}[(\partial_n a)\gamma^+F_0 + F_1]dS = 0. \qquad 3.64$$

Proof:

We take $\Psi_* = T^+F_0 - F_1$ and apply Lemma 3.3 to $F_* = F_0 - V\Psi_*$, which proves representation (3.61) if and only if

$$\int_{\partial\Omega} T_\Delta^+[a(F_0 - V(T^+F_0 - F_1))]dS = 0. \qquad 3.65$$

Taking in to account the jump property of single layer potential and that $aVg \equiv V_\Delta g$ is a harmonic function in the bounded domain Ω^-, condition (3.65) reduces to

$$0 = \int_{\partial\Omega}[(T_\Delta^+a)\gamma^+F_0 + aT_\Delta^+F_0 - T^+F_0 + F_1]dS - \int_{\partial\Omega} T_\Delta^+V_\Delta(T^+F_0 - F_1)dS$$

$$= \int_{\partial\Omega}[(\partial_n a)\gamma^+F_0 + F_1]dS.$$

One can check on the example $F_1 = T^+F_0$ that condition (3.65) and thus (3.64) is satisfied not for all $(F_0, F_1) \in H^{1,0}(\Omega; A)xH^{\frac{-1}{2}}(\partial\Omega)$.

Theorem 3.7

If condition (2.12) and (2.18) hold, then the operator $M^{21} : H \to F^{21}$ is a continuous Freedom operator with zero index. It has one dimensional null-space spanned over the element (u^0, ψ^0, φ^0) defined in Theorem 3.4(i) and the cokernel spanned over the function f defined by (3.64).

Proof:

The claim about the null-space particularly that its dimension is 1, follows from Theorem 3.4(i)

Let us now consider the equation $M^{21}U = (F_0, F_1)^T$,

i.e., $u + Ru - V\psi + W\varphi = F_0$ in Ω,

$$\frac{1}{2}\psi + T^+Ru - w'\psi + L^+\varphi = F_1 \ \ on \ \ \partial\Omega.$$

With arbitrary $(F_0, F_1) \in F^{21}$ for $(u, \psi, \varphi) \in H$. By corollary 3.2 , if $f(F_0, F_1) = 0$, where the linear functional $f \in F^{21*}$ is defined in (3.64), then the right hand side is represent able in form (3.62)-(3.63) and the equation is solvable due to Theorem 3.4 (ii) .

On the other hand, we have from (3.8), the jump Theorem 2.2 and Lemma

$$f(M^{21}(u,\psi,\varphi)^T) = \int_{\partial\Omega} T_\Delta^+\left\{a\left[u + Ru - V\psi + W\varphi - V\left(T^+(u + Ru - V\psi + W\varphi) - (\frac{1}{2}\psi + T^+Ru - w'\psi + L^+\varphi)\right)\right]\right\}dS$$
$$= \int_{\partial\Omega} T_\Delta^+\{a[u + Ru + W\varphi - VT^+u]\}dS = \int_{\partial\Omega} T_\Delta\{apAu\}dS. \qquad 3.66$$

Since $u \in H^{1,0}(\Omega; A)$, by Theorem 2.2 we have $apAu \in H^{1,0}(R^3; A)$ and thus $apAu \in H^{1,0}(R^3; \Delta) \subset H^2_{loc}(R^3)$. This implies that $T_\Delta^+\{apAu\} = T_\Delta^-\{apAu\}$ on $\partial\Omega$ and the last integral (3.66) is zero because $apAu$ is harmonic in the bounded domain Ω^- . Thus the range of the operator $M^{21}: H \to F^{21}$ coincides with the elements of $(F_1, F_2) \in F^{21}$ such that $f(F_1, F_2) = 0$, which implies that the dimension of the Coker $M^{21}: H \to F^{21}$ is 1. Since the dimension of the null-space is also 1, we conclude that the operator is Fredholm with zero index.

3.3.2. Properties of operators $M^{\alpha\beta}: X \to Y^{\alpha\beta}$

To prove in [2] the invertibility of the operators $M^{\alpha\beta}: X \to Y^{\alpha\beta}$ for bounded domains, we essentially used the compactness of the operator $R: H^1(\Omega) \to H^1(\Omega)$ based on the Rellich compactness theorem. However, the latter theorem does not hold for unbounded domains with compact boundaries, and to cope with this, we will split the operator R into two parts, one of which can be made arbitrarily small while the other one is compact, if the PDE coefficient satisfies the additional condition

$$\lim_{x\to\infty} \rho(x)\nabla a(x) = 0 \qquad 3.67$$

Lemma 3.4

Let conditions (2.12) and (3.67) hold. Then for any $\varepsilon > 0$ the operator R can be represented as $R = R_c + R_s$, where $\|R_s\|_{H^1(\Omega)\to H^1(\Omega)} < \varepsilon$, while $R_c : H^1(\Omega) \to H^1(\Omega)$ is compact.

Proof:

Let B_η be a ball centered at 0 with a radius η such that $\partial\Omega \subset B_\eta$ and let $\mu \in D(R^3)$ be a cut-off function such that $\mu = 1$ in B_η, $\mu = 0$ in $R^3 \setminus B_{2\eta}$ and $0 \le \mu(x) \le 1$ in R^3 Denote $R_s g := R(\mu g)$ $R_s g := R((1-\mu)g)$.

Bf (??) we have for arbitrary $g \in H^1(\Omega)$,

$$\|R_s g\|_{H^1(\Omega)} = \left\|\sum_{j=1}^{3} p\partial j[(1-\mu)g\partial ja]\right\|_{H^1(\Omega)} \le Q\|p\|_{\overline{H}^{-1}(\Omega)\to H^1(\Omega)},$$

Where

$$Q := \sum_{j=1}^{3}\|\partial j[(1-\mu)g\partial ja]\|_{\overline{H}^{-1}(\Omega)} \le \sum_{j=1}^{3}\|(1-\mu)g\partial ja\|_{L_2(\Omega)}$$

$$\le 3\|g\|_{L_2(\rho^{-1};\Omega)}\|\rho\nabla a\|_{L_\infty(R^3\setminus B_\eta)} \le 3\|\rho\nabla a\|_{L_\infty(R^3\setminus B_\eta)}\|g\|_{H^1(\Omega)}$$

Thus for the norm of the operator R_s we have

$$\|R_s\|_{H^1(\Omega)\to H^1(\Omega)} \le 3\|\rho\nabla a\|_{L_\infty(R^3\setminus B\eta)}\|p\|_{\overline{H}^{-1}(\Omega)\to H^1(\Omega)} \to 0 \text{ as } \eta\to\infty, \text{ as claimed.}$$

Let us prove the claim about the operator $R_{c.}$. Since the support of μ belongs to $\overline{B_\eta}$, for any fixed η the operator $R_c : H^1(\Omega) \to H^1(\Omega)$ can be represented as $R_c g = R_{\Omega_{2\eta}}[\mu\ r_{\Omega_{2\eta}}]$,where $\Omega_{2\eta} = \Omega \cap B_{2\eta}$ and the operator $R_{\Omega 2\eta}$ is given by the second relation in (4.6) with Ω replaced by $\Omega_{2\eta}$. The operator $R_{\Omega 2\eta} : L_2(\Omega_{2\eta}) \to H^1(\Omega_{2\eta})$ is continuous by (4.13) since $L_2(\Omega_{2\eta}) = L_2(\rho^{-1};\Omega_{2\eta})$ for the bounded domain $\Omega_{2\eta}$.

On the other hand, the restriction operator $r_{\Omega_{2\eta}} : H^1(\Omega) \to H^1(\Omega_{2\eta}) = H^1(\Omega_{2\eta})$ is continuous while the embedding of $H^1(\Omega_{2\eta})$ in $L_2(\Omega_{2\eta})$ is compact, which implies that the operator $R_c : H^1(\Omega) \to H^1(\Omega)$ is compact.

Lemma 3.4 implies the following corollary.

Corollary 3.3

Let conditions (2.12) and (3.67) hold. Then the operator $I + R : H^1(\Omega) \to H^1(\Omega)$ is Fredholm with zero index.

Proof:

Representing $R = R_c + R_s$ by Lemma 7.4 so that $\|R_s\|_{H^1(\Omega)} < 1$ and $R_c : H^1(\Omega) \to H^1(\Omega)$ is compact, we obtain that $I + R_s : H^1(\Omega) \to H^1(\Omega)$ is invertible, which implies the Lemma claim.

Theorem 3.8

If the conditions (2.12),(2.18) and (3.67) hold, then the operators

$$M^{11} : X \to Y^{11} \qquad 3.68$$

is continuous and continuously invertible.

Proof:

By the mapping properties of the potentials, operators (3.68) are continuous and we now prove the invertibility of operator M^{11}. Let us consider the operator

$$M_0^{11} : X \to Y^{11}, \qquad 3.69$$

Where

$$M_0^{11} := \begin{bmatrix} I & -V & W \\ 0 & r_{\partial_D\Omega}\gamma^+ v & r_{\partial_D\Omega} w \\ 0 & 0 & r_{\partial_N\Omega}\hat{L} \end{bmatrix}, \qquad 3.70$$

Evidently operator (3.68) is continuous. The diagonal operators of the triangular matrix operator M_0^{11} are continuously invertible, implying that the operator $(M_0^{11})^{-1}$ inverse to (3.69) is continuous.

Let us now represent $R = R_c + R_s$ by Lemma 3.4 so that the operator R_s is sufficiently small for the operator

$$M_s^{11} := \begin{bmatrix} R_s & 0 & 0 \\ r_{\partial_D\Omega}\gamma^+ R_s & 0 & 0 \\ r_{\partial_N\Omega} T^+ R_s & 0 & 0 \end{bmatrix} \qquad 3.71$$

To satisfy the inequality

$$\left\| M_s^{11} \right\|_{X \to Y^{11}} < 1 \Big/ \left\| (M_0^{11})^{-1} \right\|_{Y^{11} \to X} .$$

Then the operator $M_0^{11} + M_s^{11} : X \to Y^{11}$ is continuously invertible, while the operator $M_c^{11} := M^{11} - M_s^{11} : X \to Y^{11}$ is compact by Lemma 3.4 and by the properties of the operators w' and $L^+ - \hat{L}$. This implies that operator $M^{11} : X \to Y^{11}$ is a Fredholm operator with zero index. Since by Theorem (3.1) it is also injective, we conclude that it is invertible.

Theorem 3.9

If the conditions (2.12), (2.18) and (3.67) hold, then the operator

$$M^{12} : X \to Y^{12} \qquad 3.72$$

is continuous and continuously invertible.

Proof :

By the mapping properties of the potentials, operator (3.72) is continuous and we now prove the invertibility of operator M^{12} . Let us consider the auxiliary operator

$$M_0^{12} := \begin{bmatrix} I & -V & W \\ & & \\ 0 & -v & \frac{1}{2} I \end{bmatrix} : X \to Y^{12}. \qquad 3.73$$

Evidently operator (3.73) is continuous. Any solution $U = (u, \psi, \varphi)^T \in X$ of the equation

$M_0^{12} U = F$, Where $F = (F_0, F_1)^T \in H^1(\Omega) x H^{\frac{1}{2}}(\partial\Omega)$ will solve also the following extended system of three equations,

$$u + W\varphi - V\psi = F_0 \text{ in } \Omega \qquad 3.74$$

$$\frac{1}{2}\varphi - v\psi = F_1 \text{ on } \partial\Omega \qquad 3.75$$

$$-r_{\partial_D\Omega} v\psi = r_{\partial_d\Omega} F_1 \text{ on } \partial_D\Omega \tag{3.76}$$

and vice-versa. Taking into account that the invertibility of the operator $r_{\partial_D\Omega} v$ follows form the first relation in (2.35), the diagonal operators of the system,

$$I : H^1(\Omega) \to H^1(\Omega)$$

$$\frac{1}{2} I : H^{\frac{-1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

$$-r_{\partial_D\Omega} v : \tilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \to H^{\frac{1}{2}}(\partial_D\Omega)$$

are continuously invertible implying that the triangular matrix operator of the system is also invertible.

If $\psi \in \tilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$ solves equation (3.76), then $\varphi = 2(F_1 + v\psi) \in \tilde{H}^{\frac{1}{2}}(\partial_D\Omega)$ by equation (3.75), and we arrive at invertibility of the operator (3.73). The rest of the proof for the operator M^{12} is similar to the one for M^{11}.

Theorem 3.10

If the conditions (2.12), (2.18) and (3.67) hold, then the operator

$$M^{22} : X \to Y^{22} \tag{3.77}$$

is continuous and continuously invertible.

Proof:

By the mapping properties of the potentials, operator (3.77) is continuous and we now prove the invertibility of operator M^{22}. Let us consider the auxiliary operator

$$M_0^{22} : X \to Y^{22}, \tag{3.78}$$

Where

$$M_0^{22} := \begin{bmatrix} I & -V & W \\ 0 & r_{D\Omega}\left(\frac{1}{2}I - w'_\Delta\right) & r_{\partial_D\Omega}\hat{L} \\ 0 & -r_{\partial_N\Omega} v & r_{\partial_N\Omega}\left(\frac{1}{2}I + w\right) \end{bmatrix},$$

Operator (3.78) is evidently continuous and can be considered as a matrix block-triangle operator with the lower diagonal block

$$M_0^{22} := \begin{bmatrix} r_{\partial_D\Omega}\left(\frac{1}{2}I - w'_{\Delta}\right) & r_{\partial_D\Omega}\hat{L} \\ -r_{\partial_N\Omega}v & r_{\partial_N\Omega}\left(\frac{1}{2}I + w\right) \end{bmatrix}.$$

Taking into account relations (2.35) and (2.37), we can represent

$$M_0^{22}g = diag\left(1, \frac{1}{a}\right)\hat{M}_{\Delta}^{22}\left[diag(1, a)g\right],$$

For any $g = (g_1, g_2)^T \in \tilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, Where $diag(1, \frac{1}{a})$ and $diag(1, a)$ are diagonal

2x2 matrices, while the operator $\hat{M}_{\Delta}^{22}$ given by (3.52) is invertible by Theorem 3.6.
Since $0 < a_0 < a(x) < a_1 < \infty$, this implies the invertibility of the operator

$\hat{M}_0^{22} : \tilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^{\frac{-1}{2}}(\partial_D\Omega) \times H^{\frac{1}{2}}(\partial_N\Omega)$ and thus of operator (3.78). The rest of the proof for the operator M^{22} is similar to the one for M^{11} .

Theorem 3.11

If conditions (2.12),(4.18) and (3.67) hold, then the operator $M^{21} : X \to Y^{21}$ is continuous Fredholm operator with zero index. It has one-dimensional null-space spanned over the element (u^0, ψ^0, φ^0) defined in Theorem 3.4 (i).

Proof:

The claim about the null-space, particularly that its dimension is 1, follows from Theorem (3.4)(i).

Let us consider the auxiliary operator

$$M_0^{21} := \begin{bmatrix} I & -V & W \\ 0 & -\frac{1}{2}I & \hat{L} \end{bmatrix} : X \to Y^{21}. \qquad 3.79$$

Evidently operator (3.79) *is continuous . Any solution* $U=(u,\psi,\varphi)^T \in X$ *of the equation* $M_0^{21}=F$, *where* $F=(F_0,F_1)^T \in H^1(\Omega)xH^{\frac{-1}{2}}(\partial\Omega)$ Will also solve the following extended system of three equations,

$$u - V\varphi + W\psi = F_0 \qquad in\ \Omega$$

$$-\frac{1}{2}\psi + \hat{L}\varphi = F_1 \qquad on\ \partial\Omega$$

$$-r_{\partial_N\Omega}\hat{L}\varphi = r_{\partial_N\Omega}F_1 \qquad on\ \partial_N\Omega,$$

and vice-versa. Taking into account that invertibility of the operator $r_{\partial_N\Omega}\hat{L}$ follows from relation (2.42), the diagonal operators of the system,

$$I : H^1(\Omega) \to H^1(\Omega)$$

$$\frac{1}{2}I : H^{\frac{-1}{2}}(\partial\Omega) \to H^{\frac{-1}{2}}(\partial\Omega)$$

$$r_{\partial_N\Omega}\hat{L} : \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega) \to \widetilde{H}^{\frac{-1}{2}}(\partial_D\Omega)$$

are continuously invertible implying that the triangular matrix operator of the system is also invertible. If $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial_D\Omega)$ solves the third equation of the system, then $F - \hat{L}\varphi \in \widetilde{H}^{\frac{-1}{2}}(\partial_N\Omega)$, and we arrive at invertibility of the operator (3.79).

Then the reasoning similar to the second paragraph of the proof for operator M^{11} in Theorem 3.8 implies that operator $M^{21} : X \to Y^{21}$ is Fredholm with zero index.

Finally, the last sentence of the theorem follows from Theorem (3.4) (i).

Concluding remarks

Four different segregated direct boundary-domain integral equation systems associated with the mixed (Dirichlet-Neumann) BVP for the scalar “Laplace” PDE with variable coefficient on a three-dimensional unbounded domain, have been formulated and analyzed in this paper. Equivalence of the three BDIE systems to the original BVPs was proved in case when right hand side of the PDE is from $L_2(\rho;\Omega)$ and the Dirichlet and the Neumann data are from the spaces $H^{\frac{1}{2}}(\partial_D\Omega)$ and $H^{\frac{-1}{2}}(\partial_N\Omega)$, respectively. The inertibility of the BDIE operators these three systems were proved in the corresponding Weighted Sobolev spaces. Fredholm properties of the fourth system were studied as well. This analysis was based

on the invertibility in the Weighted Sobolev spaces of the variable-coefficient BVPs in unbounded domains also proved in the paper.

Using the approach of [17], the united direct boundary-domain integro-differential systems can be also formulated and analyzed for the BVPs in exterior domains. The approach can be extended also more general PDEs and to systems of PDEs, while smoothness of the boundary can be essentially relaxed, and the PDE right hand side can be considered in more general spaces cf.[16].

Employing methods of [3], one can consider also the localized counterparts of the BDIEs for BVPs in exterior domains.

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