A STRING THEORY APPROACH TO THE THREE-DIMENSIONAL ISING MODEL

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ABSTRACT

The three-dimensional Ising model (3DIM) has been one of the unsolved problems in physics. Polyakov has suggested that it may be possible to solve this model via the string theory methods. The 3DIM in the critical regime can be assumed to correspond to the fermionic string which spans a 2D world-sheet. The 3DIM also corresponds to the conventional ϕ^2 and ϕ^4 Lagrangian field theories. It can be shown that the scaling dimensions of these two and three-dimensional fields are proportional. The constant of proportionality is heuristically identified with the Hausdorff dimension of the world-sheet. Computer calculations of the Hausdorff dimension and the scaling dimensions of the 3D conventional field, the results of which are available in the literature, enable numerical evaluation of the critical exponents α and β of the Ising model. These agree well with experimental results.

INTRODUCTION

The Ising model first appeared in the thesis of Ernst Ising in 1925. This model was, of course, one dimensional and, to Ising's disappointment, did not exhibit a phase transition. The two and three-dimensional Ising models are indeed only two of statistical models which undergo phase transitions.

The two-dimensional Ising model was first considered by Peierls in 1936. In 1941, Kramers and Wannier determined the critical temperature (the transition point) by assuming its existence. The year 1944 could be thought of as the begining of a new era in the study of the physics of phase transitions. In this year, L.Onsager presented a paper in which it has been shown rigorously that the two-dimensional Ising model exhibited a phase transition.

The partition function for the two-dimensional Ising model was obtained by converting the model into a system of free Majorana fermions. This shows that the two-dimensional Ising model is equivalent to a free quantum field theory of fermions.[1,2] The model is also treated by using combinatorial methods.[3]

The three-dimensional Ising model (3DIM) still remains one of the problems in physics for which no exact solutions have been possible yet. Even if the partition function has not been found the critical exponents have been determined by the series and renormalisation group methods. The values of the critical exponents obtained in this manner have been compared with those obtained experimentally. Recent developments in the string theory have led to the expectation that the 3DIM might be solvable in the not too distant future by techniques that are being developed currently. It, however, appears that the current level of development has not yet reached the stage where one could apply the string theory to calculate the partition function or even the critical exponents. However, as we shall see, by a combination of the ideas of the string theory with the results of renormalisation group studies one could predict some of the critical exponents.

This paper is organized as follows:

Chapter I is a review of the 3DIM and its possible representation by the string theory. Ofcourse, here, we made use of analogy to the two-dimensional Ising model in deriving the partition function. Polyakov's interpretation is mentioned.

Chapter II deals with the bosonic string. Also, in this chapter, the supersymmetrised form of the string — that is to say the fermionic string — is considered.

Chapter III is a discussion of the attempts at the calculation of the values of the critical exponent. In Chapter IV we have a short discussion of the application of the renormalization group to the Ising model.

A heuristic application of the ideas of the string model is made to some result of the renormalization group for calculating the numerical values of the critical exponents. This is done in Chapter V. Finally, we discuss briefly the results and the future possibilities.

CHAPTER I

THE ISING MODEL

The two dimensional Ising model.

We find it convenient to start with the two-dimensional Ising model (2DIM). The partition function for the 2DIM is given by*

$$Z = \sum_{\{\sigma_{\underline{x}} = \pm 1\}} \exp\left[\beta \sum_{\underline{x}} \sum_{\delta} \sigma_{\underline{x}} \sigma_{\underline{x}+\delta}\right], \qquad \beta = \frac{1}{k_B T} \qquad (I.1)$$

where $\underline{x}$ is a lattice site and the neighboring sites are indicated by the unit vector $\underline{\delta}$.

At low temperature (large β) the configuration of the two-dimensional lattice is such that all spins are mutually parallel. When the temperature is increased slowly "drop" fluctuations (i.e., domain fluctuations) will appear here and there. All spins in a drop have a reversed direction and hence we get antiparallel spin pairs along the periphery of the drop. It is obvious from eq. (I.1) that there is energy loss only along the periphery of a drop:

$$\sum_{(\text{drop})} \sigma_{\underline{x}} \sigma_{\underline{x}+\underline{\delta}} - \sum_{\substack{(\text{ground}) \\ \text{state}}} \sigma_{\underline{x}} \sigma_{\underline{x}+\underline{\delta}} = -2L \,, \qquad (I.2)$$

we thus write

$$Z = \sum_{\{\text{drops}\}} e^{-2\beta L} \,. \qquad (I.3)$$

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* See Appendix A for a brief review of the algebraic methods.

where L is the peripheral lèngth and the summation is over all possible drops.

In the above sum, eq.(I.3), are included drops which touch each other at a point like those shown in the figure below.

Suppose we identify an intersection[3] as shown in fig.1a . The above figure is then equivalent to a set of three closed paths which could be traced piecewise continuously With each path is associated a weighting factor

$$(-1)^{\text{nº of intersections}}$$

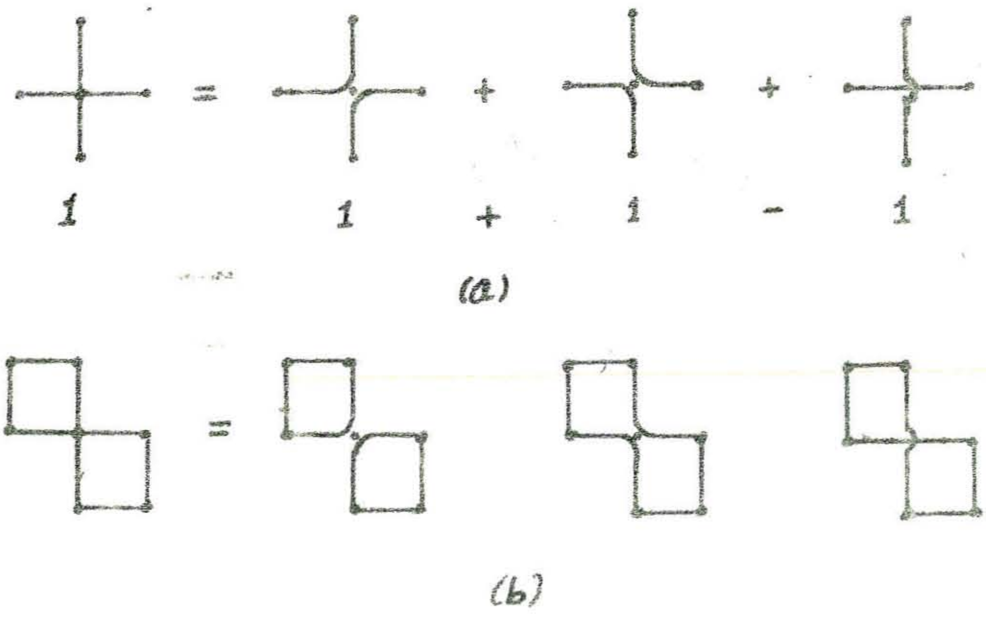


Fig. 1. An intersection point and a graphical representati

Now, if there are $\nu(p)$ intersections in a closed path, P, then we take into account the weight $(-1)^{\nu(p)}$ in eq. (I.3). The partition function then becomes

$$Z = \sum_{(p)} e^{-2\beta L(p)} (-1)^{\nu(p)}, \qquad (I.4)$$

where p labels a closed curve.

With this brief account of the 2DIM we now pass to the 3DIM.

I.2. The three-dimensional Ising model.

The drops in the three-dimensional case are bounded by two-dimensional surfaces. By going through similar arguments as in the case of the 2DIM, one has the following expression for the partition function of the 3DIM:

$$Z = \sum_{S} e^{-2\beta A(S)} (-1)^{\ell(S)}, \qquad (I.5)$$

in which S labels a closed surface whose area is A(S); $\ell(S)$ is the total length of the self-intersection line for the surface S in units of the lattice parameter.

Eq. (I.5) has been interpreted by Polyakov in terms of the string-theory language. An ordinary bosonic string could be visualized as a collection of free bosonic partons. These partons are joined together and move transversally to the string. One parton gives a contribution e^{-L} to the partition function whereas the collection as a whole contributes e^{-A}. Here L is identified as the length of a path traced out by a parton in motion. The motion of the "collection" is of course the motion of the string and A is the area of the surface spanned by the string's motion. In general, we could have a swarm of strings, their motion describing topological surfaces. Suppose now the partons carry spin. This gives them fermionic character and we may assign a factor $(-1)^{\nu}$ from each parton whereas the string, which is now endowed with spin structure, acquires the factor $(-1)^{\ell}$. On the other hand, as we shall see in the next chapter, the modes of the fermionic string carry a spin $\frac{1}{2}$.

These are the reasons to believe that the fermionic string theory describes the 3DIM.

CHAPTER II

THE STRING THEORY

Among extended objects which are endowed with some internal structure one is the string. A string could be visualized as a one-dimensional extended relativistic object. It is possible to imagine the motion of this one-dimensional extended object in a D-dimensional specetime. One then defines a functional integral (the action) and studies the dynamics of the string. We begin with the Nambu-Goto action.

II.1. The Nambu-Goto action

Let us first say a few words about a relativistic (free) point particle. A free point particle moving in spacetime traces out a time-like cure which is known as a world-line. A single variable, say τ, could serve as a parameter of the world-line and hence the equation

$$X^{\mu} = x^{\mu}(\tau) \qquad \text{(II.1)}$$

describes the position of the particle on the world-line for each specified value of τ. The relativistic action is then taken to be proportional to the length of the path traversed between given initial and final values of τ. That is,

$$S = -m\int\sqrt{dX^2} = -m\int_{\tau_1}^{\tau_2} \left[\left(\frac{dX}{d\tau}\right)^2\right]^{\frac{1}{2}} d\tau \qquad \text{(II.2)}$$

where $(dX/d\tau)^2$ stands for

$$\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}.$$

The Nambu-Goto action[4] is a natural generalization of the action (II.2) to the one-dimensional extended object—the string. Consider a fixed background spacetime with coordinates X^μ, $\mu = 1,2, \ldots, D$, D being the spacetime dimension (we do not restrict ourselves to the Minkowski space). A string which moves in this spacetime sweeps out a bidimensional surface. As usual, we parametrize the bidimensional surface by $\xi^a = (\tau,\sigma)$, $a = 0,1$. We call the bidimensional surface a world-sheet.

Let $G_{\mu\nu}$ be a metric for the background spacetime and let h_{ab} be the world-sheet metric. These two metrics, which are second-rank tensors, are related by the familiar transformation rule for tensors, i.e.,

$$h_{ab}(\xi) = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X(\xi)) \qquad \text{(II.3)}$$

The two-dimensional surface spanned between the initial and final shapes of the string has the area

$$\int d^2\xi\sqrt{-h} \qquad \text{(II.4)}$$

in which $h = \det h_{ab}$. The Nambu-Goto action is then proportional to this area:[4]

$$S_N = -\frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{-h} \qquad \text{(II.5a)}$$

$$= -\frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{-\left(\frac{\partial X}{\partial\tau}\right)^2\left(\frac{\partial X}{\partial\sigma}\right)^2+\left(\frac{\partial X}{\partial\tau}\frac{\partial X}{\partial\sigma}\right)^2} \qquad \text{(II.5b)}$$

In eq. (II.5b) we made use of the transformation rule (II.3).

Using the notations

$$\frac{\partial X}{\partial \tau} = \dot{X} \ , \ \frac{\partial X}{\partial \sigma} = X' ,$$

we rewrite eq. (II.5b) as

$$S_N = -\frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{-\dot{X}^2 X'^2 + (\dot{X}X')^2} \qquad \text{(II.5c)}$$

The constant $T = \frac{1}{2\pi\alpha'}$ is known as the string tension. The geometrical construction of this action is responsible for invariance under reparametrization of the world-sheet.[4]

The equations of motion follow from the action S_N by using the principle of least action. It turns out that a different string action, which we consider in the following section, gives the same equations of motion.

II.2. Polyakov's action and quantization

Quantization becomes manageable if we use the following action

$$S_p = -\frac{1}{2\pi} \int d^2\xi \sqrt{-g}\, g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \qquad \text{(II.6)}$$

The metric tensor g_{ab} in Polyakov's action is taken to be an independent dynamical variable. The conformal gauge

$$g_{ab}(\xi) = e^{\phi(\xi)} h_{ab}(\xi), \qquad \text{(II.7)}$$

where ϕ is arbitrary, connects the two metrics h_{ab} and g_{ab}. In the literature the background metric $G_{\mu\nu}$ is restricted to the flat Minkowski metric, $G_{\mu\nu} = \eta_{\mu\nu}$, for the sake of simplicity.

The discussion upto now is true for a bosonic string. A particularly interesting class of string theories arise when the string is endowed with spin structure; the structure of the string is enriched by the polarization information carried by the spins. As already mentioned such a string is believed to represent the 3DIM.

For the moment, however, we proceed with the bosonic string as given by the Polyakov action, eq.(II.6). The problem before calculating the partition function was the quantization of the action. The equations of motion which follow from the action (II.6) are seen to be invariant under the transformation $g \rightarrow e^{\phi} g$. This is a conformal invariance which is crucial in string theory. This symmetry makes it possible for one to work in a particular conformal gauge, namely

$$g_{ab}(\xi) = e^{\phi(\xi)} \delta_{ab} \quad . \qquad \text{(II.8)}$$

The possibility of this gauge can also be seen from a well-know theorem in differential geometry which states that for a 2D real manifold the above gauge is permissible. Polyakov[5] performed the quantization of the action (II.6) by using the conformal gauge (II.8) and integrating out X^{μ}. His result was

$$S_{eff} = \frac{26-D}{48\pi} \int d^2\xi \left[(\partial\phi)^2 + \mu^2 e^{2\phi} \right] . \qquad \text{(II.9)}$$

In eq (II.9) μ is a constant. The second term is known to represent the Liouville mode which is an addition to the free mode. Thus we have a two-dimensional quantum field theory with the Liouville mode which has not yet been solved.

In order to be able to use the string theory for our purpose one needs to do the followings:

(i) solve the Liouville theory which is twodimensional and renormalisable,

(ii) convert the purely bosonic theory, mentioned above, into a fermionic string theory.

Both of the above procedures are, of course, among the most recent aspects of research in string theory. We shall briefly indicate them in the following.

It is seen from eq. (II.9) that the effective action is nonzero for $D \neq 26$. For the Ising problem we are now considering $D = 3$, so one has to find a way of treating the Liouville mode. A recent proposal has been to attack the problem of the two dimensional field theory with Liouville mode by casting it in a form that bears strong resemblance to the two-dimensional

Wess-Zumino model[6]. This model has already been solved and the critical exponent γ was calculated by this method for certain values of D.

The problem in doing this kind of calculation with a fermionic theory is considerably more complicated than that with a bosonic field theory. Since the 3DIM is believed to be the same as the fermionic theory it is necessary to convert the bosonic theory into the fermionic theory, and of course, this conversion has already been carried out[7]. The analytic procedure involved in the conversion is what is known as supersymmetrisation and by this procedure the equation for fermions corresponding to eq(II.9) has been found. The theory expounded in ref.7 has given the following result for the string critical exponent:

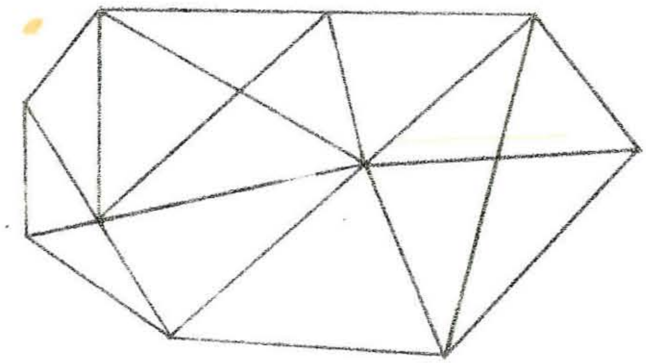
$$\gamma(D) = \frac{D-1 \pm \sqrt{(1-D)(9-D)}}{4} .$$

It is easily seen that the above expression gives a nonsensical result for our case, D = 3. For the time being, we simply comment that with substantial modification of this approach we may yet use this analytic method for calculating the critical exponents of the 3DIM.

CHAPTER III

THE MODEL OF TRIANGULATED SURFACES

In our discussion in chapter I we have mentioned, when the temperature is increased from its value T = 0, that three-dimensional "drops" appear in the spin system. The surfaces of these drops can be approximated by triangulated random surfaces. The model of triangulated random surfaces may be defind as the collection of randomly constructed neighboring triangles embedded in a D-dimensional euclidean space. We have mentioned in



the Introduction that for regular lattices there is equivalence between the Ising model and Majorana fermion model(refs. 1,2). It has been conjectured that an equivalence relation between the two models with some proper definitions occurs for an arbitrary surface.

The triangulated random surface model can be viewed as a suitable representation of Polyakov's[5] string model where the integration over metrics is replaced by summation over triangulations. This summation has been carried out by a combination of analytic and numerical methods and has been confined to "purely" bosonic surfaces. That is no account has been taken of the Ising spins on the lattice. The problem with these bosonic surfaces has been that the worldsheet becomes extremely rough and the Hausdorff dimension d_F of the surface spanned by the string has been found to tend infinity in some early work.

Two approaches have been made to overcome this difficulty in the model of triangulated random surface.[9,10,11] The approach of refs. 9 and 10 is to take into account the extrinsic curvature of the surface. It appears that the inclusion of the extrinsic curvature term improves the pathological features of a bosonic surface. In refs. 11 and 12 the model of triangulated random surface endowed with fermionic degrees of freedom has been discussed. In ref. 11 the partition function of the triangulated surface contains Ising variables whcih now reside at the vertices of the triangulated surface. In the same paper the authors have done numerical work to calculate the critical parameters. A study of table 1 of ref. 11 for the three-dimensional case can give us some idea of β, γ and d_F for this model. Here β and γ are the usual critical exponents and d_F is the Hausdorff dimension of the surface. The values of β and γ are not in good agreement with the values shown in our table 1 (Chapter V).

The numerical evaluation of the Hausdorff dimension d_F gives us a handle to evaluate the critical exponents independently by using results from renormalisation group calculations and heuristic notions about the string theory.

CHAPTER IV

THE RENORMALISATION GROUP

The renormalisation group theory is applicable to such areas of physics as the relativistic field theory and critical phenomena in statistical mechanics. Indeed, in statistical mechanics the theory has been eminently successful in calculating numerically the critical exponents by making use of the scaling ideas.[13] Also, in its field theory incarnation, the renormalisation group has been used to study critical phenomena.[14] It was early pointed out by K.Wilson, who advanced the renormalisation group theory, that both the ϕ^2 and ϕ^4 theories can be used to represent the Ising model in the continuum limit. Strong arguments have been given which indicate that the long-distance properties of physical systems such as ferromagnets, fluids, binary mixtures in the neighborhood of a second order phase transition can be described by a continuous Euclidean field theory.

Systems which we meet both in statistical mechanics and quantum field theory consist of infinite degrees of freedom. This is understood because in the former case systems like the Ising model are treated in the thermodynamic limit where their size is very large, and in the latter the fields $\phi(x)$ assume values at each spacetime point so that even in a very small region of spacetime there are infinite degrees of freedom . In both cases standard techniques of Green function, Feynman diagrams etc are used These techniques may be derived from functional integrals with weight exp(iS) in quantum field theory and exp (-βH) in statistical mechanics. For instance, consider a scalar field theory described by the Lagrangian density

$$L(x) = \sum_{\mu=1}^{4} \tfrac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \qquad \text{(IV.1)}$$

One can see that Feynman rules can be derived from the positive weight

$$\exp\{-\int d^4x(\sum_{\mu=1}^{4} \tfrac{1}{2}(\partial_\mu \phi)^2 + V(\phi))\} . \qquad \text{(IV.2)}$$

Several properties of quantum field theory may be understood through this connection, but it is near the critical point that we expect the similarities to be more apparent. Let us consider a periodic lattice with lattice spacing a. At each lattice site $\underline{n}$ we have a discrete variable $S_{\underline{n}}$

(In Chapter I we have indicated Lattice sites by $\underline{x}$. Here we prefer to use $\underline{n}$). A spin at a given site interacts with its immediate neighbors. The range of correlation between spins increases when we approach the critical point. (Mathematically, the critical regime is characterize by the singularities of the thermodynamic functions such as specific heat and magnetic succeptibility). Infact, a system at the critical point has an infinite correlation length, a fact supported by the scaling hypothesis. Near the critical point the correlation length, denoted by $\xi(T)$ is not infinite but is much much larger than the lattice spacing. In this case detailed microscopic interactions, whose scale is of the order of the lattice spacing, are irrelevant in a correct explanation of critical phenomena, and we can replace the discrete variables by continuous ones (the equivalent of a field in QFT) by averaging over regions whose size is very small compared to ξ but still large with respect to the lattice spacing a.

It has been pointed out that the observed critical characters are large scale phenomena. This means that the fluctuations near the critical point involve wavelengths which are at least of the order of a. To put it in a different way, one needs to consider only those spin fluctuations $S(\underline{k})$ whose wave number $\underline{k}$ is less than a certain cut-off value Λ. The minimum length Λ^{-1} is a basic element of the renormalisation group.

We start from the ϕ^4 theory and see how the cut-off Λ is introduced in the renormalisation procedure.

As already mentioned we can represent the Ising model in the critical regime by the ϕ^2 and ϕ^4 theory. In eq.(IV.1) we can write the function $V(\phi)$ explicitly and have

$$L = \tfrac{1}{2}\,(\partial_\mu \phi(x))^2 - \tfrac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x), \qquad \text{(IV.3)}$$

where the coupling constant λ describes the self-interaction. We try to understand how the renormalisation procedure is applied via the Feynman rules.

In quantum field theory the Green function in coordinate space is given by the vacuum expectation value of the time ordered product of field operators $\phi(x)$. The Fourier transform of this Green function is

$$G^{(n)}(p_1, p_2, \dots, p_{n-1})$$

$$= \int \prod dx_i \exp(i \sum_i p_i x_i) \langle 0 | T[\phi(x_1) \dots \phi(x_{n-1}) \phi(0)] | 0 \rangle \qquad \text{(IV.4)}$$

where the p's are momenta of particles involved in the interaction.

If $G^{(n)}$ is expanded in powers of the coupling constant λ, the rule is to write down all possible connected graphs with n external lines and any number of internal lines, called propagators, joined together by four-point vertice The various lines have momenta assigned to them by lettin the external lines carry the momenta p_i and insisting that momentum be conserved at each vertex. This leaves a certain number of internal loop momenta which must be integrated over. Further the integration over internal momenta must cover only truely independent configurations Below are given graphical elements and their mathematical equivalent according to Feynman rules.

Propagator:  $\equiv \dfrac{i}{\ell^2 - m^2 + i\varepsilon}$

Vertex: $\quad = \quad -i\lambda$

loop integration: $\quad \int \frac{d^4p}{(2\pi)^4}$

It is convenient to introduce the notion of a one-particle Green function $\Gamma^{(n)}$ corresponding to $G^{(n)}$ by throwing away all graphs which decompose into two disconnected pieces on cutting one internal propagator and then removing the external propagator from what remains.

We try to understand why some renormalisation procedure is needed to make sense of the above rules. Let a particular graph of $\Gamma^{(n)}$ have n external lines, I internal lines and p vertices. Then the number of independent internal momenta, K_i, which must be integrated over is

$$L = I - p + 1. \tag{IV.5}$$

One then writes the general form of $\Gamma^{(n)}$ as

$$\int \prod_{i=1}^{L} \frac{d^4K_i}{(2\pi)^4} \prod_{j=1}^{I} \left(\frac{1}{\ell_j^2 - m^2 + i\varepsilon} \right) \tag{IV.6}$$

In eq. (IV.6) each ℓ_j is a linear combination of the external momenta and the loop momenta. The naive over-all degree of divergence of this integral is

$$D(I) = 4L - 2I. \tag{IV.7}$$

It has been shown that this integral or its subintegral diverges for $n = 2, 4, \ldots$. The problem is to make some sense of these divergent integrals.

In ϕ^4 theory it is known that the over-all degree of divergence of $\Gamma^{(n)}$ is positive for finite n and is a function of the external lines and not of the order in λ. The spacetime dimensionality of a theory also determines whether the theory is renormalisable or not. In order to obtain the renormalisation group equation we have to learn how to define the renormalization points, the points where the divergences are subtracted out. One method is the so called multiplicative renormalisation. Here one proceeds in two steps. First, the propagators are modified so as to make them vanish so rapidly at infinity so that all loop integrations are explicitly convergent. For instance, one might make the replacement

$$\frac{1}{p^2 - m^2} \rightarrow \frac{1}{p^2 - m^2} - \frac{1}{p^2 - m^2 - \Lambda^2}$$

$$= \frac{\Lambda^2}{(p^2 - m^2)\,(p^2 - m^2 - \Lambda^2)}$$

In the limit $\Lambda \rightarrow \infty$, the divergent expression for the Feynman graph is recovered. On the other hand, it is argued that since the divergences are associated with just a finite number of Green functions in ϕ^4 theory, they might be removed by adding counter terms to the original Lagrangian density, i.e.,

$$L \to L + \delta L,$$

where L is given by eq. IV.3 and

$$\delta L = \frac{\delta Z}{2} (\partial_\mu \phi)^2 - \tfrac{1}{2}\delta m^2 \phi^2 - \frac{\delta\lambda}{4!} \phi^4 . \qquad (IV.8)$$

It is imagined that δZ, $\delta m, \delta\lambda$ all diverge as $\Lambda \to \infty$ in such a way that the divergences due to loop integrations are precisely cancelled. Since $L + \delta L$ has the same form as L the divergence analysis remains the same. We expect the divergence cancellation scheme to be self consistent. If we combine like terms in eqs. (IV.3) and (IV.8) and rescale the field we get

$$L_o = \tfrac{1}{2} (\partial_\mu \phi_o)^2 - \tfrac{1}{2} m_o^2 \phi_o^2 - \frac{\lambda_o}{4!} \phi_o^4 , \qquad (IV.9)$$

We note here that any arbitrary rescaling of the field ϕ in the kinetic term is inconsequential. The quantities ϕ_o, m_o and λ_o are referred to as the bare field, bare mass and bare coupling constant, respectively. They are given by

$$m_o^2 = (m^2 + \delta m^2) Z,$$
$$\lambda_o = (\lambda + \delta\lambda) Z^2,$$

$$\phi_o = \phi Z^{-\frac{1}{2}}.$$

The Green functions

$$\Gamma^{(n)}(p;\ \lambda_o,\ m_o)$$

calculated from the Lagrangian (IV.9) are related to the finite Green functions

$$\bar{\Gamma}^{(n)}(p;\lambda,\ m)$$

of finite parameters by

$$\Gamma^{(n)}(\lambda_o,\ m_o) = Z^{\ n/2}\ \bar{\Gamma}^{(n)}(\lambda,\ m). \qquad \text{(IV.10)}$$

We omitted the momenta p for simplicity.

We now introduce the cut-off Λ explicitly. Of course, the bare parameters λ_o and m_o as well as the parameter Z are Λ-dependent and diverge when $\Lambda \rightarrow \infty$. The finite parameters are precisely defined in theory.

Eq.(IV.10) can be rewritten explicitly as

$$\Gamma^{(n)}(\lambda_o,\ \Lambda) = Z^{n/2}\ (\frac{\Lambda}{\mu},\ \lambda_o)\ \bar{\Gamma}^{(n)}\ (\lambda,\ \mu), \qquad \text{(IV.11)}$$

where, for simplicity, we ignore the renormalisation of mass. The new parameter μ introduced in eq. (IV.11), is called the renormalisation point. The coupling constant λ depends on the renormalization point:

$$\lambda = \lambda\ (\ \frac{\Lambda}{\mu}\ ,\ \lambda_o).$$

When the limit $\Lambda \to \infty$ is taken, λ is supposed to approach a function of μ alone which is called the running coupling constant $\lambda(\mu)$. The entire content of eq.(IV.11) rests on the choice of arguments in the various functions and the assertion that $\bar{\Gamma}^{(n)}$ and λ, m approach finite limits as $\Lambda \to \infty$ while $\Gamma^{(n)}$, Z, and λ_o, m_o may diverge. We note that the right-hand side of eq. (IV.11) does not depend on the parameter μ, and taking the derivative with respect to μ gives

$$\frac{d}{d\mu}\left[Z^{n/2}\left(\frac{\Lambda}{\mu}, \lambda_o\right)\bar{\Gamma}^{(n)}(\lambda, \mu)\right] = 0$$

or

$$\frac{n}{2}Z^{n/2}\,\frac{1}{Z}\,\frac{dZ}{d\mu}\,\bar{\Gamma}^{(n)} + \left(\frac{\partial\bar{\Gamma}^{(n)}}{\partial\mu} + \frac{\partial\lambda}{\partial\mu}\,\frac{\partial\bar{\Gamma}^{(n)}}{\partial\lambda}\right)Z^{n/2} = 0$$

which, after multiplying by μ and cancelling $Z^{n/2}$, becomes

$$\left(\frac{n}{2}\,\mu\,\frac{d}{d\mu}\,\ln Z + \mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda}{\partial\mu}\,\frac{\partial}{\partial\lambda}\right)\bar{\Gamma}^{(n)} = 0 \qquad \text{(IV.12)}$$

We introduce two dimensionless factors β and η via the definitions

$$\beta(\lambda(\mu)) = \mu\frac{\partial}{\partial\mu}\;\lambda\left(\frac{\Lambda}{\mu}, \lambda_o\right)$$

$$\eta(\lambda(\mu)) = \mu\frac{d}{d\mu}\;\ln Z\left(\frac{\Lambda}{\mu}, \lambda_o\right)$$

We now write the renormalisation group equation in terms of β and λ:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \frac{n}{2} \eta \right) \bar{\Gamma}^{(n)}(\mu, \lambda) = 0 \qquad \text{(IV.13)}$$

The infra-red zero of λ is defined by

$$\beta(\lambda^*) = 0$$

and ω, the exponent that governs the leading correction to scaling theory, is defined by

$$\left. \frac{\partial}{\partial \lambda} \beta(\lambda) \right|_{\lambda = \lambda^*} = \omega$$

By further analysis, numerical and otherwise, the critical exponents have been calculated for ϕ^2 and ϕ^4 theories. We however restrict ourselves in the following to an investigation of how by using the ideas of string the with some input from the renormalisation group the critical exponents can be calculated.

CHAPTER V

CALCULATION OF CRITICAL EXPONENTS

We have seen in Chapter II that for D=3 the critical exponents can not be calculated at the present level of development of the string theory. As mentioned in Chapter III the inclusion of the extrinsic curvature term in the fermionic string theory is possibly one way to proceed in this matter. We shall, however, content ourselves by doing calculations less rigorously and by applying heuristically ideas from the previous chapters. Before we go directly to the calculations, we find it appropriate to review some important aspects of critical behavior. We do this in the following section.

V.1. Review of critical phenomena and scaling hypothesis

There are many materials whose statistical feature can be explained by studying the Ising model. Let us, for instance, consider a ferromagnetic sample. At a particular temperature $T=T_c$, called the critical temperature, the sample undergoes a phase transition, i.e., its state changes from ferromagnetic to paramagnetic or vice versa according as T is below or above T_c. Below T_c there is a spontaneous magnetization, but above T_c there is not. As mentioned in Chapter IV, the phase transition is characterized by the singularities of some observable thermodynamic quantities at the critical temperature. All divergences occuring at $T=T_c$ are critical behaviors and are explained by the theory of critical phenomena.

The amount of ordering which is built up in the regime of $T=T_c$ is measured by the order parameter which, in the case of ferromagnets, is the spontaneous magnetization $M_o(T)$. Very near the critical point $M_o(T)$ behaves like

$$M_o(T) \sim |t|^{\beta} \quad \text{as } t \to 0^- ,$$

where $t = \frac{T-T_c}{T_c}$, and β is one of the critical exponents. Such a power-law behavior stems from the idea of scaling. Let us see the critical behaviors of some other quantities. The magnetic susceptibility

$$\chi(t) = \left.\frac{\partial M(H,T)}{\partial H}\right|_{H=0} \sim \begin{cases} t^{-\gamma} & \text{as } t \to 0^+ \\ |t|^{\gamma'} & \text{as } t \to 0^- \end{cases}$$

the prime over γ distinguishes exponents below and above T_c. Similarly, the critical exponent α characterizes the specific heat at constant volume near T_c:

$$C_v(T) \sim \begin{cases} t^{-\alpha} & \text{as } t \to 0^+ \\ |t|^{-\alpha'} & \text{as } t \to 0^- . \end{cases}$$

A very important quantity in critical phenomena is the correlation length $\xi(T)$. Its critical behavior is governed by the critical exponent ν, i.e.,

$$\xi(T) \sim \begin{cases} t^{-\nu} & \text{as } t \to 0^+ \\ |t|^{-\nu'} & \text{as } t \to 0^- \end{cases}$$

(We have already seen the symbol ξ as world-sheet parameter. The difference is understood.)

In addition to the above critical exponents we have two more, namely the critical exponents for magnetization δ, and for correlation function η. Totally, there are nine critical exponents. These exponents are not all independent: the scaling hypothesis predicts certain relations between them. We simply write down the scaling predictions. (For the derivation see Appendix B)

$$\begin{aligned} \gamma' &= \beta(\delta-1)\ , \quad \gamma = \gamma'\ , \\ 2 &= \alpha + 2\beta + \gamma', \quad \alpha = \alpha', \\ \gamma &= (2-\eta)\nu, \qquad \nu = \nu', \\ D\nu &= 2-\alpha, \end{aligned}$$

where D is the dimensionality of the system.

V.2. Calculations

Bilal and Gervais[15] tried to connect critical behaviors in two and three dimensions. Here their approach is of help to us. As before we consider a two-dimensional world-sheet embedded in a three-dimensional space. Then, the basic idea in this approach is that, due to quantum fluctuations of the string position operators X_μ, the corresponding two-dimensional world-sheet embedded in the three-dimensional space is certainly very irregular for the most important string configurations. Such an irregular string surface may possibily have a nontrivial Hausdorff dimension in the large distance limit. Now imagine measuring the distance between two points in the world-sheet. The three-dimensional distance and the parameter-space distance are related through the general form

$$< \sum_{\mu=1}^{3} \left[X^{\mu}(\xi_1) - X^{\mu}(\xi_2)\right]^2 > \sim \left[(\xi_1 - \xi_2)^2\right]^{\frac{1}{\sqrt{d_F}}} \qquad (V.1)$$

where ξ_1 and ξ_2 are the parameters of the world-sheet, d_F the Hausdorff dimension of the world-sheet. The left-hand side of eq. (V.1) is the square of the euclidean norm in three dimensions. The fluctuations have been smoothed out by taking some quantum average over the X - field. Similarly, the right-hand side contains the square of the two-dimensional euclidean norm. Hereafter, we shall denote the norms simply by vertical bars.

Let us consider the critical features of the 3DIM in term of the ϕ^4 field theory in three dimensions, i.e., we take the field operators

$$\phi = \phi(X^{\mu}) \ ,$$

where X^{μ} is a point in the three-dimensional euclidean space. We now take the following large distance behavior

$$< \phi^{p}(X_1)\phi^{p}(X_2) > \sim |X_1 - X_2|^{-2d} \qquad (V.2)$$

where d is defined to be the Wilson scaling dimension of ϕ^p which are the scaling operators of the three-dimensional theory, and p, appearing on the left-hand side, is an integer. Th three-dimensional fermionic string theory may be regarded formally as defining an embedding of the two-dimensional world-sheet into the three-dimensional space, that is, we have the mapping

$$\xi \quad \to \quad X^{\mu}(\xi).$$

We have mentioned $\phi^{p}(x)$ as the scaling operators in three-dimensional theory. The inverse of the mapping just mentioned may be assumed to convert these operators into primary fields in the world-sheet. Thus under this assumption $\phi^{p}(\xi)$ is as much a field in the two-dimensional world-sheet as $X(\xi)$ is. If this assumption is correct and further if $\phi^{p}(\xi)$ are considered as primary fields in two dimensional quantum field theory, then we also have

$$<\phi^{p}(X^{\mu}(\xi_1))\phi^{p}(X^{\mu}(\xi_2))> \xrightarrow{X^{\mu}\to\ \xi} <\phi^{p}(\xi_1)\phi^{p}(\xi_2)>$$

$$\sim|\xi_1-\xi_2|^{-2\delta} \qquad (V.3)$$

where δ is by definition one of the scaling dimensions of the two-dimensional field theory. By combining eqs. (V.1),(V.2) and (V.3) we get

$$\frac{d}{\sqrt{d_F}} = \delta \qquad (V.4)$$

As mentioned in Chapter III, the value of the Hausdorff dimension d_F for the fermionic surface calculated numerically is in the range 7.3 ± 0.2 to 8.3 ± 0.4. Since the Hausdorff dimension for a purely bosonic surface has been found to be 8.3 ± 0.1, it appears probable that the actual d_F for the fermionic surface is closer to 7.3 than to 8.3. For calculating the critical exponents we prefer to take $d_F = 7.3$. We also have the values of d, the Wilson scaling dimension, calculated[16] for the ϕ^2 and ϕ^4 theories by Brezin and Coworkers[14],

$$d(\phi^4) = D + \omega \text{ and } d(\phi^2) = D - \frac{1}{\nu}, \qquad (V.5)$$

for a general spacetime dimension D. Here ω gives the correction to scaling and ν is a critical exponent. Numerical calculations for D = 3 by the same methods give

$$\omega = 0.81 \text{ and } \nu = 0.631.$$

Using these values in eqs. (V.5), together with D = 3, we obtain

$$d(\phi^4) = 3.81, \quad d(\phi^2) = 1.415 \qquad (V.6)$$

Now using eq.(V.4) we obtain two values of δ for $d_F = 7.3$:

$$\delta_1 = 1.410$$
$$\delta_2 = 0.5237 \qquad (V.7)$$

As we have already mentioned, δ_1 and δ_2 can be identified with the scaling dimensions of the two-dimensional conformal quantum field theory. It is interesting that the almost heuristic arguments given above enables us to obtain from the three-dimensional scaling dimensions the two-dimensional scaling dimensions via the string picture.

We use now the standard scaling relations[16, 17]

$$\alpha = \frac{D - 2\delta_1}{D - \delta_1}, \quad \beta = \frac{\delta_2}{D - \delta_1} \qquad (V.8)$$

In the above relations α and β are, respectively, the specific heat and spontaneous magnetization critical exponents.

In the original work of Fisher[16] δ_1 and δ_2 are, respectively the scaling dimensions of the energy density and magnetization fields. In eq. (V.7) above we have derived δ_1 and δ_2 as the scaling dimensions of the 2D ϕ^4- and ϕ^2 - field theories respectively. By this indentification we have tacitly assumed the ϕ^4 and ϕ^2 theories to correspond to the energy density and magnetization fields, respectively. In the discussions that were given by Wilson[13] regarding the correspondence between the Ising model and the Lagrangian field theories ϕ^2 - field was identified with the Ising model in the continuum limit. At a higher level of approximation ϕ^4- theory was also identified with the Ising model. It is, however, obvious that while ϕ^2- theory contains only the Kinetic energy term, the ϕ^4- theory contains a self-interaction term. Thus, an identification of the ϕ^2- theory with only the magnetization field and of the ϕ^4- theory with the energy density field appears physically reasonable. That is what we have done in calculating α and β by using eqs. (V.7) and (V.8).

$$\alpha = 0.113, \qquad \beta = 0.329$$

We have also calculated the values of α and β using $d_F = 8.3$ just for the sake of comporison. As we have mentioned before in Chapter III early work on the bosonic surface gave very large Hausdorff dimensions while the latest work (ref.11) gave the above value of d_F. Of the two sets of results available for the fermionic surface in ref.11 the higher one coincides with the above value while the lower one is 7.3. Considering this circumstance and in agreement with the general belief that the Hausdorff dimension of the fermionic surface should be less than that bosonic surface, we expect 7.3 to be the more correct value and of interest to us.

The values of α and β calculated with $d_F = 8.3$ are $\alpha = 0.212$, $\beta = 0.293$. As shown in Table 1 the agreement with other known values is less satisfactory than that with $d_F = 7.3$.

The table below compares our results with results obtained by other methods.

Table 1

Critical exponent	Expt.	3DIM		ϕ^4 model	Our results	
		series exp	RG	D = 3	$d_F = 7.3$	$d_F = 8.3$
α	0 - 0.2	0.125	0.08	0.17	0.113	0.212
β	0.3 - 0.4	0.313	0.34	0.33	0.329	0.293

Source: H. Eyring, Statistical Mechanics and dynamics, 2d ed., Wiley Interscience, New York, 1982. P.475.

L.E.Reichl, A Modern Course in Statistical Physics, University of Texas Press, Austin, 1980, P. 344.

The other exponents could be calculated by using the scaling relations mentioned previously.

In conclusion, we have calculated the critical exponents α and β by using heuristi notions of the string theory. Of course, these exponents have been calculated before by other methods with great accuracy, eq., series expansion and renormalisation group. However, these methods are not capable of calculating the free energy. There is a hope that the free energy can eventually be calculated by the string theory approach. If that becomes possible, it will be a great achievement of this theory.

APPENDIX A

THE TWO-DIMENSIONAL ISING MODEL

In this appendix we briefly review the analytic methods used to solve the 2DIM in the absence of an external magnetic field.

(a) The partition function in terms of the Ising variables. consider an N X M square lattice. At each lattice site we have an Ising spin interacting with its nearest neighbors. The Hamiltonian of the spin system is then

$$H(\sigma_{11},\ldots,\sigma_{NM}) = -J_1\Sigma\ \sigma_{nm}\sigma_{n+1,m}\quad -J_2\Sigma\ \sigma_{nm}\sigma_{n,m+1}, \qquad (A.1)$$

where the σ_{nm}'s are the Ising variables each taking the values ±1. J_1 and J_2 are, respectively, the vertical and horizontal interaction strengths. We now write the partition function:

$$Z = \sum_{\sigma_{11}=\pm 1}\cdots\sum_{\sigma_{NM}=\pm 1} \exp\{-\beta H(\sigma_{11},\ldots,\sigma_{NM})\} \qquad (A.2)$$

(b) Transformation to Pauli spin matrices

A vital observation shows that the partition function (A.2) is the trace of the N^{th} power of a transfer matrix V. The transfer matrix can conveniently be expressed in terms of Pauli spin matrices: τ^x,τ^y,τ^z, and the unit matrix I. There are 2^M possible configurations in a given row as it contains M spins. We need, therefore,

a 2^M X 2^M matrix to describe this state of affairs.
We define

$$\tau_m^z = I \times \ldots \times I \times \tau^z \times I \times \ldots \times I,$$
$$\tau_m^x = I \times \ldots \times I \times \tau^x \times I \times \ldots \times I. \qquad \text{(A.3}$$

In terms of these direct-product matrices we can write the generalized matrices V_1 and V_2 as

$$V_1 = (2 \sinh 2K_1)^{M/2} \exp K_1^* \Sigma \tau_m^x,$$
$$V_2 = \exp K_2 \Sigma \tau_m^z \tau_{m+1}^z, \qquad \text{(A.4}$$

where $K_1 = \beta J_1$, $K_2 = \beta J_2$ and K_1^* is defined by

$$\tanh K_1^* = \exp(-2K_1).$$

The partition function is then

$$Z = \mathrm{tr}(V_1 V_2)^N = \mathrm{tr}\ V^N. \qquad \text{(A.5}$$

(c) Introduction of fermion operators through the Jordan-Wig transformation.

The direct-product matrices τ_m obey the same commutation relations as the Pauli matrices. One then constructs 'raising' and 'lowering' operators, $\tau_m^{\pm}$, in the usual manner:

$$\tau_m^{\pm} = \tfrac{1}{2}(\tau_m^x \pm i \tau_m^y). \qquad \text{(A.6}$$

These spin raising and lowering operators obey mixed commutation-anticommutation rules which are neither bos nor fermionic in character. But this is not a serious problem as one could introduce a transformation (called the Jordan-Wigner transformation) to fermion operators which obey anticommutation rules only. Denoting the fermion operators by C_m for the m^{th} site in a given row, we write the JW transformation as

$$C_m = \{\exp(\pi i \sum_{l=1}^{m-1} \tau_l^+ \tau_l^-)\} \tau_m^- ,$$

$$C_m^+ = \{\exp(\pi i \sum_{l=1}^{m-1} \tau_l^+ \tau_l^-)\} \tau_m^+ ,$$

and the anticommutation rules are

$$\{C_m, C_n\} = \{C_m^+, C_n^+\} = 0,$$

$$\{C_m, C_n^+\} = \delta_{mn}$$

The number operator N_m is given by

$$N_m = C_m^+ C_m = \tau_m^+ \tau_m^-, \qquad (A.8)$$

with eigenvalues 0 or 1.

After considerable algebra, the matrices V_1 and V_2 in equation (A.4) can be expressed in terms of the fermion operators C_m, C_m^+. (The interested reader may consult refs. 1 and 2).

The partition function is calculated by classifying the states of the system according to the eigenvalues of the total number operator $N = \sum_m C_m^+ C_m$. V_1 are bilinear in the fermion operators and states of even or odd n may be considered separately.

(d) Eigenvalues of the transfer matrix

The next step is to diagonalize the transfer matrix V. One might think of a linear transformation to new fermion operators of the type

$$\xi_q = \sum_m (A_{qm} C_m + B_{qm} C_m^+) \tag{A.9}$$

such that V attains the simplest form

$$V \propto \exp(-\sum_q \varepsilon_q \xi_q^+ \xi_q + \text{const.})$$

where ε_q are the single-fermion 'energies' and are defined by

$$\text{Cosh } \varepsilon_q = \text{Cosh } 2K_2 \text{ Cosh } 2K_1^* - \sinh 2K_2 \sinh 2K_1^* \text{ Cos } q.$$

This could be achieved by introducing running wave operators which is permissible due to translational symmetry. (Again one can see the details in ref. 1)

The thermodynamic properties of the 2DIM are all contained in the partition function is just the Nth power of the largest eigenvalue of V in the limit M, N → ∞.

This eigenvalue is given by

$$\Lambda_{max} = (2 \sinh 2K_1)^{M/2} \exp(\frac{M}{4\pi} \int_{-\pi}^{\pi} \varepsilon_q dq). \tag{A.11}$$

The free energy per spin is

$$F = - \frac{K_B T}{MN} \ln Z \text{ , as M, N} \to \infty$$

$$= -K_B T \left[\ln (2 \sinh 2K_1)^{\frac{1}{2}} + \frac{1}{4\pi} \int_{-\pi}^{\pi} \varepsilon_q dq\right] \tag{A.12}$$

which is the final result.

In the purely combinatoric method Kac and Ward devised a method to write the partition function as a sum over closed directed paths. See reference 3.

APPENDIX B

THE CRITICAL EXPONENT RELATIONS

The free energy per spin can be decomposed into two parts: the regular part $F_r(T,B)$, and the singular part $F_s(t,B)$. The singular part is responsible for the observed critical behavior of the system as the system's temperature approaches the transition point. We write

$$F(T,B) = F_r(T,B) + F_s(t,B) , \qquad (B.1)$$

where $t = (T-T_c)/T_c$ and B is the external magnetic field.

The singular part in eq. (B.1) is assumed to be a homogeneous function of the parameters t and B, i.e.,

$$F_s(\lambda^p t, \lambda^q B) = \lambda F_s(t,B). \qquad (B.2)$$

Differentiate eq.(B.2) w.r.t. B and set $\lambda = (-t)^{-1/p}$ and $B = 0$. The result is

$$M(t,0) = (-t)^{(1-q)/p} M(-1,0). \qquad (B.3)$$

But near T_c $M(t,0) \sim (-t)^{\beta}$. Thus, we see that

$$\beta = \frac{1-q}{p} . \qquad (B.4)$$

To determine δ, differentiate eq.(B.2) w.r.t. B and then put $t = 0$, $\lambda = B^{-1/q}$ to get

$$M(0,B) = B^{(1-q)/q} M(0,1) .$$

Comparing this to the power-law behavior

$$M(0,B) \sim B^{1/\delta}$$

we get

$$\frac{1}{\delta} = \frac{1 - q}{q} . \qquad (B.5)$$

Differentiating eq (B.2) twice w.r.t. B gives the magnetic susceptibilit
If we put $\lambda = t^{-1/p}$ and $B = 0$ we have

$$\chi(t,0) = t^{(1 - 2q)/p} \chi(1,0).$$

But we know the power-law behavior for χ (page 27). Comparison yields

$$\gamma = \frac{2q - 1}{p} , \qquad (B.6)$$

similarly, $\gamma' = \gamma$.

The exponent α is determined by exactly similar arguments:

$$\alpha = 2 - \frac{1}{p} , \quad \alpha' = \alpha.$$

Combining eqs. (B.4-7) we get

$$\gamma = \gamma' = \beta (\delta - 1)$$

$$2 = \alpha + 2\beta + \gamma$$

Near the critical point the magnetic susceptibility and the correlation length are related by (see ref.18)

$$\chi \sim \xi^{2 - \eta} .$$

Comparing this with the power laws for χ and ξ (pages 27, 28) we see that

$$\nu = \nu', \ (2 - \eta)\nu = \gamma.$$

Similar argument give the other relations between the critical exponent See, for example, ref. 18 and the first source mentioned under Table 1 on page 34.

References

1. L.Schultz, D.Mattis and E.Lieb,
Rev. Mod.Phys, 36 (1964) 856.
2. J.Callaway, Quantum theory of the Solid State, 1974,
Acadamic Press, Inc., New York, P.132
3. M.Kac and J.C.Ward, Phys.Rev., 88 (1952) 1332
M.L.Glasser, Am.J.Phys, 38 (1970) 1033.
4. J.Scherk, Rev.Mod.Phys, 47 (1975) 123
C.Rebbi, Phys. Rep. 12C (1974) 1.
5. A.M. Polyakov, Phys. Lett., 103B (1981) 207.
6. A.M. Polyakov, Mod.Phys. Lett., 2A (1987) 893.
7. A.M. Polyakov and A.B.Zamolodchikov,
Mod, Phys. Lett., 3A (1988) 1213.
8. A.A. Migdal, Nucl. Phys. 189B (1981) 149.
9. J. Ambjorn, B.Durhuus, and T. Jonsson,
Phys.Rev. Lett., 58 (1987) 2619.
10. A.M. Polyakov, Nucl. Phys, 268B (1986) 406.
11. J.Jurkiewicz, A.Krzywicki, B.Petersson and B.Soderberg
Phys. Lett., 213B (1988) 511.
12. M.A. Bershadsky and A.A.Migdal,
Phys.Lett., 174B (1986) 393.
13. K.Wilson and J.Kogut, Phys.Rep. 12C (1974) 75.
14. E.Brezin, J.C.Le Guillon and J.Zinn-Justin, in phase
transitions and critical phenomena, ed.by C.Domb
and M.S. Green (Acadamic Press, New York, 1976)Vol.6.
15. A.Bilal and J-L.Gervais,
Nucl.Phys., 295B [FS21] (1988) 277.

16. M.E.Fisher, Rep.Prog. Phys. 30 (1967) 615.

17. A.M. Polyakov, Mod.Phys. Lett. A.3 , (1988) 819.

18. R.J.Baxter, Exactly Solved Models in Statistical Mechanics,
Acadamic Press, 1982.

DECLARATION

I hereby declare that this thesis is my original work, done under the guidance of Dr. M.A. Mojumder. All relevant sources used for the thesis are duly acknowleged.

Mesfin Tadesse

This thesis has been submitted for examination with my approval as a university advisor.

Dr. M.A. Mojumder