GRADUATE SEMINAR REPORT

On

VECTOR (MULTICRITERIA) OPTIMIZATION

(submitted in partial fulfillment of M.Sc. degree in Mathematics)

BY

ASSAYE WALELIGN

ADVISOR

Dr. SEMU MITIKU

SCHOOL OF GRADUATE STUDIES

ADDIS ABABA UNIVERSITY

June, 2004

Addis Ababa

CONTENTS

Content **Page**

PREFACE

We have said that multiobjective optimization is concerned with minimizing or maximizing problems and giving a chance for decision makers to have alternatives for their decision making problems. Unlike single objective optimization problems, in solving, multiobjective optimization problems we have solution sets that is called efficient set. It is from this set decision is made by taking elements of efficient set as alternatives, which is given by analysts.

Today, the problem of multiobjective optimization is a central problem in optimization due to its importance in different fields such as marketing, transport, engineering, environmental control, managing pollution problems and so on.

The main instrument in solving multiobjective optimization problems is the concept of domination structure (ordering cone): using the ordering cone as a preference we usually find a solution set. Also multipliers have a great help in solving multiobjective optimization problems.

This report is a completion of the two seminars I have delivered, for the qualification for M.Sc. in mathematics. In this report, discussions about solution concepts and some properties solutions, multiplier rules, and necessary and sufficient conditions in the sense of Kuhn-Tucker are made in five chapters.

I take this opportunity to thank my advisor: Dr. Semu Mitiku for his continuous follows up and made suggestions for the improvement of the report. And finally I would like to thank my friends who helped me in typing manuscripts and my family for their help and encouragement.

Assaye Walelign
July 2004

CHAPTER ONE

Introduction

We want to have a good life, which may mean more wealth, more power, more respect and more time for ourselves, together with good health and a good second generation, etc. to fulfill these and other unlimited needs we made multi-criteria decision making problems in our everyday life even though might not be perceived explicitly.

Optimization problems occur, for example, when buying things (like a car or a house) or selecting a means of transport to work, in engineering when designing focusing systems, space craft structures, bridges, in economics when planning production systems or pricing products, and in environment control when managing pollution problems. What is common in most of these optimization problems is that they have several at least partly conflicting criteria to be taken in to consideration at the same time. A number of different goals are desired to attain simultaneously. In this case, methods of traditional (single objective) optimization are not enough, but we need new ways of thanking, new concepts, and new methods.

Although analysis of multicriteria problems have been scattered in human history, putting this analysis into a formal mathematical getting is fairly new. Although there are a number of scholars and scientists who have made contributions in this mathematical analysis of multicriteria decision-making problems, Pareto is perhaps one of the most recognized pioneers.

In treating multicriteria decision making problems, constructing models used to know the structure of the problem, what the problem is, which factors. Comprise the problem, how they interrelate and so on.

Through the process, the objective of the problem and alternatives to perform it are specified. Hereafter, we shall use the notation **O** for the objective and X for the set of alternatives, which is supposed to be a subset of an n-dimensional vector space.

To solve decision-making problems by some systems, analytical methods, we usually required that degrees of objectives be represented in numerical terms, which may be of multiple kinds even for one objective. We restrict numerical terms to physical measures (money, weight, length, time...). As a performance index, for the objective O_i, an objective function

$$f_i : X \rightarrow R^1$$

is introduced . The values of $f_i(x)$ indicates how much impact is given on the objective O_i by performing an alternative x. In this paper, we assume that a smaller value for each objective function is preferred to a larger one.

Now we formulate our decision-making problems as a multiobjective optimization problem:

$$\text{(P)} \qquad \text{minimize} \quad f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \text{ over } x \in X$$

This kind of problem is also called a vector optimization. In some cases, some of the objective functions are required to be maintained under given levels prior to minimize other objective functions. We require these objective functions $g_j(x) \leq 0, j=1\ldots m$, which are called constraint functions. We will consider the problem (P) itself or (P) accompanied by the constraint $g_j(x) \leq 0$, $(j=1\ldots m)$, of course, an equality constraint $h_k(x) = 0$ may be replaced by two inequalities $h_k(x) \leq 0$ and $- h_k(x) \leq 0$.

Unlike a single objective function, an optimal solution in the sense of one that minimizes all objective functions simultaneously may not exist in multiobjective problems, and, hence we are troubled with conflicts among objectives in decision-making problems with multiple objectives. Taking the total balance of the objectives into account should make the final decision. Here is the responsibility of the decision maker to give final decision among suggested alternatives.

The decision maker's value is usually represented by saying whether or not an alternative x is preferred to an other alternative x', or equivalently whether or not f(x) is preferred to f(x'). This means the decision value is represented by some binary relation defined over X or f(X). Such a binary relation representing the decision maker's preference becomes an o rder. It i s c alled p reference o rder. T he d ecision m aker's p reference o rder s hall b e defined on the so-called criteria space Y, w hich includes the set f(X).Several kinds of preference orders is possible, sometimes, the decision-maker can not judge whether or not f(x) is preferred to f(x'). Such an order that admits incomparability for a pair of objects is called a partial order, whereas the order requiring the comparability for every pair of objects is a weak (or a total order). In practice, we often observe a partial order for the decision maker's preference. Unfortunately, however, an optimal solution in the sense of one that is most preferred with respect to the other, whence the notion of optimality do not necessarily exist for partial orders. So, instead of strict optimality, we introduce the notion of efficiency in multi-objective optimization. A vector f(x') is said to be efficient if there is no f(x) preferred to f(x') with respect to the preference order. The final decision is usually made among the set of efficient solutions.

This paper is mainly concerned with some of the theoretical aspects in vector optimization problems; in particular we will focus on existence, necessary and sufficient conditions of efficient solutions.

Chapter two is devoted to some mathematical notations and preliminaries. The first section gives a review of convex sets, cones, convex functions and other properties related to convexity, which have important role in multi-objective optimization. The second section introduces preference order as a binary relation and the concept of ordering cone. The ordering cone that is the main tool in solving multiobjective optimization problem will be formulated.

Chapter three begins with the introduction of several concepts for solutions in multiobjective optimization. Above all, efficient solutions will be the subject of primary considerations in subsequent theories. Next, some properties of efficient solutions and existence of efficient solutions will be discussed in the second section.

Chapter four will discuss on multipliers in vector optimization. Some concepts that can be used in the formulation of multipliers will be considered in the first section. The second section discuss as to how the idea of multipliers can be extended from single objective to multiobjective optimization case, alternative ways of proving theorem of multipliers rule and regularity conditions.

In chapter five we see Kuhn-Tucker conditions for multiobjective problems. First section introduces the concept of characterization efficiency and proper efficiency by scalarization that shall be used in the proof of necessary and sufficient conditions for efficient solution. The second section that is the main target of the paper is discussed on necessary and sufficient conditions in the sense of Kuhn-Tucker. That means deals with conditions for efficient solution with multipliers.

CHAPTER 2

NOTATIONS AND PRELIMINARIES

The chapter is devoted mainly to mathematical preliminaries. First, some basic properties of convex sets and functions are introduced. Second, the concept of preference order and ordering cones are discussed.

2.1. **Some Properties of Convexity**

As convexity plays an important role in the theory of optimization with a single objective it is fundamental in the theory of multi objective optimization, so the concepts of convex sets and convex functions will be introduced. In this section all spaces considered are finite-dimensional Euclidean spaces.

2.1.1. Convex Sets

Convex sets, which are fundamental in convex analysis, will be introduced.

Definition 2.1.1.

A subset X of R^n is called convex if $\alpha x + (1-\alpha)y \in X$ for any x, $y \in X$ and $\alpha \in [0,1]$.

Definition 2.1.2.

A subset K of R^n is said to be a cone if $\alpha x \in K$ whenever $x \in K$ and $\alpha > 0$. Moreover it is called convex cone if it is convex. From the above definition it follows immediately:

Proposition 2.1.1.

A set K in R^n is convex if and only if

i) $\alpha x \in K$ for all $x \in K$ and $\alpha > 0$.

ii) $x + y \in K$ for each $x, y \in K$.

Definition 2.1.3.

A cone K in R^n is called pointed if $-x \notin K$ whenever $x \neq 0$, $x \in K$, that is, $K \cap (-K) = \{0\}$. Moreover K is acute if there is an open half space

$\overset{0\,+}{H} = \{x \in R^n : \langle x, x^* \rangle > 0; x^* \neq 0\}$ Such that $clK \subset \overset{0\,+}{H} \cup \{0\}$.

From Definition 2.1.3 it can be easily seen that a convex cone K is acute if and only if clK is pointed. That means the two concepts coincide if K is closed convex cone.

Definition 2.1.4 (Polar and Strict Polar)

For a subset X in IR^n, its positive polar X^o is defined by

$$X^o = \{x^* \in R^n : \langle x, x^* \rangle \geq 0 \text{ for any } x \in X\}.$$

The strict positive polar X^{so} of X is defined by

$$X^{so} = \{x^* \in R^n : \langle x, x^* \rangle > 0 \text{ for any nonzero } x \in X\}.$$

Proposition 2.1.2

Let X, X_1, and X_2 be sets in R^n. Then

(i) The polar X^o is a closed convex cone,

(ii) The strict polar X^{so} is a convex cone,

(iii) $X^o = (clX)^o$,

(iv) If X is open, $X^{so} \cup \{0\} = X^o$,

(v) $X_1 \subset X_2$ implies $X_2^o \subset X_1^o$ and $X_2^{so} \subset X_1^{so}$, and

(vi) If K is nonempty convex cone, $K^{oo} = clK$.

Definition 2.1.5

A set X in IR^n is said to be a polyhedral convex set if it can be expressed as the intersection of some finite collection of half spaces,

i.e. $X = \{x : \langle b^i, x\rangle \leq \beta_i \ \ (i = 1, 2, ..., n)\}$ where $b^i \in IR^n, \beta_i \in IR$. If $\beta_i = 0$ for all i, then X is called polyhedral convex cone.

Definition 2.1.6

A set X in IR^n is said to be finitely generated convex set if there exist vectors a^1, a^2, …, a^m such that for a fixed integer k ($0 \leq k \leq m$). X can be expressed as:

$$X = \{x : x = \sum_{i=1}^{m} \alpha_i a^i,\ \alpha_i \geq 0\ (i = 1, 2, ..., m), \sum_{i=1}^{k} \alpha_i = 1\}$$

If $X = \{x : x = \sum_{i=1}^{m} \alpha_i a^i,\ \alpha_i \geq 0\ (i = 1, 2, ..., m)\}$, then X is said to be a finitely generated convex cone and $\{a^1, a^2, …, a^m\}$ is called the set of generators for the cone.

Proposition 2.1.3.

Let X be a polyhedral convex set in IR^n and let f be a linear vector valued function from R^n into R^p then the set f(X) is a polyhedral convex set in $\mathbf{R}^p$.

i.e. if $X = \{x : x = \sum_{i=1}^{m} \alpha_i a^i,\ \alpha_i \geq 0\ (i = 1, 2, ..., m), \sum_{i=1}^{k} \alpha_i = 1\}$, then

$$f(X) = \{x : x = \sum_{i=1}^{m} \alpha_i b^i,\ \alpha_i \geq 0\ (i = 1, 2, ..., m), \sum_{i=1}^{k} \alpha_i = 1\},$$

where $b^i = f(a^i)$ (i = 1, 2, …, m)

Corollary 2.1.1

If X_1 and X_2 are polyhedral convex sets, then $X_1 + X_2$ is polyhedral.

Definition 2.1.7

Given a set X and a convex cone K in $\mathbf{R}^n$, X is said to be K- convex if X + K is convex set.

Remark 2.1.1

A set X is convex if and only if X is {0}-convex. Moreover, if X is a convex set it is D-convex for any arbitrary, non-empty convex cone D.

2.1.2. **Convex Functions**

This subsection is concerned with convex functions, which are instrumental in optimization problems. Recall that a function f from a convex set X in to $(-\infty,\infty)$ is called convex on X if and only if

$$\alpha f(x)+(1-\alpha)f(y) \geq f(\alpha x+(1-\alpha)f(y)),$$ Which can be rewritten as

$$\alpha f(x)+(1-\alpha)f(y) - f(\alpha x+(1-\alpha)y) \in R_+$$

Note that R_+ is a convex cone. we extend this i dea in to multi-objective optimization problems.

Definition 2.1.8

Let X be a convex set in R^n, D be a convex cone in R^p. Then a function f from X in to R^p is said to be D-convex if

$$\alpha f(x)+(1-\alpha)f(y) - f[\alpha x+(1-\alpha)y] \in D,$$ for any x,y$\in$ X and $\alpha \in [0,1]$.

We state the following two propositions to be used later in chapter five.

Proposition 2.1.4

Let X be a convex set in R^n and let f be a function from R^n in to R^p, and D be a convex cone in R^p.

If the function f is D-convex, then the set f(X) is D-convex.

Proposition 2.1.5

Let X be a convex set in R^n and $f = (f_1, f_2, \ldots, f_p)$ be functions from R^n in to R^p. The function f is R^p_+-convex if and only if each f_i is convex, in this case f(X) is R^p_+-convex.

Proof:

i) Let f is R^p_+-convex function.

If $x, y \in R^n$, then $\lambda f(x) + (1-\lambda) f(y) - f[\lambda x + (1-\lambda)y] \in R^p_+$, for $\lambda \in [0,1]$, which implies

$$\lambda(f_1(x), f_2(x), \ldots, f_p(x)) + (1-\lambda)(f_1(y), f_2(y), \ldots, f_p(y)) -$$

$$(f_1[\lambda x + (1-\lambda)y], f_2[\lambda x + (1-\lambda)y], \ldots, f_p[\lambda x + (1-\lambda)y]) \in R^p_+$$

$$(\lambda f_1(x) + (1-\lambda) f_1(y), \ldots, \lambda f_p(x) + (1-\lambda) f_p(y) -$$

$$(f_1[\lambda x + (1-\lambda)y], f_2[\lambda x + (1-\lambda)y], \ldots, f_p[\lambda x + (1-\lambda)y]) \in R^p_+$$

$$\Rightarrow \lambda f_i(x) + (1-\lambda) f_i(y) - f_i[\lambda x + (1-\lambda)y)] \in R_+$$

Therefore f_i is convex for each i

ii) If f_i is convex for each i, then

$$\lambda f(x) + (1-\lambda)(f(y)) - f[\lambda x + (1-\lambda)y) =$$

$$(\lambda f_1(x) + (1-\lambda) f_1(y), \ldots, \lambda f_p(x) + (1-\lambda) f_p(y)$$

$$-(f_1[\lambda x + (1-\lambda)y], \ldots, f_p[\lambda x + (1-\lambda)y]$$

by convexity of f_i, $\lambda f_i(x) + (1-\lambda) f_i(y) - f_i[\lambda x + (1-\lambda)y) \in R_+$ for each i.

Therefore $\lambda f(x) + (1-\lambda) f(y) - f[\lambda x + (1-\lambda)y] \in R^p_+$.

2.2. Preference orders and Ordering Cones

In single objective optimization problems, the word optimality means the minimization of a certain objective function under given constraints. On the other hand, in multi objective optimization problems, it is not clear, since the objectives usually conflict with one another. Therefore, the objectives must be traded off.

In multi objective optimization problems, the preference attitudes of the decision maker play an essential role that specifies the meaning of optimality. These are often represented as binary relations on the objective space and are called preference orders. This section deals with preference orders and their representations by set-valued maps which can be called ordering cones.

2.2.1. **Preference Orders**

As mentioned above the preference attitude of the decision maker represents a preference order in the objective space. That means it is a binary relation on a set $Y = f(X) \subset R^p$, where f is the vector valued objective function and X is the feasible decision set.

The basic binary relation $\succ$ means strict preference; i.e., $y \succ z$ for y, $z \in Y$ means the result (objective value) y is preferred to z. We define two binary relations $\sim$ and $\succsim$, in relation with strict preference ($\succ$), as

$$y \sim z \text{ if and only if not } y \succ z \text{ and not } z \succ y,$$

$$y \succsim z \text{ if and only if } y \succ z \text{ or } z \sim y.$$

The relation $\sim$ is called indifference ($y \sim z$ as y is indifference to z), and $\succsim$ is called preference-indifference ($y \succsim z$ as z is not preferred to y).

The binary relations used as preference or indifference relations have interesting properties. Some of these properties of a binary relation are listed below:

A binary relation R on a set Y is

1) Reflexive if yRy for every $y \in Y$.
2) Irreflexive if not yRy.
3) Symmetric if yRz implies zRy.
4) Asymmetric if yRz implies not zRy.
5) Antisymmetric if yRz, zRy imply $y = z$.
6) Transitive if yRz, zRw imply yRw.
7) Negatively transitive if not yRz, not zRw imply not yRw.
8) Connected or complete if yRz or zRy (possibly both).
9) Weakly connected if $y \neq z$ implies yRz or zRy for every y, z, w $\in Y$.

Lemma 2.2.1

Let R be a binary relation on Y.

i) If R is transitive and irreflexive, it is asymmetric.
ii) If R is negatively transitive and asymmetric, it is transitive.
iii) If R is transitive, irreflexive, and weakly connected, it is negatively transitive.

Definition 2.2.1

A binary relation R on a set Y is said to be

i) a strict partial order if R is irreflexive and transitive.
ii) a weak order if R is asymmetric and Negatively transitive.
iii) a total order if R is irreflexive, transitive, and weakly connected.

From lemma 2.2.1 and definition 2.2.1, it can be redefined that a binary relation R is:

i) a strict partial order if it is asymmetric.
ii) A weak order if it is transitive.
iii) A total order if it is negatively transitive.

Remark 2.2.1

If R is a total order, then it is a weak order; and if R is a weak order then it is a strict partial order.

The preference order is usually assumed to be at least a strict partial order; that is, irreflexivity of preference (y is not preferred to itself) and transitivity of preference.

Definition 2.2.2

Let Y be a feasible set in the objective space R^p and $\succ$ be a preference order on Y. Then an element y` $\in$ Y is said to an efficient (noninferior) element of Y with respect to the order $\succ$ if there does not exist an element $y \in Y$ such that $y \succ y`$. The set of all efficient elements is denoted by $\xi(Y,D)$={y` $\in$ Y: there is no $y \in Y$ such that $y \succ y`$.}

Remark 2.2.2

The set of efficient elements in the decision space X may be defined in an analogous way as:

$\xi(X, f^{-1}(\succ)) = \{x` \in X : \textit{there is no } x \in X \textit{ such that } f(x) \succ f(x`), \textit{where } f^{-1}(\succ)$

is the order on X induced by $\succ$ as x $f^{-1}(\succ)$ x`  if and only if f(x) $\succ$ f(x`).

2.2.2 Ordering cones

Preference order (binary relations) on a set Y can be represented by a set-valued map from Y in to Y. In fact, a binary relation may be considered to be a subset of the product set TxT, and so it can be regarded as a graph of a set-valued map from Y in to Y. That means we identify the preference order $\succ$ with the graph of the set-valued map P:

P(y)={y` $\in$ Y :y $\succ$ y`}, (p(y) is a set of elements in Y which are less preferred to y).

Graph P: ={(y,y`) $\in$ YxY, y` $\in$ p(y)}={(y,y`) $\in$ YxY: y $\succ$ y`}.

An other way of representing preference orders by set-valued maps is the concept of domination structure (ordering cone).

For each y $\in$ Y $\subset R^p$, we define the set of domination factors

D(y):={d $\in R^p$:y $\succ$ y+d} $\cup$ {0}.

Clearly the set-valued map D from Y to R^p represents the given preference orders. We call D the domination structure (ordering cone).

We need to have asymmetric domination structure D(.).That means if d $\in$ D(y), $d \neq 0$, then -d $\notin$ D(y+d) for all y. In other words if y $\in$ y`+D(y`), y` $\in$ y+D(y), then y=y`.

As D is a preference orders (ordering cone), the set of efficient elements of a set Y in R^p be restated as:

$\xi(Y,D) = \{y` \in Y : \textit{there is no } y \neq y` \in Y \textit{ such that } y` \in y + D(y)\}$

Remark 2.2.3

We induce a domination structure D` on X from a given domination structure D on Y as:

D`(x)={d` $\in R^n$: f(x+d`) $\in$ f(x)+D(f(x))\{0}} $\cup$ {0}.

The most important and interesting case of the domination structure is when D(.) is a constant set-valued map, particularly when D(y) is a constant cone for all y. i.e., when D(y)=D (a cone).

D as preferences order is:

asymmetric if and only if d $\in$ D, $d \neq 0$, which implies -d $\notin$ D. Again this is true if and only if D is pointed.

transitive if and only if d,d` $\in$ D implies d+d` $\in$ D and again this is true if and only if D is convex.

We usually take a pointed convex cone to define domination structure and write y $\leqq_D$ y` for y,y` $\in R^p$ if and only if y`-y $\in$ D for a convex cone D in $R^p$.Also y $\leqq_D$ y` means that y`-y $\in$ D but y-y` $\notin D$.

When D is pointed y $\leqq_D$ y` if and only if y`-y$\in$D\{0}.
The subscript D is omitted from y $\leqq_D$ y` and written simply as y $\leqq$ y` if D=R_+^p.
There are different kinds of preference orders. Some of them are listed below:
i) Pareto order: when the domination structure is the cone R_+^p, it is usually denoted by $\succ$,which is equivalent to $\leq$.
ii) Weak Pareto order: when the domination cone is D\{0}=$\overset{o}{R_+^p}$ ={y$\in$ R_+^p : y>0}.

iii) Lexicographic order: usually denoted by $\succ_{\ell}$ and y $\succ_{\ell}$ y` if and only if there is k$\in$ {1,2…p} such that $y_i$=$y_i^{`}$ for all i<k and y_k<$y_k^{`}$.
iv) Order by a polyhedral convex cone: when D={d$\in$ R^p : $Ad \geq 0$} with an lxp matrix.

CHAPTER THREE

SOLUTION CONCEPTS AND SOME PROPERTIES OF SOLUTIONS

In this chapter we discuss solution concepts for multiobjective optimization problems and investigate some properties of solution. Efficiency and proper efficiency are introduced as solution concepts in the first section. In the second section, existence of efficient solution is discussed

3.1. Solution Concepts

The concepts of optimal solutions to multiobjective optimization problems are closely related to the preference attitudes of the decision makers. The most fundamental solution concept is that of efficient solutions with respect to the domination structure of the decision maker.

3.1.1 **Efficient Solutions**

We consider the multiobjective optimization problem

(p) minimize $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$ subjected to $x \in X \subset R^n$ and

let $Y = f(X) = \{y: y = f(x), x \in X\}$.

A domination structure representing a preference attitude of the decision maker is supposed to be given as a set-valued map D from Y to R^P

Definition 3.1.1:

A decision vector $x` \in X$ is said to be an efficient solution to the multiobjective optimization problem (p) with respect to the domination structure D if

$f(x`) \in \xi(Y, D)$. This means if there is no $x \in X$ such that $f(x`) \in f(x) + D$ and

$f(x`) \neq f(x)$ $\quad (f(x`) \in f(x) + D\backslash\{0\})$.In other words,

$(Y - f(x`)) \cap (-D) = \{0\}$. Or equivalently $(f(x`) - D) \cap Y = \{f(x`)\}$.

Definition 3.1.2

A decision vector $x` \in X$ is Pareto optimal ($D = R^p_+$) if there does not exist an other decision vector $x \in X$ such that $f_i(x) \leq f_i(x`)$ for all $i = 1, 2, \ldots, p$ and

$f_i(x) < f_i(x`)$ for at least one objective function. In this case,

$(f(x`) - R^p_+) \cap (Y) = \{f(x`)\}$ or equivalently $(Y - f(x`)) \cap (-R^p_+) = \{0\}$

Definition 3.1.3.

A point x` $\in$ X is said to be a weak pareto optimal solution to the problem (p) if there is no x $\in$ X such that f(x) < f(x`). That means,

$$(Y - f(x`)) \cap (-intD) = (Y-f(x`)) \cap (-D\backslash\{0\}) = (Y - f(x`) \cap (-\overset{o}{R^p_+}) = \varnothing$$

(if there is no another x $\in$ X such that $f_i(x) < f_i(x`)$ for all i = 1,2,…,p.)

Proposition 3.1.1.

Let D_1 and D_2 be domination structures, then D_1 is said to be included by D_2 if $D_1(y) \subset D_2(y)$ for all y $\in$ Y. In this case $\xi(Y, D_2) \subset \xi(Y, D_1)$.

When D is a constant set valued map (whose value is convex cone), we identify the map (domination structure) with the cone D. Then x`$\in$ X is an efficient solution to the problem (p) if and only if there is no x$\in$X such that f(x`) – f(x) $\in$ D\{0}; namely, x` is efficient if and only if

(f(X) - f(x`) ) $\cap$(-D) = {0}. This can be expressed also as (f(x`) – D) $\cap$ f(X)={f(x`)}.

The usual notation for this is that $(y`- D) \cap Y = \{y'\}$ as Y=f(X) , $y`= f(x`)$.

Proposition 3.1.2

Let D be nonempty cone containing 0, then $\xi(Y + D, D) \subset \xi(Y, D)$. Moreover $\xi(Y, D) = \xi(Y + D, D)$ if D pointed and convex.

Proof:

Suppose y $\in$ $\xi(Y + D, D)$ but y $\notin$ $\xi(Y, D)$.

If y$\notin$ Y, then there exist y`$\in$Y and d $\in$D such that y = y` + d.

Sine 0 $\in$D, Y $\subset$ Y+D implies y $\notin$ ξ(Y + D, D), which is a contradiction. If y $\in$Y and $y \notin \xi(Y, D)$, then there is y` such that y$\in$y`+D(y`) which implies y=y`+d, that means y $\notin$ ξ(Y + D, D) again a contradiction to our supposition.

To prove the second part of the proposition, let D be pointed and convex. We have shown that $\xi(Y + D, D) \subset \xi(Y, D)$, now it left to show $\xi(Y, D) \subset \xi(Y + D, D)$.

Suppose y $\in$ $\xi(Y, D)$ but y $\notin$ ξ(Y + D, D), then there exists y`$\in$Y + D with y –y` = d` $\in$D\{0}. Then y` = y`` + d`` with y``$\in$Y, d``$\in$D.

Hence y = y` + d = y``+ d``+ d` =y``+(d``+d`)=y``+d``` $\in$ Y+D, since D is convex (d`+d``=d``` $\in$D) and d`+d`` $\neq$ 0 because D is pointed.

i.e y = y``+d``` implies y$\notin$ ξ(Y , D), contradicts to our supposition.

This shows a larger set has less efficient set with respect to the same ordering cone.

Proposition 3.1.3.

Let Y and Z be two sets in R^p, and let D be a constant ordering cone on R^p then ξ(Y + Z, D) $\subset$ $\xi(Y,D)+\ \xi(Z,D)$

Proof:

Let $\hat{y}$ $\in \xi$(Y +Z, D), then $\hat{y}$ = y + z for some y$\in$ Y, z$\in$Z. We have to show y$\in$ $\xi(Y,D)$ and z $\in \xi(Z,D)$.

Suppose not, there is y` $\in$Y and a nonzero d $\in$D such that y = y` +d

Then $\hat{y}$ = y` + z + d and y` +z $\in$ Y+ Z which contradicts the supposition

$\hat{y}$ $\in \xi$(Y +Z, D). Similarly for Z

Remark 3.1.1.

The converse inclusion of Proposition 3.1.3. is not always true.

Example: -

Let Y = Z = $\{(y_{1,}\ y_2): y_1^2+y_2^2\le 1\}\subset R^2$ *and* $D = R_+^2$, then

y = (-1,0) $\in$ $\xi(Y,D)$ and z = (0, -1) $\in \xi(Z,D)$. However

$$\text{y +z} = (-1,-1) > (-\sqrt{2}\,, -\sqrt{2}\,) = (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}) + (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}) \in Y{+}D.$$

Proposition 3.1.4.

Let Y be a set in R^p, D be a cone in R^p and α be a positive number, then

$\xi(\alpha Y,D)\ =\ \alpha\ \ \xi(Y,D)$.

Proof:

If α = 1 the statement holds trivially.

If $\alpha \neq 1$, then y$\in$ $\xi(\alpha Y,D)$ implies there is y`$\in$Y such that y = $\alpha$ y` it follows α y`$\in \alpha$ $\xi(Y,D)$ implies y`$\in$ $\xi(Y,D)$.

3.1.2 Properly Efficient Solutions

Recall that ξ(Y,D) is a set, that means, the decision maker has to choose an alternative among infinitely many optimal solutions. So we need to have a relatively smaller size optimal solution set. Although weakly (pareto) optimal solutions are important for theoretical considerations, they are not always useful in practice because of its big size. Therefore it needs a more restricted concept than efficient (pareto optimal) solutions, which are properly efficient solutions.

The concept of properly efficient solutions is studied and developed by different scholars such as Borwein, Benson, Henig, Geoffrion, Kuhn-Tucker and others.

Definition 3.1.4 (Tangent cone)

Let $S \subset R^p$ and $y \in S$. The tangent cone to S at y, denoted by T(S,y), is the set of limits of the form $h=\lim t_k(y^k-y)$, where $\{t_k\}$ is a sequence of nonnegative numbers and $\{ y^k\}$ is a sequence in S with limit y.

It can easily verified that T(S,y) is always a closed cone.

Definition 3.1.5 (Borwein's proper efficiency)

A point $x` \in X$ is said to be a properly efficient solution of (P) if $T(Y+D,f(x`)) \cap (-D)=\{0\}$.

Proposition 3.1.5

If a point $x` \in X$ is a properly efficient solution of (P) by definition of Borwein, then it is also an efficient solution of (P).

Proof: (proof by contra positive)

If x' is not efficient, then there exists a nonzero vector $d \in D$ such that $d = f(x')-y$ for some $y \in Y$.

Let $d^k = (1-\frac{1}{k})d \in D$ and $t_k = k$ for $k = 1, 2, \ldots$. Then

$$y+d^k = f(x')-d+(1-\frac{1}{k})d = f(x')-(\frac{1}{k})d \to f(x') \ \ as\ k \to \infty \text{ and}$$

$$t_k(y+d^k - f(x')) = k(-d+(1-\frac{1}{k})d) = -d \to -d \ \ as\ k \to \infty$$

Hence, $T(Y+D, f(x')) \cap (-D) \neq \{0\}$, and x' is not properly efficient.

The converse of proposition 3.1.5 is not always true. (See example 3.1.1)

Definition 3.1.6 (projection cone)

Let $S \subset R^p$. The projection cone of S, denoted by P(S), is the set of all points h of the form $h = \alpha y$, where $\alpha \geq 0$ and $y \in S$.

Definition 3.1.7 (Benson's proper efficiency)

A point $x' \in X$ is said to be a properly efficient solution of the problem (P) if $cl\, P(Y + D - f(x')) \cap (-D) = \{0\}$

Since $T(Y + D, f(x')) \subset cl\, P(Y + D - f(x'))$, Benson's proper efficiency strengthens Borwein's proper efficiency.

Lemma 3.1.1

Let S be a convex set and $y \in S$, then

$T(S,y) = clP(S-y)$, which is a closed convex cone.

Proof:

It is obvious that clP(S-y) is a closed convex cone. So it suffices to show that $clP(S-y) \subset T(S,y)$, since the converse follows directly from the definitions. Now since T(S,y) is closed, we have to prove $P(S-y) \subset T(S,y)$

Let $h \in P(S-y)$. Then $h = \beta(y'-y)$ for some $\beta \geqq 0$ and $y' \in S$.

Let $y^k = [1-(1/k)]y + (1/k)y'$ and $t_k = \beta k \geqq 0$. Then

$t_k\ (y^k - y) = \beta(y'-y)$. Hence $y^k \in S$ from convexity of S, and $y^k \to y$ and $t_k(y^k - y) \to h$ as $k \to \infty$.

Thus $h \in T(S,y)$, and the proof is completed. ---//---

Theorem 3.1.1

If x' is a properly efficient solution if (P) in the sense of Benson, it is also a properly efficient solution of (P) in the sense of Borwein.

If X is a convex set and if f is a D-convex function on X, then the converse also holds; that is, they are equivalent.

Proof:

If D is a convex cone and f is D-convex on the convex set X, the set Y+D is a convex set (by proposition 2.1.3), hence, $clP(Y+D- f(x')) = T(Y+D, f(x'))$ by lemma 3.1.1. Therefore, the proper efficiency in the sense of Borwein is equivalent to that of Benson.

Example 3.1.1

Let $X = \{ (x_1, x_2): \ x_1^2 + x_2^2 \leqq 1\} \subset R^2$

$$f_1(x) = x_1,\ f_2(x) = x_2, \text{ and } D = R^2_+, \text{ i.e. } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : R^2 \rightarrow R^2.$$

Then, (-1, 0) and (0, -1) are efficient solutions but not properly efficient solutions (in the sense of Benson or Borwein).

Let us see the example in the sense of Benson,

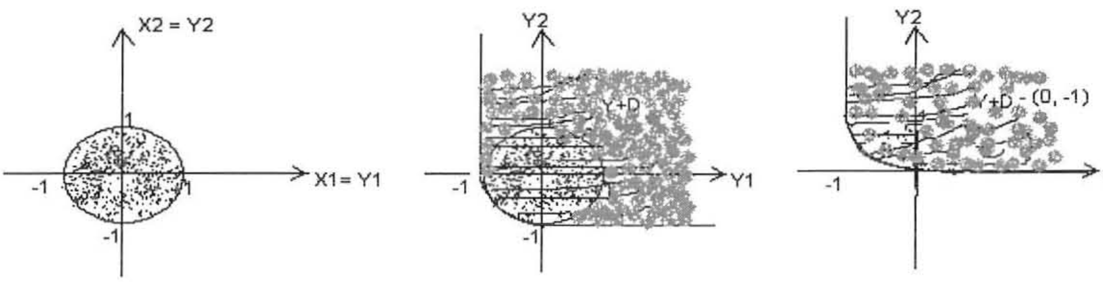
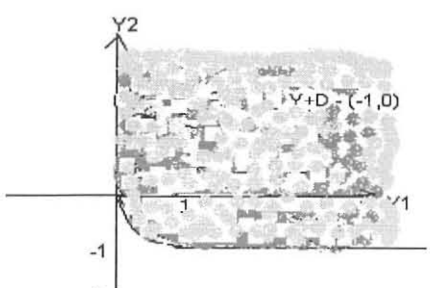


Fig. 1 a) b) c)



d)

The projection cone of Y+D in c) is the first and the second quadrant except the non-positive Y_1 axis (of course its closure includes this axis)

$$\text{cl } P(Y+D)\text{- } (0, -1)) \cap (-D) = \text{cl } P(Y+R^2_+ - (0, -1)) \cap (-R^2_+)$$

$$= \{(y_1, y_2): y_2 = 0, y_1 \leq 0\} \neq \{0\}$$

Similarly in d)

$$\text{cl } P(Y+D)\text{- } (-1, 0)) \cap (-D) = \text{cl } P(Y+R^2_+ - (-1, 0)) \cap (-R^2_+)$$

$$= \{(y_1, y_2): y_1 = 0, y_2 \leq 0\} \neq \{0\}.$$

Definition 3.1.8 (Geoffrion's proper efficiency)

Let $D = R^p_+$, a point x' is said to be a properly efficient solution of (P) if it is efficient and if there is some real $M > 0$ such that for each i and each $x \in X$ satisfying $f_i(x) < f_i(x')$, there exists at least one j such that $f_j(x') < f_j(x)$ and

$$\frac{f_i(x') - f_i(x)}{f_j(x) - f_j(x')} \leq M.$$

Theorem 3.1.2.

When $D = R^p_+$, Geoffrion's proper efficiency is equivalent to Benson's proper efficiency, and therefore stronger than Borwein's efficiency.

Proof (see [1]).

We consider another type of proper efficiency, taking a multiobjective problems:

(P') minimize $f(x) = (f_1(x), \ldots, f_p(x))$ subject to

$$x \in X = \{x: g(x) = (g_1(x), \ldots, g_m(x)) \leqq 0\}.$$

where all f_i and g_j are assumed to be continuously differentiable.

Definition 3.1.9 (Kuhn-Tucker proper efficiency)

A point x' is said to be a properly efficient solution of (P`) if it is efficient and there is no $h \in R^n$ such that

$$\langle \nabla f_i(x'), h \rangle \leqq 0 \text{ for any } i = 1, \ldots, p;$$

$$\langle \nabla f_i(x'), h \rangle < 0 \text{ for some } i,$$

and $\langle \nabla g_j(x'), h \rangle \leqq 0$ for any $j \in J(x') = \{j: g_j(x') = 0\}$.

Theorem 3.1.3

Suppose that the functions f_i and g_j are convex. If x' is a properly efficient solution of (P`) in the sense of Kuhn-Tucker, then it is also properly efficient in the sense of Geoffrion.

Proof:

This Theorem can be established as a corollary of Theorem 5.4 and 5.5, which will be given in Chapter 5.

Definition 3.1.10 (Kuhn-Tucker constraint qualification)

Problem (P`) is said to satisfy the Kuhn-Tucker constraint qualification at $x' \in X$ if, for any $h \in R^n$ such that $\langle \nabla g_j(x'), h\rangle \leqq 0$ and for any $j \in J(x')$, there exist $\bar{t} > 0$, a vector-valued function θ on $[0, \bar{t}]$ differentiable at $t = 0$, and a real number $\alpha > 0$, such that $\theta(0) = x'$, $g(\theta(t)) \leqq 0$ for any $t \in [0, \bar{t}]$, $\dot{\theta}(0) = \alpha h$

Theorem 3.1.4

Suppose that (P`) satisfies the Kuhn-Tucker constraint qualifications at x'. If x' is a properly efficient solution of (P`) in the sense of Geoffrion, then it is also properly efficient in the sense of Kuhn-Tucker.

Proof:

Suppose that x' efficient but not properly efficient in the sense of Kuhn-Tucker. Then there exists an $h \in R^n$ such that

$\langle \nabla f_1(x'), h\rangle < 0,$

$\langle \nabla f_i(x'), h\rangle \leqq 0$ for any $i = 2, \ldots, p$;

$\langle \nabla g_j(x'), h\rangle \leqq 0$ for any $j \in J(x')$ (Reorder the objective functions, if necessary).

From Kuhn-Tucker constraint qualification, there exists a continuously differentiable arc $\theta(t)$ such that $\theta(0) = x'$, $g(\theta(t)) \leqq 0$, and $\dot{\theta}(0) = \alpha h$.

Consider a sequence of positive numbers $\{t_k\} \to 0$. By taking a subsequence if necessary, we may assume that

$I = \{i:\ f_i(\theta(t_k)) > f_i(x')\}$ is constant.

Since, for $i \in I$,

$f_i(\theta(t_k)) - f_i(x') = t_k \langle \nabla f_i(x'), \alpha h\rangle + o(t_k) > 0$

and $\langle \nabla f_i(x'), h\rangle \leqq 0,$

we have $\langle \nabla f_i(x'), \alpha h\rangle = 0,\ i \in I.$

Then, since $\langle \nabla f_1(x'), h\rangle < 0,$

$$\frac{f_1(x') - f_1(\theta(t_k))}{f_i(\theta(t_k)) - f_i(x')} = \frac{-\langle \nabla f_1(x'), \alpha h\rangle + o(t_k) / t_k}{\langle \nabla f_i(x'), \alpha h\rangle + o(t_k) / t_k}$$ diverges to $+\infty$ as

$k \to \infty$. This implies that x' is not properly efficient in the sense of Geoffrion.

Any properly efficient solution is, in general, an efficient solution, but not vice versa. However, they coincide when Y is a polyhedral convex set.

Lemma 3.1.2

Let Y be a polyhedral convex set, i.e.,

$Y = \{y: \langle b^i, y\rangle \leqq \beta_i,\ i = 1, 2, \ldots, m\}$, $y' \in Y$ and

$I(y') = \{i: \langle b^i, y'\rangle = \beta_i\}$. Then $T(Y,y') = P(Y-y') = \{h: \langle b^i, h\rangle \leqq 0, \text{ for } i \in I(y')\}$.

Theorem 3.1.5

Let Y be a polyhedral convex set and D be a pointed closed convex cone, then any efficient solution is properly efficient.

Proof:

Suppose that $x' \in X$ is not a properly efficient solution. Then we can prove that there exists a non-zero $h \in D$ such that $-h \in T(Y, f(x'))$, i.e.,

$\langle b^i, -h\rangle \leqq 0$ for all $i \in I(f(x'))$.

Then for sufficiently small $\alpha > 0$, $\langle b^i, f(x') - \alpha h\rangle \leqq \beta_i$, for all $i = 1, 2, \ldots, m$.

Thus, $f(x') - (f(x') - \alpha h) \in D\backslash\{0\}$ since $\alpha h > 0$ implies $f(x') > f(x') - \alpha h$, therefore and $f(x') - \alpha h \in Y$ by definition of Y above.

Therefore, $f(x') \notin \xi(Y, D)$

3.2 Existence of Efficient Solution

Recall that in ordinary optimization problem,

minimize f(x) subject to $x \in X \subset R^n$,

the existence of optimal solution x' is guaranteed if X is compact and the objective function f is lower semi continuous. This idea can be extended to vector optimization problem (P).

The existence of efficient solutions requires acyclicity of the domination structure.

Definition 3.2.1

A domination structure D is said to be acyclic if it has no cycle;

That means, for n=1,2,... it never occurs that

$y^1 \in y^2+D(y^2)\setminus\{0\}$, $y^2 \in y^3+D(y^3)\setminus\{0\}$,..........., $y^n \in y^1+D(y^1)\setminus\{0\}$.

In other words, ($y^1 \prec y^2 \prec \ldots\ldots\ldots \prec y^n \prec y^1$ should not hold.)

Remark 3.2.1

A domination structure D is asymmetric if it is acyclic. Conversely, every transitive and asymmetric domination structure is acyclic.

Theorem 3.2.1

If a domination structure D on Y is acyclic, the sets $D(y)\setminus\{0\}$ are open and Y is nonempty and compact, then

$\xi(Y,D) \neq \phi$

Proof:

Suppose the contrary, that is

$\xi(Y,D)=\phi$, then for any $y \in Y$ there exists $y' \in Y$ such that $y \in y'+D(y')\setminus\{0\}$.

This shows $Y \subset \bigcup_{y \in Y}(y+D(y)\setminus\{0\})$.

Since the sets $D(y)\setminus\{0\}$ are open, the family of the sets $\{y+D(y)\setminus\{0\}\}$ forms an open cover of Y and Y is compact, there is a finite sub cover $\{y^i+D(y^i)\setminus\{0\}\}$ (i=1,2,..,n).

Then for any $i \in \{1,2,..,n)$,

$y^i \in y^j+D(y^j)\setminus\{0\}$ for some $j \in \{1,2,..,n)$.

This contradicts that D is acyclic.

Hence

$\xi(Y,D) \neq \phi$. //.

If the ordering cone D(y) is a constant convex cone D for all $y \in Y$, the elements of $\xi(Y,D)$ are called cone extreme points of Y. They are characterized as

$y' \in \xi(Y,D)$ if and only if $(y'-D \cup \{0\}) \cap Y=\{y'\}$.

In ordinary scalar optimization problem ($D=R_+^1$), the existence of minimal element is guaranteed under the condition that Y is bounded from below and $Y+R_+^1$ is closed. That means, under a kind of semicompactness condition. We extend this condition to multiobjective optimization problems.

Definition 3.2.2.

Let Y be a set and D be a cone in R^p.

Y is said to be D-semicompact if every open cover of Y of the form

$\{(y^\gamma - cl\,D)^C : y^\gamma \in Y, \gamma \in \tau\}$ has a finite sub cover.

Theorem 3.2.2.

If D is acute convex cone and Y is a nonempty D-semicompact set in R^p, then $\xi(Y, D) \neq \phi$.

Proof:

$D \subset cl\,D$ implies $\xi(Y, cl\,D) \subset \xi(Y, D)$.(by Proposition 3.1.2.)

It is enough to show the case in which D is a pointed closed convex cone. In this case, D defines a particular order $\leqq_D$ on Y as

$y^1 \leqq_D y^2$ if and only if $y^2 - y^1 \in D$

An element in $\xi(Y, D)$ is a minimal element with respect to $\leqq_D$. Therefore we can show that Y is inductively ordered and applying Zorn's lemma to establish the existence of minimal element.

Now, suppose the contrary that Y is not inductively ordered. Then there exists a totally ordered set $\overline{Y} = \{y^\gamma : \gamma \in \tau\}$ in Y, which has no lower bound in Y. Thus,

$\bigcap_{\gamma \in \tau} \left[(y^\gamma - D) \cap Y\right] = \phi$.

Otherwise any element of this intersection is a lower bound of $\overline{Y}$ in Y .Now it follows that for any $y \in Y$, there exists $y^\gamma \in \overline{Y}$ such that $y \notin y^\gamma - D$. Since $y^\gamma - D$ is closed, the family $\{(y^\gamma - D)^C : \gamma \in \tau\}$ forms an open cover of Y. Moreover, $y^\gamma - D \subset y^{\gamma'} - D$ if and only if $y^\gamma \leqq_D y^{\gamma'}$, and so they are totally ordered by inclusion.

Since Y is D-compact, the cover has a finite sub cover, and hence there exists a single $y^{\overline{\gamma}} \in Y$ such that $Y \subset (y^{\overline{\gamma}} - D)^C$.

However this contradicts the fact that $y^{\bar{y}} \in Y$. Therefore Y is inductively ordered by $\leqq_D$ and $\xi(Y, D) \neq \phi$ by Zorn's lemma.

It is a bit difficult to check whether Y is D-semi compact or not, we shall introduce a more stronger concept that is called cone compactness.

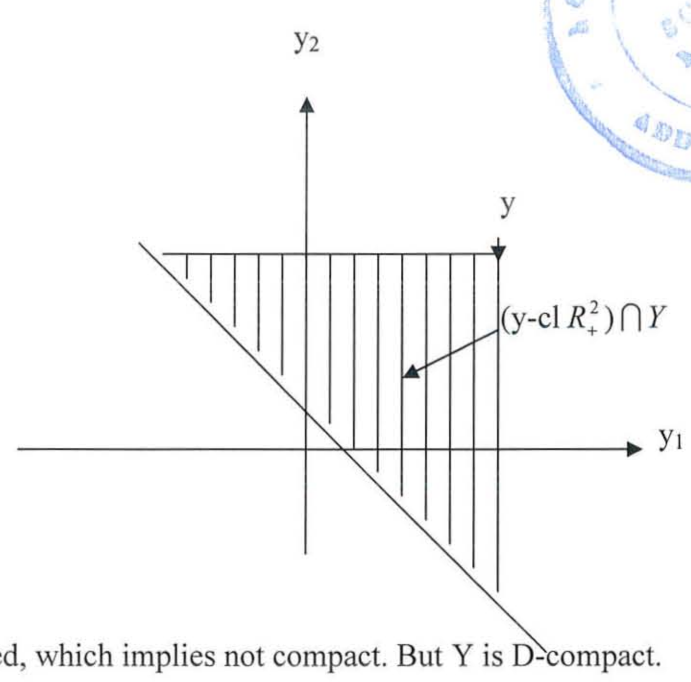
Definition 3.2.3.

Let D be a cone and Y be a set in R^p. Y is said to be D-compact if the set $(y - clD) \cap Y$ is compact for any $y \in Y$.

Remark 3.2.2

1) If Y is D-compact, then Y is D-semicompact.
2) A compact set is D-compact and so D-semicompact. However D-compact set is not necessarily compact.

Example: let D= $R_+{}^2$ and $Y = \{y \in R^2 : y_1 + y_2 \geqq 0\}$



Here Y is not bounded, which implies not compact. But Y is D-compact.

Theorem 3.2.3.

Let D be an acute convex cone in R^p. If Y$\subset R^p$ is nonempty and D-compact, then $\xi(Y, D) \neq \phi$.

Proof:

Follows immediately from theorem 3.2.2. and remark 3.2.2.

Sometimes terms of recession cones of feasible sets are used since Y is not necessarily convex.

Definition 3.2.4 (Recession cone)

Let Y be a convex set in $\mathbf{R}^n$. Its recession cone O^+Y is defined by

$O^+Y=\{y` \in \mathbf{R}^n: y+\alpha y` \in Y \text{ for any } y \in Y \text{ and } \alpha>0\}$.

Of course O^+Y is a convex cone containing the origin.

This definition can be extended to an other concept as

$Y^+ = \{y`:$ there exist sequences $\{\alpha_k\} \subset R$ and $\{y^k\} \subset Y$ such that $\alpha_k > 0, \alpha_k \to 0$ and $\alpha_k y^k \to y`$.

Remark 3.2.3

It is clear that Y^+ is a closed cone and $O^+Y \subset Y^+$.

On the other hand, O^+Y is not necessarily closed even if Y is convex.

Lemma 3.2.1

Let Y be a nonempty set. Then

i) Y is bounded if and only if $Y^+ = \{0\}$,

ii) if Y is closed and convex, $Y^+ = O^+Y$.

Lemma 3.2.2

Let $\{Y_i\}_{i\in I}$ be an arbitrary collection of nonempty closed sets. Then

$$(\bigcap_{i\in I} Y_i)^+ \subset \bigcap_{i\in I} Y_i^+$$

Definition 3.2.5.

Let Y be a nonempty set in R^p, then Y is said to be

1) D-closed if Y+clD is closed.

2) D-bounded if $Y^+ \cap (-clD) = \{0\}$

Lemma 2.2.3.

Let D be a pointed, closed, convex cone and Y be a nonempty set in R^p. Then, $Y^+ \cap (-D) = \{0\}$ if and only if $(Y+D)^+ \cap (-D) = \{0\}$

Proof:

$(\Leftarrow)$: Let $(Y+D)^{+} \cap(-D)=\{0\}$

Since $Y^{+} \subset (Y+D)^{+}$, it is clear that $Y^{+} \cap(-D)=\{0\}$

$(\Rightarrow)$: Let $Y^{+} \cap(-D)=\{0\}$ but $(Y+D)^{+} \cap(-D) \neq\{0\}$.

Then there exist sequences $\{\alpha_k\} \subset R$ and $\{y^k+d_k\}$ such that $\alpha_k \to 0, \alpha_k>0, y^k \in Y, d^k \in D, \alpha_k(y^k+d_k) \to -d \neq 0 \in -D$. there are two cases.

1) $\{\alpha_k d^k\}$ has a convergent subsequence and

2) $\{\alpha_k d^k\}$ has no convergent subsequence.

Case 1: by taking a subsequence we may assume without lose of generality that $\alpha_k d^k \to d'$. Since D is closed d' $\in$ D. Therefore, $\alpha_k y^k \to -d-d'$, which is a nonzero vector in $Y^{+} \cap(-D)$. Since D is pointed and convex. Hence $Y^{+} \cap(-D) \neq\{0\}$.

Case 2: $\{\alpha_k d^k\}$ is bounded, so $\{\alpha_k d^k\}^{+} \neq\{0\}$. By taking a subsequence of $\{\alpha_k d^k\}$ we may assume that there exists another subsequence $\{\beta_k\}$ such that $\beta_k>0$, $\beta_k \to 0$ and $\beta_k(\alpha_k d^k) \to \bar{d} \neq 0$,

Since D is closed $\bar{d} \in D$. Then

$(\beta_k \alpha_k) y^k \to -\bar{d}$ with $\beta_k \alpha_k>0$, $\beta_k \alpha_k \to 0$

$$\left\|(\beta_k \alpha_k) y^k+\bar{d}\right\|=\left\|\beta_k[\alpha_k(y^k+d^k)+d)-(\beta_k \alpha_k d^k-\bar{d})-\beta_k d\right\|$$

$$\leq \beta_k\left\|\alpha_k(y^k+d^k)+d\right\|+\left\|\beta_k \alpha_k d^k-\bar{d}\right\|+\beta_k\|d\|$$

$$\to 0 \; as \; k \to \infty$$

Thus we have proved that $Y^{+} \cap(-D) \neq\{0\}$

Lemma 3.2.4. Let D be a convex cone and Y be a set in R^p. If Y+ cl D is D-semicompact, then Y is also D-semicompact.

Proof:

Let $\{(y^{\gamma} - clD)^{c} : y^{\gamma} \in Y, \gamma \in \tau\}$ be an open cover of Y.

Let $y \in (y^{\gamma} - clD)^{c}$ with $y^{\bar{\gamma}} \in Y$.

Since clD is a convex cone, y+clD $\subset (y^{\bar{\gamma}} - clD)^{c}$

if y+d$\in y^{\bar{\gamma}} - clD$,for some d$\in$clD, y$\in y^{\bar{\gamma}} - clD$,which is a contradiction. Hence $\{(y^{\gamma} - clD)^{c} : y^{\gamma} \in Y, \gamma \in \tau\}$ is also an open cover of Y+clD.

Since Y +clD is D-semicompact this cover has a finite subcover, which is of course a subcover of Y.

Hence Y is D-semicompact.

Proposition 3.2.1.

Let D be an acute convex cone in R^{p} and Y be a nonempty set in R^{p}. If Y is D-closed and D-bounded, then Y+clD is D-compact and Y is D-semicompact.

Proof:

Let $y \in Y + clD$.

Since y-clD and y+clD are both nonempty closed sets y-clD $\cap$ y+clD is closed. Therefore, from Lemma 3.2.2 and Lemma 3.2.3,

$$((\text{y-clD}) \cap (\text{y+clD}))^{+} \subset (\text{y - clD})^{+} \cap (\text{y + clD})^{+}$$

$$= (\text{-clD}) + (\text{y+clD})^{+}$$

$$= \{0\} \text{ since Y is D-bounded.}$$

Thus y-clD $\cap$ y+clD is bounded. Therefore y+clD is D-compact, and so y+clD is D-semi compact from Proposition 3.2.1

Hence from Lemma 3.2.4, Y is D-semi compact.

Theorem 3.2.4.

Let D be an acute convex cone in R^{p} and Y be a nonempty, D-closed, D-bounded set in R^{p}, then $\xi(Y, D) \neq \phi$.

Proof:

Follows immediately from Proposition 3.2.1 and Theorem 3.2.4.

CHAPTER FOUR
MULTIPLIERS IN VECTOR OPTIMIZATION

Necessary optimality conditions expressed as multiplier rules are fundamental in the theory of optimization. Lagrange and Euler started such conditions by means of the differential calculus for the case in which a real-valued function is optimized subject to equality constraints.

Since then much work has been done to extend these classical results to more general optimization problems involving both equality and inequality constraints in one hand and functions that are not necessarily differentiable on the other hand.

There are different methods to obtain multipliers. In the case of scalar optimization problems containing finitely many inequality and equality constraints it needs to shift, the analysis of the given problem from the space in which the domain of the objective and constraints functions lies to the image space of the vector-valued function.

In this paper Breckner's multiplier rule-so called derived sets is used. There are some optimality conditions related with the theory of derived sets. We deal with a vector-minimization problem by taking K as ordering cone and introducing new term called K-derived sets.

This topic constitutes two subtopics. The first introduces new concepts such as K-derived sets and optimality conditions. The second deals with existence of multipliers in vector optimization.

4.1. Basic Concepts of Multipliers

In ordinary non-linear optimization Lagrange multipliers play a central role for finding optimal solution of a given minimization problem.

(P) minimize $f(x)$ subject to $x \in S$

$S = \{x \in X \colon g(x) \leq 0 \text{ and } h(x) = 0\}$.

If x' is a solution for (P) and be regular with respect to S, then there exists a $\lambda \in R_+^m$ and a $\mu \in R^p$ such that

$$[f`(x')]^T + \lambda^T g`(x') + \mu^T h`(x') = 0 \quad \text{and}$$

$$\lambda_i g_i(x') = 0, \; i \in \{1, 2, \ldots, m\}.$$

Where g: $U \subseteq R^n \rightarrow R^m$, and h: $U \subseteq R^n \rightarrow R^p$ are differentiable

and

$$f'(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}, g'(x) = \begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m(x)}{\partial x_1} & \cdots & \frac{\partial g_m(x)}{\partial x_n} \end{pmatrix}, h'(x) = \begin{pmatrix} \frac{\partial h_1(x)}{\partial x_1} & \cdots & \frac{\partial h_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_p(x)}{\partial x_1} & \cdots & \frac{\partial h_p(x)}{\partial x_n} \end{pmatrix}$$

(see [5]: Optimization and Theory of Approximations:Section4.4.3. Theorem 1)

Here we see the vector-minimization optimization problem in a way different from the above forms. In previous chapters we have seen that the objective function f: $R^n \rightarrow R^p$ without constraint functions. Now we turn our discussion to these functions and multipliers. And we consider the problem as a whole as follows.

Consider the optimization problem.

Let X be a non-empty subset of a topological space and f(x) = $\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} : X \rightarrow R^m$, where,

f_1:X$\rightarrow R^{m_1}$, f_2:X$\rightarrow R^{m_2}$, f_3:X$\rightarrow R^{m_3}$, are functions with m = $m_1+m_2+m_3$ and $K_1 \subseteq R^{m_1}$, $K_2 \subseteq R^{m_2}$, $K_3 \subseteq R^{m_3}$, are convex cones (ordering cones) , for which int$K_1 \neq \phi$, int$K_2 \neq \phi$, while K_2 and K_3 are closed convex cones.

The problem is:

$$f_1(x) \rightarrow K_1\text{- min} \quad \text{subject to } S = \{x \in X: f_2(x) \in -K_2, f_3(x) \in -K_3\},$$

(where $f_1(x) \rightarrow K_1$-min is equivalent to mean $f_1(x) \rightarrow$min with respect to the ordering cone K).

Let L:= { $y \in K_2^*$; <y, $f_2(x)$> = 0}, with K_1^*, K_2^*, K_3^* describe the respective dual cones of K_1, K_2 and K_3.

According to the convention constructed above every vector $y \in R^m$ is interpreted as the triple $y = (y_1, y_2, y_3) \in R^{m_1} \times R^{m_2} \times R^{m_3}$ and $K := K_1 x\, K_2 x\, K_3 \subseteq R^m$.

Definition 4.1.

A point $x_o \in X$ is said to be

i) a weakly pareto K_1-minimal point for the function f_1 over S if $x_o \in S$ and $(f_1(x_o) - \text{int}K_1) \cap f_1(S) = \phi$;

ii) a local weakly pareto K_1-minimal point for the function f_1 over S if $x_o \in S$ and if there is a neighborhood V of x_o such that $(f_1(x_o) + \text{int}K_1) \cap f_1(S \cap V) = \phi$.

We try to formulate multipliers in relation with new terms called K-derived cones.

Definition 4.2.

The tuple $\{d_1, d_2, \ldots, d_{m+1}\}$ of points in R^m is called a D-derived convex cone for f at x_0 if there exists a number $r > 0$, a function w: $B_+^{m+1}(r) \rightarrow X$

(where $B_+^{m+1}(r) := \{x \in B_+^{m+1}, \|x\| \le r\})$, and a function $\rho = (\rho_1, \rho_2, \rho_3)$: $B_+^{m+1}(r) \rightarrow R^{m_1} x R^{m_2} x R^{m_3}$ such that the following conditions are fulfilled:

1) $f(w(t)) - f(x_0) - (t^1 d_1 + t^2 d_2 + \ldots + t^{m+1} d_{m+1}) - K \;\; \forall t = (t^1, \ldots, t^{m+1}) \in B_+^{m+1}(r)$
2) $w(0) = x_o$ and w is continuous at 0.
3) There exists $y_1^0 \in K_1, y_2^0 \in K_2$ so that for each $\varepsilon > 0$ there is a number $r_\varepsilon \in (0, r]$ such that $\rho_1(t) \in \varepsilon y_1^0 - K_1$, $\rho_2(t) \in \varepsilon y_2^0 - K_2$ (respectively) $\forall t \in B_+^{m+1}(r_\varepsilon)$
4) $\rho_3(0) = 0$ and ρ_3 is continuous.

Multipliers rule can be deduced using a simple separation theorem.

For this let us see the concepts of separation of sets.

Definition 4.3

A subset H of R^n is called a hyper plane if it is represented as

$H = \{x \in R^n : \langle x, x^* \rangle = \beta\}$, for nonzero $x^* \in R^n and \beta \in R$ .In this case, x^* is called a normal to the hyperplane H. Moreover, the sets

$$H^{-}(res.\overset{o}{H}{}^{-}) = \{x \in R^n : \langle x, x^* \rangle \le (resp. <)\beta\}$$

$$H^{+}(resp.\overset{o}{H}{}^{+}) = \{x \in R^n : \langle x, x^* \rangle \ge (resp. >)\beta\}$$

are called closed(resp. open) half spaces associated with hyperplane H.

Definition 4.4

Let X_1 and X_2 be nonempty sets in R^n. A hyperplane H is said to separate X_1 and X_2 if $X_1 \subset H^+ and\ X_2 \subset H^- (or\ X_1 \subset H^- and\ X_2 \subset H^+)$.

It is said to separate X_1 and X_2 properly if, in addition, $X_1 \bigcup X_2 \not\subset H$. It is said to separate X_1 and X_2 strongly if, there exists a positive number $\varepsilon > 0$ such that $X_1 + \varepsilon B \subset H^+$ and $X_2 + \varepsilon B \subset H^-$ ($X_1 + \varepsilon B \subset H^-$ and $X_2 + \varepsilon B \subset H^+$).

Proposition 4.1

Let X_1 and X_2 be nonempty sets in R^n. There exists a hyperplane separating X_1 and X_2 properly if and only if there exists a vector x^* such that

i) $\inf\{\langle x, x^* \rangle : x \in X_1\} \ge \sup\{\langle x, x^* \rangle : x \in X_2\}$

ii) $\sup\{\langle x, x^* \rangle : x \in X_1\} > \inf\{\langle x, x^* \rangle : x \in X_2\}$

There exists a hyperplane separating X_1 and X_2 strongly if and only if there exists a vector x^* such that

$\inf\{\langle x, x^* \rangle : x \in X_1\} > \sup\{\langle x, x^* \rangle : x \in X_2\}$

Theorem 4.1

Let $X_1$and X_2 be nonempty convex subsets of R^n. There exists a hyperplane separating $X_1$and X_2 properly if and only if $riX_1 \cap riX_2 = \phi$.

Theorem 4.2

Let $X_1$and X_2 be nonempty, disjoint convex sets. If $X_1$is compact and X_2 is closed, then there exists a hyperplane separating X_1 and X_2 strongly.

4.2. Existence of Multipliers

In optimization problems multipliers play a central role to get optimal element (set). As we indicated earlier we are trying to extend the existence of multipliers of single objective problems to multicriteria problems. Concerning this idea we have a theorem that states the existence of a multiplier vector in solving vector optimization problems.

The proof of this theorem can be given in two ways: proof without using the theory of nonlinear inequalities and applying separation theorem. The main target of this chapter is to prove the following:

Theorem 4.3

Let $x_o \in X$ be a local weakly minimal point for the function f_1 over S and let $D \subset R^m$ be K-derived convex cone for f at x_o. Then there exists a multiplier vector $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in K_1^* x K_2^* x K_3^*, \lambda \neq 0$, such that

$\langle \lambda, d \rangle \geq 0 \forall d \in D$,

$\langle \lambda_2, f_2(x_o) \rangle = 0$

For the sake of proving the multiplier rule (theorem 4.3), first let us state the following proposition (called Breckner`s assertion)

Proposition 4.2.

Let to be a vector in R^m, let k be a closed convex cone in the space R^p, and let F: $R^m \to R^p$, be a continuous function, differentiable at the origin, satisfies the following conditions:

i) $F(0) \in -K$,

ii) $F'(0)(t_0) \in -K$,

iii) $F'(0)[R^m] - K = R^p$.

Then there exists a sequence $\{a_n\}$ (n = 1,2, ...) of positive numbers with $\lim\limits_{n\to\infty} a_n = 0$, and a sequence $\{t_n\}$ (n = 1, 2, ...) of vectors in the space R^m with $\lim\limits_{n\to\infty} t_n = t_o$ such that

$F(a_n t_n) \in -K \ \forall n$.

We state and prove the following lemma to be used in the proof of the Proposition4.2.

Lemma 4.1.

Let $U \subset R^m$ be a non-empty convex and compact set, let A:U $\to R^p$ be a continuous function, B: $R^m \to R^p$ be a linear function, and let K be a convex closed cone (ordering cone) of the space R^p. Then the set-valued mapping $\tau : U \to 2^U$ given by $\tau(u) = \{\bar{u} \in U : A(u) \in B(\bar{u}) - K\}$ possesses a fixed point for every $u \in U$, the set τ (u) is non-empty.

Proof:

We have to check that the assumption of KAKUTANI'S fixed-point theorem is fulfilled. That is the mapping τ to be closed, and the image τ (u) to be convex and closed for every $u \in U$.

a) For closedness of τ :

Consider two converging sequences $\{u_n\} \subset U$ and $\{\overline{u}_n\} \subset U$ with

$\overline{u}_n \in \tau(u_n)\ \forall n$ Or equivalently $A(u) \in B(\overline{u}_n) - K \quad \forall n$.

Let $u := \lim_{n\to\infty} u_n$ *and* $\overline{u} = \lim_{n\to\infty} \overline{u}_n$.

Then $u \in U$ and $\overline{u} \in U$ (since U is closed)

By continuity of A and B and by the closedness of K,

$A(u) \in B(\overline{u})-K$,

That means $\overline{u} \in \tau$ (u).

b) For convexity of τ (u):

Let $u \in U$ be fixed. Consider two arbitrary points $\overline{u}_1$ and $\overline{u}_2$ in τ (u).

Since U is convex, $\lambda \overline{u}_1 + (1-\lambda)\overline{u}_2 \in U \quad \forall \lambda \in [0,1]$. Furthermore

$A(u) = \lambda A(u) + (1-\lambda)A(u)$

$\in \lambda B(\overline{u}_1) + (1-\lambda)B(\overline{u}_2) - K$

$= B(\lambda u_1 + (1-\lambda)u_2) - K \quad \forall \lambda \in [0,1]$ (Because B is linear)

Hence $\lambda(\overline{u}_1) + (1-\lambda)(\overline{u}_2) \in \tau(u)$ for all $\lambda \in [0,1]$

c) For closedness of τ (u):

Let $u \in U$ be fixed. Consider a converging sequence $\{\overline{u}_n\} \subset \tau(u)$ with $\overline{u} = \lim_{n\to\infty} \overline{u}_n$.

Since $\overline{u}_n \in U$ for all n and by closedness of U, we have $\overline{u} \in U$.

From $A(u) \in B(\overline{u}_n) - K$, by continuity of B and closedness of K, it follows $A(u) \in B(\overline{u}) - K$, which means $\overline{u} \in \tau$ (u).

Remark 4.2:

The above lemma 4.2 is still valid, even if the linearity of B is replaced by K-concavity.

Proof:_ (of Proposition 4.2)

If $t_0 = 0$, the proposition is fulfilled trivially taking $\{t_n\} = \{0, 0, \ldots\}$ and arbitrary sequence $\{a_n\}$ of positive numbers converging to zero. Now we assume $t_o \neq 0$.

Let $U_n = \{u \in R^m : \|u\| \leq \frac{1}{2n}\}$ for every natural number n = 1, 2, ... It is easy to see that U_n is convex, closed, and bounded for each n, and hence compact.

By linearity of $F'(t_0)(.)$ and condition (iii), the set $v_n = -(F'(t_0)[u_n] + K)$ is a neighbourhood of the origin of R^p. Taking into account

$\lim_{x \to 0} \frac{1}{\|x\|}[F(x) - F(0) - F'(0)(x)] = 0$, we find a constant $\varepsilon_n > 0$ for every n, such that

$$F(x) - F(x_0) - F'(0)(x) \in \|x\| v_n \quad \forall x : \|x\| < \varepsilon_n .$$

Hence we have

$$\begin{aligned} F(x) &\in F(0) + F'(0) - \|x\|(F'(t_0)[u_n] - K) \\ &= F(0) + F'(0)(x) - \|x\| F'(t_o)[u_n] - K \quad \forall x : \|x\| < \varepsilon_n, \ \forall n \end{aligned}$$

By (i), this implies

$$F(x) \in F'(0)(x) - \|x\| F'(t_0)[u_n] - K \ \forall x : \|x\| < \varepsilon_n, \ \forall n \qquad (1)$$

Let $\{\alpha_n\}$ be a sequence of positive numbers converging to zero with

$0 < \alpha_n \leq \varepsilon_n \quad \forall n$. From

$$\left\| \frac{t_0}{2\|t_0\|} + u \right\| \leq \frac{\|t_0\|}{2\|t_0\|} + \|u\| \leq \frac{1}{2} + \frac{1}{2n} < 1 \quad \forall u \in u_n, \forall n \qquad (2)$$

we get $\left\| \alpha_n \left(\frac{t_0}{2\|t_0\|} + u \right) \right\| < \varepsilon_n \quad \forall u \in u_n, \forall n$. So from (1) we have

$$F\left(\alpha_n \left(\frac{t_0}{2\|t_0\|} + u \right) \right) \in F'(0)\left(\alpha_n \left(\frac{t_0}{2\|t_0\|} + u \right) \right) - \alpha_n \left\| \left(\frac{t_0}{2\|t_0\|} + u \right) \right\| F'(t_0)[u_n] - K$$

$$= \frac{\alpha_n}{2\|t_0\|} F'(0)(t_0) + \alpha_n F'(0)(u) - \alpha_n F'(t_0)\left[\left\| \left(\frac{t_0}{2\|t_0\|} + u \right) \right\| u_n \right] - K \qquad \forall u \in u_n, \forall n$$

By (ii), we have

$$\frac{\alpha_n}{2\|t_0\|}F'(0)(t_0) \in -K \quad \forall n;$$

and by (2) $\left\|(\frac{t_0}{2\|t_0\|}+u)\right\|U_n \subseteq U_n \quad \forall n \in u_n, \ \forall n$,

Hence it holds that

$$F\left(\alpha_n(\frac{t_0}{2\|t_0\|}+u)\right) \in \alpha_n F'(0)u - \alpha_n F'(t_0)[u_n] - K \quad \forall n \in U_n, \ \forall n \qquad (3)$$

Now let $\tau_n : u_n \rightarrow 2^{u_n}$ be the set valued mapping for every natural number n, defined by:

$$\tau_n(u) = \left\{ \overline{u} \in u_n : F\left(\alpha_n(\frac{t_0}{2\|t_0\|}+u)\right) \in \alpha_n F'(0)(u-\overline{u}) - K \right\},$$

Due to (3), each $u \in U_n$ for every n, the set $\tau_n(u)$ is non-empty.

In addition the mapping $F\left(\alpha_n(\frac{t_0}{2\|t_0\|}+.)\right) - \alpha_n F'(0)(.)$ is continuous and the mapping $-\alpha_n F'(0)(.)$ is linear

Applying lemma 4.1, for every n the mapping τ_n has a fixed point $u_n \in U_n$ fulfilling $u_n \in \tau_n(u_n)$, that means

$$F\left(\alpha_n(\frac{t_0}{2\|t_0\|}+u_n)\right) \in -K \quad \forall n \qquad (4)$$

Therefore, the sequences $\{a_n\}$ and $\{t_n\}$ defined by

$$a_n := \frac{\alpha_n}{2\|t_0\|} \qquad \forall n$$

$t_n := t_0 + 2\|t_0\|u_n \quad \forall n$ have the property claimed in the proposition.

Remark 4.3

An additional complimentarity c ondition $\langle \lambda_3, f_3(x_o)\rangle = 0$ can be derived, which is not given in theorem 4.3.In this case theorem 4.3 can be reformulated as:

Let $x_0 \in X$ be a local weakly K_1-minimal point for the function f_1 over S and let $D \in R^m$ be an K-derived convex cone for f at x_0. Then there exist a multiplier vector $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in K_1^* x K_2^* x K_3^*$, $\lambda \neq 0$ such that

$$\langle \lambda, d\rangle \geq 0 \forall d \in D,$$
$$\langle \lambda_2, f_2(x_o)\rangle = 0,$$
$$\langle \lambda_3, f_3(x_o)\rangle = 0.$$

A Direct Approach by Using a Separation Theorem

Breckner's Multiplier rule can be deduced using a simple separation theorem. In proving for the existence of a multiplier vector the main idea we have to show is that the derived cones constructed at an optimal point never contain a vector belonging to the interior of the ordering cone.

For simplicity of the proof, we take $K_3 = \{o^{m_3}\}$, any problem that fulfills the requirements stated initially can be reformulated equivalently. As we have mentioned above the main task we have to do is that the following proposition is true.

Proposition 4.3

Let $x_0 \in X$ be a local weakly K_1-minimal point for the function f_1 over S and let $D \in R^m$ be an K-derived cone for f at x_0. Then there exists a sequence $\{b_k\} \subset R^m$ (k = 1, 2, ...) converging to zero, such that

$$D \cap (-b_k - (K_1 x L^* x \{0^{m_3}\})) = \phi \qquad \forall k \qquad (5)$$

Remark 4.4:

Property (5) implies the origin of R^m being an extremal point of the system $\{D, (-K_1)x(-L^*)x\{0^{m_3}\}\}$

Since $D \cap (-b_k - (K_1 x L^* x \{0^{m_3}\})) = \phi \qquad \forall k$,

there is a closed hyperplane that separates the convex sets D and $-b_k + (-K_1)x(-L^*)x\{0^{m_3}\}$.

Hence, there exists a multiplier vector λ_k for each k

$\lambda_k = (\lambda_{k_1}, \lambda_{k_2}, \lambda_{k_3}) \in R^{m_1} x R^{m_2} x R^{m_3}$, $\|\lambda_k\| = 1$, fulfilling

$\langle \lambda_k, d \rangle \geq 0 \quad \forall d \in D$

$\langle \lambda_{k_1}, y \rangle \geq \langle \lambda_{k_1}, b_{k_1} \rangle \quad \forall y \in K_1$

$\langle \lambda_{k_2}, y \rangle \geq \langle \lambda_{k_2}, b_{k_2} \rangle \quad \forall y \in L^*$

Since the unit ball in finite dimensional space is compact, the sequence $\{\lambda_k\}$ is convergent. Let it converges to $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in R^{m_1} x R^{m_2} x R^{m_3}$, $\lambda \neq 0$ with

$\langle \lambda, d \rangle \geq 0 \quad \forall d \in D$

$\langle \lambda_1, y \rangle \geq 0 \quad \forall y \in K_1$

$\langle \lambda_2, y \rangle \geq 0 \quad \forall y \in L^*$, Since $b_k \to 0, k \to \infty$

Hence, $\lambda_1 \in K_1^*, \lambda_2 \in L \subseteq K_2^* \; and \; \langle \lambda_2, f_2(x_0) \rangle = 0$

If so, it only remains to prove the assertion of Proposition 4.3, but to do this we require some additional conditions. Whence we state the following lemma with out proof (see [6]).

Lemma 4.2:

Let $D \in R^m$ be a K-derived convex cone for a function f: X $\rightarrow R^m$ at $x_0 \in S$. If there are some $d_1 \in -\text{int}K_1$, and $d_2 \in -\text{int}L^*$ with $(d_1, d_2, d_3) \in D \; \forall d_3 \in R^m$, then there exists a sequence $\{x_n\} \in X$ (n = 1,2, …) converging to x_o such that

$f_1(x_n) - f_1(x_0) \in -\text{int}K_1 \quad \forall n$

$f_2(x_n) \in -K_2 \quad \forall n$

$f_3(x_n) = 0 \qquad \forall n$.

Now we are in a position to proof the main proposition.

Proof: (of proposition 4.3)

Taking a vector $b_1 \in \mathrm{int}K_1$, and a vector $b_2 \in \mathrm{int}L^*$ one can always find a vector $b_3 \in R^{m_3}$ to b_1 and b_2 such that

$$-\begin{pmatrix} b_1 + K_1 \\ b_2 + L^* \\ b_3 \end{pmatrix} \bigcap D = \phi \text{, if not so, we would have}$$

$$-\begin{pmatrix} b_1 + K_1 \\ b_2 + L^* \\ b_3 \end{pmatrix} \bigcap D \neq \phi \quad \forall b_3 \in R^{m_3}$$

Thus, there is $d_1 \in -b_1 - K_1 \subseteq -\mathrm{int}K_1$, and $d_2 \in -b_2 - L^* \subseteq -\mathrm{int}L^*$ with $\begin{pmatrix} d_1 \\ d_2 \\ -b_3 \end{pmatrix} \in D \quad \forall b \in R^{m_3}$.

But by lemma 4.2, there must exist a sequence $\{x_n\} \in X$ converging to x_o such that

$$f_1(x_n) - f_1(x_0) \in -\mathrm{int}K_1 \quad \forall n$$
$$f_2(x_n) \in -K_2 \quad \forall n$$
$$f_3(x_n) = 0 \qquad \forall n.$$

But this contradicts the local K_1- minimality of x_o. Hence we have $-\begin{pmatrix} b_1 + K_1 \\ b_2 + L^* \\ b_3 \end{pmatrix} \bigcap D = \phi$.

Then the sequence $\{b_k\} \subset R^m$ defined by $b_k = \dfrac{1}{k}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ (k =1, 2, …) converges to zero and fulfills property (5) in proposition 4.3.

Corollary 4.1

Let $x_o \in X$ real or complex normed linear space that does not reduce to its origin) be a local weakly Pareto solution of f satisfying the following conditions:

i) x_o is interior to X;

ii) f is Frechet differentiable at x_o;

iii) there exists a number r>0 such that f_3 is continuous on $B(x_o,r_o) \cap X$.

Then there exists a vector $\lambda^* \in R^m \setminus \{0\}$ *such that*

$$\lambda_1^* \geq 0_1 ,\ \lambda_2^* \geq 0_2 ;$$
$$\langle f_2(x_o), \lambda_2^* \rangle = 0;$$
$$\sum_{i=1}^{m_1} \lambda_{1i}^* f_{1i}'(x_o) + \sum^{m_2} \lambda_{2i}^* f_{2i}'(x_o) + \sum_{i=1}^{m_3} \lambda_{3i}^* f_{3i}'(x_o) = 0$$

Proof: See[7]

4.3. Regularity Conditions

A multiplier rule is usually associated with a condition called regularity. A regularity condition ensures that the multiplier corresponding with the objective function does not vanish.

We did not have this condition in theorem 4.3. Next we state Hestenes regularity condition.

Theorem 4.4

Let the assumptions of theorem 4.3 be fulfilled.

a) If there is a vector $c \in \text{int}K_1$ with

(-c, 0, 0) $\in D + K_1 x L^* x K_3$, then for the multiplier λ in theorem 4.3,the first component, can be chosen as $\lambda_1 = 0$

b) If there is an arbitrary vector $c \in R^{m_1}$ with

(-c, 0, 0) $\notin D + K_1 x L^* x K_3$, then a multiplier λ in theorem 4.3 can be chosen as $\lambda_1 \neq 0$

Proof:

a) Since (-c, 0, 0) $\in D + K_1 x L^* x K_3$, there is a vector $(x_1, x_2, x_3) \in K_1 x L^* x K_3$ such that

(-c, 0, 0) - $(x_1, x_2, x_3) \in D$

Let λ be the multiplier in theorem 4.3. Then it follows that $\langle \lambda, (\text{-c}, 0, 0) - (x_1, x_2, x_3) \rangle \geq 0$

Since $\lambda_1 \in K_1^*, \lambda_2 \in K_2^*, \lambda_3 \in K_3^*$, we have

$$
\begin{aligned}
\langle \lambda_1, c \rangle &= -\langle \lambda, (-c,0,0) \rangle \\
&\leq -\langle \lambda, (x_1, x_2, x_3) \rangle \\
&= -\langle \lambda_1, x_1 \rangle - \langle \lambda_2, x_2 \rangle - \langle \lambda_3, x_3 \rangle \\
&\leq 0
\end{aligned}
$$

On the other hand $\langle \lambda_1, x \rangle \geq 0 \quad \forall x \in K_1$, hence $\langle \lambda_1, c \rangle = 0$

$c \in \text{int}K_1$ implies that $\lambda_1 = 0$.

b) Every such vector can be written as

$(\text{- c}, 0, 0) = \text{x - } \lambda$ with

$x \in D + K_1 x L^* x K_3$

$\text{-}\lambda \in D + K_1 x L^* x K_3]^*$,

and $\langle \lambda, x \rangle = 0$.

It follows $\lambda \neq 0$, otherwise $(\text{- c}, 0, 0) = x \in D + K_1 x L^* x K_3$.

Due to $D \subseteq D + K_1 x L^* x K_3$ or $[D + K_1 x L^* x K_3]^* \subseteq D^*$ *implies* $\lambda \in D^*$ hence $\langle \lambda, d \rangle \geq 0 \quad \forall d \in D$

Due to $K_1 x L^* x K_3 \subseteq D + K_1 x L^* x K_3$ or $[D + K_1 x L^* x K_3]^* \subseteq K_1^* x L\ x K_3^*$,

we have $\lambda \in K_1^* x L\ x K_3^*$ Moreover it is clear that

$$\langle \lambda_1, c \rangle = -\langle \lambda, (-c,0,0) \rangle = -\langle \lambda, x \rangle + \langle \lambda, \lambda \rangle = \|\lambda\|^2 > 0$$

Assuming $\lambda_1 = 0$, we would get the contradiction.

CHAPTER FIVE

KUHN-TUKER CONDITIONS FOR MULTIOBJECTIVE PROBLEMS

The chapter deals with the Kuhn-Tucker conditions for a given multiobjective optimization problem:

(P) minimize $f(x) = (f_1(x), \ldots\ldots\ldots, f_p(x))^T$

Subject to $x \in X = \{x \in R^n : g(x) = (g_1(x), \ldots\ldots\ldots\ldots, g_m(x))^T \leqq 0\}$ with ordering cone $D = R_+^P$

We prefer to see the chapter in two parts. The first section discusses some concepts that are needed in the proof of the main idea of the chapter. The necessary and sufficient conditions for existence of proper efficiency (pareto optimality) will be discussed in the second part of the chapter.

5.1 Characterization of Efficiency and Proper Efficiency by Scalarization

In this section we develop the concept of scalarization for efficient and properly efficient points. The efficient point of Y with respect to a convex cone D is denoted by $\xi(Y,D)$ as before.

That is $y` \in \xi(Y,D)$ if and only if $(Y-y`) \cap (-D) = \{0\}$ (or empty when D does not contain the origin).

The set of all properly efficient points of Y in the sense of Benson is denoted by

$\rho(Y,D)$ where D is a closed convex cone.

$y` \in \rho(Y,D)$ if and only if $\mathrm{cl}P(Y+D-y`) \cap (-D) = \{0\}$.

The characterization of efficient (properly efficient) points is scalarization by vectors in polar or strict polar cone of the domination cone.

Let $S(\mu,Y) = \{y` : <\mu,y`> = \inf\{<\mu,y> : y \in Y\}$.

$S(\mu,Y)$ is the set supporting points of Y with the inner normal vector μ. It is obvious that $S(\alpha\mu,Y) = S(\mu,Y)$ for any $\alpha>0$. Hence we may normalize μ as $\|\mu\| = 1$.

Let

$$\psi(Y,D)=\bigcup_{\mu\in D^{SO}} S(\mu,Y) \qquad \text{and} \qquad \psi^{'}(Y,D)=\bigcup_{\mu\in D^{O}\setminus\{0\}} S(\mu,Y)$$

From these

$$\psi\ (\mathrm{Y, D}) \subset \psi\,`(\mathrm{Y, D}).$$

For investigation of the relationships between ρ(Y, D) (or ξ (Y,D)) and ψ (Y,D) (or ψ `(Y,D) we state the following

Theorem 5.1

If D is closed convex cone in R^P, then for arbitrary Y in R^P,

$$\psi\ (\mathrm{Y, D}) \subset \rho(\mathrm{Y, D})$$

Proof:

Let $\mu \in D^{so}$ and y` $\in$ S($\mu$,Y). If we suppose that -d $\in$ clP(Y+D-y`) and d $\in$ D, then there exist sequences $\{y^k\} \subset$ Y, $\{d^k\} \subset$ D and $\{b_k\} \subset$ R such that $b_k \geq 0$ and $b_k(y^k+d^k-y`) \to -d$ as $k \to \infty$.

Taking the inner product with μ ,we have

$$\lim_{k\to\infty} b_k\{(\langle\mu,y^k\rangle-\langle\mu,y'\rangle+\langle\mu,d^k\rangle\}=-\langle\mu,d\rangle.$$

Since y` $\in$ S (μ, Y), $d^k \in$ D, and $\mu \in D^{so}$, the left hand side is nonnegative. So

$<\mu$, d $>\leqq 0$ and hence d = 0

Thus we have established that

clP(Y+D-y`)$\cap$ (-D)={0}

That means y` $\in \rho$ (Y, D) . //.

Theorem 5.2:

If D is a convex cone and Y is a set in R^P, then

ψ (Y, D) $\subset$ ξ (Y, D).

Proof:

Suppose y\` $\in \psi$ (Y, D) but y\` $\notin \xi$(Y, D).

If y\` $\notin$ Y, then it is clear that y\` $\notin \psi$ (Y,D),so assume y\` $\in$ Y\\$\xi$(Y,D), and there exists a y* $\in$ Y such that y\`-y* $\in$ D\\{0}

Then for any $\mu \in$ D^{so},

$$\langle \mu, y'-y^* \rangle > 0$$

Thus

y\` $\notin \psi$ (Y, D), as was to be proved.//.

Theorem 5.3

If D is an acute convex cone and Y is a D-convex set in R^P, then

ξ (Y, D) $\subset \psi$ ' (Y, D).

Proof:

Let y\` $\in$ $\xi$ (Y, D). From proposition 3.1.2, y\` $\in \xi$ (Y+D, D) and so (Y+D-y\`) $\cap$ (-D)={0} (or ϕ when D does not contain the origin).

Applying the separation theorem, we have the existence of a nonzero vector $\mu \in R^P$ such that

$\langle \mu$,y+d-y\` $\rangle \geqq 0$ for all y $\in$ Y and d $\in$ D,

$\langle \mu$,-d $\rangle \leqq 0$ for all d\` $\in$ D.

From the latter, $\mu \in$ D^o and from the former, $\langle$u, y$\rangle \geqq \langle$u, y\`$\rangle$ for all y $\in$Y, since we can take d=0(or as nearly equal to zero as desired). Thus we have established that y\` $\in$ S(μ,Y) $\subset \psi$ \`(Y, D).

This completes the proof.//.

Corollary 5.1:

Let D be an acute, open, convex cone and Y be a D-convex set in R^P. Then

ξ(Y, D) = ψ \`(Y,D)

Proof:

In view of proposition.2.1.2, D^{so}=D^o\\{0} when D is open.

Hence ψ (Y, D) $=\psi$ ` (Y, D), and the corollary directly follows from theorem 5.

5.2 Necessary and Sufficient Conditions

In chapter four we have seen certain conditions for the existence of multipliers. Now it is time to derive necessary and sufficient c onditions for optimality under the assumption that every f_i(i=1...p) and g_j(j=1,....,m) is continuously differentiable.

Theorem 5.4

Let x' $\in$ X be a properly efficient solution to (P) i n the sense of Kuhn-Tucker, then there exist $\mu' \in R^p$ and $\lambda' \in R^m$ such that

i) $\langle \mu', \nabla f(x') \rangle + \langle \lambda', \nabla g(x') \rangle = 0$

ii) $\langle \lambda', g(x') \rangle = 0$

iii) $\mu' > 0$, $\lambda' \geqq 0$

where $\langle \mu', \nabla f(x') \rangle$ and $\langle \lambda', \nabla g(x') \rangle$ stands for $\sum_{i=1}^{p} \mu_i \nabla f_i(x')$ and $\sum_{j=1}^{m} \lambda_j \nabla g_j(x')$, respectively.

Proof:

From Kuhn-Tucker definition of proper efficiency, there is no $h \in R^n$ such that $\langle h, \nabla f_i(x') \rangle \leqq 0$ (i=1,...,p) with strict inequality for at least one i and

$\langle h, \nabla g_j(x') \rangle \leqq 0$ $j \in J(x')=\{j:g_j(x')=0\}$

Then, from Kuhn-Tucker theorem of alternatives, there exist

$\mu'_i > 0$ (i=1,...,p) and $\lambda'_j \geqq 0$ ($j \in J(x')$)

such that

$$\sum_{i=1}^{p} \mu_i \nabla f_i(x') + \sum_{j \in J(x')} \lambda'_j \nabla g_j(x') = 0$$

Letting $\lambda'_j=0$ for $j \notin J(x')$, we can immediately establish the theorem.//.

In ordinary optimization problems we know that the sufficient condition for $x` \in X$ to be the efficient solution is that X is convex, and the objective function is convex. In case of vector objective optimization problems (P) a sufficient condition for $x` \in X$ to be a properly efficient solution is given in the following theorems.

Theorem 5.5

Let $x' \in X$, the functions f and g (for all i ,j) are convex, and there exist $\mu' \in R^p$ and $\lambda' \in R^m$ such that

i) $\langle \mu', \nabla f(x') \rangle + \langle \lambda', \nabla g(x') \rangle = 0$

ii) $\langle \lambda', g(x') \rangle = 0$

iii) $\lambda' \geqq 0$, $\mu' > 0$

Then x' is a properly efficient solution of (P).

Proof:

The condition in theorem 5.4 implies that $f(x') \in S(\mu', Y)$ so also $f(x') \in \psi(Y, D)$ and hence, from

Theorem 5.1, $f(x') \in \psi(Y, D) \subset \rho(Y, D)$

That means x' is a properly efficient solution of (P).//.

Weak optimality conditions play an interesting role in the theory of vector optimization. The necessary and sufficient conditions for $x' \in X$ to be a weak pareto optimal solution to (P) is discussed as follows;

Theorem 5.6

Suppose that (P) satisfies the Kuhn-Tucker constraint qualification. Then, a necessary condition for x' to be a weak Pareto optimal solution to (P) is that there exist $\mu' \in R^P$ and $\lambda' \in R^m$ such that

i) $\langle \mu', \nabla f(x') \rangle + \langle \lambda', \nabla g(x') \rangle = 0$

ii) $\langle \lambda', g(x') \rangle = 0$,

iii) $\lambda' \geqq 0$, $\mu' \geq 0$

Proof:

Let x' be a weak Pareto optimal solution of (P),

We proof that there is no h such that

$\langle \nabla f_i(x'), h \rangle < 0 \quad i=1,\ldots,p$

$\langle \nabla g_j(x'), h \rangle \leqq 0 \quad j \in J(x') = \{j : g_j(x') = 0\}$.

Suppose to the contrary, that there exist an h satisfying the above inequalities. From Kuhn-Tucker constraint qualification, there exists a continuously differentiable arc □(t) ($0 \leqq t \leqq \bar{t}$) such that □(0) =x, g(□(t)) $\leqq 0$ and $\dot{\theta}(t) = \alpha h$ $(\alpha > 0)$ as

$f_i(\theta(t)) = f_i(x') + t\langle \nabla f_i(x'), \alpha h \rangle + 0(t)$

Since $\langle \nabla f_i(x'), h \rangle < 0$ for all i = 1, 2, …, p, then $f_i(\theta(t)) < f_i(x')$ for t sufficiently small. This contradicts the weak Pareto optimality of x'.//.

Theorem 5.7.

If every function f_i (i = 1, 2, …, p) and g_j (j = 1, 2, …, m) is convex, the condition in theorem 5.6 is also sufficient for x' $\in$ X to be a weak Pareto optimal solution to (P).

Proof:

The condition in theorem 5.6 implies that f(x') $\in S(\mu', Y)$, where $\mu' \in R_+^p \setminus \{0\}$. Since in view of corollary 5.1, $S(\mu', Y) \subset \xi(Y, \text{int}\, R_+^p)$ under the convexity assumption, x' is a weak Pareto optimal solution of (P).//.

References:

[1]: Sawaragi Y., Nakayama H. and Tanino T.: Theory of Multiobjective Optimization, 1985, Academic Press.

[2]: Sekatzek M.: Contributions to Multiplier Rule for Multiobjective Optimization problems, 1998, Halle/Saale, Germany.

[3]: W.W. Breckner and A. Gopfert: Multipliers for Weak Pareto Optimization Problems, 1996, Halle (sale), Germany.

[4]: Miettinen: On the Methodology of Multiobjective Optimization with Applications, 1994.

[5]: Dinh The Luc: Theory of vector optimization, Lecture Notes in Economics and Mathematical systems, 1989.

[6] Deumlich R.: Optimization and theory of approximation, Textbook, 1997, Addis Ababa.

[7] Deumlich R.: Functional Analysis I, Textbook, 1997, Addis Ababa.

[8] Deumlich R.: Functional Analysis II, Textbook, 1998, Addis Ababa.

[9] P.L. Yu: Multiple criteria Decision Making, Five Basic Concepts, 1989, U.S.A.