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ON

GREEN'S FUNCTIONS AND STURM-LIOUVILLE PROBLEMS

(SUBMITTED IN THE PARTIAL FULLFILMENT OF THE MSc DEGREE IN MATHEMATICS)

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1.1 The concept of Green's Functions

Boundary value problems are almost inevitable consequences of using mathematics to study problems arising in the real world and it is not at all surprising that their solution has been the major concern of many mathematicians. The result of all this endeavors is that now a day the approach towards a solution of a specific boundary value problem can be made in any one of many ways. In this topic we shall examine in detail that one particular method which requires the construction of an auxiliary function known as Green's function. To show how such functions arise, and to initiate a further study of the method we will first solve, elementary method, a typical boundary value problem.

Consider the problem of the forced, transverse vibrations of a taut string of length l.If the time dependent part of the solution are first removed by the usual separation of variables technique, we obtain the following differential equation containing the transverse displacement of the string, u, as unknown;

$$\frac{d^2u}{dx^2}+k^2u=-f(x), \quad 0<x<l \tag{1.1}$$

If the ends of the string are kept fixed ,then this equation must be solved for u subject to the boundary conditions :

$$u(0)=u(l)=0 \tag{1.2}$$

To solve the boundary value problem posed by ordinary second order differential equation (1.1) and associated boundary conditions (1.2), we will employ the method of variation of parameters, that is we will assume that a solution of the problem actually exists, and that, furthermore it has the precise form:

$$u(x)=A(x)\cos kx+B(x)\sin kx \tag{1.3}$$

If we differentiate (1.3) twice with respect to x, and assuming that

$$A'(x)\cos kx+B'(x)\sin kx=0 \tag{1.4}$$

Then we find that (1.3) has a solution provided that

$$-kA'(x)\sin kx+kB'(x)\cos kx=-f(x) \tag{1.5}$$

Although assumption (1.4) was introduced primarily to ease the ensuring algebra, equation (1.4) and (1.5) are two linear algebraic equations for the unknowns $A'(x)$ and $B'(x)$.

Solving these equations

$$A'(x) = \frac{f(x)\sin kx}{k}, \quad B'(x) = \frac{-f(x)\cos kx}{k} \tag{1.6}$$

Thus, formally we can write the solution of (1.1) in the form

$$u(x) = \frac{\cos kx}{k}\int_{c_1}^{x} f(y)\sin kydy - \frac{\sin kx}{k}\int_{c_2}^{x} f(y)\cos kydy \tag{1.7}$$

where c_1 and c_2 are constants which must be so chosen as to ensure that the boundary conditions (1.2) are satisfied.

Inserting the conditions $u(0) = 0$ in to (1.7) we find that we must choose c_1 such that

$$\int_{c_1}^{0} f(y)\sin kydy = 0 \tag{1.8}$$

Since *f(y)* is assumed arbitrary, this implies that we must choose $c_1=0$. The condition $u(l)=0$ when inserted in to (1.7) will require that

$$u(l) = \frac{\cos kl}{k}\int_{0}^{l} f(y)\sin kydy - \frac{\sin kl}{k}\int_{c_2}^{l} f(y)\cos kydy = 0$$

This can be written as

$$\frac{\cos kl}{k}\int_{0}^{l} f(y)\sin kydy - \frac{\sin kl}{k}\int_{0}^{l} f(y)\cos kydy - \frac{\sin kl}{k}\int_{c_2}^{0} f(y)\cos kydy = 0$$

which implies

$$-\frac{\sin kl}{k}\int_{c_2}^{0} f(y)\cos kydy + \frac{1}{k}\int_{0}^{l} f(y)\sin k(y-l)dy = 0 \tag{1.9}$$

Combining the results (1.8) and (1.9), we see that the solutions (1.7) can now be written in the form

$$u(x) = \frac{1}{k}\int_{0}^{x} f(y)\sin k(y-x)dy - \frac{\sin kx}{k\sin kl}\int_{0}^{l} f(y)\sin k(y-l)dy \tag{1.10}$$

$$= \int_{0}^{x} f(y)\frac{\sin ky\sin k(l-x)}{k\sin kl}dy + \int_{x}^{l} f(y)\frac{\sin kx\sin k(l-y)}{k\sin kl}dy$$

$$= \int_0^l f(y)G(x,y)dy \qquad 1.11$$

Where

$$G(x,y) = \begin{cases} \dfrac{\sin ky \sin k(l-x)}{k \sin kl}, & 0 \le y \le x \\ \dfrac{\sin kx \sin k(l-y)}{k \sin kl}, & x \le y \le l \end{cases} \qquad 1.12$$

This function $G(x,y)$ is a two point function of position known as the Green's function for the equation (1.1) and associated boundary condition (1.2) .Its existence is assured in this particular problem , provided that $\sin kl \neq 0$ .The case when $\sin kl = 0$ will be examined in much more detail later.

One of the main advantages of the representation (1.11) of the solution to our boundary value problem is that the Green's function $G(x,y)$,is independent of the forcing term $f(x)$,and depends only upon the particular differential equation being examined and the boundary conditions which are imposed . Consequently, the solution to all such problems as that examined above but having different forcing term f , is known in the form (1.11) once $G(x,y)$ has been determined; always provided that the resulting integral in (1.11) exists.

The fundamental equation we must be able to answer is : 'Given a differential equation and associated boundary conditions , is it possible to construct a function $G(x,y)$,the Green's function and obtain the required solution in the form (1.11) , without first having to solve the given boundary value problem ?' This question leads us to the method of constructing of the Green's function and the study of further properties of Green's function.

With a view to constructing Green's functions for more general problems, let us examine more closely the particular Green's function we have already constructed .We notice that $G(x,y)$ as defined in(1.12) the following properties:

1. It satisfies the homogeneous form of the given differential equation. That is

$$G'' + \quad k^2 G = 0$$

in each of the interval $0 \le y < x, \quad x < y \le l$ .The behavior of G at $x = y$ is ,at the moment, uncertain.

2.The function G is continuous at $x = y$ since

$$\lim_{y \to x^-} G(x,y) = \frac{\sin kx \sin k(l-x)}{k \sin kl} = \lim_{y \to x^+} G(x,y)$$

3. The derivative of G with respect to y is discontinuous at $x=y$ .This can be seen as

$$G'(x,x^-) = \lim_{y \to x^-} G'(x,y) = \frac{\cos kx \sin k(l-x)}{\sin kl}$$

$$G'(x,x^+) = \lim_{y \to x^+} G'(x,y) = \frac{-\sin kx \cos k(l-x)}{\sin kl}$$

where $G'(x,y)$ represents the differentiation with respect to y. Hence

$$G'(x,x^+) - G'(x,x^-) = -1$$

4. The function $G(x,y)$ satisfies the relations $G(x,0) = G(x,l) = 0$ and thus satisfies the boundary conditions of the problem.

5. The function $G(x,y)$ is symmetric in its arguments, hence

$$G(x,y) = G(y,x)$$

With this several properties of the Green's function in mind, let us now try to solve the given boundary value problem, (1.1) and (1.2), by assuming from the out set existence of a Green's function, $G(x,y)$. If such an assumption is valued, we should be able to recover directly from the differential equation, not only the representation (1.11) but also the several properties of $G(x,y)$ and integrating with respect to x over the range $0 \le x \le l$, we obtain

$$\int_0^l (u''+k^2u)G(x,y)dx = -\int_0^l f(x)G(x,y)dx \qquad 1.13$$

The only assumption we shall make regarding the behavior of $G(x,y)$ is that it is possibly not a well- behaved function of x when x approaches the value y. Consequently, since the above integral may be improper; we exclude the point $x=y$ from the range of integration and write:

$$\int_0^l (u''+k^2u)G(x,y)dx = \lim_{\xi \to y^-} \int_0^{\xi} (u''+k^2u)G(x,y)dx + \lim_{\eta \to y^+} \int_{\eta}^{l} (u''+k^2u)G(x,y)dx \qquad 1.14$$

Treating each integral on the right hand side separately, we have, on integrating twice by part:

$$\int_0^{\xi}(u''+k^2u)G(x,y)dx = [Gu'-G'u]_0^{\xi} + \int_0^{\xi}u(G''+k^2G)dx$$

and

$$\int_{\eta}^{l}(u''+k^2u)G(x,y)dx = [Gu'-G'u]_{\eta}^{l} + \int_{\eta}^{l}u(G''+k^2G)dx$$

If we choose $G(x,y)$ to satisfy, as a function of *x*, $G''+k^2G=0$ in the range $0 \le x \le \xi$ and $\eta \le x \le l$, then the integral on the right hand side vanish . Inserting the remaining integrated terms in to (1.14) taking the appropriate limits, and employing the boundary condition (1.2) we find that (1.13) reduces

$$-\int_0^{l} f(x)G(x,y)dx = \{G(y,y^-)u'(y^-) - G'(y,y^-)u(y^-) - G(0,y)u'(0)$$
$$+ G(l,y)u'(l) - G(y,y^+)u'(y^+) + G'(y,y^+)u(y^+)\ \}$$

Provided

$$G(x,y) = G(y,x).$$

If now we assume that $G(x,y)$ satisfies the boundary conditions (1.2) we can effect a further reduction, and obtain:

$$\int_0^{l} f(x)G(x,y)dx = -u(y)\{G'(y,y^+) - G'(y,y^-)\} + u'(y)\{G(y,y^+) - G(y,y^-)\} \qquad 1.15$$

Where we have used the assumed continuity of *u(x)* and written $u(y^+) = u(y^-) = u(y)$ and $u'(y^+) = u'(y^-) = u'(y)$.Finally, if we assert that $G(x,y)$ is continuous at *x=y*, whereas $G'(x,y)$ has discontinuity there specified by property 3, then (1.15) reduces to

$$u(y) = \int_0^{l} f(x)G(x,y)dy \qquad 1.16$$

And we thus obtain the desired representation of the solution to the given boundary value problem. The representation (1.16) is in terms of a two-point function of position $G(x,y)$ which satisfies

$$G'+\ \ k^2G = 0 \qquad x \neq y$$

$$G(0,y) = G(l,y) = 0$$

$$G(x,y) \text{ continuous at } x=y$$

$$G'(y,y^+) - G'(y,y^-) = 1$$

and

$$G(x,y) = G(y,x)$$

Clearly these are the five properties that we established earlier.

2 Differential Operators

The differential equation (1.1) can be rewritten in the form

$$Lu = -f$$

where we introduce as a separate quantity for our consideration the differential operator L. In that specific example

$$L \equiv (\frac{d^2}{dx^2} + k^2).$$

Definition 2.1: A transformation or operator L on a vector space V to a vector space W is an operation on a subset $D(L)$ of V, called the domain of definition of the operator L, which assigns to each vector y in $D(L)$ another vector Ly in W called the image of y under L.

Two operators L_1 and L_2 are said to be equal if

(a) $D(L_1) = D(L_2)$

(b) $L_1 y = L_2 y$ for all y in $D(L_1)$ and $D(L_2)$.

The range, $R(L)$, of a transformation L is the set of all vectors x of the form $x = Ly$ for all y in $D(L)$.

Definition 2.2: The operator L is said to be linear if $D(L)$ is a linear manifold of V and if

$$L(ay_1 + by_2) = aLy_1 + bLy_2.$$

Where y_1 and y_2 in $D(L)$ and a and b are scalars.

Now we are at the time at which we define the differential operator and differential equations. Suppose $a_0, a_1, ..., a_n$ are continuous functions of x defined on a real interval $[a,b]$ and let L_n denote the formal differential operator of order n, (n is a natural number) then

$$L_n = a_0 \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + ... + a_{n-1} \frac{d}{dx} + a_n$$

The associated differential equation for any function y which is a function of x which possesses n derivatives (and the lower derivatives) on the interval $[a,b]$ can be written as

$$L_n y = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + a_{n-1} \frac{dy}{dx} + a_n y$$

Where $a_0(x) \neq 0$ for all x in $[a,b]$.

The typical problem (1.1) can be rewritten now in the operator form as

$$Lu = -f$$

where $L = \frac{d^2}{dx^2} + k^2 \frac{d}{dx}$, u the unknown function which must satisfy this equation and certain imposed boundary conditions, and f a known function.

One method of determining the function u is to find the operator which is inverse to L. That is find the operator which is such that $LL^{-1} = L^{-1}L = I$, the identity operator. When such an inverse operator exists we have seen that it takes the form of an integral operator, the kernel of which is known as the Green's function for the operator L. We have already found one such kernel for the problem we examined in the previous topic and to enable us to find the Green's function for more general operator we will first define the Dirac delta function δ.

2.1 Inverse Operator and the δ –Function

Let L be a linear, ordinary differential operator acting on the space of functions $u(x)$. The inverse operator to L is L^{-1} and it is such that $L\ L^{-1} = L^{-1}\ L = I$. We assume that L^{-1} is an integral operator with kernel $k(x,t)$ so that

$$L^{-1}u(x) = \int k(x,t)u(t)dt$$

Then

$$u(x) = Iu(x) = LL^{-1}u(x) = L\int k(x,t)u(t)dt$$

Since L is a differential operator with respect to the variable x, we see that

$$u(x) = \int Lk(x,t)u(t)dt \qquad 2.1$$

We may write the kernel of this integral operator in the form

$$Lk(x,t) = g(x,t)$$

and we obtain

$$u(x) = \int g(x,t)u(t)dt \qquad 2.2$$

Now, if this result is to be true for all continuous functions u it follows that $g(x,t)$ must be zero whenever $x \neq t$ and when $x = t$ the integral on the write must reduce identically to $u(x)$. To ensure that this was always the case Dirac introduced the so called δ function in place of $g(x,t)$ and obtain

$$u(x) = \int \delta(t-x)u(t)dt \qquad 2.3$$

Where $\delta(x) = 0$ if $x \neq 0$. Such a function is zero every where except at the origin, where it becomes infinite in such a way as to ensure

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

This expression states that the 'surface area' under the delta function is equal to zero. As the width of this function goes to zero, the value of the function must become infinite to ensure that

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

is satisfied. Important property of the δ- function, and that which makes it so useful, is: for every continuous function ϕ,

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0) \qquad 2.4$$

That is the delta function picks out the value of a continuous function at the origin. Consequently

$$\delta(x_i - x) = \delta(x - x_i)$$

That is the delta function operates like an even function. This means that the delta function may be used without worrying about whether $\delta(x_i - x)$ or $\delta(x - x_i)$ appears. The Dirac delta function is also the derivative of the Heaviside unit step function $H(x - x_i)$

$$H(x - x_i) = \begin{cases} 0, x < x_i \\ 1, x > x_i \end{cases} \quad 2.5$$

$$\delta(x_i - x) = \frac{d}{dx} H(x - x_i) \quad 2.6$$

$$H(x - x_i) = \int_{-\infty}^{x} \delta(x_0 - x_i) dx_0 \quad 2.7$$

Another important property of the delta function is that

$$\delta[c(x_i - x)] = \frac{1}{|c|} \delta(x - x_i)$$

To show this substitute $y=cx$ we have

$$\int_{-\infty}^{\infty} \delta[c(x - x_i) f(x) dx = \int_{-\infty}^{\infty} \delta(y - cx_i) f(\frac{y}{c}) \frac{1}{c} dy$$

Carrying the y- integration using property 2.3 of the delta function we have

$$\int_{-\infty}^{\infty} \delta[c(x - x_i)] f(x) dx = \frac{1}{c} f(x_i), \quad \text{when } c>0 \quad 2.8$$

and

$$\int_{-\infty}^{\infty} \delta[c(x - x_i)] f(x) dx = -\frac{1}{c} f(x_i), \quad \text{when } c<0 \quad 2.9$$

Combining 2.8 and 2.9 we have

$$\int_{-\infty}^{\infty} \delta[c(x - x_i)] f(x) dx = \frac{1}{|c|} f(x_i) \quad 2.10$$

The right hand side of equation 2.10 can also be written as

$$\frac{1}{|c|}\int_{-\infty}^{\infty}\delta(x-x_i)f(x)dx \qquad 2.11$$

Combining 2.10 and 2.11 gives us that

$$\delta[c(x_i-x)]=\frac{1}{|c|}\delta(x-x_i) \qquad 2.12$$

as desired. In similar manner we can show that

$$\int_{-\infty}^{\infty}\delta(ax-b)f(x)dx=|a^{-1}|f(ba^{-1}) \qquad 2.13$$

2.2 The Domain of a Linear Differential Operator

Before we can continue with the problem of inverting a differentiation operator, we should specify the domain of the operator where it acts.

Let

$$L_n=a_0(x)\frac{d^n}{dx^n}+a_1(x)\frac{d^{n-1}}{dx^{n-1}}+...+a_{n-1}(x)\frac{d}{dx}+a_n(x)$$

be our operator where $a_0(x),...,a_n(x)$ are continuous functions and $a_0(x)\neq 0$ for all x in $[a,b]$. Let the space S of all real – valued functions which are (Lebesgue) square integrable over $[a,b]$ .That is , S contains all real functions u defined for $a\leq x\leq b$ and such that

$$\int_a^b (u(x))^2 dx<\infty$$

Then the operator L acts on the subspace S' such that $S'\subseteq S$ and for $u\in S'$, $Lu(x)$ is also in S'. Because we have no reason to suppose that every function in S is in fact differentiable. Also,even when a certain function u is in S, but Lu may not be. For example $u(x)=x\sin x^{-1}$ is in S, where S is a vector space of all real valued functions which are Lebesgue integrable over [0, 1], and is differentiable, but its derivative

$$u'(x)=\sin x^{-1}-x^{-1}\cos(x^{-1})$$

is not in S. Moreover in order to obtain a unique solution of the differential equation

$$Lu(x)=f(x) \qquad 2.14$$

it is not sufficient simply to specify L, we also require conditions on $u(x)$ itself. These conditions to be the boundary conditions

$$B_1(u):\ \alpha_{10}u(a)+\alpha_{11}u'(a)+\beta_{10}u(b)+\beta_{11}u'(b)=0$$

$$B_2(u):\ \alpha_{20}u(a)+\alpha_{21}u'(a)+\beta_{21}u(b)+\beta_{22}u'(b)=0 \qquad 2.15$$

where the α_{ij} and β_{ij} are known constants.

We can now define the domain of the operator L as follows: it is the set of all functions u in S which have piecewise continuous n^{th} order derivatives which satisfy the boundary conditions (2.15) and ensure that Lu is also in S.

The domain of L, which is clearly a linear manifold of S, need not necessarily, be a subspace (i.e. a complete linear manifold) of S. That is there could exist a sequence of functions u_n in the domain, which converges to a limit u in S, although this limit, u is not in the domain of L.

2.3 Adjoint Differential Operators

Let S be the linear vector space of all real valued functions u defined on $a \le x \le b$ such that

$$\int_a^b (u(x))^2 dx < \infty$$

If u and v are functions belong to S, we define an inner product in to S, by

$$(u,v) = \int_a^b u(x)v(x)dx$$

In order that we apply the theory of linear operator to the differential operator L, we need to be able to define adjoint operator L^*.

Definition 2.3: The adjoint of a differential operator L is defined to be the operator L^* for which

$$(Lu,v)=(u,L^*v)$$

for all u in the domain of L and v in the domain of L^*.

This definition determines not only the operational definition of L^*, but its domain as well. Consider the example

$$Lu = \frac{du}{dx}$$

with boundary conditions $u(0) = 2u(1)$ with inner product

$$(Lu,v)= \int_0^1 u'(x)v(x)dx$$

$$= [uv]_0^1 - \int_0^1 u(x)v'(x)dx$$

$$=u(1)[v(1)-2v(0)] - \int_0^1 u\frac{dv}{dx}dx$$

In order to make $(Lu,v)=(u,L^*v)$ we must have $L^*v= -v'$ with the boundary condition $v(1)=2v(0)$. We notice that in this example L acts on the manifold of square integrable function u which satisfies $u(0)=2u(1)$, but L^* acts on the manifold of square integrable function $v(x)$ which are such that $v(0)= \frac{1}{2}v(1)$. In general the manifold on which L^* acts is differential from that on which L acts.

Definition 2.4: If $L=L^*$ the differential operator is said to be formally self- adjoint .If in addition the boundary conditions for L and L^* are equivalent in the sense that they define the same manifold, then the differential operator is said to be self- adjoint .

Example 2.1 Let

$$L \equiv e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx}$$

On the manifold defined by $u'(0)=0$, $u(1)=0$ then

$$(v, Lu) = \int_0^1 v\left\{e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx}\right\}udx$$

$$= \int_0^1 v(e^x u')'dx$$

$$= [ve^x u']_0^1 - [v'e^x u]_0^1 + \int_0^1 u(e^x v')' dx$$

$$= u'(1)v(1)e^x + v'(0)u(0) + \int_0^1 u(e^x v'' + e^x v')dx$$

Thus we see that the differential operator in the adjoint operator has the form

$$e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx}$$

And L is consequently formally self-adjoint. In order that

$$(v, Lu) = (L^* v, u)$$

for all u, the boundary conditions satisfied by v must be $v'(0)=0$; $v(1)=0$. Therefore since $u(x)$ and $v(x)$ now satisfy the same boundary conditions, it follows that L is self – adjoint.

Notice that the general second order operator

$$Lu = a(x)\frac{d^2 u}{dx^2} + b(x)\frac{du}{dx} + c(x)u \qquad 2.16$$

where $a(x), b(x)$ and $c(x)$ are continuous functions , $a \neq 0$ for all x in the given interval , has a formal adjiont defined by

$$L^* v = \frac{d^2}{dx^2}(av) - \frac{d}{dx}(bv) + cv \qquad 2.17$$

Multiplying (2.16) by v and (2.17) by u and then subtracting (2.17) from (2.16) and integrating on [α,β] gives

$$\int_\alpha^\beta [vLu - uL^* v]dx = [J(v,u)]_\alpha^\beta \qquad 2.18$$

Where

$$J(v,u) = avu' - u(av)' + buv.$$

$J(v,u)$ is called the conjunct of the functions v and u .Equations (2.18) is called Green's identity.

2.4 Self Adjoint Second- Ordered Differential Operators

In this section we consider second order differential operator in some detail. Consider the general form of second order differential operator

$$Lu \equiv -\frac{1}{w}(pu')' + qu \qquad 2.19$$

This operator is self adjoint, provided that the scalar product is chosen to be

$$(u,v) = \int_0^1 u(x)v(x)w(x)dx \qquad 2.20$$

The minus sign in the definition of L is to ensure that the operator is positive definite. That this is the case can be seen as follows:

$$(u, Lu) = \int_0^1 u(-\frac{1}{w}(pu')' + qu)wdx$$

$$= \int_0^1 (pu'^2 + qwu^2)dx + puu'\Big|_0^1$$

Consequently, if $p>0, q>0, w>0$, and the boundary conditions are such that the integrated terms vanish, we have that

$$(u,v) > 0$$

as required.

Apart from this one mention of boundary conditions, all we have achieved so far is the requirement for L to be formally self adjoint. We now wish to find the conditions which will ensure that L is self adjoint. To this end, if we examine the difference $(u, Lv) - (Lv, u)$, we see that

$$(u, Lv) - (Lv, u) = J(v,u)\Big|_0^1 = -p(x)(vu' - uv')\Big|_0^1$$

Therefore, L will be self adjoint if $J(v,u)$ vanishes identically when u and v are in the same manifold. Two particular special cases arise which will have important application later.

1. Unmixed boundary condition

Boundary conditions are said to be unmixed if they involve the function and its derivatives at either $x=0$ or $x=1$, but not at both. For example

$$au(0) + bu'(0) = 0$$

If u satisfies an unmixed condition at $x=0$ and an unmixed condition at $x=1$ then L is self adjoint .

2. Periodic boundary conditions

Boundary conditions are said to be periodic if they have the form

$$u(0) = u(1), \qquad u'(0) = u'(1).$$

Again if u satisfies periodic boundary conditions, L is self adjoint (provided that p is periodic, i.e $p(0)=p(1)$).

Theorem 2.1: If u is any solution of $Lu=0$ and v is any solution of $L^*v=0$, then the conjunct of u and v is a constant whose value depends on u and v.

Corollary 2.2: If L is a formally self adjoint operator and u_1 and u_2 are two solutions of $Lu=0$, then the conjunct of u_1 and u_2 is a constant whose value depends on u_1 and u_2.

Corollary 2.3: If L is self adjoint and u_1 and u_2 are two solutions of $Lu=0$ and if $J(u_1, u_2)$ vanishes for some value of x for which $p(x)\neq 0$ then u_1 and u_2 linearly dependent.

Proof: from corollary 2.2 it follows that the conjunct must vanish for all x. Consequently

$$u_1 u'_2 - u_1' u_2 = 0$$

This implies that

$$\frac{u_1 u'_2 - u'_1 u_2}{u_2^2} = 0$$

$$\left(\frac{u_1}{u_2}\right)' = 0$$

$$\left(\frac{u_1}{u_2}\right) = c$$

Hence u_1 and u_2 are linearly dependent

Theorem 2.4: Consider the differential equation

$$Ly = a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad 2.21$$

where a_0 has continuous second derivative, a_1 has continuous first derivative and a_2 is continuous and $a_0 \neq 0$ for all x in $[a,b]$. A necessary and sufficient condition that the differential equation be self adjoint is that

$$\frac{d}{dx}(a_0(x)) = a_1(x) \quad \text{on } [a,b].$$

Proof: the adjoint equation is given by equation 2.7 i.e.

$$L^* y = \frac{d^2}{dx^2}(a_0 y) - \frac{d}{dx}(a_1 y) + a_2 y = 0$$

$$= a_0(x)\frac{d^2 y}{dx^2} + [2a_0'(x) - a_1(x)]\frac{dy}{dx} + [a_0''(x) - a_1'(x) + a_2(x)]y = 0 \qquad 2.22$$

Now if

$$\frac{d}{dx}(a_0(x)) = a_1(x)$$

on $[a,b]$ we have

$$2a_0'(x) - a_1(x) = a_1(x)$$

$$a_0''(x) - a_1'(x) + a_2(x) = a_2(x)$$

This shows that equations (2.21) and (2.22) are identical. Conversely if (2.21) and (2.22) are identical then

$$2a_0'(x) - a_1(x) = a_1(x)$$

$$a_0''(x) - a_1'(x) + a_2(x) = a_2(x)$$

The second of these equations shows that

$$a_0'(x) = a_1(x) + c$$

where c is arbitrary constant. From the first condition we see that

$$a_0'(x) = a_1(x).$$

Thus $c = 0$ and we have

$$\frac{d}{dx}[a_0(x)] = a_1(x), \qquad \text{for all } x \text{ in } [a,b].$$

Corollary 2.5: suppose the second order differential equation (2.21) is self adjiont. Then it can be rewritten in the form

$$\frac{d}{dx}[a_0(x)\frac{dy}{dx}]+a_2(x)y=0$$

Proof: Let

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$$

is self adjoint by theorem 2.4 we have that

$$\frac{d}{dx}[a_0(x)]=a_1(x)$$

substituting this in (2.10) gives

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$$

This implies

$$\frac{d}{dx}[a_0(x)\frac{dy}{dx}]+a_2(x)y=0\ .$$

Theorem 2.6: If the coefficients $a_{0,}a_{1,}a_2$ in (2.21) are continuous on $[a,b]$ and $a_0(x)=0$ for all x in $[a,b]$, then (2.21) can be transformed in to equivalent self adjoint equation

$$\frac{d}{dx}[p(x)\frac{dy}{dx}]+q(x)y=0$$

where

$$p(x)=\exp(\int\frac{a_1(x)}{a_0(x)}dx)$$

and

$$q(x)=\frac{a_2(x)}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx)$$

by multiplying throughout the equation by the factor

$$\frac{1}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx).$$

Proof: Consider

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad (2.23)$$

multiplying (2.23) by the factor

$$\frac{1}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx)$$

gives

$$\exp(\int\frac{a_1(x)}{a_0(x)}dx)y''+\frac{a_1(x)}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx)y'+\frac{a_2(x)}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx)y=0$$

If we let

$$\overline{a}_0(x)=\exp(\int\frac{a_1(x)}{a_0(x)}dx)$$

and

$$\overline{a}_1(x)=\frac{a_1(x)}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx)$$

we observe that $\overline{a}_0'(x)=\overline{a}_1(x)$ .From this we have

$$\frac{d}{dx}[\exp(\int\frac{a_1(x)}{a_0(x)}dx)\frac{dy}{dx}]+\frac{a_2(x)}{a_0(x)}\exp(\int\frac{a_1(x)}{a_0(x)}dx)y=0$$

That is

$$\frac{d}{dx}[p(x)\frac{dy}{dx}]+q(x)y=0.$$

3 Sturm-Liouville Problems

Partial differential equations can be solved using the so called method of separation of variables. Under separable conditions the second order homogeneous partial differential equations are transformed into ordinary differential equations of the form

$$a_1(x)\frac{d^2y}{dx^2}+a_2(x)\frac{dy}{dx}+[a_3(x)+\lambda]y=0 \qquad 3.1$$

If we introduce

$$p(x)=\exp[\int\frac{a_2(x)}{a_1(x)}dx],\ q(x)=\frac{a_3}{a_1}p(x),\ \ s(x)=\frac{p(x)}{a_1(x)}p(x) \qquad 3.2$$

Into equation 3.1, we obtain

$$\frac{d}{dx}[p(x)\frac{dy}{dx}]+[q(x)+\lambda s(x)]y=0 \qquad 3.3$$

This is known as the Sturm-Liouville problem. In terms of the operator if

$$L\equiv\frac{d}{dx}[p\frac{d}{dx}]+q\,,$$

equation 3.3 can be written as

$$Ly+\lambda s(x)y=0 \qquad 3.4$$

where λ is a parameter independent of x, *and* p, q and s are real-valued functions of x. To ensure the existence of solutions, we let q and s be continuous and p be continuously differentiable in a closed finite interval $[a,b]$.

The Sturm-Liouville problem is called regular in the interval $[a,b]$ if the function *p(x)* and *s(x)* are positive in the interval $[a,b]$. Thus, for a given λ there exist two linearly independent solutions of a regular Sturm-Liouville equations in the interval $[a,b]$.

The Sturm-Liouville equation

$$Ly+\lambda s(x)y=0\,,\ \ a\le x\le b$$

together with the separated boundary conditions

$$\begin{cases}\alpha_1 y(a)+\alpha_2 y'(a)=0\\ \beta_1 y(b)+\beta_2 y'(b)=0\end{cases} \qquad 3.5$$

where the constants α_1 and α_2, and likewise β_1 and β_2 are not both zero and are given real numbers, is called a regular Sturm-Liouville boundary value problem. The value of λ for which the Sturm-Liouville problem has a nontrivial solution are called the eigenvalues, and the corresponding solutions are called the eigenfunctions

Example 3.1. Consider the regular Sturm–Liouville problem

$$y'' + \lambda y = 0,\ 0 \le x \le \pi,$$

$$y(0) = 0,\quad y'(\pi) = 0.$$

When $\lambda \le 0$, it can be readily shown that λ is not an eigenvalue. However, when $\lambda > 0$, the solution of the Sturm–Liouville equation is

$$y(x) = A\cos\sqrt{\lambda}\ x + B\sin\sqrt{\lambda}\, x.$$

Applying the condition $y(0) = 0$, we obtain $A = 0$. The condition $y'(\pi) = 0$ yields

$$B\sqrt{\lambda}\ \cos\sqrt{\lambda}\ \pi = 0.$$

Since $\lambda \neq 0$ and B = 0 yields a trivial solution, we must have

$$\cos\sqrt{\lambda}\ \pi = 0,\ B \neq 0.$$

This equation is satisfied if

$$\sqrt{\lambda} = \frac{(2n-1)}{2}, \qquad n = 1, 2, 3, \ldots.$$

and hence, the eigenvalues are $\lambda_n = \dfrac{(2n-1)^2}{4}$, and the corresponding eigenfunctions are

$$\phi_n = \sin\frac{(2n-1)}{2}x, \qquad n = 1, 2, 3, \ldots.$$

Example 3.2: Consider the Euler equation

$$x^2 y'' + x y' + \lambda y = 0,\ 1 \le x \le e$$

with the end condition

$$y(1) = 0,\ y(e) = 0.$$

By using the transformation (3.2), the Euler equation can be put into the Sturm–Liouville form:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{1}{x}\lambda y = 0$$

The solution of the Euler equation is

$$y(x) = c_1 x^{i\sqrt{\lambda}} + c_2 x^{-i\sqrt{\lambda}}$$

Noting that $x^{ia} = e^{ia} \ln x = \cos(a \ln x) + i \sin(a \ln x)$, the solution $y(x)$ becomes

$$y(x) = A\cos(\sqrt{\lambda}\ \ln x) + B\sin(\sqrt{\lambda}\ \ln x),$$

where A and B are constants related to c_1 and c_2. The end condition $y(1) = 0$ gives $A = 0$, and the end condition $y(e) = 0$ gives

$$\sin\sqrt{\lambda} = 0,\ B \neq 0,$$

which in turn yields the eigenvalues

$$\lambda_n = n^2\pi^2,\ n = 1, 2, 3, \ldots,$$

and the corresponding eigenfunctions

$$\sin(n\pi \ln x), \qquad n = 1, 2, 3, \ldots.$$

Another type of problem that often occurs in practice is the periodic Sturm–Liouville system. The Sturm–Liouville equation

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] + [q(x) + \lambda s(x)]y = 0,\ \ a \leq x \leq b,$$

in which $p(a) = p(b)$, together with the periodic end conditions

$$y(a) = y(b),\ \ y'(a) = y'(b)$$

is called a periodic Sturm–Liouville system.

Example 3.3. Consider the periodic Sturm–Liouville system

$$y'' + \lambda y = 0,\ -\pi \leq x \leq \pi,$$

$$y(-\pi) = y(\pi),\ \ y'(-\pi) = y'(\pi).$$

Here we note that $p(x) = 1$, hence $p(-\pi) = p(\pi)$. When $\lambda > 0$, we see that the solution of the Sturm–Liouville equation is

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Application of the periodic end conditions yields

$$(2\sin\sqrt{\lambda}\,\pi)B = 0,$$

$$(2\sqrt{\lambda}\ \sin\sqrt{\lambda}\,\pi)A = 0.$$

Thus, to obtain a nontrivial solution, we must have

$$\sin\sqrt{\lambda}\,\pi = 0,\ A\neq 0,\ B\neq 0.$$

Consequently,

$$\lambda_n = n^2,\ n = 1, 2, 3, \ldots.$$

Since $\sin\sqrt{\lambda}\,\pi = 0$ is satisfied for arbitrary *A* and *B*, we obtain two linearly independent eigenfunctions *cos(nx)*, and *sin(nx)* corresponding to the same eigenvalue n^2.

It can be readily shown that if $\lambda < 0$, the solution of the Sturm–Liouville equation does not satisfy the periodic end conditions. However, when $\lambda = 0$ the corresponding eigenfunction is 1. Thus, the eigenvalues of the periodic Sturm–Liouville system are 0, & n^2, and the corresponding eigenfunctions are 1, {*cos(nx)*}, {*sin(nx)*}, where *n* is a positive integer.

3.1 Eigenvalues and Eigenfunctions

In examples 1 and 2 of the regular Sturm–Liouville systems in the preceding section, we see that there exists only one linearly independent eigenfunction corresponding to the eigenvalue λ, which is called an eigenvalue of multiplicity one (or a simple eigenvalue). An eigenvalue is said to be of multiplicity *k* if there exist *k* linearly independent eigenfunctions corresponding to the same eigenvalue. In Example 3.3 of the periodic Sturm–Liouville system, the eigenfunctions *cosnx, sinnx* correspond to the same eigenvalue n^2. Thus, this eigenvalue is of multiplicity two.

In the preceding examples, we see that the eigenfunctions are *cosnx* and *sin nx* for $n = 1, 2, 3, \ldots$.. It can be easily shown by using trigonometric identities that

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0,\ m\neq n,$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0,\ \text{for all integers } m,\ n,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0,\ m\neq n.$$

We say that these functions are orthogonal to each other in the interval $[-\pi, \pi]$. The orthogonality relation holds in general for the eigenfunctions of Sturm–Liouville systems

Let $\varphi(x)$ and $\psi(x)$ be any real-valued integrable functions on an interval I. Then φ and ψ are said to be orthogonal on I with respect to a weight function $\rho(x) > 0$, if and only if,

$$(\varphi, \psi) = \int_I \varphi(x)\, \psi(x)\, \rho(x)\, dx = 0. \tag{3.6}$$

The interval I may be of infinite extent or it may be either open or closed at one or both ends of the finite interval. When $\varphi = \psi$ in (3.6) we define the norm of φ by

$$\|\varphi\| = \left[\int_I \varphi^2(x)\, \rho(x)\, dx\right]^{\frac{1}{2}} \tag{3.7}$$

Theorem 3.1. Let the coefficients p, q, and s in the Sturm–Liouville system be continuous in $[a, b]$. Let the eigenfunctions φ_j and φ_k, corresponding to λ_j and λ_k, be continuously differentiable. Then φ_j and φ_k are orthogonal with respect to the weight function $s(x)$ in $[a, b]$.

Proof. Since φ_j corresponding to λ_j satisfies the Sturm–Liouville equation, we have

$$\frac{d}{dx}(p\varphi'_j) + (q + \lambda_j s)\, \varphi_j = 0 \tag{3.8}$$

and for the same reason

$$\frac{d}{dx}(p\varphi'_k) + (q + \lambda_k s)\, \varphi_k = 0. \tag{3.9}$$

Multiplying equation (3.8) by φ_k and equation (3.9) by φ_j, and subtracting, we obtain

$$(\lambda_j - \lambda_k)\, s\varphi_j\varphi_k = \varphi_j \frac{d}{dx}(p\varphi'_k) - \varphi_k \frac{d}{dx}(p\varphi'_j)$$

$$= \frac{d}{dx}[(p\varphi'_k)\, \varphi_j - (p\varphi'_j)\varphi_k]$$

and integrating yields

$$(\lambda_j - \lambda_k) \int_a^b s\varphi_j\varphi_k dx = [p(\varphi_j\varphi'_k - \varphi'_j\varphi_k]_a^b$$

$$= p(b)[\varphi_j(b)\, \varphi'_k(b) - \varphi'_j(b)\, \varphi_k(b)]$$

We say that these functions are orthogonal to each other in the interval $[-\pi, \pi]$. The orthogonality relation holds in general for the eigenfunctions of Sturm–Liouville systems

Let $\varphi(x)$ and $\psi(x)$ be any real-valued integrable functions on an interval I. Then φ and ψ are said to be orthogonal on I with respect to a weight function $\rho(x) > 0$, if and only if,

$$(\varphi, \psi) = \int_I \varphi(x)\,\psi(x)\,\rho(x)\,dx = 0. \tag{3.6}$$

The interval I may be of infinite extent or it may be either open or closed at one or both ends of the finite interval. When $\varphi = \psi$ in (3.6) we define the norm of φ by

$$\|\varphi\| = \left[\int_I \varphi^2(x)\,\rho(x)\,dx\right]^{\frac{1}{2}} \tag{3.7}$$

Theorem 3.1. Let the coefficients p, q, and s in the Sturm–Liouville system be continuous in $[a, b]$. Let the eigenfunctions φ_j and φ_k, corresponding to λ_j and λ_k, be continuously differentiable. Then φ_j and φ_k are orthogonal with respect to the weight function $s(x)$ in $[a, b]$.

Proof. Since φ_j corresponding to λ_j satisfies the Sturm–Liouville equation, we have

$$\frac{d}{dx}(p\varphi'_j) + (q + \lambda_j s)\,\varphi_j = 0 \tag{3.8}$$

and for the same reason

$$\frac{d}{dx}(p\varphi'_k) + (q + \lambda_k s)\,\varphi_k = 0. \tag{3.9}$$

Multiplying equation (3.8) by φ_k and equation (3.9) by φ_j, and subtracting, we obtain

$$(\lambda_j - \lambda_k)\,s\varphi_j\varphi_k = \varphi_j\frac{d}{dx}(p\varphi'_k) - \varphi_k\frac{d}{dx}(p\varphi'_j)$$

$$= \frac{d}{dx}[(p\varphi'_k)\,\varphi_j - (p\varphi'_j)\varphi_k]$$

and integrating yields

$$(\lambda_j - \lambda_k)\int_a^b s\varphi_j\varphi_k dx = [p(\varphi_j\varphi'_k - \varphi'_j\varphi_k]_a^b$$

$$= p(b)[\varphi_j(b)\,\varphi'_k(b) - \varphi'_j(b)\,\varphi_k(b)]$$

$$-p(a)[\varphi_j(a)\,\varphi'_k(a) - \varphi'_j(a)\,\varphi_k(a)] \quad (3.10)$$

the right side of which is called the boundary term of the Sturm–Liouville system. The end conditions for the eigenfunctions φ_j and φ_k are

$$b_1\varphi_j(b)+b_2\varphi'_j(b) = 0,$$
$$b_1\varphi_k(b)+b_2\varphi'_k(b) = 0.$$

If $b_2 \neq 0$, we multiply the first condition by $\varphi_k(b)$ and the second condition by $\varphi_j(b)$, and subtract to obtain

$$[\varphi_j(b)\,\varphi'_k(b) - \varphi'_j(b)\,\varphi_k(b)] = 0. \quad (3.11)$$

In a similar manner, if $a_2 \neq 0$, we obtain

$$[\varphi_j(a)\,\varphi'_k(a) - \varphi'_j(a)\,\varphi_k(a)] = 0. \quad (3.12)$$

We see by virtue of (3.11) and (3.12) that

$$(\lambda j - \lambda k) \int_a^b s\varphi_j\,\varphi_k\,dx = 0. \quad (3.13)$$

If λ_j and λ_k are distinct eigenvalues, then

$$\int_a^b s\varphi_j\,\varphi_k\,dx = 0. \quad (3.14)$$

Theorem 3.2.The eigenfunctions of a periodic Sturm–Liouville system in $[a, b]$ are orthogonal with respect to the weight function $s(x)$ in $[a, b]$.

Proof. The periodic conditions for the eigenfunctions φ_j and φ_k are

$$\varphi_j(a) = \varphi_j(b),\ \varphi'_j(a) = \varphi'_j(b),$$
$$\varphi_k(a) = \varphi_k(b),\ \varphi'_k(a) = \varphi'_k(b).$$

Substitution of these into equation (3.10) yields

$$(\lambda_j - \lambda_k) \int_a^b s\varphi_j\,\varphi_k\,dx = [p(b) - p(a)][\varphi_j(a)\,\varphi'_k(a) - \varphi'_j(a)\,\varphi_k(a)]$$

Since $p(a) = p(b)$, we have

$$(\lambda_j - \lambda_k) \int_a^b s\varphi_j\,\varphi_k\,dx = 0. \quad (3.15)$$

For distinct eigenvalues $\lambda_j \neq \lambda_k$, $(\lambda_j - \lambda_k) \neq 0$ and thus,

$$\int_a^b s\varphi_j \varphi_k \, dx = 0. \tag{3.16}$$

Theorem 3.3. For any $y, z \in D(L)$, we have the Lagrange identity

$$yL[z] - zL[y] = \frac{d}{dx}[p(yz' - zy')]. \tag{3.17}$$

Proof. Using the definition of the Sturm–Liouville operator, we have

$$yL[z] - zL[y] = y\frac{d}{dx}\left(p\frac{dz}{dx}\right) + qyz - z\frac{d}{dx}\left(p\frac{dy}{dx}\right) - qyz$$

$$= \frac{d}{dx}[p(yz' - zy')].$$

Theorem 3.4: The Sturm–Liouville operator L is self-adjoint. In other words, for any $y, z \in D(L)$, we have

$$(L[y], z) = (y, L[z]), \tag{3.18}$$

where (,) is the inner product in $L^2([a, b])$ defined by

$$(f, g) = \int_a^b f(x)\, \overline{g(x)}\, dx. \tag{3.19}$$

Proof. Since all constants involved in the boundary conditions of Sturm Liouville system are real, if $z \in D(L)$, then $\bar{z} \in D(L)$. Also since p, q and s are real-valued, $\overline{L[z]} = L[\bar{z}]$. Consequently, we have

$$(L[y], z) - (y, L[z]) = \int_a^b (\bar{z}\, L[y] - yL[\bar{z}])\, dx$$

$$= [p(z\,y' - y\,z')]_a^b, \text{ by (3.17)}. \tag{3.20}$$

We next show that the right hand side of this equality vanishes for a regular RSL system. If $p(a) = 0$, the result follows immediately. If $p(a) > 0$, then y and z satisfy the boundary conditions of the form (3.5) at $x = a$. That is,

$$\begin{bmatrix} y(a) & y'(a) \\ \bar{z}(a) & \bar{z}'(a) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

Since a_1 and a_2 are not both zero, we have

$$\bar{z}(a)\,y'(a) - y(a)\,\bar{z}'(a) = 0.$$

A similar argument can be used to the other end point $x = b$, so that the right-hand side of (3.10) vanishes. This proves the theorem.

Theorem 3.5: All the eigenvalues of a regular Sturm–Liouville system with $s(x) > 0$ are real.

Proof. Let λ be an eigenvalue of a RSL system and let $y(x)$ be the corresponding eigenfunction. This means that $y \neq 0$ and

$$L[y] = -\lambda sy.$$

Then

$$0 = (L[y], y) - (y, L[y]) = (\bar{\lambda} - \lambda) \int_a^b s(x)\,|y(x)|^2\,dx.$$

Since $s(x)>0$ in $[a,b]$ and $y \neq 0$, the integrand is a positive number. Thus, $\bar{\lambda} = \lambda$, and hence, the eigenvalues are real. This completes the proof.

Theorem 3.6: If $\varphi_1(x)$ and $\varphi_2(x)$ are any two solutions of the equation

$$L[y] + \lambda sy = 0 \text{ on } [a,b],$$

then

$$p(x)W(\varphi_1, \varphi_2)(x) = \text{constant},$$

where W is the Wronskian.

Proof. Since φ_1 and φ_2 are solutions of $L[y]+\lambda sy = 0$, we have

$$\frac{d}{dx}(p\frac{d}{dx}\varphi_1) + (q + \lambda s)\,\varphi_1 = 0,$$

$$\frac{d}{dx}(p\frac{d}{dx}\varphi_2)(q + \lambda s)\,\varphi_2 = 0.$$

Multiplying the first equation by φ_2 and the second equation by φ_1, and subtracting, we obtain

$$\varphi_1\frac{d}{dx}(p\frac{d}{dx}\varphi_2) - \varphi_2\frac{d}{dx}(p\frac{d}{dx}\varphi_1) = 0$$

Integrating this equation from a to x, we obtain

$$p(x)\,[\varphi_1(x)\,\varphi'_2(x) - \varphi'_1(x)\,\varphi_2(x)] = p(a)\,[\varphi_1(a)\,\varphi'_2(a) - \varphi'_1(a)\,\varphi_2(a)]$$

$$= \text{constant}. \qquad (3.21)$$

This is called Abel's formula where W is the Wronskian.

Theorem 3.7 An eigenfunction of a regular Sturm–Liouville system is unique except for a constant factor.

Proof. Let $\varphi_1(x)$ and $\varphi_2(x)$ be eigenfunctions corresponding to an eigenvalue λ. Then, according to Abel's formula (3.21) we have

$$p(x)W(\varphi_1,\varphi_2)(x) = \text{constant},\ p(x) > 0,$$

where W is the Wronskian. Thus, if W vanishes at a point in $[a, b]$, it must vanish for all $x \in [a, b]$.Since φ_1 and φ_2 satisfy the end condition at $x = a$, we have

$$a_1\varphi_1(a)+a_2\varphi'_1(a) = 0,$$

$$a_1\varphi_2(a)+a_2\varphi'_2(a) = 0.$$

Since a_1 and a_2 are not both zero, we have

$$\varphi_1(a)\,\varphi'_2(a) - \varphi'_1(a)\,\varphi_1(a) = W(\varphi_1,\varphi_2)(a) = 0.$$

Therefore, $W(\varphi_1,\varphi_2)(x) = 0$ for all $x \in [a, b]$, which is a sufficient condition for the linear dependence of two functions φ_1 and φ_2. Hence, $\varphi_1(x)$ differs from $\varphi_2(x)$ only by a constant factor.

Theorem 3.5 states that all eigenvalues of a regular Sturm–Liouville system are real, but it does not guarantee that any eigenvalue exists. However, it can be proved that a self-adjoint regular Sturm–Liouville system has a denumerably infinite number of eigenvalues. To illustrate this, we consider the following example.

Example 3.4 . Consider the Sturm–Liouville system

$$y'' + \lambda y = 0, \quad 0 \le x \le 1,$$

$$y(0) = 0,$$

$$y(1) + hy'(1) = 0, \quad h>0 \text{ a constant}.$$

Here $p = 1,\ q = 0,\ s = 1$. The solution of the Sturm–Liouville equation is

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Since $y(0) = 0$, gives $A = 0$, we have

$$y(x) = B\sin\sqrt{\lambda}x$$

Applying the second boundary condition, we have

$$\sin\sqrt{\lambda} + h\sqrt{\lambda}\cos\sqrt{\lambda} = 0,\ B \neq 0$$

which can be rewritten as

$$\tan\sqrt{\lambda} = -h\sqrt{\lambda}$$

If $\alpha = \sqrt{\lambda}$ is introduced in this equation, we have

$$\tan\alpha = -h\alpha.$$

This equation does not possess an explicit solution. Thus, we determine the solution graphically by plotting the functions $\xi = \tan\alpha$ and $\xi = -h\alpha$ against α, as shown in Figure 3.1. The roots are given by the intersection of two curves, and as is evident from the graph, there are infinitely many roots α_n for $n = 1, 2, 3, \ldots$. To each root α_n, there corresponds an eigenvalue

$$\lambda_n = \alpha_n^2,\ n = 1, 2, 3, \ldots.$$

Thus, there exists an ordered sequence of eigenvalues

$$\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \text{ with}$$

$$\lim_{n\to\infty} \lambda_n = \infty$$

The corresponding eigenfunctions are

$$\phi_n = \sin(\sqrt{\lambda}\, n\, x)$$

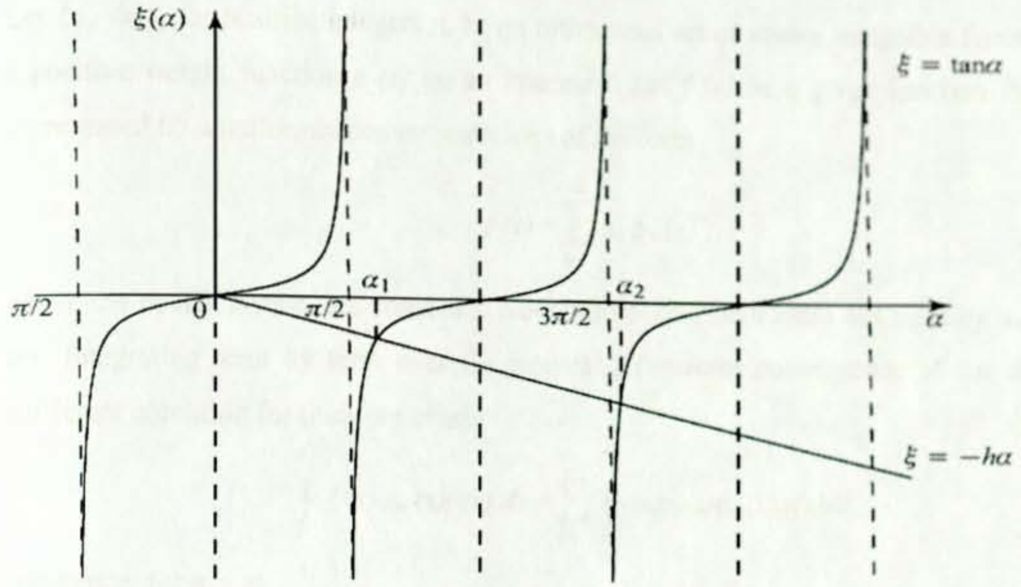

Fig. 3.1

Theorem 3.8: A self-adjoint regular Sturm–Liouville system has an infinite sequence of real eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots \text{with}$$

$$\lim_{n\to\infty} \lambda_n = \infty$$

For each n the corresponding eigenfunction $\varphi_n(x)$, uniquely determined up to a constant factor, has exactly n zeros in the interval (a, b).

3.2 Eigenfunction Expansions

A real-valued function $\varphi(x)$ is said to be square-integrable with respect to a weight function $\rho(x) > 0$, if, on an interval I,

$$\int_I \varphi^2(x)\,\rho(x)\,dx < +\infty. \tag{3.22}$$

An immediate consequence of this definition is the Schwarz inequality

$$\left| \int_I \varphi(x)\,\psi(x)\,\rho(x)\,dx \right|^2 \leq \int_I \varphi^2(x)\,\rho(x)\,dx \int_I \psi^2(x)\,\rho(x)\,dx \tag{3.23}$$

for square-integrable functions $\varphi(x)$ and $\psi(x)$.

Let $\{\varphi_n(x)\}$, for positive integers n, be an orthogonal set of square integrable functions with a positive weight function $\rho(x)$ on an interval I. Let $f(x)$ be a given function that can be represented by a uniformly convergent series of the form

$$f(x)=\sum_{n=1}^{\infty} c_n \varphi_n(x), \tag{3.24}$$

where the coefficients c_n are constants. Now multiplying both sides of (3.24) by $\varphi_m(x)\,\rho(x)$ and integrating term by term over the interval I (uniform convergence of the series is a sufficient condition for this), we obtain

$$\int_I f(x)\varphi_m(x)\rho(x)\,dx=\sum_{n=1}^{\infty}\int_I c_n\varphi_n(x)\varphi_m(x)\rho(x)dx,$$

and hence, for $n=m$,

$$\int_I f(x)\varphi_n(x)\rho(x)dx=c_n\int_I \varphi_n^{\,2}(x)\rho(x)dx.$$

Thus,

$$c_n=\frac{\int_I f\varphi_n\rho dx}{\int_I \varphi_n^{\,2}\rho dx} \tag{3.25}$$

Hence, we have the following theorem:

Theorem 3.9: If f is represented by a uniformly convergent series

$$f(x)=\sum_{n=1}^{\infty} c_n\varphi_n(x)$$

on an interval I, where φ_n are square-integrable functions orthogonal with respect to a positive weight function ρ (x), then c_n are determined by

$$c_n=\frac{\int_I f\varphi_n\rho dx}{\int_I \varphi_n^{\,2}\rho dx}$$

Example 3.5: Expand the function $f(x)=x$ over the interval $0<x<\pi$ using the solution to the regular Sturm- Liouville problem of

$$y'' + \lambda y=0, \qquad y(0)=y(\pi)=0$$

The eigenfunctions are $y_n(x)=\sin(nx),\ n=1,2,3,\ldots\ s(x)=1,\ a=0,\ b=\pi$. Hence

$$C_n=\frac{\int_0^\pi x\sin(nx)dx}{\int_0^\pi \sin^2(nx)dx}=\frac{-x\cos(nx)/n+\sin(nx)/n^2\ \big|_0^\pi}{\frac{x}{2}-\sin(2nx)/4n\ \big|_0^\pi}=-\frac{2}{n}\cos n\pi=\frac{2}{n}(-1)^n$$

Hence

$$f(x)=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx),\quad 0\le x<\pi$$

4. Green's Functions for Sturm-Liouville Problems

4.1 Problems with homogeneous boundary conditions

Suppose that the operator L is defined by

$$Ly\equiv\frac{d}{dx}[p(x)\frac{dy}{dx}]+q(x)y\equiv(p(x)y')'+q(x)y \tag{4.1}$$

where y is twice continuously differentiable, p is continuously differentiable, q is continuous and $p(x)>0$ for all $x\in[a,b]$ .Clearly L is the Sturm-Liouville operator .Suppose further $Ly=0$ has the only trivial solution when subject to the that boundary conditions

$$B_1[y]:\ \alpha_1 y(a)+\alpha_2 y'(a)=0$$

$$B_2[y]:\ \beta_1 y(b)+\beta_2 y'(b)=0$$

Lemma 4.1. There is a solution $y=u$ of $Ly=0$ satisfying $B_1[y]:\alpha_1 y(a)+\alpha_2 y'(a)=0$, $|\alpha_1|+|\alpha_2|\neq 0$ and a solution $y=v$ of $Ly=0$ satisfying $B_2[y]:\beta_1 y(b)+\beta_2 y'(b)=0$ $|\beta_1|+|\beta_2|\neq 0$ such that u and v are linearly independent over $[a,b]$.

Proof: Noting that $Ly=0$ is the linear equation

$$p(x)y''+p'(x)y'+q(x)y=0 \qquad (x\in[a,b])$$

where p , p' and q are continuous on $[a,b]$ and $p(x)>0$ for each x in $[a,b]$, we see that we may apply the existence part of the well known "basic existence and uniqueness theorem" of the initial value problem to find functions u, v solving

$$Lu = 0, \qquad u(a) = \alpha_2,\ u'(a) = -\alpha_1,$$

$$Lv = 0, \qquad v(b) = \beta_2,\ v'(b) = -\beta_1.$$

Then, certainly $y = u$ solves $Ly = 0$ taken with $\alpha_1 y(a) + \alpha_2 y'(a) = 0$ and $y = v$ solves $Ly = 0$ taken with $\beta_1 y(b) + \beta_2 y'(b) = 0$. Further, u, v must both be not identically zero in $[a, b]$, because of the conditions placed on α_1, α_2, β_1 and β_2.

Suppose that C, D are constants such that

$$Cu(x) + Dv(x) = 0, \text{ for each } x \text{ in } [a, b]. \tag{4.2}$$

As u, v solve $Ly = 0$, they must be differentiable and hence we may deduce

$$Cu'(x) + Dv'(x) = 0, \text{ for each } x \text{ in } [a, b]. \tag{4.3}$$

Multiplying (4.2) by β_1 and (4.3) by β_2, adding and evaluating at $x = b$ gives

$$C(\beta_1 u(b) + \beta_2 u'(b)) = 0.$$

If $C \neq 0$, u satisfies $\beta_1 y(b) + \beta_2 y'(b) = 0$, as well as $Ly = 0$ and $\alpha_1 y(a) + \alpha_2 y'(a) = 0$, and must be the trivial solution $u \equiv 0$. This contradicts the fact that u is not identically zero in $[a, b]$. Similarly, we can show that

$$D(\alpha_1 v(a) + \alpha_2 v'(a)) = 0$$

and hence, if $D \neq 0$, that $v \equiv 0$ in $[a, b]$, also giving a contradiction. So, C and D must both be zero and u, v must be linearly independent over $[a, b]$.

Theorem 4.2: Suppose that the operator Ly is the Sturm-Liouville problem defined by (4.1). Suppose further that the homogeneous equation

$$Ly = 0, \qquad (x \in [a, b]) \tag{4.4}$$

has only the trivial solution (that is, the solution $y = 0$ identically on $[a, b]$) when subject to both boundary conditions

$$B_1[y]:\ \alpha_1 y(a) + \alpha_2 y'(a) = 0$$

$$B_2[y]:\ \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{4.5}$$

where

$$|\alpha_1| + |\alpha_2| \neq 0$$

and

$$|\beta_1| + |\beta_2| \neq 0 \ .$$

If f is continuous on $[a, b]$, then the non-homogeneous equation

$$Ly = f(x), \qquad (x \in [a, b]) \tag{4.6}$$

taken together with both conditions $B_1[y]$ and $B_2[y]$, has a unique solution which may be written in the form

$$y(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi, \quad (x \in [a, b]) \tag{4.7}$$

the Green's function G is continuous on $[a, b]^2$, is twice continuously differentiable on $\{(x, \xi) \in [a, b]^2 : x \neq \xi\}$, and satisfies the jump condition

$$\frac{\partial G}{\partial x}(\xi^+, \xi) - \frac{\partial G}{\partial x}(\xi^-, \xi) = \frac{1}{p(\xi)} \qquad (\xi \in [a,b])$$

Where the Green's function $G(x,\xi)$ satisfies $L(G(x, \xi)) = \delta(x - \xi)$ subject to

$$B_1[G] = B_2[G] = 0.$$

Proof: Let y_1 and y_2 be the functions given by lemma 4.1. The Green's function satisfies the homogeneous equation for $x \neq \xi$ and satisfies the homogeneous boundary condition. Thus it must have the following form:

$$G(x.\xi) = \begin{cases} c_1(\xi) y_1(x) & for \quad a \leq x \leq \xi \\ c_2(\xi) y_2(x) & for \quad \xi \leq x \leq b \end{cases}$$

Here c_1 and c_2 are unknown functions of ξ. The first constrain on c_1 and c_2 comes from the continuity condition at $x = \xi$. Hence

$$\begin{aligned} G(\xi^-, \xi) &= G(\xi^+, \xi) \\ c_1(\xi) y_1(\xi) &= c_2(\xi) y_2(\xi) \\ c_1(\xi) y_1(\xi) - c_2(\xi) y_2(\xi) &= 0 \end{aligned} \tag{4.8}$$

Now since

$$L(G(x, \xi)) = \delta(x - \xi)$$

we have that

$$p(x) G''(x, \xi) + p'(x) G'(x, \xi) + q(x) G(x, \xi) = \delta(x - \xi) \tag{4.9}$$

Writing this in standard form we have

$$G''(x,\xi)+\frac{p'(x)}{p(x)}G'(x,\xi)+\frac{q(x)}{p(x)}G(x,\xi)=\frac{\delta(x-\xi)}{p(x)} \tag{4.10}$$

Integrating both sides from ξ^- to ξ^+ we obtain

$$\int_{\xi^-}^{\xi^+}\frac{p'}{p}G'(x,\xi)dx=0=\int_{\xi^-}^{\xi^+}\frac{q}{p}G(x,\xi)dx$$

and

$$\int_{\xi^-}^{\xi^+}\frac{\delta(x-\xi)}{p(x)}dx=\frac{1}{p(\xi)}$$

since G(x, ξ) is continuous . Putting this together, we have

$$G'(\xi^+,\xi)-G'(\xi^-,\xi)=\frac{1}{p(\xi)} \tag{4.11}$$

$$c_2(\xi)y'_2(\xi)-c_1(\xi)y'_1(\xi)=\frac{1}{p(\xi)} \tag{4.12}$$

which is the second constraint on c_1 and c_2 . Now equations (4.8) and (4.12) give us the system

$$\begin{cases} c_1(\xi)y_1(\xi)-c_2(\xi)y_2(\xi)=0 \\ c_1(\xi)y_1'(\xi)-c_2(\xi)y_2'(\xi)=-\dfrac{1}{p(\xi)} \end{cases}$$

Solving this system using Kramer's rule

$$c_1(\xi)=-\frac{y_2(\xi)}{p(\xi)(-W(\xi))},\qquad c_2(\xi)=-\frac{y_1(\xi)}{p(\xi)(-W(\xi))}$$

where $W(x)$ is the Wronskian of $y_1(x)$ and $y_2(x)$. By theorem 3.6 we have $pW=A$ for some constant A, Then the Green's function is

$$G(x,\xi)=\begin{cases} \dfrac{y_1(x)y_2(\xi)}{A}, & \text{for } a\le x\le\xi \\ \dfrac{y_2(x)y_1(\xi)}{A}, & \text{for } \xi\le x\le b \end{cases}$$

Hence G is continuous on $[a, b]^2$, as can be seen by letting ξ to x both from above and below. Clearly, G is also twice continuously differentiable when $x<\xi$ and when $x>\xi$.

Now it is time to prove that, for x in $[a,b]$

$$y(x) = \int_a^b G(x, \xi)f(\xi)\, d\xi \quad \text{and } B_1[y]=B_2[y]=0$$

that is

$$L[y]=L[\int_a^b G(x, \xi)f(\xi)\, d\xi],$$

$$= \int_a^b L[G(x, \xi)]f(\xi)\, d\xi,$$

$$= \int_a^b \delta(x- \xi)f(\xi)\, d\xi,$$

$$= f(x).$$

The integral also satisfies the boundary conditions:

$$B_i[\int_a^b G(x, \xi)f(\xi)\, d\xi]= \int_a^b B_i[G(x, \xi)]f(\xi)\, d\xi$$

$$= \int_a^b (0)f(\xi)\, d\xi$$

$$=0, \quad i=1,2,3....$$

Now it remains to prove that the solution is unique. Suppose that y_1, y_2 solve

$$Ly=f(x)$$

together with $B_1$and B_2.Then ,it is easy to see that $y=y_1$-y_2 solve $Ly=0$ together with $B_1$and B_2. By hypothesis of the theorem y must be the trivial solution (y is identically zero). So

$$y_1 - y_2=0$$

which implies

$$y_1=y_2.$$

Thus if a solution to $Ly=f(x)$ together with B_1 and B_2 exists, it must be unique. Hence the proof is completed.

Note that the Green's function $G=G(x, \xi)$ as constructed in the above proof, is independent of the function f on the right hand side of (4.6). That is the same G works for every continuous function f.

The method of the proof of theorem 4.2 actually gives a technique for solving equations of the form (4.6), with boundary conditions B_1 and B_2, which are now outline and exemplify.

(i) Find y_1 satisfying $Ly=0$, B_1 and y_2 satisfying $L_y=0$, B_2 which are linearly independent.

(ii) Calculate $W=W(y_1,y_2)$ which is non-zero and $A=pW$ is also non-zero.

(iii) Write down the unique solution

$$y(x) = \int_a^b G(x, \xi)f(\xi)\, d\xi, \qquad (x \in [a, b])$$

where

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{A}, & for \quad a \le x \le \xi \\ \dfrac{y_2(x)y_1(\xi)}{A}, & for \quad \xi \le x \le b \end{cases}$$

Example 4.1: Find the Green's function $G(x, \xi)$ which allows the unique solution of the problem

$$y''+y = f(x), \quad y(0)=y(\pi/2)=0, \qquad (x \text{ in } [0, \pi/2])$$

where f is continuous on $[0, \pi/2]$ to be expressed in the form

$$y(x) = \int_0^{\pi} G(x, \xi)f(\xi)\, d\xi, \qquad (x \text{ in } [0, \pi/2])$$

Solution *(i)* The function $y_1=\sin x$ satisfies

$$y''+y = 0, \qquad y(0)=0$$

and $y_2=\cos x$ satisfies

$$y''+y = 0\ , \quad y(\pi/2)=0$$

(ii) The Wronskian $W=W(y_1,y_2)$ is given by

$$W(y_1,y_2)(x)=\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} =-1$$

So that

$$A = pW = -1, \text{ (as } p=1)$$

*(iii)*The required Green's function is

$$G(x,\xi) = \begin{cases} -\sin x \cos\xi, & \text{for} \quad 0 \le x \le \xi \le \pi/2 \\ -\cos x \sin\xi, & \text{for} \quad 0 \le \xi \le x \le \pi/2 \end{cases}$$

4.2 Problems with Nonhomogeneuos Boundary Conditions

In the preceding section the boundary conditions B_1, B_2 are homogeneous. If the boundary conditions B_1 and B_2 are nonhomogeneuos, what will happen?

Consider $L[y]$ be given by (4.1) subject to the boundary conditions:

$$B_3[y]:\ \alpha_1 y(a) + \alpha_2 y'(a) = \gamma_1$$

$$B_4[y]:\ \beta_1 y(b) + \beta_2 y'(b) = \gamma_2 \tag{4.13}$$

The solution will be $y=u+v$ where

$$(pu')' + qu = f(x),\ \alpha_1 u(a) + \alpha_2 u'(a) = 0,\quad \beta_1 u(b) + \beta_2 u'(b) = 0 \tag{4.14}$$

and

$$(pv')' + qv = f(x),\quad \alpha_1 v(a) + \alpha_2 v'(a) = \gamma_1,\quad \beta_1 v(b) + \beta_2 v'(b) = \gamma_2 \tag{4.15}$$

The problem for v may have no solution, a unique solution or an infinite number of solutions. We consider only the case that there is a unique solution for v. In this case the homogeneous equation subject to homogeneous boundary condition has only the trivial solution.
We observe that equation (4.14) is identical with equation (4.1) subject to boundary conditions (4.5) and by theorem 4.2 the solution for u is given by

$$u(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi,$$

where

$$G(x,\xi) = \begin{cases} \dfrac{u_1(x)u_2(\xi)}{A}, & a \le x \le \xi \le b \\ \dfrac{u_2(x)u_1(\xi)}{A}, & a \le \xi \le x \le b \end{cases}$$

Thus if there is a unique solution for *v*, the solution for *y* is

$$y = v + \int_a^b G(x, \xi) f(\xi)\, d\xi$$

4.3 Eigenfunction Expansion for Green's Function

Consider the general Sturm-Liouville non-homogeneous ordinary differential equation:

$$Lu = (pu')' + qu = f(x) \tag{4.16}$$

subject to two homogeneous boundary conditions (4.5). We introduce a related eigenvalue problem:

$$L\varphi = -\lambda s \varphi \tag{4.17}$$

Subject to the same homogeneous boundary conditions (4.5).The weight function s can be chosen arbitrarily. We solve (4.16) by seeking $u(x)$ as a generalized Fourier series of the eigenfunctions:

$$u(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{4.18}$$

we differentiate this twice term by term since both φ_n and $u(x)$ solve the same homogeneous boundary conditions

$$\sum_{n=1}^{\infty} a_n L\varphi_n = -\sum_{n=1}^{\infty} a_n \lambda_n s \varphi_n = f(x).$$

where (4.17) has been used. The orthogonality of the eigenfunctions (with weight s) implies that

$$-a_n \lambda_n = \frac{\int_a^b f(x)\varphi_n dx}{\int_a^b \varphi_n^2 s dx} \tag{4.19}$$

The solution of the boundary value problem for the non-homogeneous ordinary differential equation is thus (after interchanging summation and integration)

$$u(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(x_0)}{-\lambda_n \int_a^b \varphi_n^2 s dx} dx_0 = \int_a^b f(x_0) G(x, x_0) dx_0 \tag{4.20}$$

For this problem, the Green's function has the representation in terms of the eigenfunctions:

$$G(x,x_0)=\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(x_0)}{-\lambda_n\int_a^b\varphi_n^2 s dx} \tag{4.21}$$

Again the symmetry is explicitly shown. The Green's function does not exist if one of the eigenvalues is zero. So for this reason $\lambda_n \neq 0$ for each n=1,2,3,….

Example 4.2: For the boundary value problem

$$\frac{d^2u}{dx^2}=f(x),\qquad u(0)=0 \text{ and } u(L)=0$$

The related eigenvalues problems,

$$\frac{d^2\varphi}{dx^2}=-\lambda\varphi,\quad \varphi(0)=0 \text{ and } \varphi(L)=0$$

is well known. The eigenvalues are $\lambda_n=\left(n\pi/L\right)^2$, n=1,2,3,… and the corresponding eigenfunctions are $\sin\left(n\pi x/L\right)$. Hence the Green's function is given by

$$G(x,x_0)=\frac{-2}{L}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi x/L\right)\sin\left(n\pi x_0/L\right)}{\left(n\pi/L\right)^2}$$

and the solution

$$u(x)=\int_0^L f(x)G(x,x_0)dx$$

4.4 Fredholm Alternative and Generalized Green's Functions

As we have seen in the preceding section if $\lambda=0$ is an eigenvalues, then the Green's function does not exist. In order to understand the difficulty, we reexamine the non-homogeneous problem

$$Lu=f(x) \tag{4.22}$$

subject to homogeneous boundary conditions. By the method of eigenfunctions expansion, in the preceding section we obtained

$$u(x)=\sum_{n=1}^{\infty} a_n\varphi_n(x) \tag{4.23}$$

where by substitution

$$-a_n\lambda_n=\frac{\int_a^b f(x)\varphi_n dx}{\int_a^b \varphi_n^2 s dx} \tag{4.24}$$

If $\lambda_n=0$ (for some n, often the lowest eigenvalue), there may not be any solutions to the non-homogeneous boundary value problem. In particular, if

$$\int_a^b f(x)\varphi_n(x)dx \neq 0 \ .$$

for the eigenfunctions corresponding to λ_n=0, then (4.24) can not be satisfied.

Example 4.3: Consider the following simple non-homogeneous boundary value problem:

$$\frac{d^2u}{dx^2}=e^x \quad \text{with } \frac{du}{dx}(0)=0 \text{ and } \frac{du}{dx}(L)=0$$

We attempt to solve this problem by integrating

$$\frac{du}{dx}=e^x+c$$

The two boundary conditions can not be satisfied as they are contradictory:

$$0=1+c$$

$$0=e^L+c$$

There is no guarantee that there are any solutions to a homogeneous boundary value problem when $\lambda=0$ is an eigenvalue for the related eigenvalue problem

$$\frac{d^2\varphi_n}{dx^2}=-\lambda_n\varphi_n \quad \text{with } \frac{d\varphi_n}{dx}(0)=0 \text{ and } \frac{d\varphi_n}{dx}(L)=0$$

If $\lambda=0$ is an eigenvalue, we have shown that there may be difficulty in solving

$$Lu=f(x)$$

subject to homogeneous boundary conditions. The eigenfunctions φ_n

$$L\varphi_n=-\lambda_n s\varphi_n$$

subject to the same homogeneous boundary conditions. Thus if $\lambda=0$ is an eigenvalue, the corresponding eigenfunctions $\varphi_h(x)$ satisfies

$$L\varphi_h=0$$

with the same homogeneous boundary condition. Thus, $\varphi_h(x)$ is a non trivial homogeneous solution of (4.22). This is important: Non-trivial homogeneous solutions of (4.22) solving the same homogeneous boundary conditions are equivalent to eigenfunctions corresponding to the zero eigenvalue. If there are no non-trivial homogeneous solutions (solving the same homogeneous boundary conditions), then $\lambda=0$ is not an eigenvalue. If there are non-trivial homogeneous solutions, then $\lambda=0$ is an eigenvalue.

The notion of a homogeneous solution is less confusing than can be a zero eigenvalue. For example consider

$$\frac{d^2u}{dx^2}+u=e^x \quad \text{with } u(0)=0 \text{ and } u(\pi)=0$$

The homogeneous solution for this problem is

$$\varphi=\sin x$$

However, it may cause some confusion to say $\lambda=0$ is an eigenvalue (although it is true). The definition of the eigenvalue for this problem is

$$\frac{d^2\varphi}{dx^2}+\varphi=-\lambda\varphi \quad \text{with } \varphi(0)=0 \text{ and } \varphi(\pi)=0$$

This is best written as

$$\frac{d^2\varphi}{dx^2}+(\lambda+1)\varphi=0$$

Therefore, $\lambda+1=(n\pi/L)^2=n^2$, $n=1,2,3,\ldots$and it is now clear that $\lambda=0$ is an eigenvalue ($n=1$).

4.4.1 Fredholm Alternative

Important conclusions can be reached form (4.19) obtained by the method of eigenfunctions expansion. The Fredholm alternative summarizes these results for non-homogeneous problems:

$$Lu=f(x) \tag{4.22}$$

subject to homogeneous boundary conditions (of the self adjoint type). Either

(i) $u=0$ is the only homogeneous solution (i.e. $\lambda=0$ is not an eigenvalue) in which case the non-homogeneous problem has a unique solution or

(ii) There are nontrivial homogeneous solutions $\varphi_h(x)$ (i.e. $\lambda=0$ is an eigenvalue), in which case the non-homogeneous problem has no solutions or an infinite number of solutions.

Let us describe in more detail what occurs if $\varphi_n(x)$ is a nontrivial homogenous solution. By (4.19) there is an infinite number of solutions of (4.22) if

$$\int_a^b f(x)\varphi_h(x)dx = 0 \tag{4.25}$$

because the corresponding a_n is arbitrary. Thus non-unique solutions correspond to an arbitrary additive multiple of a homogeneous solution $\varphi_h(x)$. Equation (4.25) corresponds to the forcing function being orthogonal to the homogeneous solution (with weight 1). If

$$\int_a^b f(x)\varphi_h(x)dx \neq 0 \tag{4.26}$$

then the non-homogeneous problem (with homogeneous boundary condition) has no solution. Here is the summary in table form

Condition	Number of Solutions	$\int_a^b f(x)\varphi_n(x)dx$
$\varphi_h(x)=0 \quad (\lambda\neq 0)$	1	0
$\varphi_h(x)\neq 0 \quad (\lambda=0)$	∞	0
$\varphi_h(x)\neq 0 \quad (\lambda=0)$	0	$\neq 0$

A different phrasing of the Fredholm alternative states that for the homogeneous problem (4.22) with homogeneous boundary conditions, solutions exist only if the forcing function is orthogonal to all homogeneous solutions.

Note that if $u=0$ is the only homogeneous solution, then $f(x)$ is automatically orthogonal to it (in a somewhat trivial way), and there is a solution.

Part of the Fredholm alternative can be shown without using an eigenfunction expansion. If the non-homogeneous problem has a solution, then

$$Lu=f(x)$$

All homogeneous solution, $\varphi_h(x)$, satisfy $L\varphi_h=0$ .We now use Green's formula with $v=\varphi_h$ and obtain

$$\int_a^b [uL\varphi_h - \varphi_h Lu] = 0$$

$$\int_a^b [u.0 - f(x)\varphi_h(x)dx] = 0$$

or

$$\int_a^b f(x)\varphi_h(x)dx = 0$$

Since u and φ_h satisfy the same homogeneous boundary condition.

Example 4.4: consider $\frac{d^2u}{dx^2} = e^x$ with $\frac{du}{dx}(0)=0$ and $\frac{du}{dx}(L)=0$

$u=1$ is a homogeneous solution. According to Fredholm alternative theorem, there is a solution to this problem only if e^x is orthogonal to this homogeneous solution, since

$$\int_a^b e^x.1dx \neq 0,$$

there are no solution.

Example 4.5: consider

$$\frac{d^2u}{dx^2} + 2u = e^x \quad \text{with } u(0)=0 \text{ and } u(\pi)=0$$

Since there are no solution of the corresponding homogeneous problem other than $u=0$, the Fredholm alternative implies that there is a unique solution.

Example 4.6: consider again

$$\frac{d^2u}{dx^2} + \left(\frac{\pi}{L}\right)^2 u = \beta + x \quad \text{with } u(0)=0 \text{ and } u(L)=0$$

Since $\varphi_h(x)=\sin \pi x/L$ is a solution of the homogeneous problem, the non-homogeneous problem only has a solution if the right-hand side is orthogonal to $\sin \pi x/L$:

$$0= \int_0^L (\beta + x)\sin\frac{\pi x}{L}dx$$

This can be used to determine the only value of β for which there is a solution:

$$\beta = \frac{-\int_0^L x\sin\frac{\pi x}{L}dx}{\int_0^L \sin\frac{\pi x}{L}dx} = -\frac{L}{2}.$$

4.4.2 Generalized Green's Functions

In this section, we analyze

$$Lu=f(x) \tag{4.22}$$

subject to homogeneous boundary conditions when $\lambda=0$ is an eigenvalue. If a solution to (4.22) exists, we will produce a particular solution of (4.22) by defining and constructing a modified or generalized Green's function.

If $\lambda=0$ is not an eigenvalue, then there is a unique solution of the non-homogeneous boundary value problem, (4.22), subject to homogeneous boundary condition. In (preceding section) we represent the solution using a Green's function $G(x,x_0)$ satisfying

$$L[G(x,x_0)]=\delta(x-x_0) \tag{4.27}$$

subject to the same homogeneous boundary conditions. Here we analyze the case in which $\lambda=0$ is an eigenvalue. There are non-trivial homogeneous solutions $\varphi_h(x)$ of (4.22) $L\varphi_h=0$. We will assume that there are solutions of (4.22) that is

$$\int_a^b f(x)\varphi_h(x)dx = 0 \tag{4.28}$$

However, the Green's function defined by (4.27) does not exist for all x_0 following from (4.26) since $\delta(x-x_0)$ is not orthogonal to solutions of the homogeneous problem for all x_0:

$$\int_a^b \delta(x-x_0)\varphi_h(x)dx = \varphi_h(x_0)$$

where $\varphi_h(x_0)$ is not identically zero. Instead we introduce a simple comparison problem that has a solution. $\delta(x-x_0)$ is not orthogonal to $\varphi_h(x)$ because it has a component in the direction $\varphi_h(x)$. However, there is a solution of (4.22) for all x_0 for the forcing function:

$$\delta(x-x_0) + c\,\varphi_h(x)$$

if c is properly chosen. In particular, we determine c easily such that this function is orthogonal to $\varphi_h(x)$:

$$0= \int_a^b \varphi_h(x)[\delta(x-x_0)+c\varphi_h(x)]dx = \varphi_h(x_0)+c\int_a^b \varphi_h^2(x)dx$$

Thus, we introduce the generalized Green's function $G_m(x,x_0)$, which satisfies

$$L[G_m(x,x_0)] = \delta(x-x_0) - \frac{\varphi_h(x)\varphi_h(x_0)}{\int_a^b \varphi_h^2(x)dx} \tag{4.29}$$

subject to the same homogeneous boundary conditions. Since the right hand side of (4.29) is orthogonal to $\varphi_h(x)$, unfortunately there are an infinite number of solutions.

Using Green's formula we can easily show that the generalized Green's function is symmetric:

$$G_m(x,x_0)=G_m(x_0,x) \tag{4.30}$$

If $g_m(x,x_0)$ is one symmetric generalized Green's function then the following is also a symmetric generalized Green's function.

$$G_m(x,x_0)=g_m(x,x_0)+\beta\varphi_h(x_0)\varphi_h(x)$$

For any constant β (independent of x and x_0). Thus there are an infinite number of symmetric generalized Green's functions.

We use Green's formula to derive a representation formula for $u(x)$ using the generalized Green's function. Letting $u=u(x)$ and $v=G_m(x,x_0)$, green's formula states that

$$\int_a^b \{u(x)L[G_m(x,x_0)\} - G_m(x,x_0)L[u(x)]\}dx = 0$$

since both $u(x)$ and $G_m(x,x_0)$ satisfy the same homogeneous boundary conditions. The differential equation (4.22) and (4.29) imply that

$$\int_a^b \left\{ u(x)\left[\delta(x-x_0) - \frac{\varphi_h(x)\varphi_h(x_0)}{\int_a^b \varphi_h^2(x)dx} \right] - G_m(x,x_0)f(x) \right\} dx = 0$$

Using the fundamental Dirac delta property (and reversing the role of x and x_0) yields:

$$u(x) = \int_a^b f(x_0)G_m(x,x_0)dx_0 + \frac{\varphi_h(x)}{\int_a^b \varphi_h^2(x)dx}\int_a^b u(x_0)\varphi_h(x_0)dx_0$$

where the symmetry of $G_m(x,x_0)$ has also been utilized. The last expression is a multiple of the homogeneous solution, and thus a simple particular solution of (4.22) is

$$u(x) = \int_a^b f(x_0)G_m(x,x_0)dx \tag{4.31}$$

The same form as occurs when λ=0 is not an eigenvalue.

Example 4.7: consider the problem with nontrivial homogeneous solution:

$$\frac{d^2u}{dx^2} = f(x), \text{ with } \frac{du}{dx}(0)=0 \text{ and } \frac{du}{dx}(L)=0 \tag{4.32}$$

A constant is a homogeneous solution (eigenfunction corresponding to the zero eigenvalue). For a solution to exist, by the Fredholm alternative, $\int_0^L f(x)dx = 0$. We assume *f(x)* is of this type [example *f(x) =x-L/2*]. The generalized Green's function $G_m(x,x_0)$ satisfies:

$$\frac{d^2G_m}{dx^2} = \delta(x - x_0) + c \tag{4.33}$$

$$\frac{dG_m}{dx}(0)=0 \text{ and } \frac{dG_m}{dx}(L)=0 \tag{4.34}$$

since a constant is the eigenfunction. For there to be such a generalized Green's function, the right hand side must be orthogonal to the homogeneous solutions:

$$\int_a^b [\delta(x - x_0) + c]dx = 0, \text{ or c= -1/L}$$

We use properties of the Dirac delta function to solve (4.33) with (4.34). For $x \neq x_0$

$$\frac{d^2G_m}{dx^2} = -\frac{1}{L}$$

By integration

$$\frac{dG_m}{dx} = \begin{cases} -\dfrac{x}{L}, & x < x_0 \\ -\dfrac{x}{L} + 1, & x > x_0 \end{cases} \tag{4.35}$$

where the constants of integration have been chosen to satisfy the boundary conditions at $x=0$ and at $x=L$. The jump condition for the derivative, $\left.\frac{dG_m}{dx}\right|_{x_0^-}^{x_0^+} = 1$, obtained by integrating

(4.33), is already satisfied by (4.35). We integrate again to obtain $G_m(x,x_0)$. Assume that $G_m(x,x_0)$ is continuous at $x=x_0$ yields:

$$G_m(x,x_0)=\begin{cases} -\dfrac{x^2}{2L}+x_0+c(x_0), & x<x_0 \\ -\dfrac{x^2}{2L}+x+c(x_0), & x>x_0 \end{cases}$$

where $c(x_0)$ is an arbitrary additive constant that depends on x_0 and corresponds to an arbitrary multiple of the homogeneous solution. This is the representation of all possible generalized Green's functions. Often we desire $G_m(x,x_0)$ to be symmetric. For example $G_m(x,x_0)=G_m(x_0,x)$ for $x<x_0$ yields:

$$-\frac{1}{L}\frac{x_0^2}{2}+x_0+c(x)=-\frac{1}{L}\frac{x^2}{2}+x_0+c(x_0) \text{ or } \qquad c(x)=-\frac{1}{L}\frac{x_0^2}{2}+\beta$$

where β is arbitrary constant. Thus finally we obtain the generalized Green's function:

$$G(x,x)=\begin{cases} -\dfrac{1}{L}\dfrac{(x^2+x_0^2)}{2}+x_0+\beta, & x<x_0 \\ -\dfrac{1}{L}\dfrac{(x^2+x_0^2)}{2}+x+\beta, & x>x_0 \end{cases}$$

A solution of (4.32) is given by

$$u(x)=\int_0^L f(x_0)G_m(x,x_0)dx\,.$$

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