N = 2 SUPERSYMMETRIC APPROACH TO THE STRING THEORY TREATMENT OF THREE DIMENSIONAL ISING MODEL.

by

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ABSTRACT

In this work a string theoretic approach is made to the calculation of the critical exponents α and β of the 3D Ising Model. This approach which has been pursued by polyakov and collaborators has been pushed upto N = 1 supersymmetry (SUSY) of the world sheet. No realistic results have been obtained for N = 1 SUSY. Since it is now known that SUSY in space-time is given by N = 2 SUSY in world sheet, an N = 2 supersymmetric approach has been pursued in this work. Polyakov has given arguments showing the identity of the string theory away from the critical dimension and 2D superconformal field theory. It has been found here that for N = 2 SUSY the gauge coupling of the 2D super conformal field theory (or the Wess-Zumino-Witten model) with the induced 2D super gravity does not renormalise. This enables an ansätz to be made for α and β in terms of the conformal dimensions of 2D superconformal field theory. The latter are known now and the ansätz, which is analogous to Polyakov's ansatz for γ, seem to give excellent values for α and β.

INTRODUCTION

Ising model was introduced in 1925 as the simplest model of a magnetic system which was expected to show second order phase transition. In the event it was found by analytic methods of statistical mechanics that this model in one dimension (1D) does not show phase transition. The solution of the two-dimensional Ising model (2D) which was considered intractable for about twenty years was finally achieved by Onsager in 1944 in what has been considered a 'tour de force'. Solution of the three-dimensional Ising model (3DIM) has proved still more intractable.

Though progress has been made towards an eventual solution of the problem by using numerical methods, a fully analytic solution has not yet been achieved. It appears that conventional methods of statistical mechanics is yet incapable of treating the problem; new methods in string theory have been suggested.

Since the application of string theory is mostly focused on the area of high energy phenomenology, success of the effort towards a solution of 3DIM will confirm the belief in the universality of string theory. The methods and techinques developed in this connection may open the way to more extensive use of string theory techinques in statistical mechanics.

The plan, as given in the table of contents, is as follows. Chapter I and II are basically introductory in nature. Chapter III discusses the elements of string theory and the simplest approach to supersymmetry via superspace. In this chapter it is shown that the 2D superconformal quantum field theory which is equivalent to a Wess-Zumino-Witten model has a gsℓ(2/1) current algebra symmetry. In chapter IV after an introduction to the idea of critical exponents arguments are given leading to an ansätz for α and β for the 3D Ising model. This ansätz which is in line with polyakov's ansatz for γ seems to lead to excellent values.

CHAPTER - I

INTRODUCTION TO ONE AND TWO DIMENSIONAL ISING MODELS

The statistical study of systems of interacting particles is besset by many problems of largely mathematical nature. These difficulties have motivated theorists to devote a great deal of effort to devising and studying the simplest sorts of model systems which show any resemblance to those occurring in nature. The most sucessful of these models is one introduced by E. Ising in an attempt to explain the ferromagnetic phase transition.

Many aspects of the Ising model have been investigated by exact methods since the rigorous results obtained by Onsager for two dimensional Ising system. In addition to their direct significance for the theory of phase transitions, the results obtained were important as a test for the various approximation methods.

We shall begin our discussion with the one dimensional Ising model (1DIM). The solution [1] for this case is given here because it demonstrates the general feature of the model.

Sec A: One dimensional Ising model.

Consider an Ising system on a line of N sites with periodic boundary conditions as shown in fig. 1.

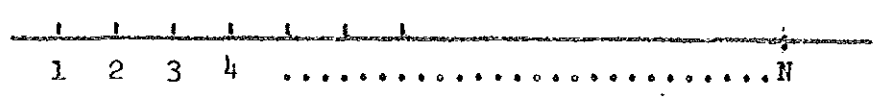

Fig.1. The one dimensional lattice.

Let the state of each site be described by a spin variable σ with values ± 1. If we assume each unit to interact with its two direct neighbors, the energy of interaction is given by

$$E_1 = -J \Sigma \sigma_n \sigma_{n+1}$$

where J is the exchange coupling constant periodic boundary condition means $\sigma_{N+1} = \sigma_1$.

If the system is in an external magnetic field H, the associated energy is given by

$$E_2 = -H \Sigma \sigma_n$$

Therefore, the total energy of the system is

$$\begin{aligned} E &= E_1 + E_2 \\ &= -J \Sigma \sigma_n \sigma_{n+1} - H \Sigma \sigma_n \end{aligned}$$

The thermodynamic variable of interest is the partition function Z.

$$\begin{aligned} Z &= \sum_{\text{Conf.}} \exp(-\beta E) \\ &= \sum_{\sigma} \exp\{\beta J \Sigma \sigma_n \sigma_{n+1} + \beta H \Sigma \sigma_n\} \end{aligned}$$

If we introduce the abbreviations

$$K = \beta J \text{ and } h \equiv \beta H$$

the partition function takes the form

$$Z = \Sigma_\sigma \exp\{K \Sigma \sigma_n \sigma_{n+1} + h \Sigma \sigma_n\} \qquad (1)$$

Let us now construct the transfer matrix formalism for the one dimensional Ising model.

Expanding the Sum in eq. (1) yields

$$Z = \sum_{\sigma} \exp K \{ \sigma_1\sigma_2 + \sigma_2\sigma_3 + \ldots + \sigma_N\sigma_1 \}$$

$$X \exp h \{ \sigma_1 + \sigma_2 + \ldots + \sigma_N \}$$

Now define

$$V(\sigma_n, \sigma_{n+1}) = \exp K \sigma_n\sigma_{n+1} + h \left(\frac{\sigma_n + \sigma_{n+1}}{2}\right)$$

then

$$Z = \sum_{\sigma} \prod_{n=1}^{N} V(\sigma_n, \sigma_{n+1})$$

It is easy to see that this is the trace of the N th power of a 2 X 2 matrix T, where the rows (columns) of T are labelled by the possible configurations of the initial (final) member of a neighboring pair of spins, ie.

$$T = \begin{pmatrix} V(+,+) & V(+,-) \\ V(-,+) & V(-,-) \end{pmatrix}$$

$$= \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix}$$

and

$$Z = \text{trace } T^N$$

The matrix T is called the transfer matrix.

The value of the trace is unchanged if we replace T by V_2V_1 so that

$$Z = \text{trace } (V_2V_1)^N$$

where

$$V_1 = \begin{pmatrix} e^{k} & e^{-k} \\ e^{-k} & e^{k} \end{pmatrix}$$

and

$$V_2 = \begin{pmatrix} e^{h} & 0 \\ 0 & e^{-h} \end{pmatrix}$$

The eigen values are determined by the equation

$$\begin{vmatrix} e^{K+h} - \lambda & e^{-k} \\ e^{-k} & e^{k-h} - \lambda \end{vmatrix} = 0$$

with solutions

$$\lambda_{1,2} = e^{k} \operatorname{Cosh} h \pm (e^{2k} \operatorname{Sinh}^{2} h + e^{-2k})^{\frac{1}{2}}$$

Hence the partition function is simply

$$Z = \lambda_1^N + \lambda_2^N$$

In practice, we assume N to be very large so that our chain is nearly infinite in length. Therefore only the largest eigenualue of T is at all important.

$$\text{As } N \to \infty, \left(\frac{\lambda_2}{\lambda_1}\right)^N \to 0$$

Therefore, the free energy per site

$$F = -\frac{1}{\beta} \ln \lambda_1$$

$$= -\frac{1}{\beta} \ln \left| e^{k} \cosh h + (e^{2k} \sinh^2 h + e^{-2k})^{\frac{1}{2}} \right|$$

The magnetization per spin is

$$M = -\frac{\partial F}{\partial H}$$

$$= e^{k} \sinh h \{ e^{2k} \sinh^2 h + e^{-2k} \}^{-\frac{1}{2}}$$

For any temperature $T>0$, zero magnetic field corresponds to $h = 0$. The equation for the magnetization asserts that in this limit $M \to 0$, so that there is no spontaneous magnetization at finite temperatures. Thus the one-dimensional Ising system does not exhibit a phase transition.

Sec B: Two dimensional Ising model.

There are two main approaches to the exact solution of two-dimentional Ising model (2DIM). The first exact solution for the square lattice was obtained by Onsager in 1944 by a spinor algebric method. The second one is a combinatorial solution of the problem found by Kac and Ward. Other simpler and more direct modified forms of these two approaches are now available. Here we shall follow the method of Schultz [2].

Consider a set of spin arranged on a square lattice of M columns and N rows, interacting only with nearest neighbours. Since this problem has not been solved in the presence of an external magnetic field, we shall put $H=0$. The figure below shows a two dimensional spin system.

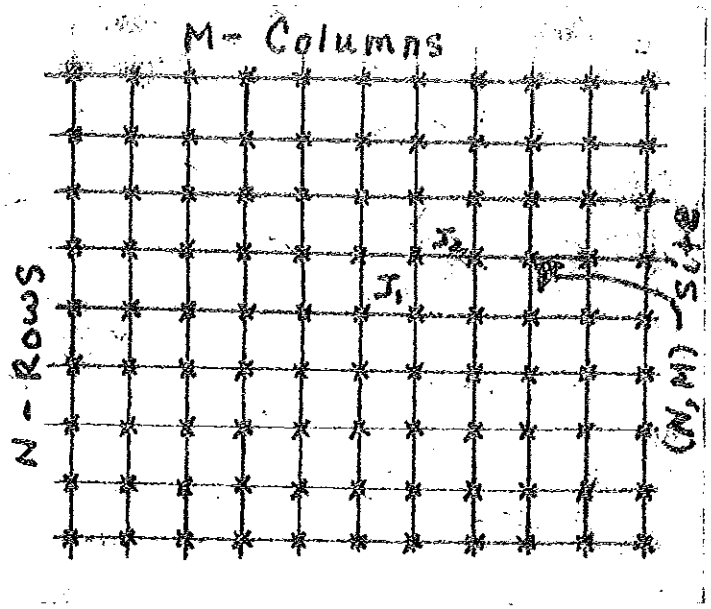

fig. 2. Square lattice

The energy of the system is given by

$$E = - J_1 \Sigma \sigma_{nm} \sigma_{n+1,m} - J_2 \Sigma \sigma_{nm} \sigma_{n,m+1}$$

J_1 and J_2 are, respectively, the exchange couplings within the columns and within the rows.

To find the transfer matrix for the 2D case we write the matrices V_1 and V_2 for the 1D case in terms of the pauli matrices τ^α.

We first note that V_1 and V_2 can be expressed as

$$V_1 = \left[\exp K \right] I + \left[\exp (-K) \right] \tau^x$$

$$V_2 = I \cosh h + \tau^z \sinh h$$

Since

$$(\tau^{\alpha})^2 = I,$$

$$\exp(a\,\tau^{\alpha}) = I \cosh a + \tau^{\alpha} \sinh a$$

$$= \cosh a\,(I + \tau^{\alpha} \tanh a)$$

which means that

$$V_2 = \exp(h\,\tau^{Z})$$

Similarly by defining a quantity

K^* such that

$$\tan h\, K^* = \exp(-2K)$$

we can write V_1 as

$$V_1 = (2 \operatorname{Sin} h\, 2K)^{\frac{1}{2}} \exp(K^* \tau^{X})$$

The matrix V_1 and V_2 can be generalised for the 2D case. Now instead of summing over two configurations of each spin, we sum over the 2^M configuration of each row. The matrix V_2 is diagonal and can be written as

$$V_2 = \exp(K \Sigma\, \tau_m^{Z}\, \tau_{m+1}^{Z})$$

Similary

$$V_1 = (2 \sin h\, 2K)^{\frac{M}{2}} \exp(K^* \Sigma\, \tau_m^{X})$$

The matrices τ_m^{X} and τ_m^{Z} are $2^M \times 2^M$ matrices defined by

$$\tau_m^{Z} = I \times \ldots \times I \times \tau^{Z} \times I \times \ldots I$$

$$\tau_m^{X} = I \times \ldots \times I \times \tau^{X} \times I \times \ldots I$$

and

$$Z = \operatorname{trace} V^N$$

where

$$V = V_1 V_2$$

To determine the eigen values of the matrix V, we introduce a spin raising and spin lowering operators σ^+ and σ^- respectively as

$$\sigma^z = \sigma^+ + \sigma^- \; ; \; \sigma^x = \sigma^+\sigma^- - \sigma^-\sigma^+$$

which obey

$$\left[\sigma_j^{\pm}, \sigma_\ell^{\pm} \right] = 0 \qquad \text{for } j \neq \ell$$

$$\{\sigma_j^+, \sigma_j^-\} = 1$$

$$\left[\sigma_j^+, \sigma_j^+\right] = \left[\sigma_j^-, \sigma_j^-\right] = 0.$$

The operators obey the mixed set of commutation and anticommutation rules. However it is possible to introduce new operators which convert the commutation into anticommutation

$$C_m^+ = \sigma_m^+ \prod_{\nu=1}^{m-1}{}' \sigma_\nu^x$$

$$C_m = \sigma_m^- \prod_{\nu=1}^{m-1} \sigma_\nu^x$$

The C's all anticommute with each other and even with themselves, except for the one non-vanishing anticommutator

$$C_m^+ C_m + C_m C_m^+ = -1$$

In terms of these Fermion operators, V_1 becomes

$$V_1 = (2 \sinh 2K_1)^{M/2} \exp K^* \sum_{\nu=1}^{m} (C_\nu^+ C_\nu - C_\nu C_\nu^+)$$

Similarly

$$V_2 = \exp\left[K_2 \sum_{\nu=1}^{M} (C_\nu - C_\nu^+)\,(C_{\nu+1} + C_{\nu+1}) \right]$$

It is possible to impose cyclic boundary conditions so that the system will acquire translational invariance. However we shall ignore this point here since it has no influence on the final result. We now make a further transformations

$$C_n = \frac{e^{i\Pi/4}}{\sqrt{M}} \sum_{\mu=0}^{M-1} \exp\left[-\frac{2\pi i n\mu}{M} \right] \eta_\mu$$

$$C_n^+ = \frac{e^{-i\Pi/4}}{\sqrt{M}} \sum_{\mu=0}^{M-1} \exp\left[\frac{2\Pi i n\mu}{M} \right] \eta_\mu^+$$

In this case we get

$$V1 = (2\sin h\ 2k_1) \left[K^* \sum_{\mu=-\frac{1}{2}}^{\frac{1}{2}M-1} (\eta_\mu^+ \eta_\mu - \eta_\mu \eta_\mu^+) \right]$$

$$V_2 = \exp\left[K_2 \sum_{\mu=-\frac{1}{2}M}^{\frac{1}{2}M-1} \{ie^{\frac{2\Pi i\mu}{M}} \eta_\mu \eta_{-\mu} + ie^{-2\Pi i\mu/M} \eta_\mu^+ \eta_\mu^+ + e^{\frac{2\Pi i\mu}{M}} \eta_\mu \eta_\mu^+ - e^{-\frac{2\Pi i\mu}{M}} \eta_\mu^+ \eta_\mu \} \right]$$

Since we treat μ and $-\mu$ as independent variables we can write the exponential of the sum as a product of exponentials. We set therefore.

$$V_1 = (2 \sinh 2K_1) \sum_{\mu=0}^{\frac{1}{2}M} \bar{V}_1(\mu)$$

$$V_2 = \prod_{\mu=0}^{\frac{1}{2}M} V_2(\mu)$$

with

$$V_1(\mu) = \exp\{ 2K_1^* \left[\eta_\mu^+ \eta_\mu - \eta_{-\mu} \eta_{-\mu}^+ \right] \}$$

$$V_2(\mu) = \exp\{ 2K_2 \Big[- \cos \frac{2\Pi\mu}{M} (\eta_\mu^+ \eta_\mu - \eta_{-\mu} \eta_{-\mu}^+)$$

$$+ \sin \frac{2\Pi\mu}{M} (\eta_\mu^+ \eta_{-\mu}^+ + \eta_{-\mu} \eta_\mu) \Big] \}$$

With these relations we can write the corresponding matrix as

$$V = (2 \sinh 2K_1)^{\frac{M}{2}} \prod_{\mu=0}^{\frac{1}{2}M} V_1^{\frac{1}{2}}(\mu)\, V_2(\mu)\, V_1^{\frac{1}{2}}(\mu)$$

Since each μ factor in this product commutes with any other μ factor, the problem is decomposed into a direct product of independent matrix problems. Any product of individual eigenvalues will be an eigenvalue of the matrix.

Rigorous analysis shows that all eigenvalues will have the form

$$\lambda = (2 \sinh 2 K_1)^{M/2} \exp \sum_\mu \varepsilon(\mu)$$

where $\varepsilon(\mu)$ is defined by the relation

$$\cosh \varepsilon(\mu) = \cosh 2K_1^* \cosh 2K_2 - \sinh 2K^* \sinh 2K_2 \cos \frac{2\Pi\mu}{M}$$

In the limit $N, M \to \infty$, the other eigenvalues have negligible effect. The partition function is just the N th power of the largest eigenvalue of V_1, in analogy with the 1D Case.

$$\lambda_{max} = (2 \sinh 2K_1)^{M/2} \exp \frac{M}{2\Pi} \int_0^\Pi \varepsilon(q)dq.$$

CHAPTER II

Three-dimensional Ising Model, Random Surfaces and String Theory

Sec. A: Polyakov's Arguments.

The simple arguments connecting Ising Model with strings was given by Polyakov [3]. Here we shall review his arguments.

One starts by considering 2D Case. Since there are only two orientations in the Ising model, we denote 'up' spins by a cross and 'down' spins by a small circle. If one starts at high temperature, the crosses and circles on a two dimensional structure will be randomly distributed. This is the paramagnetic phase. When the temperature is lowered towards the the critical temperature it is believed that small drops of spin appears in the system. All spins in one drop have the same orientation. The shape, size and location of these drops are random. As the critical temperature T_c is approached these drops become bigger and eventually at T_c they coalesce form a ferromagnetic system. The situation at $T \gtrsim T_c$ and $T<T_c$ are shown in Figs. 3 and 4.

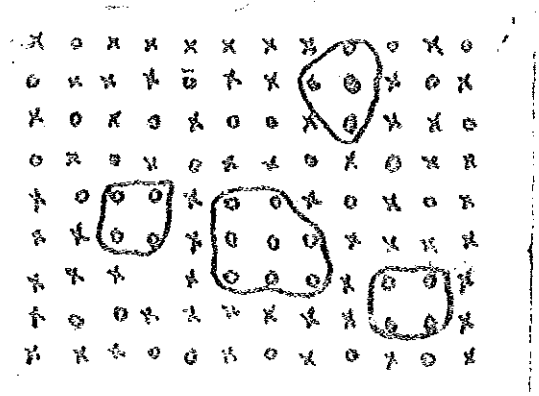

Fig. 3. Drop of flipped spins for $T \gtrsim T_c$.

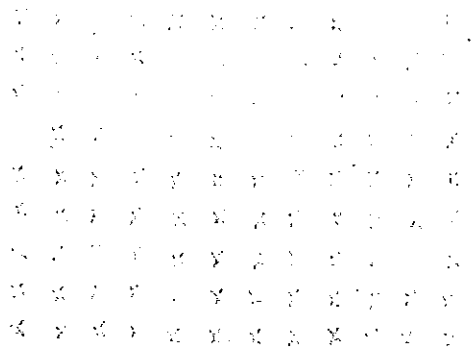

Fig.4. Configuration with all spins up for $T<T_c$

he other hand, when the temperature is raised individual ns are allowed to fluctuate. In the region $T \lesssim T_c$, the ns in a drop flip as depicted in Fig. 5.

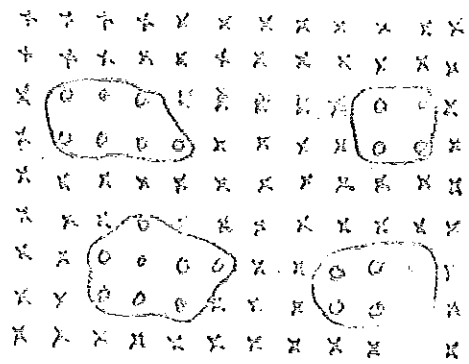

Fig 5. Drops of flipped spins for $T \lesssim T_c$,

et x denotes a lattice point and $\hat{\ell}$ a unit vector. The partition function for the 2D case is given by

$$Z = \sum_{\sigma_{x=\pm 1}} \exp \beta \cdot \sum_{x,\ell} \sigma_x \, \sigma_{x+\ell}$$

for any temperature.

It is clear from this relation that to form a single drop we pay in energy γ times the boundary of the drop where the

have anti-parallel spins. This applies for $T \lesssim T_c$ (Fig.5)

$$\sum_{\text{drops}} \sigma_x \sigma_{x+\delta} - \sum_{\text{groundstate}} \sigma_x \sigma_{x+\delta} = (1)(-1)L - (1)(1)L$$
$$= -2L$$

where L is the perimeter of the drop.

Therefore a configuration with a single drop contributes a ter

$$Z = \exp(-2L)$$

to the partition function.

It is evident that such a drop may appear anywhere on the lattice. Moreover the sizes and shapes of the drops may also vary. Summing over shapes and number of drops gives

$$Z = \sum_{\text{drop}} e^{-2L}$$

The sum over drops can be changed to a sum over paths. This is done by giving weights to the intersection points. If we have a self-intersection drops as shown below, it has three possible paths.

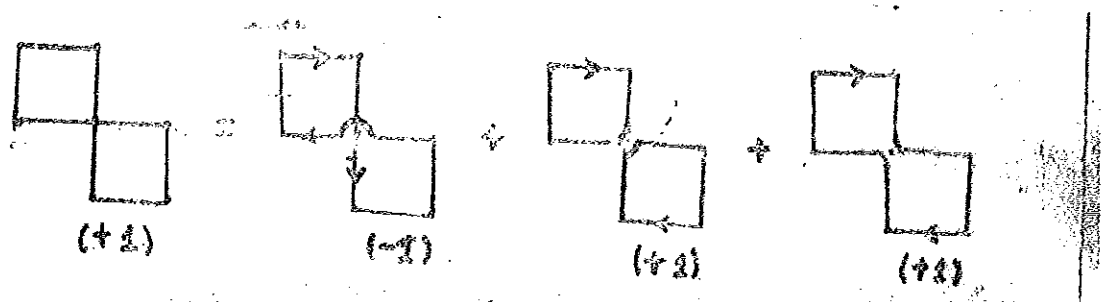

Fig. 6. Possible paths for a self-intersection drops.

When a single parton moves, it describes a path of length L and contributes e^{-L} to the partition function. The string which is a collection of free bosonic partons in its motion spans an area A and therefore contributes e^{-A} to the partition function. If the partons have spin, then we have to add an extra factor $(-1)^{\nu}$ for each parton, while the string acquires a factor $(-1)^{\ell}$.

Since it is known that a fermionic string can be viewed as a collection of spin ½ partons, the 3DIM can be explained interms of string theory.

As we have seen, the calculation of the partition function of a 3DIM is reduced to the summation over random surfaces. These sums are similar to the sum over random paths in ordinary quantum mechanics. It will be seen in the following that even though Z as given by equation (2) cannot be reliably calculated by analytic methods, the analogy with the string theory permits calculations of at least the critical exponents.

Sec B: Theoretical Justification for $d \leq 1$.

Until very recently the link between the critical properties of random surfaces and the continuum string theories has been much weaker than in the case of field theory. However, in a new remarkable work Knizhnik et al [4] have calculated the critical exponent γ for random surfaces for dimensions of the embedding space $d \leq 1$. Although such embedding dimensions are not physically, they are very

important from the conceptual point of view. The reason is γ can be exactly calculated for triangulated bosonic random surfaces for a number of these unphysical dimensions [5]. In fact the values are

$$d = -2, \quad \gamma = -1$$

$$d = 0, \quad \gamma = -\tfrac{1}{2}$$

$$d = \tfrac{1}{2}, \quad \gamma = -1/3$$

$$d = 1, \quad \gamma = 0$$

These values compare well with those calculated using the formula obtained in ref [4],

$$\gamma(d) = \frac{d-1 - \sqrt{(d-1)(d-25)}}{12}$$

The agreement can be considered a proof of the identity of the model of random surfaces and the continuum string theory for embedding dimensions $d \leq 1$, [6].

It will be seen in ChIII that this restriction on d seems to the disappear in the N = 2 supersymmetric case. Thus further strengthens the identification mentioned above.

CHAPTER III

String Theory and Two Dimensional Superconformal Quantum Field Theory

Sec.A: Bosonic String Theory

The string theory originated in the analysis of elementary particles. Instead of assuming elementary particles as point like objects with no internal structure a theory is devised in which elementary particles are thought of as one dimensional curves with infinitesimal thickness or so-called strings which interact by joining and splitting.

Since there is a belief that string theory can explain the four fundamental forces of nature, it is studied much extensively in modern physics. In this section we develop the equation of motion for a free string [4].

The motion of a particle along its path is described by stating the functional dependence of the three space coordinates on the time t.

$$X^i = f^i(t)$$

This kind of description is possible in the theory of relativity as well as in non-relativistic physics. But in the theory of relativity, it is useful to choose a description in which the time is not set apart from the spatial coordinates. Therefore we have to describe its motion in Minkowski space.

The path followed by the particle in this particular space is called the "world-line" and its parametric representation is given by

$$X^{\mu} = f^{\mu}(\tau)$$

where τ is the parameter which is called the proper time.

The relativistic action for a free particle is given by

$$S = -m \int_{\tau_1}^{\tau_2} \left| \left(\frac{\partial X^{\mu}}{\partial \tau} \right)^2 \right|^{\frac{1}{2}} d\tau$$

The motion of the string can be studied in a similar fashion

Mathematically a string is a one-dimensional finite curve in space which changes its shape and position as a function of time. As the string moves in space - time a two - dimensional strip is generated. In analyogy with world-line we shall call such a configuration a world sheet. If the intrinsic parameter of the string is σ, then its parametric equation is given by

$$X^{\mu} = X^{\mu}(\sigma)$$

Any range of variation can be taken for σ and we fix $0 \leq \sigma \leq \pi$.

A second variable τ is required which can parametrize the evolution of the state of the string. From the view point of World-sheets the variables σ and τ are merely a pair of co-ordinates labelling the points of the sheet. The dynamical variables $X^{\mu}(\sigma, \tau)$ introduced define the space time location of (σ, τ) and therefore the configuration of the

World-sheet.

The area element spanned by two infinitesimal displacements is given by

$$d^2A = \{(\frac{\partial x}{\partial \sigma}\frac{\partial x}{\partial \tau})^2 - (\frac{\partial x}{\partial \sigma})^2(\frac{\partial x}{\partial \tau})^2\}^{\frac{1}{2}}\, d\sigma\, d\tau$$

The dynamics of the string is determined by the action.

Nambu (1970) suggested that the relativistic action for a free string be proportional to the area of the surface spanned by the string. We therefore define the action to be

$$S = \frac{-1}{2\Pi\alpha^1}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\Pi} d\sigma \sqrt{(\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \sigma})^2 - (\frac{\partial x}{\partial \tau})^2(\frac{\partial x}{\partial \sigma})^2} \quad ---- (3)$$

Let us define a lagrangian density

$$L = \frac{-1}{2\Pi\alpha^1}\{-\dot{x}^2 x'^2 + (\dot{x}\cdot x')^2\}^{\frac{1}{2}}$$

where $\dot{x} \equiv \frac{\partial x}{\partial \sigma}$, $x' \equiv \frac{\partial x}{\partial \tau}$

The equation of motion of the string follows from the principle of least action

$$\delta s = 0$$
$$= \int_{\tau_1}^{\tau_2} d\tau \int_0^{\Pi} d\sigma \;\; \delta L$$

Since L is a function of $\dot{x}$ and x'

$$\delta L = \frac{\partial L}{\partial \dot{x}}\frac{\partial}{\partial \tau}\delta x + \frac{\partial L}{\partial x'}\frac{\partial}{\partial \sigma}\delta x$$

But

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \dot{\chi}}\,\delta\chi\right) = \frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \dot{\chi}}\right)\delta\chi + \frac{\partial}{\partial \dot{\chi}}\,\frac{\partial}{\partial\tau}(\delta\chi)$$

and hence

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \dot{\chi}}\,\delta\chi\right) - \frac{\partial}{\partial \dot{\chi}}\left(\frac{\partial}{\partial \dot{\chi}}\right)\delta\chi = \frac{\partial L}{\partial \dot{\chi}}\,\frac{\partial}{\partial\tau}\,(\delta\chi)$$

Similar equation for $\frac{\partial L}{\partial \chi'}\,\frac{\partial}{\partial\sigma}\,(\delta\chi)$

Therefore

$$\delta s = \int_0^{\pi} d\sigma\,\frac{\partial L}{\partial \dot{\chi}}\,\delta\chi\,\Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau\,\frac{\partial L}{\partial \chi'}\,\Big|_{\sigma=0}^{\sigma=\pi}$$

$$- \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma\left(\frac{\partial}{\partial\tau}\,\frac{\partial L}{\partial \dot{\chi}} + \frac{\partial}{\partial\sigma}\,\frac{\partial L}{\partial \chi'}\right)\delta\chi$$

Since $\delta\chi$ is arbitrary, we get the equations

$$\frac{\partial}{\partial\tau}\,\frac{\partial}{\partial \dot{\chi}} + \frac{\partial}{\partial\sigma}\,\frac{\partial L}{\partial \chi'} = 0$$

and

$$\frac{\partial L}{\partial \chi'} = 0 \quad \text{at} \quad \sigma = 0,\ \pi$$

We can see that the dynamics of free string is analogous to quantum field theory in which X^{μ} replaces the field and σ represents the space and τ the time.

We can introduce a more symmetric notation

$$(\xi^1,\ \xi^2) = (\tau, \sigma)$$

The metric tensor is then given by

$$g_{\alpha\beta}\,(\xi) = \partial_{\alpha}X^{\mu}\,\partial_{\beta}X_{\mu}\ ;\ \alpha,\beta = 1,2$$

By construction the action for the string is invariant under any reparametrization of the variables

$$\xi \to f^{\mu}(\xi^{\mu})$$

Consequently the two dimensional stress-energy tensor vanishes. This symmetry is called reparametrisation symmetry.

There is another symmetry for string theory which is known as conformal symmetry. This arises from the require- that strings do not appear spontaneously from the vacuum. This non-trivial condition on 2D field theory appears if the theory is conformally invariant. Mathematically the following transformation

$$g_{ab}(\sigma,\tau) \to e^{\phi(\sigma,\tau)} \; g_{ab}(\sigma,\tau)$$

is called a conformal transformation. For a conformally invariant theory, the Lagrangian should be invariant under this transformation.

Sec. B: Conformal Field Theory in Two Dimensions

In this section we shall discuss two dimensional conformal field theory [8].

To begin with, consider the general coordinate transformation

$$\xi^a = \eta^a(\xi)$$

Under this transformation the metric tensor g_{ab} transform as

$$g_{ab} \to g'_{ab} = \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} g_{a'b'}$$

The conformal group is then defined as those coordinate transformations which leave the metric (4) invariant up to a scale factor, ie

$$g_{ab}(\xi) \to g'_{ab}(\xi') = e^{\phi(\xi_1,\xi_2)} g_{ab(\xi_1,\xi_2)} \quad --- (4)$$

In two dimensions this conformal group is infinite dimensional. To describe the group in two dimensions, it is common to introduce complex variables.

Let $z = x^0 + ix'$

and

$\bar{z} = x^0 - ix'$

The two dimensional conformal group consists of all transformations of the form

$$z \to f(z)$$
$$z \to \bar{f}(\bar{z})$$

Where f and $\bar{f}$ are arbitrary analytical functions.

The metric in the complex coordinate will be

$$ds^2 = dz d\bar{z}$$

The infinitesimal generators of the group can be determined by considering the infinitesimal coordinate transformation

$$\begin{aligned} z &\to z' = z + \varepsilon(z) \\ \bar{z} &\to \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}) \end{aligned} \tag{5}$$

where $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are infinitesimal analytical function. Equation (5) constitute local conformal transformations.

Making Laurent expansions of $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ we get

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1}$$

$$\bar{\varepsilon}(\bar{z}) = \sum_{m=-\infty}^{\infty} \bar{\varepsilon}_m \bar{z}^{m+1}$$

If we introduce generators

$$L_n \equiv z^{n+1} \frac{d}{dz} \quad \text{and} \quad \bar{L}_m = \bar{z}^{m+1} \frac{d}{d\bar{z}}$$

Then

$$\left[L_n, L_m\right] = (n-m)\, L_{n+m}$$

$$\left[\bar{L}_n, \bar{L}_m\right] = (n-m)\, \bar{L}_{n+m}$$

$$\left[L_n, \bar{L}_m\right] = 0 \; ; \; n, m \in z$$

This classical algebra is sometimes called loop algebra. In the quantam case, the algebras will be corrected to include an extra term. Since this algebra is defined locally, they are local conformal algebra in two dimensions. We can also have the global conformal group which is defined to be the group of conformal transformations. The infinitesimal generators are globally defined and the generators are

$$\{ L_{-1}, L_0, L_1 \} \text{ and } \{ \bar{L}_{-1}, \bar{L}_0, \bar{L}_1 \}.$$

The global conformal algebra is useful for characterizing properties of physical states. If we work in a basis of eigenstates of the two operators L_0 and $\bar{L}_0$, and denote their eigenvalues by h and $\bar{h}$ respectively, then h and $\bar{h}$ are known as the conformal weight of the state.

We define now a field theory with conformal invariance in d-dimensions. One assumes the existence of a set of fields $\{A_i\}$, where the index i specifies different fields This set of fields in general is infinite and contains in particular the derivatives of all the fields $A_i(x)$.

We now consider the case of two dimensions. The line element $ds^2 = dz d\bar{z}$ transforms under

$$z \to f(z) \text{ and } \bar{z} \to \bar{f}(\bar{z}) \text{ as}$$

$$ds^2 \to \left(\frac{\partial f}{\partial z}\right) \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) ds^2$$

we shall generalize this transformation law to the form

$$\phi\ (z,\ \bar{z}) \rightarrow \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi\ (f(z),\ \bar{f}(\bar{z})) \tag{6}$$

where h and $\bar{h}$ are real valued positive quantities and $\{\phi_i\}$ is a subset of $\{A_i\}$. The transformation property (6) defines what is known as a primary field ϕ of conformal weight $(h,\bar{h})$. Not all fields in conformal field theory will turn out to have this transformation property. The rest of the fields are known as secondery fields.

In the conformal field theory the scale transformation is also important. The scale transformation is given by

$$\xi^a \rightarrow \lambda \xi^a$$

where ξ^a's are coordinates.

In the quantum field theory this scale symmetry takes place provided the stress-energy tensor is traceless

$$T^a_a = 0$$

The stress-energy tensor infact satisfy the usual conservation equation

$$\partial_a T^{ab}(\xi) = 0$$

In two dimensions these two equations can be reduced to

$$\partial_{\bar{z}} T = 0 \quad , \quad \partial_z \bar{T} = 0$$

where

$$\bar{T} = T_{11} - T_{12} - 2iT_{12}$$

$$\bar{T} = T_{11} - T_{22} - 2i\ T_{12}$$

The fields $T(z)$ and $\bar{T}(\bar{z})$ represent the generations of the infinitesimal conformal transformations.

The operator product expansion of the stress-energy tensor with itself is given by

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2}\, T(\omega) + \frac{1}{(z-\omega)}\, \partial T(\omega) + \text{regular terms} \qquad (7)$$

This equation is derived by noting the general variation of the field $T(z)$

$$\delta_\varepsilon T(z) = \varepsilon(z)\, T^{1}(z) + 2\varepsilon^{1}(z)\, T(z) + \tfrac{1}{2}\, C\, \varepsilon^{111}(z)$$

Under

$$z \to z+\varepsilon(z).$$

This expression combined with the general definition of any conformal field $A(z,\bar{z})$,

$$\delta\varepsilon A(z,\bar{z}) = \oint T(\xi)\,\varepsilon(\xi)\, A(z,\bar{z})\, d\xi$$

gives eq (7)

Expanding the stress-energy tensor in a Laurent series we get

$$T(z) = \Sigma\, z^{-n-2}\, L_n$$
$$\bar{T}(\bar{z}) = \Sigma\, \bar{z}^{-n-2}\, \bar{L}_n \qquad (8)$$

Equation (8) can be inverted using cauchy's theorem to give

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

$$\bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z})$$

To derive the algebra for L_n, consider

$$[L_n, T(\omega)] = L_n T(\omega) - T(\omega) L_n$$

$$= \oint \frac{dz}{2\pi i} z^{n+1} T(z) T(\omega) - \oint \frac{dz}{2\pi i} z^{n+1} T(\omega) T(z)$$

using Cauchy's theorem we get

$$[L_n, T(\omega)] = \frac{C}{12} n(n^2 - 1) \omega^{n-2} + 2(n+1)\omega^n T(\omega) + \omega^{n+1} \partial_\omega T(\omega)$$

The expansion (8) yields

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{C}{12} n(n^2-1) \delta_{n+m,0}$$

Similarly

$$[\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m} + \frac{C}{12} n(n^2-1) \delta_{n+m,0}$$

$$[L_n, \bar{L}_m] = 0$$

This is the virasoro algebra for two-dimensional conformal field theory. The constant C is called the central charge and its value in general depends on the particular theory under consideration.

Sec C: The String Theory Effective Action

The arguments adduced by Polyakov Connecting the 3DIM with the String theory have been mentioned in the previous chapter. This argument requires the intermediation of the theory of random surfaces. If one could solve the theory of random surfaces then the 3DIM problem could have been solved. But no satisfactory analytic solution has not been found yet.

At this point it is necessary to give a short discussion about Supersymmetry (SUSY) and its implication for String theory. The concept of SUSY belongs to the realm of high energy physics where it is believed that at very high energies the fermion and the boson have the same properties. For the purpose here, this concept is immaterial. Instead, one is interested in finding a suitable mathematical procedure for converting a theory which describes bosonic objects to one describing fermionic objects. It is very well known now that the world sheet of the bosonic String can be converted into a world sheet for the fermionic string by introducing an appropriate number of anticommuting fermionic coordinates for parametrizing the World sheet.

The question then arises as to how many such fermionic coordinates are needed as parameters of the fermionic World sheet. In this connection one needs to make a distinction between the coordinate parameters transforming locally or globally on the World sheet under reparametrisation. This matter will be discussed in greater detail in the next section. One notes that basically one is interested in supersymmetri-

sation of the 4D space-time properties of a theory. The procedure of introducing fermionic parameters on the World sheet is only a mathematical construction to ensure space-time SUSY. It has been proved recently [9] that space-time SUSY is obtained in String theory by introducing N = 2 (global) Supersymmetry ie by introducing two global fermionic parameters on the World sheet. It has also been noted in chapter II that the String World sheet has conformal symmetry. The combination of conformal symmetry and N=2(global) supersymmetry is called N=2(global) superconformal symmetry. The theory of a 2D superconformal symmetry has been discussed at length recently [10].

Coming back to the 3D IM, if one accepts Polyakov's arguments, then it is necessary to find the action of the fermionic String in order to calculate the partition function and/or the critical exponents. The calculation of the effective action for the bosonic String, starting from eq(3) was performed by Polyakov [11].

At this point it is necessary to mention that the fermionic String theory is no longer in use; it has bean found that it is more convenient from the phenomenological point of view to replace the fermionic String by the so-called heterotic string [12]. Both the bosonic and heterotic Strings can be quantised at their respective critical dimensions and the effective action at other than critical dimension are given by [11,13]

Bosonic:-

$$S_{eff} = \frac{26-d}{48\pi} \int d^2z \; R \; \frac{1}{\Delta} R \tag{9}$$

R = Scalar Curvature

Heterotic (N=1):-

$$S_{eff} = \frac{10-d}{16\pi} \int d^2z \; d\theta E^{-1} \Sigma^+ \frac{\nabla_+}{\Box + \frac{1}{2}\Sigma^+ \nabla_+} \Sigma^+ \tag{10}$$

where Δ in the first equation is the Laplace Beltrami Operator in 2D. These actions are called anomally actions. The meaning of the symbols in the second eq (10) are in reference [13].

These equations are quoted here mainly to point out that the effective action disappears when d = 26 and 10 respectively, only at which dimensions the theories can be quantised. It may be mentioned here that the effective action going to zero in the critical dimension does not mean absence of dynamics which are determined by the symmetries of the theory [14].

An interesting observation [15a] that made it possible to use the above forms eqs (9,10) of the effective action for the 3DIM is that even though D=3 now, is the following. One may pend on the immediate quantisation and instead notice the identity of S_{eff} above with the action of the two-dimensional quantum field theory. For eq (9) explicit identity has been proved [15a], for N=1,2 the identity is accepted. In section B 2DCFT has been discussed; the N=2 two dimensional superconformal field theory

(2DSCFT) will be discussed in the next section.

Sec. D: The Heterotic String and Two Dimensional Super Conformal Field Theory

Before launching into a discussion of the proper topics of this section, it would be appropriate to comment on the analogy between the S_{eff} (eqs. 9 and 10) and those of the corresponding two dimensional theories. As already mentioned only in the N=0 case (Bosonic String) has an explicit proof been given by showing the equivalence of the actions. The simpler path is to show, as has been done for the N=1 case [15, 16, 13] that it is possible to find a field in superspace (superfield) which satisfies the same equation of motion as that obtained from the anomaly action. This latter superfield can be replaced by an equivalent superfield which gives an Wess-Zumino-Witten (WZW) - like action. The general WZW theory is another manifestation of the 2DSCQFT. The details of this procedure will be demonstrated in the following for the case of the heterotic string on a world-sheet with N=2 global SUSY.

As already mentioned the heterotic string [12] has replaced the fermionic as having phenomenologyically desirable qualities. It is a closed string. In the critical dimension, the degrees of freedom of the string, being massless 2D free fields, can be decomposed into right and left movers, ie functions of $\tau-\sigma$ and $\tau+\sigma$ respectively. The heterotic string may be constructed with the following degrees of

freedom:

a) The right moving sector of fermionic superstring consists of 8 transverse bosonic coordinates and 8 fermionic coordinates.

b) The left moving bosonic sector consists of 8 transerse bosonic coordinates and 16 internal bosonic coordinates.

This heterotic string lives on a 2D World-sheet. These are reasons why any string, whether bosonic or heterotic should have conformal symmetry. To ensure 4D space-time SUSY one introduces two independent global fermionic parameters Θ_+ and $\bar{\Theta}_+$ on the World-sheet. On this World-sheet one can then define the covariant derivatives ∇_A, [17],

$$\begin{aligned} \nabla_A &\equiv E_A + W_A{}^M \\ &= E_A^N D_N + W_A{}^M \end{aligned}$$

with

$$\begin{aligned} \nabla_A &\equiv (\nabla_A, \bar{\nabla}_+, \nabla_{\mu}, \nabla_{\phi}) \\ D_N &\equiv (D_+, \bar{D}_+, \partial_{\mu}, \partial_{\phi}) \\ W_A &\equiv (W_+, \bar{W}_+, W_{\mu}, W_{\phi}) \\ D_+ &\equiv \frac{\partial}{\partial\Theta_+} + \bar{\Theta}_+ \frac{\partial}{\partial z} \\ &\equiv \frac{\partial}{\partial\Theta} + i\bar{\Theta}_+ \frac{\partial}{\partial X^0} \\ \bar{D} &\equiv \frac{\partial}{\partial\bar{\Theta}_+} + \Theta_+ \frac{\partial}{\partial\bar{z}} \\ &\equiv \frac{\partial}{\partial\bar{\Theta}_+} + i\Theta_+ \frac{\partial}{\partial X^0} , \end{aligned} \tag{11}$$

where

$$x^{\neq} = x^0 + x^1$$
$$x^{=} = x^0 + x^1$$
$$z = x^0 + ix^1$$
$$\bar{z} = x^0 - ix^1$$

x^0 and x^1 are the parameter space coordinates. The second forms of D_+ and $\bar{D}_+$ are obtained by analytic continuation E_A^N 's are the vielbein fields which can be expanded around the flat background;

$$E_A^N = \delta_A^N + H_A^N$$

M is the Lorentz generator and W_A's are the Spin connections. The formalism is similar to that of 2D relativity.

It is known that 2D manifold can be described locally by a single superfield. But the superfields E_A^N (or H_A^N) are large in number and redundant. By a combination of constraints on the covariant derivatives and gauge transformations of reparametrization and Lorentz invariances, the number of superfields will be reduced to one in the following.

Brooks et al [17] have given the following constraints and associated identities,

$$[\nabla_+, \nabla_+\} = 0$$

$$[\nabla_+, \bar{\nabla}_+\} = 2i\nabla_{\neq}$$

$$[\nabla_+, \nabla_{\neq}\} = 0$$

$$[\nabla_+, \nabla_=\} = iG_= \nabla_+ - 2i\bar{\Sigma}^+ M$$

$$[\nabla_{\neq}, \nabla_=\} = -\Sigma^+ \nabla_+ - \bar{\Sigma}^+ \bar{\nabla}_+ - RM$$

and

$$\nabla_{+} G_{=} = \bar{\bar{\Sigma}}^{+}$$

$$\nabla_{+} \bar{\bar{\Sigma}}^{+} = 0$$

$$\nabla_{+} \Sigma^{+} + \bar{\bar{\nabla}}_{+} \bar{\bar{\Sigma}}^{+} = R$$

$$\nabla_{+} R = 2i \nabla_{\ddot{+}} \bar{\bar{\Sigma}}^{+} \tag{13}$$

R is the scalar curvature of the world sheet.

Writing the covariant derivatives in full

$$\nabla_{+} = E_{+}^{+} D_{+} + E_{+}^{\ddot{+}} \partial_{\ddot{+}} + E_{+}^{=} \partial_{=} + W_{+} M.$$

$$\bar{\bar{\nabla}}_{+} = \bar{\bar{E}}_{+}^{+} \bar{\bar{D}}_{+} + E_{+}^{\ddot{+}} \partial_{\ddot{+}} + E_{+}^{=} \partial_{=} + \bar{\bar{W}}_{+} M.$$

$$\nabla_{\ddot{+}} = E_{\ddot{+}}^{+} D_{+} + E_{\ddot{+}}^{\ddot{+}} \partial_{\ddot{+}} + E_{\ddot{+}}^{=} \partial_{=} + W_{\ddot{+}} M.$$

$$\nabla_{=} = E_{+}^{+} D_{+} + E_{=}^{\ddot{+}} \partial_{\ddot{+}} + E_{=}^{=} \partial_{=} + W_{=} M \tag{14}$$

One may note that $+$ is a fermionic and $\ddot{+}$ and $=$ are bsonic indices with lorentz weights $\frac{1}{2}$, $+1$ and -1, respectively. Fore example,

$$ME_{=} = -E_{=}$$

$$MW_{+} = \tfrac{1}{2} W_{+}$$

$$MW_{\ddot{+}} = W_{\ddot{+}} \tag{15}$$

It may be possible to solve the constraints and identities in eqs (11) in general case. It is, however, possible to achieve simplification by noting that the derivatives ∇_A's are covariant under general super coordinates and local Lorentz transformations.

$$\nabla_A' = e^{k} \nabla_A e^{-k} \tag{16}$$

with gauge parameters K^N and Λ respectively.

From eq (16)

$$\delta \nabla_A = \nabla_A' - \nabla_A = [K, \nabla_A] \tag{17}$$

for infinitesimal gauge transformations. One can notice that the vielbein fields are

$$E_{+}^{+} , \bar{E}_{+}^{+} , E_{+}^{+} , E_{+}^{\not{+}} , E_{+}^{+} , E_{\not{+}}^{\not{+}} , E_{+}^{=} ,$$

$$E_{=}^{+} , E_{=}^{\not{+}} , E_{=}^{=} \text{ numbering ten in total.}$$

The constraints are five in number. So following Ref [18] We take the following independent fields often the constraints are applied,

$$E_{+}^{+} , E_{+}^{\not{+}} , E_{+}^{=} , E_{=}^{\not{+}} , E_{=}^{=} .$$

One can set the three fields

$$E_{+}^{+} , E_{+}^{\not{+}} \text{ and } E_{+}^{=}$$

equal to zero by adjusting the components of the gauge parameter, namely

$$K^{\not{+}}, K^{+} \text{ and } K^{=}.$$

$E_{=}^{=}$ may by eliminated by adjusting Λ.

This is a standard procedure and one is left with only one independent vielbein $E_{=}^{\not{+}}$. Thus the only independent gauge superfield is $H_{=}^{\not{+}}$ which is identified with the gravitational

superfield in 2D.

The solution of the constraints and identities in eqs(11) now becomes simpler. These have been found to be [19],

$$
\begin{aligned}
E_+ &= D_+ \\
\bar{E}_+ &= \bar{D}_+ \\
E_\ddagger &= \partial_\ddagger \\
E_= &= \partial_= + H_=^\ddagger \partial_\ddagger - \frac{i}{2}\,(D_+ H_=^\ddagger)\,\bar{D}_+ - \frac{i}{2}(\bar{D}_+ H_=^\ddagger) D_+ \\
W_+ &= 0 \\
\bar{W}_+ &= 0 \\
W_\ddagger &= 0 \\
W_= &= \partial_\ddagger H_=^\ddagger \\
R &= -\partial_\ddagger^2 H_=^\ddagger \\
\Sigma^+ &= \frac{i}{2}\,\bar{D}_+ \partial_\ddagger H_=^\ddagger \\
\bar{\Sigma}^+ &= \frac{i}{2}\,D_+\,\partial_\ddagger H_=^\ddagger \\
\nabla_+ G &= \bar{\Sigma}^+ = D_+ G \quad .
\end{aligned}
\tag{18}
$$

Transformation of $H_=^\ddagger$ is now,

$$
\begin{aligned}
\delta H_=^\ddagger &= -\nabla_= K^\ddagger \\
&= -\Big[\,\partial_= + H_=^\ddagger\,\partial_\ddagger - \frac{i}{2}\,(D_+ H_=^\ddagger)\bar{D}_+ - \frac{i}{2}\,(\bar{D}_+ H_=^\ddagger) D_+ \\
&\qquad + \partial_\ddagger H_=^\ddagger M\Big] K^\ddagger
\end{aligned}
$$

But

$$MK^\ddagger = -K^\ddagger$$

So

$$-\nabla_{=} K^{\ddagger} = \delta H^{\ddagger}_{=}$$

$$= -[\ \partial_{=} + H^{\ddagger}_{=}\partial_{\ddagger} - \frac{i}{2}\ (D_{+}H^{\ddagger}_{=})\bar{D}_{+} - \frac{i}{2}\ (\bar{D}_{+}H^{\ddagger}_{=})D_{+}$$

$$- \partial_{\ddagger} H^{\ddagger}_{=}\ |K^{\ddagger} \tag{20}$$

One can now write

$$\delta S = d\mathbf{z} \oint d\theta_{+} \int d\bar{\theta}_{+} J_{\ddagger} \delta H^{\ddagger}_{=}$$

$$= d\mathbf{z} \oint d\theta_{+} \int d\theta_{+} J_{\ddagger} \nabla_{=} K^{\ddagger}$$

$$= d\mathbf{z} \oint d\theta_{+} \int d\bar{\theta}_{+} \nabla_{=} J_{\ddagger} K^{\ddagger},$$

On integrating by part. $J_{\ddagger}$ is an appropriate supercurrent.

The equation of motion given by $\delta S = 0$ leads to

$$\nabla_{=} J_{\ddagger} = 0$$

Again consider the general relation [11,20]

$$T^{a}_{a} = T_{r}T = \frac{d}{24\pi} R \tag{22}$$

where T is the Stress-energy tensor. One notes that the Stress-energy tensor is a higher tensorial current with lorentz weight nil, so are R and $\nabla_{=} J_{\ddagger}$ and one sets,

$$T^{a}_{a} = \nabla_{=} J_{\ddagger},$$

which leads to

$$R = 0. \tag{23}$$

Thus the equation of motion is from eq. (18)

$$\partial^{2}_{\ddagger}\ H^{\ddagger}_{=} = 0. \tag{24}$$

This appears to be a new result for N=2 superspace.

Following Refs [15,13] one introduces a new scalar superfield f which is determined by $H^{*}_{=}$ through the equation,

$$\partial_{=} f = -H^{*}_{=}\, \partial_{=} f + \frac{i}{2}\,(D_{+}H^{*}_{=})\bar{D}_{+} f + \frac{i}{2}\,(\bar{D}_{+}H^{*}_{=})D_{+} f \ . \tag{25}$$

Using eq (25) one now finds the transformation of f which induces eq(20). In other words, the intention is to look for the classical equation of motion of the superfield f and try to obtain the quantum analogue in a generalised form. To this end, from eq (25),

$$\begin{aligned}\partial_{=}\delta f = & -\delta H^{*}_{=}\partial_{\pm} f - H^{*}_{=}\,\partial_{*}\delta f + \frac{i}{2}\,(D+\delta H^{*}_{=})\bar{D}_{+} f \\ & + \frac{i}{2}\,(D_{+}H^{*}_{=})\bar{D}_{+}\delta f + \frac{i}{2}\,(\bar{D}_{+}\delta H^{*}_{=})D_{+} f \\ & + \frac{i}{2}\,(\bar{D}_{+}H^{*}_{=})D_{+}\delta f \ .\end{aligned} \tag{26}$$

Now using eq (20)

$$\begin{aligned}\partial_{=}\delta f = & \nabla_{=}K^{*}\,\partial_{=} f - H^{*}_{=}\partial_{\pm}\delta f - \frac{i}{2}\,(D_{+}\nabla_{=}K^{*})\bar{D}_{+} f \\ & + \frac{i}{2}\,(D_{+}H^{*}_{=})\bar{D}_{+}\delta f - \frac{i}{2}\,(\bar{D}_{+}\nabla_{=}K^{*})D_{+} f \\ & + \frac{i}{2}\,(\bar{D}_{+}H^{*}_{=})D_{+}\delta f\end{aligned}$$

or

$$\begin{aligned}\Big[\,\partial_{=} + H^{*}_{=}\partial_{\pm} - \frac{i}{2}\,(D_{+}H^{*}_{=})\bar{D}_{+} - \frac{i}{2}\,(\bar{D}_{+}H^{*}_{=})D_{+}\Big]\delta f \\ = \nabla_{=}K^{*}\,\partial_{*} f - \frac{i}{2}\,(D_{+}\nabla_{=}K^{*})\bar{D}_{+} f \\ - \frac{i}{2}\,(\bar{D}_{+}\nabla_{=}K^{*})D_{+} f\end{aligned} \tag{27}$$

Solution of equation (27) leads to [19] ,

$$\delta f = K^{\ddagger}\partial_{\ddagger} f - \frac{i}{2} D_{+}K^{\ddagger}\bar{D}_{+} f - \frac{i}{2} \bar{D}_{+}K^{\ddagger}D_{+} f \tag{28}$$

Setting $\delta f = 0$, one would get the classical equation of motion for the superfield. It is not necessary, however, to find the equation of motion, and following Polyakov [13] the assumption is made that the quantum analogue of eq(28) is

$$\delta f = K\, \partial_{\ddagger} f - \frac{i}{2} D_{+}K^{\ddagger}\bar{D}_{+} f - \frac{i}{2} \bar{D}_{+}K^{\ddagger}D_{+} f \tag{29}$$

Here the assumption essentially is that due to gauge interaction of the superconformal field with the gravitational (gauge) field the form of the transformation law is not changed. More generally, the eq (29) is assumed to be of the form [15,13]

$$\delta f = K\partial_{\ddagger} f - \frac{i}{2} : D_{+}K^{\ddagger}\bar{D}_{+} f : - \frac{i}{2} : \bar{D}_{+}K^{\ddagger}D_{+} f : - \lambda(\partial_{\ddagger}K^{\ddagger}) f \tag{30}$$

where the symbol : : means that the operator products at coincident points have been suitably defined. The constants K and λ could be determined from consistency conditions arising from the properties of the superconformal field f.

Sec. E: N = 2 Superconformal Ward Identity and the Graded Algebra gsl (2/1).

Ward identities are identities that correspond to the symmetry properties of a theory. In the case of N = 1

superconformal symmetry the ward identities for general variation of 2DQFT are known [15, 13], for the N = 2 case also they have been calculated [19]. Only the result is quoted here for N = 2

$$< ja\ y > = \sum_{i=1}^{N} \frac{L^{a}_{i}}{x^{\mp}_{oi}} < y > \tag{31}$$

where $y = f(z_1) \ldots f(z_N)$; N is the number of poles in a given domain of z, and

$$x^{=}_{oi} \equiv x^{=}_{o} - x^{=}_{i} \quad .$$

j^{a}'S are defined by,

$$H^{\mp}_{=} \equiv x^{\mp 2} j^{-1} - 2x^{\mp} j^{o} + j^{1} - \theta^{+} x^{\mp} j^{-\frac{1}{2}}$$

$$+ \theta^{+} j^{\frac{1}{2}} - \bar{\theta}^{+} x^{\mp} j_{-\frac{1}{2}} + \bar{\theta}^{+} j^{\frac{1}{2}} + q_{o} \theta^{+} \bar{\theta}^{+} \tag{32}$$

and L^{a}'S are defined by,

$$L^{-1} \equiv \partial_{\mp}$$

$$L^{o} \equiv x^{\mp} \partial_{\mp} - \frac{i}{2} \theta^{+} D_{+} - \frac{i}{2} \bar{\theta}^{+} \bar{D}_{+} - \lambda$$

$$L^{+1} \equiv x^{\mp 2} \partial_{\mp} - i x^{\mp} \theta^{+} D_{+} - i x^{\mp} \bar{\theta}^{+} \bar{D}_{+} - 2\lambda x^{=}$$

$$L^{\frac{1}{2}} \equiv G^{\frac{1}{2}} = 2x^{\mp} \theta^{+} \partial_{\mp} + i x^{\mp} D_{+} + \theta^{+} \bar{\theta}^{+} \frac{\partial}{\partial \theta^{+}} - 2\lambda \theta^{+}$$

$$L^{-\frac{1}{2}} \equiv G^{-\frac{1}{2}} = 2\theta^{+} \partial_{\mp} + i D_{+}$$

$$\bar{L}^{\frac{1}{2}} \equiv G^{-\frac{1}{2}} = 2x^{\mp} \bar{\theta}^{+} \partial_{\mp} + i x^{\mp} \bar{D}_{+} - \theta^{+} \bar{\theta}^{+} \frac{\partial}{\partial \bar{\theta}^{+}} - 2\lambda \bar{\theta}^{+}$$

$$\bar{L}^{-\frac{1}{2}} \equiv \bar{G}^{-\frac{1}{2}} = 2\bar{\theta}^{+} \partial_{\mp} + i \bar{D}_{+}$$

$$\tilde{L} \equiv J^{o} = \bar{\theta}^{+} \frac{\partial}{\partial \bar{\theta}^{+}} - \theta^{+} \frac{\partial}{\partial \theta^{+}} \tag{33}$$

It turns out that $L^{\pm 1, o}$ are the generators of the projective transformation in superspace, $G_{\pm\frac{1}{2}}$, $\bar{G}_{\pm\frac{1}{2}}$ are the generators of SUSY called superchanges and $\tilde{L}$ is the zero mode J^{o} of an

U(1) current which has charge q_o.

The L'S in eq (33) satisfy the N = 2 graded algebra of the group SL (2,R) ie gsl(2/1), given by ,

$$[\, L^m, L^n \,] = (n-m)\, L^{m+n}$$

$$[\, L^m, G^n \,] = -\, (\tfrac{1}{2}m-m)\, G^{m+n}$$

$$\{\, G^m, G^n \,\} = \{\, \bar{G}^m, \bar{G}^n \,\} = 0$$

$$[\, L^m, \bar{G}^n \,] = -(\tfrac{1}{2}m-n)\, \bar{G}^{m+n}$$

$$[\, J^o, G^r \,]\, G^r$$

$$[\, J^o, \bar{G}^r \,] = \bar{G}^r$$

$$[\, L^m, J^o \,] = 0 \qquad (34)$$

CHAPTER IV

An Ansätz for Critical Exponents of 3D Ising model

Sec A: Critical Exponents

At present there is no quantitative theory of phase transition. But it is known that phase transitions are characterized by the appearance of some non-zero quantity in the ordered state. In a ferromagnetic this quantity is the spontaneous magnetization. Such a quantity is called the order parameter. The order parameter is expected to be zero above the critical temperature and non zero below T_c. The study of critical phenomena has come to focus, in recent years, more and more on the value of a set of indices called critical exponents which describe the behaviour near the critical region of the various quantities of interest [1].

The critical exponent for a general thermodynamic function f (ε) is defined in terms of a dimensionless variable ε called the reduced temperature

$$\varepsilon = \frac{T - T_c}{T_c}$$

If for the function f (ε) which is assumed to be positive and continuous for sufficiently small positive values of ε the limit

$$\lambda = \lim_{\varepsilon \to 0} \frac{\ln f(\varepsilon)}{\ln \varepsilon}$$

exist, then λ is called the critical exponent of the function $f(\varepsilon)$.

In short we frequently write this relation in the form

$$f(\varepsilon) \sim \varepsilon^{\lambda}$$

which does not imply

$$f(\varepsilon) = A\varepsilon^{\lambda}$$

But in general there are correction terms and the relation is replaced by

$$f(\varepsilon) = A\varepsilon^{\lambda}(1+B\varepsilon^{y}+\ldots) \ , \ y>0$$

Here we shall consider some of the critical exponents associated with a magnetic system.

Consider the magnetization M as a function of T and H. At fixed T, let H tend to zero. If M remains non-zero, the system is said to experience spontaneous magnetization. It is known that the qualitative plot for the spontaneous magnetization as a function of temperature in zero magnetic field is as shown in fig 1.

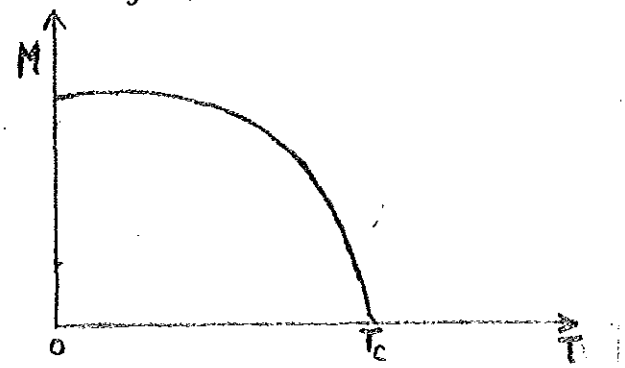

Fig 7. Magnetization as a function of temperature.

For a region near T_c, M should vanish. Therefore, we can approximate this behaviour by

$$M(T) \sim (\varepsilon)^{\beta}$$

where β is the critical exponent for the magnetization. Also, the susceptibility χ is found to diverage as T approaches T_c,

$$\chi \sim (\varepsilon)^{-\gamma}$$

Another quantity of interest is the specific heat

$$c = -T \frac{\partial^2 F}{\partial T^2}$$

It also may diverge as the temperature is reduced to T_c

$$c \sim (\varepsilon)^{-\alpha}$$

where a is the specific heat critical exponent.

A similar power law can be obtained for other variables by analyzing their corresponding qualitative graphs. In general however, the critical exponents may differ above and below the critical temperature. In this case we must define two different critical exponents. For example, the susceptibility and the specific heat have each two critical exponents.

$$\chi \sim (\varepsilon)^{-\gamma} \quad - \quad T \to T_c \text{ from above}$$

$$(-\varepsilon)^{-\gamma'} \quad T \to T_c \text{ from below}$$

$$C \sim (\varepsilon)^{-\alpha} \quad T \to T_c \text{ from above}$$

$$(-\varepsilon)^{-\alpha'} \quad T \to T_c \text{ from below}$$

Interest in critical exponents is due to the fact that, although the corresponding exponents may differ slightly from material to material they seem to depend primarely on fundamental parameters such as the dimensionality of the system. This independence of the critical exponents on the nature of the system is referred to as "universality

The exact solution for the 2DIM is obtained in zero magnetic field. Therefore it is not possible to obtain directly the spontaneous magnetization and the exponents β, γ, and γ^1. The spontaneous magnetization and the exponen β was , however, obtained later by Yang indirectly. The susceptibility of the 2DIM has not yet been calculated, we have only approximate values for the exponent γ^1. Since no one has yet succeeded in solving the three dimensional Isin Model exactly all exponents are obtained by approximation techniques till now.

Sec B: <u>Nonrenormalisation of Coupled N = 2 SCQFT and the Ansatz for α and β</u>

One notices that the 2D gravity superfield H is the gauge field of SUSY and according to the rules of gauge interaction it couples with the matter field which is the N = 2 2DSCQFT. It is expected in general that due to this coupling the characterstices of the coupled field will be different from the original fields. This has been shown to be the case for N = 1 SUSY in ref |15b|. In the same reference it has been shown that the coupled superfield has

inherent local currents which take on values in the N=1 super lie algebra gsl (2).

In ch.III above it has been shown that for N=2 SUSY the coupled superfield contains currents which take on values in the N=2 super lie algebra gsl (2/1). In general the lie algebra of currents, in addition to conformal symmetry, characterises the Wess-Zumino Witten (WZW) model and as is well known that the 2DSCQFT has a one-to-one correspondence with this model [21]. The WZW model is characterized by k, a positive integer, which is known as the central charge of the model. The 2DSCQFT, as in the case of ordinary 2DCFT, is characterised by the conformal dimension h. From which ever point of view one looks at the coupled gauge-matter field, it has been found that both h and k are renormalized for N=0 and 1 [15]. For the case of N = 2 SUSY it is expected that such renormalisation would not occur but has not been explicitly demonstrated. This is done in the following for the conformal dimension h for N=2 2D superconformal primary field.

Since the specific formalism for N = 2 is rather complicated [23, 24] it has been found that an approach via N = 0 formalism followed by a generalisation at the end suffice to demonstrate the nonrenormalisation of h. One may start from the variation of the fields $j^a(z)$ under the gauge group of the WZW model [8],

$$\delta_\omega j^a(z) = if^{abc}\,\omega^b(z)\,j^c(z) + k\frac{\partial}{\partial z}\omega^a(z).$$

which combined with the relation

$$\delta_\omega A(z,\bar{z}) = \int_{C_z} J^a(\xi)\, \omega^a(\xi)\, A(z,\bar{z})\, d\xi$$

gives the operator product expansion (OPE),

$$j^a(z)\, j^b(\omega) = \frac{k\,\delta^{ab}}{(z-\omega)^2} + \frac{if^{abc}}{(z-\omega)}\, j^c(\omega) + \text{regular terms} \tag{35}$$

This is the OPE for what is known as an affine, or affine Kac - Moody (KM) algebra or 2D current algebra for WZW Model. This is an exactly solvable model of 2DQFT. For physical reasons k is required to be a positive integer. In terms of mode expansion

$$j^a(z) = \sum_{n\,z} J^a_n\, z^{-(n+1)} ,$$

which with eq. (35) gives

$$[\, j^a_m , J^b_n \,] = if^{abc}\, J^c_{m+n} + km\delta^{ab}\delta_{m+n,\,0} \tag{36}$$

where f^{abc} are structure constants of the gauge groups of the WZW model. m, nϵz and a,b,c, run over $|G| \equiv$ dim G.

The representation theory of KM algebra has many features similar to a virasoro algebra. There exists the notion of primary fields $\phi^\ell(r)$, a multiplet of fields with index ℓ, with respect to the affine KM algebra, for which the OPE has the leading singularity,

$$j^a(z)\, \phi_{(r)}(\omega) = \frac{t^a_{(r)}}{z-\omega}\, \phi^\ell_{(r)}(\omega) + \tag{37}$$

This should be recognised as the statement that $\phi_{(r)}$ transform as some representation (r) of G where the right side numerator is shorthand for

$$[\, t^a_{(r)} \,]^{\ell k} \, \phi^k_{(r)}$$

and $t^a_{(r)}$ are representation matrices for G in the representation (r). The representation metnioned above may be taken as the irreducible representation. The primary fields $\phi_{(r)}$ create states out of vacuum called the highest weight states,

$$\phi_{(r)}(0)|0> \; = \; |(r)> \; ,$$

a multiplet of states that provide a representation of the zero-mode algebra,

$$J^a_0 \; |(r)> \; = \; t^a_{(r)} |(r)>$$

$$J^a(z)|(r)> \; = \; 0 \; , \; n>0 \qquad (38)$$

The algebric structure characterising an affine or current algebra turns out to incorporate a natural definition of T(z), the stress-energy tensor,

$$T(z) = \frac{1}{\beta} \sum_{a=1}^{|G|} : J^a(z) \, J^a(z) :$$

$$= \lim_{z \to \omega} \sum_{a=1}^{|G|} J^a(z) \, J^a(\omega) - \frac{k|G|}{(z-\omega)^2} \qquad (39)$$

Mode expansion on either side of (39) gives

$$L_n = \frac{1}{\beta} \sum_{m=\infty}^{\infty} : J^a_{m+n} \; J^a_{-m} : \tag{40}$$

β has been evaluated in ref[8],

$$\beta = 2k + C_A,$$

$$C_A \delta^{ab} = f^{acd} f^{bcd}$$

ie C_A is the casimir operator in the adjoint representation of G.

Thus

$$L_o \; |(r)> = \frac{1}{2k + C_A} \sum_{a=1}^{|G|} \sum_{m=-\infty}^{\infty} : J^a_m \; J^a_{-m} : |(r)> \tag{41}$$

Generalisation of this procedure to superalgebra is given in ref |25 | and that a direct generalisation is possible is obvious. Using eq. (38), (40) and (41),

$$L_o \; |(r) = \frac{1}{2k + C_A} \sum_{a=1}^{|G|} t^a_{(r)} \; t^a_{(r)} |(r)>$$

Since by definition

$$\sum_{a=1}^{|G|} t^a_{(r)} \; t^a_{(r)}$$

is the quadratic casmir operator of the representation (r),

$$L_o \; |(r)> = \frac{C_r}{2k + C_A} \; |(r)> , \tag{43}$$

ie. the eigenvalues of L_0 is

$\frac{C_r}{2k+C_A}$. Again, by definition, the conformal highest weight h_r of a primary multiplet $\phi_{(r)}(z)$ is given by,

$$L_0|(r)> = h_r|(r)> ,$$

where h_r is also the conformal dimension of the field $\phi_{(r)}(z)$. Thus

$$h_r = \frac{C_r}{2k+C_A} \tag{44}$$

gives the dimension of the (super) conformal field. C_r for the N = 2 graded $S\ell(2,R) \equiv gs\ell(2/1)$ which appears in the notation of ref [22] as $sp\ell$ (2/1) has been found in the same reference as $\mu^2-\beta^2$, where μ is the highest weight of the irreducible representation of $gs\ell(2/1)$,

$$\mu = 0, \tfrac{1}{2}, 1, 3/2, \tag{45}$$

and β is any real number. In order to find C_A one notices that it has been pointed out in ref [24] that for both Lie algebra and superalgebras $C_a = 2g$ where g is called the dual coexter number. g has been calculated for the super-group SU(m/n) in ref. [24] as equal to m-n where m and n are, respecively, the number of bosonic and fermionic super Lie generators. Such calculation does not appear to exist for $gs\ell(2/1)$ but it is safe to take that the relevant g is a positive or negative integer. However, C_A for Osp (2/2) has been calculated [25] by a method very different from that of ref. [24] and found to be 1.

Since $gs\ell(2/1)$ and Osp (2/2) are isomorphic the casimir

operators are expected to be the same. Remembering $C_A = 2g$, the only appropriate choice of $C_A \doteq 2$. The departure from the result of ref[25] is presumably due to difference in normalisation in the two cases.

Finally one sets $\beta = 0$ and obtains from eq (44),

$$h_r = \frac{\mu^2}{2(k+1)} \tag{46}$$

Again, in ref [15b], k has been calculated as a a function of d for the N = 2 supersymmetric case,

$$d - 3 = 2k \tag{47}$$

Since WZW models can have only k = 0,1,2,3,... etc., one can see from eq (47) that the values of d consistent with this constraint is

$$d = 3,5,7,9,11,....$$

It may noted in this connection that Gervais and Neveu [26] have found that away from the critical dimension of 10, a fermionic string can be quantised only at dimension= 3,5 and 7. Also at k = 0 corresponding to d = 3 above the WZW theory has properties which are qualitatively different from those at higher k values. These matters need further investigation.

To continue with the case of 3D Ising model corresponding to k = 0, one finds from eq. (46),

$$h_r = \frac{\mu^2}{2} = 0, \frac{1}{8}, \frac{1}{2}, \tag{48}$$

One may compare these values of h_r with those of the free N = 2 2DSC primary fields, quoted for example, in Ref [10] , fig. 2.,

$$h_r = 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \tag{49}$$

It hus appears that in the permitted spectrum of primary fields, three out of four dimensions from the relevant Kac spectrum appear in nonrenormalised from. The reason why the other ie $\frac{1}{4}$ does not appear is not obvious; it may have something to do with the fact that the three that appear, namely, 0, $\frac{1}{8}$, and $\frac{1}{2}$ are enough to provide excellent values for α and β for 3D Ising model.

For N = 1 global SUSY on the world sheet polyakov et al [15] made the ansätz $\gamma = h_r$ and arrived at the value

$$\gamma(d) = \frac{d-1 \pm \sqrt{(1-d)(9-d)}}{4}$$

An equation which is meaningless in realistic dimensions. An ansatz is necessary, however, as even though the critical exponents are functions of h_r the exact functional form is unknown in general. What is more is that, unlike the case of N = 0, the identification of the superconformal

primary fields for N = 1, 2, ... etc is also unknown. Finally, using the two nontrivial values of primary field dimensions in eq. (48) one may make the ansätz

$$\beta = \frac{1}{2} - \frac{1}{8} = \frac{3}{8} = 0.375$$

and $$\alpha = \frac{1}{8} = 0.125 \ ,$$

and compare them with other available values in table 1.

Table 1.

	Expt	3DIM		ϕ^4 - theory	This calculation
		Series expan	RG		
α	0∿0.2	0.125	0.08	0.17	0.125
β	0-3-0.4	0.313	0.34	0.33	0.375

Source: * (1) L.E Reichl, A modern course in statistical physics (Univ. of Texas press, Austin 1980) P 344.

(2) H. Eyring, Statistical mechanics and dynamics (Wiley, Newyork 1982) P 475

CONCLUSION

Though this calculation is not very rigorous because of the current state of development of the theory of 2D superconformal fields and the algebra of supergroups the agreement appears excellent. It is important to point out that while the theory of 2D conformal fields has been useful in the study of critical systems in 2D, the approach of ref |15| is exceptional in the sense that for the first time a 3D critical system has been treated via the theory of 2D superconformal fields. This has been made possible by the use of string theory methods.

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