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THE NONDEGENERATE PARAMETRIC OSCILLATOR COUPLED TO SQUEEZED VACUUM RESERVOIRS

A Thesis

Submitted to the

School of Graduate Studies

of Addis Ababa University

in Partial Fulfillment

of the Requirements for the

Degree of Master of Science in

Physics

by

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May 1998

Addis Ababa

Dedication

This work is totally dedicated to my beloved kids: Meazet and Eden.

Acknowledgements

I am very much indebted and would like to thank from the bottom of my heart to my instructor and advisor, Dr. Fesseha Kassahun, head, Department of physics, AAU, for his sustained advice, careful follow up and fatherly treatment, and without his help this thesis would not have had its present form. I would like also to express my deepest gratitude to my beloved wife, w/o Mulu Hailu, not only for her indefinite moral and financial assistance during my stay here for two years but also for her decisions to confront the difficulties of loneliness and look after alone our two kids, Meazet and Eden, when I was supposed to join this program. My mother, father, brothers and sisters, Ato Gizaw Mengistu, Ato Misrak Gethun, Ato Tesfu Kassaye, Ato Daniel Bekele, my families, friends and intimates also deserve my heart felt thanks for their contributions to the success of the thesis in one way or another.

Lastly, I would like to acknowledge the National Regional Government of Tigrai for sponsoring me to join the School of Graduate Studies.

Abstract

Employing the pertinent quantum Hamiltonian describing the interaction of a two-mode light with two uncorrelated squeezed vacuum reservoirs, we derive the equation governing the time evolution of the reduced density operator. With the help of the resulting equation, we obtain the master equation for the signal-idler modes produced by a nondegenerate parametric oscillator coupled to two uncorrelated squeezed vacuum reservoirs. The corresponding Fokker-Planck equation for the Q-function is then solved employing the method of evaluating the propagator developed by Fesseha [1]. Finally, applying this Q-function, we calculate the quadrature fluctuations and the photon number distributions for the signal mode as well as the signal-idler modes.

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1. Introduction

In recent times, the subject of squeezing of light has received a great deal of attention by several authors [2-11]. These nonclassical states of light (squeezed states) are characterized by a reduction of quantum fluctuations (noise) in one quadrature component of the light below the vacuum level, or below that achievable in a coherent state, at the expense of increased fluctuations in the other component such that the product of these fluctuations still obeys the uncertainty relation [11-21].

It was Takahashi [16] who first pointed out that a parametric amplifier enhances the noise in one quadrature component of the signal mode and attenuates the noise in the other quadrature. This prediction has been confirmed by several authors [11-15,17,18,22,23]. In the initial experiments carried out to observe squeezing, a noise reduction of 4-17% relative to the quantum standard limit has been obtained [3]. In order to increase the gain, the parametric medium may be placed inside an optical cavity where it is coherently pumped and becomes a parametric oscillator [4,8,9,24,25]. Optical parametric oscillators are quantum devices with a definite threshold for self sustained oscillations [26-28].

Because of the inherent two photon nature of the interaction, the parametric processes have been studied as a source of squeezed states [4,8,9,26-28]. A quantum analysis of the parametric oscillation was first given by Graham and Haken [17]. It had been experimentally reported that, squeezing amounting to a noise reduction of greater than 60% below the quantum limit, has been achieved in a degenerate parametric oscillator operating below threshold [4,18]. This simple dessipative quantum system has played an important role in the studies of squeezing.

In a parametric oscillator a strong pump photon interacts with a nonlinear-medium (crystal) and is down converted into two photons of smaller frequencies. If the two photons produced in the down conversion have the same frequencies, the oscillator is referred to as a degenerate parametric oscillator [25,27], otherwise it is called a nondegenerate parametric oscillator [9] and the two different photons are called the signal and idler photons. The down conversion of the pump photon into highly correlated signal and idler photons was observed for the first time in parametric amplification by Burham and Weinberg [29] and later by Friberg et al [30]. A quantum-mechanical treatment of the nondegenerate parametric oscillator is essential since it generates squeezed states with nonclassical properties which have potential applications in optical communications, gravitational wave detection [31-33], interferometry [10,34,35], spectroscopical measurements [36] and for the study of fundamental concepts.

In quantum mechanics, one can tackle the problem of system-reservoir interaction based on either the schrödinger formalism which leads to the so called master equation or the Heisenberg approach which leads to the Quantum Langevin equations. One can determine the evolution of expectation values of various observables using the master equation. However, this equation does not provide much physical intuition into the evolution of the system. As a result, the analysis of such physical systems is often carried out by converting the master equation into a corresponding c-number partial differential equation called the Fokker-Planck equation . Depending on the nature of the system, this can be done applying the Glauber-Sudarshan P-function, the positive P-function, or the Q-function. For systems with nonclassical features such as the nondegenerate parametric oscillator, for which the Glauber- Sudarshan P-function is highly singular, one may use the Q-function.

The main objective of this thesis is to calculate, using the Q-function, the quadrature fluctuations and the photon number distribution for the signal mode as well as the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two uncorrelated squeezed vacuum reservoirs.

The Q-function is expressible in terms of the Q-function propagator and the initial Q-function. It is possible to determine the Q-function propagator using the path integral methods [37] or by directly solving the Fokker-Planck equation. However, we find it to be convenient to evaluate the Q-function propagator applying the method developed by Fesseha [1].

The thesis is organized as follows: In chapter two we derive the master equation for the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two uncorrelated squeezed vacuum reservoirs. The solution of the pertinent Fokker-Planck equation for the Q-function is obtained using the propagator method discussed in Ref. [1]. From the resulting Q-function, we also obtain the Q-function for the signal mode as well as for two special cases of interest. In chapter three we calculate, using the Q-function, the quadrature fluctuations of the signal mode as well as the signal-idler modes. We also determine the quadrature fluctuations of the signal mode as well as the signal idler modes for three special cases of interest. In chapter four we first derive a general expression for the photon number distribution for a two-mode light in terms of the Q-function. This is then used to obtain the photon number distribution for the signal mode as well as the signal-idler modes. Finally, in chapter five we present a summary of the important results and discuss certain issues of interest.

2. The Q-function

In this chapter we first derive the equation of evolution for the reduced density operator for a two-mode light in a cavity coupled to two independent squeezed vacuum reservoirs. We then obtain the master equation and the corresponding Fokker-Plank equation for the Q-function for signal-idler modes produced by a nondegenerate parametric oscillator coupled to the two uncorrelated squeezed vacuum reservoirs. We finally solve the Fokker-Planck equation and obtain the explicit form of the Q-function applying the propagator method developed by Fesseha [1].

2.1 The Master Equation

In this section we seek to derive the equation of evolution for the reduced density operator of a system of two-mode light and two uncorrelated squeezed vacuum reservoirs applying the Hamiltonian describing the interaction between the system and the reservoirs. Considering the system to be a nondegenerate parametric oscillator that generates signal-idler modes, we obtain the corresponding master equation.

Denoting the density operator for the system and the reservoirs by $\hat{\chi}(t)$, the density operator for the system alone is defined by

$$\hat{\rho}(t) = Tr_R(\hat{\chi}(t)), \tag{2.1}$$

where Tr_R indicates that the trace is taken over the reservoir variables only. The density operator $\hat{\chi}(t)$ evolves in time according to

$$\frac{d\hat{\chi}(t)}{dt} = \frac{1}{i\hbar}\left[\hat{H}_{SR}(t), \hat{\chi}(t)\right], \tag{2.2}$$

where $\hat{H}_{SR}(t)$ is the Hamiltonian describing the interaction between the system and the reservoir. A formal solution of this equation can be written as

$$\hat{\chi}(t) = \hat{\chi}(0) + \frac{1}{i\hbar}\int_0^t dt' \left[\hat{H}_{SR}(t'), \hat{\chi}(t')\right], \tag{2.3}$$

where $\hat{\chi}(0)$ is the density operator at the initial time.

Since the system is not correlated with the reservoir at the initial time [12], we can write that

$$\hat{\chi}(0) = \hat{\rho}(0)\hat{R},$$

in which $\hat{\rho}(0)$ and $\hat{R}$ are respectively the density operators of the system and the reservoir at the initial time. Then in view of this relation, (2.3) becomes

$$\hat{\chi}(t) = \hat{\rho}(0)\hat{R} + \frac{1}{i\hbar}\int_0^t dt' \left[\hat{H}_{SR}(t'), \hat{\chi}(t')\right]. \tag{2.4}$$

Substituting this result into expression (2.2), we get

$$\frac{d\hat{\chi}(t)}{dt} = \frac{1}{i\hbar}\left[\hat{H}_{SR}(t), \hat{\rho}(0)\hat{R}\right] - \frac{1}{\hbar^2}\int_0^t dt' \left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}(t'), \hat{\chi}(t')\right]\right]$$

and applying the Born approximation for which

$$\hat{\chi}(t') = \hat{\rho}(t')\hat{R},$$

we see that

$$\frac{d\hat{\rho}(t)}{dt} = \frac{1}{i\hbar}Tr_R\left[\hat{H}_{SR}(t), \hat{\rho}(0)\hat{R}\right] - \frac{1}{\hbar^2}\int_0^t dt' Tr_R\left[\hat{H}_{SR}(t), [\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}]\right]. \tag{2.5}$$

We now consider the system to be a two-mode light with frequencies ω_a and ω_b in a cavity (one of the end mirrors of this cavity is taken to be partially transmittive) coupled to two independent squeezed vacuum reservoirs. The interaction between the two-mode light and the squeezed vacuum reservoirs is describable, in the interaction picture, by the Hamiltonian

$$\hat{H}_{SR}(t) = i\hbar\Bigg[\sum_j \lambda_j(\hat{a}^\dagger \hat{A}_j\, e^{i(\omega_a-\omega_j)t} - \hat{a}\hat{A}_j^\dagger\, e^{-i(\omega_a-\omega_j)t})$$
$$+ \sum_k \lambda_k(\hat{b}^\dagger \hat{B}_k\, e^{i(\omega_b-\omega_k)t} - \hat{b}\hat{B}_k^\dagger\, e^{-i(\omega_b-\omega_k)t})\Bigg], \tag{2.6}$$

in which $\hat{a}\,(\hat{a}^\dagger)$ and $\hat{b}\,(\hat{b}^\dagger)$ are the annihilation (creation) operators for the cavity modes, $\hat{A}_j\,(\hat{A}_j^\dagger)$ and $\hat{B}_k\,(\hat{B}_k^\dagger)$ are the annihilation (creation) operators for the reservoir modes with frequencies ω_j and ω_k, respectively and λ_j andλ_k are the coupling constants describing the interaction between the intracavity modes and the reservoir modes. Applying the cyclic property of trace and the fact that

$$Tr_R\left(\hat{R}\hat{H}_{SR}(t)\right) = \langle\hat{H}_{SR}(t)\rangle_R,$$

one can easily see that

$$\frac{1}{i\hbar}Tr_R\left[\hat{H}_{SR}(t),\hat{\rho}(0)\hat{R}\right]=\frac{1}{i\hbar}\left(\langle\hat{H}_{SR}(t)\rangle_R\,\hat{\rho}(0)-\hat{\rho}(0)\langle\hat{H}_{SR}(t)\rangle_R\right).$$

In view of (2.6), we note that

$$\langle\hat{H}_{SR}(t)\rangle_R=i\hbar\left[\sum_j\lambda_j(\hat{a}^\dagger\langle\hat{A}_j\rangle_R\,e^{i(\omega_a-\omega_j)t}-\hat{a}\langle\hat{A}_j^\dagger\rangle_R\,e^{-i(\omega_a-\omega_j)t})\right.$$

$$\left.+\sum_k\lambda_k(\hat{b}^\dagger\langle\hat{B}_k\rangle_R\,e^{i(\omega_b-\omega_k)t}-\hat{b}\langle\hat{B}_k^\dagger\rangle_R\,e^{-i(\omega_b-\omega_k)t})\right].$$

However, for squeezed vacuum reservoirs

$$\langle\hat{A}_j\rangle_R=\langle\hat{A}_j^\dagger\rangle_R=\langle\hat{B}_k\rangle_R=\langle\hat{B}_k^\dagger\rangle_R=0.$$

Thus

$$\frac{1}{i\hbar}Tr_R\left[\hat{H}_{SR}(t),\hat{\rho}(0)\hat{R}\right]=0,\tag{2.7}$$

and as a result, expression (2.5) takes the form

$$\frac{d\hat{\rho}(t)}{dt}=-\frac{1}{\hbar^2}\int_0^t dt'\,Tr_R\left[\hat{H}_{SR}(t),[\hat{H}_{SR}(t'),\hat{\rho}(t')\,\hat{R}]\right].$$

Applying the Markov approximation, in which $\hat{\rho}(t')$ is replaced by $\hat{\rho}(t)$, and using the cyclic property of the trace in the first term, we have

$$\frac{d\hat{\rho}(t)}{dt}=-\frac{1}{\hbar^2}\int_0^t dt'\left[Tr_R\left(\hat{R}\,\hat{H}_{SR}(t)\hat{H}_{SR}(t')\right)\hat{\rho}(t)-Tr_R\left(\hat{H}_{SR}(t)\,\hat{\rho}(t)\hat{R}\,\hat{H}_{SR}(t')\right)\right.$$

$$\left.-Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)\hat{R}\,\hat{H}_{SR}(t)\right)+\hat{\rho}(t)Tr_R\left(\hat{R}\,\hat{H}_{SR}(t')\hat{H}_{SR}(t)\right)\right].$$

This can be rewritten as

$$\frac{d\hat{\rho}(t)}{dt}=-\frac{1}{\hbar^2}\int_0^t dt'\langle\hat{H}_{SR}(t)\hat{H}_{SR}(t')\rangle_R\hat{\rho}(t)-\frac{1}{\hbar^2}\int_0^t dt'\hat{\rho}(t)\langle\hat{H}_{SR}(t')\hat{H}_{SR}(t)\rangle_R$$

$$+\frac{1}{\hbar^2}\int_0^t dt'\,Tr_R\left(\hat{H}_{SR}(t)\hat{\rho}(t)\hat{R}\hat{H}_{SR}(t')\right)+\frac{1}{\hbar^2}\int_0^t dt'\,Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)\hat{R}\,\hat{H}_{SR}(t)\right).$$

We now set for convenience

$$\hat{\Gamma}_1=\frac{1}{\hbar^2}\int_0^t dt'\langle\hat{H}_{SR}(t)\hat{H}_{SR}(t')\rangle_R\hat{\rho}(t),\tag{2.8a}$$

$$\hat{\Gamma}_2=\frac{1}{\hbar^2}\int_0^t dt'\,\hat{\rho}(t)\langle\hat{H}_{SR}(t')\hat{H}_{SR}(t)\rangle_R,\tag{2.8b}$$

$$\hat{\Gamma}_3 = \frac{1}{\hbar^2}\int_0^t dt'\, Tr_R\left(\hat{H}_{SR}(t)\hat{\rho}(t)\hat{R}\,\hat{H}_{SR}(t')\right), \tag{2.8c}$$

$$\hat{\Gamma}_4 = \frac{1}{\hbar^2}\int_0^t dt'\, Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t)\hat{R}\,\hat{H}_{SR}(t)\right). \tag{2.8d}$$

On account of (2.6), we see that

$$\begin{aligned}
\hat{H}_{SR}(t)\hat{H}_{SR}(t') = -\hbar^2 &\left[\sum_j \lambda_j \left(\hat{a}^\dagger \hat{A}_j\, e^{i(\omega_a-\omega_j)t} - \hat{a}\hat{A}_j^\dagger\, e^{-i(\omega_a-\omega_j)t}\right)\right.\\
&\left. + \sum_k \lambda_k \left(\hat{b}^\dagger \hat{B}_k\, e^{i(\omega_b-\omega_k)t} - \hat{b}\hat{B}_k^\dagger\, e^{-i(\omega_b-\omega_k)t}\right)\right]\\
&\times\left[\sum_l \lambda_l \left(\hat{a}^\dagger \hat{A}_l\, e^{i(\omega_a-\omega_l)t'} - \hat{a}\hat{A}_l^\dagger\, e^{-i(\omega_a-\omega_l)t'}\right)\right.\\
&\left. + \sum_m \lambda_m \left(\hat{b}^\dagger \hat{B}_m\, e^{i(\omega_b-\omega_m)t'} - \hat{b}\hat{B}_m^\dagger\, e^{-i(\omega_b-\omega_m)t'}\right)\right],
\end{aligned}$$

from which follows

$$\begin{aligned}
\langle \hat{H}_{SR}(t)\hat{H}_{SR}(t')\rangle_R = -\hbar^2 &\left[\sum_{j,l} \lambda_j\lambda_l \left(\hat{a}^{\dagger 2}\langle \hat{A}_j\hat{A}_l\rangle_R\, e^{i(\omega_a-\omega_j)t+i(\omega_a-\omega_l)t'}\right.\right.\\
&\left. - \hat{a}^\dagger\hat{a}\langle \hat{A}_j\hat{A}_l^\dagger\rangle_R\, e^{i(\omega_a-\omega_j)t-i(\omega_a-\omega_l)t'}\right)\\
&+ \sum_{j,m} \lambda_j\lambda_m \left(\hat{a}^\dagger\hat{b}^\dagger\langle \hat{A}_j\hat{B}_m\rangle_R\, e^{i(\omega_a-\omega_j)t+i(\omega_b-\omega_m)t'} - \hat{a}^\dagger\hat{b}\langle \hat{A}_j\hat{B}_m^\dagger\rangle_R\, e^{i(\omega_a-\omega_j)t-i(\omega_b-\omega_m)t'}\right)\\
&- \sum_{j,l} \lambda_j\lambda_l \left(\hat{a}\hat{a}^\dagger\langle \hat{A}_j^\dagger\hat{A}_l\rangle_R\, e^{-i(\omega_a-\omega_j)t+i(\omega_a-\omega_l)t'} - \hat{a}^2\langle \hat{A}_j^\dagger\hat{A}_l^\dagger\rangle_R\, e^{-i(\omega_a-\omega_j)t-i(\omega_a-\omega_l)t'}\right)\\
&- \sum_{j,m} \lambda_j\lambda_m \left(\hat{a}\hat{b}^\dagger\langle \hat{A}_j^\dagger\hat{B}_m\rangle_R\, e^{-i(\omega_a-\omega_j)t+i(\omega_b-\omega_m)t'} - \hat{a}\hat{b}\langle \hat{A}_j^\dagger\hat{B}_m^\dagger\rangle_R\, e^{-i(\omega_a-\omega_j)t-i(\omega_b-\omega_m)t'}\right)\\
&+ \sum_{k,l} \lambda_k\lambda_l \left(\hat{b}^\dagger\hat{a}^\dagger\langle \hat{B}_k\hat{A}_l\rangle_R\, e^{i(\omega_b-\omega_k)t+i(\omega_a-\omega_l)t'} - \hat{b}^\dagger\hat{a}\langle \hat{B}_k\hat{A}_l^\dagger\rangle_R e^{i(\omega_b-\omega_k)t-i(\omega_a-\omega_l)t'}\right)\\
&+ \sum_{k,m} \lambda_k\lambda_m \left(\hat{b}^{\dagger 2}\langle \hat{B}_k\hat{B}_m\rangle_R\, e^{i(\omega_b-\omega_k)t+i(\omega_b-\omega_m)t'} - \hat{b}^\dagger\hat{b}\langle \hat{B}_k\hat{B}_m^\dagger\rangle_R\, e^{i(\omega_b-\omega_k)t-i(\omega_b-\omega_m)t'}\right)\\
&- \sum_{k,l} \lambda_k\lambda_l \left(\hat{b}\hat{a}^\dagger\langle \hat{B}_k^\dagger\hat{A}_l\rangle_R\, e^{-i(\omega_b-\omega_k)t+i(\omega_a-\omega_l)t'} - \hat{b}\hat{a}\langle \hat{B}_K^\dagger\hat{A}_l^\dagger\rangle_R\, e^{-i(\omega_b-\omega_k)t-i(\omega_a-\omega_l)t'}\right)\\
&- \sum_{k,m} \lambda_k\lambda_m \left(\hat{b}\hat{b}^\dagger\langle \hat{B}_k^\dagger\hat{B}_m\rangle_R\, e^{-i(\omega_b-\omega_k)t+i(\omega_b-\omega_m)t'}\right.\\
&\left.\left. - \hat{b}^2\langle \hat{B}_k^\dagger\hat{B}_m^\dagger\rangle_R\, e^{-i(\omega_b-\omega_k)t-i(\omega_b-\omega_m)t'}\right)\right].
\end{aligned} \tag{2.9}$$

We note that for squeezed vacuum reservoirs,

$$\langle \hat{A}_j \hat{A}_l \rangle_R = -M_A \, \delta_{l,2j_a-j}, \tag{2.10a}$$

$$\langle \hat{B}_k \hat{B}_m \rangle_R = -M_B \, \delta_{m,2k_b-k}, \tag{2.10b}$$

$$\langle \hat{A}_j^\dagger \hat{A}_l \rangle_R = N_A \, \delta_{j,l}, \tag{2.10c}$$

$$\langle \hat{B}_k^\dagger \hat{B}_m \rangle_R = N_B \, \delta_{k,m}, \tag{2.10d}$$

$$\langle \hat{A}_j \hat{A}_l^\dagger \rangle_R = (N_A + 1)\delta_{j,l}, \tag{2.10e}$$

$$\langle \hat{B}_k \hat{B}_m^\dagger \rangle_R = (N_B + 1)\delta_{k,m}, \tag{2.10f}$$

$$\langle \hat{A}_j \hat{B}_m \rangle_R = \langle \hat{A}_j \hat{B}_m^\dagger \rangle_R = \langle \hat{A}_j^\dagger \hat{B}_m^\dagger \rangle_R = 0, \tag{2.10g}$$

where the subscripts A and B denote the two squeezed vacuum reservoirs, and the parameters N_A, N_B, M_A, M_B describe the effects of squeezing of these reservoirs. Actually, the parameters N and M represent the mean photon number and the phase property of these reservoirs respectively. For a squeezed vacuum reservoir, the relation between N and M can be expressed as

$$|M|^2 = N(N+1).$$

On account of (2.10), expression (2.9) takes the form

$$\begin{aligned}
\langle \hat{H}_{SR}(t)\hat{H}_{SR}(t')\rangle_R = -\hbar^2 \Bigg[\sum_{j,l} \lambda_j \lambda_l \Big(& -M_A \, \delta_{l,2j_a-j} \, e^{i(\omega_a-\omega_j)t+i(\omega_a-\omega_l)t'} \hat{a}^{\dagger 2} \\
& -(N_A+1)\delta_{j,l} \, e^{i(\omega_a-\omega_j)t-i(\omega_a-\omega_l)t'} \hat{a}^\dagger \hat{a} - N_A \, \delta_{j,l} \, e^{-i(\omega_a-\omega_j)t+i(\omega_a-\omega_l)t'} \hat{a}\hat{a}^\dagger \\
-M_A \, \delta_{l,2j_a-j} \, e^{-i(\omega_a-\omega_j)t-i(\omega_a-\omega_l)t'} \hat{a}^2 \Big) & + \sum_{k,m} \lambda_k \lambda_m \Big(-M_B \, \delta_{m,2k_b-k} \, e^{i(\omega_b-\omega_k)t+i(\omega_b-\omega_m)t'} \hat{b}^{\dagger 2} \\
& -(N_B+1)\delta_{k,m} \, e^{i(\omega_b-\omega_k)t-i(\omega_b-\omega_m)t'} \hat{b}^\dagger \hat{b} - N_B \, \delta_{k,m} \, e^{-i(\omega_b-\omega_k)t+i(\omega_b-\omega_m)t'} \hat{b}\hat{b}^\dagger \\
& -M_B \, \delta_{m,2k_b-k} \, e^{-i(\omega_b-\omega_k)t-i(w_b-w_m)t'} \hat{b}^2 \Big) \Bigg],
\end{aligned}$$

so that applying the properties of the Kronecker delta symbol

$$\delta_{n,m} = \begin{cases} 1 & \text{for n = m,} \\ 0 & \text{for n different from m,} \end{cases} \tag{2.11}$$

we get

$$\langle \hat{H}_{SR}(t)\hat{H}_{SR}(t')\rangle_R = \hbar^2 \Bigg[\sum_j \Big(\lambda_j \lambda_{2j_a-j} \, M_A \, e^{i(\omega_a-\omega_j)t+i(\omega_a-\omega_{2j_a-j})t'} \hat{a}^{\dagger 2}$$

$$+\lambda_j^2 (N_A+1) e^{i(\omega_a-\omega_j)(t-t')} \hat{a}^\dagger \hat{a} + \lambda_j^2 N_A\, e^{-i(\omega_a-\omega_j)(t-t')} \hat{a}\hat{a}^\dagger$$

$$+\lambda_j \lambda_{2j_a-j}\, M_A\, e^{-i(\omega_a-\omega_j)t - i(\omega_a-\omega_{2j_a-j})t'} \hat{a}^2 \Big) +$$

$$\sum_k \Big(\lambda_k \lambda_{2k_b-k}\, M_B\, e^{i(\omega_b-\omega_k)t + i(\omega_b-\omega_{2k_b-k})t'} \hat{b}^{\dagger 2}$$

$$+\lambda_k^2 (N_B+1) e^{i(\omega_b-\omega_k)(t-t')} \hat{b}^\dagger \hat{b} + \lambda_k^2 N_B\, e^{-i(\omega_b-\omega_k)(t-t')} \hat{b}\hat{b}^\dagger$$

$$+\lambda_k \lambda_{2k_b-k}\, M_B\, e^{-i(\omega_b-\omega_k)t - i(\omega_b-\omega_{2k_b-k})t'} \hat{b}^2 \Big) \Big] .$$

Now (2.8a) can be written as

$$\hat{\Gamma}_1 = \left(I_{1a} M_A \hat{a}^{\dagger 2} + I_{2a}(N_A+1)\hat{a}^\dagger \hat{a} + I_{3a} N_A \hat{a}\hat{a}^\dagger + I_{4a} M_A \hat{a}^2 \right) \hat{\rho}(t)$$

$$\left(I_{1b} M_B \hat{b}^{\dagger 2} + I_{2b}(N_B+1)\hat{b}^\dagger \hat{b} + I_{3b} N_B \hat{b}\hat{b}^\dagger + I_{4b} M_B \hat{b}^2 \right) \hat{\rho}(t) \tag{2.12}$$

where we have set

$$I_{1a} = \int_0^t dt' \sum_j \lambda_j \lambda_{2j_a-j}\, e^{i(\omega_a-\omega_j)t + i(\omega_a-\omega_{2j_a-j})t'}, \tag{2.13a}$$

$$I_{2a} = \int_0^t dt' \sum_j \lambda_j^2\, e^{i(\omega_a-\omega_j)(t-t')}, \tag{2.13b}$$

$$I_{3a} = \int_0^t dt' \sum_j \lambda_j^2\, e^{-i(\omega_a-\omega_j)(t-t')}, \tag{2.13c}$$

$$I_{4a} = \int_0^t dt' \sum_j \lambda_j \lambda_{2j_a-j}\, e^{-i(\omega_a-\omega_j)t - i(\omega_a-\omega_{2j_a-j})t'}, \tag{2.13d}$$

with similar expressions for I_{1b}, I_{2b}, I_{3b} and I_{4b}. Upon introducing the density of states $g(\omega)$, in which

$$\sum_j \lambda_j \lambda_{2j_a-j} \to \int_0^\infty d\omega\, g(\omega)\lambda(\omega)\lambda(2\omega_a-\omega),$$

we get

$$I_{1a} = \int_0^\infty d\omega\, g(\omega)\lambda(\omega)\lambda(2\omega_a-\omega) \int_0^t dt'\, e^{i(\omega_a-\omega)(t-t')}, \tag{2.14a}$$

$$I_{2a} = \int_0^\infty d\omega\, g(\omega)\lambda^2(\omega) \int_0^t dt'\, e^{i(\omega_a-\omega)(t-t')}, \tag{2.14b}$$

$$I_{3a} = \int_0^\infty d\omega\, g(\omega)\lambda^2(\omega) \int_0^t dt'\, e^{-i(\omega_a-\omega)(t-t')}, \tag{2.14c}$$

$$I_{4a} = \int_0^\infty d\omega\, g(\omega)\lambda(\omega)\lambda(2\omega_a-\omega) \int_0^t dt'\, e^{-i(\omega_a-\omega)(t-t')}. \tag{2.14d}$$

Furthermore, setting $t - t' = \tau$, we see that

$$\int_0^t dt'\, e^{\pm i(\omega_a - \omega)(t-t')} = \int_0^t d\tau\, e^{\pm i(\omega_a - \omega)\tau}. \tag{2.15}$$

Since the exponential function is a rapidly oscillating function of time, the upper limit of integration can be extended to infinity. Thus making use of the approximate relation

$$\int_0^\infty e^{\pm i(\omega_a - \omega)\tau} = \pi\, \delta(\omega_a - \omega) \tag{2.16}$$

one finds

$$I_{1a} = \int_0^\infty d\omega\, g(\omega)\lambda(\omega)\, \lambda(2\omega_a - \omega)\pi\, \delta(\omega_a - \omega), \tag{2.17}$$

and applying the property of the Dirac delta function we obtain

$$I_{1a} = \pi g(\omega_a)\lambda^2(\omega_a) = \frac{\gamma_A}{2}, \tag{2.18}$$

where

$$\gamma_A = 2\pi g(\omega_a)\lambda^2(\omega_a)$$

is the cavity damping rate for mode A. It is easy to check that

$$I_{1a} = I_{2a} = I_{3a} = I_{4a} = \frac{\gamma_A}{2}. \tag{2.19}$$

Similarly, one can also show that

$$I_{1b} = I_{2b} = I_{3b} = I_{4b} = \frac{\gamma_B}{2}, \tag{2.20}$$

in which

$$\gamma_B = 2\pi g(\omega_b)\lambda^2(\omega_b)$$

is the cavity damping rate for mode B.

Substituting (2.19) and (2.20) into (2.12), we have

$$\begin{aligned}\hat{\Gamma}_1 = {} & \frac{\gamma_A}{2}\left(M_A\, \hat{a}^{\dagger 2} + (N_A + 1)\hat{a}^\dagger \hat{a} + N_A\, \hat{a}\hat{a}^\dagger + M_A\, \hat{a}^2\right)\hat{\rho}(t) \\ & \frac{\gamma_B}{2}\left(M_B\, \hat{b}^{\dagger 2} + (N_B + 1)\hat{b}^\dagger \hat{b} + N_B\, \hat{b}\hat{b}^\dagger + M_B\, \hat{b}^2\right)\hat{\rho}(t).\end{aligned} \tag{2.21}$$

Moreover, using the fact that

$$\int_0^t dt'\, \hat{H}_{SR}(t)\hat{H}_{SR}(t') = \int_0^t dt'\, \hat{H}_{SR}(t')\hat{H}_{SR}(t),$$

one readily gets the result

$$\hat{\Gamma}_2 = \frac{\gamma_A}{2}\left(M_A\,\hat{\rho}(t)\hat{a}^{\dagger 2} + (N_A+1)\hat{\rho}(t)\hat{a}^{\dagger}\hat{a} + N_A\,\hat{\rho}(t)\hat{a}\hat{a}^{\dagger} + M_A\,\hat{\rho}(t)\hat{a}^2\right)$$
$$\frac{\gamma_B}{2}\left(M_B\,\hat{\rho}(t)\hat{b}^{\dagger 2} + (N_B+1)\hat{\rho}(t)\hat{b}^{\dagger}\hat{b} + N_B\,\hat{\rho}(t)\hat{b}\hat{b}^{\dagger} + M_B\,\hat{\rho}(t)\hat{b}^2\right). \qquad (2.22)$$

Applying once more (2.6), expression (2.8c) can be rewritten as

$$\hat{\Gamma}_3 = -\int_0^t dt' Tr_R \left[\sum_j \lambda_j \left(\hat{a}^{\dagger}\hat{A}_j e^{i(\omega_a-\omega_j)t} - \hat{a}\hat{A}_j^{\dagger}\, e^{-i(\omega_a-\omega_j)t}\right)\right.$$
$$\left. + \sum_k \lambda_k \left(\hat{b}^{\dagger}\hat{B}_k\, e^{i(\omega_b-\omega_k)t} - \hat{b}\hat{B}_k^{\dagger}\, e^{-i(\omega_b-\omega_k)t}\right)\right]\hat{\rho}(t)\hat{R}$$
$$\times \left[\sum_l \lambda_l \left(\hat{a}^{\dagger}\hat{A}_l\, e^{i(\omega_a-\omega_l)t'} - \hat{a}\hat{A}_l^{\dagger}\, e^{-i(\omega_a-\omega_l)t'}\right)\right.$$
$$\left. + \sum_m \lambda_m \left(\hat{b}^{\dagger}\hat{B}_m\, e^{i(\omega_b-\omega_m)t'} - \hat{b}\hat{B}_m^{\dagger}\, e^{-i(\omega_b-\omega_m)t'}\right)\right],$$

from which follows

$$\hat{\Gamma}_3 = -\int_0^t dt'\, Tr_R \left[\sum_{j,l} \lambda_j\lambda_l \left(\hat{a}^{\dagger}\hat{\rho}(t)\hat{a}^{\dagger}\hat{A}_j\hat{R}\,\hat{A}_l\, e^{i(\omega_a-\omega_j)t+i(\omega_a-\omega_l)t'}\right.\right.$$
$$\left. -\hat{a}^{\dagger}\hat{\rho}(t)\hat{a}\hat{A}_j\hat{R}\,\hat{A}_l^{\dagger}\, e^{i(\omega_a-\omega_j)t-i(\omega_a-\omega_l)t'}\right)$$
$$+\sum_{j,m} \lambda_j\lambda_m \left(\hat{a}^{\dagger}\hat{\rho}(t)\hat{b}^{\dagger}\hat{A}_j\hat{R}\,\hat{B}_m\, e^{i(\omega_a-\omega_j)t+i(\omega_b-\omega_m)t'} - \hat{a}^{\dagger}\hat{\rho}(t)\hat{b}\hat{A}_j\hat{R}\,\hat{B}_m^{\dagger}\, e^{i(\omega_a-\omega_j)t-i(\omega_b-\omega_m)t'}\right)$$
$$-\sum_{j,l} \lambda_j\lambda_l \left(\hat{a}\hat{\rho}(t)\hat{a}^{\dagger}\hat{A}_j^{\dagger}\hat{R}\hat{A}_l\, e^{-i(\omega_a-\omega_j)t+i(\omega_a-\omega_l)t'} - \hat{a}\hat{\rho}(t)\hat{a}\hat{A}_j^{\dagger}\hat{R}\,\hat{A}_l^{\dagger}\, e^{-i(\omega_a-\omega_j)t-i(\omega_a-\omega_l)t'}\right)$$
$$-\sum_{j,m} \lambda_j\lambda_m \left(\hat{a}\hat{\rho}(t)\hat{b}^{\dagger}\hat{A}_j^{\dagger}\hat{R}\,\hat{B}_m\, e^{-i(\omega_a-\omega_j)t+i(\omega_b-\omega_m)t'} - \hat{a}\hat{\rho}(t)\hat{b}\hat{A}_j^{\dagger}\hat{R}\,\hat{B}_m^{\dagger}\, e^{-i(\omega_a-\omega_j)t-i(\omega_b-\omega_m)t'}\right)$$
$$+\sum_{k,l} \lambda_k\lambda_l \left(\hat{b}^{\dagger}\hat{\rho}(t)\hat{a}^{\dagger}\hat{B}_k\hat{R}\,\hat{A}_l\, e^{i(\omega_b-\omega_k)t+i(\omega_b-\omega_l)t'} - \hat{b}^{\dagger}\hat{\rho}(t)\hat{a}\hat{B}_k\hat{R}\,\hat{A}_l^{\dagger}\, e^{i(\omega_a-\omega_k)t-i(\omega_a-\omega_l)t'}\right)$$
$$+\sum_{k,m} \lambda_k\lambda_m \left(\hat{b}^{\dagger}\hat{\rho}(t)\hat{b}^{\dagger}\hat{B}_k\hat{R}\,\hat{B}_m\, e^{i(\omega_b-\omega_k)t+i(\omega_b-\omega_m)t'} - \hat{b}^{\dagger}\hat{\rho}(t)\hat{b}\hat{B}_k\hat{R}\,\hat{B}_m^{\dagger}\, e^{i(\omega_b-\omega_k)t-i(\omega_b-\omega_m)t'}\right)$$
$$-\sum_{k,l} \lambda_k\lambda_l \left(\hat{b}\hat{\rho}(t)\hat{a}^{\dagger}\hat{B}_k^{\dagger}\hat{R}\,\hat{A}_l\, e^{-i(\omega_b-\omega_k)t+i(\omega_a-\omega_l)t'} - \hat{b}\hat{\rho}(t)\hat{a}\hat{B}_K^{\dagger}\hat{R}\,\hat{A}_l^{\dagger}\, e^{-i(\omega_b-\omega_k)t-i(\omega_a-\omega_l)t'}\right)$$
$$-\sum_{k,m} \lambda_k\lambda_m \left(\hat{b}\hat{\rho}(t)\hat{b}^{\dagger}\hat{B}_k^{\dagger}\hat{R}\,\hat{B}_m\, e^{-i(\omega_b-\omega_k)t+i(\omega_b-\omega_m)t'}\right.$$
$$\left.\left. -\hat{b}\hat{\rho}(t)\hat{b}\hat{B}_k^{\dagger}\hat{R}\,\hat{B}_m^{\dagger}\, e^{-i(\omega_b-\omega_k)t-i(\omega_b-\omega_m)t'}\right)\right]. \qquad (2.23)$$

Applying the cyclic property of the trace and making use of relations (2.10), (2.11) and (2.14), expression (2.23) can be put in the form

$$\hat{\Gamma}_3 = \Big(I_{1a}\, M_A\, \hat{a}^\dagger \hat{\rho}(t)\hat{a}^\dagger + I_{2a} N_A\, \hat{a}^\dagger \hat{\rho}(t)\hat{a} + I_{3a}(N_A+1)\hat{a}\hat{\rho}(t)\hat{a}^\dagger + I_{4a}\, M_A\, \hat{a}\hat{\rho}(t)\hat{a}\Big)$$

$$\Big(I_{1b}\, M_B\, \hat{b}^\dagger \hat{\rho}(t)\hat{b}^\dagger + I_{2b} N_B\, \hat{b}^\dagger \hat{\rho}(t)\hat{b} + I_{3b}(N_B+1)\hat{b}\hat{\rho}(t)\hat{b}^\dagger + I_{4b}\, M_B\, \hat{b}\hat{\rho}(t)\hat{b}\Big)$$

so that, in view of (2.19) and (2.20), we have

$$\hat{\Gamma}_3 = \frac{\gamma_A}{2}\Big(M_A\, \hat{a}^\dagger \hat{\rho}(t)\hat{a}^\dagger + N_A\, \hat{a}^\dagger \hat{\rho}(t)\hat{a} + (N_A+1)\hat{a}\hat{\rho}(t)\hat{a}^\dagger + M_A\, \hat{a}\hat{\rho}(t)\hat{a}\Big)$$

$$\frac{\gamma_B}{2}\Big(M_B\, \hat{b}^\dagger \hat{\rho}(t)\hat{b}^\dagger N_B\, \hat{b}^\dagger \hat{\rho}(t)\hat{b} + (N_B+1)\hat{b}\hat{\rho}(t)\hat{b}^\dagger + M_B\, \hat{b}\hat{\rho}(t)\hat{b}\Big). \tag{2.24}$$

Similarly, one can easily establish that

$$\hat{\Gamma}_4 = \frac{\gamma_A}{2}\Big(M_A\, \hat{a}^\dagger \hat{\rho}(t)\hat{a}^\dagger + N_A\, \hat{a}^\dagger \hat{\rho}(t)\hat{a} + (N_A+1)\hat{a}\hat{\rho}(t)\hat{a}^\dagger + M_A\, \hat{a}\hat{\rho}(t)\hat{a}\Big)$$

$$\frac{\gamma_B}{2}\Big(M_B\, \hat{b}^\dagger \hat{\rho}(t)\hat{b}^\dagger N_B\, \hat{b}^\dagger \hat{\rho}(t)\hat{b} + (N_B+1)\, \hat{b}\hat{\rho}(t)\hat{b}^\dagger + M_B\, \hat{b}\hat{\rho}(t)\hat{b}\Big). \tag{2.25}$$

Now combining expressions (2.21), (2.22), (2.24) and (2.25) we arrive at

$$\frac{d\hat{\rho}(t)}{dt} = \frac{\gamma_A}{2}(N_A+1)\Big(2\,\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger\hat{a}\Big) + \frac{\gamma_A N_A}{2}\Big(2\,\hat{a}^\dagger\hat{\rho}(t)\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}(t)$$

$$-\hat{\rho}(t)\hat{a}\hat{a}^\dagger\Big) + \frac{\gamma_A M_A}{2}\Big(2\,\hat{a}^\dagger\hat{\rho}(t)\hat{a}^\dagger + 2\,\hat{a}\hat{\rho}(t)\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^{\dagger 2} - \hat{a}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^2\Big)$$

$$\frac{\gamma_B}{2}(N_B+1)\Big(2\,\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^\dagger\hat{b}\Big) + \frac{\gamma_B N_B}{2}\Big(2\,\hat{b}^\dagger\hat{\rho}(t)\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho}(t) - \hat{\rho}(t)\hat{b}\hat{b}^\dagger\Big)$$

$$+\frac{\gamma_B M_B}{2}\Big(2\,\hat{b}^\dagger\hat{\rho}(t)\hat{b}^\dagger + 2\,\hat{b}\hat{\rho}(t)\hat{b} - \hat{b}^{\dagger 2}\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^{\dagger 2} - \hat{b}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^2\Big). \tag{2.26}$$

This is the equation of evolution for the reduced density operator of the two-mode light in a cavity coupled to two independent squeezed vacuum reservoirs. We note that the effects of the reservoirs are incorporated via the parameters N_A, M_A, N_B and M_B.

We now consider the two-mode light in the cavity to be the signal-idler modes produced by a nondegenerate parametric oscillator. In a nondegenerate parametric oscillator a strong pump light of frequency ω_0 interacts with a nonlinear-medium (crystal) inside the cavity and gives rise to a two-mode squeezed light (the signal-idler modes) with frequencies ω_a and ω_b such that $w_0 = w_a + w_b$. We take w_a and w_b to be the frequencies of the signal and idler modes, respectively. With the pump mode treated classically (in which the amplitude of the pump mode is assumed to be real and constant), the nondegenerate parametric oscillator is described in the interaction picture by the Hamiltonian

$$\hat{H}_s = i\hbar\kappa\gamma_0 \left(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger\right), \tag{2.27}$$

where κ is the coupling constant and γ_0 is the amplitude of the pump mode. Hence the master equation for the signal-idler modes coupled to two independent squeezed vacuum reservoirs, in view of (2.26) and (2.27), has the form

$$\begin{aligned}
\frac{d\hat{\rho}(t)}{dt} &= \kappa\gamma_0 \left(\hat{a}\hat{b}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}\hat{b} + \hat{\rho}(t)\hat{a}^\dagger\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}(t)\right) \\
&+\frac{\gamma_a}{2}(N_A+1)\left(2\,\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger\hat{a}\right) + \frac{\gamma_A N_A}{2}\left(2\,\hat{a}^\dagger\hat{\rho}(t)\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}(t) - \hat{\rho}(t)\hat{a}\hat{a}^\dagger\right) \\
&+\frac{\gamma_A M_A}{2}\left(2\,\hat{a}^\dagger\hat{\rho}(t)\hat{a}^\dagger + 2\,\hat{a}\hat{\rho}(t)\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^{\dagger 2} - \hat{\rho}(t)\hat{a}^2 - \hat{a}^2\hat{\rho}(t)\right) \\
&\frac{\gamma_B}{2}(N_B+1)\left(2\,\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^\dagger\hat{b}\right) + \frac{\gamma_B N_B}{2}\left(2\,\hat{b}^\dagger\hat{\rho}(t)\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho}(t) - \hat{\rho}(t)\hat{b}\hat{b}^\dagger\right) \\
&+\frac{\gamma_B M_B}{2}\left(2\,\hat{b}^\dagger\hat{\rho}(t)\hat{b}^\dagger + 2\,\hat{b}\hat{\rho}(t)\hat{b} - \hat{b}^{\dagger 2}\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^{\dagger 2} - \hat{b}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{b}^2\right).
\end{aligned} \tag{2.28}$$

This master equation, which is the basis of our analysis, describes the interaction inside the cavity as well as the interaction of the signal-idler modes produced by the nondegenerate parametric oscillator and the squeezed vacuum reservoirs via the partially transmitting mirror.

2.2 The Fokker-Planck Equation

In this section we derive the Fokker-Plank equation for the Q-function and then obtain the solution of the resulting equation. In order to obtain the Fokker-Planck equation for the Q-function corresponding to the master equation (2.28), we first put all terms in normal order. Applying the commutation relations

$$\left[\hat{a}, f(\hat{a},\hat{a}^\dagger)\right] = \frac{\partial f(\hat{a},\hat{a}^\dagger)}{\partial\hat{a}^\dagger}, \tag{2.29a}$$

$$\left[\hat{a}^\dagger, f(\hat{a},\hat{a}^\dagger)\right] = -\frac{\partial f(\hat{a},\hat{a}^\dagger)}{\partial\hat{a}}, \tag{2.29b}$$

we see that

$$\hat{a}\hat{\rho} = \hat{\rho}\hat{a} + \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}, \tag{2.30a}$$

$$\hat{\rho}\hat{a}^\dagger = \hat{a}^\dagger\hat{\rho} + \frac{\partial\hat{\rho}}{\partial\hat{a}}, \tag{2.30b}$$

where the density operator

$$\hat{\rho} = \hat{\rho}(\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger, t)$$

is considered to be in normal order. On the basis of (2.30) and the relation

$$[\hat{a}, \hat{b}\hat{c}] = \hat{b}[\hat{a}, \hat{c}] + [\hat{a}, \hat{b}]\hat{c},$$

we see that

$$\hat{a}\hat{b}\hat{\rho} = \hat{a}\hat{\rho}\hat{b} + \hat{a}\frac{\partial\hat{\rho}}{\partial\hat{b}^\dagger}. \tag{2.31a}$$

Furthermore, employing the identity

$$\hat{a}\frac{\partial\hat{\rho}}{\partial\hat{b}^\dagger} = \frac{\partial\hat{\rho}}{\partial\hat{b}^\dagger}\hat{a} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}^\dagger\partial\hat{b}^\dagger} \tag{2.31b}$$

along with (2.30a), one can put (2.31a) in the normal order as

$$\hat{a}\hat{b}\hat{\rho} = \hat{\rho}\hat{a}\hat{b} + \frac{\partial\hat{\rho}}{\partial\hat{b}^\dagger}\hat{a} + \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}\hat{b} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}^\dagger\partial\hat{b}^\dagger}. \tag{2.31c}$$

Similarly, applying (2.30b) and the relation

$$\frac{\partial\hat{\rho}}{\partial\hat{a}}\hat{b}^\dagger = \hat{b}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{b}}, \tag{2.32a}$$

one readily obtains

$$\hat{\rho}\hat{a}^\dagger\hat{b}^\dagger = \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} + \hat{a}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{b}} + \hat{b}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{b}}. \tag{2.32b}$$

Then combination of expressions (2.31c) and (2.32b) leads to the result

$$\hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} = \frac{\partial\hat{\rho}}{\partial\hat{b}^\dagger}\hat{a} + \hat{a}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{b}} + \hat{b}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial\hat{\rho}}{\partial\hat{a}^\dagger}\hat{b} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{b}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}^\dagger\partial\hat{b}^\dagger}. \tag{2.33}$$

Applying (2.29) once more, one gets

$$\hat{a}\frac{\partial\hat{\rho}}{\partial\hat{a}} = \frac{\partial\hat{\rho}}{\partial\hat{a}}\hat{a} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{a}^\dagger}, \tag{2.34a}$$

$$\frac{\partial\hat{\rho}}{\partial\hat{a}}\hat{a}^\dagger = \hat{a}^\dagger\frac{\partial\hat{\rho}}{\partial\hat{a}} + \frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{a}^\dagger}, \tag{2.34b}$$

from which we can easily show that

$$\hat{\rho}\hat{a}^\dagger\hat{a} = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\partial}{\partial\hat{a}^\dagger}(\hat{a}^\dagger\hat{\rho}) - \frac{\partial^2\hat{\rho}}{\partial\hat{a}\partial\hat{a}^\dagger}, \tag{2.34c}$$

$$\hat{a}^\dagger \hat{a}\hat{\rho} = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) - \frac{\partial^2 \hat{\rho}}{\partial \hat{a}\partial \hat{a}^\dagger}. \tag{2.34d}$$

In view of the last two expressions, we obtain

$$2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger \hat{a} = \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}) + 2\frac{\partial^2 \hat{\rho}}{\partial \hat{a}\partial \hat{a}^\dagger}. \tag{2.35}$$

Since

$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1, \tag{2.36}$$

we get

$$\hat{a}\hat{a}^\dagger \hat{\rho} + \hat{\rho}\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger \hat{a} + 2\hat{\rho}. \tag{2.37a}$$

On account of (2.30), the above expression can be put in the form

$$\hat{a}\hat{a}^\dagger \hat{\rho} + \hat{\rho}\hat{a}\hat{a}^\dagger = 2\hat{a}^\dagger \hat{\rho}\hat{a} + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}) + \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}), \tag{2.37b}$$

which can be rearranged to give

$$2\hat{a}^\dagger \hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger \hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger = -\left(\frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}) + \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a})\right). \tag{2.38}$$

Employing the relations

$$\hat{a}^\dagger \hat{\rho}\hat{a}^\dagger = \hat{a}^{\dagger 2}\hat{\rho} + \hat{a}^\dagger \frac{\partial \hat{\rho}}{\partial \hat{a}}, \tag{2.39a}$$

$$\hat{a}\hat{\rho}\hat{a} = \hat{\rho}\hat{a}^2 + \frac{\partial \hat{\rho}}{\partial \hat{a}^\dagger}\hat{a}, \tag{2.39b}$$

$$\hat{\rho}\hat{a}^{\dagger 2} = \hat{a}^{\dagger 2}\hat{\rho} + 2\hat{a}^\dagger \frac{\partial \hat{\rho}}{\partial \hat{a}} + \frac{\partial^2 \hat{\rho}}{\partial \hat{a}^2}, \tag{2.39c}$$

$$\hat{a}^2 \hat{\rho} = \hat{\rho}\hat{a}^2 + 2\frac{\partial \hat{\rho}}{\partial \hat{a}^\dagger}\hat{a} + \frac{\partial^2 \hat{\rho}}{\partial \hat{a}^{\dagger 2}}, \tag{2.39d}$$

one can readily show that

$$2\hat{a}^\dagger \hat{\rho}\hat{a}^\dagger + 2\hat{a}\hat{\rho}\hat{a} - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 = -\left(\frac{\partial^2 \hat{\rho}}{\partial \hat{a}^2} + \frac{\partial^2 \hat{\rho}}{\partial \hat{a}^{\dagger 2}}\right). \tag{2.40}$$

Now combining expressions (2.33), (2.35), (2.38), (2.40) and similar expressions associated with $\hat{b}$ and $\hat{b}^\dagger$, the master equation (2.28) has the form

$$\frac{d\hat{\rho}(t)}{dt} = \kappa\gamma_0 \left[\frac{\partial \hat{\rho}}{\partial \hat{b}^\dagger}\hat{a} + \hat{a}^\dagger \frac{\partial \hat{\rho}}{\partial \hat{b}} + \hat{b}^\dagger \frac{\partial \hat{\rho}}{\partial \hat{a}} + \frac{\partial \hat{\rho}}{\partial \hat{a}^\dagger}\hat{b} + \frac{\partial^2 \hat{\rho}}{\partial \hat{a}\partial \hat{b}} + \frac{\partial^2 \hat{\rho}}{\partial \hat{a}^\dagger \partial \hat{b}^\dagger}\right]$$

$$+\frac{\gamma_A}{2}(N_A + 1)\left[\frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a}) + \frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}) + 2\frac{\partial^2 \hat{\rho}}{\partial \hat{a}\partial \hat{a}^\dagger}\right] - \frac{\gamma_A N_A}{2}\left[\frac{\partial}{\partial \hat{a}^\dagger}(\hat{a}^\dagger \hat{\rho}) + \frac{\partial}{\partial \hat{a}}(\hat{\rho}\hat{a})\right] -$$

$$\frac{\gamma_A}{2}M_A\left[\frac{\partial^2\hat{\rho}}{\partial\hat{a}^2}+\frac{\partial^2\hat{\rho}}{\partial\hat{a}^{\dagger 2}}\right]+\frac{\gamma_B}{2}(N_B+1)\left[\frac{\partial}{\partial\hat{b}}(\hat{\rho}\hat{b})+\frac{\partial}{\partial\hat{b}^\dagger}(\hat{b}^\dagger\hat{\rho})+2\frac{\partial^2\hat{\rho}}{\partial\hat{b}\partial\hat{b}^\dagger}\right]$$
$$-\frac{\gamma_B N_B}{2}\left[\frac{\partial}{\partial\hat{b}^\dagger}(\hat{b}^\dagger\hat{\rho})+\frac{\partial}{\partial\hat{b}}(\hat{\rho}\hat{b})\right]-\frac{\gamma_B}{2}M_B\left[\frac{\partial^2\hat{\rho}}{\partial\hat{b}^2}+\frac{\partial^2\hat{\rho}}{\partial\hat{b}^{\dagger 2}}\right]. \tag{2.41}$$

In order to transform the above master equation into a c-number Fokker-Planck equation for the Q-function, we multiply (2.41) on the left by $\langle\alpha,\beta\mid$ and on the right by $\mid\alpha,\beta\rangle$, so that

$$\frac{\partial Q}{\partial t}=\left[\kappa\gamma_0\left(\frac{\partial^2}{\partial\alpha\partial\beta}+\frac{\partial^2}{\partial\alpha^*\partial\beta^*}+\frac{\partial}{\partial\beta^*}\alpha+\frac{\partial}{\partial\beta}\alpha^*+\frac{\partial}{\partial\alpha^*}\beta+\frac{\partial}{\partial\alpha}\beta^*\right)\right.$$
$$+\gamma_A(N_A+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}+\frac{\gamma_A}{2}\left(\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right)-\frac{\gamma_A M_A}{2}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\alpha^{*2}}\right)$$
$$\left.+\gamma_B(N_B+1)\frac{\partial^2}{\partial\beta\partial\beta^*}+\frac{\gamma_B}{2}\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)-\frac{\gamma_B M_B}{2}\left(\frac{\partial^2}{\partial\beta^2}+\frac{\partial^2}{\partial\beta^{*2}}\right)\right]Q, \tag{2.42}$$

where

$$Q=Q(\alpha^*,\alpha,\beta^*,\beta,t)=\frac{1}{\pi^2}\langle\alpha,\beta\mid\hat{\rho}(\hat{a}^\dagger,\hat{a},\hat{b}^\dagger,\hat{b},t)\mid\alpha,\beta\rangle.$$

This is the Fokker-Planck equation for the Q-function for the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two uncorrelated squeezed vacuum reservoirs.

We now proceed to obtain the solution of this equation. Introducing Cartesian coordinates defined by

$$\alpha=x_1+iy_1, \tag{2.43a}$$

$$\alpha^*=x_1-iy_1, \tag{2.43b}$$

$$\beta=x_2+iy_2, \tag{2.43c}$$

$$\beta^*=x_2-iy_2, \tag{2.43d}$$

we note that

$$x_1=\frac{1}{2}(\alpha+\alpha^*), \tag{2.43e}$$

$$y_1=\frac{1}{2}(\alpha-\alpha^*), \tag{2.43f}$$

$$x_2=\frac{1}{2}(\beta+\beta^*), \tag{2.43g}$$

$$y_2=\frac{1}{2}(\beta-\beta^*). \tag{2.43h}$$

One can easily show that

$$\frac{\partial}{\partial\alpha}=\frac{1}{2}(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial y_1}), \tag{2.44a}$$

$$\frac{\partial}{\partial\alpha^*}=\frac{1}{2}(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial y_1}), \tag{2.44b}$$

$$\frac{\partial}{\partial\beta}=\frac{1}{2}(\frac{\partial}{\partial x_2}-i\frac{\partial}{\partial y_2}), \tag{2.44c}$$

$$\frac{\partial}{\partial\beta^*}=\frac{1}{2}(\frac{\partial}{\partial x_2}+i\frac{\partial}{\partial y_2}). \tag{2.44d}$$

Making use of these relations, it can be verified that

$$\frac{\partial^2}{\partial\alpha\partial\beta}=\frac{1}{4}\left(\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial y_1\partial y_2}-i\frac{\partial^2}{\partial x_1\partial y_2}-i\frac{\partial^2}{\partial y_1\partial x_2}\right), \tag{2.45a}$$

$$\frac{\partial}{\partial\beta^*}\alpha=\frac{1}{2}\left(\frac{\partial}{\partial x_2}x_1-\frac{\partial}{\partial y_2}y_1+i\frac{\partial}{\partial y_2}x_1+i\frac{\partial}{\partial x_2}y_1\right), \tag{2.45b}$$

$$\frac{\partial}{\partial\alpha^*}\beta=\frac{1}{2}\left(\frac{\partial}{\partial x_1}x_2-\frac{\partial}{\partial y_1}y_2+i\frac{\partial}{\partial x_1}y_2+i\frac{\partial}{\partial y_1}x_2\right), \tag{2.45c}$$

$$\frac{\partial^2}{\partial\alpha\partial\alpha^*}=\frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}\right), \tag{2.45d}$$

$$\frac{\partial}{\partial\alpha}\alpha=\frac{1}{2}\left(\frac{\partial}{\partial x_1}x_1+\frac{\partial}{\partial y_1}y_1+i\frac{\partial}{\partial x_1}y_1-i\frac{\partial}{\partial y_1}x_1\right), \tag{2.45e}$$

$$\frac{\partial^2}{\partial\alpha^2}=\frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial y_1^2}-2i\frac{\partial^2}{\partial x_1\partial y_1}\right), \tag{2.45f}$$

$$\frac{\partial^2}{\partial\beta\partial\beta^*}=\frac{1}{4}\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2}\right), \tag{2.45g}$$

$$\frac{\partial}{\partial\beta}\beta=\frac{1}{2}\left(\frac{\partial}{\partial x_2}x_2+\frac{\partial}{\partial y_2}y_2+i\frac{\partial}{\partial x_2}y_2-i\frac{\partial}{\partial y_2}x_2\right), \tag{2.45h}$$

$$\frac{\partial^2}{\partial\beta^2}=\frac{1}{4}\left(\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial y_2^2}-2i\frac{\partial^2}{\partial x_2\partial y_2}\right). \tag{2.45i}$$

Thus combination of the these results and their complex conjugates along with expression (2.43), one readily obtains

$$\begin{aligned}\frac{\partial Q}{\partial t}=&\left[\frac{\kappa\gamma_0}{2}\left(\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial y_1\partial y_2}\right)+\kappa\gamma_0\left(\frac{\partial}{\partial x_1}x_2+\frac{\partial}{\partial x_2}x_1-\frac{\partial}{\partial y_1}y_2-\frac{\partial}{\partial y_2}y_1\right)\right.\\&+\frac{\gamma_A(N_A+1)}{4}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}\right)-\frac{\gamma_A M_A}{4}\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial y_1^2}\right)\\&\left.+\frac{\gamma_B(N_B+1)}{4}\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2}\right)-\frac{\gamma_B M_B}{4}\left(\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial y_2^2}\right)\right]Q,\end{aligned} \tag{2.46}$$

where $Q=Q(x_1,x_2,y_1,y_2,t)$.

Next introducing the transformation defined by

$$x_1 = x + u, \tag{2.47a}$$

$$x_2 = x - u, \tag{2.47b}$$

$$y_1 = y + v, \tag{2.47c}$$

$$y_2 = v - y, \tag{2.47d}$$

one easily gets

$$x = \frac{1}{2}(x_1 + x_2), \tag{2.48a}$$

$$u = \frac{1}{2}(x_1 - x_2), \tag{2.48b}$$

$$y = \frac{1}{2}(y_1 - y_2), \tag{2.48c}$$

$$v = \frac{1}{2}(y_1 + y_2). \tag{2.48d}$$

In view of these relations, we have

$$\frac{\partial}{\partial x_1} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \tag{2.49a}$$

$$\frac{\partial}{\partial x_2} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\right), \tag{2.49b}$$

$$\frac{\partial}{\partial y_1} = \frac{1}{2}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial v}\right), \tag{2.49c}$$

$$\frac{\partial}{\partial y_2} = \frac{1}{2}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial y}\right), \tag{2.49d}$$

from which one readily obtains

$$\frac{\partial^2}{\partial x_1 \partial x_2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial u^2}\right), \tag{2.50a}$$

$$\frac{\partial^2}{\partial y_1 \partial y_2} = \frac{1}{4}\left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial y^2}\right), \tag{2.50b}$$

$$\frac{\partial}{\partial x_1}x_2 = \frac{1}{2}\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial x}u + \frac{\partial}{\partial u}x - \frac{\partial}{\partial u}u\right), \tag{2.50c}$$

$$\frac{\partial}{\partial x_2}x_1 = \frac{1}{2}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}u - \frac{\partial}{\partial u}x - \frac{\partial}{\partial u}u\right), \tag{2.50d}$$

$$\frac{\partial}{\partial y_1}y_2 = \frac{1}{2}\left(\frac{\partial}{\partial v}v - \frac{\partial}{\partial v}y + \frac{\partial}{\partial y}v - \frac{\partial}{\partial y}y\right), \tag{2.50e}$$

$$\frac{\partial}{\partial y_2}y_1 = \frac{1}{2}\left(\frac{\partial}{\partial v}v + \frac{\partial}{\partial v}y - \frac{\partial}{\partial y}v - \frac{\partial}{\partial y}y\right), \tag{2.50f}$$

$$\frac{\partial}{\partial x_1}x_1 = \frac{1}{2}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}u + \frac{\partial}{\partial u}x + \frac{\partial}{\partial u}u\right), \tag{2.50g}$$

$$\frac{\partial}{\partial y_1}y_1 = \frac{1}{2}\left(\frac{\partial}{\partial y}y + \frac{\partial}{\partial y}v + \frac{\partial}{\partial v}y + \frac{\partial}{\partial v}v\right), \tag{2.50h}$$

$$\frac{\partial}{\partial x_2}x_2 = \frac{1}{2}\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial x}u - \frac{\partial}{\partial u}x + \frac{\partial}{\partial u}u\right), \tag{2.50i}$$

$$\frac{\partial}{\partial y_2}y_2 = \frac{1}{2}\left(\frac{\partial}{\partial y}y - \frac{\partial}{\partial y}v - \frac{\partial}{\partial v}y + \frac{\partial}{\partial v}v\right), \tag{2.50j}$$

$$\frac{\partial^2}{\partial x_1^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial x \partial u}\right), \tag{2.50k}$$

$$\frac{\partial^2}{\partial y_1^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial v^2} + 2\frac{\partial^2}{\partial y \partial v}\right), \tag{2.50l}$$

$$\frac{\partial^2}{\partial x_2^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} - 2\frac{\partial^2}{\partial x \partial u}\right), \tag{2.50m}$$

$$\frac{\partial^2}{\partial y_2^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial y \partial v}\right). \tag{2.50n}$$

On account of expressions (2.49) and (2.50), the Fokker-Planck equation given by (2.46) can be written as

$$\begin{aligned}\frac{\partial Q}{\partial t} = \Bigg[&\frac{\kappa\gamma_0}{8}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) + \kappa\gamma_0\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y - \frac{\partial}{\partial u}u - \frac{\partial}{\partial v}v\right)\\
&+\left(\frac{\gamma_A(N_A+1)+\gamma_B(N_B+1)}{16}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) +\\
&\left(\frac{\gamma_A(N_A+1)-\gamma_B(N_B+1)}{8}\right)\left(\frac{\partial^2}{\partial x\partial u} + \frac{\partial^2}{\partial y\partial v}\right) + \left(\frac{\gamma_A+\gamma_B}{4}\right)\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial u}u +\right.\\
&\left.\frac{\partial}{\partial y}y + \frac{\partial}{\partial v}v\right) + \left(\frac{\gamma_A-\gamma_B}{4}\right)\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial u}x + \frac{\partial}{\partial y}v + \frac{\partial}{\partial v}y\right) - \left(\frac{\gamma_A M_A + \gamma_B M_B}{16}\right)\\
&\times\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial v^2}\right) + \left(\frac{\gamma_A M_A - \gamma_B M_B}{8}\right)\left(\frac{\partial^2}{\partial x\partial u} - \frac{\partial^2}{\partial y\partial v}\right)\Bigg]Q,\end{aligned} \tag{2.51}$$

where $Q = Q(x, y, u, v, t)$.

Now setting

$$\gamma_A = \gamma_B = \gamma, \tag{2.52a}$$

$$N_A = N_B = N, \tag{2.52b}$$

$$M_A = M_B = M, \tag{2.52c}$$

expression (2.51) takes the form

$$\frac{\partial Q}{\partial t} = \Bigg[\frac{\kappa\gamma_0 + \gamma(N-M+1)}{8}\frac{\partial^2}{\partial x^2} + \frac{\kappa\gamma_0 + \gamma(N+M+1)}{8}\frac{\partial^2}{\partial y^2}$$

$$-\frac{\kappa\gamma_0 - \gamma(N-M+1)}{8}\frac{\partial^2}{\partial u^2} - \frac{\kappa\gamma_0 - \gamma(N+M+1)}{8}\frac{\partial^2}{\partial v^2}$$

$$+\frac{2\kappa\gamma_0 + \gamma}{2}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right) - \frac{2\kappa\gamma_0 - \gamma}{2}\left(\frac{\partial}{\partial u}u + \frac{\partial}{\partial v}v\right)\Bigg] Q. \tag{2.53}$$

In order to solve this differential equation using the propagator method discussed in Ref. [1], we need to transform the above equation into a schrödinger-type equation. This can be achieved upon replacing $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, x, y, u, v\right)$ and $Q(x,y,u,v,t)$ by $(i\hat{p}_x, i\hat{p}_y, i\hat{p}_u, i\hat{p}_v, \hat{x}, \hat{y}, \hat{u}, \hat{v})$ and $\mid Q(t)\rangle$. Hence equation (2.53) becomes

$$i\frac{d \mid Q(t)\rangle}{dt} = i\Bigg[-\frac{\lambda_1}{8}\hat{p}_x^2 - \frac{\lambda_2}{8}\hat{p}_y^2 + \frac{\lambda_3}{8}\hat{p}_u^2 + \frac{\lambda_4}{8}\hat{p}_v^2 + i\frac{\lambda_5}{2}(\hat{p}_x\hat{x} + \hat{p}_y\hat{y})$$

$$-i\frac{\lambda_6}{2}(\hat{p}_u\hat{u} + \hat{p}_v\hat{v})\Bigg] \mid Q(t)\rangle = \hat{H} \mid Q(t)\rangle, \tag{2.54}$$

in which

$$\lambda_1 = \kappa\gamma_0 + \gamma(N-M+1), \tag{2.55a}$$

$$\lambda_2 = \kappa\gamma_0 + \gamma(N+M+1), \tag{2.55b}$$

$$\lambda_3 = \kappa\gamma_0 - \gamma(N-M+1), \tag{2.55c}$$

$$\lambda_4 = \kappa\gamma_0 - \gamma(N+M+1), \tag{2.55d}$$

$$\lambda_5 = 2\kappa\gamma_0 + \gamma, \tag{2.55e}$$

$$\lambda_6 = 2\kappa\gamma_0 - \gamma. \tag{2.55f}$$

A formal solution of (2.54) can be put in the form

$$\mid Q(t)\rangle = \hat{U}(t) \mid Q(0)\rangle, \tag{2.56a}$$

where

$$\hat{U}(t) = exp(-i\hat{H}t) \tag{2.56b}$$

and

$$\hat{H} = -i\frac{\lambda_1}{8}\hat{p}_x^2 - i\frac{\lambda_2}{8}\hat{p}_y^2 + i\frac{\lambda_3}{8}\hat{p}_u^2 + i\frac{\lambda_4}{8}\hat{p}_v^2 - \frac{\lambda_5}{2}(\hat{p}_x\hat{x} + \hat{p}_y\hat{y}) + \frac{\lambda_6}{2}(\hat{p}_u\hat{u} + \hat{p}_v\hat{v}). \tag{2.56c}$$

Upon multiplying (2.56a) by $\langle x, y, u, v \mid$ on the left, we have

$$Q(x, y, u, v, t) = \langle x, y, u, v \mid \hat{U}(t) \mid Q(0)\rangle, \tag{2.57a}$$

where

$$Q(x, y, u, v, t) = \langle x, y, u, v \mid Q(t)\rangle. \tag{2.57b}$$

Introducing the four-dimensional completeness relation for the position eigenstates

$$\hat{I} = \int dx'\, dy'\, du'\, dv' \mid x', y', u', v'\rangle\langle x', y', u', v' \mid, \tag{2.58}$$

in expression (2.57a), we see that

$$Q(x, y, u, v, t) = \int dx'\, dy'\, du'\, dv'\, Q(x, y, u, v, t|x', y', u', v', 0)Q_o(x', y', u', v'), \tag{2.59a}$$

in which

$$Q_0(x', y', u', v') = \langle x', y', u', v'|Q(0)\rangle \tag{2.59b}$$

is the initial Q-function and

$$Q(x, y, u, v, t|x', y', u', v', 0) = \langle x, y, u, v|\hat{U}(t)|x', y', u', v'\rangle \tag{2.59c}$$

is the Q-function propagator.

According to Fesseha [1], the propagator associated with a quadratic Hamiltonian of the form

$$\hat{H}(\hat{x}_1, ...\hat{x}_n, \hat{p}_1, ...\hat{p}_n, t) = \sum_{i=1}^{n} \left[a_i\hat{p}_i^2 + b_i(t)\hat{p}_i\hat{x}_i + c_i(t)\hat{x}_i^2\right] \tag{2.60}$$

is expressible as

$$Q(x_1, ...x_n, t|x_1', ...x_n', 0) = \left[\frac{i}{2\pi}\right]^{\frac{n}{2}} \prod_{j=1}^{n} \sqrt{\frac{\partial^2 S_c}{\partial x_j \partial x_j'}} \exp\left[-\xi \int_0^t b_j(t')dt' + iS_c\right], \tag{2.61}$$

where S_c is the classical action, ξ is a parameter connected with operator ordering and a_j is a constant different from zero. On comparing (2.60) and (2.56c), we see that

$$(x_1, x_2, x_3, x_4) = (x, y, u, v),$$

$$c_x = c_y = c_u = c_v = 0,$$

$$b_x = b_y = -\frac{\lambda_5}{2},$$

$$b_u = b_v = \frac{\lambda_6}{2}$$

and for the antistandard operator ordering $\xi = \frac{1}{2}$. Thus the Q-function propagator associated with the Hamiltonian (2.56c) is expressible as

$$Q(x,y,u,v,t|x',y',u',v',0) = \frac{1}{4\pi^2}\left[\frac{\partial^2 S_c}{\partial x'\partial x}\frac{\partial^2 S_c}{\partial y'\partial y}\frac{\partial^2 S_c}{\partial u'\partial u}\frac{\partial^2 S_c}{\partial v'\partial v}\right]^{\frac{1}{2}} e^{iS_c + \frac{(\lambda_5-\lambda_6)}{2}t}. \tag{2.62}$$

In order to obtain the explicit form of this expression, we should first determine the classical action. To this end, we note that the Hamiltonian function corresponding to the quantum Hamiltonian (2.56c) is given by

$$H = -i\frac{\lambda_1}{8}p_x^2 - i\frac{\lambda_2}{8}p_y^2 + i\frac{\lambda_3}{8}p_u^2 + i\frac{\lambda_4}{8}p_v^2 - \frac{\lambda_5}{2}(p_x x + p_y y) + \frac{\lambda_6}{2}(p_u u + p_v v). \tag{2.63}$$

With the help of the Lagrangian

$$L = \sum_i \dot{x}_i p_i - H \tag{2.64}$$

and the Hamilton equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \tag{2.65}$$

one can readily show that

$$L = \frac{2i}{\lambda_1}(\dot{x} + \frac{\lambda_5}{2}x)^2 + \frac{2i}{\lambda_2}(\dot{y} + \frac{\lambda_5}{2}y)^2 - \frac{2i}{\lambda_3}(\dot{u} - \frac{\lambda_6}{2}u)^2 - \frac{2i}{\lambda_4}(\dot{v} - \frac{\lambda_6}{2}v)^2. \tag{2.66}$$

Applying the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \tag{2.67}$$

along with (2.66), it can be easily verified that

$$\ddot{x} - (\frac{\lambda_5}{2})^2 x = 0, \tag{2.68a}$$

$$\ddot{y} - (\frac{\lambda_5}{2})^2 y = 0, \tag{2.68b}$$

$$\ddot{u} - (\frac{\lambda_6}{2})^2 u = 0, \tag{2.68c}$$

$$\ddot{v} - (\frac{\lambda_6}{2})^2 v = 0. \tag{2.68d}$$

The solutions of these differential equations can be written as

$$x(t) = a_1 e^{\frac{\lambda_5}{2}t} + a_2 e^{-\frac{\lambda_5}{2}t}, \tag{2.69a}$$

$$y(t) = b_1 e^{\frac{\lambda_5}{2}t} + b_2 e^{-\frac{\lambda_5}{2}t}, \tag{2.69b}$$

$$u(t) = c_1 e^{\frac{\lambda_6}{2}t} + c_2 e^{-\frac{\lambda_6}{2}t}, \tag{2.69c}$$

$$v(t) = d_1 e^{\frac{\lambda_6}{2}t} + d_2 e^{-\frac{\lambda_6}{2}t}. \tag{2.69d}$$

Now substituting these expressions and the corresponding first order time derivatives into (2.66), the Lagrangian takes the form

$$L = 2i\lambda_5^2 \left(\frac{a_1^2}{\lambda_1} + \frac{b_1^2}{\lambda_2}\right) e^{\lambda_5 t} - 2i\lambda_6^2 \left(\frac{c_2^2}{\lambda_3} + \frac{d_2^2}{\lambda_4}\right) e^{-\lambda_6 t}. \tag{2.70}$$

On account of the above result, the classical action defined by

$$S_c = \int_0^T L(t)dt$$

takes the form

$$S_c = 2i\lambda_5 \left(\frac{a_1^2}{\lambda_1} + \frac{b_1^2}{\lambda_2}\right) \left(e^{\lambda_5 T} - 1\right) + 2i\lambda_6 \left(\frac{c_2^2}{\lambda_3} + \frac{d_2^2}{\lambda_4}\right) \left(e^{-\lambda_6 T} - 1\right). \tag{2.71}$$

Applying the boundary conditions $x_i(0) = x_i'$ and $x_i(T) = x_i''$ in (2.69), one gets

$$a_1 = \frac{x'' e^{\frac{\lambda_5}{2}T} - x'}{e^{\lambda_5 T} - 1}, \tag{2.72a}$$

$$b_1 = \frac{y'' e^{\frac{\lambda_5}{2}T} - y'}{e^{\lambda_5 T} - 1}, \tag{2.72b}$$

$$c_2 = \frac{u'' e^{\frac{-\lambda_6}{2}T} - u'}{e^{-\lambda_6 T} - 1}, \tag{2.72c}$$

$$d_2 = \frac{v'' e^{\frac{-\lambda_6}{2}T} - v'}{e^{-\lambda_6 T} - 1}. \tag{2.72d}$$

Inserting the above expressions into (2.71) and replacing (x'', y'', u'', v'', T) by (x, y, u, v, t) we get

$$S_c = 2i\lambda_5 \left[\frac{\left(x' - x\, e^{\frac{\lambda_5}{2}t}\right)^2}{\lambda_1 \left(e^{\lambda_5 t} - 1\right)} + \frac{\left(y' - y\, e^{\frac{\lambda_5}{2}t}\right)^2}{\lambda_2 \left(e^{\lambda_5 t} - 1\right)}\right]$$
$$+ 2i\lambda_6 \left[\frac{\left(u' - u\, e^{\frac{-\lambda_6}{2}t}\right)^2}{\lambda_3 \left(e^{-\lambda_6 t} - 1\right)} + \frac{\left(v' - v\, e^{\frac{-\lambda_6}{2}t}\right)^2}{\lambda_4 \left(e^{-\lambda_6 t} - 1\right)}\right], \tag{2.73}$$

so that employing this relation, we obtain the following results:

$$\frac{\partial^2 S_c}{\partial x \partial x'} = -\frac{4i\lambda_5\, e^{\frac{\lambda_5}{2}t}}{\lambda_1 \left(e^{\lambda_5 t} - 1\right)}, \tag{2.74a}$$

$$\frac{\partial^2 S_c}{\partial y \partial y'} = -\frac{4i\lambda_5\, e^{\frac{\lambda_5}{2}t}}{\lambda_2\,(e^{\lambda_5 t}-1)}, \tag{2.74b}$$

$$\frac{\partial^2 S_c}{\partial u \partial u'} = -\frac{4i\lambda_6\, e^{-\frac{\lambda_6}{2}t}}{\lambda_3\,(e^{-\lambda_6 t}-1)}, \tag{2.74c}$$

$$\frac{\partial^2 S_c}{\partial v \partial v'} = -\frac{4i\lambda_6 e^{-\frac{\lambda_6}{2}t}}{\lambda_4\,(e^{-\lambda_6 t}-1)}. \tag{2.74d}$$

Thus, in view of (2.74), the Q-function propagator (2.62) takes the form

$$Q(x,y,u,v,t|x',y',u',v',0) = \frac{4\lambda_5\lambda_6\, e^{(\lambda_5-\lambda_6)t}}{\pi^2\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}(e^{\lambda_5 t}-1)(e^{-\lambda_6 t}-1)}$$

$$\times\, exp\left(-\frac{2\lambda_5}{(e^{\lambda_5 t}-1)}\left[\frac{x'^2-2x\, e^{\frac{\lambda_5}{2}t}x'+x^2e^{\lambda_5 t}}{\lambda_1}+\frac{y'^2-2y\, e^{\frac{\lambda_5}{2}t}y'+y^2e^{\lambda_5 t}}{\lambda_2}\right]\right.$$

$$\left.-\frac{2\lambda_6}{(e^{-\lambda_6 t}-1)}\left[\frac{u'^2-2u\, e^{\frac{-\lambda_6}{2}t}u'+u^2e^{-\lambda_6 t}}{\lambda_3}+\frac{v'^2-2v\, e^{\frac{-\lambda_6}{2}t}v'+v^2e^{-\lambda_6 t}}{\lambda_4}\right]\right). \tag{2.75}$$

Considering the signal-idler modes produced by the nondegenerate parametric oscillator to be initially in a two-mode vacuum state, we see that the initial Q-function is

$$Q_0(\alpha',\beta') = \frac{1}{\pi^2}\langle\alpha',\beta'|0,0\rangle\langle 0,0|\alpha',\beta'\rangle = exp(-\alpha'^*\alpha'-\beta'^*\beta'), \tag{2.76}$$

and in terms of the Cartesian variables (2.43a-2.43d), this equation becomes

$$Q_0(x_1',x_2',y_1',y_2') = \frac{1}{\pi^2}exp\left[-\left(x_1'^2+x_2'^2+y_1'^2+y_2'^2\right)\right]. \tag{2.77}$$

Furthermore, in terms of x', y', u' and v', one can write

$$\int dx_1'\, dx_2'\, dy_1'\, dy_2'\, Q_0(x_1',x_2',y_1',y_2') = \int dx'\, dy'\, du'\, dv'\, Q_0(x',y',u',v'),$$

in which

$$Q_0(x',y',u',v') = \frac{|J|}{\pi^2}exp\left[-2\left(x'^2+y'^2+u'^2+v'^2\right)\right]$$

and J is the Jacobian of the transformation given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial x} & \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} & \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial y} & \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix}.$$

Making use of (2.47) in the above expression, one can easily show that $|J| = 4$. Hence

$$Q_0(x',y',u',v') = \frac{4}{\pi^2}\, exp\left[-2\left(x'^2+y'^2+u'^2+v'^2\right)\right]. \tag{2.78}$$

Moreover, combination of (2.75) and (2.78) along with (2.59a) leads to

$$Q(x,y,u,v,t) = \frac{16\lambda_5\lambda_6\, e^{(\lambda_5-\lambda_6)t}}{\pi^4\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}(e^{\lambda_5 t}-1)(e^{-\lambda_6 t}-1)} exp\left[\frac{-2\lambda_5}{e^{\lambda_5 t}-1}\left(\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}\right)\right.$$
$$\left. -\frac{2\lambda_6}{e^{-\lambda_6 t}-1}\left(\frac{u^2}{\lambda_3}+\frac{v^2}{\lambda_4}\right)\right] \int_{-\infty}^{\infty} dx'\, exp\left[-2\left(1+\frac{\lambda_5}{\lambda_1(e^{\lambda_5 t}-1)}\right)x'^2 + \frac{4x\lambda_5\, e^{\frac{\lambda_5}{2}t}}{\lambda_1(e^{\lambda_5 t}-1)}x'\right]$$
$$\times \int_{-\infty}^{\infty} dy'\, exp\left[-2\left(1+\frac{\lambda_5}{\lambda_1(e^{\lambda_5 t}-1)}\right)y'^2 + \frac{4y\lambda_5\, e^{\frac{\lambda_5}{2}t}}{\lambda_2(e^{\lambda_5 t}-1)}y'\right]$$
$$\times \int_{-\infty}^{\infty} du'\, exp\left[-2\left(1+\frac{\lambda_6}{\lambda_1(e^{-\lambda_6 t}-1)}\right)u'^2 + \frac{4u\lambda_6\, e^{-\frac{\lambda_6}{2}t}}{\lambda_1(e^{-\lambda_6 t}-1)}u'\right]$$
$$\times \int_{-\infty}^{\infty} dv'\, exp\left[-2\left(1+\frac{\lambda_6}{\lambda_1(e^{-\lambda_6 t}-1)}\right)v'^2 + \frac{4v\lambda_6\, e^{-\frac{\lambda_6}{2}t}}{\lambda_1(e^{-\lambda_6 t}-1)}v'\right]. \tag{2.79}$$

Then carrying out the integrations applying the relation

$$\int_{-\infty}^{\infty} dx'\, exp\left[-kx'^2 + dx'\right] = \sqrt{\frac{\pi}{k}}\, exp\left[\frac{d^2}{4k}\right], \quad k>0 \tag{2.80}$$

the Q-function (2.79) turns out to be

$$Q(x,y,u,v,t) = \frac{4}{\pi^2\sqrt{a_1a_2a_3a_4}} exp\left[-\frac{2}{a_1}x^2 - \frac{2}{a_2}y^2 - \frac{2}{a_3}u^2 - \frac{2}{a_4}v^2\right], \tag{2.81}$$

where

$$a_1 = \frac{\lambda_1\left(e^{\lambda_5 t}-1\right)+\lambda_5}{\lambda_5 e^{\lambda_5 t}}, \tag{2.82a}$$

$$a_2 = \frac{\lambda_2\left(e^{\lambda_5 t}-1\right)+\lambda_5}{\lambda_5 e^{\lambda_5 t}}, \tag{2.82b}$$

$$a_3 = \frac{\lambda_3\left(e^{-\lambda_6 t}-1\right)+\lambda_6}{\lambda_6 e^{-\lambda_6 t}} \tag{2.82c}$$

and

$$a_4 = \frac{\lambda_4\left(e^{-\lambda_6 t}-1\right)+\lambda_6}{\lambda_6 e^{-\lambda_6 t}}. \tag{2.82d}$$

It can be easily verified that the Jacobian of the inverse transformation is $|J'| = \frac{1}{4}$. We can then write

$$\int dx\, dy\, du\, dv\, Q(x,y,u,v,t) = \int dx_1\, dx_2\, dy_1\, dy_2\, Q'(x_1,x_2,y_1,y_2,t), \tag{2.83}$$

in which

$$Q'(x_1, x_2, y_1, y_2, t) = |J'|\frac{4}{\pi^2\sqrt{a_1a_2a_3a_4}}exp\left[-\frac{1}{2a_1}(x_1 + x_2)^2 - \frac{1}{2a_2}(y_1 - y_2)^2\right.$$

$$\left.-\frac{1}{2a_3}(x_1 - x_2)^2 - \frac{1}{2a_4}(y_1 + y_2)^2\right] \tag{2.84}$$

is obtained from (2.81) employing the inverse transformations (2.48). Now taking into account expressions (2.43e-2.43h) along with (2.84), one can readily verify that

$$Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{1}{\pi^2\sqrt{a_1\,a_2\,a_3\,a_4}}exp\left[-\frac{1}{8a_1}(\alpha + \alpha^* + \beta + \beta^*)^2 + \frac{1}{8a_2}(\alpha - \alpha^* - \beta + \beta^*)^2\right.$$

$$\left.-\frac{1}{8a_3}(\alpha + \alpha^* - \beta - \beta^*)^2 + \frac{1}{8a_4}(\alpha - \alpha^* + \beta - \beta^*)^2\right].$$

Finally, the Q-function for the signal-idler modes produced by the the nondegenerate parametric oscillator coupled to two squeezed vacuum reservoirs is given by

$$Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{D}{\pi^2}exp\left[-b_1(\alpha^*\alpha + \beta^*\beta) + b_2(\alpha\beta + \alpha^*\beta^*)\right.$$

$$\left.+b_3(\alpha\beta^* + \alpha^*\beta) + \frac{1}{2}b_4(\alpha^2 + \alpha^{*2} + \beta^2 + \beta^{*2})\right], \tag{2.85}$$

where

$$D = \frac{1}{\sqrt{a_1\,a_2\,a_3\,a_4}}, \tag{2.86}$$

$$b_1 = \frac{1}{4}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right), \tag{2.87a}$$

$$b_2 = \frac{1}{4}\left(-\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right), \tag{2.87b}$$

$$b_3 = \frac{1}{4}\left(-\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - \frac{1}{a_4}\right), \tag{2.87c}$$

$$b_4 = \frac{1}{4}\left(-\frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3} + \frac{1}{a_4}\right). \tag{2.87d}$$

It can be readily verified that the Q-function (2.85) is normalized.

We now proceed to obtain the expression for the Q-function for some special cases of interest. When there are no squeezed vacuum reservoirs ($r = 0$), that is, when the external environment is an ordinary vacuum, the quantities given by (2.55) reduce to

$$\lambda_1 = \lambda_2 = \kappa\gamma_0 + \gamma, \tag{2.88a}$$

$$\lambda_3 = \lambda_4 = \kappa\gamma_0 - \gamma, \tag{2.88a}$$

$$\lambda_5 = 2\kappa\gamma_0 + \gamma, \tag{2.88a}$$

$$\lambda_6 = 2\kappa\gamma_0 - \gamma. \tag{2.88a}$$

and in view of these results, (2.82) becomes

$$a_1 = a_2 = \frac{\lambda_1\left(e^{\lambda_5 t} - 1\right) + \lambda_5}{\lambda_5 e^{\lambda_5 t}}, \tag{2.89a}$$

$$a_3 = a_4 = \frac{\lambda_3\left(e^{-\lambda_6 t} - 1\right) + \lambda_6}{\lambda_6 e^{-\lambda_6 t}}. \tag{2.89b}$$

Furthermore, applying these relations in (2.86), one readily gets

$$D = \frac{1}{a_1 a_3}, \tag{2.90}$$

$$b_1 = \frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_3}\right), \tag{2.91a}$$

$$b_2 = \frac{1}{2}\left(\frac{1}{a_3} - \frac{1}{a_1}\right), \tag{2.91b}$$

$$b_3 = b_4 = 0. \tag{2.91c}$$

Thus, the Q-function (2.85) for this case takes the form

$$Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{1}{\pi^2 a_1 a_3} exp\left[-\frac{1}{2}\left(\frac{a_1 + a_3}{a_1 a_3}\right)(\alpha^*\alpha + \beta^*\beta)\right.$$
$$\left. + \frac{1}{2}\left(\frac{a_1 - a_3}{a_1 a_3}\right)(\alpha\beta + \alpha^*\beta^*)\right]. \tag{2.92}$$

This is the Q-function for the nondegenerate parametric oscillator coupled to ordinary vacuum [2]. On the other hand, in the absence of damping ($\gamma = 0$), expressions (2.55) and (2.82) reduce to

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \kappa\gamma_0, \tag{2.93a}$$

$$\lambda_5 = \lambda_6 = 2\lambda_1 = 2\kappa\gamma_0, \tag{2.93b}$$

$$a_1 = a_2 = e^{-\kappa\gamma_0 t}\cosh\kappa\gamma_0 t, \tag{2.93c}$$

$$a_3 = a_4 = e^{\kappa\gamma_0 t}\cosh\kappa\gamma_0 t. \tag{2.93d}$$

Then applying these results, one can easily obtain

$$D = \frac{1}{a_1 a_3} = sech^2\kappa\gamma_0 t, \tag{2.94a}$$

$$b_1 = \frac{a_1 + a_3}{2a_1 a_3} = 1, \tag{2.94b}$$

$$b_2 = \frac{a_1 - a_3}{2a_1a_3} = -\tanh \kappa\gamma_0 t, \tag{2.94c}$$

$$b_3 = b_4 = 0. \tag{2.94d}$$

Making use of (2.93) and (2.94), one can put expression (2.85) in the form

$$Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{sech^2 \kappa\gamma_0 t}{\pi^2} exp\left[-\alpha^*\alpha - \beta^*\beta - (\tanh \kappa\gamma_0 t)(\alpha\beta + \alpha^*\beta^*)\right]. \tag{2.95}$$

This is the Q-function for the nondegenerate parametric amplifier.

We next seek to obtain the Q-function for the signal mode. We note that the Q-function for the signal mode is given by

$$Q(\alpha, \alpha^*, t) = \int d^2\beta\, Q(\alpha, \alpha^*, \beta, \beta^*, t),$$

so that using (2.85) and the relation

$$\int \frac{d^2\alpha}{\pi} exp\left[-a'\alpha^*\alpha + b'\alpha + c'\alpha^* + A'\alpha^2 + B'\alpha^{*2}\right] = \frac{1}{\sqrt{a'^2 - 4A'B'}}$$

$$\times exp\left[\frac{a'b'c' + A'c'^2 + B'b'^2}{a'^2 - 4A'B'}\right], \quad a' > 0 \tag{2.96}$$

one finds the Q-function for the signal mode to be

$$Q(\alpha, \alpha^*, t) = \frac{D}{\pi\sqrt{Y}} exp\left[-a\alpha^*\alpha + \frac{A}{2}\left(\alpha^2 + \alpha^{*2}\right)\right], \tag{2.97}$$

in which

$$Y = b_1^2 - b_4^2, \tag{2.98a}$$

$$a = \frac{1}{Y}\left((b_1 + b_4)\left[b_1(b_1 - b_4) + 2b_2b_3\right] - b_1\left[b_2 + b_3\right]^2\right), \tag{2.98b}$$

$$A = \frac{1}{Y}\left((b_1 + b_4)\left[b_4(b_1 - b_4) + 2b_2b_3\right] + b_4\left[b_2 + b_3\right]^2\right). \tag{2.98c}$$

Upon integrating (2.92) and (2.95) with respect to β by employing (2.96), one readily finds the Q-function for signal mode in the absence of a squeezed vacuum reservoirs and in the absence of damping to be

$$Q(\alpha, \alpha^*, t) = \frac{2}{\pi(a_1 + a_3)} exp\left[-\frac{2}{a_1 + a_3}\alpha^*\alpha\right] \tag{2.99}$$

and

$$Q(\alpha, \alpha^*, t) = \frac{sech^2 \kappa\gamma_0 t}{\pi} exp\left[-(sech^2 \kappa\gamma_0 t)(\alpha^*\alpha)\right], \tag{2.100}$$

respectively.

3. Quadrature Squeezing

In this chapter we seek to analyze the intracavity quadrature fluctuations for the signal mode as well as the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two squeezed vacuum reservoirs using the Q-functions (2.85) and (2.97).

3.1 Quadrature Squeezing of the Signal Mode

The squeezing properties of a signal-mode light are described by two Hermitian operators defined as

$$\hat{a}_1 = \hat{a}^\dagger + \hat{a}, \tag{3.1a}$$

$$\hat{a}_2 = i(\hat{a}^\dagger - \hat{a}). \tag{3.1b}$$

These quadrature operators obey the commutation relation

$$[\hat{a}_1, \hat{a}_2] = 2i. \tag{3.1c}$$

The variance of these quadrature operators can be put in the form

$$(\Delta\hat{a}_1)^2 = 1 + 2\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^2\rangle - \langle\hat{a}^\dagger\rangle^2 - \langle\hat{a}\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle, \tag{3.2a}$$

$$(\Delta\hat{a}_2)^2 = 1 + 2\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}^2\rangle + \langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle. \tag{3.2b}$$

We now proceed to calculate the expectation values involved in expression (3.2). Applying the relation

$$\langle\hat{A}(\hat{a}, \hat{a}^\dagger)\rangle = \int_{-\infty}^{\infty} d^2\alpha\, Q(\alpha, \alpha^*, t) A_a(\alpha, \alpha^*), \tag{3.3}$$

in which $A_a(\alpha, \alpha^*)$ is the c-number equivalent of the operator $\hat{A}(\hat{a}, \hat{a}^\dagger)$ for the antinormal ordering, one can write that

$$\langle\hat{a}\rangle = \int_{-\infty}^{\infty} d^2\alpha\, Q(\alpha, \alpha^*, t)\, \alpha. \tag{3.4}$$

In view of (2.97), this can be written as

$$\langle\hat{a}\rangle = \frac{D}{\sqrt{Y}}\frac{\partial}{\partial b}\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\, exp\left[-a\alpha^*\alpha + \frac{A}{2}\left(\alpha^2 + \alpha^{*2}\right) + b\alpha\right]\Bigg|_{b=0},$$

and on the basis of (2.96) for which $c' = 0$ and $A' = B' = \frac{A}{2}$, one can verify that

$$\langle \hat{a} \rangle = \frac{D}{\sqrt{Y}} \frac{\partial}{\partial b} \left[\frac{exp\left(\frac{Ab^2}{2(a^2-A^2)}\right)}{\sqrt{a^2 - A^2}} \right] \Bigg|_{b=0} = 0. \tag{3.5a}$$

Similarly, one can show that

$$\langle \hat{a}^\dagger \rangle = 0. \tag{3.5b}$$

Now on account of (3.5), expression (3.2) reduces to

$$(\Delta \hat{a}_1)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle, \tag{3.6a}$$

$$(\Delta \hat{a}_2)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle. \tag{3.6b}$$

Making use of the fact that the c-number equivalent of $\hat{a}^\dagger \hat{a}$ for the antinormal ordering is $\alpha^* \alpha - 1$ and applying (2.97), one can express (3.6a) in the form

$$(\Delta \hat{a}_1)^2 = \frac{D}{\sqrt{Y}} \left[1 + 2\left(-\frac{\partial}{\partial a} - 1 \right) + \frac{\partial}{\partial b} + \frac{\partial}{\partial c} \right]$$

$$\times \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} \, exp\left[-a\alpha^*\alpha + (\frac{A}{2} + b)\alpha^2 + (\frac{A}{2} + c)\alpha^{*2} \right] \Bigg|_{b=c=0},$$

from which, with the help of (2.96), one obtains

$$(\Delta \hat{a}_1)^2 = \frac{D}{\sqrt{Y}} \left[-1 - 2\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + \frac{\partial}{\partial c} \right] \frac{1}{\sqrt{a^2 - 4(\frac{A}{2} + b)(\frac{A}{2} + c)}} \Bigg|_{b=c=0},$$

so that upon carrying out the differentiation and setting $b = c = 0$, we get

$$(\Delta \hat{a}_1)^2 = \frac{D}{\sqrt{Y(a^2 - A^2)}} \left[-1 + \frac{2a}{a^2 - A^2} + \frac{2A}{a^2 - A^2} \right]. \tag{3.7}$$

It can be easily shown that

$$\frac{D}{\sqrt{Y(a^2 - A^2)}} = 1. \tag{3.8}$$

Hence the variance of $\hat{a}_1$ given by (3.7) reduces to

$$(\Delta \hat{a}_1)^2 = \frac{2}{a - A} - 1. \tag{3.9a}$$

Similarly, one can verify that

$$(\Delta \hat{a}_2)^2 = \frac{2}{a + A} - 1. \tag{3.9b}$$

Now on account of (2.87) and (2.98), expression (3.9) takes the form

$$(\Delta\hat{a}_1)^2 = a_1 + a_3 - 1, \tag{3.10a}$$

$$(\Delta\hat{a}_2)^2 = a_2 + a_4 - 1, \tag{3.10b}$$

so that in view of (2.82) we see that

$$(\Delta\hat{a}_1)^2 = \frac{\lambda_1\left(e^{\lambda_5 t} - 1\right) + \lambda_5}{\lambda_5 e^{\lambda_5 t}} + \frac{\lambda_3\left(e^{-\lambda_6 t} - 1\right) + \lambda_6}{\lambda_6 e^{-\lambda_6 t}} - 1, \tag{3.11a}$$

$$(\Delta\hat{a}_2)^2 = \frac{\lambda_2\left(e^{\lambda_5 t} - 1\right) + \lambda_5}{\lambda_5 e^{\lambda_5 t}} + \frac{\lambda_4\left(e^{-\lambda_6 t} - 1\right) + \lambda_6}{\lambda_6 e^{-\lambda_6 t}} - 1. \tag{3.11b}$$

We note that the nondegenerate parametric oscillator is in a steady state for $\gamma > 2\kappa\gamma_0$. Hence at steady state ($t \to \infty$), the variances described by (3.11) reduce to

$$(\Delta\hat{a}_1)^2 = \frac{\lambda_1}{\lambda_5} + \frac{\lambda_3}{\lambda_6} - 1, \tag{3.12a}$$

$$(\Delta\hat{a}_2)^2 = \frac{\lambda_2}{\lambda_5} + \frac{\lambda_4}{\lambda_6} - 1, \tag{3.12b}$$

and with the aid of (2.55) one can rewrite these expressions as

$$(\Delta\hat{a}_1)^2 = \frac{2(N-M)+1}{1-(\frac{2\kappa\gamma_0}{\gamma})^2}, \tag{3.13a}$$

$$(\Delta\hat{a}_2)^2 = \frac{2(N+M)+1}{1-(\frac{2\kappa\gamma_0}{\gamma})^2}, \tag{3.13b}$$

We recall that for a squeezed vacuum reservoir

$$N = sinh^2 r, \tag{3.14a}$$

$$M = \sinh r \cosh r. \tag{3.14b}$$

where r is the squeeze parameter taken to be real and positive for convenience. On account of these relations, expressions (3.13) take the form

$$(\Delta\hat{a}_1)^2 = \frac{e^{-2r}}{1-(\frac{2\kappa\gamma_0}{\gamma})^2}, \tag{3.15a}$$

$$(\Delta\hat{a}_2)^2 = \frac{e^{2r}}{1-(\frac{2\kappa\gamma_0}{\gamma})^2}. \tag{3.15b}$$

Using (3.15) one can show that

$$(\Delta\hat{a}_1)^2 < 1, \tag{3.16a}$$

when

$$r > -\frac{1}{2}ln\left(1-\left(\frac{2\kappa\gamma_0}{\gamma}\right)^2\right) \tag{3.16b}$$

and

$$(\Delta\hat{a}_2)^2 > 1, \tag{3.16c}$$

for any value r. This shows that the squeezing of the signal mode occurs in the first quadrature for the value of r specified by (3.16b).

We now consider some special cases of interest. When $r = 0$, substitution of (2.88) and (2.89) into (3.11) leads to

$$(\Delta\hat{a}_1)^2 = (\Delta\hat{a}_2)^2 = \frac{1 - \frac{\kappa\gamma_0}{\gamma}e^{-(\gamma-2\kappa\gamma_0)t}\left[(1-e^{-4\kappa\gamma_0 t}) + \frac{2\kappa\gamma_0}{\gamma}(1+e^{-4\kappa\gamma_0 t})\right]}{\left[1-(\frac{2\kappa\gamma_0}{\gamma})^2\right]}. \tag{3.17a}$$

At steady state, when the parametric oscillator is operating below threshold, this equation reduces to

$$(\Delta\hat{a}_1)^2 = (\Delta\hat{a}_2)^2 = \frac{1}{\left[1-(\frac{2\kappa\gamma_0}{\gamma})^2\right]} \tag{3.17b}$$

which is greater than unity. Hence the signal mode in this case is not in a squeezed state.

In the absence of damping, application of (2.93) and (2.94) in expression (3.11) yields

$$(\Delta\hat{a}_1)^2 = (\Delta\hat{a}_2)^2 = 2\overline{n} + 1, \tag{3.18a}$$

where

$$\overline{n} = shin^2\kappa\gamma_0 t \tag{3.18b}$$

is the mean photon number for the signal mode in this case. The variances (3.18a) indicate that the signal mode is in a chaotic state as expected.

Furthermore, in the absence of parametric interaction ($\kappa = 0$), combination of (2.55) and (3.14) yields

$$\lambda_1 = \gamma(N - M + 1) = \frac{\gamma}{2}\left(e^{-2r}+1\right), \tag{3.19a}$$

$$\lambda_2 = \gamma(N + M + 1) = \frac{\gamma}{2}\left(e^{2r}+1\right), \tag{3.19b}$$

$$\lambda_3 = -\lambda_1, \tag{3.19c}$$

$$\lambda_4 = -\lambda_2, \tag{3.19d}$$

$$\lambda_5 = -\lambda_6 = \gamma, \tag{3.19e}$$

so that in view of these results expression (2.82) becomes

$$a_1 = \frac{1}{2}\left(e^{-2r}+1\right) + \frac{1}{2}e^{-\gamma t}\left(1-e^{-2r}\right), \tag{3.20a}$$

$$a_2 = \frac{1}{2}\left(e^{2r}+1\right) + \frac{1}{2}e^{-\gamma t}\left(1-e^{2r}\right), \tag{3.20b}$$

$$a_3 = a_1, \tag{3.20c}$$

$$a_4 = a_2. \tag{3.20d}$$

Employing these values in (3.11), one obtains

$$\left(\Delta \hat{a}_1\right)^2 = 2a_1 - 1 = 1 - \left(1-e^{-\gamma t}\right)\left[1-e^{-2r}\right] < 1, \tag{3.21a}$$

$$\left(\Delta \hat{a}_2\right)^2 = 2a_2 - 1 = 1 + \left(1-e^{-\gamma t}\right)\left[e^{2r}-1\right] > 1, \tag{3.21b}$$

and at steady state these relations reduce to

$$\left(\Delta \hat{a}_1\right)^2 = e^{-2r}, \tag{3.22a}$$

$$\left(\Delta \hat{a}_2\right)^2 = e^{2r}, \tag{3.22b}$$

which are the quadrature fluctuations of the squeezed vacuum reservoir A.

3.2 Quadrature Squeezing of the Signal-Idler Modes

In this section we seek, applying the Q-function (2.85), to investigate the squeezing properties of the signal-idler modes produced by the nondegenerate parametric oscillator coupled to the two squeezed vacuum reservoirs.

The squeezing properties of a two-mode light are described by two quadrature operators defined by

$$\hat{c}_1 = \frac{1}{\sqrt{2}}\left(\hat{a}_1 + \hat{b}_1\right), \tag{3.23a}$$

$$\hat{c}_2 = \frac{1}{\sqrt{2}}\left(\hat{a}_2 + \hat{b}_2\right). \tag{3.23b}$$

where

$$\hat{a}_1 = (\hat{a}^\dagger + \hat{a}), \tag{3.24a}$$

$$\hat{b}_1 = (\hat{b}^\dagger + \hat{b}), \tag{3.24b}$$

$$\hat{a}_2 = i(\hat{a}^\dagger - \hat{a}), \tag{3.24c}$$

$$\hat{b}_2 = i(\hat{b}^\dagger - \hat{b}) \tag{3.24d}$$

and $\hat{a}$ ($\hat{b}$) denotes the annihilation operator for the intracavity mode a (b). The quadrature operators $\hat{c}_1$ and $\hat{c}_2$ satisfy the commutation relation $[\hat{c}_1, \hat{c}_2] = 2i$.

On account of expression (3.23), the variances

$$(\Delta\hat{c}_1)^2 = \langle\hat{c}_1^2\rangle - \langle\hat{c}_1\rangle^2, \tag{3.25a}$$

$$(\Delta\hat{c}_2)^2 = \langle\hat{c}_2^2\rangle - \langle\hat{c}_2\rangle^2, \tag{3.25b}$$

can be written as

$$(\Delta\hat{c}_1)^2 = \frac{1}{2}(\Delta\hat{a}_1)^2 + \frac{1}{2}(\Delta\hat{b}_1)^2 + \langle\hat{a}_1, \hat{b}_1\rangle, \tag{3.26a}$$

$$(\Delta\hat{c}_2)^2 = \frac{1}{2}(\Delta\hat{a}_2)^2 + \frac{1}{2}(\Delta\hat{b}_2)^2 + \langle\hat{a}_2, \hat{b}_2\rangle, \tag{3.26b}$$

in which

$$\langle\hat{a}_i, \hat{b}_i\rangle = \langle\hat{a}_i\hat{b}_i\rangle - \langle\hat{a}_i\rangle\langle\hat{b}_i\rangle. \tag{3.27}$$

In particular, when a and b represent the signal and idler modes, respectively, it can be shown that

$$(\Delta\hat{a}_1)^2 = (\Delta\hat{b}_1)^2, \tag{3.28a}$$

$$(\Delta\hat{a}_2)^2 = (\Delta\hat{b}_2)^2. \tag{3.28b}$$

It then follows that

$$(\Delta\hat{c}_1)^2 = (\Delta\hat{a}_1)^2 + \langle\hat{a}_1, \hat{b}_1\rangle, \tag{3.29a}$$

$$(\Delta\hat{c}_2)^2 = (\Delta\hat{a}_2)^2 + \langle\hat{a}_2, \hat{b}_2\rangle. \tag{3.29b}$$

We now proceed to obtain the explicit form of (3.29). In view of expressions (3.24) and (3.3), we see that

$$\langle\hat{a}_1\hat{b}_1\rangle = \int_{-\infty}^{\infty} d^2\alpha\, d^2\beta\, (\alpha^* + \alpha)(\beta^* + \beta)\, Q(\alpha, \alpha^*, \beta^*, \beta, t). \tag{3.30}$$

Then applying the Q-function(2.85) in this equation, we have

$$\langle\hat{a}_1\hat{b}_1\rangle = D\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}(\alpha^* + \alpha)\, exp\left[-b_1\alpha^*\alpha + \frac{b_4}{2}(\alpha^2 + \alpha^{*2})\right]$$

$$\times \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi}(\beta^* + \beta)\, exp\left[-b_1\beta^*\beta + (b_2\alpha + b_3\alpha^*)\beta + (b_2\alpha^* + b_3\alpha)\beta^* + \frac{1}{2}b_4(\beta^2 + \beta^{*2})\right].$$

On setting

$$K = b_2\alpha + b_3\alpha^*,$$

$$\langle\hat{a}_1\hat{b}_1\rangle = D\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}(\alpha^*+\alpha)\,exp\left[-b_1\alpha^*\alpha + \frac{b_4}{2}(\alpha^2+\alpha^{*2})\right]\left(\frac{\partial}{\partial K}+\frac{\partial}{\partial K^*}\right)$$

$$\times\int_{-\infty}^{\infty}\frac{d^2\beta}{\pi}exp\left[-b_1\beta^*\beta + (b_2\alpha+b_3\alpha^*)\beta + (b_2\alpha^*+b_3\alpha)\beta^* + \frac{1}{2}b_4(\beta^2+\beta^{*2})\right],$$

so that performing the integration with respect to β on the basis of (2.96) and carrying out the differentiation, we obtain

$$\langle\hat{a}_1\hat{b}_1\rangle = \frac{D}{Y^{\frac{3}{2}}}(b_1+b_4)(b_2+b_3)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}(\alpha^2+\alpha^{*2}+2\alpha^*\alpha)exp\left[-a\alpha^*\alpha+\frac{A}{2}(\alpha^2+\alpha^{*2})\right],$$

from which follows

$$\langle\hat{a}_1\hat{b}_1\rangle = \frac{D}{Y^{\frac{3}{2}}}(b_1+b_4)(b_2+b_3)\left(2\frac{\partial}{\partial A}-2\frac{\partial}{\partial a}\right)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}exp\left[-a\alpha^*\alpha+\frac{A}{2}(\alpha^2+\alpha^{*2})\right].$$

Next, integrating over α and carrying out the differentiation, we get

$$\langle\hat{a}_1\hat{b}_1\rangle = \frac{D}{Y^{\frac{3}{2}}(b_1+b_4)(b_2+b_3)}\left(\frac{2A+2a}{a^2-A^2}\right)\frac{1}{\sqrt{a^2-A^2}}. \tag{3.31}$$

Making use of expression (3.8) along with (2.87) and (2.98), the above equation reduces to

$$\langle\hat{a}_1\hat{b}_1\rangle = a_1 - a_3. \tag{3.32}$$

In view of (3.5),(3.24a), and (3.24b), we see that

$$\langle\hat{a}_1\rangle\langle\hat{b}_1\rangle = 0. \tag{3.33}$$

Therefore,

$$\langle\hat{a}_1,\hat{b}_1\rangle = a_1 - a_3. \tag{3.34}$$

Similarly, one can easily show that

$$\langle\hat{a}_2,\hat{b}_2\rangle = a_2 - a_4. \tag{3.35}$$

Now application of (3.11), (3.29), (3.34) and (3.35) in (3.26) gives the result

$$(\Delta\hat{c}_1)^2 = 2a_1 - 1, \tag{3.36a}$$

$$(\Delta\hat{c}_2)^2 = 2a_4 - 1. \tag{3.36b}$$

On account of (2.82a), (2.82d), and (2.55), the quadrature fluctuations of the signal-idler modes take the form

$$(\Delta\hat{c}_1)^2 = 2\left[\frac{\kappa\gamma_0 + \gamma(N-M+1)}{2\kappa\gamma_0+\gamma}\left(1-e^{-(2\kappa\gamma_0+\gamma)t}\right) + e^{-(2\kappa\gamma_0+\gamma)t}\right] - 1, \tag{3.37a}$$

$$(\Delta\hat{c}_2)^2 = 2\left[\frac{\kappa\gamma_0 - \gamma(N+M+1)}{2\kappa\gamma_0-\gamma}\left(1-e^{-(\gamma-2\kappa\gamma_0)t}\right) + e^{-(\gamma-2\kappa\gamma_0)t}\right] - 1. \tag{3.37b}$$

Finally, in view of (3.14), we have

$$(\Delta\hat{c}_1)^2 = 1 - \left(1-e^{-(\gamma+2\kappa\gamma_0)t}\right)\left[1 - \frac{\gamma\, e^{-2r}}{\gamma+2\kappa\gamma_0}\right] < 1 \tag{3.38a}$$

and

$$(\Delta\hat{c}_2)^2 = 1 + \left(1-e^{-(\gamma-2\kappa\gamma_0)t}\right)\left[\frac{\gamma\, e^{2r}}{\gamma-2\kappa\gamma_0} - 1\right] > 1. \tag{3.38b}$$

Hence the signal-idler modes of the nondegenerate parametric oscillator coupled to two independent squeezed vacuum reservoirs are in a squeezed state.

We now proceed to consider some cases of interest regarding the above quadrature fluctuations. At steady state ($t \to \infty$), (3.38) can be put in the form

$$(\Delta\hat{c}_1)^2 = \left(\frac{\gamma}{\gamma+2\kappa\gamma_0}\right)e^{-2r}, \tag{3.39a}$$

$$(\Delta\hat{c}_2)^2 = \left(\frac{\gamma}{\gamma-2\kappa\gamma_0}\right)e^{2r}. \tag{3.39b}$$

In addition, at threshold ($\gamma = 2\kappa\gamma_0$) one easily obtains

$$(\Delta\hat{c}_1)^2 = \frac{1}{2}e^{-2r}, \tag{3.40a}$$

$$(\Delta\hat{c}_2)^2 \to \infty. \tag{3.40b}$$

These results indicate that, for large value of r, a squeezing approaching 100% below the vacuum level can be achieved in the first quadrature whereas an infinitely enhanced fluctuations occurs in the second quadrature.

In the absence of squeezed vacuum reservoirs ($r = 0$), expression (3.38) becomes

$$(\Delta\hat{c}_1)^2 = \frac{\gamma + \left(2\kappa\gamma_0\, e^{-(\gamma+2\kappa\gamma_0)t}\right)}{\gamma+2\kappa\gamma_0} < 1, \tag{3.41a}$$

$$(\Delta\hat{c}_2)^2 = \frac{\gamma - \left(2\kappa\gamma_0\, e^{-(\gamma-2\kappa\gamma_0)t}\right)}{\gamma-2\kappa\gamma_0} > 1. \tag{3.41b}$$

This shows that the signal-idler modes produced by the nondegenerate parametric oscillator in the absence of squeezed vacuum reservoirs is also in a squeezed state. At steady state and at threshold, these relations reduce to

$$(\Delta\hat{c}_1)^2 = \frac{1}{2}, \tag{3.41c}$$

$$(\Delta\hat{c}_2)^2 \rightarrow \infty. \tag{3.41d}$$

We see that for this case there is only a 50% reduction of noise below the vacuum level in the first quadrature. This result is in complete agreement with result obtained by B. Daniel and K. Fesseha [2]. Moreover, in the absence of damping ($\gamma = 0$), expression (3.38) reduces to

$$\Delta\hat{c}_1^2 = e^{-2\kappa\gamma_0 t} < 1, \tag{3.42a}$$

$$\Delta\hat{c}_2^2 = e^{2\kappa\gamma_0 t} > 1, \tag{3.42b}$$

which are the quadrature fluctuations of the the signal-idler modes produced by the nondegenerate parametric amplifier.

Finally, when there is no parametric interaction inside the cavity ($\kappa = 0$), expression (3.38) reduce to

$$(\Delta\hat{c}_1)^2 = 1 - \left(1 - e^{-\gamma t}\right)\left[1 - e^{-2r}\right] < 1, \tag{3.43a}$$

$$(\Delta\hat{c}_2)^2 = 1 + \left(1 - e^{-\gamma t}\right)\left[e^{2r} - 1\right] > 1, \tag{3.43b}$$

which leads, at steady state, to the quadrature fluctuations of the reservoir modes A and B given by

$$(\Delta\hat{c}_1)^2 = e^{-2r}, \tag{3.44a}$$

$$(\Delta\hat{c}_2)^2 = e^{2r}. \tag{3.44b}$$

When we compare relations (3.41c), (3.41d) and (3.44) with (3.39), we see that the quadrature variances at steady state is the product of the variances of the nondegenerate parametric oscillator coupled to ordinary vacuum and the the variances pertaining to the squeezed vacuum reservoirs. Furthermore upon comparing expressions (3.22) and (3.44) we observe that, at steady state the variances of a single-mode squeezed vacuum reservoir as well as those of two independent squeezed vacuum reservoirs are the same.

4. The Photon Number Distribution

In this chapter we seek to derive a general expression for the photon number distribution in terms of the Q-function for a two-mode light. Employing this result we next determine the explicit form of the photon number distribution for the signal mode as well as the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two independent squeezed vacuum reservoirs. We also calculate the photon number distribution for the signal mode as well as the the signal-idler modes for three limiting cases.

The photon number distribution for a two-mode light, described by the density operator $\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t)$, is defined by

$$P(n,m,t) = \langle n,m|\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t)|n,m\rangle, \tag{4.1}$$

which represents the joint probability of finding n photons of mode a and m photons of mode b at any time t. Introducing the two-mode completeness relation for coherent states

$$\hat{I} = \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}|\alpha,\beta\rangle\langle\alpha,\beta| \tag{4.2}$$

twice in (4.1), we see that

$$P(n,m,t) = \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\frac{d^2\eta}{\pi}\frac{d^2\nu}{\pi}\langle n,m|\alpha,\beta\rangle\langle\alpha,\beta|\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t)|\eta,\nu\rangle\langle\eta,\nu|n,m\rangle. \tag{4.3}$$

We recall that

$$\langle n,m|\alpha,\beta\rangle = \langle n|\alpha\rangle\langle m|\beta\rangle, \tag{4.4a}$$

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}e^{-\frac{1}{2}\alpha^*\alpha}. \tag{4.4b}$$

In view of these results, expression (4.3) can be put in the form

$$P(n,m,t) = \frac{1}{n!\,m!}\int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\frac{d^2\eta}{\pi}\frac{d^2\nu}{\pi}\alpha^n\beta^m\eta^{*n}\nu^{*m}$$

$$\times exp\left[-\frac{1}{2}\alpha^*\alpha - \frac{1}{2}\beta^*\beta - \frac{1}{2}\eta^*\eta - \frac{1}{2}\nu^*\nu\right]\langle\alpha,\beta|\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t)|\eta,\nu\rangle. \tag{4.5}$$

Expanding the density operator in normal order we have

$$\hat{\rho}(\hat{a}^\dagger, \hat{a}, \hat{b}^\dagger, \hat{b}, t) = \sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\; \hat{a}^{\dagger i}\; \hat{b}^{\dagger k}\; \hat{a}^j\; \hat{b}^l$$

and noting

$$\langle\alpha,\beta|\hat{a}^{\dagger i}\hat{b}^{\dagger k}\hat{a}^j\hat{b}^l|\eta,\nu\rangle = \alpha^{*i}\beta^{*k}\eta^j\nu^l\langle\alpha|\eta\rangle\langle\beta|\nu\rangle,$$

$$\langle\alpha|\eta\rangle = e^{-\frac{1}{2}\alpha^*\alpha+\alpha^*\eta-\frac{1}{2}\eta^*\eta},$$

one can easily show that

$$P(n,m,t) = \frac{1}{n!\,m!}\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\alpha^n\alpha^{*i}e^{-\alpha^*\alpha}\int_{-\infty}^{\infty}\frac{d^2\eta}{\pi}\eta^j\eta^{*n}e^{-\eta^*\eta+\alpha^*\eta}$$
$$\times\int_{-\infty}^{\infty}\frac{d^2\beta}{\pi}\beta^m\beta^{*k}e^{-\beta^*\beta}\int_{-\infty}^{\infty}\frac{d^2\nu}{\pi}\nu^l\nu^{*m}e^{-\nu^*\nu+\beta^*\nu}.$$

This expression can be rewritten as

$$P(n,m,t) = \frac{1}{n!\,m!}\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\alpha^n\alpha^{*i}e^{-\alpha^*\alpha}$$
$$\times\frac{\partial^n}{\partial a^n}\frac{\partial^j}{\partial\alpha^{*j}}\left[\int_{-\infty}^{\infty}\frac{d^2\eta}{\pi}e^{-\eta^*\eta+\alpha^*\eta+a\eta^*}\right]\Bigg|_{a=0}$$
$$\times\int_{-\infty}^{\infty}\frac{d^2\beta}{\pi}\beta^m\beta^{*k}e^{-\beta^*\beta}\frac{\partial^m}{\partial b^m}\frac{\partial^l}{\partial\beta^{*l}}\left[\int_{-\infty}^{\infty}\frac{d^2\nu}{\pi}e^{-\nu^*\nu+\beta^*\nu+b\nu^*}\right]\Bigg|_{b=0}.$$

Now carrying out the integrations with respect to η and ν according to the relation (2.96), one readily obtains

$$P(n,m,t) = \frac{1}{n!\,m!}\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\alpha^n\alpha^{*i}e^{-\alpha^*\alpha}\frac{\partial^n}{\partial a^n}\frac{\partial^j}{\partial\alpha^{*j}}\left[e^{a\alpha^*}\right]\big|_{a=0}$$
$$\times\int_{-\infty}^{\infty}\frac{d^2\beta}{\pi}\beta^m\beta^{*k}e^{-\beta^*\beta}\frac{\partial^m}{\partial b^m}\frac{\partial^l}{\partial\beta^{*l}}\left[e^{b\beta^*}\right]\big|_{b=0}.$$

Then performing the differentiations with respect to α^* and β^*, we get

$$P(n,m,t) = \frac{1}{n!\,m!}\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\alpha^n\alpha^{*i}e^{-\alpha^*\alpha}\frac{\partial^n}{\partial a^n}\left[a^je^{\alpha^*a}\right]\big|_{a=0}$$
$$\times\int_{-\infty}^{\infty}\frac{d^2\beta}{\pi}\beta^m\beta^{*k}e^{-\beta^*\beta}\frac{\partial^m}{\partial b^m}\left[b^le^{\beta^*b}\right]\big|_{b=0}.$$

This can be put in the form

$$P(n,m,t) = \frac{1}{n!\,m!}\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\frac{\partial^n}{\partial a^n}\frac{\partial^m}{\partial b^m}\left[a^jb^l\frac{\partial^n}{\partial c^n}\frac{\partial^i}{\partial a^i}\left(\int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}e^{-\alpha^*\alpha+a\alpha^*+c\alpha}\right)\Bigg|_{a=c=0}\right.$$
$$\left.\times\frac{\partial^m}{\partial d^m}\frac{\partial^k}{\partial b^k}\left(\int_{-\infty}^{\infty}\frac{d^2\beta}{\pi}e^{-\beta^*\beta+b\beta^*+d\beta}\right)\Bigg|_{b=d=0}\right].$$

Furthermore, carrying out the integrations over α and β, and differentiating with respect to a (i times) and b (k times), one readily obtains

$$P(n,m,t) = \frac{1}{n!\ m!}\frac{\partial^n}{\partial a^n}\frac{\partial^n}{\partial c^n}\frac{\partial^m}{\partial b^m}\frac{\partial^m}{\partial d^m}\left[\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\ c^i a^j d^k b^l e^{ac} e^{bd}\right]\Bigg|_{a=c=b=d=0}.$$

Finally, upon replacing (a,c,b,d) by $(\alpha^*,\alpha,\beta^*,\beta)$, the photon number distribution for a two-mode light takes the form

$$P(n,m,t) = \frac{\pi^2}{n!\ m!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\frac{\partial^{2m}}{\partial\beta^m\partial\beta^{*m}}\left[Q(\alpha,\alpha^*,\beta,\beta^*,t)\ e^{\alpha^*\alpha+\beta^*\beta}\right]\big|_{\alpha=\alpha^*=\beta=\beta^*=0}, \quad (4.6)$$

where

$$Q(\alpha,\alpha^*,\beta,\beta^*,t) = \frac{1}{\pi^2}\sum_{i,j,k,l=0}^{\infty} C_{ijkl}(t)\ \alpha^i\alpha^{*j}\beta^k\beta^{*l}.$$

From (4.6), it can be verified that the photon number distribution for a single-mode light has the form

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left[Q(\alpha,\alpha^*,t)\ e^{\alpha^*\alpha}\right]\big|_{\alpha=\alpha^*=0}. \quad (4.7)$$

4.1 The Photon Number Distribution of the Signal Mode

In this section we seek to obtain the explicit form of the photon number distribution for the signal mode produced by the nondegenerate parametric oscillator. Employing the Q-function (2.97) in expression (4.7), the photon number distribution for the signal mode can be written as

$$P(n,t) = \frac{D}{n!\ \sqrt{Y}}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\exp\left[(1-a)\alpha^*\alpha + \frac{A}{2}\left(\alpha^2+\alpha^{*2}\right)\right]\Bigg|_{\alpha=\alpha^*=0}. \quad (4.8)$$

Now expanding the exponential in power series, we have

$$e^{(1-a)\alpha^*\alpha} = \sum_{i=0}^{\infty}\frac{(1-a)^i}{i!}(\alpha^*\alpha)^i, \quad (4.9a)$$

$$e^{\frac{A}{2}\alpha^2} = \sum_{j=0}^{\infty}\left(\frac{A}{2}\right)^j\frac{\alpha^{2j}}{j!}, \quad (4.9b)$$

$$e^{\frac{A}{2}\alpha^{*2}} = \sum_{k=0}^{\infty}\left(\frac{A}{2}\right)^k\frac{\alpha^{*2k}}{k!}. \quad (4.9c)$$

On account of these relations, expression (4.8) takes the form

$$P(n,t) = \frac{D}{n!\ \sqrt{Y}}\sum_{i,j,k=0}^{\infty}\frac{(1-a)^i}{i!\ j!\ k!}\left(\frac{A}{2}\right)^{j+k}\frac{\partial^n\left(\alpha^{i+2j}\right)}{\partial\alpha^n}\frac{\partial^n\left(\alpha^{*i+2k}\right)}{\partial\alpha^{*n}}\Bigg|_{\alpha=\alpha^*=0}.$$

Carrying out the differentiations using the identity

$$\frac{\partial}{\partial x^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n} \tag{4.10}$$

and applying the condition

$$\alpha = \alpha^* = 0,$$

one finds

$$P(n,t) = \frac{D}{n!\,\sqrt{Y}} \sum_{i,j,k=0}^{\infty} \frac{(1-a)^i \left(\frac{A}{2}\right)^{j+k}}{i!\;j!\;k!} \frac{(i+2j)!}{(i+2j-n)!} \frac{(i+2k)!}{(i+2k-n)!} \delta_{i+2j,n}\;\delta_{i+2k,n}. \tag{4.11}$$

Thus applying the property of the Kronecker delta symbol described by (2.11), one can easily see that

$$i + 2j = n,$$

$$i + 2k = n,$$

from which follows

$$j = k,$$

$$i = n - 2j.$$

Since factorials are defined for nonnegative integers, we note that

$$j \le \frac{n}{2} \tag{4.12a}$$

when n is even and

$$j \le \frac{n-1}{2} \tag{4.12b}$$

when n is odd. Consequently, the photon number distribution for the signal mode is expressible as

$$P(n,t) = \frac{D}{\sqrt{Y}} \sum_{j=0}^{[n]} \left(\frac{A}{2}\right)^{2j} (1-a)^{n-2j} \frac{n!}{(j!)^2\,(n-2j)!}, \tag{4.13}$$

in which

$$[n] = \begin{cases} \frac{n}{2} & \text{for n even} \\ \frac{n-1}{2} & \text{for n odd} \end{cases}.$$

This result indicates that there is a finite probability of finding an odd number of signal photons in the cavity.

Now we consider three special cases of interest. We first take the case for which $r = 0$. For this case application of (2.91) in expression (2.98) leads to

$$Y = b_1 = \frac{a_1 + a_3}{2a_1 a_3}, \tag{4.14a}$$

$$a = \frac{2}{a_1 + a_3}, \tag{4.14b}$$

$$A = 0, \tag{4.14c}$$

so that combination of these results and expression (2.90) along with (4.13) yields

$$P(n,t) = \frac{2}{(a_1 + a_3)} \sum_{j=0}^{[n]} \left(1 - \frac{2}{a_1 + a_3}\right)^{n-2j} \frac{n!}{(j!)^2 \, (n-2j)!} \delta_{0,2j}.$$

Then making use of the property of the Kronecker delta symbol again we have

$$P(n,t) = \frac{2}{(a_1 + a_3)} \left(1 - \frac{2}{a_1 + a_3}\right)^{n}. \tag{4.15a}$$

On account of relation (3.3) and the Q-function (2.99), the mean photon number for the signal mode in this case takes the form

$$\langle \hat{a}^{\dagger} \hat{a} \rangle = \overline{n} = \frac{a_1 + a_3}{2} - 1 \tag{4.15b}$$

and substituting this result into expression (4.15a) we get

$$P(n,t) = \frac{(\overline{n})^n}{(\overline{n} + 1)^{n+1}}. \tag{4.15c}$$

This is the photon number distribution for the signal mode in the absence of the squeezed vacuum reservoir.

In the absence of damping, substituting expression (2.94) into (2.98) results in

$$Y = 1, \tag{4.16a}$$

$$a = 1 - b_2^2 = 1 - \tanh^2 \kappa\gamma_0 t, \tag{4.16b}$$

$$A = 0, \tag{4.16c}$$

and in view of these results expression (4.13) can be put in the form

$$P(n,t) = sech^2 \ \kappa\gamma_0 t \sum_{j=0}^{[n]} \frac{\left(\tanh^2 \kappa\gamma_0 t\right)^n n!}{(j!)^2 \, (n-2j)!} \delta_{0,2j}.$$

Then making use of relation (2.11) we have

$$P(n,t) = \left[\frac{\tanh^n \kappa\gamma_0 t}{\cosh \kappa\gamma_0 t}\right]^2 . \tag{4.17}$$

This is the photon number distribution for the signal mode produced by the nondegenerate parametric oscillator in the absence of cavity damping.

Now we would like to calculate the photon number distribution in the absence of parametric interaction and at steady state. For this case expressions (3.20) reduce to

$$a_1 = e^{-r} \cosh r, \tag{4.18a}$$

$$a_2 = e^{r} \cosh r, \tag{4.18b}$$

$$a_3 = a_1, \tag{4.18c}$$

$$a_4 = a_2. \tag{4.18d}$$

Then substituting these results into expressions (2.86) and (2.87), one easily gets

$$D = \frac{1}{a_1 a_2} = sech^2 r, \tag{4.19a}$$

$$b_1 = \frac{a_1 + a_2}{2a_1 a_2} = 1, \tag{4.19b}$$

$$b_2 = b_3 = 0, \tag{4.19c}$$

$$b_4 = \frac{a_1 - a_2}{2a_1 a_2} = -\tanh r. \tag{4.19d}$$

Furthermore, application of these results in (2.98) yields

$$Y = 1 - \tanh^2 r = sech^2\, r, \tag{4.20a}$$

$$a = 1, \tag{4.20b}$$

$$A = -\tanh r, \tag{4.20c}$$

and on the basis of these results, (4.13) can be put in the form

$$P(n,t) = sech\, r \sum_{j=0}^{[n]} \left(-\frac{\tanh r}{2}\right)^{2j} \frac{n!}{(j!)^2\,(n-2j)!}\delta_{0,n-2j}.$$

Applying relation (2.11) once more, we see that

$$j = \frac{n}{2}$$

and hence

$$P(n,t) = sech\ r \left(-\frac{\tanh r}{2}\right)^n \frac{n!}{[(\frac{n}{2})!]^2}.$$

However, factorials are defined for nonnegative integers only and as a result we have

$$P(n) = \begin{cases} 0 & \text{if n is odd} \\ \frac{\tanh^n r}{n!\ 2^n \cosh r} \frac{(-1)^n (n!)^2}{[(\frac{n}{2})!]^2} & \text{if n is even} \end{cases}.$$

Finally, the photon number distribution can be expressed as

$$P(n) = \frac{1}{n!\ 2^n} \frac{\tanh^n r}{\cosh r} H_n^2(0), \tag{4.21a}$$

in which

$$H_n(0) = \frac{(-1)^{\frac{n}{2}} n!}{(\frac{n}{2})!} \tag{4.21b}$$

is a Hermite Polynomial. Expression (4.21a) is identical to the photon number distribution for a single-mode squeezed vacuum state . This result shows that for $\kappa = 0$ and at steady state ($t \to \infty$) what we have in the cavity is only the squeezed vacuum reservoir A. One can also see that (4.21a) has the same form as the photon number distribution of a degenerate parametric amplifier with r replaced by $\kappa\gamma_0 t$.

4.2 Photon Number Distribution of the Signal-Idler Modes

Next, we seek to obtain the photon number distribution for the signal-idler modes applying the Q-function. Then substituting expression (2.85) into (4.6), the photon number distribution for signal-idler modes produced by the nondegenerate parametric oscillator coupled to two uncorellated squeezed vacuum reservoirs can be put in the form

$$P(n,m,t) = \frac{D}{n!\ m!} \frac{\partial^{2n}\partial^{2m}}{\partial\alpha^{*n}\partial\alpha^n\partial\beta^{*m}\partial\beta^m} exp\left[(1-b_1)(\alpha^*\alpha + \beta^*\beta) + b_2(\alpha\beta + \alpha^*\beta^*)\right]$$
$$\times exp\left[a_3(\alpha\beta^* + \alpha^*\beta) + \frac{b_4}{2}(\alpha^2 + \alpha^{*2} + \beta^2 + \beta^{*2})\right]\Bigg|_{\alpha^*=\alpha=\beta^*=\beta=0}. \tag{4.22}$$

In order to carry out the differentiations, it proves to be convenient first to expand the exponential functions involved in expression (4.22) in power series:

$$e^{(1-b_1)\alpha^*\alpha} = \sum_{f=0}^{\infty} \frac{(1-b_1)^f \alpha^{*f}\alpha^f}{f!},$$

$$e^{(1-b_1)\beta^*\beta} = \sum_{g=0}^{\infty} \frac{(1-b_1)^g \beta^{*g}\beta^g}{g!},$$

$$e^{b_2\alpha\beta} = \sum_{h=0}^{\infty} \frac{b_2^h \alpha^h \beta^h}{h!},$$

$$e^{b_2\alpha^*\beta^*} = \sum_{i=0}^{\infty} \frac{b_2^i \alpha^{*i} \beta^{*i}}{i!},$$

$$e^{b_3\alpha\beta^*} = \sum_{j=0}^{\infty} \frac{b_3^j \alpha^j \beta^{*j}}{j!},$$

$$e^{b_3\alpha^*\beta} = \sum_{k=0}^{\infty} \frac{b_3^k \alpha^{*k} \beta^k}{k!},$$

$$e^{(\frac{b_4}{2})\alpha^2} = \sum_{l=0}^{\infty} \frac{(\frac{b_4}{2})^l \alpha^{2l}}{l!},$$

$$e^{(\frac{b_4}{2})\alpha^{*2}} = \sum_{p=0}^{\infty} \frac{(\frac{b_4}{2})^p \alpha^{*2p}}{p!},$$

$$e^{(\frac{b_4}{2})\beta^2} = \sum_{q=0}^{\infty} \frac{(\frac{b_4}{2})^q \beta^{2q}}{q!},$$

$$e^{\frac{b_4}{2}\beta^{*2}} = \sum_{r=0}^{\infty} \frac{(\frac{b_4}{2})^r \beta^{*2r}}{r!}.$$

Introducing these expressions into (4.22), we see that

$$P(n,m,t) = \frac{D}{n!} \sum_{f,h,i,j,k,l,p=0}^{\infty} \frac{(1-b_1)^f b_2^{h+i} b_3^{j+k} (\frac{b_4}{2})^{l+p}}{f!\ h!\ i!\ j!\ k!\ l!\ p!} \frac{\partial^n}{\partial\alpha^n}\left[(\alpha)^{f+h+j+2l}\right]\Big|_{\alpha=0}$$

$$\times \frac{\partial^n}{\partial\alpha^{*n}}\left[(\alpha^*)^{f+i+k+2p}\right]\Big|_{\alpha^*=0} \frac{1}{m!} \sum_{g,q,r=0}^{\infty} \frac{(1-b_1)^g (\frac{b_4}{2})^{q+r}}{g!\ q!\ r!}$$

$$\times \frac{\partial^m}{\partial\beta^m}\left[(\beta)^{g+h+k+2q}\right]\Big|_{\beta=0} \frac{\partial^m}{\partial\beta^{*m}}\left[(\beta^*)^{g+i+j+2r}\right]\Big|_{\beta^*=0}.$$

Application of (4.10) and the condition $\alpha = \alpha^* = \beta = \beta^* = 0$ leads to

$$P(n,m,t) = \frac{D}{n!} \sum_{f,h,i,j,k,l,p=0}^{\infty} \frac{(1-b_1)^f b_2^{h+i} b_3^{j+k} (\frac{b_4}{2})^{l+p}}{f!\ h!\ i!\ j!\ k!\ l!\ p!} \frac{(f+h+j+2l)!}{(f+h+j+2l-n)!}\delta_{f+h+j+2l,n}$$

$$\times \frac{(f+i+k+2p)!}{(f+i+k+2p-n)!}\delta_{f+i+k+2p,n} \frac{1}{m!} \sum_{g,q,r=0}^{\infty} \frac{(1-b_1)^g (\frac{b_4}{2})^{q+r}}{g!\ q!\ r!} \frac{(g+h+k+2q)!}{(g+h+k+2q-m)!}$$

$$\times \delta_{g+h+k+2q,m} \frac{(g+i+j+2r)!}{(g+i+j+2r-m)!}\delta_{g+i+j+2r,m}. \tag{4.23}$$

Making use of the property of the Kronecker delta symbol, one gets

$$f+h+j+2l-n = 0, \tag{4.24a}$$

$$f + i + k + 2p - n = 0, \tag{4.24b}$$

$$g + h + k + 2q - m = 0, \tag{4.24c}$$

$$g + i + j + 2r - m = 0, \tag{4.24d}$$

so that, in view of these results, expression (4.23) takes the form

$$P(n,m,t) = D \sum_{f,g,h,i,j,k,l,p,q,r} \frac{n!\ m!\ (1-b_1)^{f+g} b_2^{h+i} b_3^{j+k} (\frac{b_4}{2})^{l+p+q+r}}{f!\ g!\ h!\ i!\ j!\ k!\ l!\ p!\ q!\ r!}. \tag{4.25}$$

This is the photon number distribution for the signal-idler modes. .

Now we proceed to obtain the photon number distribution for the three cases previously considered. In the absence of squeezed vacuum reservoirs ($r = 0$), application of expressions (2.90) and (2.91c) in (4.25) leads to

$$P(n,m,t) = \frac{1}{a_1 a_2} \sum_{f,g,h,i,j,k,l,p,q,r} \frac{n!\ m!\ (1-b_1)^{f+g} b_2^{h+i} \delta_{j+k,0}\ \delta_{l+p+q+r,0}}{f!\ g!\ h!\ i!\ j!\ k!\ l!\ p!\ q!\ r!}, \tag{4.26}$$

from which, in view of (2.11), one easily obtains

$$j = k = l = p = q = r = 0. \tag{2.27}$$

Then substituting (4.27) into expression (4.24), we get

$$f + h - n = 0,$$

$$f + i - n = 0,$$

$$g + h - m = 0,$$

$$g + i - m = 0.$$

It then follows that

$$h = i,$$

$$f = n - h,$$

$$g = m - h,$$

and consequently expression (4.26) takes the form

$$P(n,m,t) = \frac{1}{a_1 a_2} \sum_{h=0}^{min(n,m)} \frac{n!\ m!\ (1-b_1)^{n+m-2h}\ b_2^{2h}}{h!\ (n-h)!\ h!\ (m-h)!}. \tag{4.28}$$

Furthermore, when $n = m$, we obtain [2]

$$P(n,n,t) = \frac{1}{a_1 a_2} \sum_{h=0}^{n} \left[\frac{n!}{h!\,(n-h)!}\right]^2 (1-b_1)^{2(n-h)}\, b_2^{2h}. \tag{4.29}$$

When the port-mirror is assumed to be hundred percent reflective ($\gamma = 0$), substituting (2.94) into (4.25), one can easily verify that

$$P(n,m,t) = sech^2\ \kappa\gamma_0 t \sum_{f,g,h,i,j,k,l,p,q,r=0}^{\infty} \frac{n!\ m!\ (-\tanh\kappa\gamma_0 t)^{h+i}\ \delta_{f+g,0}\delta_{j+k,0}\ \delta_{l+p+q+r,0}}{f!\ g!\ h!\ i!\ j!\ k!\ l!\ p!\ q!\ r!}. \tag{4.30}$$

Now applying relation (2.11) in this expression, we get

$$f = g = j = l = p = q = r = 0$$

and in view of these results, (4.24) reduce to

$$h - n = 0,$$

$$i - n = 0,$$

$$i - m = 0,$$

$$h - m = 0,$$

which leads to

$$h = i = n = m.$$

Consequently, expression (4.30) turns out to be

$$P(n,n,t) = \left[\frac{\tanh^n \kappa\gamma_0 t}{\cosh\kappa\gamma_0 t}\right]^2, \tag{4.31}$$

which represents the photon number distribution of the signal-idler modes produced by the nondegenerate parametric amplifier. Comparison of expressions (4.17) and (4.31) shows that the photon number distributions for the signal mode and the signal-idler modes are identical. This indicates that, in the absence of damping, the probabilities of observing n signal photons and the joint probability of observing n signal and n idler photons are the same.

Lastly, we consider the case when there is no parametric interaction inside the cavity ($\kappa = 0$). Application of (3.20) in expressions (2.86) and (2.87) yields

$$D = \frac{1}{a_1 a_2} = \left[1 + \left(1 - e^{-2\gamma t}\right) sinh^2\ r\right]^{-1}, \tag{4.32}$$

$$b_1 = \frac{a_1 + a_2}{2a_1 a_2} = \frac{(1 - e^{-\gamma t})\, sinh^2\, r}{1 + (1 - e^{-2\gamma t})\, sinh^2\, r}, \tag{4.33a}$$

$$b_2 = b_3 = 0, \tag{4.33b}$$

$$b_4 = \frac{a_1 - a_2}{2a_1 a_2} = \frac{-(1 - e^{-\gamma t}) \sinh r \cosh r}{1 + (1 - e^{-2\gamma t})\, sinh^2 r}. \tag{4.33c}$$

Then substituting (4.33b) into expression (4.25) and making use of relation (2.11), one can easily verify that

$$h = j = i = k = 0,$$

and on the basis of this result, (4.24) reduce to

$$f + 2l - n = 0,$$

$$f + 2p - n = 0,$$

$$g + 2q - m = 0,$$

$$g + 2r - m = 0,$$

from which we see that

$$l = p,$$

$$r = q,$$

$$f = n - 2l,$$

$$g = m - 2q.$$

Consequently, expression (4.25) takes the form

$$P(n, m, t) = D \sum_{l,q=0}^{[n],[m]} \frac{(1 - b_1)^{n-2l}\, (\frac{b_4}{2})^{2l}\, n!}{(h!)^2\, (n - 2l)!} \frac{(1 - b_1)^{m-2q}\, (\frac{b_4}{2})^{2q}\, m!}{(q!)^2\, (m - 2q)!}, \tag{4.34}$$

where

$$[n] = \begin{cases} \frac{n}{2} & \text{for n even,} \\ \frac{n-1}{2} & \text{for n odd,} \end{cases}$$

and with a similar expression for m. From this expression we note that there is a finite probability of finding an odd number of photons associated with each squeezed vacuum reservoir mode. This is due to the fact that, although photons in each reservoir exist in pairs, there is some probability for an odd number of photons from each mode to enter through the port-mirror.

At steady state, expressions (4.32) and (4.33) reduce to

$$D = sech^2\, r, \tag{4.35}$$

$$b_1 = 1, \tag{4.35a}$$

$$b_2 = b_3 = 0, \tag{4.35b}$$

$$b_4 = -\tanh r. \tag{4.35c}$$

Applying once more (2.11), one can readily obtain

$$n - 2l = 0,$$

$$m - 2q = 0.$$

Thus, in view of these results, expression (4.34) takes the form

$$P(n,m) = \frac{1}{n!\; m!\; 2^{n+m}} \frac{tanh^{n+m}\, r}{\cosh r} H_n^2(0)\, H_m^2(0), \tag{4.36}$$

in which $H_n(0)$ or ($H_m(0)$ is the Hermit Polynomial described by (4.21b). From the property of the Hermite Polynomial, we see that the joint probability of finding an odd number of n photons of the squeezed vacuum reservoir A and m photons of the squeezed vacuum reservoir B is zero. Expression (4.36) represents the photon number distribution for two independent squeezed vacuum reservoirs. When we made a comparison between expressions (4.21a) and (4.36) we see that, at steady state, the photon number distribution for two independent squeezed vacuum reservoirs is the product of the photon number distributions of each squeezed vacuum reservoir.

5. Conclusion

We have derived the time evolution of the reduced density operator for a two- mode light coupled to two uncorrelated squeezed vacuum reservoirs. Employing the resulting equation, we have obtained the master equation for the signal-idler modes produced by a nondegenerate parametric oscillator coupled to two independent squeezed vacuum reservoirs and consequently the pertinent Fokker-Planck equation for the Q-function. We have solved this Fokker-Planck equation applying the propagator method developed by Fesseha [1].

We have calculated the intracavity quadrature fluctuations for the signal mode and signal-idler modes using the pertinent Q-functions. We have also obtained the quadrature fluctuations for the signal mode and the signal-idler modes for three cases of interest.

We have seen that the signal mode at steady state is in a squeezed state for certain values of r. We have also shown that the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two squeezed vacuum reservoirs are in a two-mode squeezed state at any time t and for any value of r. Furthermore, we have shown that at steady state and at threshold it is possible to produce an arbitrarily large squeezing (approaching 100%) in the first quadrature with enhanced noise in the second quadrature. In the absence of squeezed vacuum reservoirs, at steady state and at threshold, we have seen that a maximum of 50% suppression of noise below the quantum limit can be achieved. This is in complete agreement with results obtained in Ref. [2].

We have derived a general expression for the photon number distribution for a two-mode light in terms of the Q-function. Applying the resulting expression, we have obtained the photon number distributions for the signal mode and for the signal-idler modes.

In the absence of squeezed vacuum reservoirs, we have shown that the photon number distribution for equal number of signal and idler photons has the same form as the result obtained in Ref. [2]. We have also seen that photon number distribution, in the absence of cavity damping, has the same form as the photon number distribution of a two-mode squeezed vacuum reservoir [38] with r replaced by $\kappa\gamma_0 t$. For the case when there is no parametric interaction inside the cavity, we have found that there is a finite joint probability of finding odd numbers of photons associated with the squeezed vacuum reservoirs. This indicates the possibility for odd numbers of photons to enter into the port-mirror. However, at steady state the joint probability of finding odd numbers of photons is zero and this joint probability is simply the product of the photon number distributions of the squeezed vacuum reservoirs A and B.

Finally, we would like to point out that one of the main tasks in the this thesis has been solving the pertinent Fokker-Planck equation for the Q-function. This has been done by transforming the the Fokker-Planck equation in Cartesian Coordinates (2.53) into a schrödinger-type equation and then determining the corresponding Q-function propagator using the method discussed in Ref. [1]. This method is applicable for obtaining an explicit form of the propagator associated with a quadratic Hamiltonian of arbitrary form. The task of evaluating the propagator using this method essentially reduces to the problem of solving the Euler-Lagrange equations.

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