

**ELECTROMAGNETIC FIELDS
AND
THE RADIATION FROM
A SYSTEM OF TWO CHARGES**

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ABSTRACT

The classical two - body problem has been considered and the exact solution of Kepler's problem has been obtained for the general case.

Using this solution the spectrum of the radiation from the system of two charged particles with Coulomb interaction is briefly discussed. In addition the angular distribution of the average power radiated by the system is precisely calculated. The effect of the radiation on the system and the spectrum is deeply investigated. Relativistic circular motion of a charge and the spectrum of the radiation is considered in the second part in which the direct way of evaluation of the average power radiated by the system is considered. The interference of the radiation from a system of two charges in a circular motion is also considered. The Lagrange expansions of the electromagnetic potentials of a relativistic charge in a circular motion is considered for far zone. The exact expression of the simultaneous electromagnetic potentials of the charge for any distance is also considered as well as their spectrum for a circular motion.

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Introduction

It is well known that the two-body problem is a basic problem in physics. It can be used as a model for the simplest possible structure of an atom. The classical two-body problem for charged particles in their electromagnetic interaction has no exact solution due to the radiation from the two particles and the impossibility to determine the simultaneous interaction of the particles. The radiation causes the instability of the classical model of hydrogen atom, which led N.Bohr to formulate his famous semi-quantum theory for such a system. He developed his theory for circular motion of an electron around a proton assuming that the circumference of the circle is quantized. Yet, it is to be noted that he took for granted that an electron in its lowest orbit does not radiate. While the Bohr theory correctly predicts the spectral series of hydrogen atom it is incapable of being extended to treat the spectra of complex atoms having two or more electrons each. A more general theory, Quantum theory, was then developed to give detailed explanations of atoms including their interactions. The theory, however, had some preconditions (assumptions) which enable it to overcome the question of stability of atoms. Some of the assumptions are the wave function of quantum particle must be continuous and finite every where and should vanish at infinity. With these assumptions in mind Quantum theory could be used to solve the

two-body problem for charged particles even to the extent of arguing the stability of atoms.

The mystery of non-radiating electron while it is accelerating is till left open for question. For this reason the classical solution of the two-body problem failed to describe the structure of a simple atom, thus, the paradox of radiative collapse of hydrogen atom in classical electrodynamics could be solved only in terms of quantum theory.

Therefore we strongly believe that the deep investigation of the Kepler problem for charged particles may help in understanding the whole physics of atoms, particularly in accounting for the stability of an atom. The problem is equally important in plasma physics too, since collisions between two charged particles play crucial role in the establishment of an equilibrium state.

It is evident from classical electrodynamics that an electron initially set to revolve around a certain attractive force center can not repeatedly trace its original orbit due to radiation. The radiation carries away with it energy and angular momentum from the system. This loss to radiation brings about the decrease of the energy and angular momentum of the system that can keep the electron in flight. Hence the electron will be forced to deviate from its original orbit. However, if the electron is moving non-relativistically justifiable approximations show that the electron nearly retraces the orbit many times before falling to the center. Relativistic charges (electrons) would rather deviate more from their original orbit

because relativistic charges radiate more. On the other hand, electrons can be made rotating around a given circular orbit by applying external fields such as magnetic field as in the case of an electron in a synchrotron radiation. Still, despite the presence of magnetic field, a relativistic electron may deviate from its original orbit due to radiation reaction if the field is homogeneous as is discussed by R Lieu(1987).

In fact the Kepler problem with drag has been considered by P GL Leach (1987) to describe and estimate the deviation from an original orbit due to drag. His considerations are mainly devoted to neutral bodies with drag law proportional to the velocity and inversely proportional to the radial distance. The result he got shows that due to the drag the body keeps on spiralling and sooner or later falls into the center of force. Although spiralling is the natural consequence of the radiation, his results can not be used for charged particles because the law of drag he considered is different from the law of drag for the charged particles. That the law of drag for charged particles is not proportional to its inverse radial distance for one thing, and is not also proportional to its velocity for another.

The direct measurement of the deviation in the atomic scale is rather unthinkable. In this case an indirect measurement is possible by investigating the spectrum of their radiation. When an electron is maintained to move in a perfectly periodic motion, the radiation fields will have precisely repeated values in every period $T=2\pi/\omega_0$, and its Fourier analysis would have

consisted of discrete radiated frequencies $w=n(2\pi/T)=nw_0$ that are integer multiples of the fundamental w_0 . Any deviation from the original (preceding) orbit would spoil the periodicity of the motion and abandon the use of Fourier-series expansion. In such cases the discreteness of the spectrum disappears.

As a matter of fact no spectrum is discrete in physically realized situations. Each of the so-called discrete frequencies emitted by atoms, for instance, is broadened in accordance with $w=2\pi/\Delta t$ by the finite duration Δt of the states of motion that supply the radiated energy.

There is at least a so-called natural broadening just because the losses to radiation must change the energy of the motion (some estimates will be made in sec.(2.5)). For similar reasons, electrons in a synchrotron tend to spiral out of their orbits and this may broaden each of the harmonics $w=nw_0$ sufficiently to result in a continuous spectrum.

The angular distribution of the power radiated for non-relativistic circular motion is the example given in every standard electrodynamics books. It is found to have simple dependence on the angle of radiation and the plane of the orbit, but does not depend on the angular position of the position vector of the charge. Although elliptic orbit is a possible orbit, the angular distribution of the radiation from such an orbiting charge is not easy to find. Unlike the circular motion the power radiated by the charge depends on the angle of its position vector. Its radiation is also discrete.

Normally when calculating the power radiated from a non-

relativistic charge in circular orbit, for example, it turns out that the dipole radiation is the only term considered and we usually say the dipole radiation corresponds to the non-relativistically moving charge. In spite of their presence, higher multipole radiations are neglected compared to the first multipole radiation, the dipole radiation. This result follows from the following conditions that must be fulfilled:

- i) the point of observation be in the far zone ($r \gg r'$) and in the radiation zone ($kr \gg 1$) and,
- ii) the particle's speed be much less than the speed of light ($wr' \ll c$).

If, however, elliptic motion of the charge is considered every multipole radiation becomes comparable to its neighbours and all harmonics $w = nw_0$ appear distinctively despite the above conditions.

Interference effects lead to another aspect of the radiation from the system of more than one charge. It is an experimental fact that a circular current-carrying wire does not radiate despite the presence of accelerated charges constituting the current. This, we believe should be explained in terms of the interference of the fields produced by the charges. Specifically, the fields must be interfering destructively in this case.

Another thing one would like to know is the electromagnetic

interaction of two charges. Yet, one may ask for the simultaneity of their interaction. It is always necessary to know where one particle is when the field is emitted by the other and vice versa. The simultaneous expansion of the fields produced due to a relativistic charge is a more convenient method to describe the electromagnetic interaction of charges. For this the Lagrange expansion method can be of great help as is done in the papers of A.N Gordeyev (1975) and ch G van Weert (1976).

The aim of this research is the relativistic generalization of two-body problem for charged particles. This is very important for the consideration of electromagnetic radiation from such a system. It is also important to investigate the interference effects in the coherent radiation from both particles constituting the system. The progress in that direction will be also of great value for the understanding of relativistic particles with electromagnetic interaction.

In this thesis work we shall divide our study into four main chapters. In the first chapter a short, precise and concise survey of classical Kepler problem will be made. The two particles will be considered to have different masses and considerations will be limited to bounded motion. With the two-body problem reduced to single body problem, the orbit equation will be derived using vector analysis only and will be shown to be elliptic. The additional constant of motion peculiar to potentials proportional to $1/r$ will get little emphasis. Followingly we shall give the derivation of Kepler's problem

which will be solved using Fourier-series expansion. The solution of Kepler's problem will permit us to find the explicit time dependence of the coordinates of the position vector of the particle. Finally the "quantization" of energy and angular momentum of the system will be made apparent in terms of the fourier components of the rectangular components of the position vector.

The second chapter will be mainly devoted to radiation from two non-relativistic charges that are interacting with coulomb-interaction. In the first section of this chapter we develop the basics behind the theory of radiation from a non-relativistically accelerated charge. In particular we shall first find the fields from the periodic motion of a charge using Fourier-series expansion and evaluate the power and angular momentum radiated into the m th-multipole using the Fourier components of the fields. The assumptions (approximations) made in evaluating the radiation are the distance of the observation point is much greater than the size of the charge distribution (for zone) and the wavelength of the emitted wave (radiation zone). Above and all non-relativistic motion is considered in this chapter . The approximations made will permit us to evaluate the radiation in terms of the fourier component of the dipole moment of the system. The mathematical facilities laid in chapter one will enable us to evaluate the fourier component of the dipole moment of the system. Then it will be shown that the spectrum of the radiation is discrete. Further more the spectrum in some different cases (having same energy but different

initial angular momentum) will be given. Consequently comparison of which harmonics radiate more will be made. The effect of radiation in the absence of an external field is then discussed in the subsequent two sections. The first will be on the effect of radiation on the orbit, the gradual collapse of the system and some estimates of the collapsing time of the system will be made. Of course results will be compared with some known results. The second will have brief discussions of the broadening of each spectral lines due to radiation.

The radiation from a relativistic charge in circular motion will be considered in the third chapter. The magnetic field produced by the charge will be expanded in Fourier-series and the exact expression for the angular distribution^{of} the power radiated into the n th multipole of w_0 will be determined. In the process, of course, we come across a long and difficult derivations. Some of them are derived there and some postponed to appendix. Following this, attempt will be made to find the radiation from two charges and the^{interference} effect will be considered. More attention will be given to non-relativistic charges moving on the same circle but on the opposite ends of a diameter. We shall also investigate the destructive / constructive nature of the interference for same / opposite charges.

The last chapter is devoted to simultaneous expansions of the fields from a relativistic charge. The main advantage of this is the possibility to exclude independent consideration of the radiation field of the particles since it will be expressed completely in terms of the characteristics of the motion of the

particles. The first half of the chapter considers the evaluation of the fields in far zone. The second half is devoted to evaluation of the exact simultaneous fields for any point outside the charge, which is true irrespective of its velocity. This will enable one to deal with the electromagnetic interaction of charged particles.

Finally appendix is given in which much of the mathematical derivations that are necessary for this work are given. We come to the end this work by making some conclusions and giving main results.

CHAPTER ONE

CLASSICAL KEPLER PROBLEM

1. 1 Review Of Classial Two-Body Problem

To understand and describe the motion of two interacting bodies, what we almost always do is we reduce the two body problem into a single body problem for certain suitable interaction potentials. In this respect, the non-relativistic motion of two particles, that are interacting with attractive potentials which are inversely proportional to the distance between the particles, can be described by the following constants of motion:

$$\langle E \rangle = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r} \quad (1.1.1)$$

$$\langle \underline{L} \rangle = \underline{r} \times \mu \underline{v} = \mu \dot{r}^2 \theta \hat{e}_3 \quad (1.1.2)$$

$$\underline{C} = \underline{v} \times \underline{L} - k \hat{e}_r \quad (1.1.3)$$

Where $\langle E \rangle$ and $\langle \underline{L} \rangle$ are the total energy and angular momentum of the system. \underline{C} is a third additional constant of motion peculiar to the above mentioned type of potential. It is called the Laplace-Runge-Lenz vector. The existence of the further integral of motion is due to the degeneracy of the motion associated with potentials of the coulomb form k/r . In the above equations μ is the reduced mass, r is the distance between the particles, k is a constant, \hat{e}_3 is a unit vector normal to the plane of motion, \hat{e}_r is a unit vector in the direction of r and the velocity \underline{v} is the time

derivative of r . The dots in (1.1.1) and (1.1.2) indicate the time derivative of the variables. The time average sign $\langle \rangle$ in eqns (1.1.1) and (1.1.2) are to be kept just to facilitate the discussion in the next section. However, it is worth while to note that E and L are actually constants for non-dissipating systems and they can replace $\langle E \rangle$ and $\langle L \rangle$ in eqns (1.1.1) and (1.1.2) respectively.

Eqn: (1.1.1) contains three different terms on its right hand side. The first is the kinetic energy of the system, the second is the centrifugal potential energy and the last term is the interaction potential energy of the two particles.

The orbit equation can be obtained easily if we take the scalar product of \underline{c} with \underline{r} :

$$\underline{r} \cdot \underline{c} = rC \cos\theta \quad (1.1.4)$$

Where $r = |\underline{r}|$, $C = |\underline{c}|$ and θ is the angle measured from \underline{c} to \underline{r} . On the other hand, from eqn (1.1.3) we can write

$$\underline{r} \cdot \underline{c} = \underline{r} \cdot (\underline{V} \times \underline{L}) - kr. \quad (1.1.5)$$

Noting that

$$\underline{r} \cdot (\underline{V} \times \underline{L}) = \underline{L} \cdot (\underline{r} \times \underline{V}) = \frac{L^2}{\mu}$$

and substituting it in eqn(1.1.5) and rearranging together with eqn (1.1.4) gives

$$r(\theta) = \frac{L^2 / \mu k}{1 + \frac{C}{k} \cos(\theta)} \quad (1.1.6)$$

The magnitude of \underline{c} can be found from the scalar product of

eqn (1.1.3) with itself :

$$C^2 = |\underline{V} \times \underline{L}|^2 + k^2 - \frac{2k}{\mu r} L^2. \quad (1.1.7)$$

But

$$|\underline{V} \times \underline{L}|^2 = V^2 L^2$$

because \underline{V} and \underline{L} are perpendicular to each other. Where $\underline{V} = |\dot{\underline{r}}|$ is the magnitude of $\dot{\underline{r}}$. Substituting this in eqn(1.1.7) gives

$$C = k \left(1 + \frac{2EL^2}{\mu k^2} \right)^{\frac{1}{2}} \quad (1.1.8)$$

after some rearrangements. Therefore the orbit equation takes the form :

$$r(\theta) = \frac{A}{1 + \epsilon \cos(\theta)} \quad (1.1.9)$$

Where $A = L^2 / \mu k$ and $\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k}}$

Eqn (1.1.9) is the equation of a conic section with one focus at the origin. The eccentricity ϵ of the orbit is given by

$$\epsilon = \frac{C}{k} = \left(1 + \frac{2EL^2}{\mu k^2} \right)^{\frac{1}{2}} \quad (1.1.10)$$

In the equivalent problem of two particles interacting according to the attractive potential mentioned before, the orbit of each particle is a conic section with one focus at the center of mass of the two particles.

It can be seen from (1.1.10) that, if $-\mu k^2 / 2L^2 < E < 0$ then $0 < \epsilon < 1$, i.e the orbit is an ellipse with semi-major axis a and semi-minor axis b given by (see fig 1.1.)

$$a = \frac{k}{2|E|} = \frac{A}{1-\epsilon^2} \quad (1.1.11)$$

$$b = a\sqrt{1-\epsilon^2} = \frac{L}{\sqrt{2\mu|E|}} \quad (1.1.12)$$

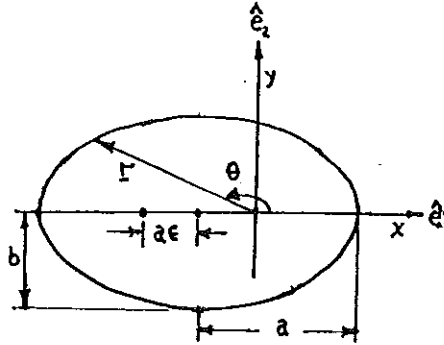


fig.(1.1)

Since the motion is finite ($0 < \epsilon < 1$), then it will be a periodic one with a period given by

$$T = 2\pi \left(\frac{\mu a^3}{k^2} \right)^{1/2} \quad (1.1.13)$$

The corresponding angular frequency of the motion is then related to a or E by

$$\omega^2 = \frac{k}{\mu a^3} = - \frac{8E^3}{\mu k^2} \quad (1.1.14)$$

The more is the semi-major axis, the less frequent the motion repeats itself. It may be useful to note that ω depends only on the energy of the system and not on its angular momentum. But ϵ depends on both E and L .

The two body problem (Kepler Problem) is said to be completely solved if one can determine the coordinates (r, θ) or (x, y) as an explicit function of time. The determination can be facilitated if we make the following transformations:

$$x = a(\cos\phi - \epsilon) \quad (1.1.15)$$

$$Y = b \sin\phi \quad (1.1.16)$$

Squaring and adding eqns (1.1.15) and (1.1.16) and substituting it in $r = (x^2 + y^2)^{\frac{1}{2}}$ yield

$$r = a (1 - \epsilon \cos\phi). \quad (1.1.17)$$

The relation between θ and ϕ can be found from eqns (1.1.9) and (1.1.17) (noting that $A = a (1 - \epsilon^2)$ from (1.1.11)) to be

$$\tan \frac{\theta}{2} = \frac{\sqrt{1+\epsilon}}{\sqrt{1-\epsilon}} \tan \frac{\phi}{2} \quad (1.1.18)$$

We learn from eqns (1.1.15)-(1.1.18) that if we can determine ϕ as an explicit function of time, then we can determine x, y, r and θ as explicit functions of time at least in principle. Using the orbit equation, eqns (1.1.17), (1.1.18) and the statement of Kepler's second law that

$$\frac{\pi ab}{T} dt = \frac{1}{2} r^2 d\theta$$

We can find ϕ as an implicit function of time given to be

$$w_0 t = \phi(t) - \epsilon \sin\phi(t) \quad (1.1.19)$$

Eqn(1.1.19) is called Kepler's problem. In the next section we investigate the behavior of $\epsilon \sin\phi(t)$ and expand it in Fourier-series to find $\phi(t)$ as an explicit function of time t .

1.2 Fourier-Series Expansion Of Kepler's Problem
Interms Of Bessel Functions.

Before trying to expand $\epsilon \sin\phi$ into a Fourier series let's re-write eqn (1.1.19) as

$$U(t') = t' + \epsilon \sin U(t') \quad (1.2.1)$$

just to make writing economical. Where

$$t' = \omega_0 t \quad \text{and} \quad U(t') = \phi(t(t')).$$

Then we shall show that $\epsilon \sin U(t')$ is a periodic function of t' of period 2π .

Eqn (1.2.1) defines $U(t')$ as a strictly increasing function of t' , because upon differentiating (1.2.1) with respect to t' we get

$$\frac{du}{dt'} = \frac{1}{1 - \epsilon \cos U} \quad (1.2.2.)$$

Which is always greater than zero. conversely, t' can be also considered as a strictly increasing function of u for a similar reason.

It is clear that at a certain fixed $t' = t'_0$

$$U(t'_0) - \epsilon \sin U(t'_0) = t'_0 \quad (1.2.3.)$$

and

$$U(t'_0 + 2\pi) - \epsilon \sin U(t'_0 + 2\pi) = t'_0 + 2\pi$$

But also

$$U(t'_0) + 2\pi - \epsilon \sin(U(t'_0) + 2\pi) = t'_0 + 2\pi$$

since t' is a strictly increasing function of U , the solution of (1.2.3) is uniquely determined, so that

$$U(t'_0 + 2\pi) = U(t'_0) + 2\pi \quad (1.2.4)$$

Hence, $\sin U(t')$ is a periodic function of t' of period 2π . We can also demonstrate that $\sin U(t')$ is an odd function of t' . From eqn (1.2.3) it is clear that

$$U(-t'_0) - \epsilon \sin U(-t'_0) = -t'_0$$

and

$$-U(t'_0) - \epsilon \sin(-U(t'_0)) = -t'_0$$

The uniqueness of the solution of (1.2.3) requires that

$$U(-t'_0) = -U(t'_0) \quad (1.2.5)$$

Then it follows that

$$\sin U(-t'_0) = -\epsilon \sin U(t'_0)$$

Showing that $\sin U(t'_0)$ is an odd function of t'_0 . In addition to the above two properties we also have $\sin U(\pi) = 0$ because $U(\pi) = \pi$; see eqn (1.2.1)

We can now make use of the theory of Fourier-series to expand $\epsilon \sin U(t')$ in a uniformly convergent sine series as follows:

$$\epsilon \sin U(t') = \sum_{n=1}^{\infty} A_n \sin(nt') \quad (1.2.6)$$

with

$$A_n = \frac{2}{\pi} \int_0^{\pi} \epsilon \sin U(t') \sin(nt') dt' \quad n = 1, 2, 3, \dots$$

Integrating by parts yields

$$A_n = \frac{-2\epsilon \sin U(t') \cos(nt')}{n\pi} \Big|_0^{\pi} + \frac{2}{n\pi} \int_0^{\pi} \cos(nt') \frac{d(\epsilon \sin U)}{dt} dt \quad (1.2.7)$$

Which reduces to

$$A_n = \frac{2}{n\pi} \int_0^{\pi} \left(\frac{dU}{dt} - 1 \right) \cos(nt') dt'$$

if we use eqn (1.2.1). Expanding the integrand yields

$$A_n = \frac{2}{n\pi} \int_0^{\pi} \frac{dU}{dt} \cos(nt') dt' - \frac{2}{n\pi} \int_0^{\pi} \cos(nt') dt'$$

The last integral vanishes and in the first integral we replace t' by $U - \epsilon \sin U$ to get

$$A_n = \frac{2}{n} \left[\frac{1}{\pi} \int_0^{\pi} \cos(nu - n\epsilon \sin U) du \right] = \frac{2}{n} J_n(n\epsilon) \quad (1.2.8)$$

The expression in the square bracket, $[\]$, is the integral representation of Bessel functions, $J_n(n\epsilon)$, of integral order n and argument $n\epsilon$. Eqn (1.2.6) can therefore be written as

$$\epsilon \sin U(t') = \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\epsilon) \sin(nt') \quad (1.2.9a)$$

and eqn (1.2.1) is solved by the series

$$U(t') = t' + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\epsilon) \sin(nt') \quad (1.2.9b)$$

Going back to our original notations $w_0 t = t'$, $\phi(t) = U(t'(t))$ we

can write the solution of Kepler's problem (i.e. ϕ as an explicit function of time t) as

$$\phi(t) = W_0 t + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\epsilon) \sin(nw_0 t) \quad (1.2.10)$$

If we do similar steps to expand $\epsilon \cos U(t')$ as was done for $\epsilon \sin U(t')$ we arrive at

$$\epsilon \cos U(t') = \sum_{n=1}^{\infty} \frac{2\epsilon}{n} J'_n(n\epsilon) \cos(nt') - \frac{\epsilon^2}{2} \quad (1.2.11)$$

Where $J'_n(n\epsilon)$ is the derivative of Bessel function with respect to its argument. The explicit time dependence of the coordinate x on t can be found by substituting eqn (1.2.11) in eqn (1.1.15)

$$x(t) = \sum_{n=1}^{\infty} \frac{a_n}{n} \cos(nw_0 t) - \frac{3\epsilon a}{2} \quad (1.2.12)$$

where

$$a_n = a (J_{n-1}(n\epsilon) - J_{n+1}(n\epsilon)) = 2a J'_n(n\epsilon) \quad (1.2.13)$$

Similarly substituting (1.2.9a) in (1.1.16) yields

$$y(t) = \sum_{n=1}^{\infty} \frac{b_n}{n} \sin(nw_0 t) \quad (1.2.14)$$

where

$$b_n = a\sqrt{1-\epsilon^2} (J_{n-1}(n\epsilon) + J_{n+1}(n\epsilon)) = \frac{2a\sqrt{1-\epsilon^2}}{\epsilon} J_n(n\epsilon) \quad (1.2.15)$$

It is also possible to find $r(t)$ using eqn (1.2.11) in eqn(1.1.17)

$$r(t) = a \left[1 - \frac{\epsilon^2}{2} + \sum_{n=1}^{\infty} \frac{2}{n} J'_n(n\epsilon) \cos(nw_0 t) \right] \quad (1.2.16)$$

Note that in expressing (1.2.13) and (1.2.15) as the difference and sum of Bessel functions we've used the recurrence relations for Bessel function. It is also to be noted that we also come across expressions containing $1/r(t)$ in dealing with Kepler's problem. To find $1/r(t)$ in a rather easier form than taking the inverse of (1.2.16), let's substitute $(1-\epsilon \cos u)$ by $1/(dU/dt')$ from eqn(1.2.2) in eqn (1.1.17) to get

$$\frac{1}{r(t)} = \frac{1}{a} \frac{du}{dt'}$$

Differentiating eqn(1.2.9b) with respect to t' ; changing t' by $w_0 t$ and substituting it in the preceding equation yields

$$\frac{1}{r(t)} = \frac{1}{a} \left[1 + \sum_{n=1}^{\infty} 2J'_n(n\epsilon) \cos(nw_0 t) \right] \quad (1.2.17)$$

Now that we have got $x(t)$, $y(t)$ and $r(t)$ as explicit functions of time let's make some comparisons and remarks on averaging over the angle ϕ and time t before we close this section. A comparison between (1.1.15) and (1.2.12) shows that while the ϕ -average of x gives $-a\epsilon$, the time average of x gives $-3a\epsilon/2$ i.e.

$$\langle x \rangle_{\phi} = -a\epsilon$$

$$\langle x \rangle_t = -\frac{3a\epsilon}{2}$$

The physical meaning of the time average of x is that the particle spends much of its time to the left of the origin

than to the right of it. This is justifiable from the point of view of the conservation of the energy of the system. The constant total energy of the system is the sum of kinetic and potential energies. Since the potential energy is proportional to $-1/r$ then, the smaller is r the higher is the potential energy (negatively) and the higher is its kinetic energy corresponding to faster velocity v . So when v is high the particle spends shorter time in the region where r is small (see. fig 1.1) and vice versa.

So it has to be noted that the time t -averaging and the angle ϕ averaging are different. This difference is expected because

$$\frac{d\phi}{dt} = \frac{w_0}{1-\epsilon\cos\phi} \quad (1.2.18)$$

To take the time average of functions involving ϕ only like eqn(1.1.15) one should change dt by $((1-\epsilon\cos\phi)/w_0)dU$ in the process of averaging

If we look at eqns (1.1.16) and (1.2.14), however, the ϕ -average and the t -average of y give equal value that is zero.

$$\langle y(t) \rangle_t = \langle y(\phi) \rangle_\phi = 0 \quad (1.2.19)$$

This is natural because the motion is symmetric about the x -axis.

1.3 "Quantization" Of Energy And Angular Momentum Of The System

Consider the energy of the system written in the form

$$E = \frac{1}{2} \mu (\dot{x}^2 + \dot{y}^2) - \frac{k}{r} \quad (1.3.1)$$

Where the dots indicate the time derivatives of the variables. The first term of eqn (1.3.1) is the kinetic energy and the second is the potential energy of the system. Differentiating eqns(1.2.12) and (1.2.14) with time t , squaring them, using eqn (1.2.17) and substituting it in eqn(1.3.1) yield

$$E = \frac{\mu \omega_0^2}{2} \sum_{n,m=1}^{\infty} (a_n a_m \sin(n\omega_0 t) \sin(m\omega_0 t) + \sum_{n,m=1}^{\infty} (b_n b_m \cos(n\omega_0 t) \cos(m\omega_0 t) - \frac{k}{a} (1 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t))) \quad (1.3.2)$$

Where

$$C_n = 2 J_n'(n\epsilon)$$

But does this mean that ϵ depend on time ? To answer this consider the following from eqns (1.1.15) and (1.1.16)

$$\dot{x} = -a \dot{\phi} \sin\phi$$

$$y = b \dot{\phi} \cos\phi$$

This after being squared followed by the substitution in to the kinetic energy term of eqn(1.3.1) yield

$$T = \frac{\mu a^2 \omega_0^2 (1 - \epsilon^2 \cos^2 \phi)}{2(1 - \epsilon \cos \phi)^2} \quad (1.3.3)$$

In which we have substituted $\omega_0^2 / (1 - \epsilon \cos \phi)^2$ by $\dot{\phi}^2$. The potential energy term of (1.3.1) can be also expressed as

$$U = \frac{-k}{r} = - \frac{\mu \omega_0^2 a^2}{(1 - \epsilon \cos \phi)} \quad (1.3.4)$$

if we use eqns (1.1.17) and (1.1.14). This together with (1.3.3) yield

$$E = T + U = \frac{-k}{2a} \quad (1.3.5)$$

Which is actually a constant. On the other hand, the time average of (1.3.2) gives

$$\langle E \rangle = - \sum_{n=1}^{\infty} \frac{\mu \omega_0^2}{4} (a_n^2 + b_n^2) = - \sum_{n=1}^{\infty} \langle E_n \rangle \quad (1.3.6)$$

$$= - \mu \omega_0^2 a^2 \sum_{n=1}^{\infty} J_n^2 + \frac{1 - \epsilon^2}{\epsilon^2} J_n^2 \quad (1.3.7a)$$

Where

$$\langle E_n \rangle = \mu \omega_0^2 a^2 \left(J_n'^2 + \frac{1 - \epsilon^2}{\epsilon^2} J_n^2 \right) \quad (1.3.7b)$$

The series on the right of (1.3.7a) can be summed (see appendix) and it is exactly $\frac{1}{2}$. This and eqn (1.1.14) reduces (1.3.7a) to

$$\langle E \rangle = \frac{-k}{2a} \quad (1.3.8)$$

Comparison between eqns (1.3.5) and (1.3.7a) shows that, with or without averaging, E remains the same $-k/2a$ and does not show any time dependence. Hence, eqn (1.3.2) can be said to

show the apparent (fictitious) dependence of \underline{L} on t .

We can also write \underline{L} in terms of x , y and their time derivatives as

$$\underline{L} = \mu (x \dot{y} - y \dot{x}) \hat{e}_3$$

(1.3.9) Similar to (1.3.2) the apparent dependence of \underline{L} on t is given by

$$\begin{aligned} \underline{L} = & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n w_0 a_n b_m \left(\frac{\cos(n w_0 t) \cos(m w_0 t)}{n} + \frac{\sin(n w_0 t) \sin(m w_0 t)}{m} \right) \\ & - \frac{3 \epsilon a w_0 \mu}{2} \sum_{n=1}^{\infty} b_n \cos(n w_0 t) \hat{e}_3 \end{aligned} \quad (1.3.10)$$

The same procedure as was done for E using eqns (1.1.15), (1.1.16) and (1.2.18) in eqn (1.3.9) can be made to see that \underline{L} is also a constant given by

$$\underline{L} = \mu w_0 a^2 \sqrt{1 - \epsilon^2} \hat{e}_3$$

On the other hand the time average of (1.3.10) yields

$$\langle \underline{L} \rangle = \sum_{n=1}^{\infty} \left(\frac{\mu w_0 a_n b_n}{n} \right) \hat{e}_3 = \sum_{n=1}^{\infty} \langle L_n \rangle \hat{e}_3 \quad (1.3.11)$$

$$= \frac{4 \mu w_0 a^2 \sqrt{1 - \epsilon^2}}{\epsilon} \sum_{n=1}^{\infty} \frac{J_n J'_n}{n} \quad (1.3.12)$$

Where

$$\langle L_n \rangle = \frac{4 \mu w_0 a^2 \sqrt{1 - \epsilon^2}}{n \epsilon} J_n(n \epsilon) J'_n(n \epsilon) \quad (1.3.13)$$

The series on the right of (1.3.12) can be summed and it is

equal to $\epsilon/4$. Therefore (1.3.11) reduces to

$$\langle \underline{L} \rangle = \mu W_0 a^2 \sqrt{1-\epsilon^2} \hat{e}_3 .$$

With or without averaging \underline{L} also remains a constant. This shows that \underline{L} in (1.3.10) is just seemingly dependent on t being actually a constant

Eqns(1.3.6) and (1.3.11) reflect the fact that the energy and angular momentum of the system are "quantized". The energy E of the system, for example, is the sum of the discrete energy levels $\langle E_n \rangle$. The discreteness of E or \underline{L} of the system, however, disappears for $\epsilon \rightarrow 0$. This shows that the origin of "quantization" is the presence of ϵ . But it has to be remembered that, the energy E and angular momentum \underline{L} of the system can take any value as far as classical mechanics is concerned. And that value of E or \underline{L} can be expressed as the sum of discrete levels. There is no forbidden E or \underline{L} unlike in quantum mechanics. More over the dependence of E or L on n is different from that in quantum theory. In short, this is to say that the sense of quantization in our case and that in quantum theory are different both in origin and dependence on n .

CHAPTER TWO

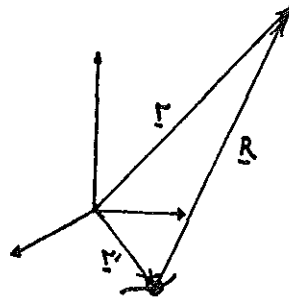
RADIATION BY NON-RELATIVISTIC CHARGESAND ITS EFFECTS2.1 Introduction

Up to now we have been discussing the mechanical properties of the motion, of two bodies that are interacting with each other with forces that are central and proportional to the inverse square of the distance between them. The developed properties are based on the assumptions that the system is isolated, does not have any interactions with other systems, and under no circumstances loses/gain energy or angular momentum. In other words, the energy and angular momentum of the system is conserved. Now it will be the subject of this chapter to give detailed study of the non-relativistic motion of the system if it loses energy and angular momentum due to radiation.

Among all possible motions governed by the law of force mentioned above, coulombic attraction of two oppositely charged particles is a case in point. We shall assume only non-relativistic motion. We begin to study the motion by first establishing the basis behind the radiation of an accelerated charge.

Consider a charged body in arbitrary motion. The retarded vector potential due to the moving charge (see fig 2.1) is given by

$$\underline{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{\underline{J}(\mathbf{r}', t - R/c)}{R} dv' \quad (2.1.1)$$



(Fig.2.1)

Where \underline{J} is the current density, dv' is the volume element and $R = |\underline{r} - \underline{r}'|$. If Fourier-Transform is made such that

$$\underline{A}(\underline{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{A}(\underline{r}, \omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{A}_\omega(\underline{r}) e^{-i\omega t} d\omega \quad (2.1.2a)$$

and

$$\underline{J}(\underline{r}', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{J}(\underline{r}', \omega) e^{-i\omega t'} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{J}_\omega(\underline{r}') e^{-i\omega t'} d\omega \quad (2.1.2b)$$

then substitution in eqn (2.1.1) gives

$$\underline{A}_\omega(\underline{r}) = \frac{1}{c} \int \frac{\underline{J}_\omega(\underline{r}') e^{ikR}}{R} dv' \quad (2.1.2c)$$

Where $t' = t - R/c$, $k = \omega/c$ is the magnitude of the wave vector and $\underline{A}_\omega(\underline{r}) = \underline{A}(\underline{r}, \omega)$ and $\underline{J}_\omega(\underline{r}') = \underline{J}(\underline{r}', \omega)$ are the Fourier Components. For $r \gg r'$ (in far zone)

$$R = r \left(1 - \frac{2\hat{n} \cdot \underline{r}'}{r} + \frac{r'^2}{r^2} \right)^{1/2} \approx r - \hat{n} \cdot \underline{r}' + \frac{r'}{2} \left(\frac{r'}{r} [1 - (\hat{n} \cdot \hat{n}')^2] \right) \quad (2.1.3)$$

Therefore :-

$$e^{ikR} \approx e^{ikr} e^{-ik \cdot \underline{r}'} \left[1 + ikr' \frac{r'}{2r} (1 - (\hat{n} \cdot \hat{n}')^2) \right] \quad (2.1.4)$$

Where

$$\underline{k} = k\hat{n} = k\hat{n}$$

is a wave vector directed toward the field point and \hat{n}' is a unit vector along \underline{r}' . The last term of (2.1.4) is to be discriminated if we assume that $r \gg (a/\lambda)a$. Where $\lambda = 1/k$ and $a = \max |\underline{r}'|$. Under this condition we write eqn(2.1.4) for a non-relativistic motion ($ka \ll 1$) as

$$e^{ikR} \approx e^{ikr}(1 - i\underline{k} \cdot \underline{r}') \quad (2.1.5)$$

All these being considerations regarding the retardation factor e^{ikR} in (2.1.2c) we now turn our attention to the denominator of the integrand of eqn(2.1.2c)

$$\frac{1}{R} = \frac{1}{r} \left(1 - \frac{2\underline{n} \cdot \underline{r}'}{r} + \frac{r'^2}{r^2} \right)^{-1/2} \quad (2.1.6)$$

In far zone

$$\frac{1}{R} \approx \frac{1}{r} \left(1 + \frac{\hat{n} \cdot \underline{r}'}{r} \right)$$

Multiplying eqns(2.1.5) and (2.1.6) yield

$$\begin{aligned} \frac{e^{ikR}}{R} &= \frac{e^{ikr}}{r} \left(1 + \frac{\hat{n} \cdot \underline{r}'}{r} \right) (1 - i\underline{k} \cdot \underline{r}') \\ &\approx \frac{e^{ikr}}{r} \left(1 - i\underline{k} \cdot \underline{r}' + \frac{\hat{n} \cdot \underline{r}'}{r} \right) \end{aligned} \quad (2.1.7)$$

Where we have neglected a term proportional to $(r'/\lambda)r'$ in accordance to the condition $r \gg (a/\lambda)a$. When this is used in (2.1.2c) we find the Fourier-component of the vector potential as follows :

$$\underline{A}_w(\underline{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{rc} \left\{ \int \underline{J}_w(\underline{r}') dv' + \left(1 - \frac{1}{ikr}\right) \int (-i\mathbf{k}\cdot\underline{r}') \underline{J}_w(\underline{r}') dv' \right\} \quad (2.1.8)$$

This expression consists of two terms. The first term is called the electric dipole term and the second contains terms called the magnetic dipole and electric quadrupole combined together. In the non-relativistic case ($kr' \ll 1$) the first term is dominant over the second. The radiation that would be calculated from the first term of (2.1.8) only is called the dipole radiation. Therefore the dipole radiation correspond to a non-relativistic motion and higher multipole radiation arise due to relativistic corrections. In what will follow more emphasis will be given to the former case.

Let's consider the vector resulting from the divergence of the dyadic tensor (outer product) in

$$\nabla' \cdot (\underline{J}_w \underline{r}') = \underline{r}' \cdot (\nabla' \cdot \underline{J}_w) + (\underline{J}_w \cdot \nabla') \underline{r}' = i\omega \rho \underline{r}' + \underline{J}_w \quad (2.1.9)$$

Where the operator ∇' is to operate on the space variables (x', y', z'). The complete divergence on the left has a zero resultant upon integration over an entire source, and so $\underline{J}_w(\underline{r}')$ in the first term of eqn(2.1.8) can be replaced by $-i\omega \rho \underline{r}'$ to yield

$$\underline{A}_w(\underline{r}) = -ik \underline{D}_w \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r} \quad (2.1.10)$$

corresponding to non-relativistic cases. Where

$$\underline{D}_w = \int \underline{r}' \rho_w(\underline{r}') dv' \quad (2.1.11)$$

is the Fourier component of the dipole moment. Now that we

have evaluated $\underline{A}_w(\underline{r})$ that is valid in the far zone ($r \gg r'$), let's also find the fields \underline{B}_w and \underline{E}_w for $kr \gg 1$ (radiation zone).

With \underline{D}_w independent of the field point \underline{r} the curl operation in $\underline{B}_w = \nabla \times \underline{A}_w$ yields

$$\underline{B}_w(\underline{r}) = k^2 (\hat{k} \times \underline{D}_w) \left(1 - \frac{1}{ikr} \right) \frac{e^{ikr}}{r} \quad (2.1.12)$$

The second curl operation needed for $\underline{E}_w = (\nabla \times \underline{B}_w) / (-ik)$ can be also carried out to yield

$$\underline{E}_w(\underline{r}) = k^2 [(\hat{k} \times (\underline{D}_w \times \hat{k})) \frac{e^{ikr}}{r} + \frac{3\hat{n}(\hat{n} \cdot \underline{D}_w) - \underline{D}_w}{r^3} (1 - ikr) e^{ikr}] \quad (2.1.13)$$

It can be seen that the long range term proportional to r^{-1} is just the radiation field for the case of $kr' \ll 1$ and the shorter range terms reduce to the static form of a dipole field in the limit $k = \omega/c \rightarrow 0$

The resulting outward flux of energy radiated by a single monochromatic component averaged for the period of the motion is given by

$$\langle \underline{S}_w \rangle = \frac{ck^4}{8\pi r^2} \hat{k} |\hat{k} \times \underline{D}_w|^2 + \frac{ck^3}{8\pi r^3} \left(1 + \frac{1}{k^2 r^2} \right) i\hat{k} \times (\underline{D}_w^* \times \underline{D}_w) \quad (2.1.14)$$

This is the result of electric dipole radiation consisting of two major terms. The first has radial (longitudinal) direction $\hat{n} = \hat{k}$ and is inversely proportional to r^2 . The second is transverse to the radial direction and is inversely proportional to r^3 . In calculating the power radiated per unit solid angle by a single monochromatic component only the first

term will contribute

$$\langle \frac{dP_w}{d\Omega} \rangle = r^2 \hat{k} \cdot \langle \underline{S}_w \rangle = \frac{ck^4}{8\pi} |\hat{k} \times \underline{D}_w|^2 \quad (2.1.15)$$

Only the second term of eqn (2.1.14) will be involved in the evaluation of the angular momentum radiated per unit time per unit solid angle

$$\left\langle \frac{d}{d\Omega} \left(\frac{dL_w}{dt} \right) \right\rangle = \underline{r} \times r^2 \langle \underline{S}_w \rangle / c \quad (2.1.16)$$

We've established the general results for monochromatic sources. It remains now to use the results developed for our considerations of two-body problem.

2.2. Radiation In The Case Of Coulomb Attraction.

Consider now the motion of two opposite charges with coulomb inter-action. If the masses of the two charges are different then their non-relativistic motion is well described by the classical kepler problem discussed in chapter one. We can therefore use the results developed in chapter one, for example, the coordinates as expressed as explicit functions of time and the energy and angular momentum expressed as the sum of discrete levels. These together with the results of section (2.1) will help us to calculate the energy radiated by the two charges. Eqns(2.1.15) and (2.1.16) reveal the fact that to evaluate the radiated energy and angular momentum of a system requires the evaluation of fourier components of the dipole moment of the system. To simplify the evaluation let's shift the origin of the coordinates of Fig(1.1) by $3\epsilon a/2$ to the

left. See fig(2.2)

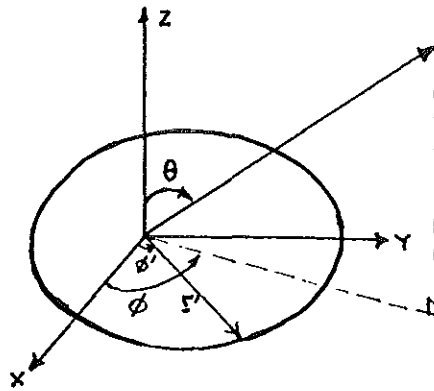


fig. (2.2)

In the new coordinates the corresponding eqns of (1.2.12) and (1.2.14) are given by

$$x'(t) = \sum_{n=1}^{\infty} \frac{a_n}{n} \cos(n\omega_0 t) \quad (2.2.1)$$

$$y'(t) = \sum_{n=1}^{\infty} \frac{b_n}{n} \sin(n\omega_0 t) \quad (2.2.2)$$

The dipole moment of the system is

$$\underline{D} = e (x' \hat{e}_1 + y' \hat{e}_2) \quad (2.2.3)$$

Where e is the magnitude of the charge carried by each particles and \hat{e}_1 and \hat{e}_2 are unit vectors along x' and y' respectively. Using eqns (2.2.1) and (2.2.2) in (2.2.3) we find the required fourier component of \underline{D} as

$$\underline{D}_w = e \left(\frac{a_n}{n} \hat{e}_1 + i \frac{b_n}{n} \hat{e}_2 \right) \quad (2.2.4)$$

Since the motion considered is periodic, $w = n\omega_0$ is the frequency of the n th harmonics. From now on we shall represent the Fourier components by a subscript "n" to indicate the n th harmonics for periodic motions.

To evaluate the power radiated per unit solid angle of (2.1.15) we need to evaluate $|\hat{\mathbf{k}} \times \underline{\mathbf{D}}_n|^2$. But let's first express $\hat{\mathbf{k}}$ in terms of spherical coordinates and the unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ as

$$\hat{\mathbf{k}} = \sin\theta \cos\phi \hat{\mathbf{e}}_1 + \sin\theta \sin\phi \hat{\mathbf{e}}_2 + \cos\theta \hat{\mathbf{e}}_3 \quad (2.2.5)$$

This together with (2.2.4) yield

$$|\hat{\mathbf{k}} \times \underline{\mathbf{D}}_n|^2 = \frac{e^2}{n^2} \left\{ (a_n^2 + b_n^2) + [a_n^2 \cos^2\phi + b_n^2 \sin^2\phi] \sin^2\theta \right\} \quad (2.2.6)$$

We can calculate the power radiated into the n th harmonics by substituting eqn (2.2.6) into (2.1.15) and after some rearrangements yield

$$\left\langle \frac{dP_n}{d\Omega} \right\rangle = \frac{e^2}{8\pi c^3} \frac{(n\omega_0)^4}{n^2} \left\{ (a_n^2 + b_n^2) - (a_n^2 \cos^2\phi + b_n^2 \sin^2\phi) \sin^2\theta \right\} \quad (2.2.7)$$

Expressing this in terms of Bessel functions by using the definitions of a_n and b_n we get

$$\begin{aligned} \left\langle \frac{dP_n}{d\Omega} \right\rangle = \frac{a^2 e^2}{2\pi c^3} (n\omega_0)^4 \left\{ \frac{1}{n^2} \left[(J_n')^2 + \frac{1-\epsilon^2}{\epsilon^2} (J_n)^2 \right] \right. \\ \left. - \frac{1}{n^2} \left[\cos^2\phi (J_n')^2 + \frac{1-\epsilon^2}{\epsilon^2} \sin^2\phi (J_n)^2 \right] \sin^2\theta \right\} \quad (2.2.8) \end{aligned}$$

The angular distribution of the entire power radiated can then be evaluated just by summing up the preceding

equation for all possible n as follows:

$$\left\langle \frac{dp}{d\Omega} \right\rangle = \left(\frac{aew_0^2}{2\pi c^3} \right)^2 \left\{ \sum_{n=1}^{\infty} n^2 \left[(J_n')^2 + \frac{1-\epsilon^2}{\epsilon^2} J_n^2 \right] \right. \\ \left. - \left[\cos^2 \phi \sum_{n=1}^{\infty} n^2 (J_n')^2 + \frac{1-\epsilon^2}{\epsilon^2} \sin^2 \phi \sum_{n=1}^{\infty} n^2 J_n^2 \right] \sin^2 \theta \right\}$$

The serieses on the right of this equation can be summed (see appendix) to yield

$$\frac{dp}{d\Omega} = \frac{(aew_0^2)^2}{32\pi c^3 (1-\epsilon^2)^{5/2}} \left\{ 4(2+\epsilon^2) - (4 + \epsilon^2 + 2\epsilon^2 \cos^2 \phi) \sin^2 \theta \right\}$$

This shows that the more is ϵ the more is radiation . More over we see from eqn (2.2.8) here that the spectrum is discrete. It is useful to note that the Bessel functions in (2.2.8) do not depend on the polar angles ϕ and θ . Therefore integration of (2.2.8) over the solid angle is possible for each harmonics. Integrating (2.2.7) over the solid angle yield

$$\langle P_n \rangle = 2\tau w_0^2 n^2 \left[\frac{\mu a w_0^2}{4} (a_n^2 + b_n^2) \right] \quad (2.2.9a)$$

because a_n and b_n are independent of the angles θ and ϕ .

Or

$$\langle P_n \rangle = 2\tau w_0^2 \left(\frac{\mu w_0^2 a^2}{4} \right) n^2 \left[(J_n')^2 + \frac{1-\epsilon^2}{\epsilon^2} (J_n)^2 \right] \quad (2.2.9b)$$

Which could also be expressed as

$$\langle P_n \rangle = 2\tau w_0^2 n^2 \langle E_n \rangle \quad (2.2.9c)$$

The last equality follows from the use of eqn(1.3.7a)

in (2.2.9b). Here $\tau = 2e^2/3\mu c^3 \approx 6.3 \times 10^{-24}$ sec. is a time

constant for $\mu = m_e$ (mass of an electron). Eqns (2.2.9a)-(2.2.9c) represent the power radiated in to the nth harmonics averaged over the period of the motion. It is also to be noted that $\langle P_n \rangle$ is proportional to $\langle E_n \rangle$, the average energy of the nth harmonics. The power radiated as to for which component it is stronger is to be discussed later in this chapter.

There is also angular momentum radiated by the system. It can be seen from eqn(2.1.14) that in the evaluation of the radiated angular momentum the contribution comes only from the second term. This consists of two terms: one varying as $1/r^3$ and the other as $1/r^5$. The r dependence of the first vanishes in view of eqn(2.1.16) but the second remains to vary as $1/r^2$. Thus the second is not dissociated from the source and can not be considered as radiation. The evaluation of the radiated angular momentum also requires the evaluation of $\hat{k} \times (\underline{D}_n^* \times \underline{D}_n)$ and $\hat{k} \times [\hat{k} \times (\underline{D}_n^* \times \underline{D}_n)]$. It can be shown that

$$\hat{k} \times (\underline{D}_n^* \times \underline{D}_n) = \frac{2e^2}{n^2} a_n b_n \sin\theta \hat{e}_\phi$$

and

$$\begin{aligned} & \hat{k} \times [\hat{k} \times (\underline{D}_n^* \times \underline{D}_n)] \\ &= \frac{2e^2}{n^2} a_n b_n \sin\theta [-\cos\theta (\cos\phi \hat{e}_1 + \sin\phi \hat{e}_2) + \sin\theta \hat{e}_3] \end{aligned}$$

if we use eqns (2.2.4) and (2.2.5). Therefore the radiated angular momentum per unit solid angle per unit time is given by

$$\left\langle \frac{d}{d\Omega} \left(\frac{dL_n}{dt} \right) \right\rangle = \frac{(n\omega_0)^3}{8\pi c^3} \frac{(2e^2 a_n b_n)}{n^2} \sin\theta \left[(\sin\theta \hat{e}_3 - \cos\theta (\cos\phi \hat{e}_1 + \sin\phi \hat{e}_2)) \right]$$

Integrating this over the total solid angle yield

$$\left\langle \frac{dL_n}{dt} \right\rangle = -\mu \tau w_0^3 n a_n b_n \hat{e}_3 \quad (2.2.11)$$

$$= -\frac{4\mu w_0 a^2 \sqrt{1-\epsilon^2}}{\epsilon} (\tau w_0^2 n) J_n' J_n \hat{e}_3 \quad (2.2.12)$$

$$= -\tau w_0^2 n^2 \langle \underline{L}_n \rangle \quad (2.2.13)$$

Eqn (2.2.13) shows that the average radiated angular momentum in to each harmonics is proportional to, and oppositely directed to its average angular momentum.

The results of eqns(2.2.8) and (2.2.10) could be used to deduce what one would expect if the orbit was circular instead of elliptic. This can be seen by taking the limit $\epsilon \rightarrow 0$. To see this clearly it is useful to note that when $\epsilon \rightarrow 0$, $(J_n'(n\epsilon))^2 \rightarrow \frac{1}{4}$ and $((1-\epsilon^2)/\epsilon^2)(J_n(n\epsilon))^2 \rightarrow \frac{1}{4}$ (see appendix). Therefore,

$$\lim_{\epsilon \rightarrow 0} \left\langle \frac{dP_n}{d\Omega} \right\rangle \rightarrow \left\langle \frac{dP_1}{d\Omega} \right\rangle = \frac{(eaw_0^2)^2}{8\pi c^3} (1+\cos^2 \theta) \quad (2.2.14)$$

and

$$\lim_{\epsilon \rightarrow 0} \left\langle \frac{d}{dt} \left(\frac{dL_n}{d\Omega} \right) \right\rangle \rightarrow \left\langle \frac{d}{dt} \left(\frac{dL_1}{dt} \right) \right\rangle = \frac{2w^3}{3c^3} (ea)^2 \quad (2.2.15)$$

for circular non-relativistic motion. These are well known results which can be found in any standared books on Electrodynamics.

2.3 The Spectrum Of The Radiation

It was mentioned in the last section that the spectrum of the radiation is discrete. The discreteness is a direct consequence of the presence of ϵ . This can be understood from eqns(2.2.14) and (2.2.15) in which the discreteness disappears in the limit $\epsilon \rightarrow 0$. To make some comparisons of the strength of the radiation from different harmonics let's write eqn. (2.2.9b) as

$$\langle P_n \rangle = n^2 \alpha_n \quad (2.3.1)$$

Where

$$\lambda = \tau W_0^2 \left(\frac{\mu W_0^2 a^2}{4} \right) \quad (2.3.2)$$

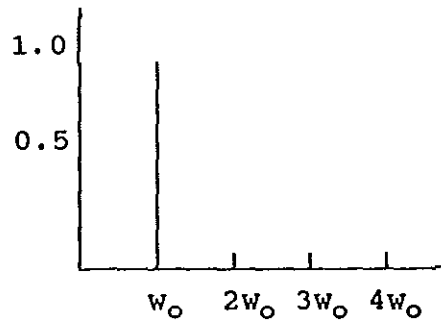
and

$$\alpha_n = (J_{n-1})^2 + (J_{n+1})^2 - 2J_n^2 \quad (2.3.3)$$

The form of α_n is obtained after using the recurrence relations for Bessel functions of eqn (2.2.9b). We shall consider λ to be a constant by choosing particular value for a or W_0 . This will make the energy of the system to be fixed, see eqs(1.1.11) and (1.1.14), but \underline{L} to be arbitrary. We shall fix \underline{L} by giving different values for ϵ . Hence, the spectrum we are going to compare is for systems of the same energy but different angular momentum. Note that the more is ϵ the less is the angular momentum of the system for fixed a , see eqn (1.3.14).

$\epsilon = 0.1$

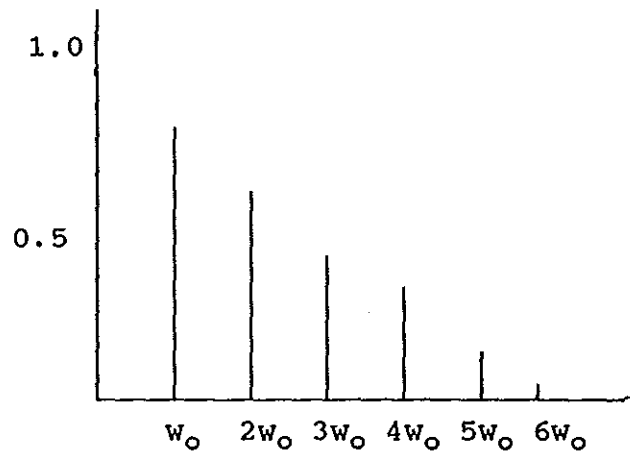
| n | P_n/λ |
|---|---------------|
| 1 | 0.99 |
| 2 | 0.04 |
| 3 | 0.001 |



fig(2.3)

 $\epsilon = 0.5$

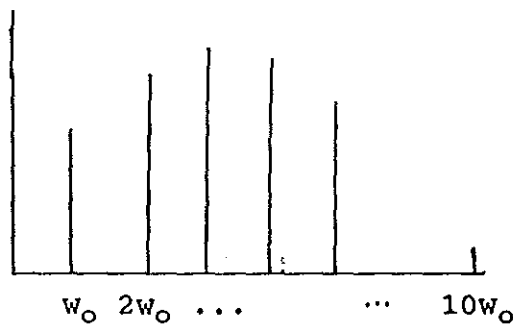
| n | P_n/λ |
|---|---------------|
| 1 | 0.76 |
| 2 | 0.68 |
| 3 | 0.41 |
| 4 | 0.23 |
| 6 | 0.01 |



fig(2.4)

 $\epsilon = 0.7$

| n | P_n/λ |
|---|---------------|
| 1 | 0.56 |
| 2 | 0.83 |
| 3 | 0.90 |
| 4 | 0.84 |



fig(2.5)

The spectrum shown have some qualitative features. For $\epsilon=0.1$ the radiation from lower harmonics is stronger. The first harmonics is practically dominant. For $\epsilon=0.5$ the radiation from the lower harmonics is still dominant. Unlike the $\epsilon=0.1$ case the strength of the radiation decreases gently as one goes from lower harmonics to higher harmonics. For $\epsilon=0.5$ the power radiated by the third harmonics is the strongest. Unlike the $\epsilon=0.1$ and $\epsilon=0.5$ cases, the $\epsilon=0.7$ case the strength of radiation keeps on increasing, reaches a maximum and then falls off for higher harmonics.

2.4 Effects Of Radiation On The System

It is clear that as long as there is a loss of energy and angular momentum, however small, the system can not exist indefinitely on its initial state. If the system is to remain unaffected, in the presence of radiation, there should exist a forcing agency to off set the losses due to radiation. In this case the forcing agency can keep the trajectory of the orbit unchanged. In the absence of the forcing agency, however, the orbit of the motion will be changed. This change can be seen from the change of the parameters describing the orbit. If we come to our particular case of elliptic orbit, the change can be seen from the change of the semi-major axis and the eccentricity ϵ of the orbit. The radiation takes away with it energy and angular momentum. This brings the change on the semi-major and semi-minor axes and the eccentricity, see eqns (1.1.10), (1.1.11) and (1.1.12)

Since our treatment is completely non-relativistic, the

losses due to radiation during one period of the motion is so small that we can consider the changes of the parameters to be adiabatic. In this case the new orbit will not go into a different type of orbit, rather it will have almost the same form but with small change of the size of the orbit. In our particular case of elliptic orbit, after one period we will get another elliptic orbit with slight decrease of the semi-major axis a , semi-minor axis b and the eccentricity ϵ of the orbit.

The qualitative description of the change of the orbit is now to be made quantitative by first evaluating the average rate of change of the energy and angular momentum of the system and then evaluate how ϵ varies with time.

The average rate of change of the energy of the system is equal to the negative of the average total power radiated by the system. So we write

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle &= - \langle P \rangle = - \sum_{n=1}^{\infty} \langle P_n \rangle \\ &= -2\tau W^2 \sum_{n=1}^{\infty} n^2 \langle E_n \rangle \end{aligned} \quad (2.4.1)$$

We shall no more use the subscript "0" for W because W is no more a constant in the absence of the forcing agency. Expressed interms of Bessel functions eqn(2.4.1) takes form

$$\left\langle \frac{dE}{dt} \right\rangle = -2\tau W^2 \left(\frac{\mu W^2 a^2}{4} \right) \sum_{n=1}^{\infty} 4n^2 \left[J_n'^2 + \frac{1-\epsilon^2}{\epsilon^2} J_n^2 \right] \quad (2.4.2)$$

The series on the right of (2.4.2) can be summed to yield

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{\tau W^2 (2 + \epsilon^2)}{(1 - \epsilon^2)^{5/2}} \langle E \rangle \quad (2.4.3)$$

Where

$$\sum_{n=1}^{\infty} n^2 \left[J_n'^2 + \frac{1 - \epsilon}{\epsilon^2} J_n^2 \right] = \frac{2 + \epsilon^2}{4(1 - \epsilon^2)^{5/2}}$$

(see appendix), and

$$\langle E \rangle = - \frac{\mu W^2 a^2}{2}$$

is the instantaneous average energy of the system. One can also perform the summation over all harmonics of eqn (2.2.12) so that

$$\left\langle \frac{dL}{dt} \right\rangle = -\tau W^2 \mu w a^2 (1 - \epsilon^2)^{1/2} \frac{4}{\epsilon} \sum_{n=1}^{\infty} n J_n' J_n \quad (2.4.4)$$

is the average rate of change of angular momentum of the system. Where L is the magnitude of its angular momentum. The preceding equation can be written as

$$\frac{dL}{dt} = - \frac{\tau W^2}{(1 - \epsilon^2)^{3/2}} \langle L \rangle . \quad (2.4.5)$$

Where we have used the mathematical relation $\sum_{n=1}^{\infty} n J_n' J_n = \epsilon/4(1 - \epsilon^2)^{3/2}$ to substitute the series on the right of (2.4.4) (see appendix) and

$$\langle L \rangle = \mu w a^2 \sqrt{1-\epsilon^2}$$

is the instantaneous angular momentum of the system. Note that (2.4.3) does not have minus sign unlike eqn (2.4.5). This is so because $\langle \epsilon \rangle$ is negative. Therefore the energy increases magnitudewise but its angular momentum decreases with time due to radiation. We shall assume that E and L of the system don't appreciably change during one period of the motion. In view of this we shall drop average sign in eqns(2.4.3) and (2.4.5). Of course our assumption is quite reasonable for systems of atomic size. The reasonability of our assumption in atomic scale can be seen from the following: consider the ratio of the radiated energy during one period to that of the instantaneous energy E

$$\left(\frac{\Delta E}{E} \right)_{\text{period}} = 2\pi\tau w \frac{2+\epsilon^2}{(1-\epsilon^2)^{5/2}} \quad (2.4.6)$$

in atomic scale where $w \approx 4 \times 10^{16} \text{S}^{-1}$, $\tau = 6.3 \times 10^{-24} \text{sec}$. and for $0 < \epsilon < 0.9$ the ratio is always less than 4.87×10^{-4} . But our assumption fails for $\epsilon \rightarrow 1$. So, for large part of the range of ϵ ($0 < \epsilon < 1$) our assumption remains reasonable.

With this in mind let us re-write eqns (2.4.3) and (2.4.5) as

$$\frac{dE}{dt} = \frac{\tau w^2 (2+\epsilon^2)}{(1-\epsilon^2)^{5/2}} E \quad (2.4.7)$$

$$\frac{dL}{dt} = - \frac{\tau w^2}{(1-\epsilon^2)^{3/2}} L \quad (2.4.8)$$

It should be clear that eqns(2.4.7) and (2.4.8) can not be integrated at once because ϵ and W are also functions of time.

We can find the rate of change of the eccentricity by differentiating eqn(1.1.10) with respect to time and noting that

$$\frac{d\epsilon}{dt} = \frac{\epsilon}{E} \frac{dE}{dt} + \frac{\epsilon}{L} \frac{dL}{dt} \quad . \quad (2.4.9)$$

Using eqn(1.1.10) to find the partial derivatives of ϵ with respect to E and L and using the results of eqns(2.4.7) and (2.4.8) the preceding equation becomes

$$\frac{d\epsilon}{dt} = - \frac{3\tau W^2 \epsilon}{2(1-\epsilon^2)^{3/2}} \quad (2.4.10)$$

We can also find the rate of change of w using eqns (1.1.14) and (2.4.7) as

$$\frac{dw}{dt} = \frac{w}{E} \frac{dE}{dt} = \frac{3}{2} \frac{\tau W^2 (2+\epsilon^2)}{(1-\epsilon^2)^{5/2}} w \quad (2.4.11)$$

It remains now to find the explicit time dependence of ϵ , w , E and L . To facilitate these we divide one of the equations (2.4.7), (2.4.8), (2.4.10) and (2.4.11) by the other and make evaluation (integration) simple. To start with, divide (2.4.8) by (2.4.7) which gives

$$\frac{dE}{dL} = - \left(\frac{2+\epsilon^2}{1-\epsilon^2} \right) \frac{E}{L}$$

Using the expression for ϵ from eqn (1.1.10) we write the preceding differential equation as an equation involving E and L and their differential only,

$$\frac{dE}{dL} = \frac{\mu k^2}{2L^3} \left(3 + \frac{2EL}{\mu k^2} \right) \quad (2.4.12)$$

Which when integrated yield

$$E = -\frac{\mu k^2}{2L^2} \left(1 - \left(\frac{L}{L_0} \right)^3 \right) + \left(\frac{E_0}{L_0} \right) L \quad (2.4.13)$$

Where E_0 and L_0 are the initial values of the energy and angular momentum of the system respectively. corresponding to E_0 and L_0 there are also initial values of the eccentricity ϵ_0 and angular frequency W_0 . Intermis of ϵ and ϵ_0 eqn(2.4.13) can be re-expressed as

$$\frac{E}{E_0} = \left(\frac{\epsilon}{\epsilon_0} \right)^{2/3} \left[1 + \frac{(\epsilon_0/\epsilon)^2 - 1}{(1 - \epsilon_0^2)} \right] \quad (2.4.14)$$

Dividing eqn (2.4.7) by (2.4.11) also gives

$$\frac{dE}{dw} = \frac{2}{3} \left(\frac{E}{w} \right) \quad (2.4.15)$$

Which, upon integration becomes

$$\left(\frac{E}{E_0} \right)^3 = \left(\frac{w}{w_0} \right)^2 \quad (2.4.16)$$

Similarly, dividing eqn(2.4.8) by (2.4.10) and integrating the differential equation that would arise containing ϵ and L gives

$$\left(\frac{L}{L_0}\right)^3 = \left(\frac{\epsilon}{\epsilon_0}\right)^2 \quad (2.4.17)$$

Using eqn (2.4.14) in (2.4.16) we can get $(w/w_0)^2$ expressed in terms of ϵ/ϵ_0 as

$$\left(\frac{W}{W_0}\right)^2 = \left(\frac{\epsilon_0}{\epsilon}\right)^4 \left(\frac{1-\epsilon^2}{1-\epsilon_0^2}\right)^3 \quad (2.4.18)$$

Eqn (2.4.17) expressed as

$$\epsilon = \epsilon_0 \left(\frac{L}{L_0}\right)^{3/2}$$

Shows that if the initial eccentricity is zero ($\epsilon_0 = 0$) then ϵ always remains to be zero independent of the value of L . That is if the initial orbit is circular then the subsequent orbits will also be circular but with smaller radius. This is the result of our assumption that the energy loss in one period is very small compared to the instantaneous energy. Similarly $\epsilon=0$ for $\epsilon_0=0$ unless for $L=0$. Substituting eqn(2.4.18) in (2.4.10) to eliminate the dependence of $d\epsilon/dt$ on W yield

$$\frac{d\epsilon}{dt} = -A \frac{(1-\epsilon^2)^{3/2}}{\epsilon^3} \quad (2.4.19)$$

Where

$$A = \frac{3\tau W_0^2}{2} \frac{\epsilon_0^4}{(1-\epsilon_0^2)^3} \quad (2.4.20)$$

The expression on the right of (2.4.19) depends only on ϵ . It's integration with time is so simple that it gives

$$\sqrt{1-\epsilon^2} + \frac{1}{\sqrt{1-\epsilon^2}} = \gamma - At \quad (2.4.21)$$

Where

$$\gamma = \frac{2-\epsilon_0^2}{\sqrt{1-\epsilon_0^2}} \quad (2.4.22)$$

One can also re-write eqn (2,4,21) as

$$\left(\frac{\epsilon}{\epsilon_0}\right)^2 = \frac{(\gamma - At)^2 - 4}{2\epsilon_0^2} \left\{ [1 + 4((\gamma - At)^2 - 4)^{-1}]^{1/2} - 1 \right\} \quad (2.4.23)$$

To show the explicit time dependence of ϵ on t . It is useful to note that A and γ depend on the initial values of ϵ_0 and w_0 . The explicit time dependence of E and L can be also obtained by substituting eqn (2.4.23) in (2.4.14) and (2.4.17) respectively.

2.5 The Collapse Of The Classical Atom

In trying to apply our results in atomic scale we consider the positively charged particle to be the nucleus and the negatively charged particle to be the electron of just a hydrogen atom. Our non-relativistic results are applicable here because electrons in atoms have very low speed compared to the speed of light. The effects of radiation in the first few revolutions is very small, but their gradual accumulation over

many revolutions gives them great importance to the basic concepts of physics. They lead to the problem of accounting for the stability of atoms. On the classical basis, orbiting electrons must be expected to continue radiating and should eventually fall into the oppositely charged nucleus as they lose the energy that can keep them in flight.

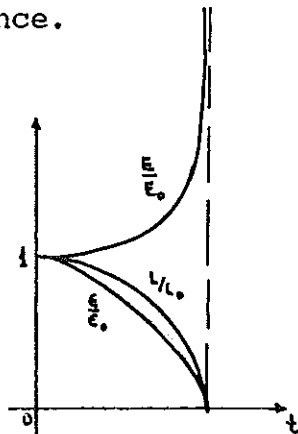
The time of fall of an orbiting electron from different possible orbits have been determined in many different electrodynamics books (see for example ref 8). There, it is shown that the time of fall from outer orbits is longer than that from inner orbits. In our part here we shall consider the time of fall from any orbit with different initial conditions specified by w_0 and ϵ_0 . The time of fall of the electron into the nucleus is termed as the time of collapse of the atom.

It can be seen from eqn(2.4.21) that ϵ becomes zero after a certain time t_c given by

$$t_c = \frac{\gamma - 2}{A} \quad (2.5.1)$$

$$= \frac{2}{3\tau} \frac{1}{w_0^2} \frac{(1 - \epsilon_0^2)^{5/2}}{\epsilon_0^4} (2 - \epsilon_0^2 - 2(1 - \epsilon_0^2)^{1/2}) \quad (2.5.2)$$

At $t=t_c$ when ϵ becomes zero, L/L_0 becomes zero and $E/E_0 \rightarrow \infty$ (see eqns(2.4.17) and (2.4.14)). see also fig(2.6) for the feature of their time dependence.



This shows that the system sooner or later will collapse due to radiation. The time t_c is called the collapse time of the system.

The collapse time t_c is different for different initial values of ϵ_0 and w_0 . For a given value of ϵ_0 , the collapse time is longer for small values of w_0 and is short for large w_0 . This is what should be expected because fast moving charges radiate more and the more is radiation the sooner the atom collapses. On other hand, for a given w_0 , t_c takes different values for different values of ϵ_0 . When ϵ_0 is large t_c is very short. The large ϵ_0 corresponds to small initial angular momentum for given w_0 (see eqn(1.3.14)). That t_c is short for large ϵ_0 can be also seen from eqn (2.5.2). The more is ϵ the more is radiation and hence the faster the atom collapses. In geometric terms, more elliptic orbits collapse sooner for a given w_0 .

Note that eqn(2.5.2) is seemingly divergent as $\epsilon_0 \rightarrow 0$. To see the actual dependence of t_c on ϵ_0 , when ϵ_0 is small, let's expand $(1-\epsilon_0^2)^{1/2}$ and $(1-\epsilon_0^2)^{5/2}$ for small ϵ_0 :

$$(1-\epsilon_0^2)^{1/2} \approx 1 - \frac{\epsilon_0^2}{2} - \frac{\epsilon_0^4}{8} - \frac{\epsilon_0^6}{16} \dots$$

$$(1-\epsilon_0^2)^{5/2} \approx 1 - \frac{5\epsilon_0^2}{2} + \frac{15\epsilon_0^4}{8} \dots$$

Substituting these in eqn (2.5.2) yield

$$t_c \approx \frac{1}{6\tau} \frac{1}{w_0^2} \left(1 - \frac{5\epsilon_0^2}{2}\right) = \frac{a_c}{4C} \left(\frac{a_0}{a_c}\right)^3 \left(1 - \frac{5\epsilon_0^2}{2}\right) \quad (2.5.2)$$

Where $a_c = e^2/\mu c^2$ is the classical electron radius (for $\mu \approx m_e$; mass of an electron) and a_o is the semi-major axis corresponding to w_o . t_c in eqn(2.5.3) tends to a constant value when $\epsilon_o \rightarrow 0$:

$$t_c \rightarrow \frac{a_c}{4c} \left(\frac{a_o}{a_c} \right)^3 \quad (2.5.4)$$

This is the time of collapse for an initially circular orbit of initial radius a_o .

The following table shows values of t_c for different values of ϵ_o and constant $w_o = 4.12 \times 10^{16} \text{ s}^{-1}$ (a_o which corresponds to chosen w_o is of the order of 1st Bohr's radius).

| $w_o = 4.12 \times 10^{16} \text{ sec}^{-1}$ | |
|--|-----------------------------------|
| ϵ_o | $t_c \times 10^{-11} \text{ Sec}$ |
| 0.000 | 2.39 |
| 0.001 | 1.56 |
| 0.100 | 1.53 |
| 0.500 | 0.87 |
| 0.750 | 0.54 |
| 0.900 | 0.05 |

The above table shows that the time of collapse t_c gets shorter and shorter when ϵ_o is made larger and larger. That is, if the orbit is more of circular then it collapses later and if it is more of elliptic it collapses sooner.

The time of collapse is of the order of 10^{-11} sec . This time is a very short time compared to the multi-billion-year-long

life of universe. But on the atomic scale, this time is enough for the charge to make about 7×10^4 revolutions.

A little more is to be said about transition time from n th to $(n-1)$ th state predicted by correspondence principle for large quantum number of circular orbits. The principle states that for large quantum number n the transition time from n th state to $(n-1)$ th state is given by

$$\hbar \omega / P \quad (2.5.5a)$$

Where P is the power radiated by the fundamental. For circular orbits of hydrogen like atoms

$$\frac{1}{\tau} = \frac{2}{3} \frac{e^2}{\hbar c} \left(\frac{e^2}{\hbar c} \right)^4 \left(\frac{\mu c^2}{\hbar} \right) \frac{1}{n^5} \quad (2.5.5b)$$

Where τ is the transition time. Can the result of our model give the same result? To see this let's evaluate P of (2.5.5a) from eqn (2.2.9b) for $\epsilon=0$. The only term surviving is only the $n=1$ term

$$P_1 = \frac{2}{3} \frac{e^2}{c^3} a^2 \omega^4 \quad (2.5.6)$$

Let's make the substitution

$$a = r_n = \frac{n^2 \hbar^2}{\mu e^2}$$

and

$$\omega^2 = \frac{e^2}{\mu a^3}$$

into (2.5.6'), then

$$P_1 = \frac{2}{3} \frac{e^2}{c^3} \left(\frac{n^2 \hbar^2}{\mu e^2} \right)^2 \left(\frac{e^2}{\mu a^3} \right)^4$$

substitution in to (2.5.5a) yield

$$\tau = \frac{h\nu}{p} = \left[\frac{2e^2}{3c} \left(\frac{n^2 \hbar^2}{\mu e^2} \right)^2 \left(\frac{e^2}{\mu a^3} \right)^3 \right]^{-1} \hbar$$

or

$$\frac{1}{\tau} = \frac{2}{3} \frac{e^2}{hc} \left(\frac{e^2}{\hbar c} \right)^4 \left(\frac{\mu c^2}{\hbar} \right) \frac{1}{n^5}$$

exactly the same as (2.5.5b). So our model also gives the same result.

2.6. Radiation Widths Of Spectral Lines

As treated in Sec(2.3) an orbiting charge yield radiations of ideally discrete frequencies $w_0, 2w_0, \dots$ because it was presumed that a periodic motion of orbital frequency w_0 was perfectly maintained throughout each radiation process. In atoms, such well-defined periodicities can be maintained by the internal forces like the energy-conserving nuclear coulomb force only in so far as the mechanical system remains unperturbed by the radiative reactions (or by other conceivable dissipative effects) and as the constant energy needed to maintain a perfect periodicity is conserved. AT least the losses to radiation will not permit this, having consequences like the continuous inward spiraling of the charge, discussed in the preceding section. The radiatively perturbed motion becomes aperiodic, and its Fourier

decomposition will have a continuum of frequencies replacing the unperturbed discrete frequencies.

Since radiative effects on motion are very small (proportional to τ), the continuum departures from each unperturbed frequency will have a small effective range $\Delta\omega$, with the result a small broadening of each spectral line. Direct measurements of such radiation width have value for conclusions about the dynamics of radiating systems.

Evaluations of the effect are generally approached through the solution of the non-relativistic equation of motion

$$\mu \ddot{\underline{r}} = - \frac{e^2}{r^3} \underline{r} + \mu \tau \dddot{\underline{r}} \quad (2.6.1)$$

Where the first term on the right hand side of (2.6.1) is the usual coulomb attraction force and the second is the radiation reaction. Eqn(2.6.1) is equivalent to the two equations

$$\ddot{x} - \tau \dddot{x} = - \frac{e^2}{\mu r^3} x \quad (2.6.2)$$

$$\ddot{y} - \tau \dddot{y} = - \frac{e^2}{\mu r^3} y \quad (2.6.3)$$

From the solutions of the kepler problem eqns(1.1.15)-(1.1.17) and eqn(1.1.19) it can be shown that

$$\frac{x}{r^3} = \frac{1}{a} \frac{\partial}{\partial \epsilon} \left(\frac{1}{r} \right) \quad (2.6.4)$$

and

$$\frac{y}{r^3} = - \frac{\sqrt{1-\epsilon^2}}{a\omega\epsilon} \frac{\partial}{\partial t} \left(\frac{1}{r} \right) \quad (2.6.5)$$

Substituting these into (2.6.2-3) yield

$$\ddot{x} - r\ddot{x} = \frac{e^2}{\mu a} \frac{1}{r} \quad (2.6.6)$$

$$\ddot{y} - r\ddot{y} = - \frac{\sqrt{1-\epsilon^2}}{a\omega\epsilon} \frac{\partial}{\partial t} \left(\frac{1}{r} \right) \quad (2.6.7)$$

Leaving this as it is, let's also re-write eqns(1.2.12), (1.2.14) and (1.2.17) after substituting (1.2.13) and (1.2.15) for a_n and b_n as

$$x(t) = -\frac{3a\epsilon}{2} + a \sum_{n=1}^{\infty} \frac{J_n'(n\epsilon)}{n} 2\cos(n\omega t)$$

$$y(t) = \frac{a\sqrt{1-\epsilon^2}}{\epsilon} \sum_{n=1}^{\infty} \frac{J_n(n\epsilon)}{n} 2\sin(n\omega t)$$

$$\frac{1}{r(t)} = \frac{1}{a} \left[1 + \sum_{n=1}^{\infty} J_n'(n\epsilon) 2 \cos(n\omega t) \right]$$

These can be also expressed interms of exponentials as

$$x(t) = -\frac{3a\epsilon}{2} + a \left[\sum_{n=1}^{\infty} \frac{J_n'(n\epsilon)}{n} e^{in\omega t} + \sum_{n=1}^{\infty} \frac{J_n'(n\epsilon)}{n} e^{-in\omega t} \right] \quad (2.6.8)$$

$$y(t) = \frac{a\sqrt{1-\epsilon^2}}{\epsilon} \left[\sum_{n=1}^{\infty} \frac{J_n(n\epsilon)}{in} e^{in\omega t} - \sum_{n=1}^{\infty} \frac{J_n(n\epsilon)}{in} e^{-in\omega t} \right] \quad (2.6.9)$$

$$\frac{1}{r(t)} = \frac{1}{a} \left[1 + \sum_{n=1}^{\infty} J_n(n\epsilon) e^{in\omega t} + \sum_{n=1}^{\infty} J_n(n\epsilon) e^{-in\omega t} \right] \quad (2.6.10)$$

If we make the change of variable n by $-n$ in the first serieses of the preceding equations we arrive at

$$x(t) = \frac{3a\epsilon}{2} + a \left[\sum_{n=1}^{\infty} \frac{J_{-n}(-n\epsilon)}{-n} e^{-inwt} + \sum_{n=1}^{\infty} \frac{J_n'(n\epsilon)}{n} e^{-inwt} \right] \quad (2.6.11)$$

$$y(t) = \frac{a\sqrt{1-\epsilon^2}}{\epsilon} \left[\sum_{n=1}^{\infty} \frac{J_{-n}(-n\epsilon)}{-n} e^{-inwt} - \sum_{n=1}^{\infty} \frac{J_n(n\epsilon)}{n} e^{-inwt} \right] \quad (2.6.12)$$

$$\frac{1}{r(t)} = \frac{1}{a} \left[1 + \sum_{n=1}^{\infty} J_{-n}(-n\epsilon) e^{-inwt} + \sum_{n=1}^{\infty} J_n(n\epsilon) e^{-inwt} \right] \quad (2.6.13)$$

Using the properties of the Bessel functions

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{and} \quad J_n(-x) = (-1)^n J_n(x)$$

it is easy to see that

$$J_{-n}(-n\epsilon) = J_n(n\epsilon) \quad \text{and}$$

$$J_{-n}'(-n\epsilon) = -J_n'(n\epsilon)$$

The latter can be proved easily if one uses the recurrence relation

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Therefore we can write

$$x(t) = \sum_{n=-\infty}^{\infty} a'_n e^{-inwt} \quad ; \quad a'_n = \frac{aJ_n'(n\epsilon)}{n} \quad \text{for } n \neq 0$$

$$a'_0 = \frac{-3\epsilon a}{2} \quad \text{for } n=0 \quad (2.6.14)$$

$$y(t) = \sum_{n=-\infty}^{\infty} b'_n e^{-inwt} \quad ; \quad b'_n = \frac{J_n(n\epsilon)}{in} \quad \text{for } n \neq 0$$

$$b'_0 = 0 \quad \text{for } n=0 \quad (2.6.15)$$

$$\frac{1}{r(t)} = \sum_{n=-\infty}^{\infty} c'_n e^{-inwt} \quad ; \quad c'_n = \frac{J_n(n\epsilon)}{a} \quad \text{for all } n$$

(2.6.16)

We consider w as an unknown frequency which must be found by substituting the above equations into the equations of motion. Substituting (2.6.14) and (2.6.16) in (2.6.2) yield

$$\sum_{n=-\infty}^{\infty} (-inw)^2 a'_n e^{-inwt} - \tau \sum_{n=-\infty}^{\infty} (-inwt)^3 a'_n e^{-inwt} =$$

$$\sum_{n=-\infty}^{\infty} \left(\frac{-e^2}{\mu a} \frac{\partial c'_n}{\partial \epsilon} \right) e^{-inwt}$$

Which can be satisfied only when

$$(-n^2 w^2 - i\tau (nw)^3) \frac{a J'_n}{n} = \frac{-e^2}{\mu a^2} \frac{\partial J_n(n\epsilon)}{\partial \epsilon} = \frac{-e^2 n}{\mu a^2} J'_n$$

or

$$n^2 w^2 + i\tau n^3 w^3 = \frac{e^2 n^2}{\mu a^3}$$

That is

$$w^2 = w_0^2 - i\tau w^3 \tag{2.6.17}$$

Where

$$w_0^2 = e^2 / \mu a^3$$

is the frequency defined in eqn (1.1.14). It is worth while to note that we can also substitute eqns (2.6.15) and (2.6.16) in (2.6.3) and will result in the same equation eqn (2.6.17). The unperturbed ($\tau \rightarrow 0$) roots of (2.6.17) are $w \rightarrow \pm w_0$, and at these values the radiative correction adds only a fraction of magnitude w_0 . In all relevant situations it is more than

adequate to evaluate the radiative term of (2.6.17) as $+iw_0^3\tau$ for the respective unperturbed roots $\pm w_0$, and then the results are

$$w \approx w_0(1 + iw_0 n \tau)^{1/2} \approx +w_0 - \frac{iw_0^2 \tau n}{2} \quad (2.6.18)$$

Substituting this into eqn (2.6.15) yields

$$x(t) = \sum_{n=-\infty}^{\infty} A_n(t) \exp(-inw_0 t)$$

Where

$$A_n(t) = a'_n \exp(-\Gamma_n t/2)$$

Thus the solutions of each harmonics in eqns (2.4.14-16) will have oscillations proportional to $\exp(\pm inw_0 t)$ modulated by an exponentially decreasing amplitude proportional to $\exp(-\Gamma_n t/2)$ with

$$\Gamma_n = n^2 w_0^2 \tau \quad (2.6.19)$$

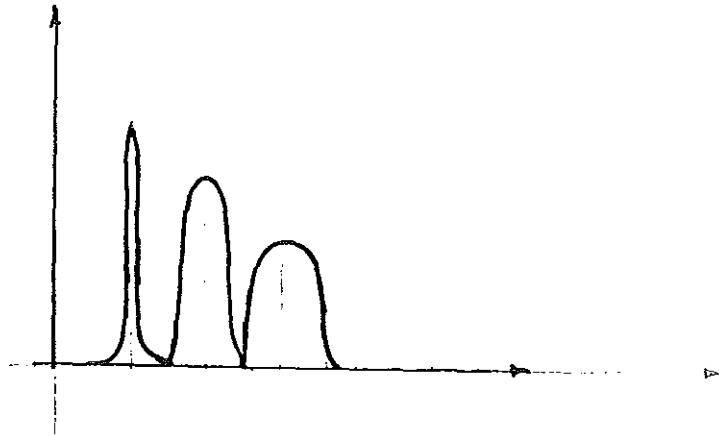
This defines the width of each spectral line and shows that the higher harmonics have larger width. Thus, the spectrum becomes continuous due to widening of the spectral lines when,

$$w^2 \tau \approx w_0$$

that is for

$$n \approx \frac{1}{\sqrt{\tau w_0}} \approx 2000$$

When $w_0 = 4.12 \times 10^{16-1} \text{ s}$. Therefore, the spectrum instead of being consisted of sharp lines as was shown in the previous section will show some broadening in which the broadening is highly pronounced for $n \gg 1$ see fig.(2.7).



fig(2.7)

CHAPTER THREERADIATION FROM RELATIVISTIC CHARGEMOVING IN A CIRCULAR ORBIT3.1 Fields From A Charge In Circular Orbit

Accelerations of a particle perpendicular to its velocity are characteristics of deflections by an imposed centripetal force. The centripetal force could be due to an imposed magnetic field on a moving charge. These occur, for example, in the synchrotron devices for accelerating particles, and emissions arising from deflections of high velocities by any magnetic field in any circumstances are customarily referred to as synchrotron radiation. It is to be noted that our consideration of circular path is not a matter of choice, but on the realization of its applicability. In addition, towards the middle of this chapter we shall see the analogy of the spectrum from elliptic and circular paths.

For accelerated charge in non-relativistic circular motion, the angular distribution of the radiated power shows a simple $\cos^2\theta$ behaviour, as seen on page eqn(2.2.14), where θ is the angle measured from z' to \underline{r} , see fig(2.2). Its dependence for relativistic charges is, however, complicated due to relativistic effects.

If we remove the last restriction, $Ka \ll 1$, imposed in deriving the radiation of non-relativistic charges leaving the

restriction only $r \gg a$ and $r \gg (a/\lambda)a$, then our consideration will be more general which includes both relativistic and non-relativistic cases. But still we have to impose a new restriction: $Ka < 1$ or $a < \lambda$. That is the size of source should be strictly less than $\frac{1}{2}\pi$ of the wave length of the emitted field. In other words this means that the speed of the particle must be less than the speed of light. Combining all the restrictions in a simple form is $r \gg \lambda \gg a^2/r$. The field point is still in a far zone and $kr \gg 1$ (radiation zone)

For particles under going periodic motion we expand the field variables in a Fourier-Series and corresponding to eqns (2.1.2a), (2.1.2b) and (2.1.2c) we have

$$\underline{A}(\underline{r}, t) = \sum_{n=-\infty}^{\infty} \underline{A}_n(\underline{r}) \exp(-in\omega_0 t) \quad (3.1.1a)$$

$$\underline{J}(\underline{r}', t') = \sum_{n=-\infty}^{\infty} \underline{J}_n(\underline{r}') \exp(-in\omega_0 t') \quad (3.1.1b)$$

and

$$\underline{A}_n(\underline{r}) = \frac{e^{ikr}}{cr} \int \underline{J}_n(\underline{r}') e^{-ik \cdot \underline{r}'} dv' \quad (3.1.1c)$$

Where $k = n\omega_0/c$. The last equation is the result of the substitution of eqns (3.1.1a) and (3.1.1b) in to eqn (2.1.1) and analyzing it for $r \gg \lambda \gg a^2/r$. The Fourier component of the current density is

$$\underline{J}_n(\underline{r}') = \frac{1}{T} \int_0^T \underline{J}(\underline{r}', t') \exp(-in\omega_0 t') dt' \quad (3.1.2)$$

It must be clear that the fields can be also expanded in

Fourier-Series and their Fourier components are given by

$$\underline{B}_n = \nabla \times \underline{A}_n$$

and
$$\underline{E}_n = \frac{\nabla \times \underline{B}_n}{-ik}$$

For r satisfying the condition $r^2 \gg \lambda \gg a^2/r$, the Fourier components B_n and E_n take the form

$$\underline{B}_n(\underline{r}) = i\mathbf{k} \times \underline{A}_n(\underline{r}) \quad (3.1.3)$$

$$\underline{E}_n(\underline{r}) = -\hat{\mathbf{k}} \times \underline{B}_n(\underline{r}) \quad (3.1.4)$$

This shows that both the electric and magnetic vectors are transverse to the propagation direction \mathbf{k} through the field point, have equal magnitudes, and are transverse to each other (with $\underline{B}_n = \hat{\mathbf{k}} \times \underline{E}_n$) exactly as in free space plane waves moreover, the phase factor is $\exp(i\mathbf{k}r) = \exp(i\mathbf{k} \cdot \underline{r})$, exactly as in plane wave, and has magnitudes that are constant over large sectors of the spherical front at the very remote distances. Electromagnetic radiation has thus been generated and the energy it carries away from the source is evaluated next.

3.2. Angular Distribution Of The Radiation.

The energy flux-density carried away by any electromagnetic wave is

$$\underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$$

Using eqn (3.1.4) in the preceding equation one can find

$$\underline{S} = \frac{c}{4\pi} |\underline{B}|^2 \hat{k} \quad (3.2.1)$$

and it is directed precisely out ward, in the direction of $\hat{k}=\hat{r}$ because of the asymptotic transversality of the field (3.1.3) and (3.1.4) to the radial direction. It measures the intensity of the radiation as energy per unit time passing through unit areas on a very large sphere of radius r , centered on the source. Thus the power radiated into a solid-angle element $d\Omega$ that subtends an area $\hat{r} \cdot d\underline{s} = r^2 d\Omega$ on the sphere is given by

$$\frac{dP}{d\Omega} = \frac{cr^2}{4\pi} |\underline{B}|^2 \quad (3.2.2)$$

Since \underline{B} can be written in Fourier-series as

$$\underline{B} = \sum_{n=-\infty}^{\infty} \underline{B}_n(\underline{r}) \exp(-in\omega_0 t) ; \quad \underline{B}_n = \frac{1}{T} \int_0^T \underline{B}(\underline{r}, t) \exp(in\omega_0 t) dt$$

then

$$|\underline{B}|^2 = \sum_{n, m=-\infty}^{\infty} \underline{B}_n \cdot \underline{B}_m^* \exp[-i(n-m)\omega_0 t] \quad (3.2.3)$$

And

$$\underline{B}_n = \underline{B}_{-n}^*$$

(3.2.4)

because the magnetic field is real. The power radiated per unit solid angle is then

$$\frac{dP}{d\Omega} = \frac{cr^2}{4\pi} \sum_{n,m=-\infty}^{\infty} \underline{B}_n \cdot \underline{B}_m^* \exp[-i(n-m)\omega_0 t]$$

This when averaged over time gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{4\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \underline{B}_n \cdot \underline{B}_m^* \frac{1}{T} \int_0^T \exp[-i(n-m)\omega_0 t] dt$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{4\pi} \left\{ \sum_{n=1}^{\infty} |\underline{B}_n|^2 + \sum |\underline{B}_{-n}|^2 + B_0^2 \right\} \quad (3.2.5)$$

Assuming that the average of the field

$$\underline{B}_0 = \frac{1}{T} \int_0^T \underline{B}(\underline{r}, t) dt$$

to be zero and using the fact that $B_n = B_{-n}^*$, eqn (3.2.4) may be written as follows

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{2\pi} \sum_{n=1}^{\infty} |\underline{B}_n|^2 = \sum_{n=1}^{\infty} \frac{dP_n}{d\Omega}$$

where

$$\frac{dP_n}{d\Omega} = \frac{cr^2}{2\pi} |\underline{B}_n|^2 \quad n=1, 2, 3 \dots \quad (3.2.6)$$

is the power radiated per unit solid angle into the n th multipole

To give the complete description of the power radiated, we have to express \underline{B}_n in terms of the given quantities, the charge

of the particle, its velocity and the distance from the origin to the point of observation. Finally we compute the power radiated in the far zone. Substituting eqn (3.1.2) into (3.1.1c) and substituting the final result into eqn (3.1.3) yield

$$\underline{B}_n = \frac{n\omega_0}{4\pi r c^2} i e^{ikr} \hat{k} \times \int_0^T e^{in\omega_0 t'} dt' \int \underline{J}(\underline{r}', t') \exp(-i\mathbf{k} \cdot \underline{r}') dv'$$

For a point charge in motion the current density is given by

$$\underline{J}(\underline{r}', t') = e \underline{v}(t') \delta(\underline{r}' - \underline{r}_0')$$

Where e is the charge carried and $\underline{v}(t')$ is the velocity of the particle. Therefore

$$\underline{B}_n = \frac{e}{r} \frac{n\omega_0^2}{2\pi c^2} i e^{ikr} \int_0^T \exp(i(n\omega_0 t' - \mathbf{k} \cdot \underline{r}')) \mathbf{k} \times \underline{v} dt' \quad (3.2.7)$$

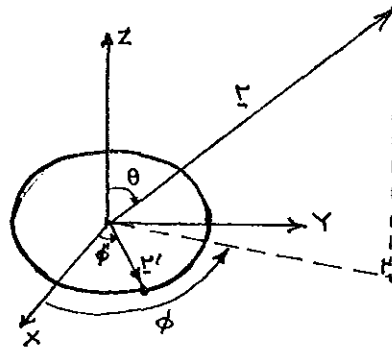
Substituting this into (3.2.6) yield

$$\frac{dP_n}{d\Omega} = \frac{e^2 n^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^T (\mathbf{k} \times \underline{v}) \exp[i(n\omega_0 t' - \mathbf{k} \cdot \underline{r}'_0)] dt' \right|^2 \quad (3.2.8)$$

Eqn(3.2.8) can be used to find the spectrum of radiation from an accelerating charge in any possible type of motion following any kind of trajectory as far as we can integrate the integral in that equation. In particular it can be used to find the spectrum of the radiation from a relativistic point charge moving in a circular orbit. The non-relativistic circular motion case, has

already been deduced from the general type of orbit, elliptic orbit, in chapter two. In this section we shall investigate, the exact expression for the spectrum of radiation from a relativistic charge in circular motion.

For a circular orbit of radius a and period of motion $T = 2\pi/\omega_0$ (see fig (3.1))



fig(3.1)

$$\hat{k} \times \underline{V}(t') = -\omega_0 a \{ \cos\theta \cos\phi \hat{e}_1, + \cos\theta \sin\phi' \hat{e}_2 - \sin\theta \cos(\phi - \phi') \hat{e}_3 \}$$

(3.2.9)

and

$$\begin{aligned} \underline{K} \cdot \underline{r}'_0 &= \frac{n\omega_0 a}{c} \sin\theta \cos(\phi - \phi') \\ &= n\beta \sin\theta \cos(\phi - \phi') \end{aligned}$$

or

$$\underline{K} \cdot \underline{r}'_0 = x \cos(\phi - \phi') \quad (3.2.10)$$

where

$$\beta = \frac{\omega_0 a}{c} \quad \text{and} \quad x = n\beta \sin\theta$$

Let's evaluate

$$\mathbf{j} = \int_0^T \mathbf{k} \times \mathbf{y} \exp i[n\omega_0 t' - \mathbf{k} \cdot \mathbf{r}_0'] dt' \quad (3.2.11)$$

for the circular motion. Using eqns (3.2.9) and (3.2.10) into (3.2.11)

$$\begin{aligned} \mathbf{j} = & -a \cos \theta \int_0^T \exp i[n\omega_0 t' - x \cos(\phi - \phi')] \cos \phi' d(\omega_0 t') \hat{e}_1 \\ & - a \cos \theta \int_0^T \exp i[n\omega_0 t' - x \cos(\phi - \phi')] \sin \phi' d(\omega_0 t') \hat{e}_2 \\ & + a \sin \theta \int_0^T \exp i[n\omega_0 t' - x \cos(\phi - \phi')] \cos(\phi - \phi') d(\omega_0 t') \hat{e}_3 \end{aligned} \quad (3.2.12.)$$

If $\phi' = 0$ at $t = 0$ then $\phi'(t) = \omega_0 t'$ and $d\phi = d(\omega_0 t')$. using this fact we shall evaluate (3.2.12) component by component

$$\begin{aligned} j_x &= -a \cos \theta \int_0^{2\pi} \exp i[n\phi - x \cos(\phi - \phi')] \cos \phi' d\phi' \\ &= \frac{-a \cos \theta}{2} \int_{-\phi}^{2\pi - \phi} \{ \exp i[(n+1)\phi' - x \cos \alpha] + \exp i[(n-1)\phi' - x \cos \alpha] \} d\alpha \end{aligned} \quad (3.2.13)$$

The last equality is obtained by substituting $\frac{1}{2} (e^{i\phi'} + e^{-i\phi'})$ for $\cos \phi'$ and changing the variable : $\phi - \phi' = -\alpha$

Let's once more pay attention only to the first integral of eqn(3.2.13) leaving the coefficient for a moment.

$$I_1 = \int_{-\phi}^{2\pi - \phi} e^{i(n+1)(\phi + \alpha) - x \cos \alpha} d\alpha = e^{i(n+1)\phi} \int_{-\phi}^{2\pi - \phi} e^{i[(n+1)\alpha - x \cos \alpha]} d\alpha$$

If we make the change of variable $\alpha \rightarrow \alpha - \pi/2$ then the last integral becomes

$$I_1 = \frac{e^{i(n+1)\phi}}{i^{(n+1)}} \int_{-\pi/2-\phi}^{\pi/2-\phi} \exp[i((n+1)\alpha - x\sin\alpha)] d\alpha$$

or

$$I_1 = \frac{e^{i(n+1)\phi}}{i^{(n+1)}} \left\{ \int_{-\pi/2-\phi}^0 \exp[i((n+1)\alpha - x\sin\alpha)] d\alpha + \int_0^{2\pi} \exp[i((n+1)\alpha - x\sin\alpha)] d\alpha - \int_{\pi/2-\phi}^{2\pi} \exp[i((n+1)\alpha - x\sin\alpha)] d\alpha \right\}$$

If we make once more the change of variable $\alpha \rightarrow \alpha - 2\pi$ in the last integral of the preceding equation and noting that $\exp(-2\pi(n+1)i) = 1$ then it can be seen that it is the same as the first integral of the preceding equation. Hence the first and the last terms of I_1 add up to zero leaving

$$I_1 = 2\pi \frac{e^{i(n+1)\phi}}{i^{(n+1)}} \left[\frac{1}{2\pi} \int_0^{2\pi} e^{[i(n+1)\alpha - x\sin\alpha]} d\alpha \right] \quad (3.2.14)$$

The expression in the square bracket of (3.2.14) is the standard representation of Bessel function of order $(n+1)$

$$I_1 = \frac{e^{i(n+1)\phi}}{i^{(n+1)}} 2\pi J_{n+1}(x) \quad (3.2.15)$$

Where $J_{n+1}(x)$ is Bessel function of order $(n+1)$. Similar procedures can be traced to evaluate the second integral of eqn(3.2.13) to give

$$I_2 = \frac{e^{i(n-1)\phi}}{i^{(n-1)}} 2\pi J_{n-1}(x) \quad (3.2.16)$$

Combining all these and substituting into (3.2.13) yields

$$j_x = -\pi a \cos \theta \left[\frac{e^{i(n+1)\phi}}{i^{(n+1)}} J_{n+1}(x) + \frac{e^{i(n-1)\phi}}{i^{(n-1)}} J_{n-1}(x) \right] \quad (3.2.17)$$

The only difference between the first component and the second component in eqn (3.2.12) is that the first has $\cos \phi'$ in its integrand and the second $\sin \phi'$. This will bring a sign of minus between the first and the second integrands of (3.2.13) and a factor of $1/i$ for that equation and hence exactly the same expressions as eqns (3.2.15) and (3.2.16) can be found so that the second component of (3.2.12) will be

$$j_y = \frac{-\pi a \cos \theta}{i} \left[\frac{e^{i(n+1)\phi}}{i^{(n+1)}} J_{n+1}(x) - \frac{e^{i(n-1)\phi}}{i^{(n-1)}} J_{n-1}(x) \right] \quad (3.2.18)$$

What remains now is to do similar procedure for the third component of eqn (3.2.12). If we make the change of variable $\phi - \phi' = -\alpha$ in the third component then

$$j_z = a \sin \theta e^{in\phi} \int_{-\phi}^{2\pi-\phi} e^{i(n\alpha - x \cos \alpha)} \cos \alpha \, d\alpha \quad (3.2.19)$$

$$= a \sin \theta i e^{in\phi} \frac{\partial}{\partial x} \int_{-\phi}^{2\pi-\phi} e^{i(n\alpha - x \cos \alpha)} \, d\alpha \quad (3.2.20)$$

The integral in eqn (3.2.20) can be again rearranged so as to reduce it to the standard form of Bessel function. Therefore

$$j_z = 2\pi i a \sin \theta e^{in(\phi-\pi/2)} J_n'(x) \quad (3.2.21)$$

Where $J_n'(x)$ is the usual derivative of Bessel function with respect to its argument. Now we have to find j_x^2 , j_y^2 and j_z^2 to find $|j|^2$. Let's see the first component

$$j_x^2 = j_x \cdot j_x^*$$

so

$$j_x^2 = \pi^2 a^2 \cos^2 \theta [J_{n+1}^2 + J_{n-1}^2 - 2 \cos(2\phi) J_{n-1} J_{n+1}]$$

Similarly we have

$$j_y^2 = \pi^2 a^2 \cos^2 \theta [J_{n+1}^2 + J_{n-1}^2 + 2 \cos(2\phi) J_{n-1} J_{n+1}]$$

and

$$j_z^2 = 4\pi^2 a^2 \sin^2 \theta J_n'^2$$

Therefore

$$|j|^2 = 2\pi^2 a^2 \cos^2 \theta (J_{n+1}^2 + J_{n-1}^2) + 4\pi^2 a^2 \sin^2 \theta J_n'^2 \quad (3.2.22)$$

Note that the dependence of $|j|^2$ on ϕ disappears and hence we shall see that the spectrum does not depend on ϕ . Eqn(3.2.22) can be also written as

$$|j|^2 = 4\pi^2 a^2 \left(J_n'^2 + \frac{\cot^2 \theta}{\beta^2} J_n^2 \right) \quad (3.2.23)$$

if we make use of the recurrence relations for Bessel functions

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

and make the substitution $x = n\beta\sin\theta$. The spectrum of the radiation can thus be given as

$$\frac{dP_n}{d\Omega} = \frac{e^2 w_0^4 a^2}{2\pi c^3} n^2 \left[[J_n'(n\beta\sin\theta)]^2 + \frac{\cot^2\theta}{\beta^2} [J_n(n\beta\sin\theta)]^2 \right] \quad (3.2.24)$$

after substituting eqn (3.2.23) into (3.2.8). When $\beta \rightarrow 0$ the only term out of (3.2.24) that can exist is the first term, $n=1$

$$\frac{dP_n}{d\Omega} \rightarrow \frac{dP_1}{d\Omega} = \frac{e^2 w_0 a^2}{8\pi c^3} (1 + \cos^2\theta) \quad (3.2.25)$$

which is the dipole radiation. Like that in (2.2.8) the spectrum is discrete. The origin of the discreteness in (3.2.24) is the presence of higher multipole radiation. We have dipole radiation, quadruple radiation and so on corresponding to $w_0, 2w_0, \dots$ frequencies. Eqns(2.2.8) and (3.2.24) have similar form but quite different contents. Unlike (2.2.8), (3.2.24) depend on ϕ . The most interesting thing of this is that one can distinguish between elliptic and circular orbits of a charge just by observing the spectrum of the radiation with out actually observing the trajectory. The dependence of the spectrum of the radiation on ϕ is charactorstics of the radiation from elliptic motion of a charge and independence on ϕ is that of circular motion.

The arguments of the Bessel functions in (3.2.24) contain a factor of $\sin\theta$ but those of (2.2.8)'s do not. As the result of this (3.2.24) can not be easily integrated over the solid

angle but (2.2.8) can. If we perform the summation over all n of eqn (3.2.24) we get

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 w_0^2 a^2}{2\pi C^3} \left[\sum_{n=1}^{\infty} n^2 [J_n'(nx)]^2 + \frac{\cot^2 \theta}{\beta^2} \sum_{n=1}^{\infty} n^2 [J_n(nx)]^2 \right] \quad (3.2.26)$$

This is an exact expression for the angular distribution of the average power radiated. The serieses on the right of (3.2.26) can be summed up to yield

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 a^2 w_0}{32\pi C^3} \left[\frac{4(1+\cos^2 \theta) - \beta^2(1+3\beta^2) \sin^4 \theta}{(1-\beta^2 \sin^2 \theta)^{7/2}} \right] \quad (3.2.27)$$

Where we've used the mathematical relations (see appendix)

$$\sum_{n=1}^{\infty} n^2 [J_n'(nx)]^2 = \frac{4 + 3x^2}{16(1-x^2)^{5/2}}$$

and

$$\sum_{n=1}^{\infty} n^2 [J_n(nx)]^2 = \frac{x^2(4+x^2)}{16(1-x^2)^{7/2}}$$

Expression (3.2.27) is again exact for relativistic and non-relativistic motion. It is reasonably simple to integrate eqn (3.2.27) than (3.2.24). Integrating (3.2.27) over the solid angle, thus, gives

$$\langle P \rangle = \frac{2e^2}{3C^3} \frac{a^2 w_0}{(1-\beta^2)^2} \quad (3.2.28)$$

Which means fast moving charges radiate more.

3.3 Interference Effects In The Radiation From A System Of Two Charges

Consider two charges moving in a circular orbit being on opposite ends of a diameter. see fig (3.2)

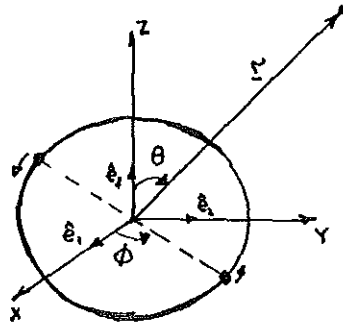


fig.(3.2)

The n^{th} Fourier-component of the magnetic field due to the two charges is given by

$$\underline{B}_n = \underline{B}_{1n} + \underline{B}_{2n} \quad n = \text{integer} \quad (3.3.1)$$

Where \underline{B}_{1n} is the fourier-component of the field due to q_1 and \underline{B}_{2n} is due to q_2 . Eqn(3.2.7) can be used to find the fields produced by both charges to yield the net n^{th} multipole field as follows :

$$\begin{aligned} \underline{B}_n = & \frac{q_1}{r} \frac{nw_0^2}{2\pi C^2} i e^{ikr} \int_0^T dt' (\hat{k} x y_1) \exp i(nw_0 t' - \underline{K} \cdot \underline{r}'_{01}) \\ & + \frac{q_2}{r} \frac{nw_0^2}{2\pi C^2} i e^{ikr} \int_0^T dt' (\hat{k} x y_2) \exp i(nw_0 t' - \underline{K}_0 \cdot \underline{r}'_{02}) \end{aligned} \quad (3.3.2)$$

The two terms represent the fields produced by each charges independently. q_1 and q_2 are in general diferent but we shall

consider two possible cases: $q_1 = -q_2$ and $q_1 = q_2$ and leave them moving being on the two ends of a diameter so that

$$\underline{r}'_{o1} = -\underline{r}'_{o2} \quad \text{and} \quad \underline{v}_1 = -\underline{v}_2. \quad (3.3.3)$$

$$\text{Case I-.....} q_1 = -q_2 \quad (3.3.4)$$

In this case \underline{E}_n will be

$$\underline{E}_n = \frac{qnw_o^2}{2\pi c^2} \frac{ie^{ikr}}{r} \left[\int_0^T dt' (\hat{k} \times \underline{v}) \exp i(nw_o t' + k \cdot \underline{r}'_o) \right. \\ \left. + \int_0^T dt' (\hat{k} \times \underline{v}) \exp i(nw_o t' + k \cdot \underline{r}'_o) \right] \quad (3.3.5)$$

Which is obtained after substituting q for q_1 and $-q$ for q_2 ; $\underline{r}'_{o1} = \underline{r}'_o$ and $-\underline{r}'_o$ for \underline{r}'_{o2} and \underline{v} for \underline{v}_1 and $-\underline{v}$ for \underline{v}_2 in eqn(3.3.2) as required by (3.3.3) and (3.3.4) . The first integral of the preceding equation is exactly the expression in eqn (3.2.11) which is integrated in sec.(3.2) component by component. See eqns(3.2.17) (3.2.18) and (3.2.21). After some rearrangements the first integral of eqn (3.3.2) may be written as

$$\underline{j}_1 = -2\pi i a e^{i(\phi - \pi/2)n} \left[\left(\cos\phi J'_n(n\epsilon) - (i/\epsilon) \sin\phi J_n(n\epsilon) \cos\theta \right) \hat{e}_1 \right. \\ \left. + \left(\sin\phi J'_n(n\epsilon) + (i/\epsilon) \cos\phi J_n(n\epsilon) \cos\theta \right) \hat{e}_2 \right. \\ \left. - \sin\theta J'_n(n\epsilon) \hat{e}_3 \right] \quad (3.3.6)$$

Where \hat{e}_1 , \hat{e}_2 and \hat{e}_3 are unit vectors as indicated in fig.(3.2) and $\epsilon = \beta \sin \theta$. The second integral of (3.3.5) is the same as the first one, except that ϵ is to be replaced by $-\epsilon$ because $\underline{r}_{o1} = -\underline{r}_{o2}$

$$\begin{aligned} \underline{j}_2 = -2\pi i a e^{i(\phi - \pi/2)n} \left[(\cos\phi J_n'(-n\epsilon) - (i/-\epsilon) \sin\phi J_n(-n\epsilon)) \cos\theta \hat{e}_1 \right. \\ \left. + (\sin\phi J_n'(-n\epsilon) + (i/-\epsilon) \cos\phi J_n(-n\epsilon)) \cos\theta \hat{e}_2 \right. \\ \left. - \sin\theta J_n'(-n\epsilon) \hat{e}_3 \right] \end{aligned} \quad (3.3.7)$$

Using the properties of Bessel function that

$$J_n(-x) = -(-1)^n J_n(x) \quad ; \quad J_n'(-x) = (-1)^n J_n'(x)$$

the second integral can be re-written as

$$\underline{j}_2 = -(-1)^n \underline{j}_1 \quad (3.3.8)$$

so that the resultant Fourier component of the magnetic field is

$$\begin{aligned} \underline{B}_n(\underline{r}) = \\ (1 - (-1)^n) n \frac{q a \omega^2}{rc^2} \exp\left[i\left(\frac{\phi - \beta r - \pi}{a} n\right)\right] \left[\left(\cos\phi J_n'(n\epsilon) - \frac{i \sin\phi J_n(n\epsilon)}{\epsilon} \right) \cos\theta \hat{e}_1 \right. \\ \left. + \left(\sin\phi J_n'(n\epsilon) + \frac{i \cos\phi J_n(n\epsilon)}{\epsilon} \right) \cos\theta \hat{e}_2 \right. \\ \left. - \sin\theta J_n'(n\epsilon) \hat{e}_3 \right] \end{aligned} \quad (3.3.9)$$

It is clear from this that $\underline{E}_n(\underline{r})$ is non-zero only for odd n . We then write it as

$$\begin{aligned} \underline{E}_{2m+1}(\underline{r}) = & \\ & \frac{2qaw^2}{rc^2} (2m+1) \exp\left[i(2m+1) \left(\phi - \frac{\beta r - \pi}{a} \right) \right] \left[\left(\cos\phi J'_{2m+1} - \frac{i \sin\phi J_{2m+1}}{\epsilon} \cos\theta \hat{e}_1 \right. \right. \\ & \left. \left. + \left(\sin\phi J'_{2m+1} + \frac{i \cos\phi J_{2m+1}}{\epsilon} \right) \cos\theta \hat{e}_2 - \sin\theta J'_{2m+1} \hat{e}_3 \right) \right] \end{aligned} \quad (3.3.10)$$

In which the argument of the Bessel functions is $(2m+1)\beta\sin\theta$ and m is an integer.

The power radiated per unit solid angle into the m th-multipole is obtained by substituting this into eqn (3.2.6)

$$\frac{dP_{2m+1}}{d\Omega} = 4 \left(\frac{qaw_0}{2\pi C^3} \right)^2 (2m+1)^2 \left([J_{2m+1}((2m+1)\epsilon)]^2 + \frac{\cot^2\theta}{\beta^2} [J_{2m+1}(2m+1)\epsilon]^2 \right) \quad (3.3.11)$$

where $m = 0, 1, 2, \dots$. The average total power radiated can then be calculated by summing up the contributions from all multipoles

$$\left\langle \frac{dP}{d\Omega} \right\rangle = 4 \left(\frac{qaw_0}{2\pi C^3} \right)^2 \sum_{m=0}^{\infty} (2m+1)^2 \left((J'_{2m+1})^2 + \frac{\cot^2\theta}{\beta^2} (J_{2m+1})^2 \right) \quad (3.3.12)$$

Eqn(3.3.12) can not be integrated over the solid angle to give the angular distribution of the power radiated by the system of the two charges. Because, unlike the series in eqn (3.2.26), the series in eqn (3.3.12) can not be summed to give

the explicit dependence of $\langle dp/d\Omega \rangle$ on the polar angles. But just to make some feeling of the effect of interference let's evaluate the series for small β . Let's first find

$$\lim_{\beta \rightarrow 0} \left[(J_{2m+1})^2 + \frac{\cot^2 \theta}{\beta^2} (J_{2m+1}) \right] =$$

$$\lim_{\beta \rightarrow 0} \left[\frac{1}{4} (J_{2m} - J_{2m+2})^2 + \frac{\cos^2 \theta}{4} \frac{\beta^2 \sin^2 \theta}{\beta^2 \sin^2 \theta} (J_{2m} + J_{2m+2})^2 \right]$$

The only term that would survive is that for $m=0$. Therefore the limit approaches to

$$\frac{1}{4} (1 + \cos^2 \theta)$$

Thus

$$\left\langle \frac{dP}{d\Omega} \right\rangle \rightarrow 4 \frac{(qaw_0^2)^2}{8\pi c^3} (1 + \cos^2 \theta) \quad (3.3.13)$$

Comparison of this with (2.2.14) shows that when charges are in a circular motion the power radiated in the Larmor-radiation for the two equal and opposite charges is four times that due to a single charge.

Case II..... $q_1 = q_2$ (3.3.14)

In this case one would have one more negative sign in eqn (3.3.7) because q_2 has now the same charge as q_1 . The resultant magnetic field component would then be

$$\begin{aligned} \underline{B}_n = & (1+(-1)^n) \frac{nqaw_0^2}{rc^2} \exp \left[i \left(\phi + \frac{\beta r}{a} - \pi \right) n \right] \left[(\cos \phi J_n'(n\epsilon) - \frac{i \sin \phi J_n(n\epsilon)}{\epsilon}) \cos \theta \hat{e}_1 \right. \\ & \left. + \left(\sin \phi J_n'(n\epsilon) + \frac{i \cos \phi J_n(n\epsilon)}{\epsilon} \right) \cos \theta \hat{e}_2 - \sin \theta J_n'(n\epsilon) \hat{e}_3 \right] \end{aligned} \quad (3.3.15)$$

The non-zero contribution comes only from even n . Therefore we write the expression for \underline{B}_n as

$$\begin{aligned} \underline{B}_{2m} = & (2m) \frac{qaw_0^2}{rc^2} \exp \left[i \left(\phi + \frac{\beta r}{a} - \tau \right) 2m \right] \left[(\cos \phi J_m' - \frac{i \sin \phi J_{2m}}{\epsilon}) \cos \theta \hat{e}_1 \right. \\ & \left. + \left(\sin \phi J_m' + \frac{i \cos \phi J_{2m}}{\epsilon} \right) \cos \theta \hat{e}_2 - \sin \theta J_{2m}' \hat{e}_3 \right] \end{aligned} \quad (3.3.16)$$

substituting this into (3.2.6) yield

$$\frac{dP_{2m}}{d\Omega} = 4(2m)^2 \left(\frac{qaw_0^2}{2\pi c^3} \right)^2 \left[[J_{2m}'(2m\epsilon)]^2 + \frac{\cot^2 \theta}{\beta^2} [J_{2m}(2m\epsilon)]^2 \right] \quad (3.3.17)$$

where m is an integer, $m=1,2,3,\dots$. Note also that the angular distribution of the radiated power is independent of ϕ in both cases.

Let's also evaluate the average total power radiated per unit solid angle:

$$\left\langle \frac{dp}{d\Omega} \right\rangle = 4 \left(\frac{qaw_0^2}{2\pi c^3} \right)^2 \sum_{m=1}^{\infty} (2m)^2 \left(J_{2m}^2 + \frac{\cot^2 \theta}{\beta^2} J_{2m}^2 \right) \quad (3.3.18)$$

Again the sum on the right-hand side of the preceding equation can not be summed. But to make some feeling of the effect of the interference when $q_1 = q_2$, let's evaluate the series on the right

for small β . To do so let's also first see

$$\begin{aligned} & \lim_{\beta \rightarrow 0} \left(J_{2m}^2 + \frac{\cot^2 \theta}{\beta^2} J_{2m}^2 \right) \\ &= \lim_{\beta \rightarrow 0} \frac{1}{4} (J_{2m-1} - J_{2m+1})^2 + \frac{\cos^2 \theta}{4} \frac{\beta^2 \sin^2 \theta}{\beta^2 \sin^2 \theta} (J_{2m-1} + J_{2m+1})^2 \end{aligned}$$

When the argument tends to zero ($\beta \rightarrow 0$), the survival of Bessel function is guaranteed only when the order is zero. In this case none of the above Bessel functions can have order zero. Thus the limit tends to zero in proportion to β^2 . This means that the dipole radiation by the two like charges which are moving on the same circle with the same non-relativistic speed being on opposite ends of a diameter is absent. Comparison between the two cases, $q_1 = -q_2$ and $q_1 = q_2$ shows that in the Larmor radiation the radiation from the former is constructively interfered and that from the later is destructively interfered.

CHAPTER FOUR

SIMULTANEOUS EXPANSION
FOR THE POTENTIALS OF A CHARGE
IN CIRCULAR MOTION

4.1. Lagrange Expansion For Far Zone Fields
Produced By A Charge In Circular Motion

Consider a charge moving on the x-y plane as shown in fig(4.1)

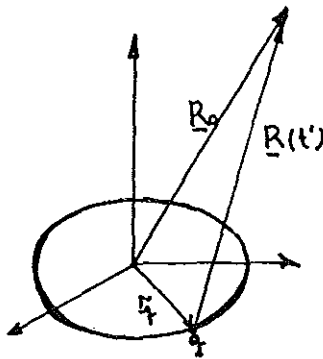


fig.(4.1)

The Lienard-Wiechert vector potential for $R = |\underline{R}_0 - \underline{r}_q| \neq 0$ is given by

$$\underline{A} = \frac{q\underline{\beta}(t')}{R(t') - \underline{R}(t') \cdot \underline{\beta}(t')} \quad (4.1.1)$$

Where

$$t' = t - \frac{R(t')}{c}$$

and

$$\underline{\beta} = \frac{1}{c} \frac{d}{dt'} \underline{r}_q(t')$$

and; \underline{r}_q and $\underline{R}(t')$ are vectors as shown in fig (4.1). In far zone where $R_0 \gg |\underline{r}_q|$, we can approximate $R(t')$ as

$$R(t') = \left(R_0^2 + r_q^2 - 2\underline{R}_0 \cdot \underline{r}_q \right)^{1/2} \approx R_0 - \hat{n}_0 \cdot \underline{r}_q(t') \quad (4.1.2)$$

Where

$$\hat{n}_0 = \underline{R}_0 / R_0$$

is a unit vector. This means that

$$t' \approx t - R_0/c + \hat{n}_0 \cdot \underline{r}_q(t')/c = t^* - (\hat{n}_0 \cdot \underline{r}_q(t')/c) \quad (4.1.3)$$

Where

$$t^* = t - R_0/c \quad (4.1.4)$$

In the denominator of (4.1.1), however, we can approximate $R(t') \approx R_0$ so that the retarded vector potential may be written as

$$\underline{A}(\underline{R}_0, t) = \frac{q\underline{\beta}(t')}{R_0[1 - \hat{n}(t') \cdot \underline{\beta}(t')]} \quad (4.1.5)$$

for $R_0 \gg |\underline{r}_q|$. If the charge is moving in a circular orbit of radius a we can write

$$\begin{aligned} \underline{\beta}(t') &= \underline{\beta}[t^* + 1/c \hat{n}_0 \cdot \underline{r}_q(t')] \\ &= \frac{a\omega_0}{c} \left[-\sin(\omega_0 t^* + \underline{k} \cdot \underline{r}_q) \hat{e}_1 + \cos(\omega_0 t^* + \underline{k} \cdot \underline{r}_q) \hat{e}_2 \right] \end{aligned}$$

where ω_0 is the angular frequency of the motion and $\underline{k} = (\omega_0/c) \hat{n}_0$

Using the properties of the trigonometric functions we can re-express this as

$$\begin{aligned} \underline{\beta}(t') &= \underline{\beta}_q(t^*) \cos[\underline{k} \cdot \underline{r}_q(t')] - k \underline{r}_q(t^*) \sin[\underline{k} \cdot \underline{r}_q(t')] \end{aligned} \quad (4.1.6)$$

Thus the retarded vector potential of a circulating charge in the far zone takes form

$$\underline{A}(\underline{R}_0, t) = \frac{q}{R_0} \left[\frac{\underline{\beta}_q(t^*) \cos[\underline{k} \cdot \underline{r}_q(t')]}{[1 - \hat{n}(t') \cdot \underline{\beta}(t')]} - k \underline{r}_q(t^*) \frac{\sin[\underline{k} \cdot \underline{r}_q(t')]}{[1 - \hat{n}(t') \cdot \underline{\beta}(t')]} \right] \quad (4.1.7)$$

Eqn(4.1.7) is the retarded vector potential that should be calculated at an earlier time $t' = t - R/c + \hat{n}_0 \cdot \underline{r}_q/c = t^* + (\hat{n}_0 \cdot \underline{r}_q)/c$. To evaluate this at a simultaneous time we shall use the lagrange expansion which states that any arbitrary function $u(t')$ of the retarded time t' can be expressed as

$$U(t') = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{d^m}{dt^m} \left[\left(\frac{R}{c} \right)^m U(t) (1 - \hat{n}(t) \cdot \underline{\beta}(t)) \right] \quad (4.1.8)$$

Where the series on the right is a function of t only and

$$t' = t - \frac{R(t')}{c} \quad (4.1.9)$$

We can also use the lagrange expansion for our purpose if we replace $R(t')/c$ by $-\hat{n}_0 \cdot \underline{r}_q(t')/c$ and t' by t^* . The Lagrange expansion of (4.1.7) is then given by

$$\underline{A} = \frac{q}{R_0} \underline{\beta}_q(t^*) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{d}{dt^*} \right)^m \left[\left(\frac{-\underline{k} \cdot \underline{r}_q(t^*)}{w_0} \right)^m \cos[\underline{k} \cdot \underline{r}(t^*)] \right] \\ - \underline{k} \underline{r}_q(t^*) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{d}{dt^*} \right)^m \left[\left(\frac{-\underline{k} \cdot \underline{r}_q(t^*)}{w_0} \right)^m \sin[\underline{k} \cdot \underline{r}_q(t^*)] \right]$$

(4.1.10)

Since

$$\hat{n}_0 = \sin\theta \cos\phi \hat{e}_1 + \sin\theta \sin\phi \hat{e}_2 + \cos\theta \hat{e}_3$$

and

$$\underline{r}_q(t^*) = a [\cos(w_0 t^*) \hat{e}_1 + \sin(w_0 t^*) \hat{e}_2]$$

We can write

$$\frac{\hat{n}_0 \cdot \underline{r}_q(t^*)}{c} = \frac{1}{w_0} \underline{k} \cdot \underline{r}_q(t^*) \\ = \frac{a}{c} \sin\theta \cos(w_0 t^* - \phi) = \frac{\beta \sin\theta}{w_0} \cos(w_0 t^* - \phi) = \frac{\epsilon \cos\alpha}{w_0}$$

(4.1.11)

where $\beta = w_0 a/c$; $\epsilon = \beta \sin\theta$ and $\alpha = w_0 t^* - \phi = w_0 t - w_0 R_0/c - \phi$.

Substituting these in (4.1.10) yield

$$\underline{A} = \frac{q}{R_0} \underline{\beta}_q(t^*) \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{d}{dt^*} \right)^m \left[\left(\frac{\epsilon \cos\alpha}{w_0} \right)^m \cos(\epsilon \cos\alpha) \right] \\ - \frac{w_0 \underline{r}_q(t^*)}{c} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{d}{dt^*} \right)^m \left[\left(\frac{\epsilon \cos\alpha}{w_0} \right)^m \sin(\epsilon \cos\alpha) \right]$$

(4.1.12)

Noting that

$$\left(\frac{d}{dt^*} \right)^m = w_0^m \left(\frac{d}{d\alpha} \right)^m$$

(4.1.13)

We can also re-write eqn (4.1.12) as

$$\underline{A} = \frac{q}{R_0} \left[\beta_q(t^*) \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} \frac{d^m}{d\alpha^m} (\cos^m \alpha \cos(\epsilon \cos \alpha)) \right. \\ \left. - \frac{kr_q}{c} \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} \frac{d^m}{d\alpha^m} (\cos^m \alpha \sin(\epsilon \cos \alpha)) \right] \quad (4.1.14)$$

Note that for the case of positronium the first sum should be taken over even values of m only, while the second sum over odd values only. On the other hand if the charges are the same, unlike in the positronium the first sum should be taken over odd values of m only and the second over even values of m only.

Using the evident relations

$$\cos^{2m} \alpha \sin(\epsilon \cos \alpha) = i^{2m} \frac{\partial^{2m}}{\partial \epsilon^{2m}} \sin(\epsilon \cos \alpha)$$

$$\cos^{2m+1} \alpha \sin(\epsilon \cos \alpha) = -i^{2m+1} \frac{\partial^{2m+1}}{\partial \epsilon^{2m+1}} \cos(\epsilon \cos \alpha)$$

$$\cos^{2m} \alpha \cos(\epsilon \cos \alpha) = i^{2m+1} \frac{\partial^{2m}}{\partial \epsilon^{2m}} \cos(\epsilon \cos \alpha)$$

and

$$\cos^{2m+1} \alpha \cos(\epsilon \cos \alpha) = i^{2m+1} \frac{\partial^{2m+1}}{\partial \epsilon^{2m+1}} \sin(\epsilon \cos \alpha)$$

We can re-write eqn (4.1.14) as follows

$$\underline{A} = \frac{q}{R_0} \left[\sum_{m=0}^{\infty} \frac{(i\epsilon)^{2m}}{2m!} \frac{\partial^{2m}}{\partial \alpha^{2m}} \frac{\partial^{2m}}{\partial \epsilon^{2m}} \beta_q(t^* + (\epsilon/w) \cos \alpha) \right. \\ \left. - ik \sum_{m=0}^{\infty} \frac{(i\epsilon)^{2m+1}}{(2m+1)!} \frac{\partial^{2m+1}}{\partial \alpha^{2m+1}} \frac{\partial^{2m+1}}{\partial \epsilon^{2m+1}} r_q(t^* + (\epsilon/w) \cos \alpha) \right] \quad (4.1.15)$$

This expression is equivalent to the Fourier-Series but reminds the double Taylor series. This is another representation of the vector potential due to a circulating charge in the far zone.

4.2 The Exact Expression For The Field Of A Particle In A Circular Motion

Consider the retarded electromagnetic potentials produced by a moving point charge of charge q given by

$$\phi(\underline{r}, t) = 2q \int_{-\infty}^{\infty} \theta(\tau) \delta[|\underline{r} - \underline{r}_q(t')|^2 - \tau^2] d\tau \quad (4.2.1a)$$

$$\underline{A}(\underline{r}, t) = 2q \int_{-\infty}^{\infty} \theta(\tau) \underline{\beta}_q(t') \delta[|\underline{r} - \underline{r}_q(t')|^2 - \tau^2] d\tau \quad (4.2.1b)$$

Where \underline{r} is the observation point at time, t , $\underline{r}_q(t')$ is the space trajectory of the charge at an earlier time $t' = t - \tau/c$ and $\underline{\beta}_q(t')$ is its velocity divided by the speed of light, also at an earlier time t'

Since $r_q^2 = a^2$ is a constant for a point charge moving in a circular orbit, one may write the expressions for the potentials as

$$\phi(\underline{r}, t) = 2q \int_{-\infty}^{\infty} \theta(\tau) \delta[z + g(\tau) - \tau^2] d\tau \quad (4.2.2a)$$

$$\underline{A}(\underline{r}, t) = 2q \int_{-\infty}^{\infty} \theta(\tau) \underline{\beta}_q(t - \tau/c) \delta[z + g(\tau) - \tau^2] d\tau \quad (4.2.2b)$$

Where

$$z = r^2 + r_q^2$$

is a constant independent of τ and

$$g(\tau) = -2\underline{r} \cdot \underline{r}_q(t - \tau/c) \quad (4.2.3)$$

Now let's expand $\delta[z + g(\tau) - \tau^2]$ in a Taylor Series around $g(\tau)=0$

$$\delta[z + g(\tau) - \tau^2] = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k}{\partial g^k} \delta \right)_{g=0} [g(\tau)]^k$$

or

$$\delta[z + g(\tau) - \tau^2] = \sum_{k=0}^{\infty} \frac{[g(\tau)]^k}{k!} \frac{\partial^k}{\partial z^k} \delta[z - \tau^2] \quad (4.2.4)$$

Substitution into (4.2.2a,b) yield

$$\phi = 2q \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial z^k} \int_{-\infty}^{\infty} \theta(\tau) [g(\tau)]^k \delta(z - \tau^2) d\tau \quad (4.2.5a)$$

$$\underline{A} = 2q \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial z^k} \int_{-\infty}^{\infty} \theta(\tau) \underline{\beta}_q(t - \tau/c) [g(\tau)]^k \delta[z - \tau^2] d\tau \quad (4.2.5b)$$

Using the property of δ -function that

$$\int_{-\infty}^{\infty} \theta(\tau) f(\tau) \delta[F(\tau)] d\tau = \sum \frac{\theta(\tau_0) f(\tau_0)}{|dF/dt|} \quad (4.2.6)$$

(Where τ_0 is a root of $F(\tau) = 0$) Eqns (4.2.5a) and (4.2.5b) can

be re-expressed as

$$\phi = q \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \frac{\partial^k}{\partial z^k} \left[\frac{r \cdot r_q(t - \sqrt{z}/c)}{\sqrt{z}} \right]^k \quad (4.2.7a)$$

$$\underline{A} = q \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \frac{\partial^k}{\partial z^k} \left[\frac{\beta_q(t - \sqrt{z}/c)}{\sqrt{z}} [r \cdot r_q(t - \sqrt{z}/c)] \right] \quad (4.2.7b)$$

In obtaining eqns(4.2.7a) and (4.2.7b) we have used $F(\tau) = z - \tau^2$ which has two roots $\pm\sqrt{z}$. The contribution to the sum on the right of eqn (4.2.6) comes only from the positive root of $F(\tau)=0$, due to the presence of the step function which is zero for $\tau_0 < 0$.

Using the Leibnitz rule of differentiation for product of two functions and rearranging the index of variables the potentials may be written as

$$\phi = q \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} z^{-n-1/2} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \frac{\partial^k}{\partial z^k} (r \cdot r_q(t'))^{k+n} \quad (4.2.8a)$$

$$\underline{A} = q \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} z^{-n-1/2} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \frac{\partial^k}{\partial z^k} \left(\beta_q(t') [r \cdot r_q(t')]^{k+n} \right) \quad (4.2.8b)$$

Where we have used the fact that

$$\frac{\partial^n}{\partial z^n} \frac{1}{\sqrt{z}} = \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{n!} z^{-n-1/2} \quad (4.2.8c)$$

and make the change of variable

$$t' = t - \sqrt{z}/c$$

Let's expand the potentials in the Fourier-Series.

$$\phi = \sum_{m=-\infty}^{\infty} \phi_m \exp(-imw_0 t) \quad (4.2.9a)$$

$$A = \sum_{m=-\infty}^{\infty} A_m \exp(-imw_0 t) \quad (4.2.9b)$$

Where w_0 is the frequency of the motion of the charge.

The Fourier-Component of (4.2.9a) is then

$$\phi_m = \frac{q}{T\sqrt{z}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} \frac{1}{(\sqrt{z})^{2n}} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \int_0^T \frac{\partial^k}{\partial z^k} [\underline{r} \cdot \underline{r}_q(t')]^{k+n} \exp(imw_0 t) dt \quad (4.2.10)$$

The integral on (4.2.10) can be written as

$$\begin{aligned} \frac{\partial^k}{\partial z^k} \int_0^T [\underline{r} \cdot \underline{r}_q(t')]^{k+n} \exp(imw_0 t) dt = \\ \left(\frac{-1}{2c\sqrt{z}} \right)^k \int_0^T \frac{\partial^k}{\partial t^k} [\underline{r} \cdot \underline{r}_q(t')]^{k+n} e^{-imw_0 t} dt \end{aligned}$$

if we note that

$$\frac{\partial}{\partial z} = -\frac{1}{2c\sqrt{z}} \frac{\partial}{\partial t}$$

and

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t}$$

Integrating (4.2.11) by parts k times yield

$$\phi_m = \frac{q}{T\sqrt{z}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \int_0^T \left(\frac{2\mathbf{r} \cdot \mathbf{r}_q(t')}{z} \right)^n \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-imw_0 \mathbf{r} \cdot \mathbf{r}_q}{c\sqrt{z}} \right)^k \exp(imw_0 t) dt \quad (4.2.12)$$

Since

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{(2n)! x^n}{2^{2n}(n!)^2}$$

and

$$e^{-ix} = \sum_{k=0}^{\infty} \frac{(-ix)^k}{k!}$$

We may write (4.2.12) as

$$\phi_m(\underline{r}) = \frac{q}{T} \int_0^T \frac{\exp\{imw_0[t - \mathbf{r} \cdot \mathbf{r}_q(t')/c\sqrt{z}]\}}{|\underline{r} - \underline{r}_q(t')|} dt \quad (4.2.13a)$$

Similar procedure can be made to show that

$$\underline{A}_m(\underline{r}) = \frac{q}{T} \int_0^T \frac{\beta(t') \exp\{imw_0(t - \mathbf{r} \cdot \mathbf{r}_q(t'))\}}{|\underline{r} - \underline{r}_q(t')|} dt \quad (4.2.13b)$$

These expressions are exact for any circular motion of charge, valid for any $\beta \lesssim 1$ and any distance r for which $\underline{r} \neq \underline{r}_q(t')$.

Differentiating these expressions (with $\dot{\underline{A}}_m = -imw_0 \underline{A}_m$) we may find the exact expressions for the fields \underline{E} and \underline{B} of the moving charge. By substituting the fields into the equation of motion

$$\frac{d}{dt} (m \gamma \underline{V}) = q [\underline{E} + 1/c \underline{V} \times \underline{B}] + \underline{F}_{\text{react.}}$$

We can verify whether circular motion is possible for a system of two relativistic charged particles with

electromagnetic interaction only. It should be noted that without simultaneous expansion it is rather difficult to obtain such expressions.

If we use the expansion

$$\frac{1}{|\underline{r} - \underline{r}'|} = \sum_{l=0}^{\infty} \sum_{m'=-l}^l \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm'}(\theta', \phi') Y_{lm'}(\theta, \phi) \quad (4.2.14)$$

Where $r_{<}(r_{>})$ is the smaller (the larger) of r and r' ; θ' and ϕ' are the polar coordinates of \underline{r}' and θ and ϕ are of \underline{r} ; Y_{lm} 's are the spherical harmonics. Using this we may write the expressions in (4.2.13 a,b) as expansions in Bessel functions. Omitting the intermediate steps we give the final expressions

$$\phi_m = qe^{im(\phi+\alpha-\pi/2)} \sum_{l=0}^{\infty} \sum_{m'=-l}^l C_{lmm'} P_{lm'}(\cos\theta) J_{m-m'} \left\{ \begin{array}{c} \frac{a^l}{r^{l+1}} \\ \\ \frac{r^l}{a^{l+1}} \end{array} \right\} \quad (4.2.14a)$$

$$\underline{A}_m = qe^{im(\phi+\alpha-\pi/2)} \sum_{l=0}^{\infty} \sum_{m'=-l}^l C_{lmm'} P_{lm'}(\cos\theta) \left\{ \begin{array}{c} \frac{a^l}{r^{l+1}} \\ \\ \frac{r^l}{a^{l+1}} \end{array} \right\} \left[\begin{array}{l} \frac{(m-m') \sin\theta J_{m-m'}}{m\epsilon} \hat{e}_r \\ \\ + \frac{(m-m') \cos\theta J_{m-m'}}{m\epsilon} \hat{e}_\theta + i\beta J_{m-m'} \hat{e}_\phi \end{array} \right] \quad (4.2.14b)$$

Where

$$J_{m-m'} = J_{m-m'}(m\beta\epsilon)$$

$$c_{1mm'} = \frac{(1-m')!}{(1+m')!} P_{1m'}(\cos\theta') \exp(im'\pi/2)$$

$$\alpha = \frac{\beta\sqrt{r^2+a^2}}{a} \quad \text{and} \quad a = |r_q|, \text{ radius of the circle and}$$

$$c = \frac{r \sin \theta}{\sqrt{r^2+a^2}}$$

In our case $\theta'=\pi/2$ and $\phi'=w_0t$. The potentials have different dependence on r for $r<a$ and $r>a$ but are continuous at $r=a$. It should be noted that the series in eqn (4.2.14) is convergent only for $r_< = r_>$. However, the serieses in the expressions for the potentials are convergent including for $r=a(r_<=r_>)$ due to the presence of the Bessel functions. Unlike the potentials the fields \underline{E}_m and \underline{B}_m are discontinuous at $r=a$.

Conclusions And Main Results

This thesis work can be said to consist of two main parts: Non-relativistic and Relativistic periodic motions of charges and the radiation from them. In the first chapter the non-relativistic kepler problem is deeply considered to the extent of solving kepler's problem completely. Kepler's problem which seems rather difficult to be solved has been exactly solved using Fourier-Series expansion. Using this the explicit time dependence of the coordinates, which permit us to locate where the particle is, at any time, is clearly determined. This not only helped as to determine the coordinates at any time t but also, to express the energy and angular momentum of the system as the sum of discrete levels which deserves the name "Quantization". "Quantization" here does not mean that the system can have only quantized energy or angular momentum as in the case of Quantum theory.

The radiation from two charges moving non-relativistically in the case of coulomb interaction in which the fourier components of the dipole moment of this system is expressed interms of the fourier components of the coordinates is briefly considered in the second chapter. Even though our consideration is totaly non-relativistic the spectrum of the radiation we got is discrete. The higher harmonics, however, disappear if the eccentricity ϵ , of the orbit tends to zero. Then, the radiation

becomes exactly that of the monochromatic dipole radiation from non-relativistic circular motion of a charge. The angular distribution of the time average of the entire power radiated is also determined to be

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{(eaw_0)^2}{32\pi c^3 (1-\epsilon^2)^{5/2}} \left(4(2+\epsilon^2) - (4 + \epsilon^2 + 2\epsilon^2 \cos^2 \phi) \sin^2 \theta \right)$$

It seems that this expression has not been considered before probably because it follows from the mathematical result

$$\sum_{n=1}^{\infty} n^2 J_n^2(n\epsilon) = \frac{\epsilon^2 (4 + \epsilon^2)}{16(1-\epsilon^2)^{7/2}}$$

(See Appendix) that contains a misprint in main mathematical books available. Unlike in circular motion, the angular distribution of the power radiated depends on ϕ (the polar angle). This shows that the distribution is maximum parallel to the major axis of the orbit and is minimum parallel to the minor axis for a given angle θ . Apart from this it also shows that charges moving in more elliptic orbits radiate more than those close to a circle.

Furthermore the effects of radiation have been considered deeply. It has been demonstrated that the system collapses due to radiation. This explains the radiative collapse of the classical model of a hydrogen atom. The theoretical collapse time estimated for atomic scale systems is in the range of 10^{-12} - 10^{-11} sec. This is in good agreement with those given in different electrodynamics books. The broadening of each spectral line is the other aspect of the effect of radiation. Although

the radiation from non-relativistic charge is very small, the gradual broadening of each spectral line lead the spectrum to become contineous at least for higher harmonics.

In the third chapter the discreteness of the spectrum of the radiation from a relativistic circular motion of a charge is observed. This demonstrates that in circular motion the higher multipole radiation arise only due to relativistic corrections. It would also be worth noting that higher multipole radiation arise only due to relativistic corrections is not in general true. To prove this it is enough to remember that the spectrum from non-relativistic elliptic motion of a charge is discrete. One of our achievements in this is we have found the exact expression for the angular distribution of the time average of the entire power radiated in a rather obvious and direct method. Before the completion of that chapter we have also considered the interference effects of the radiation field from two charges. Unfortunately, we faced a great difficulty in summing up the serieses in eqns (3.3.12) and (3.3.18) which prevented us from discussing the interference effects of radiation field from relativistic charges. The only chance remaining was to evaluate the serieses for non-relativistic motion ($\beta \rightarrow 0$). There we found that the radiation from the same charges interfere distructively and that from opposite charges interfere constructively.

The simultaneous expansion of the fields which would make life easier in investigating the interaction of charges particularly to those in circular motion irrespective of the

magnitude of their speed ($0 \leq \beta \leq 1$) has been considered in chapter four. The main results of this are :

- i) The simultaneous fields in far zone due to a circulating charge have been obtained. The main advantage of these expression is that they do not need independent considerations to evaluate the radiation field due to more than one charge. In this we have taken a first step towards the application of the Lagrange expansion of fields first considered by A.N Gordeyev to evaluate the fields at asimultaneous time
- ii) The exact simultaneous expression for the fields of a charge in circular motion, at every point outside from the particle has been found. By differentiating these expressions for the scalar and rector potentials one can find the fields \underline{E} and \underline{B} due to the charge in circular motion. From this one can also find the electromagnetic interaction of two charges and justify the possibility of such relativistic motion.

A P P E N D I X

This section is meant here to prove some of the mathematical relationships used in this thesis work.

We first re-write the equations we already know.

$$\phi = w_0 t + \epsilon \sin \phi \quad (\text{A.1.1}) ; \quad x(t) = \sum_{n=1}^{\infty} \frac{2aJ_n'(n\epsilon)}{n} \cos(nw_0 t) - \frac{3\epsilon a}{2} \quad (\text{A.1.5})$$

$$x = a(\cos \phi - \epsilon) \quad (\text{A.1.2}) ; \quad y(t) = \sum_{n=1}^{\infty} \frac{2bJ_n(n\epsilon)}{n} \sin(nw_0 t) \quad (\text{A.1.6})$$

$$y = b \sin \phi \quad (\text{A.1.3}) ; \quad \frac{\partial \phi}{\partial t} = \frac{w_0}{1 - \epsilon \cos \phi} \quad (\text{A.1.7})$$

$$r = a(1 - \epsilon \cos \phi) \quad (\text{A.1.4}) ; \quad \frac{\partial \phi}{\partial \epsilon} = \frac{\sin \phi}{1 - \epsilon \cos \phi} \quad (\text{A.1.8})$$

Differentiating (A.1.2) and (A.1.5) with respect to time yield respectively

$$\frac{\partial x}{\partial t} = - \frac{aw_0 \sin \phi}{1 - \epsilon \cos \phi} \quad (\text{A.1.9})$$

$$\frac{\partial x(t)}{\partial t} = -2aw_0 \sum_{n=1}^{\infty} J_n' \sin(nw_0 t) \quad (\text{A.1.10})$$

Multiplying (A.1.9) by (A.1.3) and (A.1.10) by (A.1.6) and equating the results obtained gives

$$\sum_{n,m=1}^{\infty} \frac{4abw_0}{\epsilon} \frac{J_n J_m'}{n} \sin(nw_0 t) \sin(mw_0 t) = \frac{abw_0 \sin^2 \phi}{(1-\epsilon \cos \phi)}$$

The time average of this yield

$$\frac{2}{\epsilon} \sum_{n=1}^{\infty} \frac{J_n J_n'}{n} = \frac{w_0}{2\pi} \int_0^{\pi} \frac{\sin^2 \phi dt}{1-\epsilon \cos \phi} \quad (\text{A.1.11})$$

Using eqn(A.1.7) the integral over time can be changed to integral over ϕ so that

$$\frac{2}{\epsilon} \sum_{n=1}^{\infty} \frac{J_n J_n'}{n} = \frac{1}{2\pi} \int_0^{\pi} \sin^2 \phi d\phi = 1/2$$

Therefore

$$\sum_{n=1}^{\infty} \frac{J_n(n\epsilon) J_n'(n\epsilon)}{n} = \frac{\epsilon}{4} \quad (\text{A.1.12})$$

Differentiating this with respect to ϵ yield

$$\sum_{n=1}^{\infty} (J_n')^2 + (J_n J_n'') = \frac{1}{4} \quad (\text{A.1.13})$$

Using Bessel's differential equation

$$J_n''(n\epsilon) + \frac{1}{n\epsilon} J_n'(n\epsilon) + \left(1 - \frac{n^2}{n^2 \epsilon^2}\right) J_n(n\epsilon) = 0$$

we can re-write (A.1.13) as

$$\sum_{n=1}^{\infty} \left[J_n'^2 - J_n \left(\frac{1}{n\epsilon} J_n' + \left(1 - \frac{1}{\epsilon^2}\right) J_n \right) \right] = \frac{1}{4}$$

or

$$\sum_{n=1}^{\infty} \left(J_n'^2 + \frac{1-\epsilon^2}{\epsilon^2} J_n^2 \right) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{J_n J_n'}{n\epsilon}$$

Substituting (A.1.12) in the series on the right of the preceding equation we get the first important relation we need:

$$\sum_{n=1}^{\infty} (J_n'^2 + \frac{1-\epsilon^2}{\epsilon^2} J_n^2) = \frac{1}{2} \quad (\text{A.1.14})$$

Let's now try to evaluate $\sum_{n=1}^{\infty} n^2 J_n^2 (n\epsilon)$.

Consider

$$\frac{1}{r} = \frac{1}{a} \left[1 + \sum_{n=1}^{\infty} 2J_n(n\epsilon) \cos(nw_0 t) \right]$$

Differentiating this with time gives

$$\frac{\partial}{\partial t} \left(\frac{1}{r} \right) = -\frac{2w_0}{a} \sum_{n=1}^{\infty} n J_n \sin(nw_0 t) \quad (\text{B.1.1})$$

Similarly from (A.1.4)

$$\frac{\partial}{\partial t} \left(\frac{1}{r} \right) = \frac{w_0 \epsilon \sin \phi}{a(1-\epsilon \cos \phi)^3} \quad (\text{B.1.2})$$

Squaring (B.1.1) and (B.1.2) and equating yields

$$\sum_{n,m=1}^{\infty} nm J_n J_m \sin(nw_0 t) \sin(mw_0 t) = \frac{\epsilon^2 \sin^2 \phi}{4(1-\epsilon \cos \phi)^6} \quad (\text{B.1.3})$$

The time average of this equation gives

$$\frac{4\pi}{\epsilon^2} \sum_{n=1}^{\infty} n^2 J_n^2 (n\epsilon) = \int_0^T \frac{\sin^2 \phi}{(1-\epsilon \cos \phi)^5} \frac{w_0 dt}{1-\epsilon \cos \phi} \quad (\text{B.1.4})$$

Again using (A.1.7), (B.1.4) reduces to

$$\frac{4\pi}{\epsilon^2} \sum_{n=1}^{\infty} n^2 J_n^2 = \int_0^{2\pi} \frac{\sin^2 \phi d\phi}{(1-\epsilon \cos \phi)^5} \quad (\text{B.1.5})$$

Integrating by parts the right hand side of (B.1.5) gives

$$I = \frac{1}{4\epsilon} \int_0^{2\pi} \frac{\cos \phi d\phi}{(1-\epsilon \cos \phi)^4} \quad (\text{B.1.6})$$

We shall integrate this using the theory of complex variables.

Let

$$z = e^{i\phi} \quad |z| = 1$$

Which implies

$$d\phi = \frac{dz}{iz} \quad ; \quad \cos\phi = \frac{z^2 + 1}{2z} \quad (\text{B.1.7})$$

$$\text{and } (1 - \epsilon \cos\phi) = -\frac{\epsilon}{2z} (z - z_1)(z - z_2) \quad (\text{B.1.8})$$

Where

$$z_1 = \frac{1 + \sqrt{1 - \epsilon^2}}{\epsilon} \quad \text{and} \quad z_2 = \frac{1 - \sqrt{1 - \epsilon^2}}{\epsilon} \quad (\text{B.1.9})$$

and that $|z_1| > 1$ and $|z_2| < 1$ for $0 < \epsilon < 1$.

Thus

$$I = \frac{2}{i\epsilon^5} \oint \frac{z^2(z^2+1)}{(z-z_1)^4} \frac{dz}{(z-z_2)^4} \quad (\text{B.1.10})$$

The integral is around a unit circle in the complex plane.

From the theory of complex variables this is

$$I = \frac{2}{i\epsilon^5} \frac{2\pi i}{3!} \left[\frac{d^3}{dz^3} \left(\frac{z^2(z^2+1)}{(z-z_1)^4} \right) \right]_{z=z_1}$$

$$I = \frac{\pi(4 + \epsilon^2)}{4(1 - \epsilon^2)^{7/2}} \quad (\text{B.1.11})$$

In which we have used the values of z_1 and z_2 from

(B.1.9). Therefore substituting this in (B.1.5) yield

$$\sum_{n=1}^{\infty} n^2 J_n^2 = \frac{\epsilon^2(4 + \epsilon^2)}{16(1 - \epsilon^2)^{7/2}} \quad (\text{B.1.12})$$

It would be worth mentioning the miss-print in many books on Table of Serieses and Products, that (B.1.12) is usually

written as

$$\sum_{n=1}^{\infty} n^2 J_n^2 = \frac{\epsilon^2 (4 + \epsilon^2)}{16(1-\epsilon^2)^{1/2}} \quad (\text{B.1.13})$$

in which " 7/2 " is miss-printed as " 1/2 " .

The last will be the evaluation of $\sum_{n=1}^{\infty} n^2 J_n'^2$

Consider the time derivatives of (A.1.3) and (A.1.6),

$$\frac{\partial y}{\partial t} = \frac{b w_0 \cos \phi}{(1 - \epsilon \cos \phi)} \quad (\text{C.1.1})$$

$$\frac{\partial y(t)}{\partial t} = \frac{2b w_0}{\epsilon} \sum_{n=1}^{\infty} J_n \sin (n w_0 t) \quad (\text{C.1.2})$$

Squaring and equating the preceding two equations yield

$$\frac{4b^2 w_0^2}{\epsilon^2} \sum J_n J_m \sin(n w_0 t) \sin(m w_0 t) = \frac{b^2 w_0^2 \cos^2 \phi}{(1 - \epsilon \cos \phi)^2}$$

The time average of this equation yield

$$\frac{2b^2 w_0^2}{\epsilon^2} \sum_{n=1}^{\infty} J_n^2 = \frac{b^2 w_0^2}{T} \int_0^T \frac{\cos^2 \phi dt}{(1 - \epsilon \cos \phi)^2} = \frac{w_0 b^2}{T} \int_0^{2\pi} \frac{\cos^2 \phi d\phi}{(1 - \epsilon \cos \phi)^2}$$

or

$$\frac{4\pi}{\epsilon^2} \sum_{n=1}^{\infty} J_n^2 (n\epsilon) = \int_0^{2\pi} \frac{\cos^2 \phi d\phi}{(1 - \epsilon \cos \phi)^2} \quad (\text{c.1.3})$$

The integral on the right of (c.1.3) can be integrated using complex variables. Consider

$$I = \int_0^{2\pi} \frac{\cos^2 \phi d\phi}{(1 - \epsilon \cos \phi)^2}$$

If we let $z = e^{i\phi}$ then

$$\cos^2 \phi = \frac{(z^2+1)^2}{4z^2} \quad ; \quad (1-\epsilon \cos \phi) = -(\epsilon/2z)(z-z_1)(z-z_2)$$

Where z_1 and z_2 are as in (B.1.9) so that

$$I = \int_0^{2\pi} \frac{\cos^2 \phi \, d\phi}{(1-\epsilon \cos \phi)} = - \oint \frac{(z^2+1)^2 \, dz}{2i\epsilon(z-z_1)(z-z_2)z^2}$$

The contour is chosen to be a unit circle in the complex plane. Using Cauchy's theorem the preceding integral is

$$I = \frac{-2\pi i}{2i\epsilon} \left\{ \frac{(z^2+1)^2}{z^2(z-z_1)} \Big|_{z=z_2} + \frac{\partial}{\partial z} \frac{(z^2+1)^2}{(z-z_1)(z-z_2)} \Big|_{z=0} \right\}$$

$$I = -\frac{\pi}{\epsilon} \frac{2}{\epsilon} \left(1 - \frac{1}{\sqrt{1-\epsilon^2}} \right) = \frac{2\pi}{\epsilon^2} \left(\frac{1}{\sqrt{1-\epsilon^2}} - 1 \right) \quad (\text{c.1.4})$$

Substituting this in (c.1.3) yield

$$\sum_{n=1}^{\infty} J_n^2(n\epsilon) = \frac{1}{2} \left(\frac{1}{\sqrt{1-\epsilon^2}} - 1 \right) \quad (\text{c.1.5})$$

$$\sum_{n=1}^{\infty} n^2 J_n(n\epsilon) J_n'(n\epsilon) = \frac{\epsilon}{4(1-\epsilon^2)^{3/2}} \quad (\text{c.1.6})$$

Differentiating this once more result in

$$\sum_{n=1}^{\infty} n^2 J_n'^2 + \frac{1-\epsilon^2}{\epsilon^2} \sum_{n=1}^{\infty} n^2 J_n J_n'' = \frac{1+2\epsilon^2}{4(1-\epsilon^2)^{5/2}}$$

If we substitute for $J_n''(n\epsilon)$ from Bessel's differential equation the preceding equation takes form

$$\sum_{n=1}^{\infty} n^2 J_n'^2 + \frac{1-\epsilon^2}{\epsilon^2} \sum_{n=1}^{\infty} n^2 J_n^2 - \frac{1}{\epsilon} \sum_{n=1}^{\infty} n J_n J_n' = \frac{1+2\epsilon^2}{4(1-\epsilon^2)^{5/2}}$$

We can use (c.1.6) to replace the last series on the left of the preceding equation to get

$$\sum_{n=1}^{\infty} n^2 J_n'^2 + \frac{1-\epsilon^2}{\epsilon^2} \sum_{n=1}^{\infty} n^2 J_n^2 = \frac{1}{4} \frac{2+\epsilon^2}{(1-\epsilon^2)^{5/2}} \quad (\text{c.1.7})$$

Substituting the second series of (c.1.7) by (B.1.12) yield

$$\sum_{n=1}^{\infty} n^2 J_n'^2 (n\epsilon) = \frac{4+3\epsilon^2}{16(1-\epsilon^2)^{5/2}} \quad (\text{c.1.8})$$

REFERENCES

1. Arfken, G.: "Mathematical Methods for Physicist," Academic Press, New York, 1966.
2. Bellman R.: "Perturbation Technique in Mathematics, Physics and Engineering," Holt, Rinehart and Winston, New York, 1964.
3. Bronson, P. F. and Sipe J. E.: "Classical Radiative Electron Capture by a Proton," J. Phys. A: Math. Gen. Vol. 19, pp (1577 - 1582), 1986.
4. Gordeyev, A. N.: "Simultaneous Expansion for the Electro-magnetic Field of a Relativistic Charged Particle," J. Phys. A: Math. Gen., Vol. 8, p. 1048, 1963.
5. Gradshteyn, I. S. and Ryshik, I. M.: "Table of Serieses, Products and Integrals," VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
6. Heitler, W.: "The Quantum Theory of Radiation," 3rd ed., Oxford University Press, London, 1954.
7. Jackson, J. D.: "Classical Electrodynamics," 2nd ed., Wiley, New York, 1975.
8. Konopinski, E. J.: "Electromagnetic Fields and Relativistic Particles," McGraw-Hill, New York, 1981.
9. Landau, L. D. and Lifshitz, E. M.: "The Classical Theory of Fields (Course of Theoretical Physics; Vol. 2)," 4th ed., Translated from 6th rev. ed. of Teoriia Polia, Pergamon Press, Oxford, 1975.
10. Landau, L. D. and Lifshitz, E. M.: "Mechanics, (Course of Theoretical Physics, Vol. 1)" 2nd ed., Translated from the 2nd rev. enl. ed. of Mekanika, Pergamon Press, Oxford, 1965.
11. Leach PGL: "The First Integrals and Orbit Equation for the Kepler Problem with Drag," J. Phys. A: Math. Gen., Vol. 20, pp (1997 - 2002), 1987.
12. Lieu R: "Synchrotron Radiation Reaction," J. Phys. A: Math. Gen., Vol. 20, pp (2405 - 2413), 1987.
13. Marion, Jerry B.: "Classical Dynamics of Particles and Systems," 2nd ed., Academic Press, New York, 1970.

14. Page, L. and Adams, N. L.: "Electrodynamics," D. Van Nostrand, New York, 1940.
15. Watson, G. N.: "A Treatise on the Theory of Bessel Functions," 2nd ed., Cambridge University Press, London, 1944.
16. Weert, Ch G. V.: "Comment: Simultaneous Expansion for the Electromagnetic Field of a Relativistic Charge Particle," J. Phys. A: Math. Gen., Vol. 9, No 3, 1976.
17. Wichman Eyvind H.: "Quantum Physics," Berkley Physics Course - Vol. 4, McGraw-Hill Company, New York, 1971.