Addis Ababa University
School of Graduate Studies
Department of Mathematics

Graduate seminar report
on

Iterated Norms in Nikol'skiĭ-Besov spaces with Generalized Smoothness

By: Wondimu Woubante

Advisor: Dr. Tsegaye Gedif

June 2002
Addis Ababa

PREFACE

This is a report of two seminars, which is conducted, in the first and second semester.

The main objective of this seminar is to prove the theorem

$$\underbrace{\mathscr{B}_\theta^{\bar{\varphi}}(\mathscr{B}_\theta^{\bar{\varphi}}\ldots\left(\mathscr{B}_\theta^{\bar{\varphi}}(\mathscr{B}_\theta^{\bar{\varphi}}\left(L_p(G)\right)\ldots\right))}_{k} \hookrightarrow B_{p,\theta}^{\bar{\varphi}k}(G)$$

and for a special conditions

$$\underbrace{\mathscr{B}_\theta^{\bar{\varphi}}(\mathscr{B}_\theta^{\bar{\varphi}}\ldots\left(\mathscr{B}_\theta^{\bar{\varphi}}(\mathscr{B}_\theta^{\bar{\varphi}}\left(L_p(G)\right)\ldots\right))}_{k} = B_{p,\theta}^{\bar{\varphi}k}(G)$$

But to prove this theorem different expressions and propositions should be examined. For this the seminar has 4 chapters.

Chapter 1 - preliminaries

Chapter 2 - Definitions and some facts on Nikolskii-Besove space.

Chapter 3 -- Iterated norms for Nikoskii-Besove spaces.

Chapter 4 - Iterated norms in Nikolskii-Besov space with generalized smoothness.

Before all I like to thank the heavenly father almighty God, with the help of WHOM this seminar come to reality.

My special gratitude to My Advisor and teacher Dr. Tsegaye Gedif not only for his continuous advice but also to provide me his unpublished results to the completion of my seminar and to show me the bright future.

I am really lucky to have him as my advisor; with out his kind help this seminar would have been impossible.

Finally I am indebted: to all of my previous teachers and friends who wish my progress, to W/o Tobiaw Teferra to print my report.

CONTENTS

CHAPTER I

Preliminaries

I.1 Notations and basic inequalities

The following standard notations are used through out this paper.

N - the set of all natural numbers,

$\mathbf{N}_0$ - the set of all non-negative integers,

Z - the set of all integers,

$\Re$- the set of all real numbers,

C- the set of all complex numbers,

$\mathbf{N}_0^n$ - $\underbrace{\mathbf{N}_0 \times \mathbf{N}_0 \cdots \times \mathbf{N}_0}_{n}$ - the set of multi-indices (n is the natural number which will be used exclusively to denote the dimension),

$\Re^n$ - $\underbrace{\Re \times \Re \times \ldots \times \Re}_{n}$

For $\alpha \in \mathbf{N}_0^n$, $\alpha \neq 0$, we write:

$$D^\alpha f \equiv \frac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$ the derivative of the function f of order α

For an arbitrary nonempty set $\Omega \subset \Re^n$ we denote the following:

$C(\Omega)$ - the space of functions continuous on Ω.

$C_b(\Omega)$ - the Banach space of functions f continuous and bounded on Ω with the norm

$$\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|,$$

For a measurable nonempty set $\Omega \subset \Re^n$ we shall denote by:

$L_p(\Omega)$ $(1 \leq p \leq \infty)$ - the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f| \, dx \right)^{\frac{1}{p}} < \infty \quad \text{if } 1 \leq p < \infty$$

$$\|f\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf_{w:\ meas\,(w)=0}\ \sup_{x\in\Omega\setminus w} |f(x)| < \infty \quad \text{if } p=\infty$$

for the case where measure(Ω) $\neq 0$

and if measure(Ω) = 0 , then we set $\|f\|_{L_\infty(\Omega)} = 0$

We note that if $\Omega \subset \Re^n$ is an open set , then for $f \in C(\Omega)$ $\|f\|_{C(\Omega)} = \|f\|_{L_\infty(\Omega)}$.

For an open nonempty set $\Omega \subset \Re^n$ we shall denote by:

$L_p^{loc}(\Omega)\,(1 \le p \le \infty)$ - the set of functions defined on Ω such that for each compact $K \subset \Omega$ $f \in L_p(K)$[1],

$C^\ell(\Omega)(\ell \in \mathbf{N})$ - the space of functions f defined on Ω such that $\forall \alpha \in \mathbf{N}_0^n$ where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \ell$ *and* $\forall x \in \Omega$ *the derivative* $(D^\alpha f)(x)$ exists and $D^\alpha f \in C(\Omega)$,

$C^\infty(\Omega) = \bigcap_{l=0}^{\infty} C^l(\Omega)$ - the space of infinitely continuously differentiable functions on Ω.

a.e - almost every where.

I.2. Basic inequalities

Hölder's inequality.

Suppose that $\frac{1}{p} + \frac{1}{q} = 1$ i.e., $p = \frac{q}{q-1}$ for $1 < q < \infty$, $p = \infty$ for $q = 1$ and $p = 1$ for $q = \infty$.

If $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$, then $fg \in L_1(\Omega)$ and

$$\|fg\|_{L_1(\Omega)} \le \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}. \tag{1.1}$$

Minkowski's inequality. If f, $g \in L_p(\Omega)$, then $f + g \in L_p(\Omega)$ and

$$\| f + g \|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}. \tag{1.2}$$

Minkowski's inequality for integrals.

In addition, let $A \subset R^m$ be a measurable set. Suppose that f is measurable on $A \times \Omega$ and $f(\cdot, y) \in L_p(\Omega)$ for almost all $y \in A$. Then

$$\left\| \int_A f(\cdot, y) \right\|_{L_p(\Omega)} \leq \int_A \|f(\cdot, y)\, dy\|_{L_p(\Omega)} \tag{1.3}$$

if the right-hand side is finite.

Generalized Minkowski's inquality

For any finite N of elements $x^1, x^2, \ldots, x^N$ of a normed linear space **E** we have the inequality

$$\left\| \sum_{k=1}^{N} x^k \right\| \leq \sum_{k=1}^{N} \|x^k\| \tag{1.4}$$

obtained by induction from the basic inequality $\|x+y\| \leq \|x\| + \|y\|$ $(x, y \in \mathbf{E})$.

further, if x is the sum of a series $\sum_{k=1}^{\infty} x^k$ which is convergent in the norm in the space **E**, which means

$$\left\| x - \sum_{k=1}^{N} x^k \right\| \to 0 \quad \text{as } N \to \infty \quad \text{then}$$

$$\|x\| = \lim_{N\to\infty} \left\| \sum_{k=1}^{N} x^k \right\| \leq \lim_{N\to\infty} \sum_{K=1}^{n} \|x^k\| = \sum_{k=1}^{\infty} \|x^k\| \tag{1.5}$$

Now applying (1.4) and (1.5) to elements of the space l_p and L_p we obtain (Minkowski's) inequalities

$$\left(\sum_{k=1}^{\infty}\left|\sum_{i=1}^{m}a_{ki}\right|^{p}\right)^{\frac{1}{p}} \leq \sum_{i=1}^{m}\left(\sum_{k=1}^{\infty}|a_{ki}|^{p}\right)^{\frac{1}{p}} \quad (1 \leq p \leq \infty) \tag{1.6}$$

and

$$\left(\int_{\Omega}\left|\sum_{i=1}^{m}f_i(x)\right|^{p} dx\right)^{\frac{1}{p}} \leq \sum_{i=1}^{m}\left(\int_{\Omega}|f_i(x)|^{p} dx\right)^{\frac{1}{p}} \quad (1 \leq p \leq \infty) \tag{1.7}$$

Where it is allowable to put m = ∞ on condition that in this case the sums of the series $\sum_{i=1}^{\infty}a_{ki}$ are understood as numbers a_k such that

$$\sum_{k=1}^{\infty}\left|a_k - \sum_{i=1}^{m}a_{ki}\right|^{p} \to 0 \qquad as\ m \to \infty$$

and the sum $\sum_{i=1}^{\infty}f_i(x)$ is understood as F(x) $\in L_p$, to which the finite sum

$\sum_{i=1}^{m}f_i(x) \to F(x) \quad as\ m \to \infty$ with respect to the metric of L_p, i.e.

$$\int\left|F(x) - \sum_{i=1}^{m}f_i(x)\right|^{p} dx \to 0 \qquad (m \to \infty).$$

Inequality (1.7) involves summation with respect to the discrete index i = 1, 2, ... , on which $f_i(x)$ depends. As an analogue of (1.7) we can write *generalized Minkowiski's inequality*

$$\left(\int_{G}\left(\int_{\Omega}|F(x,y)|dy\right)^{p} dx\right)^{\frac{1}{p}} \leq \int_{\Omega}\left(\int_{G}|F(x,y)|^{p} dx\right)^{\frac{1}{p}} dy \tag{1.8}$$

If $F(x,y) = 0$ outside $\Omega \times G$ inequality (1.8) can be written as

$$\left(\int_{R^m} \left(\int_{R^n} | F(x,y) | dy \right)^p dx \right)^{\frac{1}{p}} \leq \int_{R^n} \left(\int_{R^m} | F(x,y) |^p dx \right)^{\frac{1}{p}} dy \qquad (1.9)$$

In generalized Minkowski's inequality (1.8) and (1.9)the parameter y takes the role of the summation index *i* in (1.7) and integration instead of summation is carried out with respect to y.

I.3 On Generalized derivatives

Definition 1. Let $\Omega \subset R^n$ be an open set. Set A is said to be *compact* (*strictly contained*) in Ω if A is bounded and (the closure of A, $\bar{A} \subset \Omega$). To denote that A is compact in Ω we write as $A \subset\subset \Omega$.

Definition2. . A measurable function *f* is said to be *finite* in Ω if it vanishes a.e. out side a certain set $\Omega' \subset\subset \Omega$. That is

$$f(x) = 0 \quad \forall x \in \Omega \setminus \Omega' \text{ and}$$

$$f(x) \neq 0 \quad \forall x \in \Omega'$$

Definition 3. The support of a continuous function φ defined on Ω is symbolized as supp φ is the closure of the set $\{x \in \Omega \mid \varphi(x) \neq 0\}$ i.e. $\overline{\{x \in \Omega \mid \varphi(x) \neq 0\}}$

Note: If Supp $\varphi \subset\subset \Omega$, then φ is finite in Ω.

Definition 4. A function f is said to be locally integrabe in Ω ($f \in L_p^{loc}(\Omega)$) if $f \in L_p^{loc}(K)$ for every compact subset $K \subset \Omega$.

Definition 5. *(The set of basic functions D(Ω))*

All functions finite or infinitely differentiable functions in Ω whose supports are compact subsets of Ω are called basic functions in Ω. i.e.

$$D(\Omega) = \left\{ \varphi \in C^{\infty}(\Omega) \mid \overline{\{x \in \Omega \mid \varphi(x) \neq 0\}} \subset\subset \Omega \right\}.$$

Definition 6. Any linear continuous functional f on a *space of basic functions* D′(Ω) is called a generalized function specified on an open set Ω. The value of the generalized function on the basic function φ ∈ D(Ω) will be written as (f, φ).

From the definition of generalized derivative the m_j^{th} generalized derivative with respect to x_j for any locally summable function f is given by

$$\begin{aligned}(D_j^{m_j} f, \varphi)(x) &= \int D_j^{m_j} f(x)\varphi(x)dx \qquad \forall \varphi \in D(\Omega)\\ &= (-1)^{m_j} \int f(x) D_j^{m_j} \varphi(x)dx \qquad (\text{using integration by parts})\\ &= (-1)^{m_j} (f, D_j^{m_j}\varphi)(x)\end{aligned}$$

Generally, the m_j^{th} generalized derivative of generalized function f with respect to x_j is defined by

$$(D_j^{m_j} f, \varphi)(x) = (-1)^{m_j} (f, D_j^{m_j}\varphi)(x)$$

1.4 Equivalent norms

Let Z_1 and Z_2 normed vector spaces.

1. Z_1 is embedded in Z_2 ($Z_1 \hookrightarrow Z_2$) iff $Z_1 \subset Z_2$ and $\|f\|_{Z_2} \leq c_2 \|f\|_{Z_1}$ for all $f \in Z_1$.

2. $\|f\|_{Z_2} \sim \|f\|_{Z_1}$ iff there exist C_1 and C_2 such that

$$\|f\|_{Z_2} \leq c_2 \|f\|_{Z_1} \text{ and } \|f\|_{Z_2} \leq c_1 \|f\|_{Z_1}.$$

CHAPTER 2

The Nikol'skii-Besov spaces

2.1. Difference of a function

Definition 2.1 Let $h \in \Re$ and $f : \Re^n \rightarrow \Re$ be a measurable function. The difference of f with respect to $x_j, j \in \{1,2,\cdots,n\}$ with step h, denoted by $(\Delta_h\, jf)(x)$ is defined by :

$$(\Delta_h\, jf)(x) = f(x_1,\cdots,x_j+h,\cdots,x_n) - f(x_1,\cdots,x_j,\cdots,x_n)$$
$$= f(x+he_j) - f(x) \qquad \textbf{(2.1)}$$

Here e_j is a unit vector $(0,\cdots,\underset{\substack{\uparrow \\ j^{th}\ component}}{1},\cdots,0)$.

Applying the above definition twice on the function f we extend our definition to the 2^{nd} difference of f with respect to x_j, with steps h.

$$\left(\Delta^2_{h,} f_{,j} f\right)(x) = f(x+2he_j) - 2f(x+he_j) + f(x) \qquad (2.2)$$

Indeed:

$$\left(\Delta^2_{h,j} f\right)(x) = \left(\Delta_{h}, j(\Delta_{h,j} f)\right)(x)$$
$$= \left(\Delta_h, j(f(x+he_j)) - f(x)\right)$$
$$= f(x+he_j+he_j) - f(x+he_j) - \left(\Delta_{h,j} f\right)$$
$$= f(x+2he_j) - f(x+he_j) - f(x+he_j) + f(x)$$
$$= f(x+2he_j) - 2f(x+he_j) + f(x).$$

The third difference of a function $f : \Re^n \rightarrow \Re$ with respect to x_j with step h $\in \Re$ is denoted by $\left(\Delta^3_h, jf\right)$ is given by:

$$\left(\Delta^3_{h,j} f\right) = \left(\Delta_h, j(\Delta^2{}_{h,j} f)\right)$$

$$= \left(f(x+2he_j+he_j)-2f(x+he_j+he_j)+f(x+he_j)\right)-\left(f(x+2he_j)-2f(x+he_j)+f(x)\right)$$

$$= \quad f(x+3hej)-3f(x+2hej)+3f(x+hej)-f(x)$$

Continuing this way by induction that we get the α^{th} difference of a function $f:\Re^n \to \Re$ with respect to x_j with step $h \in \Re$, denoted by $\left(\Delta^{\alpha}_{h},_{j}f\right)$ given by:

$$\left(\Delta^{\alpha}_{h},_{j}f\right) = \sum_{k=0}^{\alpha}(-1)^k \binom{\alpha}{k} f(x+(\alpha-k)he_j) \text{ for } \alpha \in \mathrm{N} \tag{2.3}$$

Lemma 2.1 Let f be an integrable function on (-∞ , ∞). Then

$$\int f(x)dx = \int f(x+h)dx \tag{2.4}$$

<u>Proof</u> <u>*Case i:*</u> $f(\mathrm{x}) = \chi_E(x)$ where E is measurable and mE < ∞. Since the function f is integrable,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \chi(x)dx = m\mathrm{E}$$

On the other hand

$$\int_{-\infty}^{\infty} f(x+h)dx = \int_{-\infty}^{\infty} \chi(x+h)dx = \int_{-\infty}^{\infty} \chi_{(E-h)}(x)dx = m\,(\mathrm{E} - h)$$

But Lebesgue measure is translation invariant i.e mE = m (E-h).

Hence,

$$\int f(x)dx = \int f(x+h)dx$$

<u>*Case ii:*</u> Suppose $f(x) = \sum_{i=1}^{m} a_i \chi_{E_i}(x)$, where each E_i is measurable with $mE_i < \infty$ as f is integrable.

Thus, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \sum_{i=1}^{m} a_i \chi(x)\, dx$

$$= \sum_{i=1}^{m} \int_{-\infty}^{\infty} a_i \chi_{E_i}(x)dx$$

$$= \sum_{i=1}^{m} a_i \int_{-\infty}^{\infty} \chi_{E_i}(x)dx$$

$$= \sum_{i=1}^{m} a_i \int_{-\infty}^{\infty} \chi_{E_i}(x)dx$$

$$= \sum_{i=1}^{m} a_i \int_{-\infty}^{\infty} \chi_{E_i}(x+h)dx \quad , \qquad \text{by case (i) above.}$$

$$= \sum_{i=1}^{m} \int_{-\infty}^{\infty} a_i \chi_{E_i}(x+h)\,dx$$

$$= \int_{-\infty}^{\infty} \sum_{i=1}^{m} a_i \chi(x+h)$$

$$= \int_{-\infty}^{\infty} f(x+h)dx$$

Case iii: Suppose f is a non-negative function. Then, since f is integrable and f is non-negative, there exist an increasing sequence of non-negative measurable simple functions vanishing out side a set of finite measure and such that $\varphi_n(x) \xrightarrow{n\to\infty} f$.

Thus, by Monotone convergence theorem , we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

Then

$$\int_{-\infty}^{\infty} f(x+h)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x+h)dx \text{ , by Monotone convergence theorem}$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x)dx \text{ , by case ii.}$$

$$= \int_{-\infty}^{\infty} f(x)dx \text{ , by Monotone convergence theorem.}$$

Case iv: Let f be integrable , thus $f=f^+ - f^-$ and f^+ and f^- are integrable and moreover $f^+ \geq 0$, $f^- \geq 0$. Therefore,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f^+(x)dx - \int_{-\infty}^{\infty} f^-(x)dx$$

$$= \int_{-\infty}^{\infty} f^{+}(x+h)dx - \int_{-\infty}^{\infty} f^{-}(x+h)dx \quad \text{,by } case\ (iii)$$

$$= \int_{-\infty}^{\infty} [f^{+}(x+h) - f^{-}(x+h]\ dx$$

$$= \int_{-\infty}^{\infty} f(x+h)dx$$

We shall consider the following properties of the difference of f , which will be invoked in the proof of the theorem in the following sections. These properties are stated for n = 1.

Properties:

1. $\|\Delta_{h}^{2} f\|_{L_p(R)} \leq 2\|\Delta_{h} f\|_{L_p(R)}$ (2.5)

Proof:
$$\begin{aligned}
\|\Delta_{h}^{2} f\|_{L_p(R)} &= \|f(x+2h) - 2f(x+h) + f(x)\|_{L_p(R)} \\
&\leq \|f(x+2h) - f(x+h)\|_{L_p(R)} + \| - f(x+h) + f(x)\|_{L_p(R)} \\
&= \|f((x+h)+h) - f(x+h)\|_{L_p(R)} + \|f(x+h) - f(x)\|_{L_p(R)} \\
&= \|\Delta_{h} f\|_{L_p(R)} + \|\Delta_{h} f\|_{L_p(R)} \\
&= 2\|\Delta_{h} f\|_{L_p(R)}.
\end{aligned}$$

2. $\Delta_{h} f = \frac{1}{2}\Delta_{2h} f - \frac{1}{2}\Delta_{h}^{2} f$ (2.6)

Proof:
$$\begin{aligned}
\frac{(\Delta_{2h} f)(x) - (\Delta_{h}^{2} f)(x)}{2} &= \frac{[f(x+2h) - f(x)] - [f(x+2h) - 2f(x+h) + f(x)]}{2} \\
&= \frac{2[f(x+h) + f(x)]}{2} \\
&= (\Delta_{h} f)(\mathrm{x}).
\end{aligned}$$

Remark 1. If E_h is a operator defined on the space of functions $f: \Re^n \to \Re$ by $(E_h f)(x) := f(x+h)$ and I is the identity operator, then (2.6) can be adapted to the operator equation:

$$E_h - I = \tfrac{1}{2}(E_{2h} - I) - \tfrac{1}{2}(E_h - I)^2,$$

which reminds us the property

$$x - I = \tfrac{1}{2}(x - I) - \tfrac{1}{2}(x - I)^2 \quad x \in \Re.$$

3. $$\left\|\Delta_h^\sigma f\right\|_{L_p(R)} \leq 2^\sigma \left\| f \right\|_{L_p(R)} \qquad (2.7)$$

Proof: From (2.3) $\Delta_h^\sigma f = \sum_{r=1}^{\omega} (-1)^{\sigma-r} \binom{\sigma}{r} f(x+rh)$ *for all* $\sigma \in \mathbf{N}$, where $\binom{\sigma}{r} = \dfrac{\sigma!}{r!(\sigma-r)!}$.

Thus $$\left\|\Delta_h^\sigma f\right\|_{L_p(R)} = \left\|\sum_{r=0}^{\sigma} (-1)^{\sigma-r} \binom{\sigma}{r} f(x+rh)\right\|_{L_p(R)}$$

$$\leq \sum_{r=0}^{\sigma} \binom{\sigma}{r} \left\|f(x+rh)\right\|_{L_p(R)}$$

$$\leq \sum_{r=0}^{\sigma} \binom{\sigma}{r} \left\|f(x+h)\right\|_{L_p(R)}$$

$$= \left\|f(x+h)\right\|_{L_p(R)} \sum_{r=0}^{\sigma} \binom{\sigma}{r}$$

$$= \left\|f(x)\right\|_{L_p(R)} 2^\sigma .$$

Lemma 2..2 Let $\sigma \in \mathbf{N}$; $h, \eta \in \Re$

$$\left(\Delta_h^\sigma f\right) = \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \binom{\sigma}{k} \left((\Delta_{\frac{k}{\sigma}\eta}^{\sigma} f)\ (x + (\sigma - k)h) + (-1)^{\sigma+1}\ (\Delta_{h-\frac{k}{\sigma}\eta}^{\sigma} f)(x + k\eta) \right) \qquad (2.8)$$

Proof: For $\sigma = 1$

The left hand side of (2.8) becomes $(\Delta_h f)$ and the right hand side becomes

$\left(\Delta_\eta E_{(\sigma-k)h} + \Delta_{h-\eta} E_\eta\right)$ but $\sigma - k = 0$

$= \left(\Delta_\eta f + \Delta_{h-\eta} E_\eta\right)$

$= f(x+\eta) - f(x) + f(x+h) - f(x+\eta)$

$= f(x + h) - f(x) = (\Delta_h, f)$.

thus it is true for $\sigma = 1$

Let $\sigma = 2$, starting from the right- hand side of (2.7) ,

$$\sum_{k=1}^{2} (-1)^{(2-k)} \binom{2}{k} \left[\left(\Delta^2_{\frac{k}{2}\eta}, E_{(2-k)}f \right) + (-1)^3 \left(\Delta^2_{h-\frac{k}{2}\eta}, E_{k\eta} f \right) \right]$$

$=2\, f(x + \eta + h) + 4f(x + \frac{1}{2}\eta) + 2f(x + \eta) + 2f(x + 2h) - 4f(x + h + \frac{1}{2}\eta) + 2f(x + \eta) +$

$+ f(x + 2\eta) - 2f(x + \eta) + f(x) - f(x + 2h) + 2\, f(x + h + \eta) - f(x + 2\eta)$

$= f(x + 2h) -2f (x + h) + f(x)$

$= (\Delta_h^2\, f)$.

Then the assertion is valid for $\sigma = 2$.

Therefore; proceeding by induction the lemma holds true for all $\sigma \in \mathbf{N}$.

Corollary 2.2. For $\alpha \in \mathbf{N}$; $h, \eta \in \Re$, we have

$$\left\|\Delta^{\sigma}_{h} f\right\|_{L_p(\Re)} \leq \sum_{k=1}^{\sigma} \binom{\sigma}{k} \left[\left\| \Delta^{\alpha}_{\frac{k}{\sigma}\eta} f \right\|_{L_p(\Re)} + \left\| \Delta^{\sigma}_{h-\frac{k}{\sigma}\eta,} f \right\|_{L_p(\Re)} \right] \tag{2.9}$$

This is a direct application of translation invariance of the norm $\|\cdot\|_{L_p(\mathbf{R})}$.

Lemma 2..3 Let $1 \leq p \leq \infty$, then for all $\alpha \in \mathbf{N}$, there is $A = A(\sigma)$ such that

$\forall\, h > 0$ and for all measurable functions f on $\Re$, we have,

$$\left\|\Delta^{\sigma}_{h} f\right\|_{L_p(\mathbf{R})} \leq \frac{A(\alpha)}{h} \int_0^h \left\|\Delta^{\sigma}_{h} f\right\|_{L_p(\mathbf{R})} d\eta \tag{2.10}$$

In particular,

$$\left\|\Delta_{h} f\right\|_{L_p(\mathbf{R})} \leq \frac{2}{h} \int_0^h \left\|\Delta_{\eta} f\right\|_{L_p(\mathbf{R})} d\eta \tag{2.11}$$

Proof: From the above corollary, we get for $\alpha \in \mathbf{N}$; $h, \eta \in \Re$

$$\left\|\Delta^{\sigma}_{h} f\right\|_{L_p(\mathbf{R})} \leq \sum_{k=1}^{\sigma} \binom{\sigma}{k} \left[\left\| \Delta^{\sigma}_{\frac{k}{\sigma}\eta} f \right\|_{L_p(\mathbf{R})} + \left\| \Delta^{\sigma}_{h-\frac{k}{\sigma}\eta} f \right\|_{L_p(\mathbf{R})} \right]$$

Integrating this from 0 to h with respect to η, we have

$$h\left\|\Delta_h^\sigma f\right\|_{L_p(\mathbf{R})} \le \sum_{k=1}^{\sigma}\binom{\sigma}{k}\left[\int_0^h \left\|\Delta^\sigma_{\frac{k}{\sigma}\eta} f\right\|_{L_p(\mathbf{R})} d\eta + \int_0^h \left\|\Delta^\sigma_{h-\frac{k}{\sigma}\eta} f\right\|_{L_p(\mathbf{R})} d\eta\right]$$

$$\le \sum_{k=1}^{\sigma}\frac{\sigma}{k}\binom{\sigma}{k}\left[\int_0^{\frac{k}{\sigma}h} \left\|\Delta^\alpha_\eta f\right\|_{L_p(\mathbf{R})} d\eta + \int_{h-\frac{k}{\sigma}\eta}^{h} \left\|\Delta^\sigma_\eta f\right\|_{L_p(\mathbf{R})} d\eta\right] d\eta]d\eta$$ This is so by $\frac{k}{\sigma}\to\eta$

and $h - \frac{k}{\sigma}\eta \to \eta$

$$\le 2\sigma\sum_{k=1}^{\sigma}\frac{\binom{\sigma}{k}}{k}\int_0^h \left\|\Delta^\sigma_\eta f\right\|_{L_p(\Re)} d\eta = 2\sigma\int_0^h \left\|\Delta^\sigma_\eta f\right\|_{L_p(\Re)} d\eta \sum_{k=1}^{\sigma}\frac{\binom{\sigma}{k}}{k}$$

$$\Rightarrow \quad \left\|\Delta^\sigma_h f\right\|_{L_p(\mathbf{R})} \le \frac{A(\sigma)}{h}\int_0^h \left\|\Delta^\sigma_\eta f\right\|_{L_p(\Re)} d\eta$$

where $A(\sigma) = 2\sigma\sum_{k=1}^{\sigma}\frac{\binom{\sigma}{k}}{k} \le 2\sigma 2^\sigma$, we have the lemma.

Now, in particular for $\sigma = 1$. Let $0 < \eta < h$. One can easily verify that

$\Delta_h f = \Delta_\eta f + \Delta_{h-\mu} E_\eta f$, where $(E_\eta f)(x) := f(x+\eta)$

Thus

$$f(x+h) - f(x) = f(x+\eta) - f(x) + f(x+\eta+h-\eta) - f(x+\eta)$$

$$\Rightarrow \quad \left\|\Delta_h f\right\|_{L_p(\Re)} \le \left\|\Delta_\eta f\right\|_{L_p(\Re)} + \left\|\Delta_{h-\eta} E_\eta f\right\|_{L_p(\Re)}$$

$$\Leftrightarrow \quad \left\|\Delta_h f\right\|_{L_p(\Re)} \le \left\|\Delta_\eta f\right\|_{L_p(\Re)} + \left\|\Delta_{h-\eta} f\right\|_{L_p(\Re)} \tag{2.12}$$

Integrating both sides with respect to η from *0* to *h*, we get

$$h\left\|\Delta_h f\right\|_{L_p(\Re)} \le \int_0^h \left\|\Delta_\eta f\right\|_{L_p(\Re)} d\eta + \int_0^h \left\|\Delta_{h-\eta} f\right\|_{L_p(\Re)} d\eta$$

$= 2\int_0^h \left\|\Delta_\eta f\right\|_{L_p(\Re)} d\eta$, by change of variable of $h-\eta \to \eta$ *as* $h \to 0$.

2.2 Definition and particular case on Nikolskii-Besov Space

2.2.1 Definition

Definition: Let $\ell > 0$, $1 \leq P \leq \theta$ and $1 \leq \theta \leq \infty$. Let f: $\Re^n \rightarrow \Re$ be a measurable function and h be any variable

$f \in B^{\ell}_{P,\theta}(\Re^n)$ (the Nikolskii-Besov space) iff

$f \in L_p(\Re^n)$ iff

$$\|f\|_{B^{\ell}_{p,\theta}(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{i=1}^{n} \|f\|_{B^{\ell}_{p,\theta,i}(\Re^n)} < \infty$$

where

$$\|f\|_{B^{\ell}_{p,\theta,i}(\Re^n)} = \left(\int_0^{\infty} (h^{-(\ell-\bar{\ell})} \left\| \Delta^{\sigma}_{h,\,i} D_i^{\bar{\ell}} f \right\|_{L_p(\Re^n)})^{\theta} \frac{dh}{h} \right)^{1/\theta} \text{ where } \theta < \infty$$

and

$$\|f\|_{B^{\ell}_{p,\theta,i}(\Re^n)} = \sup_{h>0} h^{-(\ell-\bar{\ell})} \left\| \Delta^{\sigma}_{h,\,i} D_i^{\bar{\ell}} f \right\|_{L_p(\Re^n)} \text{ where } \theta = \infty$$

$\bar{\ell} = [\ \ell\ ]$ is the greatest integer which is less than ℓ.

$$\sigma = \begin{cases} 1 \text{ if } \ell \text{ is not integer} \\ 2 \text{ if } \ell \text{ is an integer} \end{cases}$$

ℓ is the index of smoothness of a given function P and θ are index of summability and index of characterizing summability respectively.

Related to this definition we can consider different particular cases.

2.2.2 Particular cases

1. Let n = 1 and $0 < \ell < 1$, $1 \leq \theta < \infty$, the norm of a function f in the Nikollskill-Besov space is given by as follows

$n = 1 \Rightarrow f: \Re \rightarrow \Re$

$0 < \ell < 1 \Rightarrow \bar{\ell} = 0, \sigma = 1$

so, $\left\|\Delta_{h,j}^{\sigma} D_i^{\bar{\ell}} f\right\|_{L_p(\Re^n)} = \left\|\Delta_h f\right\|_{L_p(\Re^n)}$

$$= \left\|f(x+h) - f(x)\right\|_{L_p(\Re)}$$

$$\|f\|_{B_{p,\theta}^{\ell}(\Re)} = \|f\|_{L_p(\Re)} + \sum_{i=1}^{n} \|f\|_{B_{p,\theta,i}^{\ell}(\Re)}$$

$$= \|f\|_{L_p(\Re^n)} + \|f\|_{B_{p,\theta,i}^{\ell}(\Re)}$$

$$= \|f\|_{L_p(\Re^n)} + \left(\int_0^{\infty} \|f(x+h) - f(x)\|_{L_p(\Re)}^{\theta} \frac{dh}{h^{1+\ell\theta}} \right)^{1/\theta}$$

so $f \in B_{P,\theta}^{\ell}(\Re^n) \Rightarrow \|f\|_{B_{p,\theta}^{\ell}(\Re)} < \infty$

2. Let $n = 1$, $0 < \ell < 1$, $\theta = \infty \Rightarrow \bar{\ell} = 0$, $\sigma = 1$ and $f: \Re \to \Re$

$$\|f\|_{B_{p,\theta}^{\ell}(\Re)} = \|f\|_{L_p(\Re)} + \sup_{h>0} h^{-\ell} \left\|f(x+h) - f(x)\right\|_{L_p(\Re)}$$

3. Let $n = 1$ $\quad 0 < \ell < 1$, $p = \infty$, $\theta = \infty$

$$\Rightarrow \bar{\ell} = 0, \quad \sigma = 1$$

$$\|f\|_{L_p(\Re)} = \|f\|_{L_\infty(\Re)} = \sup_{t\in\Re} \text{vari } |f(t)|$$

$$\sup_{t\in\Re} \text{vari } |f(t)| = \text{In}\,f\ \{ M : m\{ t: f(t) > M\} = 0\}$$

m is the lebsgue measure of a set.

There fore

$$\|f\|_{B_{\infty,\infty}^{\ell}(\Re)} = \sup_{x\in\Re} \text{vari } |f(x)| + \sup_{h>0} h^{-\ell} \sup_{x\in\Re} \text{vari } |f(x + h) - f(x)|$$

So $f \in \|f\|_{B_{\infty,\infty}^{\ell}(\Re)}$ iff 1) $f \in L_\infty(\Re)$ and

2) $\sup_{h>0} h^{-\ell} \sup_{x\in\Re} \text{vari } |f(x + h) - f(x)| < \infty$

$\Rightarrow \|f\|_{L_\infty(\Re)} < \infty$

$\Rightarrow \exists M_1$ such that $\forall x \in\Re$, $|f(x)| \le M_1$ and $\forall h > 0$, $\exists M_2$ such that

$h^{-\ell} \sup_{x\in\Re} \text{vari } |f(x + h) - f(x)| \le M_2$

$\Rightarrow \sup_{x\in\Re} \text{vari } |f(x + h) - f(x)| \le M_2 h^{\ell}$

But $|f(x + h) - f(x)| \le \sup_{x\in\Re} \text{vari } |f(x + h)$, almost ,$\forall x \in \Re$

$\Rightarrow$For almost, $\forall x \in \Re$, $|f(x + h) - f(x)| \le M_2 h^{\ell}$

Let $M = \text{Max}\ \{M_1, M_2\}$

$\therefore f \in B^{\ell}_{\infty,\infty}(\Re)$ iff $\exists M$ such that for almost for all x in $\Re$, $|f(x)| \le M$ and $\forall h > 0$,

almost $\forall x \in \Re$, $|f(x + h) - f(x) \le Mh^{\ell}$

4. $n = 1, 0 < \ell < 1, 1 \le p < \infty, \theta = \infty$. it is the same as the second case

$f \in B^{\ell}_{p,\infty}(\Re) \Rightarrow f \in L_p(\Re)$ and

$\sup_{h>0} h^{-\ell} \|f(x - h) - f(x)\|_{L_p(\Re)} \le \infty$

$\Rightarrow$ $\forall h > 0, \exists M$: $h^{-\ell} \|f(x + h) - f(x)\|_{L_p(\Re)} \le M$

$\Rightarrow$ $\| f(x + h) - f(x)\|_{L_p(\Re)} \le Mh^{\ell}$

so, $f \in B^{\ell}_{p,\infty}(\Re)$ iff

1) $f \in L_p(\Re)$ 2) $\exists M$: $\|f(x + h) - f(x)\| \le Mh^{\ell}, \forall h > 0$

Let $n = 1, \ell = 1.\ \theta = \infty \Rightarrow \overline{\ell} = 0, \sigma = 2$

so $\| f \|_{B_{p,\infty}}(\Re) = \|f\|_{L_p(\Re)} + \sup_{h>0} h^{-\ell} \left\|(\Delta^2_h f)\right\|_{L_p(\Re)}$

$= \|f\|_{L_p}(\Re) + \sup_{h>0} \frac{1}{h^{\ell}} \|f(x + 2h) - 2f(x + h) + f(x)\|_{L_p(\Re)}$

$f \in B_{p,\infty}(\Re) \Leftrightarrow f \in L_p(\Re)$ and $\forall h > 0$, $\left\|\Delta^2_h f\right\|_{L_p(\Re)} \le Mh$ for some $M > 0$.

If $p = \infty$ we have

$f \in B^{\ell}_{\infty,\infty}(\Re) \Leftrightarrow \exists M > 0$,almost$\forall x \in \Re$, $|f(x)| \le M$ and $\forall h > 0$, almost$\forall x \in \Re$,

$\left|\Delta^{\sigma}_h f\right| \le Mh$.

$|f(x + 2h) - 2f(x+h) + f(x)| \le |[f(x+h) - f(x+h)] - f[x+h) - f(x)]|$

$\le |f(x+h+h) - f(x+h)| + |f(x+h) - f(x)|$

Thus $\forall x \in \Re, \forall h > 0, |f(x+h) - f(x)| \le Mh$

$\Rightarrow$ $|f(x+2h) - 2f(x+h) + f(x)| \leq |f(x+2h) - f(x+h)| + |f(x+h) - f(x)|$

$\leq Mh + Mh = 2M'h.$

6. $n = 1$, $0 < \ell < 1$, $1 \leq p < \infty$ and $1 \leq \theta < \infty$.

It is the same as case I, But let's consider the convergence.

$f \in B^{\ell}_{p,\infty}(\Re)$ iff

$f \in L_p(\Re)$ and

$$\left(\int_0^\infty (h^{-\ell} \| \Delta_h f \|_{L_p(\Re)})^\theta \frac{dh}{h}\right)^{1/\theta} < \infty.$$

$$\Leftrightarrow \int_0^\infty h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$$

claim $\int_0^\infty h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$ iff $\int_0^1 h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$

Proof $[0, 1] \leq [0, \infty)$

$\Rightarrow$ $\int_0^\infty h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$ implies $\int_0^1 h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$

To show the other side, let $\int_0^1 h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$

Thus $\int_0^\infty h^{-\ell\theta} \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} = \int_0^1 \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}} + \int_1^\infty \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}}$

Let us consider $\int_1^\infty \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}}$

$$\int_1^\infty \|f(x+h) - f(x)\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}} \leq \int_1^\infty 2^\theta \|f\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}} \leq 2^\theta \|f\|^\theta_{L_p(\Re)} \int_1^\infty \frac{dh}{h^{1+\ell\theta}}$$

But $\int_1^\infty \frac{dh}{h^{1+\ell\theta}} = \lim_{b\to\infty} \left[\frac{-1h^{-\ell\theta}}{\ell\theta}\right]_1^b = \lim_{b\to\infty}\left(\frac{1}{\ell\theta} - \frac{b^{-\ell\theta}}{\ell\theta}\right)$

$$\therefore \int_1^\infty \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}} \leq 2^\theta \frac{\|f\|^\theta_{L_p(\Re)}}{\ell\theta} < \infty$$

since $\|f\|_{L_p(\Re)} < \infty$, $2^\theta \dfrac{\|f\|^\theta_{L_p(\Re)}}{\ell\theta} < \infty$

$$\Rightarrow \int_1^\infty \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h^{1+\ell\theta}} < \infty \text{ and } \int_0^1 \|\Delta_h f\|^\theta_{L_p(\Re)} \frac{dh}{h} < \infty$$

so, $\int_0^1 \|\Delta_h f\|^\theta_{L_p(\Re)} \dfrac{dh}{h^{1+\ell\theta}} < \infty \Rightarrow \int_0^\infty \|\Delta_h f\|^\theta_{L_p(\Re)} \dfrac{dh}{h^{1+\ell\theta}} < \infty$

7. $n = 1$, $p = \theta$, $0 < \ell < 1$, $\overline{\ell} = 0$, $\theta = 1$

It $p = \theta$ then $B^\ell_{p,\theta}(\Re^n) \equiv B^\ell_p(\Re^n)$

$$\text{So } \|f\|_{B^\ell_p(\Re)} = \|f\|_{L_p(\Re)} + \left(\int_0^\infty (h^{-\ell}\|\Delta_h f\|)^p{}_{L_p(\Re)} \frac{dh}{h}\right)^{1/p}$$

$$\text{But } \left[\int_o^\infty \|\Delta_h f\|^p_{L_p(\Re)} \frac{dh}{h^{1+\ell p}}\right]^{1/p} = \left[\int_o^\infty \left(\int_\Re |f(x+h) - f(x)|^p_{dx}\right) \frac{dh}{h^{1+\ell p}}\right]^{1/p}$$

$$= \left(\int_0^\infty \int_{-\infty}^\infty |f(x+h) - f(x)|^p_{dx} \frac{dh}{h^{1+\ell p}}\right)^{1/p}$$

$$= \left(\int_{-\infty}^\infty dx \int_0^\infty \frac{|f(x+h) - f(x)|^p}{h^{1+\ell p}} dh\right)^{1/p}$$

$h > 0$, let $h + x = y$, $h = y - x$ $dh = dy$

$$= \int_{-\infty}^\infty dx \int_x^\infty \frac{|f(y) - f(x)|}{|y - x|^{1+\ell p}} dy$$

$$\text{But } \int_{-\infty}^\infty dx \int_{-x}^\infty \frac{|f(y) - f(x)|^p}{|y - x|^{1+\ell p}} dy = \frac{1}{2} \int_{-\infty}^\infty dx \int_{-\infty}^\infty \frac{|f(y) - f(x)|^p}{|y - x|^{1+\ell p}} dy$$

$$\text{So } \|f\|_{B^\ell_p(\Re)} = \|f\|_{L_p(\Re)} + \left(\frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{|f(y) - f(x)|}{|y - x|^{1+\ell p}} dx dy\right)^{1/p}$$

8. For $p = \theta$, $0 < \ell < 1$, $f: \Re^n \to \Re$ $\overline{\ell} = 0$, and $\sigma = 1$

$$\|f\|_{B^\ell_p(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{i=1}^n \|f\|_{B^\ell_{p,i}(\Re^n)}$$

$$\|f\|_{B^\ell_{p,i}(\Re^n)} = \left(\int_0^\infty h^{-\ell p} \|\Delta_h f\|^p_{L_p(\Re)} \frac{dh}{h}\right)^{1/p}$$

$$\|f\|_{B^{\ell}_{p,i}(\Re^n)} = \left(\int_0^{\infty} h^{-\ell p}\|\Delta_h f\|^p_{L_p(\Re)}\frac{dh}{h}\right)^{1/p}$$

$$= \left(\int_0^{\infty}\|f(x+hej)-f(x)\|^p_{L_p(\Re)}\frac{dh}{h^{1+\ell p}}\right)^{1/p}$$

$$= \left(\frac{1}{2}\int_{\Re^n}\int_{\Re^n}\frac{|f(y)-f(x)|^p}{\|y_i - x_i\|^{1+\ell p}}dxdy\right)^{1/p}$$

$$\therefore \quad \|f\|_{B^{\ell}_p(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{i=1}^{n}\left(\frac{1}{2}\int_{\Re^n}\int_{\Re^n}\frac{|f(y)-f(x)|^p}{\|y_i - x_i\|^{1+\ell p}}dxdy\right)^{1/p}$$

9. $P = \theta,\ \ell = 1,\ \overline{\ell} = 0,\ \sigma = 2$

$$\|f\|_{B^{\ell}_p(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{i=1}^{n}\left(\frac{1}{2}\int_{\Re^n}\int_{\Re^n}\frac{\left|f(x)-2f\left(\frac{x+y}{2}\right)f(y)\right|^p}{\|y-x\|^{1+\ell p}}\right)^{1/p}$$

$$= \|f\|_{B_p(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{i=1}^{n}\left(\frac{1}{2}\int_{\Re^n}\int_{\Re^n}\frac{|f(x)|-f\left(\frac{x+y}{2}\right)+f(y)|^p}{\|y_i - x_i\|^{1+\ell p}}dxdy\right)^{1/p}$$

CHAPTER 3

Iterated norms for Nikolskiĭ-Besov spaces

3.1 Definitions and *notations*

Definition 3.1. A vector space over $\Re$ is said to be a *semi-normed* space if and only if there exists

$$\|\bullet\| : X \to \Re \text{ such that}$$

$$\|x\| \geq 0 \qquad \forall x \in X$$

$$\|\alpha x\| = |\alpha| \, \|x\| \qquad \forall x \in X \quad \text{and} \quad \forall \alpha \in \Re$$

$$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$$

Definition of *iterated norms* :

Let $\Omega \subset \Re^n$ be an open set. For all $\delta > 0$ let $\Omega_\delta = \{ x \in \Omega \mid \rho(x, \partial\Omega) > \delta \}$

$$= \{ x \in \Omega \mid \inf_{y \in \partial\Omega} \|x - y\| > \delta \}$$

Let $Z(\Omega_\delta)$ be a *semi-normed* space of functions defined on Ω_δ, let $1 \leq \theta \leq \infty$ and let

$$\ell := (\ell_1, \ell_2, \ldots, \ell_n) \qquad \text{where } \ell_i \geq 0;$$

$\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_n)$ where σ_j are positive integers and

$m := (m_1, m_2, \ldots, m_n)$ where the m_j are non-negative integers.

We say $f \in \mathscr{B}_\theta^{\ell}(Z(\Omega)) \equiv \mathscr{B}_\theta^{\ell,\sigma,m;H}(Z(\Omega))$ if $f \in L^{Loc}(\Omega)$ (f locally integrable on Ω) and

$$\|f\|_{\mathscr{B}_\theta^{\ell}(Z(\Omega))} := \|f\|_{Z(\Omega)} + \sum_{j=1}^{n} \|f\|_{B_{\theta,j}^{\ell_j}(Z(\Omega))} < \infty \quad , \tag{3.1}$$

where the sum is taken over those j for which $\ell_j > 0$, and for $\ell_j > 0$,

$$\|f\|_{\beta^{\ell_j}_{\theta,j}(Z(\Omega))} := \left\| \; h^{-(\ell_j - m_j)} \; \left\|\Delta^{\sigma_j}_{h,j} D^{m_j}_j f\right\|_{Z(\Omega_{\sigma_j h})} \; \right\|_{L^*_\theta(o,H)} \tag{3.2}$$

Here $D^{m_j}_j f$ denotes the m_j^{th} generalized derivative of f with respect to x_j.

$\Delta^{\sigma_j}_{h,j} D^{m_j}_j f$ denotes the σ_j^{th} difference of the m_j^{th} generalized derivative of f with respect to x_j with steps h.

Moreover, we shall assume that the triple ℓ, σ, m is permissible. That is for those j $\in$ {1,2,…,n}

such that $\ell_j > 0$

$$\ell_j + m_j > \ell_j > m_j \tag{3.3}$$

Now, if $\ell_j > 0$ replacing

$$\|\bullet\|_{L_p(\Omega_{\sigma_j h})} \quad by \quad \|\bullet\|_{Z(\Omega_{\sigma_j h})}$$

In the definition of anisotropic[1] Nikol'skiĭ-Besov spaces, we set the following definition:

$$\|f\|_{B^{\ell_j}_{\theta,j}(Z(\Omega))} := \|f\|_{Z(\Omega)} + \|f\|_{\beta^{\ell_j}_{\theta,j}(Z(\Omega))}$$

and thus

$\mathscr{B}^{\ell}_{\theta}(L_p(\Omega)) \equiv B^{\ell}_{p,\theta}(\Omega)$ (the relation between these two notation will be cleared latter)

In (3.1) and (3.2) further set

$$\ell := \ell_2 := (\ell_{21}, \ell_{22}, \ldots, \ell_{2n})$$

$$\sigma := \sigma_2 := (\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n})$$

$$m := m_2 := (m_{21}, m_{22}, \ldots, m_{2n})$$

$$\text{H} := \text{H}_2, \text{ and } Z(Z(\Omega_\delta) := B^{\ell_1}_{\theta}(L_p(\Omega_\delta)) \equiv B^{\ell_1}_{p,\theta}(\Omega_\delta)$$

with parameters σ_1, m_1, ℓ_1 and H_1.

We then obtain the norm of the form $\|f\|_{B^{\ell_2}_{\theta}(B^{\ell_1}_{\theta}(L_p(\Omega)))}$.

Continuing this process, we obtain the following norms, which we naturally call them *iterated norms*. For all integers k ≥ 2

$$\|f\|_{\underbrace{B_\theta^{\ell_k}(\cdots B_\theta^{\ell_1}}_{k}(L_p(\Omega))\cdots)} := \|f\|_{B_\vartheta^{\ell_k}(\underbrace{B_\theta^{\ell_{k-1}}(\cdots B_\theta^{\ell_k}}_{k-1}(L_p(\Omega))\cdots))} \tag{3.4}$$

and the corresponding spaces $\mathscr{B}_\theta^{\ell_k}(\dots \mathscr{B}_\theta^{\ell_1}(L_p(\Omega))\dots)$

are called *Nikol'skiĭ-Besov spaces with iterated norms*. We have to point that for each spaces $B_\theta^{\ell_s}$

the remaining parameters are respectively σ_s, m_s, *and* H_s, where it is supposed that for all

$s \in \{1, 2, \dots, k\}$ the triples ℓ_s, σ_s, m_s are permissible.

3.2. Properties of iterated norms for Nikolskiĭ-Besov spaces.

Lemma 3. 1. (*On the fractional differentiation of an inequality*)

Let $\mu_0 > 0$, let $\Omega \subset \Re n$ be an open set, and for each $\mu \in [0, \mu_0)$ let a set of function $T(\Omega_\mu)$ and

semi-normed function spaces $X(\Omega_\mu)$ and $Y(\Omega_\mu)$ be defined such that

$$T(\Omega_\mu) \cap Y(\Omega_\mu) \subset T(\Omega_\mu) \cap X(\Omega_\mu) \tag{3.5}$$

and

$$\|f\|_{X(\Omega_\mu)} \leq \|f\|_{Y(\Omega_\mu)} \ \forall f \in T(\Omega_\mu) \cap Y(\Omega_\mu). \tag{3.6}$$

If $\Omega_\mu = \phi$ for some μ, we assume that $\|f\|_{X(\Omega_\mu)} := \|f\|_{Y(\Omega_\mu)} := 0$ and for $\mu = 0$

$\Omega_\mu = \Omega)$.

Further let $1 \leq \theta \leq \infty$, let ℓ, σ, μ be permissible triple and let

$$0 < H < \mu_0 \left(\max_{1 \leq j \leq n} \sigma_j \right)^{-1} \tag{3.7}$$

$$\Leftrightarrow \qquad 0 < H \max_{1 \leq j \leq n} \sigma_j < \mu_0$$

$$\Leftrightarrow \qquad 0 < \mu_0 - H\left(\max_{1 \leq j \leq n} \sigma_j \right).$$

Then

$$\forall\, \mu \in \left[0,\ \mu_0 - H \max_{1\le j\le n} \sigma_j\right) \tag{3.8}$$

and $\forall f \in T(\Omega_\mu) \cap \mathscr{B}_\theta^{\ell}(Y(\Omega_\mu))$ such that $\forall j \in \{1, 2, \ldots, n\}$

$$\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f \ \in\ T(\Omega_{\mu+\sigma_j h})\ \ \forall h \in (0,H) \tag{3.9}$$

The following inequality is satisfied

$$\| f \|_{\mathscr{B}_\theta^{\ell}(X(\Omega_\mu))} \ \le\ \| f \|_{\mathscr{B}_\theta^{\ell}(Y(\Omega_\mu))} \tag{3.10}$$

Proof: Let $\mu \in \left[0,\ \mu_0 - H \max_{1\le j\le n} \sigma_j\right)$ and $f \in T(\Omega_{\mu+\sigma_j h}) \cap \mathscr{B}_\theta^{\ell}(Y(\Omega_\mu))$ then by (3.9) for all $j \in \{1,2,\ldots,n\}$ and all $h \in (0, H)$ we have $\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} \in T(\Omega_{\mu+\sigma_j h})$.

Moreover $f \in \mathscr{B}_\theta^{\ell}(Y(\Omega_\mu)) \Leftrightarrow f \in Y(\Omega_\mu)$ and $\sum{}' \|f\|_{\beta_\theta^{\ell}(X(\Omega_\mu))} < \infty$. Then by substituting

$T(\Omega_{\mu+\sigma_j h})$ by $Y(\Omega_{\mu+\sigma_j h})$ in (3.9) we get $\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f \in Y(\Omega_{\mu+\sigma_j h})$.

Consequently, by (3.6) for almost all $h \in (0, H)$ we have that

$$\left\|\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f\right\|_{X(\Omega_{\mu+\sigma_j h})} \le \left\|\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f\right\|_{Y(\Omega_{\mu+\sigma_j h})}.$$

Multiplying both sides of this inequality by $h^{-(\ell_j-\mu_j)}$ and applying the $L_\theta^*(0, H)$ norm we obtain

$$\begin{aligned}\|f\|_{\beta_{\theta,j}^{\ell_j}(X(\Omega_\mu))} &= \left\| h^{-(\ell_j-\mu_j)} \left\|\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f\right\|_{X(\Omega_{\mu+\sigma_j h})}\right\|_{L_\theta^*(0,H)} \\ &\le \left\| h^{-(\ell_j-\mu_j)} \left\|\Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f\right\|_{Y(\Omega_{\mu+\sigma_j h})}\right\|_{L_\theta^*(0,H)} \\ &= \|f\|_{\beta_{\theta,j}^{\ell_j}(Y(\Omega_\mu))}\end{aligned}$$

and from (3.6) we have $\|f\|_{X(\Omega_\mu)} \le \|f\|_{Y(\Omega_\mu)}$. Combining these two inequalities, we get

$$\|f\|_{X(\Omega_\mu)} + \sum_{j=1}^{n}\|f\|_{\beta_{\theta,j}^{\ell_j}(X(\Omega_\mu))} \le \|f\|_{Y(\Omega_\mu)} + \sum_{j=1}^{n}\|f\|_{\beta_{\theta,j}^{\ell_j}(Y(\Omega_\mu))}$$

$$\Leftrightarrow \qquad \|f\|_{B_\theta^{\ell_j}(X(\Omega_\mu))} \le \|f\|_{B_{\theta,j}^{\ell_j}(Y(\Omega\mu))} .$$

Corollary 3. 1. If $\forall\mu \in \left[0,\ \mu_0 - \max_{1\le j\le n}\sigma_j\right)$ and $\forall f \in T(\Omega_\mu) \cap \mathscr{B}_\theta^{\ell}(Y(\Omega_\mu))$ condition (3.9) is satisfied for all j ∈ {1, 2,…,n}, then for all $\mu \in \left[0,\ \mu_0 - H\max_{1\le j\le n}\sigma_j\right)$ we have

$$\|f\|_{B_{\theta,j}^{\ell_j}(X(\Omega\mu))} \le \|f\|_{B_{\theta,j}^{\ell_j}(Y(\Omega\mu))} \tag{3.11}$$

Corollary1 claims that condition (3.9) can be established if $f \in T(\Omega\mu) \cap \mathscr{B}_\theta^{\ell}(Y(\Omega_\mu))$ ∀j∈ {1,2, ... ,n} and $\forall h\in (0, H)$ since $f \in \mathscr{B}_\theta^{\ell}(Y(\Omega_\mu))$

$$\|f\|_{\beta_{\theta,j}^{\ell_j}(Z(\Omega))} = \left\| h^{-(\ell_j - m_j)} \left\|\Delta_{h,j}^{\sigma_j} D_j^{m_j} f\right\|_{Z(\Omega_{\sigma_j h})} \right\|_{L_0^*(0,H)} < \infty$$

$$\Rightarrow \qquad \Delta_{h,j}^{\sigma_j}\ D_j^{m_j} f \in Z(\Omega_{\sigma_j h}) .$$

Then the proof follows as in Theorem 1, by assuming conditions (3.5) and (3.6)

Corollary 3. 2. Let condition (3.6) and (3.7) be satisfied. Let $1 \le \theta < \infty$; $\ell_1, \sigma_1, m_1, \ldots, \ell_k, \sigma_k, m_k$ be permissible triples and let

$$\sum_{j=1}^{k} H_s \ \underset{1\le j\le n}{mas}\sigma_j < \mu_0 \tag{3.12}$$

then

$$\forall\mu \in \left[0, \mu_o - \sum H_s \max_{1\le j_s\le n}\sigma_{j_s}\right) \tag{3.13}$$

and $\forall f \in T(\Omega_\mu) \cap \mathscr{B}_\theta^{\ell_k}(\ldots(\mathscr{B}_\theta^{\ell_1}(Y(\Omega))\ldots)$ which are such that $\forall j_s \in \{1, 2, \ldots, n\}$

$$\Delta_{h_1,j_1}^{\sigma_{j_1}} \cdots \Delta_{h_k,j_k}^{\sigma_{j_k}} D_{j_1}^{m_{j_1}} \cdots D_{j_k}^{m_{j_k}} f \in T\left(\Omega_{\mu+\sum_1^k \sigma_{j_s} h_s}\right) \quad \forall h_s \in (0, H_s) \tag{3.13}$$

the following inequality is satisfied

$$\|f\|_{\mathscr{B}_{\theta,j}^{\ell k}(\ldots\mathscr{B}_{\theta}^{\ell_1} X(\Omega\mu))\ldots)} \le \|f\|_{\mathscr{B}_{\theta,j}^{\ell k}(\ldots\mathscr{B}_{\theta}^{\ell_1} Y(\Omega\mu))\ldots)} \tag{3.14}$$

And equivalent norm on the space $B_\theta^{\ell_k}(\ldots B_\theta^{\ell_1}(L_p(\Omega))\ldots)$, which is more convenient estimates, will be introduced in the next Lemma.

We set

$$\|f\|_{B^{\ell_k}(\ldots B_\theta^{\ell_k}(L_p(\Omega))\ldots)} = \|f\|_{L_p(\Omega)} + \sum_{r=1}^{k} \sum_{\substack{i_1,\ldots,ir=1 \\ i_s \ne i_t (s\ne t)}}^{k} \sum_{i_1,\ldots,j_r=1}^{n} \|f\|_{B_{\theta,j_1}^{\ell ir,jr}(\ldots B_{\theta,j_1}^{\ell j_1,j_1}(L_p(\Omega))\ldots)} \tag{3.15}$$

Lemma 3. 2. For all considered values of the parameters

$$\|f\|_{\mathscr{B}_\theta^{\ell_k}(\ldots\mathscr{B}_\theta^{\ell_1}(L_p(\Omega))\ldots)} \sim \|f\|^*_{\mathscr{B}_\theta^{\ell_k}(\ldots\mathscr{B}_\theta^{\ell_1}(L_p(\Omega))\ldots)} \tag{3.16}$$

where ~ denotes equivalence of the norms.

Lemma 3. 3 Let $1 \le p, \theta \le \infty$ and $0 < H \le \infty$. Then

$$\|f\|_{B^{\ell_1+\ell_2}(L_p(G)}} \le C_1 \|f\|_{B_\theta^{\ell_1}(B_\theta^{\ell_2}(L_p(G)))} \tag{3.17}$$

for all permissible triples ℓ_1, σ_1, m_1 and ℓ_2, σ_2, m_2

$\left(\beta_\theta^{\ell_1+\ell_2} \equiv \beta_\theta^{\ell_1+\ell_2,\sigma_1+\sigma_2,m_1+m_2 j\ H}\right)$ and ℓ_1 depends only on min $\{\sigma_1,\sigma_2\}$

Corollary 3.3 For any natural k, and for $1 \le p, \theta \le \infty$, and $0 \le H \le \infty$.

$$\|f\|_{\beta_\theta^{k\ell}(L_p(G))} \le C_2^{k-1} \|f\|_{\beta_\theta^{\ell}(\ldots\beta_\theta^{\ell}(L_p(G))\ldots)} \tag{3.18}$$

for any permissible triple ℓ, σ, m and c_2 depends only on $\sigma\left(\left(\beta_\theta^{kl} \equiv \beta_\theta^{kl,k\sigma,km:H}\right)\right.$.

3.3 Equivalent norm to the space $\mathscr{B}_\theta^{\ell k}(\cdots\mathscr{B}_\theta^{\ell 1}(X(\Omega_\mu))\cdots)$

Note: For all considered values of parameters We then obtain that

$$\| f \|_{\mathscr{B}_\theta^{\ell 2}(\mathscr{B}_\theta^{\ell 1}(L_p(\Omega)))} \sim \| f \|_{L_p(\Omega)} + \sum_{j_1}^{n} \| f \|_{\beta_{\theta,j_1}^{\ell_1\cdot j_1}(L_p(\Omega))} + \sum_{j_2}^{n} \| f \|_{\beta_{\theta,j_2}^{\ell_2\cdot j_2}(L_p(\Omega))}$$

$$+ \sum_{j_1}^{n}\sum_{j_2}^{n} \| f \|_{\beta_{\theta,j_2}^{\ell_2\cdot j_2}(\beta_{\theta,j_1}^{\ell_1\cdot j_1}(L_p(\Omega)))}$$

$$\Leftrightarrow \quad \| f \|_{\mathscr{B}_\theta^{\ell 2}(\mathscr{B}_\theta^{\ell 1}(L_p(\Omega)))} \sim \| f \|^*_{\mathscr{B}_\theta^{\ell 2}(\mathscr{B}_\theta^{\ell 1}(L_p(\Omega)))}.$$

Then for k ≥ 2, it holds true by induction.

Note: We notice that for all considered values of parameters

$$\| f \|_{\beta_\theta^{\lambda 2}(\beta_\theta^{\lambda 1}(L_p(\Omega)))} = \| f \|_{\beta_\theta^{\lambda 1}(\beta_\theta^{\lambda 2}(L_p(\Omega)))} =$$

$$\left\| \left\| h_1^{-(\lambda_1 - m_1)} h_2^{-(\lambda_2 - m_2)} \left\| \Delta_{h_{1,j_1}}^{\sigma_1} \Delta_{h_{2,j_2}}^{\sigma_{2,j_2}} D_{j_2}^{m_{2,j_2}} f \right\|_{L_p(\Omega_{\sigma_1 h_1 + \sigma_2 h_2})} \right\|_{L_\theta^*(0,H_1)} \right\|_{L_\theta^*(0,H_2)} \tag{3.18}$$

For following sections we shall consider the one - dimensional case, i.e. n = 1. And let

G =(a , b), where $-\infty \le a < b \le \infty$.

Lemma 3.4 . Let $1 \le p \le \infty$ and $0 < H \le \infty$. Then

$$\| f \|_{\beta_\theta^{\ell_1 + \ell_2}(L_p(G))} \le c_1 \| f \|_{\beta_\theta^{\ell 1}(\beta_\theta^{\ell 2}(L_p(G)))}, \tag{3.19}$$

for all permissible triples ℓ_1, σ_1, m_1 and ℓ_2, σ_2, m_2 $\left(\beta_\theta^{\ell_1 + \ell_2} \equiv \beta_\theta^{\ell_1 + \ell_2, \sigma_1 + \sigma_2, m_1 + m_2; H}\right)$ and C_1 depends only on min $\{\sigma_1, \sigma_2\}$.

Proof: It is enough to prove (3.28) for $m_1 = m_2 = 0$. This is so because in the general case it is enough to apply this particular case to $f^{(m_1 + m_2)} := D^{m_1 + m_2} f$ .Without loss of generality, let $\sigma_1 \le \sigma_2$. We employ the inequality, which was proved in section ...

$$\left\| \Delta_h^\sigma f \right\|_{L_p(G_{\sigma h})} \le A(\sigma) \frac{1}{h} \int_0^h \left\| \Delta_\eta^\sigma f \right\|_{L_p(G_{\sigma h})} d\eta \tag{3.20}$$

The constant in this inequality does not depend on G.

Thus,

$$\left\|\Delta_h^{\sigma} f\right\|_{L_p(G_{\sigma h})} \le A(\sigma)\left(\frac{1}{h}\int_0^h \left\|\Delta_\eta^{\sigma} f\right\|^{\theta}_{L_p(G_{\sigma \eta})} d\eta\right)^{\frac{1}{\theta}} \le A(\sigma)\left(\int_0^h \left\|\Delta_\eta^{\sigma} f\right\|^{\theta}_{L_p(G_{\sigma \eta})} \frac{d\eta}{\eta}\right)^{\frac{1}{\theta}}$$

$$= A(\sigma)\left\| \left\|\Delta_h^{\sigma} f\right\|_{L_p(G_{\sigma h})}\right\|_{L_\theta^*(0,h)}.$$

Furthermore;

$$\left\|\Delta_h^{\sigma_1+\sigma_2} f\right\|_{L_p(G_{(\sigma_1+\sigma_2)h})} = \left\|\Delta_h^{\sigma_1}(\Delta_h^{\sigma_2} f)\right\|_{L_p((G_{\sigma_2 h})_{\sigma_1 h})}$$

$$\le A(\sigma_1)\left\| \left\|\Delta_h^{\sigma_1}\Delta_h^{\sigma_2} f\right\|_{L_p((G_{\sigma_2 h})_{\sigma_1 \eta})}\right\|_{L_\theta^*(0,h)}$$

$$\le A(\sigma_1)\, h^{\ell_1}\left\| \eta^{-\ell_1}\left\|\Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\right\|_{L_p(G_{\sigma_1\eta+\sigma_2 h})}\right\|_{L_\theta^*(0,H)}.$$

From which we obtain

$$\| f \|_{\beta_\theta^{\ell_1+\ell_2,\ \sigma_1 - \sigma_2;H}(G)} = \left\| h^{-(\ell_1+\ell_2)}\left\|\Delta_h^{\sigma_1+\sigma_2} f\right\|_{L_p(G_{(\sigma_1+\sigma_2)h})}\right\|_{L_\theta^*(0,H)}$$

$$\le A(\sigma_1)\, h^{\ell_2}\left\|\eta^{-\ell_1}\left\| \eta^{-\ell_1}\left\|\Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\right\|_{L_p(G_{\sigma_1\eta+\sigma_2 h})}\right\|_{L_\theta^*(0,H)}\right\|_{L_\theta^*(0,H)}$$

$$= \| f \|_{\beta_\theta^{\ell_1,\sigma_1;H}(\beta_\theta^{l_2,\sigma_2;H}(L_p(G)))}$$

Therefore ,

$$\| f \|_{\beta_\theta^{\ell_1+\ell_2}(L_p(G)))} \le c_1 \| f \|_{\beta_\theta^{\ell_1}(\beta_\theta^{\ell_2}(L_p(G)))}.$$

In general, for $m_1 \neq 0$ and $m_2 \neq 0$ we employ the substitution $f^{(m_1 + m_2)} := D^{m_1 + m_2} f$ in place of f, which will provide us the same result as done above for the particular case $m_1 = m_2 = 0$.

Corollary 3.4 . For any natural number k, and for $1 \leq p, \theta \leq \infty$, and $0 \leq H \leq \infty$.

$$\| f \|_{B_\theta^{k\ell}(L_p(G))} \leq c_2^{k-1} \| f \|_{\underbrace{B_\theta^\ell(\cdots B_\theta^\ell(L_p(G))\cdots)}_{k}} . \tag{3.21}$$

(For any permissible triple ℓ, σ and m C_2 depends only on σ).

<u>*Note :*</u> Since n = 1 G is an open set in R^1 and hence $G\mu = (a - \mu , b - \mu)$.

Now to generalize our result for $m \neq 0$ it suffices to apply the substitution $f^{(k\,m)}$ in place of the function f in (3.31).

Corollary 3.5 . Let $\Omega \subset \mathbf{R}$ be an open set. Then under the condition of Theorem 3 of section

$$\| f \|_{B_\theta^{\ell_1 + \ell_2}(L_p(\Omega))} \leq c_1 \| f \|_{B_\theta^{\ell_1}(B_\theta^{\ell_2}(L_p(\Omega)))} , \tag{3.22}$$

where c_1 is a constant as in (3.30)

Proof: Ω being a open set in R , it can be written as a union of countable collection $\{G_{(k)}\}_{k=1}^s$ of open intervals, where $s \in N$ or $s = \infty$. That is

$$\Omega = \bigcup_{k=1}^s G_{(k)} .$$

For $1 \leq p \leq \infty$ we have

$$\|\Delta_h f\|_{L_p(G_{\sigma h})} = \left(\sum_{k=1}^s \|\Delta_h^\sigma f\|_{Lp((G_{(k)})_{\sigma h})}^p \right)^{\frac{1}{p}} \leq A(\sigma) \frac{1}{h} \left(\sum_{k=1}^s \left(\int_0^h \|\Delta_\eta^\sigma f\|_{Lp((G_{(k)})_{\sigma \eta})} d\eta \right)^p \right)^{\frac{1}{p}}$$

$$= A(\sigma)\frac{1}{h}\left(\sum_{k=1}^{s}\left(\int_0^h \Phi_k(\eta)d\eta\right)^p\right)^{\frac{1}{p}}$$

$$= A(\sigma)\frac{1}{h}\left\|\int_0^h \Phi(\eta)d\eta\right\|_{l_p},$$

where $\Phi_k(\eta) = \|\Delta_h^\sigma f\|_{Lp((G_{(k)})_{\sigma h})}$ *and* $\int_0^h \Phi(\eta)d\eta = \left\{\int_0^h \Phi_k(\eta)d\eta\right\}_{k=1}^{k=s}$.

Now using generalized Minkowiski's inequality for integrals the inequality

$$\left\|\int_0^h \Phi(\eta)d\eta\right\|_{l_p} \le \int_0^h \|\Phi(\eta)\|_{l_p} d\eta$$

holds true.

Thus we obtain

$$\|\Delta_h f\|_{L_p(G_{\sigma h})} = A(\sigma)\frac{1}{h}\left(\sum_{k=1}^{s}\left(\int_0^h \|\Delta_\eta^\sigma f\|^p{}_{Lp((G_{(k)})_{\sigma\eta})} d\eta\right)^p\right)^{\frac{1}{p}}$$

$$= A(\sigma)\frac{1}{h}\left(\sum_{k=1}^{s}\left(\int_0^h \Phi_k(\eta)d\eta\right)^p\right)^{\frac{1}{p}}$$

$$= A(\sigma)\frac{1}{h}\left\|\int_0^h \Phi(\eta)d\eta\right\|_{l_p}.$$

The rest steps are then exactly the same as that of the proof of theorem 3.

Using analogous argument, our assertion holds also for $p = \infty$. This completes the proof.

Corollary 3.6 Let $1 \le p, \theta \le \infty$. Let ℓ_1, σ_1, m_1; ℓ_2, σ_2, m_2 and $\ell_1 + \ell_2, \sigma_3, m_3$ be permissible triples. Then

$$\| f \|_{\beta_\theta^{\ell_1}(\beta_\theta^{\ell_2}(L_p(G)))} \sim \| f \|_{\beta_{p,\theta}^{\ell_1+\ell_2}(G))} \tag{3.23}$$

corollary 3.7 Let $1 \le p,\theta \le \infty$, $0 < \mathrm{H} < \infty$ and ℓ, σ, m be permissible triple. Then

$$\|f\|_{\mathscr{B}_\theta^\ell(\cdots\mathscr{B}_\theta^\ell(L_p(G))\cdots)} \le C_4^{\,k}\sum_{s-0}^{[k\ell]+2}\|f^{(s)}\|_{L_p(G)}$$

where C_4 depends only on ℓ and σ.

Proof: Making use of theorem 2 and theorem 4, it suffices to show the assertion for $1 \le \theta < \infty$.

To this end,

$$I_k^{\frac{1}{\theta}} \;:=\; \|f\|_{\beta_\theta^{\ell;1}(\cdots\beta_\theta^{\ell;1}(L_p(G))\cdots)} \;\le\; A_1^k\|f^{[k\ell]+2}\|_{L_p(G)}.$$

Taking into account the symmetry of the integrand with respect to h_1, h_2, ..., h_k we get

$$I_k \;=\; k!\int_0^1 dh_1\int dh_2\cdots\int (h_1\cdots h_k)^{-1-\theta(\ell-m)}\|\Delta_{h_1}^\sigma\cdots\Delta_{h_k}^\sigma f^{(km)}\|_{L_p^\theta(G_{h_1+\cdots+h_k})}dh_k.$$

Since

$$\|\Delta_{h_1}^\sigma\cdots\Delta_{h_k}^\sigma f\|_{L_p^\theta(G_{h_1+\cdots+h_k})} \le 2^\sigma h_{k-s}^\beta h_{k-1-1}^\sigma\cdots h_k^\sigma\|f^{[kl]+2}\|_{L_p(G)}$$

where the integers s and β, $s \ge 0$, $0 \le \beta < \sigma$ are chosen so that $\sigma s + \beta = [k(\ell - m)]+2$.

Now

$$\int_0^1 h_1^{-1-\theta\ell} dh_1 \cdots \int_0^{h_{k-s-2}} h_{k-s-1}^{-1-\theta\ell} dh_{k-s-1} \int_0^{h_{k-s-1}} h_{k-s}^{-1-\theta(\beta-\ell)} dh_{k-s} \int_0^{h_{k-s}} h_{k-s+1}^{-1-\theta(\ell-\sigma)} dh_{k-s+1} \cdots \int^{h_{k-1}} h_k^{-1-\theta(\ell-\sigma)} dh_k$$

$$\leq \frac{1}{\left((\omega-1)^s \ell^{k-s} s!(k-s)\right)},$$

this completes the proof.

Theorem : Let $1 \leq p, \theta \leq \infty$, $k \in \mathbf{N}$ and $0 < H_s \leq \infty$ $\forall s \in \{1, \cdots, k+1\}$. Let the triples Let $\ell_1, \sigma_1, m_1, \ldots \ell_k\ \sigma_k, m_k$, and $(\ell_1 + \ell_2 + \ldots\ \ell_k), \sigma_{k+1}, m_{k+1}$ be permissible. Then the following are true:

1. For any open set $\Omega \in \mathbf{R}^n$.

$$\mathscr{B}_\theta^{\ell_k}(\cdots \mathscr{B}_\theta^{\ell_1}(L_p(\Omega)) \cdots) \hookrightarrow \mathscr{B}_{p,\theta}^{\ell_1+\ell_2+\cdots=\ell_k}(\Omega) \qquad (3.24)$$

i.e. there exists an embedding, and the embedding operator is bounded.

2. If, in addition, for any set $\Omega \in \Re^n$ there exists a bounded operator

$$S : \mathrm{B}_{p,\theta}^{\ell_1+\ell_2+\cdots+\ell_k}(\Omega) \rightarrow \mathrm{B}_{p,\theta}^{\ell_1+\ell_2+\cdots+\ell_k}(\mathbf{R}^n) \qquad (3.25)$$

Then in order for the equality

$$\mathscr{B}_\theta^{\ell_k}(\cdots \mathscr{B}_\theta^{\ell_1}(L_p(\Omega)) \cdots) = \mathrm{B}_{p,\theta}^{\ell_1+\ell_2+\cdots+\ell_k}(\Omega) \qquad (3.26)$$

to hold and for the corresponding norms to be equivalent, it is necessary and sufficient that

$$\sum_{s=1}^{k} \max_{1\leq i\leq n} \frac{\ell_{s_i}}{\ell_{_i} + \cdots + \ell_{k_i}} \leq 1. \ (34)$$

(if $\ell_{1_i} + \cdots + \ell_{k_i} = 0$ for some i we put $\frac{0}{0} := 0$)

Proof: According to corporally 3. 2 and the commutative property of (3.28), for any pen set $\Omega \in \mathbf{R}^1$ and for all $j \in \{ 1, 2, \dots, n \}$ we have

$$\|f\|_{b_{p,\theta,j}^{\ell_{1j}+\cdots+\ell_{kj}}(\Omega)} \leq c_1 \|f\|_{\beta_{\theta,j}^{\ell_{kj}}(\cdots\beta_{\theta,j}^{\ell_{1j}}(L_p(\Omega))\cdots)} .$$

Summing up these inequalities and adding $\|f\|_{L_p(\Omega)}$ to both sides, we obtain the desired inequality, taking into account that the sum on the right hand side does not exceed

$$\|f\|^{*}{}_{\beta_{\theta,j}^{\ell_{kj}}(\cdots\beta_{\theta,j}^{\ell_{1j}}(L_p(\Omega))\cdots)} .$$

(Since we have established most of our theorems based on the above equivalent norm introduced by theorem).

CHAPTER 4

Iterated Norms in Nikoliskii-Besov space with generalized smoothness

4.1 Definitions and common properties of related spaces

4.1.1 definitions

Definition 4. 1. Let $f \in L_p(\Re^n)$ such that ,$x = (x_1, x_2, \ldots, x_n) \in \Re^n, 1 \le p \le \infty$ and the spac $L_\infty(\Re^n)$will be usually understood as the space $C(\Re^n)$ of all bounded uniformly continuous functions .

The ℓ^{th} order difference of function f with step h in the direction of x_j, is defined by

$$\Delta^{\ell}_{he_j} f(x) = \sum_{v=0}^{\ell} (-1)^{v+\ell} C^{1}_{v} f(x+vh_{ej}) \qquad (4.1)$$

where $\ell \in N$ and $\ell \ge 1$, $j = 1, 2, 3, \ldots n$ e_j *unit* vector in the direction of x_j, $h \in \Re$

Cases

1) if $\ell = 1$, $\Delta^{1}_{h_{e_j}} f(x) = \sum_{v=0}^{1} (-1)^{v+1} C^{1}_{v} f(x+v h_{e_j}) = f(x + h_{e_j}) - f(x)$

2) if $\ell = 2$, $\Delta^{2}_{h_{e_j}} f(x) = f(x) + f(x + 2h_{e_j}) - 2f(x + h_{e_j})$

$$= f(x + 2h_{e_j}) + f(x) - 2f(x + h_{e_j}) + f(x)$$

Definition 4. 2 The ℓ^{th} order module of continuity of function $f \in L_p(\Re^n)$ in the direction of x_j is defined by

$$\omega^{\ell}_{p,x_j}(f,u) = \sup_{h\in\Re, |h|\le u} \| \Delta^{\ell}_{hej} f(.) \|_{Lp(\Re^n)}, \quad (1 \le p \le \infty, L_\infty = C) . \qquad (4.2)$$

And $\|\Delta^{\ell}_{hej} f\|_{Lp(\Re^n)} = \left(\int\int_{R^n} |\Delta^{\ell}_{hej} f|^p dx \right)^{1/p}$ for $1 \le p < \infty$.

Definition 4.3 Let $\beta > 0$ and $c \ge 1$ by $\Omega_\beta(c)$ we denote the cone of all functions $\varphi(u)$ increasing on [0, 1] such that

1) $\lim_{u \to 0} \varphi(u) = 0$

2) $\varphi(v)\, v^{-\beta} \le c\, \varphi(u)\, u^{-\beta}, \left(\frac{\varphi(v)}{v^{\beta}} \le c \frac{\varphi(u)}{u^{\beta}} \right)$, for $0 < u \le v \le 1$ (4.3)

that is $\varphi(u)u^{-\beta}$ is almost decreasing

$$\Omega_\beta := \bigcup_{c \ge 1} \Omega_\beta\,(c)$$

Note: For $f \in L_p(\Re^n)$, $\omega^{\ell}_{p,x_j}(f, u) \in \Omega_\ell(2^\ell)$

Let $1 \le p \le \infty$, $0 < \theta \le \infty$, $\vec{k} = (k_1, k_2, \ldots, k_n 1 \le k_j \in N$, and $\vec{\omega}(u) = \{\omega_j(u)\}_{j=1}^{n}$

be give continuous vector function on [0, 1] such that $\omega_j(0) = 0$, and $\vec{\omega}(u)$ is an increasing function on [0, 1] and $\omega_j(1) = 1, j \in \{1, 2, 3, \ldots n\}$.

Function $\vec{\omega}(u)$ plays a role of comparison function for module of continuity. We call this function of smoothness.

By using these conditions we have the following definition.

Definition 4. 4 Anisotropic Nikol'skii-Besov space $B^{\vec{\omega}(.)}_{p,\theta}(\Re^n)$ is the set of all functions

$f \in L_p(\Re^n)$ and for which $\|f\|_{B^{\vec{\omega}(.)}_{p,\theta}(\Re^n)} < \infty$

where

$$\|f\|_{B^{\vec{\omega}(.)}_{p,\theta}(\Re^n)} = \|f\|_{L_{p(\Re^n)}} + \sum_{j=1}^{n} \left(\int_0^1 \left[\frac{\omega^{k_j}_{p,x_j}(f,u)}{\omega_j(u)} \right]^{\theta} \frac{d\omega_j(u)}{\omega_j(u)} \right)^{1/\theta} \quad \text{if } \theta < \infty \text{ and} \qquad (4.4)$$

$$\|f\|_{B^{\vec{\omega}(.)}_{p,\theta}(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{j=1}^{n} \sup_{0<u<1} \left[\frac{\omega^{kj}_{p,xj}(f,u)}{\omega_j(u)} \right] \quad \text{if } \theta = \infty \qquad (4.5)$$

For $\theta \ge 1$ this expression becomes a norm, where for $0 < \theta < 1$ it is a quasinorm.

If we fix $\vec{\omega}(u)$ and p then the space $B^{\vec{\omega}(.)}_{p,\theta}(\Re^n)$ is becoming wide with growth of θ

The widest space we denote by $H^{\vec{\omega}(.)}_{p}(\Re^n) = B^{\vec{\omega}(.)}_{p,\infty}(\Re^n)$

<u>Claim</u> If we put $\omega_j(u) = h^{\ell_j}$, $0 < \ell_j < k_j$ for $j \in \{1,2, \ldots n\}$ then

$$B_{p,\theta}^{\bar{w}(.)}(\Re^n) = B_{p,\theta}^{\bar{\ell}}(\Re^n)$$

Proof

Let $\omega_j(u) = h^{\ell_j}$ and take $m_j = 0$

1 Suppose $f \in B_{p,\theta}^{\bar{w}(.)}(\Re^n)$, it implies $\|f\|_{B_{p,\theta}^{\bar{\omega}(.)}(\Re^n)} < \infty$

$$\Rightarrow \quad \|f\|_{L_p(\Re^n)} < \infty$$

$$\text{and} \quad \sum_{j=1}^{\infty} \left(\int_0^1 \left[\frac{\omega_{p,x_j}^{\bar{k}j} f(x)}{h^{\ell_j}} \right]^{\theta} \frac{d\left(h^{\ell_j}\right)}{h^{\ell_j}} \right)^{\frac{1}{\theta}} < \infty$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} \left(\int_0^1 \sup_{0 \le h \le 1} \left(h^{-\ell_j} \left\|\Delta_{h_{e_j}}^{k_j} f\right\|^{\theta} \frac{\ell_j dh}{h}\right)^{\frac{1}{\theta}} < \infty$$

$$\Rightarrow \quad \sum_{j=1}^{\infty} \left[\int_0^1 \left(h^{-\ell_j} \left\|\Delta_{h_j}^{\sigma_j} f\right\|^{\theta} \frac{dh}{h}\right)^{\frac{1}{\theta}} < \infty$$

From 1 and 4 $\|f\|_{B_{p,\theta}^{\ell}(\Re^n)} < \infty$

$$\Rightarrow \quad f \in B_{p,\theta}^{\ell}(\Re^n)$$

$$\Rightarrow \quad B_{p,\theta}^{\bar{\omega}(.)}(\Re^n) \subseteq B_{p,\theta}^{\ell}(\Re^n)$$

2 Suppose $f \in B_{p,\theta}^{\ell}(\Re^n)$, it implies $\|f\|_{B_{p,\theta}^{\ell}(\Re^n)} < \infty$

By using similar argument this implies

$$\|f\|_{B_{p,\theta}^{\bar{w}(.)}(\Re^n)} < \infty$$

$$\Rightarrow f \in B_{p,\theta}^{\bar{\omega}(.)}(\Re^n)$$

$$\Rightarrow B_{p,\theta}^{\ell}(\Re^n) \subseteq B_{p,\theta}^{\bar{\omega}(.)}(\Re^n)$$

$$\therefore \quad B_{p,\theta}^{\ell}(\Re^n) = B_{p,\theta}^{\bar{\omega}(.)}(\Re^n)$$

In isotropic case that is where $\omega_j(u) = \omega(u)$

$\Rightarrow k_j = k$, $j = 1, 2, \ldots n$, we obtain isotropic space $B_{p,\theta}^{\bar{\omega}(.)}(\Re^n)$

4.1.2 Common Properties

i) The condition $\omega_j(0) = 0$ guarantees the presence of ristrictions on behavior $\omega^{k_j}_{p,xj}$ (f u), as u $\to$ 0. Other wise the corresponding summands in (4.4) and (4.5) are finite for any function f $\in L_p(\Re^n)$.

In isotropic case $B^{\bar{\omega}(.)}_{p,\theta}(\Re^n) \neq L_p(\Re^n)$ iff $w_j(0) = 0$ (4.6)

ii) If f $\in L_p(\Re^n)$ and not equivalent to zero then $\exists$ c > 0: $\omega^{k_j}_{p,xj}(f, u) \geq c u^{k_j}$.

$$\text{Therefore } B^{\bar{\omega}(.)}_{p,\theta}(\Re^n) \neq \{0\} \Leftrightarrow \left(\int_0^1 \left[\frac{u^{k_j}}{\omega_{j(v)}} \right]^{\theta} \frac{d\omega_j(u)}{\omega_j} \right)^{\frac{1}{\theta}} < \infty \quad \text{if } \theta < \infty \tag{4.7}$$

$$\text{and } \sup \left[u^{k_j} \Big/ \omega_j(v) \right] < \infty \qquad \text{if } \theta = \infty$$

For j = 1, 2, …,n condition(3.7) guarantees the embedding

$$W^{\bar{k}}_p(\Re^n) \quad \subset\!> \; B^{\bar{w}(.)}_{p,\theta}(\Re^n) \tag{4.8}$$

Where $W^{\bar{k}}_p(\Re^n)$ is a soblev space with norm

$$\|f\|_{W^{\bar{k}}_p(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{j=1}^{n} \left\| \frac{\partial^{k_j} f}{\partial x_j^{k_j}} \right\|_{L_p(\Re^n)} \tag{4.9}$$

If $B^{\bar{\omega}(.)}_{p,\theta}(\Re^n)$ is non-empty then embedding (4.8) guarantees rich reserve of $B^{\bar{\omega}(.)}_{p,\theta}(\Re^n)$.

iii) Dealing with space $H^{\bar{\omega}(.)}_p(\Re^n)$ ($\theta = \infty$), without loss of generality, it is possible to assume that $\omega_j(u) \in \Omega_{k_j}(1)$, j = 1, 2, …, n.

So if $\omega_j(u) \notin \Omega_{k_j}(1)$, then it is possible to replace $\omega_j(u)$ by a function $\psi_j(u) \in \Omega_{k_j}(1)$ without changing the space.

$$H^{\bar{\omega}(.)}_p(\Re^n) = H^{\bar{\psi}(.)}_p(\Re^n), \text{ where } \bar{\psi}(u) = \{\psi_j(u)\}^n_{j=1} \tag{4.10}$$

Some authors A . S Djapharov, V.P. Illin and others for $\theta < \infty$ define the Nikolskii-Besov spaces in another way.

To be exact let $\vec{\varphi}(u) = \{\varphi_j(u)\}_{j=1}^{n}$ where $\varphi_j(u)$ be continuous function on [0, 1],

$1 \leq p \leq \infty$, $0 < \theta < \infty$.

And instead of $\vec{\omega}(.)$, by taking $\vec{\varphi}(u) = \{\varphi_j(u)\}_{j=1}^{n}$ where $\varphi_j(u)$ is a continuous function on [0, 1], $1 \leq p \leq \infty$, $0 < \theta < \infty$ we get the following definition.

Definition 4. 5 The space $B_{p,\theta}^{\vec{\varphi}(.)}(\Re^n)$ is the set of all functions such that

1. $f \in L_p(\Re^n)$
2. $$\|f\|_{B_{p,\theta}^{\vec{\varphi}(.)}(\Re^n)} = \|f\|_{L_p(\Re^n)} + \sum_{j=1}^{\infty}\left(\int_0^1\left[\frac{\omega_{p,x_j}^{k_j}(f,u)}{\varphi_j(u)}\right]^{\theta}\frac{du}{u}\right)^{1/\theta} < \infty \qquad (4.11)$$

If $\varphi_j(u) = \varphi(u)$ for $j \in \{1, 2, \ldots n\}$, then we obtain the isotropic space $B_{p,\theta}^{\varphi(.)}(\Re^n)$.

Analogies to i and ii we restrict

$$\int_0^1\left[\frac{1}{\varphi_j(u)}\right]^{\theta}\frac{du}{u} = \infty \text{ and } \int_0^1\left[\frac{\omega^{k_j}}{\varphi_j(u)}\right]^{\theta}\frac{du}{u} < \infty \; j = 1, 2, \ldots, n \qquad (4.12)$$

In particular, the conditions of non triviality in isotropic case will have the form

$$B_{p,\theta}^{\varphi(.)}(\Re^n) \neq L_p(\Re^n) \Leftrightarrow \int_0^1\left[\frac{1}{\varphi_j(u)}\right]^{\theta}\frac{du}{u} = \infty \qquad (4.13)$$

$$B_{p,\theta}^{\varphi(.)}(\Re^n) \neq \{0\} \Leftrightarrow \int_0^1\left[\frac{u^k}{\varphi(u)}\right]^{\theta}\frac{du}{u} < \infty \qquad (4.14)$$

Definition (4.4) and (4 .5) in general lead to the same spaces. Specifically

$$B_{p,\theta}^{\vec{\varphi}(.)}(\Re^n) = B_{p,\theta}^{\vec{\omega}(.)}(\Re^n) \qquad (4.15)$$

if we put $\omega_j(u) = \{ 1 + \theta \int_u^1 \varphi_j(u)^{\theta} v^{-1} dv \}^{1/\theta}$ $j = 1, 2, \ldots, n$ (4.16)

Indeed the norms (3.4) and (3.11) coincide since

$$\frac{dw_j(u)}{w_j(u)^{1+\theta}} = \frac{du}{\varphi_j(u)^{\theta} u} \quad 0 < u \le 1$$

The question when the descriptions (4.4) and (4.11) are equivalent with one and the same function of smoothness that $\omega_j(u) \sim \varphi_j(u)$ is very interesting, And the answer is as follows:

$$\omega_j(u) = \varphi_j(u) \Leftrightarrow \exists\alpha_j > 0 : \varphi_j(u)^{-\alpha_j} \text{ almost increasing on } [0, 1]$$

From now and onwards we will use definition 5 with some additional assumptions.

4.2 Definition of iterated norms for Nikol'skli-Besov Space with Generalized Smoothness

Let 1. $\Omega \subset \Re^n$ be an open set

2 $\Omega_\delta := \{x \in \Omega : \rho(x.\partial\Omega) > \delta)\}, \forall\delta>0$,

3 $Z(\Omega_\delta)$ be a semi normed space of functions defined on Ω_δ

4 $1 \le \theta < \infty$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ $\sigma_j \in N$

5. $\Phi(\vec{\sigma},\theta)$ be a class of functions such that $\vec{\varphi} = (\varphi_1 \ldots \varphi_n) \in \Phi(\vec{\sigma},\theta)$ satisfying the following conditions

a) $\varphi_j(h) > 0, \forall h>0$, b) $\varphi_j(h)$ be an increasing function

c) $\lim\limits_{h\to 0^+} \varphi_j(h) = 0$ and φ_j obeys $S_{\sigma s}$ property that is

$\exists m_j < \sigma_j$: $\varphi_j(h)\, h^{-m_j}$ decreasing almost every where

i.e. $\exists c \ge 1$: $\varphi_j(t)\, t^{-m_j} \le \varphi_j(s)\, s^{-m_j}$ for $0 < s \le t < \infty$.

Definition4.6 A function $f \in \mathscr{B}_\theta^{\vec{\varphi}(.)}(Z(\Omega)) \equiv \mathscr{B}_\theta^{\vec{\varphi}(.),\vec{\sigma},H}(Z(\Omega))$ iff

1. $f \in L^{Loc}(\Omega)$ (i.e f is integrable on all compact set A, $A \subseteq \Omega$) and

2. $$\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(Z(\Omega))} := \|f\|_{Z(\Omega)} + \sum_{j=1}^{\infty} \|f\|_{\beta_{\theta,j}^{\varphi_j(.)}(Z(\Omega))} < \infty \qquad (4.17)$$

Where $$\|f\|_{\beta_{\theta,j}^{\varphi_j(.)}(Z(\Omega))} := \left\| \varphi_j^{-1}(h) \| \Delta_{h,j}^{\sigma_j} f \|_{Z(\Omega_{\sigma_j h})} \right\|_{L_\theta^*(0,H)} \qquad (4.18)$$

and $\left(\Delta_{h,j}^{\sigma_j} f\right)$ denotes the σ_j^{th} different of f with respect to x_j with step h and

$L_\theta^*(0,H)$ with $1 \le \theta < \infty$, denotes the space of function g of one variable measurable on (0,H) for which $\|g\|_{L_\theta^*(0,H)} := \left(\int_0^H | g(h) |^\theta \frac{dh}{h} \right)^{1/\theta} < \infty$

$$L^*_{\infty(O,H)} = L_{\infty(o,H)}$$

and $\sum$ denotes that we are summing over this j for which $\varphi_j > 0$.

More over we will assume that the pair $\vec{\varphi}$, $\vec{\sigma}$ is permissible. That is for these $j \in \{1, 2, \ldots, n\}$ such that $\varphi_j > 0$.

$$\sigma_j > \varphi_j(h) > 0.$$

Comparison of ℓ - smoothness and Generalized smoothness.

	ℓ- smoothness		Generalized smoothness
1	we use $\vec{\ell} = (\ell_1, \ell_2, \ldots, \ell_n)$	1	we use $\vec{\varphi}_j(.) = (\varphi_1, \varphi_2, \ldots, \varphi_n)$
2	we use $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots \sigma_n)$	2	the same
3	we use $\vec{m} = (m_1, m_2, \ldots, m_n)$	3	$m = 0$
4	we use $h^{-(\ell_j - m_j)}$	4	we use $\varphi_j^{-1}(h)$
5	we use $\left(\Delta_{h,j}^{\sigma_j} D_j^{m_j} f\right)$	5	since $m = 0$, we have $\left(\Delta_{h,j}^{\sigma_j} f\right)$
6	$h \in (0, H)$	6	$h \in (0, H)$

Note:

1. for $m = 0$. If we define $\varphi_j^{-1}(h) = h^{-(\ell_j - m_j)} = h^{-\ell_j}$ then the generalized smoothness is the same as ℓ smoothness.
2. Since $m = 0$, on Generalized smoothness we don't consider the partial derivative of f.
3. ℓ smoothness is the particular part of the generalized smoothness.

   The above definition is obtained by replacing $\|\cdot\|_{L_p(\Omega\sigma_j h)}$ by $\|\cdot\|_{Z(\Omega\sigma_j h)}$ in the definition of anisotropic Nikol'skii-Besov Space $B^{\vec{\varphi}}_{p,\theta}(\Omega)$

   Thus $\mathcal{B}^{\vec{\varphi}}_\theta(L_p(\Omega)) = B^{\vec{\varphi}}_{p,\theta}(\Omega)$

In (4.17) and (4.18) we now set $\vec{\varphi} := \vec{\varphi}_2 = (\vec{\varphi}_{21}, \ldots, \vec{\varphi}_{2n})$

$\vec{\sigma} := \vec{\sigma}_2 := (\sigma_{21}, \ldots \sigma_{2n})$, H: = H_2 and $z(\Omega_\sigma) = \mathscr{B}_\theta^{\vec{\varphi}_1}\left(L_p(\Omega_\sigma)\right) \equiv B^{\vec{\varphi}_1}_{p,\theta}(\Omega_\sigma)$ with parameters σ_1 and H_1. We then obtain the norm of the form $\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_2}(\mathscr{B}_\theta^{\vec{\varphi}_1}(L_p(\Omega)))}$.

Continuing this process, we obtain the following norms, which we naturally call iterated norms for all integers $k \geq 2$.

$$\|f\|_{\underbrace{\mathscr{B}_\theta^{\varphi_k(.)}(\ldots\mathscr{B}_\theta^{\vec{\varphi}_1()}(L_p(\Omega))\ldots)}_{k}} := \|f\|_{\mathscr{B}_\theta^{\varphi_k()}\underbrace{(\mathscr{B}_\theta^{\varphi()_{k-1}}(\ldots\mathscr{B}^{\varphi_1()}(L_p(\Omega)\ldots))}_{k-1}} \tag{4.19}$$

and the corresponding spaces

$\mathscr{B}_\theta^{\varphi_k(.)}(\ldots(\mathscr{B}_\theta^{\varphi(.)_1}(L_p(\Omega))\ldots)$ which we call Nikol'skii – Besov spaces with iterated norms and generalized smoothness.

If $\vec{\varphi}_1(.) = \ldots = \vec{\varphi}_k(.) = \vec{\varphi}(.)$ we then obtain the norms ~~$\mathscr{B}^{\varphi(.)}(L_p(\Omega)$~~

$$\underbrace{\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(\ldots\mathscr{B}_\theta^{\vec{\varphi}(.)}(L_p(\Omega)))\ldots)}}_{k} = \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}\underbrace{(\mathscr{B}_\theta^{\vec{\varphi}(.)}\ldots\mathscr{B}_\theta^{\vec{\varphi}(.)}(L_p(\Omega)))\ldots)}_{k-1}} \tag{4.20}$$

with which the posed problem is connected.

4.3 Formulation of the problem and its results

In works of professor V.I.Burenkov iterated norms of Nikol'skii-Besolv's spaces were defined and with the help of these norms it is proved that the solutions of partial differential equations with constant coefficients are infinitely differentiable. In our work we consider iterated norms in spaces of Nikol'skii-Besov's type with generalized smoothness $\vec{\varphi}$ where $\vec{\varphi} \in \Phi\ (\vec{\sigma},\theta)$, $1 < p < \infty$ and $1 < \theta < \infty$. The results obtained by professor V.1 Burenkov.

<u>Theorem</u> 1 (about iterated norms with generalized smoothness).

Let $1 < p < \infty$, $1 < \theta < \infty$, $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\vec{\varphi} = (\varphi_1, \varphi_2, \ldots \varphi_n) \in \Phi(\vec{\sigma},\theta)$ open parallelepiped $G \subset \Re^n$ with faces parallel to the coordinate planes

and $\vec{\varphi}_k = \{\varphi_1^k, \varphi_2^k, \ldots \varphi_n^k\}$. Then the following assertions are true.

1) There exists an imbedding and the imbedding operator is bounded.

$$\underbrace{\mathscr{B}_\theta^{\bar\varphi}\left(\mathscr{B}_\theta^{\bar\varphi}\ldots\mathscr{B}_\theta^{\bar\varphi}\left(L_p(G)\right)\ldots\right)}_{k} \subset_{>} B_{p,\theta}^{\bar\varphi^k}(G)$$

2) If, in addition, for an open parallelepiped $G \subset \Re^n$ there exists bounded extension operator

$$S:\quad B_{p,\theta}^{\bar\varphi^k}(G) \to B_{p,\theta}^{\bar\varphi^k}(\Re^n) \tag{4.20'}$$

then the following holds true:

$$\underbrace{\mathscr{B}_\theta^{\bar\varphi}\left(\mathscr{B}_\theta^{\bar\varphi}\ldots\mathscr{B}_\theta^{\bar\varphi}\left(L_p(G)\right)\ldots\right)}_{k} = B_{p,\theta}^{\bar\varphi^k}(G)$$

with equivalence of norms.

Note that the conditions under such a bounded extension operator exists were described in the works of V.I.Burenkov, Yu.A Brudni, O.B. Besov,V.P.Ilin, S.M.Nikol'skii and P.A. Schurtzman.

4.4 Properties of iterated Norms with Generalized smoothness

The following Lemma will be used repeatedly

Lemma 4.1 On the frictional differentiation of an inequality.

i) Let $\mu_0 > 0$, $\Omega \subset \Re^n$ be an open set and for each $\mu \in [\,0, \mu_o)$. Let a set of functions $T(\Omega_\mu)$ and seminormed functional space $x(\Omega_\mu)$ and $y(\Omega_\mu)$ be defined such that $T(\Omega_\mu) \cap Y(\Omega_\mu) \subset T(\Omega_\mu) \cap X(\Omega_\mu)$ (4.21)

and $\|f\|_{x(\Omega_\mu)} \le \|f\|_{Y(\Omega_\mu)}, \forall f \in T(\Omega_\mu) \cap Y(\Omega_\mu)$ (4.22)

(for $\Omega_\mu = \phi \quad \|f\|_{x(\Omega_\mu)} := \|f\|_{Y(\Omega_\mu)} := 0$)

ii) Let $1 \le \theta \le \infty$ and $\vec\varphi$, $\vec\sigma$, 0 be a permissible triple and

let $0 < H < \mu_0 \left(\max\limits_{1 \le j \le n} \sigma_j\right)^{-1}$ (4.23)

then $\forall \mu \in [\,0, \mu_0 - H \max \sigma_j) := A$ (4.24)

and $\forall\, f \in T(\Omega_\mu) \cap B_\theta^{\varphi(.)}(\,Y(\Omega_\mu)) := B$

$\forall$j $\in$ {1, 2, …n} such that $\Delta_{j,h}^{\sigma_j}, f \in T\left(\Omega_{\mu+\sigma_j h}\right)$ $\forall$h $\in$ (0, H). (4.25)

Then the following inequality is satisfied.

$$\boxed{\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(X(\Omega_\mu))} \leq \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(y(\Omega_\mu))}} \quad (4.26)$$

Proof Let $\mu \in A$ and f $\in$B then by (5) for $\forall$j$\in$ { 1, 2, …n} and $\forall$h $\in$ (0, H) we have $\Delta_h^{\sigma_j} jf \in T\left(\Omega_{\mu+\sigma_j h}\right)$and by the definition of the space $\mathscr{B}_\theta^{\vec{\varphi}(.)}$ (y(Ω_μ), f $\in$ $\mathscr{B}_\theta^{\vec{\varphi}(.)}$ (y(Ω_μ)

$$\Rightarrow \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(y(\Omega_\mu))} < \infty$$

$$\Rightarrow \left\|\Delta_{h,j}^{\sigma_j} f\right\|_{y\left(\Omega_{\mu+\sigma_j h}\right)} < \infty$$

$$\Rightarrow \left(\Delta_{h,j}^{\sigma_j} f\right) \in y\left(\Omega_{\mu+\sigma_j h}\right)$$

Consequently by (4.22) almost for all h $\in$(0, H)

$$\left\|\Delta_{h,j}^{\sigma_j} f\right\|_{X\left(\Omega_{\mu+\sigma_j h}\right)} \leq \left\|\Delta_{h,j}^{\sigma_j} f\right\|_{Y\left(\Omega_{\mu+\sigma_j h}\right)} \quad (4.27)$$

By multiplying both sides of (4.27) by φ_j^{-1}(h)

gives

$$\varphi_j^{-1}(\text{h}) \left\|\Delta_{h,j}^{\sigma_j} f\right\|_{X\left(\Omega_{\mu+\sigma_j h}\right)} \leq \varphi_j^{-1}(\text{h}) \left\|\Delta_{h,j}^{\sigma_j} f\right\|_{Y\left(\Omega_{\mu+\sigma_j h}\right)}$$

and by applying the L_θ^* (0, H) we get

$$\|f\|_{\beta_{\theta,j}^{\vec{\varphi}_j(.)}(X(\Omega_\mu))} = \left\| \left|\varphi_j^{-1}(h)\right| \left\| \Delta_{h,j}^{\sigma_j} \right\|_{X\left(\Omega_{\mu+\sigma_j h}\right)} \right\|_{L_\theta^*(0,H)}$$

$$\leq \left\| \left|\varphi_j^{-1}(h)\right| \left\| \Delta_{h,j}^{\sigma_j} \right\|_{X\left(\Omega_{\mu+\sigma_j h}\right)} \right\|_{L_\theta^*(0,H)} = \|f\|_{\mathscr{B}_{\theta,j}^{\varphi_j(.)}(X\Omega_\mu))} \quad (4.28)$$

By combining (4.28) and (4.22) we get

$$\boxed{\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(X(\Omega_\mu))} \leq \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(Y(\Omega_\mu))}}$$

Corollary 4.1

If 1) $\forall \mu \in [0, \mu_0 - H \max_{1 \le j \le n} \sigma_j)$ and

2) $\forall f \in T(\Omega_\mu) \cap \mathscr{B}_\theta^{\vec{\varphi}(.)}(y(\Omega_\mu))$,

Condition (4.25) is satisfied for $\forall j \in \{1, 2, \ldots, n)$ then for all $\mu \in [0, \mu_0 - H \max_{1 \le j \le n} \sigma_j)$ we

have

$$\boxed{\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(X(\Omega_\mu))} \le \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}(.)}(Y(\Omega_\mu))}}$$

Note

If $f \in \mathscr{B}_\theta^{\vec{\varphi}2}\ \mathscr{B}_\theta^{\vec{\varphi}1}(z(\Omega_\mu))$ then we have the following.

$$\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_2}(\mathscr{B}_\theta^{\vec{\varphi}_{12}}(z(\Omega)))} = \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_1}}(z(\Omega)) + \sum_{j_2=1}^{n} \underbrace{\|f\|_{\beta_{\theta, j_2}^{\varphi_2, j_2}\left(\mathscr{B}_\theta^{\vec{\varphi}_1}(z(\Omega))\right)}}_{:=t}$$

$$t = \left\| \varphi_{2j2}^{-1}(h_2) \underbrace{\left\| \Delta_{h2,j2}^{\sigma_{2,j2}} f \right\|_{\mathscr{B}_\theta^{\vec{\varphi}_1}\left(z\left(\Omega_{\sigma_{2j_2}h_2}\right)\right)}}_{:=S} \right\|_{L_\theta^*(0,H_2)}$$

$$s = \left\| \Delta_{h_2,j_2}^{\sigma_{2j_2}} f \right\|_{z\left(\Omega_{\sigma_{2j_2}h_2}\right)} + \sum{}'' \underbrace{\left\| \Delta_{h_2,j_2}^{\sigma_{2j_2}} f \right\|_{\beta_{\theta,j_1}^{\varphi_{1j_1}}\left(\Omega_{\sigma_{2j_2}h_2}\right)\Big)}}_{:=r}$$

$$r = \left\| \varphi_{h_1j_1}^{-1}(h_1) \left\| \Delta_{h_1j_1}^{\sigma_1 j_1} \Delta_{h_2j_2}^{\sigma_2j_2} f \right\|_{\left(z\left(\Omega_{\sigma_{2j2}h_2 + \sigma_{1j1}h_1}\right)\right)} \right\|_{L_\theta^*(o,H_1)}$$

The above expansion is by using the definition

Corollary 4. 2

1. Let conditions (4. 22) and (4. 23) are satisfied i.e $0 < H < \mu_0 \left(\max_{1 \le j \le n} \sigma_j\right)^{-1}$

$$\|f\|_{X(\Omega_\mu)} \le \|f\|_{Y(\Omega_\mu)} \quad \forall f \in T(\Omega_\mu) \cap Y(\Omega_\mu)$$

2. Let $1 \leq \theta \leq \infty$ and let $\vec{\varphi}_1, \vec{\sigma}_1, 0_1, \ldots \vec{\varphi}_k, \vec{\sigma}_k, 0_k$ be permissible triples and

Let $\sum_{s=1}^{k} H_s \max_{1 \leq j \leq n} \sigma_{js} < \mu_o$ (4.29)

then $\forall \mu \in [\, 0, \mu_o - \sum_{s=1}^{k} H_s \max_{1 \leq j \leq n} \sigma_{js})$ (4.30)

and $\forall f \in T(\Omega_\mu) \cap \mathscr{B}_\theta^{\vec{\varphi}_k} (\ldots (\mathscr{B}_\theta^{\vec{\varphi}_1} (Y(\Omega_\mu))\ldots)$ such that $\forall j \in \{1, 2, \ldots n\}$

$$\Delta_{h_1, j_1}^{\sigma_{j1}} \ldots \Delta_{h_k j_k}^{\sigma_{jk}} f \in T\left(\Omega_{\mu + \sum_{\sigma_{js} h_s}^{k}}\right) \forall h_s \in (0, H_s) \tag{4.31}$$

then

$$\boxed{\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_k}(\ldots \mathscr{B}_\theta^{\vec{\varphi}_1}(X(\Omega_\mu))\ldots)} \leq \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_k}(\ldots \mathscr{B}_\theta^{\vec{\varphi}_1}(Y\Omega_\mu))\ldots)}} \tag{4.32}$$

Proof By induction

For k = 1, it is proved by the Lemma 4.1

Let us show for k = 2

For k = 2 $\vec{\varphi}_2 = (\varphi_{21}, \varphi_{22}, \varphi_{23}, \ldots, \varphi_{2n})$

$\vec{\varphi}_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13}, \ldots \varphi_{1n})$

$j_s \in \{j_1, j_2\}$. $Hs \in \{H_1, H_2\}$

Let $\mu_0 > 0$, let $\mu \in [0, \mu_0 - \sum_{s=1}^{n} H_s \max_{1 \leq j \leq n} \sigma_{js})$

$\forall f \in T(\Omega_\mu) \cap \mathscr{B}_\theta^{\vec{\varphi}_2}\left(\mathscr{B}_\theta^{\vec{\varphi}_1}\left(y(\Omega_\mu)\right)\right)$

$\forall j_1, \forall j_2 \in \{1, 2, \ldots, n\}$, $\forall h_1 \in (o, H_1)$, $\forall h_2 \in (0, H_2)$

$F_1 := \Delta_{h_1, j_1}^{\sigma_{1, j_1}} \Delta_{h_2, j_2}^{\sigma_2 j_2} f \in T(\Omega_{\mu+a})$ by (4.31) (4.33)

where $a = \sigma_{1 j_1} h_1 + \sigma_{2 j2} h_2$

and f $\in \mathscr{B}_\theta^{\vec{\varphi}_2}\left(\mathscr{B}_\theta^{\vec{\varphi}_1}\left(y(\Omega_\mu)\right)\right)$

$$\Rightarrow \|F_2\|_{\mathscr{B}_\theta^{\vec{\varphi}_1}\left(y\left(\Omega_{\mu+\sigma_{2j2}h_2}\right)\right)} < \infty$$

Where $F_2 = \left(\Delta_{h_2,j_2}^{\sigma_2 j_2} f\right)$

$$\Rightarrow F_2 \in \mathscr{B}_\theta^{\vec{\varphi}_1}\left(y\left(\Omega_{\mu+\sigma_{2j2}h2}\right)\right) \text{ and } F_2 \in y(\Omega_{\mu+\sigma_{2j2}h_2}) \tag{4.34}$$

$$\Rightarrow \|F_1\|_{y\left(\Omega_{\mu+a}\right)} < \infty$$

$$\Rightarrow F_1 \in y\,(\Omega_{\mu+a}) \tag{4.35}$$

so from (4.33), (4.35) and (4.22)

$$\|F_1\|_{X(\Omega_{\mu+a})} \leq \|F_1\|_{Y(\Omega_{\mu+a})} \tag{4.36}$$

Multiplying both sides of (4.36) by $\varphi_{1j_1}^{-1}(h_1)$

we get

$$\varphi_{1j_1}^{-1}(h_1)\,\|F_1\|_{X(\Omega_{\mu+a})} \leq \varphi_{1j_1}^{-1}(h_1)\,\|F_1\|_{Y(\Omega_{\mu+a})} \tag{4.37}$$

On (4.37) by applying the norm $L_\theta^*(0, H_1$

$$\Rightarrow \left\|\varphi_{1j_1}^{-1}(h_1)\,\|F_1\|_{X(\Omega_{\mu+a})}\right\|_{L_\theta^*(0,H} \leq \left\|\varphi_{1j_1}^{-1}(h_1)\|F_1\|_{Y(\Omega_{\mu+a})}\right\|_{L_\theta^*(0,H_1)} \tag{4.38}$$

$$\Rightarrow \|F_2\|_{\beta_{\theta,J1}^{\varphi 1 j1}(X(\Omega_{\mu+b}))} \leq \|F_2\|_{\beta_{\theta,J1}^{\varphi 1 j1}(Y(\Omega_{\mu+b}))} \tag{4.39}$$

Where $b = \sigma_{2j2}h_2$

$$\Rightarrow \quad \sum_{j_1=1}^{n}{}' \|F_2\|_{\beta_{\theta,J1}^{\varphi 1 j1}(X(\Omega_{\mu+b}))} \leq \Sigma' \|F_2\|_{\beta_{\theta,J1}^{\varphi 1 j1}(Y(\Omega_{\mu+b}))} \tag{4.40}$$

And from (3.33) $F_2 \in T(\Omega_{\mu+b}) \cap Y(\Omega_{\mu+b})$ again (22)

$$\|F_2\|_{x(\Omega_{\mu+b})} \leq \|F_2\|_{y(\Omega_{\mu+b})} \tag{4.41}$$

Combining (4.40) and (4.41) (by adding) we get

$$\|F_2\|_{\mathscr{B}_\theta^{\vec{\varphi}_1}(X(\Omega_{\mu+b}))} \leq \|F_2\|_{\mathscr{B}_\theta^{\vec{\varphi}_1}(Y(\Omega_{\mu+b}))} \tag{4.42}$$

Multiplying (4.42) by $\varphi_{2j_2}^{-1}(h_2)$ and applying the $L_\theta^*(0, H_2)$ norm

$$\left\| \varphi_{2j_2}^{-1}(h_2) \| F_2 \|_{\mathscr{B}_\theta^{\bar{\varphi}_1}} \left(X \left(\Omega_{\mu+b} \right) \right) \right\|_{L^*(0,H_2)} \leq \left\| \varphi_{2j_2}^{-1}(h_2) \| F_2 \|_{\mathscr{B}_\theta^{\bar{\varphi}_1}} \left(Y \left(\Omega_{\mu+b} \right) \right) \right\|_{L^*(0,H_2)} \tag{4.43}$$

$$\Rightarrow \|f\|_{\beta_{\theta j_2}^{\varphi_2 j_2} \mathscr{B}_\theta^{\bar{\varphi}_1}} \left(X \left(\Omega_\mu \right) \right) \leq \|f\|_{\beta_{\theta j_2}^{\varphi_2 j_2} \mathscr{B}_\theta^{\bar{\varphi}_1}} \left(Y \left(\Omega_\mu \right) \right) \tag{4.44}$$

On (4. 44) by applying $\sum''$

$$\sum \|f\|_{\beta_{\theta j_2}^{\varphi_2 j_2} \mathscr{B}_\theta^{\varphi_1}} \left(X \left(\Omega_\mu \right) \right) \leq \sum \|f\|_{\beta_{\theta_2}^{\varphi_2 j_2} \mathscr{B}_\theta^{\varphi_1}} \left(Y \left(\Omega_\mu \right) \right) \tag{4.45}$$

But $f \in \mathscr{B}_\theta^{\bar{\varphi}_1}(y(\Omega_\mu)) \cap T(\Omega_\mu)$ by the Lemma when k = 1

$$\Rightarrow \quad \|f\|_{\mathscr{B}_\theta^{\bar{\varphi}_1}} \left(X \left(\Omega_\mu \right) \right) \leq \|f\|_{\mathscr{B}_\theta^{\bar{\varphi}_1}} \left(Y \left(\Omega_\mu \right) \right) \tag{4.46}$$

By combining (4.45) and (4,46) (by adding) we get

$$\|f\|_{\mathscr{B}_{\theta j_2}^{\bar{\varphi}_2} \mathscr{B}_\theta^{\bar{\varphi}_1}} \left(X \left(\Omega_\mu \right) \right) \leq \|f\|_{\mathscr{B}_{\theta j_2}^{\bar{\varphi}_2} \mathscr{B}_\theta^{\bar{\varphi}_1}} \left(Y \left(\Omega_\mu \right) \right)$$

$\therefore$ It is true for k = 2

$\therefore$ By Induction it is true for any natural number k.

Remark I If $\mu_0 = \infty$ then the Lemma and its corollaries become simplified: it is not necessary to apply bounds on H (consequently not on Hs), and inequalities (4.21), (4.22) and (4.26) are satisfied for all $\mu \geq 0$

Remark 2 If instead of (4.22), we consider the more general inequality

$$\|f\|_{x(\Omega_\mu)} \leq \sum_{r=1}^{r_0} cr \|f\|_{y_r(\Omega_\mu)} \tag{4.47}$$

where cr ≥ 0 and $y_r(\Omega_\mu)$ is a seminormed space for r $\in \{1, \ldots, r_0\}$, then analogous statement, are valid. It is necessary to consider the seminorm

$$\|f\|_{y(\Omega_\mu)} := \sum_{r=1}^{r_0} C_r \|f\|_{y_r(\Omega_\mu)}$$

and to take into account that, by the definition,

$$\|f\|_{\mathscr{B}_\theta^{\bar{\varphi}(\cdot)}} \left(x \left(\Omega_\mu \right) \right) \leq \sum_{r=1}^{r_0} C_r \|f\|_{\mathscr{B}_\theta^{\varphi(\cdot)}} \left(Y \left(\Omega_\mu \right) \right)$$

On the following Lemma we will introduce on $\mathcal{B}_\theta^{\vec{\varphi}_k}\left(\left(\mathcal{B}_\theta^{\vec{\varphi}_{k-1}}\ldots\mathcal{B}_\theta^{\vec{\varphi}_1}\left(L_p(\Omega)\right)\ldots\right)\right.$ an equivalent norm which is more convenient for estimates.

We set

$$\|f\|^*_{B_\theta^{\vec{\varphi}_k}\left(\ldots B_\theta^{\vec{\varphi}_1}(L_p(\Omega))\ldots\right)} :=$$

$$:= \|f\|_{L_p(\Omega)} + \sum_{r=1}^{k}\sum_{i_1,\ldots,i_r=1}^{k}\sum_{j_1,\ldots,j_r=1}^{n}\|f\|_{B_{\theta,j_r}^{\varphi i_r,j_r}\ldots\left(\mathcal{B}_{\theta,j_1}^{\varphi i_1,j_1}\left(L_p(\Omega)\right)\ldots\right)} \tag{4.48}$$

Lemma 4. 2, for all consider values of the parameters

$$\|f\|_{\mathcal{B}_\theta^{\vec{\varphi}_k}\left(\ldots\mathcal{B}_\theta^{\vec{\varphi}_1}\left(L_p(\Omega)\right)\ldots\right)} \sim \|f\|^*_{\mathcal{B}_\theta^{\vec{\varphi}_k}\left(\ldots\mathcal{B}_\theta^{\vec{\varphi}_1}\left(L_p(\Omega)\right)\ldots\right)} \tag{4.49}$$

Proof: Let k = 2, Then according to the definitions,

$$\|f\|_{\mathcal{B}_\theta^{\vec{\varphi}_2}\left(\mathcal{B}_\theta^{\vec{\varphi}_1}\left(L_p(\Omega)\right)\right)} = \|f\|_{\mathcal{B}_\theta^{\vec{\varphi}_1}(L_p(\Omega))} + \sum_{j_2=1}^{n}\left\|\varphi_{2j_2}^{-1}(h_2)\left\|\Delta_{h_2,j_2}^{\sigma_2}f\right\|_{\mathcal{B}_\theta^{\vec{\varphi}_1}(L_p(\Omega))}\right\|_{L_\theta^*(0,H_2)}$$

$$=\|f\|_{L_p(\Omega)} + \sum_{j_2=1}^{n}\left\|\varphi_{1j_1}^{-1}(h_1)\left\|\Delta_{h1,j_1}^{\sigma_{1j_1}}f\right\|_{L_p\left(\Omega_{\sigma_{1,j_1}h_1}\right)}\right\|_{L_\theta^*(0,H1)} +$$

$$+\sum_{j_2=1}^{n}\left\|\varphi_{2j_2}^{-1}(h_2)\left(\Lambda_{0,j_2}+\sum_{j_2=1}^{n}\Delta_{j_1,j_2}\right)\right\|_{L_\theta^*(0,H_2)},$$

Where

$$\Lambda_{0,j_2} := \left\|\Delta_{h_2,j_2}^{\sigma_2,j_2}f\right\|_{L_p\left(\Omega_{\sigma_2 j_2 h_2}\right)},\text{ and for } j_1 \in \{1, \ldots, n\}$$

$$\Lambda_{j_1,j_2} := \left\|\varphi_{1j_1}^{-1}(h_1)\left\|\Delta_{h_1}^{\sigma_1,j_1}\Delta_{h_2,j_2}^{\sigma_2,j_2}f\right\|_{L_p\left(\Omega_{\sigma_1,j_1h_1+\sigma_2,j_2h_2}\right)}\right\|_{L_\theta^*(0,H_2)},$$

Making use of the fact that for non negative functions φ_s

$$(n+1)^{\frac{1}{\theta}-1}\sum_{s=0}^{n}\|\varphi_s\|_{L_\theta^*(0,H_2)} \le \left\|\sum_{s=0}^{n}\varphi_s\right\|_{L_\theta^*(0,H_2)} \le \sum_{s=0}^{n}\|\varphi_s\|_{L_\theta^*(0,H_2)}$$

and taking into account that

$$\left\|\varphi_{2j_2}^{-1}(h_2)\Lambda_{0,j_2}\right\|_{L_\theta^*(0,H_2)} = \|f\|_{B_{\theta,j_2}^{\varphi 2,j_2}(L_p(\Omega))},$$

and or

$$j_1 \in \{1, \ldots, n\}$$

$$\left\|\varphi_{2 j_2}^{-1}(h_2) \Lambda_{j_1, j_2}\right\|_{L_\theta^*(0, H_2)} = \|f\|_{\beta_{\theta . j_2}^{\varphi 2 . j_2}\left(\beta_{\theta . j_1}^{\varphi 1 . j_1}(L_p(\Omega))\right)}$$

We obtain that

$$\|f\|_{\beta_\theta^{\varphi_2()}\left(\beta_\theta^{\varphi_1()}(L_p(\Omega))\right)} \cong \|f\|_{L_p(\Omega)} + \sum_{j_1=1}^{n} \|f\|_{\beta_{\theta . j_1}^{\varphi 1 . j_1}(L_p(\Omega))} + \sum_{j_2=1}^{n} \|f\|_{\beta_{\theta . j_2}^{\varphi 2 . j_2}(L_p(\Omega))}$$

$$+\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \|f\|_{\beta_{\theta . j_2}^{\varphi 2 . j_2}\left(\beta_{\theta . j_2}^{\varphi 1 . j_1}(L_p(\Omega))\right)},$$

Which is (4.48) for k = 2, The case k > 2 follows by induction. We notice further that for all considered values of the parameters.

$$\|f\|_{\beta_\theta^{\varphi_{2j}()}\left(\beta_\theta^{\varphi_{1i}()}(L_p(\Omega))\right)}$$

$$= \left\|\varphi_{2j}^{-1}(h)\left\|\varphi_{1i}^{-1}(\eta)\right\|\Delta_\eta^{\sigma_{1i}} \Delta_h^{\sigma_{2j}} f\right\|_{L_p(\Omega_{\sigma 1 \eta + \sigma 2 h})}\Big\|_{L_\theta^*(0, H_1)}\Big\|_{L_\theta^*(0, H_2)} =$$

$$= \left\|\varphi_{1j}^{-1}(\eta)\left\|\varphi_{2j}^{-1}(h)\right\|\Delta_\eta^{\sigma_1} \Delta_h^{\sigma_2} f\right\|_{L_p(\Omega_{\sigma 1 \eta + \sigma 2 h})}\Big\|_{L_\theta^*(0, H_2)}\Big\|_{L_\theta^*(0, H_1)} \tag{4.50}$$

$$= \|f\|_{\beta_\theta^{\varphi_{1i}()}\left(\beta_\theta^{\varphi_{2j}()}(L_p(\Omega))\right)}.$$

Note

1. $$\|f\|_{\beta_{\theta . j_2}^{\varphi 2 . j_2}(L_p(\Omega))} := \left\|\varphi_{2, j_2}^{-1}(h_2)\right\|\Delta_{h_2 . j_2}^{\sigma_{2 . j_2}} f\Big\|_{L_p(\Omega_{\sigma_{2 j_2} h_2})}\Big\|_{L_\theta^*(0, H_2)}$$

2. $$\|f\|_{\beta_{\theta . j_1}^{\varphi 1 j_1}(L_p(\Omega))} := \left\|\varphi_{1, j_1}^{-1}(h_1)\right\|\Delta_{h_1 j_1}^{\sigma_{1 . j_1}} f\Big\|_{L_p(\Omega_{\sigma_{1 j_1} h_1})}\Big\|_{L_\theta^*(0, H_1)} \tag{4.51}$$

Remark : By using the above Lemma we can proof corollary 2 for k = 2

Proof

a) $\|f\|_{X(\Omega_\mu)} \leq \|f\|_{Y(\Omega_\mu)}$ by 22 (R1)

b) From (4.28) the proof of the Lemma 1

$$\|f\|_{\beta_{\theta . j_1}^{\varphi 1 j_1}(X(\Omega_\mu))} \leq \|f\|_{\beta_{\theta . j_1}^{\varphi 1 j_1}(Y(\Omega_\mu))}$$

$$\Rightarrow \sum_{j=1}^{n} \|f\|_{\beta_{\theta,j1}^{\varphi 1 j1}(X(\Omega_\mu))} \leq \sum_{j=1}^{n} \|f\|_{\beta_{\theta,j1}^{\varphi 1 j1}(Y(\Omega_\mu))} \tag{R2}$$

(c) From (4.34) we show that

$F_2 \in Y\left(\Omega_{\mu+\sigma_{2j2h_2}}\right)$ and $F_2 \in T\left(\Omega_{\mu+\sigma_{2j2h_2}}\right)$

$$\Rightarrow \|F_2\|_{X\left(\Omega_{\mu+\sigma_{2j2h_2}}\right)} \leq \|F_2\|_{Y\left(\Omega_{\mu+\sigma_{2j2h_2}}\right)} \text{ by (4.22)} \tag{R3}$$

By multiplying both sides of (R3) by $\varphi_{2j_2}^{-1}(h_2)$ and applying $L_\theta^*(0, H_2)$ norm we get

$$\left\|\varphi_{2j_2}^{-1}(h_2)\,\|F_2\|_{X\left(\Omega_{\mu+\sigma_{2j2h_2}}\right)}\right\|_{L_\theta^*(0,H_2)} \leq \left\|\varphi_{2j_2}^{-1}(h_2)\,\|F_2\|_{Y\left(\Omega_{\mu+\sigma_{2j2h_2}}\right)}$$

$$\Rightarrow \|f\|_{\beta_{\theta,j2}^{\varphi 2 j2}(X(\Omega_\mu))} \leq \|f\|_{\beta_{\theta\, j2}^{\varphi 2 j2}(Y(\Omega_\mu))} \tag{R4}$$

$$\Rightarrow \sum_{j_2=1}^{n} \|f\|_{\beta_{\theta,j_2}^{\varphi 2 j_2}(X(\Omega_\mu))} \leq \sum_{j_2}^{n} \|f\|_{\beta_{\theta j_2}^{\varphi 2 j2}(Y(\Omega_\mu))} \tag{R5}$$

Let $F := \Delta_{h_1 j_1}^{\sigma_{1 j1}} \; \Delta_{h_2}^{\sigma_{2 j2}} j_2 f$ and $a = \Omega_{\sigma_{1j1}h_1+\sigma_{2j2}h_2}$

From (4.38) we show that

$$\left\|\varphi_{1j_1}^{-1}(h_1)\,\|F\|_{x(\Omega_{\mu+a})}\right\|_{L_\theta^*(0,H_1)} \leq \left\|\varphi_{1j_1}^{-1}(h_1)\,\|F\|_{y(\Omega_{\mu+a})}\right\|_{L_\theta^*(0,H_1)}$$

By multiplying both sides of R6 by $\varphi_{2j_2}^{-1}(h_2)$ and applying $L_\theta^*(0, H_2)$ norm we get

$$\underbrace{\|f\|_{\beta_{\theta,j_2}^{\varphi 2 j_2}\left(\beta_{\theta\, j1}^{\varphi 1\cdot j1}(x(\Omega_\mu))\right)}}_{=mj_1 j_2} \leq \underbrace{\|f\|_{\beta_{\theta,j_2}^{\varphi 2 j_2}\left(\beta_{\theta,j1}^{\varphi 1 j1}(y(\Omega_\mu))\right)}}_{=b_{j1j2}}$$

$$\Rightarrow \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} m_{j_1 j_2} \leq \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} b_{j_1 j_2} \tag{R8}$$

From a), R1, R2, R5 and R8 and by definition we get

$$\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_2}\left(\mathscr{B}_\theta^{\vec{\varphi}_1}(x(\Omega_\mu))\right)} \leq \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}_2}\left(\mathscr{B}_\theta^{\vec{\varphi}_1}(y(\Omega_\mu))\right)}$$

Lemma 4. 3: Let $1 \le p, \theta \le \infty$; $0 < H \le \infty$; $\vec{\varphi}_1, \ \vec{\varphi}_2 \in \Phi\left((\vec{\sigma},\theta)\right)$, then

$$\|f\|_{\beta_\theta^{\varphi_{1i}(\cdot)\varphi_{2j}(\cdot)}(L_p(G))} \le C_1 \|f\|_{\beta_\theta^{\varphi_{1i}(\cdot)\varphi_{2j}(\cdot)}(L_p(G))} \tag{4.52}$$

for all permissible $\varphi_{1i}\sigma_{1i}$ and φ_{2j},σ_{2j}.

Where G is arbitrary open parallelepiped with sides parallel to coordinate axes, c_1 depends only on min $\{\sigma_{1i}, \sigma_{2j}\}$, and

$$\beta_\theta^{\varphi_{1i}(.)\varphi_{2j}(.)} \equiv \beta_\theta^{\varphi_{1i}(.)\varphi_{2j}(.),\sigma_{1i}+\sigma_{2j};H}$$

Proof. Without loss of generality let $\sigma_1 \le \sigma_2$.

From chapter 2 we have $\|\Delta_h^\sigma f\|_{L_p(\Re)} \le A(\sigma)\,\frac{1}{h}\int_0^h \|\Delta_\eta^\sigma F\|_{L_p(\Re)} d\eta$ (4.53)

from this we get

$$\|\Delta_h^\sigma f\|_{L_p(G\sigma h)} \le A(\sigma)\,\frac{1}{h}\int_0^h \|\Delta_\eta^\sigma F\|_{L_p(G\sigma h)} d\eta$$

$\Rightarrow$ But for $1 \le \theta \le \infty$

$\Rightarrow$ $\|\cdot\|_{L_1^*(0,h)} \le \|\cdot\|_{L_\theta(0,h)}$ so form (4.53) it follows that

$$\|\Delta_h^\sigma f\|_{L_p(G\sigma h)} \le A(\sigma)\left(\frac{1}{h}\int_0^h \|\Delta_h^\sigma f\|_{L_p(G\sigma\eta)}^\theta d\eta\right)^{1/\theta}$$

$$\le A(\sigma)\left(\int_0^h \|\Delta_h^\sigma f\|_{L_p(G\sigma\eta)}^\theta \frac{d\eta}{\eta}\right)^{1/\theta}$$

$$= \quad A(\sigma)\Big\|\, \|\Delta_h^\sigma f\|_{L_p(G\sigma h)} \Big\|_{L_p^*(0,h)} \quad \left(\text{since } h \ge \eta,\ \frac{1}{h} \le \frac{1}{\eta}\right)$$

Furthermore

$$\|\Delta_h^{\sigma_1+\sigma_2} f\|_{L_p(G(\sigma_1+\sigma_2)h)} = \|\Delta_h^{\sigma_1}\left(\Delta_h^{\sigma_2} f\right)\|_{L_p((G\sigma_2)\sigma_1 h)}$$

$$\le A(\sigma_1)\Big\|\, \|\Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\|_{L_p((G\sigma_2 h)\sigma_1\eta)} \Big\|_{L_\theta^*(0,H)}$$

from this we get

$$\|f\|_{\beta_\theta^{\varphi_{1i}(\cdot)\varphi_{2j}(\cdot),\sigma_{1i}+\sigma_{2j};H}(G)} = \left\|\varphi_{1i}^{-1}(h)\varphi_{2j}^{-1}(h)\left\|\Delta_h^{\sigma_1+\sigma_2} f\right\|_{L_p(G\sigma_1\eta+\sigma_2 h)}\right\|_{L_\theta^*(0,H)}$$

$$\leq \quad A(\sigma_1)\left\|\varphi_{2i}^{-1}(h)\left\|\varphi_{1j}^{-1}(\eta)\left\|\Delta_\eta^{\sigma_{1i}}\Delta_h^{\sigma_{2j}} f\right\|_{L_p(G\sigma_1\eta+\sigma_2 h)}\right\|_{L_\theta^*(0,H)}\right\|_{L_\theta^*(0,H)}$$

(since $\varphi_{1i}(h) \geq \varphi_{1i}(\eta)$, and $\eta \leq h$)

$$= \quad A(\sigma_1)\|f\|_{\beta_\theta^{\varphi_{1i}(\cdot)\sigma_{1i};H}\left(\beta_\theta^{\varphi_{2j}(\cdot),\sigma_{2j};H}(L_p(G))\right)} \tag{4.54}$$

$$\|f\|_{\beta_\theta^{\varphi_{1i}(\cdot)\varphi_{2j}(\cdot)}(L_p(G))} = C_1\|f\|_{\beta_\theta^{\varphi_{1i}(\cdot)}(\beta_\theta^{\varphi_{2j}(\cdot)}(L_p(G)))}$$

Corollary 4. 3: For any natural number k and for $I \leq p, \theta \leq \infty$ and $0 < H \leq \infty$

$$\|f\|_{\beta_\theta^{\varphi_j^k}(L_p(G))} \leq C_2^{k-1}\|f\|_{\beta_\theta^{\varphi_j}\left(\cdots\beta_\theta^{\varphi_j}(L_p(G))\cdots\right)} \tag{4.55}$$

for any permissible $\varphi(.)$ σ and c_2 depends on σ.

$$\left(\beta_\theta^{\varphi_j^k} \equiv \beta_\theta^{\varphi^k,k\sigma:0,H}\right).$$

Proof: From Lemma,4.3 , (4.53) and (4.54)

$$\|f\|_{\beta_\theta^{\varphi_j^2(\cdot)2\sigma:H}(L_p(G\mu))} \leq A(\sigma)\|f\|_{\beta_\theta^{\varphi_j,\sigma,H}\left(\beta_\theta^{\varphi_j,\sigma,H}(L_p(G\mu))\right)}$$

for $\mu \geq 0$. According to Lemma 1 (with $\mu_0 = \infty$) we

$$\|f\|_{\beta_\theta^{\varphi_j(\cdot),\sigma,H}\left(\beta_\theta^{\varphi_j^2(\cdot),2\sigma,H}(L_p(G\mu))\right)} \leq A(\sigma)\|f\|_{\beta_\theta^{\varphi_j(\cdot),\sigma,H}\left(\beta_\theta^{\varphi_j(\cdot),\sigma,H}\left(\beta_\theta^{\varphi_j(\cdot),\sigma,H}(L_p(G\mu))\right)\right)}$$

for all $\mu \geq 0$. Again for this by using (4.53) and (4.54)

$$\|f\|_{\beta_\theta^{\varphi_j^3(\cdot)3\sigma:H}(L_p(G\mu))} \leq A^2(\sigma)\|f\|_{\beta_\theta^{\varphi_j(\cdot),\sigma,H}\left(\beta_\theta^{\varphi_j(\cdot),\sigma,H}\left(\beta_\theta^{\varphi_j(\cdot),\sigma,H}(L_p(G\mu))\right)\right)}$$

So, for k = 2, and k = 3, it holds true. Therefore by mathematical induction it is true for all natural number k.

Lemma 4. 4 For any open set $\Omega \subseteq \Re$, $\Omega = \bigcup_1^s G_{(k)}$ where $s \in N$ or $s = \infty$ and $G_{(k)}$ is an open interval, and for $k \neq \ell$ $G_{(k)} \cap G_{(\ell)} = \{\ \}$

That is every open set of real numbers is a union of a countable collection of disjoint open intervals. (i.e. If O is open there exists $\{I_n\}_{n=1}^{\infty}$ open intervals such that $I_n \cap I_m = \{\ \}$ for $n \neq m$ and $O = \bigcup_{n=1}^{\infty} I_n$

Proof Let O be an open set. For each $x \in O$ let $bx = \sup\{y:(x,y) \subseteq O\} = T \neq \{\ \}$ and $a_x = \inf\{y: (y,x) \subseteq O\}$, $S = \{y: (y,x) \subseteq O\} \neq \{\ \}$

Let $Ix = (ax, bx)$

Claim 1 1) $I_x \subseteq O$, 2) $a_x, b_x \notin O$.

Proof

1) Let $w \in I_x$, then $w \geq x$ or $w < x$. If $w = x$ then $w \in O$.

Without loss of generality assume $w < x$. Since $a_x < w < x$ w is not a lower bound of $\{y: (y,x) \leq 0\}$. Hence there exist y such that $y < w$ and $(y, x) \subseteq O$.

Hence $(a_x - \varepsilon, x) \subseteq O$

$\Rightarrow$ $a_x - \varepsilon \in s$

$\Rightarrow$ $a_x - \varepsilon > a_x$ since $a_x = \inf S$ it is contradiction.

2. so $a_x \notin O$.

Similarly $b_x \notin O$.

From the above $I_x \subseteq 0$ for $\forall x \in O$

$$\Rightarrow \bigcup_{x \in O} I_x \subseteq O$$

and for each $x \in 0$, then $\exists I_x : x \in I_x$

$$\therefore O = \bigcup_{x \in O} I_x$$

Claim $I_x \cap I_y = \phi$ or $I_x = I_y$

Proof suppose $c \in I_x \cap I_y$

let $I_x = (a_x, b_x)$ and $I_y = (a_y, b_y)$ since $c \in (I_x \cap I_y)$ $a_y < c < b_x$ and $a_x < c < b_y$

from the above $a_x \notin O$

$\Rightarrow \quad a_x \notin (a_y, b_y)$ and $a_x < b_y$

$\Rightarrow \quad a_x \leq a_y$

similarly $a_y \notin (a_x, b_x)$ and $a_y < b_x$

$\Rightarrow \quad a_y \leq a_x$

$\Rightarrow \quad a_x = a_y$

similarly bx = by

$\therefore$ If $I_x \cap I_y \neq \phi$ then $I_x = I_y$.

So In the collection $\{I_x, x \in 0\}$ If $x, y \in 0$ and $x \neq y$ then $I_x \cap I_y = \phi$

Let $E = \{I_x : x \in O\}$ be the collection of all distinct intervals from $\{I_x : x \in O\}$

If $I_x, I_y \in E$ then $x \neq y \Rightarrow I_x \cap I_y = \theta$. and $O = \bigcup_{I \in E} I$

For each $I \in E$ pick a rational number $r_I \neq r_j$ consider $A = \{r_I, I \in E\}$. A is a countable set define a function $f: A \to E$ by $f(r_I) = I$

f is 1 – 1 and onto

$\Rightarrow \quad A \sim c$ and A is countable

$\Rightarrow \quad$ E is countable

$\Rightarrow \quad E = \{I_n\}_{n=1}^{\infty}$ and $0 = \bigcup_{n=1}^{\infty} I_n$

Remark Let $\Omega = \bigcup_1^s G(k)$, $\{G_{(k)}\}_1^s$ is a collection of pair wise disjoin sets.

$$\int_{\Omega} f(x)dx = \int_{\bigcup_1^s G_{(k)}} f(x) = \int_{G_{(1)}} f(x)dx + \int_{G_{(2)}} f(x)dx + \ldots + \int_{G_{(s)}} f(x)dx$$

$$= \sum_{k=1}^{s} \int_{G(k)} f(x)dx$$

Corollary4. 4 Let $\Omega \subset \Re$ be any arbitrary open set. Then under the condition of Lemma 3.

$$\|f\|_{B_\theta^{\varphi_1\varphi_2}(L_p(\Omega))} \leq \|f\|_{B_\theta^{\varphi_1}(B_\theta^{\varphi_2}(L_p(\Omega)))} \text{ with constant } C_1 \text{ as Lemma 4. 3} \quad (4.56)$$

Proof

By Lemma 4. 4, $\Omega_{\sigma h} = \bigcup_1^s G_{(k)\sigma h}$, $G_{(k)\sigma h}$ is any open interval in $\Re$, and for $k \neq \ell$, $G_{(k)\sigma h} \cap G_{(\ell)\sigma h} = \phi$ By using this and the above remark

$$\left[\left(\int_{\Omega_{\sigma h}} \left|\Delta_h^\sigma f\right|^p dx\right)^{1/p}\right]^p = \int_{\Omega_{\sigma h}} \left|\Delta_h^\sigma f\right|^p dx = \sum_{k=1}^s \int_{G(k)\sigma h} \left|\Delta_h^\sigma f\right|^p dx$$

$$= \sum_{k=1}^s \left[\left(\int_{G(k)\sigma h} \left|\Delta_h^\sigma f\right|^p dx\right)^{1/p}\right]^p$$

$$\Rightarrow \quad \left\|\Delta_h^\sigma f\right\|_{L_p(\Omega_{\sigma h})}^p = \sum_{k=1}^s \left\|\Delta_h^\sigma f\right\|_{L_p(G_{(k)\sigma h})}^p$$

$$\Rightarrow \quad \left\|\Delta_h^\sigma f\right\|_{L_p(\Omega_{\sigma h})}^p = \left(\sum_{k=1}^s \left\|\Delta_h^\sigma f\right\|_{L_p(G_{(k)\sigma h})}^p\right)^{1/p}$$

$$\leq \quad \left(\sum_{k=1}^s \left(A(\sigma)\frac{1}{\varphi(h)}\int_0^h \left\|\Delta_\eta^\sigma f\right\|_{L_p(G_{(k)\sigma\eta}}^p d\eta\right)^p\right)^{1/p}$$

$$= \quad A(\sigma)\frac{1}{\varphi(h)}\left(\sum_{k=1}^s \left(\int_0^h \left\|\Delta_\eta^\sigma f\right\|_{L_p(G_{(k)\sigma\eta}}^p d\eta\right)^p\right)^{1/p}$$

$$= \quad A(\sigma)\,\varphi^{-1}(h)\left\|\int_0^h \Phi_k(\eta)d\eta\right\|_{\ell p}$$

where $\Phi_k(\eta) := \left\|\Delta_\eta^\sigma f\right\|_{L_p(G_{(k)\sigma\eta})}$ and

$$\left(\sum_{k=1}^s \left(\int_0^h \Phi_k(\eta)d\eta\right)^p\right)^{1/p} = \left\|\int_0^h \Phi_k(\eta)d\eta\right\|_{\ell p}$$

Using a consequence of the generalized Minkolski inequality for integrals

$$\left\|\int_0^h \Phi_k(\eta)d\eta\right\|_{\ell p} \leq \int_0^h \left\|\phi_k(\eta)\right\|_{\ell p} d\eta$$

From this we obtain that

$$\left\|\Delta_h^{\sigma} f\right\|_{L_p(\Omega\sigma h)} \leq A(\sigma)\, \varphi^{-1}(h) \int_0^h \left\|\phi_k(\eta)\right\|_{\ell p} d\eta$$

$$= A(\sigma)\varphi^{-1}(h) \int_0^h \left(\sum \left\|\Delta_\eta^{\sigma} f\right\|_{L_p(G(k))\sigma\eta}\right)^{1/p} d\eta$$

$$= A(\sigma)\varphi^{-1}(h) \int_0^h \left\|\Delta_\eta^{\sigma} f\right\|_{L_p(\Omega\sigma\eta)} d\eta$$

So $\left\|\Delta_h^{\sigma} f\right\|_{L_p(\Omega\sigma h)} \leq A(\sigma)\varphi^{-1}(h) \int_0^h \left\|\Delta_\eta^{\sigma} f\right\|_{L_p(\Omega\sigma\eta)} d\eta$

After this by replacing $G_{\sigma h}$ by $\Omega_{\sigma h}$ we get

a) $\left\|\Delta_h^{\sigma_1\sigma_2} f\right\|_{L_p(G\sigma_1+\sigma_2)h)} \leq A)\sigma) \left\|\left\|\Delta_h^{\sigma} f\right\|_{L_p(\Omega\sigma h)}\right\|_{L_\theta^*(0,h)}$

b) $\left\|\Delta_h^{\sigma_1\sigma_2} f\right\|_{L_p(G\sigma_1+\sigma_2)h)} \leq A(\sigma_1)\,\varphi_2^{-1}(h) \left\|\varphi_1^{-1}(\eta)\right\| \Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\Big\|_{L_p(\Omega\sigma_1\eta+\sigma_2 h)}\Big\|_{L_\theta^*(0,H)}$

From these we get

$$\left\|f\right\|_{\beta_{\theta,p}^{\varphi 1\varphi 2,\sigma 1+\sigma 2,H}}(\Omega) \leq A(\sigma_1) \left\|f\right\|_{\beta_\theta^{\varphi 1}\left(\beta_\theta^{\varphi 2,\sigma 2,H}\left(L_p(\Omega)\right)\right)}$$

$\therefore \left\|f\right\|_{\beta_\theta^{\varphi 1\varphi 2}\left(L_p(\Omega)\right)} \leq C_1 \left\|f\right\|_{\beta_\theta^{\varphi 1}\left(\beta_\theta^{\varphi 2}\left(L_p(\Omega)\right)\right)}$

<u>Lemma 4.5</u>

Let $1 < p < \infty$, $1 \leq \theta \leq \infty$ $\vec{\varphi} \in \Phi(\vec{\sigma}, \theta)$,then

$$\|f\|_{\mathscr{B}_\theta^{\vec{\varphi}}(\mathscr{B}_\theta^{\vec{\varphi}}(L_p(\Re^n)))} \sim \|f\|_{\mathscr{B}_\theta^{\vec{\varphi}^2}(L_p(\Re^n))} \qquad (4.57)$$

Proof:

Consider function f $\qquad f = \sum_{k=0}^{\infty} f * \vartheta_k \qquad (4.58)$

where ϑ_k is a function such that Supp $\mathbf{F}\vartheta_k \subset P_{k+1} \setminus P_k$

$P_k = \{ \xi \in \Re^n : |\xi_j| < \nu_j(k)$ and $\nu_j(k): \varphi_j(\nu_j(k)) = 2^k \}$

and we use the fact that

$$\|f\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)} \sim \left(\sum_{k=0}^{\infty} (2^k \|f * \vartheta_k\|_{L_p(\Re^n)})^{\theta} \right)^{\frac{1}{\theta}} \tag{4.59}$$

and for iterated norms

$$\|f\|_{B^{\bar{\varphi}}_{p,\theta}(B^{\bar{\varphi}}_{p,\theta}(\Re^n))} \sim \left(\sum_{k=0}^{\infty} (2^k \|f * \vartheta_k\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)})^{\theta} \right)^{\frac{1}{\theta}} \tag{4.60}$$

by definition we have

$$\|f\|_{B^{\bar{\varphi}}_{p,\theta}(B^{\bar{\varphi}}_{p,\theta}(\Re^n))} = \|f\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)} + \sum_{j=1}^{n} \left\| \varphi_j^{-1}(h) \left\| \Delta_{h,j}^{\sigma_j} f \right\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)} \right\|_{L^*_\theta(0,\infty)} \tag{4.61}$$

Let $S_1 = \|f\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)}$ and

$$S_2 = \sum_{j=1}^{n} \left\| \varphi_j^{-1}(h) \left\| \Delta_{h,j}^{\sigma_j} f \right\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)} \right\|_{L^*_\theta(0,\infty)}$$

Then from (4.59)

$$S_1 \sim \left(\sum_{k=0}^{\infty} (2^k \|f * \vartheta_k\|_{L_p(\Re^n)})^{\theta} \right)^{\frac{1}{\theta}} \quad \text{and}$$

$$S_2 = \sum_{j=1}^{n} \left\| \varphi_j^{-1}(h) \left\| \Delta_{h,j}^{\sigma_j} f \right\|_{B^{\bar{\varphi}}_{p,\theta}(\Re^n)} \right\|_{L^*_\theta(0,\infty)} \sim$$

$$\sum_{j=1}^{n} \left\| \varphi_j^{-1}(h) \left(\sum_{k=0}^{\infty} \left(2^k \left\| \Delta_{h,j}^{\sigma_j} (f * \vartheta_k) \right\|_{L_p(\Re^n)} \right)^{\theta} \right)^{\frac{1}{\theta}} \right\|_{L^*_\theta(0,\infty)}$$

$$= \sum_{j=1}^{n}\left\{\int_0^\infty \varphi_j^{-\theta}(h)\left(\sum_{k=0}^{\infty}\left(2^k\left\|\Delta_{h,j}^{\sigma_j}\left(f*\vartheta_k\right)\right\|_{L_p(\Re^n)}\right)^\theta\right)\frac{dh}{h}\right\}^{\frac{1}{\theta}}$$

$$\sim\left(\sum_{K=0}^{\infty}2^{K\theta}\left(\sum_{j=1}^{n}\left\|\varphi_J^{-1}(h)\left\|\Delta_{h,j}^{\sigma_j}\left(f*\vartheta_k\right)\right\|_{L_p(\Re^n)}\right\|_{L_\theta^*(0,\infty)}\right)^\theta\right)^{\frac{1}{\theta}}$$

$$=\left(\sum_{K=0}^{\infty}\left(2^k\sum_{j=1}^{n}\left\|\varphi_J^{-1}(h)\left\|\Delta_{h,j}^{\sigma_j}\left(f*\vartheta_k\right)\right\|_{L_p(\Re^n)}\right\|_{L_\theta^*(0,\infty)}\right)^\theta\right)^{\frac{1}{\theta}}$$

then

$$S_1+S_2 \sim \left(\sum_{k=0}^{\infty}\left(2^k\|f*\vartheta_k\|_{L_p(\Re^n)}\right)^\theta\right)^{\frac{1}{\theta}} +$$

$$\left(\sum_{K=0}^{\infty}\left(2^k\sum_{j=1}^{n}\left\|\varphi_J^{-1}(h)\left\|\Delta_{h,j}^{\sigma_j}\left(f*\vartheta_k\right)\right\|_{L_p(\Re^n)}\right\|_{L_\theta^*(0,\infty)}\right)^\theta\right)^{\frac{1}{\theta}}$$

$$\sim\left(\sum_{K=0}^{\infty}2^{k\theta}\left[\|f*\vartheta_k\|^\theta_{Lp(\Re^n)}+\left(\sum_{j=1}^{n}\left\|\varphi_J^{-1}(h)\left\|\Delta_{h,j}^{\sigma_j}\left(f*\vartheta_k\right)\right\|_{L_p(\Re^n)}\right\|_{L_\theta^*(0,\infty)}\right)^\theta\right]\right)^{\frac{1}{\theta}}$$

$$\sim\left(\sum_{K=0}^{\infty}2^{k\theta}\left[\|f*\vartheta_k\|_{Lp(\Re^n)}+\sum_{j=1}^{n}\left\|\varphi_J^{-1}(h)\left\|\Delta_{h,j}^{\sigma_j}\left(f*\vartheta_k\right)\right\|_{L_p(\Re^n)}\right\|_{L_\theta^*(0,\infty)}\right]^\theta\right)^{\frac{1}{\theta}}$$

$$\sim\left(\sum_{k=0}^{\infty}\left(2^k\|f*\vartheta_k\|_{B^{\bar\varphi}_{p,\theta}(\Re^n)}\right)^\theta\right)^{\frac{1}{\theta}}$$

Since $\vartheta_k*\vartheta_m=0$ for $m\neq k$ and $\vartheta_k*\vartheta_k=\vartheta_k$ we have the following:

$$\left\| f * \vartheta_k \right\|_{B^{\vec{\varphi}}_{p,\theta}} \sim \left(2^m \left\| f * \vartheta_k * \vartheta_m \right\|_{L_p(\Re^n)}\right)^{\theta \frac{1}{\theta}} =$$

$$\left(2^k \left\| f * \vartheta_k \right\|_{L_p(\Re^n)}\right)^{\theta \cdot \frac{1}{\theta}} = 2^k \left\| f * \vartheta_k \right\|_{L_p(\Re^n)} .$$

Furthermore

$$\left(\sum_{k=0}^{\infty} 2^{k\theta} \left\| f * \vartheta_k \right\|^{\theta}_{B^{\vec{\varphi}}_{p,\theta}(\Re^n)} \right)^{\frac{1}{\theta}} = \left(\sum_{k=0}^{\infty} 2^{k\theta} \cdot 2^{k\theta} \left\| f * \vartheta_k \right\|^{\theta}_{L_p(\Re^n)} \right)^{\frac{1}{\theta}}$$

$$= \left(\sum_{k=0}^{\infty} \left(2^k \cdot 2^k \left\| f * \vartheta_k \right\|^{\theta}_{L_p(\Re^n)} \right)^{\theta} \right)^{\frac{1}{\theta}} = \left(\sum_{k=0}^{\infty} \left(2^{2k} \left\| f * \vartheta_k \right\|^{\theta}_{L_p(\Re^n)} \right)^{\theta} \right)^{\frac{1}{\theta}}$$

$$\sim \left\| f \right\|_{B^{\vec{\varphi}^2}_{p,\theta}(\Re^n)}$$

Corollary 4. 6

$$\left\| f \right\|_{B^{\vec{\varphi}}_{p,\theta}\left(B^{\vec{\varphi}}_{p,\theta}(\cdots B^{\vec{\varphi}}_{p,\theta}(\Re^n)\right)} \sim \left\| f \right\|_{B^{\vec{\varphi}^k}_{p,\theta}\left(\Re^n\right)}$$

<u>**Proof**</u> when k = 2 it is proved on Lemma 4.5. For any natural number k > 2, it can be proved by mathematical induction.

Proof of theorem 1

1) According to corollary (4.3) for every open parallepiped $G \subset \Re^n$ with faces parallel to coordinate planes and $\forall j \in \{1, 2, \ldots n\}$

$$\left\| f \right\|_{\beta^{\varphi_j^k}_{\theta}\left(L_p(G)\right)} \leq C_1 \left\| f \right\|_{\beta^{\varphi_j}_{\theta}\left(\cdots \beta^{\varphi_j}_{\theta}\left(L_p\left(\Re^n\right)\right)\right)}$$

Summing up these inequalities and adding $\|f\|_{L_p(G)}$ to both sides, we obtain the desired inequality, taking into account that the sum on the right -hand side doesn't exceed $\|f\|_{\mathcal{B}^{\vec{\varphi}}_{\theta}\left(\cdots\mathcal{B}^{\vec{\varphi}}_{\theta}(L_p(G))\cdots\right)}$

$$\Rightarrow \qquad \|f\|_{B^{\vec{\varphi}}_{p,\theta}(G)} \le C_1 \, \|f\|_{B^{\vec{\varphi}}_{p,\theta}\left(B^{\vec{\varphi}}_{p,\theta}\cdots B^{\vec{\varphi}}_{p,\theta}(G)\cdots\right)}$$

$$\Rightarrow \qquad B^{\vec{\varphi}}_{p,\theta}\left(\cdots B^{\vec{\varphi}}_{p,\theta}(G)\cdots\right) \subset\!> B^{\vec{\varphi}}_{p,\theta}(G)$$

2. By condition 5.20.1 there is a bounded extension operator

S: $B^{\vec{\varphi}^k}_{p,\theta}(G) \to B^{\vec{\varphi}^k}_{p,\theta}(\Re^n)$.

To prove existence of imbedding in the direction opposite to that of the imbedding in condition (5)

$$B^{\vec{\varphi}^k}_{p,\theta}(\Re^n) \subset\!> B^{\vec{\varphi}}_{p,\theta}\left(\cdots B^{\vec{\varphi}}_{p,\theta}(G)\cdots\right)$$

it is sufficient to carry out the proof for the case $G = \Re^n$. And it is carried out in Lemma 4.5

Therefore the theorem is proved.

Remarks

1. In these spaces if $\varphi_j(h) = h^{\ell_j}$ then we obtain the results which are described by Professor V.I Burenkov.

2. If $\vec{\varphi}(h)$ satisfy the following additional condition $\exists\varepsilon > 0$: $\dfrac{\varphi_j(h)}{h^{\varepsilon}}$ is increasing on $(0, H]$ (for example $\varphi_j(h) = h^{\ell_j}\,\ell_n\gamma_j\left(\dfrac{2H}{h}\right)$, $\alpha_j > 0$ $\gamma_j \in \Re$), that is the space $\mathcal{B}^{\vec{\varphi}}_{p,\theta}(L_p(G))$ possesses "exponential" reserve of smoothness.

Therefore the theorem ensures the increment of smoothness by interrelating norms, which permit to attain any index of smoothness of exponential order after finite steps.

Reference

1. V.I.BURENEKOV

   A theorem on iterated norms for Nikollskii-Besov space and its application, Proceedings of the Steklov Institute of Mathematics 1989, Issue 4.

2. Tsegaye Gedif AYELE

   About smoothness of solutions and condition of localization for spectral decomposition of operators with constant coefficients PhD dissertation in physico-matheamtical sciences Moscow, 1999

3. Tsegaye Gedif AYELE

   Spaces of Iterated norms in Nikollskii-Besov type with generalized smoothness. Moscow, 1999 – Dep V VINITI RAN 31.03.99 No 1027 – B99

4. Tsegaye Gedif AYELE

   Short notes on Nikollskii-Besov spaces.