Numerical Modelling of Thin Plates using the Finite Element Method

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Table of Contents

List of Figures

Abstract

In the advancement of digital computer it is becoming common to model a certain system to get a deep insight of the existing conditions. In this thesis Kirchoff's thin plate theory is considered to study static behaviour of a plate.

Based on the assumption of the theory a finite element computer program is developed in FORTRAN using a linear triangular element. This thesis is mainly concerned in validating the numerical result obtained from the computer program. Hence the result is compared with analytical solution and with a general finite element program called Ansyis for a specific problem. Finally, recommendation is made on the selection of finite element meshes for future analyses.

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This paper is the result of my academic years at Addis Ababa University, in the computational science program. And as such many people have contributed tremendously to this work in different ways.

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List of symbols:

a	length of plate
`b	width of plate
E	Young's Modulus
E_T	Tangent Modulus
ν	Poisson's ratio
σ_Y	Yield Stress
D	Flexural Rigidity
k_x	Curvature at midsurface in plane xz
k_y	Curvature at midsurface in plane yz
k_{xy}	Curvature at midsurface in plane xy
t	Thickness of plate
ε_x	Strain in the x-direction
ε_y	Strain in the y-direction
γ_{xy}	Shear-strain between x and y axes
w	Magnitude of plate displacement
$[K]$	Stiffness matrix
$[Q]$	Nodal force matrix
$[D]$	Elasticity matrix (relates stress to strain)
$[B]$	Matrix that maps strains (function of element)

Chapter One - Introduction

Mathematical modeling is the process by which one encounter a real-world situation, pose a question regarding the situation, and use mathematics to seek an answer to that question. Mathematical modeling involves identifying the relevant features of a real-world situation, representing those features symbolically, analyzing the model, and considering the accuracy and limitations of that model [1].

In the advancement of digital computer it is becoming common to model a certain system to get a deep insight of the existing conditions. In this thesis plate is considered as a system that posses certain distinct characteristic. For instance, its load-carrying action is similar to a certain extent to that of beam or cables thus, plates can be approximated by a grid work of an infinite number of beams or by a network of an infinite number of cables, depending on the flexural rigidity of the structures. This two-dimensional structural action of plates results in lighter structures, and therefore offers numerous economic advantages [2].

The plate, being originally flat, develops shear forces, bending and twisting moments to resist transverse loads. Because the loads are generally carried in both directions and the twisting rigidity in isotropic plates is quite significant, a plate is considerably stiffer than a beam of comparable span and thickness. So, thin plates combine light weight and form efficiency with high load-carrying capacity, economy, and technological effectiveness [2].

Because of the distinct advantages discussed above, thin plates are extensively used in all fields of engineering. Here are some of the application areas of thin plate, in architectural structures, bridges, hydraulic structures, pavements, containers, airplanes, missiles, ships, instruments and machine parts.

To study this useful structural material one can consider a plate as special three-dimensional solid where its thickness is very small when compared with other dimensions. Hence a complete three dimensional numerical treatment is not only costly but in addition often leads to serious numerical ill-conditioning problems [1]. To ease the solution, even long before numerical approaches became possible, several classical assumptions regarding the behavior of such structures were introduced. Thus numerical treatment will concern itself with the approximation to an already approximate theory. In which the validity of the solution heavily depend on the choice of numerical method.

Hence, in this thesis implementation of a solution from analytical and finite element method are considered for Kirchhoff's thin plate theory to study the static behavior of a plate.

1.1 Statement of the problem

In this thesis, the behavior of thin square plates of variable thickness under a concentrated and distributed load would be investigated. The maximum deflections from analytical as well as numerical methods were calculated.

Based on the assumption of thin plate theory a finite element program has been developed in which the solution obtained from this software is compared with analytical and a commercial software called ANSYS.

1.2 Objectives

The main objective of the thesis is to write software program that can be used to model a plate numerically using finite element method.

Specific Objectives

- ✓ Develop a computer program to get appropriate solution
- ✓ Compare the numerical and analytical solution
- ✓ Compare the results with that obtained from commercial FEM software
- ✓ Discuss about the result obtained for different thickness of the plate to observe how well the thin plate theory works

1.3 Significance of the study

The major significance of this thesis lies on the openness nature of the source code. It is running on linux operating system, this makes it powerful since linux is known to provide open source tools and libraries for the development as well as in the interpretation of the program. Moreover researchers can easily add or modify the source code which will make it more convenient for future use.

1.4 Organization of the thesis

This paper consists of five main chapters, in the first chapter it starts with general introduction of the thesis. This session also includes problem identification and specifications of objectives as well as description about the significance of the study are mentioned.

In chapter two literature reviews related to modeling a thin plate are listed. Significance of mathematical modeling and finite element procedures are mentioned in the first part of this session. And important concepts and classification to formulate a thin plate theory are discussed briefly.

The third chapter presents the mathematical modeling of the system. And implement analytical and finite element procedures to obtain the required expression to solve the governing equations.

In the fourth chapter discussion and result are presented. The results obtained from the developed software as well as from analytical solution and ANSYS software are compared and discussed.

Finally, chapter five draws some conclusions and recommendations for future work around this thesis.

Chapter Two- Literature Review

2.1 Modeling using Mathematics

The use of mathematics in solving real-world problems has become widespread in recent times. This is partly due to the use of the systems approach to problem solving, and to the increasing computational power of digital computers and computing methodology, both of which have made many more problems amenable to mathematical solution. There is hardly a branch of learning where mathematics and computing have made an impact[5].The steps involved in using mathematics to solve real-world problems are shown in fig 2.1

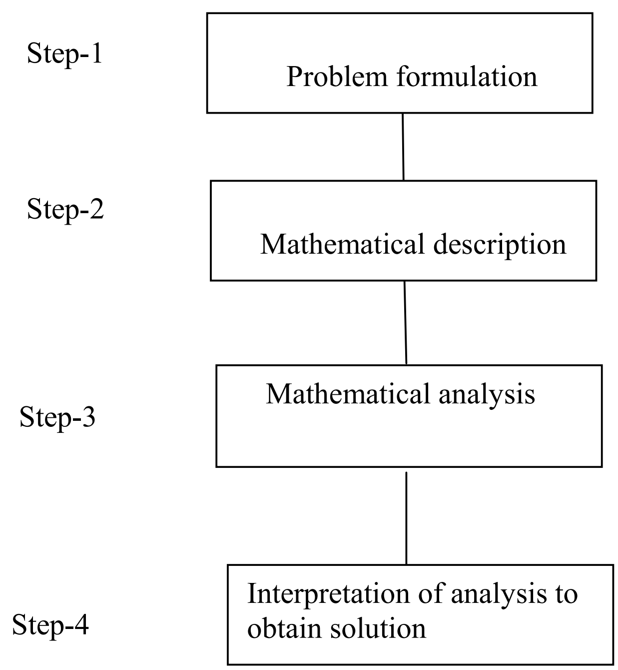

Fig 2.1 Problem solving using mathematics

The most crucial and important step is the satisfactory translation of the problem from the real physical world into a mathematical description. Once this is done, standard techniques of mathematical analysis can be used to obtain a solution to the problem. The validity of the solution depends on how well the mathematical description models the real world. The mathematical description is called mathematical model, and the process of obtaining it is called mathematical modeling.

The use of mathematics is one of many approaches to solving real-world problems. Others include experimentation either with scaled physical models or with the real world directly.

Mathematical modeling is the process by which a problem as it appears in the real world is interpreted and represented in terms of abstract symbols. This abstract description involving a mathematical formulation is called a mathematical model of the original problem. In this abstract form the problem is divorced from the real world and can be treated in mathematical terms only.

Mathematical modeling is an art as well as a science. The science aspect deals with the topics needed to execute the various steps in the modeling process. Because no two problems are the same in the real world, features such as creativity, intuition and foresight also play an important role in the modeling process. These features constitute the art aspect of mathematical modeling. This makes mathematical modeling a very challenging, and at the same times a very demanding, activity

Since the use of mathematics and computers for solving real-world problems is very widespread and has an impact in all branches of learning, hence it is adopted in this thesis to analysis an important structural material called plate.

2.2 Finite-element method

2.2.1 Background of Finite element method

The finite-element method (FEM) is regarded as relatively accurate and versatile numerical tool for solving differential equations that model physical phenomena [6]. The methodology is used in various areas of engineering in which the problems are modeled by partial differential equations. The method has found considerable application in structural engineering and related disciplines.

The finite element method is closely related to the classical variational concept of the Rayleigh-Ritz method [7, 8]. The modern finite element technique can be traced back to a paper in 1950 by Turner et al [8]. The technique was dubbed as the “finite element method” by Clough [10] and was further developed by Argyris [11]. The strong development of the method from the engineer's point of view has been led by Zienkiwicz and Taylor [12]. The mathematical theory of the finite elements has been developed and promoted by many scientists. Among them one can mention Strang and Fix [13], Babuska and Aziz [14], Oden and Reddy [15].

The plate-bending problem is one of the first problems where finite element was applied at early 1960s. Considerable effort has been devoted over the past two decades in devising efficient and accurate bending elements. The reviews compiled in References [16-20] provide a clear indication of the widespread interest elicited by the subject. They also illustrate the degree of difficulty which is inherent to formulating successful bending elements. The various difficulties encountered at that time and subsequently could not be overcome satisfactorily and this has made the research in this area alive in present days.

At the beginning, attempts have been made to develop thin-plate elements based on Kirchhoff's hypothesis where the difficulties are mostly concerned with the satisfaction of inter-elemental continuity requirement for transverse displacement w. It requires $C1$ continuity of transverse displacement (w) i.e., w and its derivatives should be continuous at the common edges between two elements. Again it has been established that the normal slope cannot be made continuous with the usual nodal unknowns i.e., $w, \frac{dw}{dx} and \frac{dw}{dy}$. In this context a number of non-conforming elements have developed, which do not satisfy the continuity requirement of normal slope. Although these elements violate the above continuity requirement but the performance of these elements are found to be very good in many cases.

2.2.2 Comparison of the FEM and FDM

Both methods, finite element method (FEM) and finite difference method(FDM) are widely used to provide approximate solution for partial differential equations (PDE). The differences between FEM and FDM are:

- The most attractive feature of the FEM is its ability to handle complicated geometries and boundaries with relative ease. While FDM in its basic form is restricted to handle rectangular shapes and simple alterations thereof, the handling of geometries in FEM is theoretically straightforward [4].

- The most attractive feature of finite differences is that it can be very easy to implement [4].

- There are several ways one could consider the FDM a special case of the FEM approach. One might choose basis functions as either piecewise constant functions or Dirac delta functions. In both approaches, the approximations are defined on the entire domain, but need not be continuous. Alternatively, one might define the function on a discrete domain, with the result that the continuous differential operator no longer makes sense, however this approach is not FEM [4].

- There are reasons to consider the mathematical foundation of the finite element approximation more sound, for instance, because the quality of the approximation between grid points is poor in FDM [4].

- The quality of a FEM approximation is often higher than in the corresponding FDM approach, but this is extremely problem dependent [4].

In this thesis FEM is used to obtain a solution as result the software developed based on this procedure can be easily applied for complicated geometries.

2.3 History of thin plate theory

Going back to history and tries to find the first mathematical statement regarding to plate was probably done by Euler, who in 1776 performed a free vibration analysis of plate problems [21]. Based on Euler J. Bernoulli [22] presented a plate as a system of mutually perpendicular strips at right angles to one another, each strip regarded as functioning as a beam. But the governing differential equation, as distinct from current approaches, did not contain the middle term.

The French mathematician Germain developed a plate differential equation that lacked the warping term [23]; by the way, she was awarded a prize by the Parisian Academy in 1816 for this work. Lagrange, being one of the reviewers of this work, corrected Germain's results (1813) by adding the missing term [24]; thus, he was the first person to present the general plate equation properly.

Cauchy [25] and Poisson [26] were first to formulate the problem of plate bending on the basis of general equations of theory of elasticity. In 1829 Poisson expanded successfully the Germain–Lagrange plate equation to the solution of a plate under static loading. In this solution, however, the plate flexural rigidity D was set equal to a constant term.

The first satisfactory theory of bending of plates is associated with Navier [27], who considered the plate thickness in the general plate equation as a function of rigidity D. He also introduced an ''exact'' method which transformed the differential equation into algebraic expressions by use of Fourier trigonometric series.

In 1850 Kirchhoff published an important thesis on the theory of thin plates [28]. In this thesis, Kirchhoff stated two basic assumptions that are now widely accepted in the plate-bending theory and are known as ''Kirchhoff's hypotheses''. Kirchhoff's theory contributed to the physical clarity of the plate bending theory and promoted its widespread use in practice. This theory is used in the thesis and will be discussed further in this chapter

2.4 Classical Thin-Plate Theory or Kirchhoff plate theory

The basic assumptions for the classical Kirchhoff's plate bending theory are very similar to those for the Euler-Bernoulli beam theory. Consider the differential slice cut from the plate by planes perpendicular to the x axis as shown in the figure below:

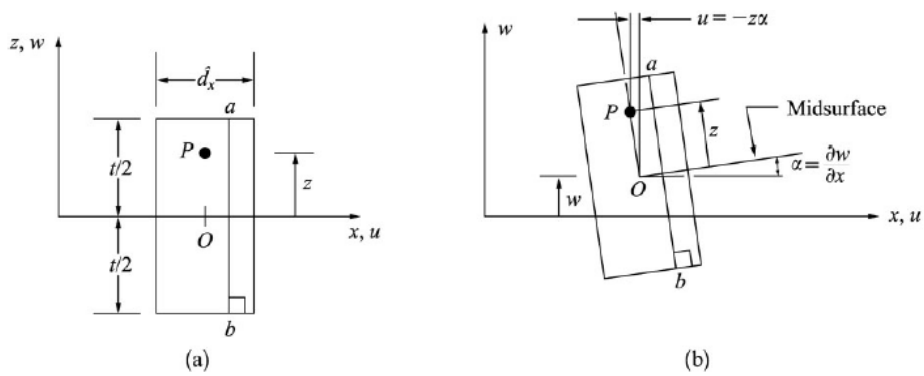

Fig. 2.2 differential slice cut of a plate [31]

Loading q causes the plate to deform laterally or upward in the z direction and, the deflection w of point P is assumed to be a function of x and y only; that is w=w(x,y) and the plate does not stretch in the z direction. The line connecting point a and b drawn perpendicular to the plate surface before loading remain perpendicular to the surface after loading. These conditions are consistent with the Kirchhoff assumptions:

1. Normals remain normal: This implies that transverse shears strains γ_{yz} =0 and γ_{xz} =0. However γ_{xy} does not equal to zero. Right angles in the plane of the plate may not remain right angles after loading. The plate may twist in the plane.
2. Thickness changes can be neglected and normals undergo no extension. This means that $\varepsilon_z = 0$.
3. Normal stress σ_z has no effect on in-plane strains ε_x and ε_y in the stress-strain equations and is considered negligible.
4. The plate is initially flat. Therefore, the in-plane deflections in the x and y directions at the midsurface, t=0 are assumed to be zero; u(x,y,0)=0 and v(x,y,0)=0.
5. The material of the plate is elastic, homogeneous and isotropic

These assumptions result in the reduction of a three-dimensional plate problem to a two-dimensional one. Consequently, the governing plate equation can be derived in a concise and straight-forward manner.

2.5 System Characterization

System characterization is the process of identifying the characteristic property of the system; in this case the system refers to a plate. Plates are common structures in engineering and they support transverse loads through bending action.

In this section different classifications and properties of plates are discussed and unique characteristics of the system will be identified.

2.5.1 Plates classification based on their boundary conditions

- ✓ Clamped, or built-in, or fixed edge
- ✓ Simply supported edge
- ✓ Free edge

2.5.2 Plates classification based on their shape

- ✓ Rectangular Plates
- ✓ Circular plates
- ✓ Elliptical Plates
- ✓ Sector-Shaped Plates
- ✓ Triangular Plates
- ✓ Skew Plates

2.5.3 Plates classification based on their thickness

One of the properties that affect the bending of a plate is the ratio between the length of a side and the thickness of the material. Based on these ratios one can identify the following classifications as shown in the figure:

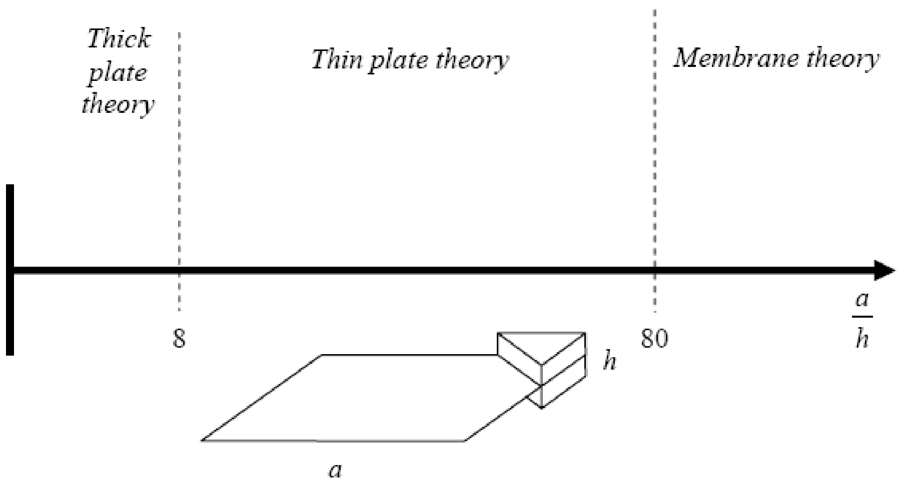

Fig 2.3 the distinguishing limits separating thick plate, thin plate and membrane theory [30]

The characterization of each stems from the ration between a given side of length a and the element's thickness, h. A plate is considered as thick if the ratio between the length of a side and the thickness is less eight. Similarly if the ratio is between eight and eighty it is taken as thin plate and if this ratio goes above 80, it will be perceived as a membrane. All of these different types of plates are characterized by the basic assumptions of their corresponding plate theory. Now a brief description about the assumptions will be discussed.

2.5.3.1 Thick plate theory

This theory considers the problem of plates as a three-dimensional problem of elasticity as a result stress analysis become more involved to find a solution.

2.5.3.2 Thin plate theory

If deflection of a plate are small in comparison with its thickness one can satisfactorily use assumptions of Kirchhoff's thin plate theory as discussed in the previous section.

2.5.3.3 Membrane theory

A plate to be considered as a true membrane, the following conditions should be satisfied

- ✓ The boundaries are free from transverse shear forces and moments. Loads applied to the boundaries must lie in planes tangent to the middle surface.

- ✓ The normal displacements and rotations at the edges are unconstrained: that is, these edges can displace freely in the direction of the normal to the middle surface
- ✓ A membrane must have a smoothly varying, continuous surface

2.6 Elasticity

Property of elasticity plays a vital role in the formulation of thin plate theory. And almost all engineering materials possess to a certain extent the property of elasticity. Hence, in this section a brief description of elasticity and some additional assumptions will be discussed.

A material is called perfectly elastic if they resume their initial form completely after a removal of the external forces. Furthermore a material can be considered as homogeneous if the matter of an elastic body is continuously distributed over its volume so that the smallest element cut from the body posses the same specific physical properties as the body. A material can also be taken as isotropic if the elastic properties are the same in all directions.

Structural materials do not satisfy the above properties completely. And they are found to contain crystals of various kinds and various orientations. And these materials are far from being homogeneous, but experience shows that solutions of the theory of elasticity based on the assumptions of homogeneity and isotropy can be applied to structures with very great accuracy.

Chapter Three - Mathematical Modeling

In this section appropriate mathematical relations are established based on the assumptions considered by thin plate theory. To meet the objective of the study two solution methods are implemented. One is analytical method where as the other one is finite element method. After obtaining the expressions using the aforementioned methods the results will be discussed in the next section. But the two solution methods use the same governing equation. Hence the first part of this section tries to formulate the equation in a Cartesian coordinates by taking into consideration the assumption given in Kirchhoff's theory.

3.1 Governing Equation for deflection of plates in Cartesian Coordinates

The foregoing assumptions of thin plate theory make it possible to drive the basic equations for thin plates. Accordingly a differential slice cut is considered from the plate by planes perpendicular to the x axis as shown in the fig.2.2 to drive the governing equation.

Based on Kirchhoff assumptions, at any point P displacement in the x direction due to a small rotation is

$$u = -z\alpha = -z\left(\frac{\partial w}{\partial x}\right) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..3.1$$

At the same point, the displacement in the y direction is:

$$v = -z\alpha = -z\left(\frac{\partial w}{\partial y}\right) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..3.2$$

The curvatures of the plate are then given as the rate of change of the angular displacements of the normal and defined as:

$$k_x = -\frac{\partial^2 w}{\partial x^2} \qquad k_y = -\frac{\partial^2 w}{\partial y^2} \qquad k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \dots\dots\dots\dots\dots\dots...3.3$$

Using the definitions for in-plane strains, along with curvature relationships, the in-plane strain/displacement equations are:

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \ldots\ldots\ldots\ldots\ldots 3.4$$

The first part of equation 3.4 is used in beam theory. The remaining two equations are new to plate theory.

Based on the third Kirchhoff assumption, the plane stress equations that relate in-plane stresses to in-plane strains for an isotropic material are:

$$\sigma_x = \frac{E}{1-v^2}\left(\varepsilon_x + v\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-v^2}\left(\varepsilon_y + \varepsilon_x\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.5$$

$$\tau_{xy} = G\gamma_{xy}$$

Where $G = \frac{E}{2(1+v)}$

The in-plane normal stresses and shear stress are shown acting on the edges of the plate as shown in the figure below:

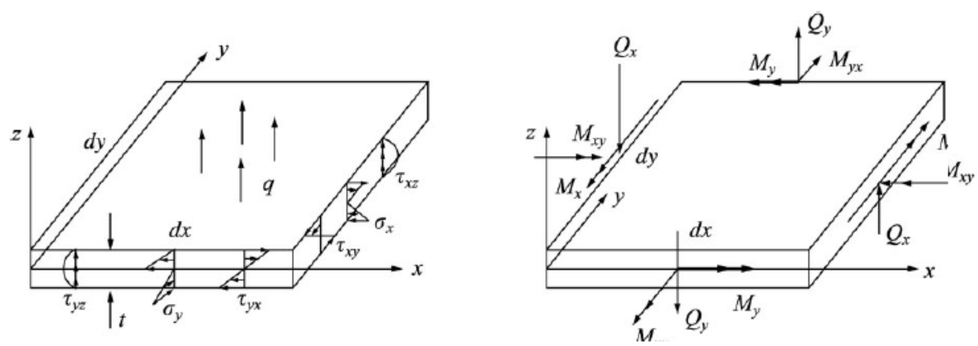

Fig 3.1 in plane normal stresses and shear stress [31]

Similar to the stress variation in a beam, the stresses vary linearly in the direction from the mid surface of the plate. The transverse shear stresses τ_{yz} and τ_{xz} are also present, even though transverse shear deformation is neglected. These stresses vary quadratically through the plate thickness.

The bending moments acting along the edge of the plate can be related to the stresses by:

$$M_x = \int_{-t/2}^{t/2} z\sigma_x dz$$

$$M_y = \int_{-t/2}^{t/2} z\sigma_y dz \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . 3.6$$

$$M_{xy} = \int_{-t/2}^{t/2} z\gamma_{xy} dz$$

Substituting strains for stresses gives:

$$M_x = \int_{-t/2}^{t/2} z\left(\frac{E}{1-v^2}\left(\varepsilon_x + v\varepsilon_y\right)\right) dz$$

$$M_x = \int_{-t/2}^{t/2} z\left(\frac{E}{1-v^2}\left(\varepsilon_y + v\varepsilon_x\right)\right) dz \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.7$$

$$M_{xy} = \int_{-t/2}^{t/2} zG\gamma_{xy} dz$$

Using the strain/curvature relationships, the moment expression become:

$$M_x = D\left(k_x + vk_y\right)$$

$$M_y = D\left(k_y + vk_x\right) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .. 3.8$$

$$M_{xy} = \frac{D(1-v)}{2} k_{xy}$$

Where $D = Et^3/[12(1-v^2)]$ is called the bending rigidity of the plate.

The equilibrium equations for plate bending are important in selecting the element displacement fields. The governing differential equations are:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0.\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.9$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

Where q is the transverse distributed loading and $Q_x$and Q_y are the transverse shear line loads as shown below.

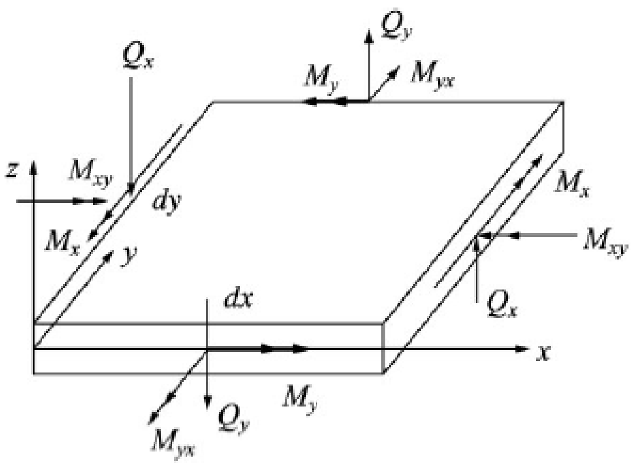

Fig 3.2 Transverse distributed load [31]

Substituting the moment/curvature expressions in the last two differential equations list above solving for $Q_x$and Q_y, and substituting the results into the first equation listed above, the governing partial differential equation for isotropic, thin-plate bending can be derived as:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.10$$

The solution of thin-plate bending is a function of the transverse displacement w. If the differentiation with respect to the y direction is neglected, the above equation simplifies to the equation for a beam and the flexural rigidity D of the plate reduces to the EI of the beam when the Poisson effect is set to zero.

3.2 Analytical Solution for rectangular thin plate

After thin plate equation is formulated one can get exact solution for different boundary and shape condition. Here Navier's method or double series solution is used to demonstrate the solution procedure for a particular problem where a concentrated force is applied on a simple supported rectangular thin plate. In addition to this the solution for distributed load and fixed plate is also considered.

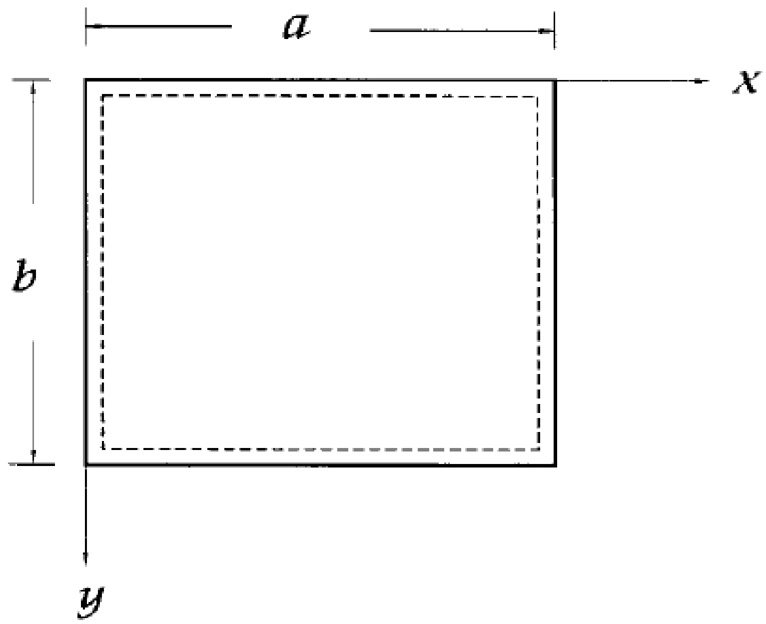

Fig 3.3 A plate of side a and b [29]

The boundary conditions for a simple supported rectangular plate of sides a and b are the following:

$$w = 0 \; and \; \frac{\partial^2 w}{\partial x^2} = 0 \; for \; x = 0 \; and \; x = a$$

$$w = 0 \; and \; \frac{\partial^2 w}{\partial y^2} = 0 \; for \; y = 0 \; and \; x = b \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 3.11$$

In this case a solution for the governing equation 3.10 can be assumed by infinite Fourier series for the deflection and distributed load as follows:

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi x}{b}$$

$$p(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi x}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 3.12$$

Where w_{mn} and p_{mn} represent coefficients to be determined. The expression assumed for the deflection can be easily showed to satisfy the prescribed boundary conditions.

Let us consider a general load configuration. To determine the Fourier coefficients p_{mn} one can multiply by $sin\frac{l\pi x}{a} sin\frac{k\pi y}{b}$ and integrated twice between the limits 0, a and 0, b.

$$\iint_0^\infty p(x,y) sin\frac{l\pi x}{a} sin\frac{k\pi y}{b} dxdy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \iint_0^\infty sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} sin\frac{l\pi x}{a} sin\frac{k\pi y}{b} dxdy \ldots \ldots 3.13$$

It can be shown by direct integration that

$$\int_0^a sin\frac{m\pi x}{a} sin\frac{l\pi x}{a} dx = \begin{Bmatrix} 0\ if\ m \neq l \\ \frac{a}{2}\ if\ m = l \end{Bmatrix}$$

$$\int_0^b sin\frac{n\pi y}{b} sin\frac{k\pi y}{b} dy = \begin{Bmatrix} 0\ if\ n \neq k \\ \frac{b}{2}\ if\ n = k \end{Bmatrix} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3.14$$

Using the property of equation 3.14, one can get a general expression for the coefficient of distributed load as shown below:

$$p_{mn} = \frac{4}{ab}\int_0^a\int_0^b p(x,y)\ sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} dxdy \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3.15$$

Using the assumed solution equation 3.15 and inserting them in the governing equation, one can obtain the following expression:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{w_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] - \frac{p_{mn}}{D}\right\} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} = 0 \ldots .3.16$$

After simplifying and rearranging one can solve for a coefficient of the deflection:

$$w_{mn}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{p_{mn}}{D} = 0$$

$$w_{mn} = \frac{1}{\pi^4 D}\frac{p_{mn}}{[(m/a)^2+(n/b)^2]^2} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 3.17$$

Inserting the expression obtained for coefficients of deflection and distributed load in equation 3.10 to get a solution for the governing equation for a general distributed load.

$$w(x,y) = \frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_{mn}}{[(m/a)^2+(n/b)^2]^2}\ sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \ldots \ldots \ldots \ldots 3.18$$

For a constant distributed load the expression in equation can be reduced further by specifying the general coefficient of distributed load

$$p_{mn} = \frac{4p_o}{\pi^2 mn}(1 - cosm\pi)(1 - cosn\pi)$$

$$p_{mn} = \frac{16p_o}{\pi^2 mn}(m, n = 1,3,5, \dots\dots) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.3.19$$

The following expression is a solution of a simple supported rectangular plate having side dimension of a and b after a constant load is applied

$$w = \frac{16p_o}{D\pi^6} \sum_{m=1,3,..}^{\infty} \sum_{n=1,3,....}^{\infty} \frac{sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}}{mn[(m/a)^2 + (n/b)^2]^2} \dots\dots\dots\dots\dots\dots\dots\dots.3.20$$

A detailed calculation to reduce equation 3.20 is clearly illustrated in Timoshenko's book [29] for a special case where the concentrated load is applied at the center of a simple supported square plate. Accordingly the maximum deflection for a square plate is given in equation 3.21

$$w_{max} = \alpha \frac{Pa^2}{D} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.21$$

Where α is 0.0116 for a square plate and P is the concentrated load. The value for α will be 0.0056 when the boundary condition changed to built in edges.

Similarly one can follow the detailed calculation presented in Timoshenko's book to obtain the maximum deflection for distributed load.

$$w_{max} = \alpha \frac{qa^4}{D} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.22$$

The value of α depends on the boundary condition of square plate as in the previous equation. As a result α will be 0.00406 for simple supported and 0.00126 for built on edges. And q is the value of distributed load.

3.3 FINITE ELEMENT FORMULATION

In the previous sections the basic assumptions of thin plate theory is discussed. And it is observed that the equation can be formulated in terms of single variable w as shown in equation 3.10. However a solution for such equation can only be available for a plate having special type of geometry. In order to overcome the limitation one can use finite element method. Here theory of elasticity is applied to obtain appropriate finite element formulation. Hence one can start from equation of equilibrium by considering the free body diagram of infinitesimal element as shown in the fig. 3.4.

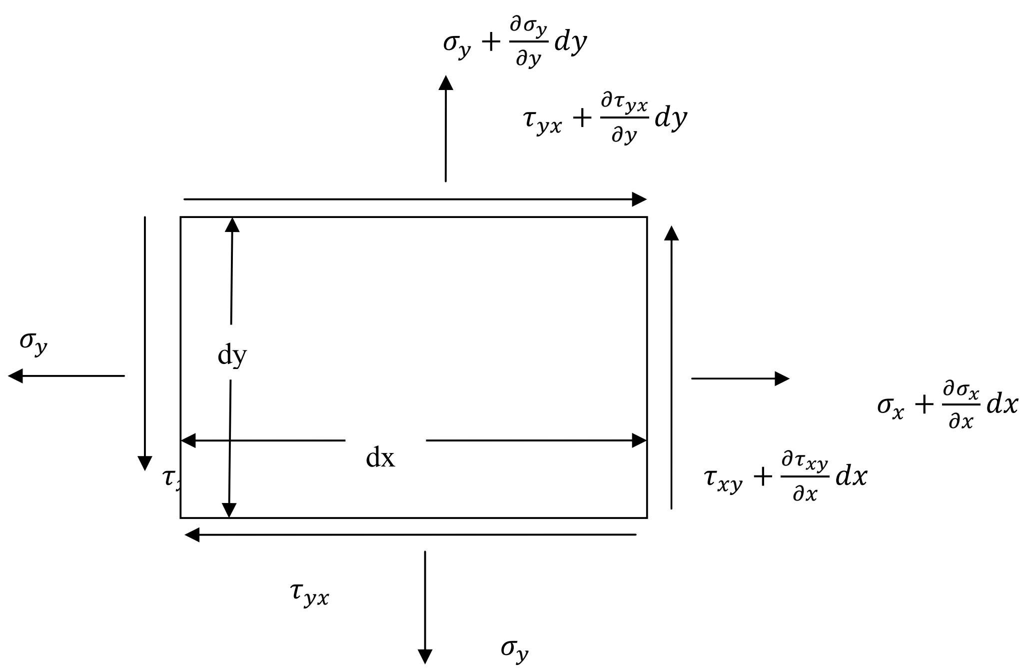

Fig. 3.4 a free body diagram of infinitesimal element

Summation of forces in the horizontal and vertical axes become

$$\sum F_x = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy - \sigma_x dy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dy\right) dx - \tau_{xy} dx + f_x dxdy = 0$$
$$\sum F_y = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dy - \tau_{xy} dy + \left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy\right) dx - \sigma_y dx + f_y dxdy = 0 \ldots 3.23$$

Where f_x and f_y are body forces per unit area (or per unit volume assuming unit thickness perpendicular to the plane) in the x and y-axes which are assumed to be positive when acted along the positive axis. All the stress components in fig. 3.4 are shown as positive. Simplifying these expressions yield equation of equilibrium as given below

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.24$$

After obtaining equilibrium equation one can use constitutive equations to get relationship between the stresses and strains.

For an isotropic material, the constitutive equation becomes

$$\{\sigma\} = [D]\{\varepsilon\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.25$$

Where $\{\sigma\} = \{\sigma_x \quad \sigma_y \quad \tau_{xy}\}^T$ denotes the stress

$\{\varepsilon\} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T$ denotes the strain

$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v^2}{2} \end{bmatrix}$ denotes the material property

Furthermore one can recall kinematic equations, which relate strains to displacements are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.26$$

Where u and v are displacements in the x and y directions respectively. In addition to body forces one should also consider boundary conditions which are either essential (or geometric or natural or traction) types. Essential conditions are prescribed displacements and natural boundary conditions are prescribed tractions which are expressed as

$$\Phi_x = \sigma_x n_x + \tau_{xy} n_y$$
$$\Phi_y = \tau_{xy} n_x + \sigma_y n_y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.27$$

Where n_x and n_y are direction cosines of the outward unit normal vector at the boundary and Φ is the given traction value.

In order to develop the finite element formulation one can apply the weighted residual method to equation 3.24 and writing them together give

$$\int_\Omega \begin{Bmatrix} w_1\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) \\ w_2\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right) \end{Bmatrix} d\Omega + \int_\Omega \begin{Bmatrix} w_1 f_x \\ w_2 f_y \end{Bmatrix} d\Omega - \int_{\Gamma_n} \begin{Bmatrix} w_1 \Phi_x \\ w_2 \Phi_y \end{Bmatrix} d\Gamma = 0 \ldots\ldots 3.28$$

Where Γ_e is the boundary for essential condition and w_i $(i = 1,2)$ is the weighting function. Applying integration by parts to the terms in the first integral in equation 3.28 yields

$$-\int_\Omega \begin{Bmatrix} \frac{\partial w_1}{\partial x}\sigma_x + \frac{\partial w_1}{\partial y}\tau_{xy} \\ \frac{\partial w_2}{\partial x}\tau_{xy} + \frac{\partial w_2}{\partial y}\sigma_y \end{Bmatrix} d\Omega + \int_\Omega \begin{Bmatrix} w_1 f_x \\ w_2 f_y \end{Bmatrix} d\Omega + \int_{\Gamma_n} \begin{Bmatrix} w_1 \Phi_x \\ w_2 \Phi_y \end{Bmatrix} d\Gamma = 0 \ldots\ldots 3.29$$

Where Γ_n is the boundary for natural conditions as described in equation 3.29 and can be rewritten as

$$\int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & 0 & \frac{\partial w_1}{\partial y} \\ 0 & \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} d\Omega = \int_{\Omega} \begin{Bmatrix} w_1 f_x \\ w_2 f_y \end{Bmatrix} d\Omega + \int_{n} \begin{Bmatrix} w_1 \Phi_x \\ w_2 \Phi_y \end{Bmatrix} d\Gamma \ldots \ldots 3.30$$

Substitution of the constitutive equation into equation 3.28 results in

$$\int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & 0 & \frac{\partial w_1}{\partial y} \\ 0 & \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} d\Omega = \int_{\Omega} \begin{Bmatrix} w_1 f_x \\ w_2 f_y \end{Bmatrix} d\Omega + \int_{\Gamma_n} \begin{Bmatrix} w_1 \Phi_x \\ w_2 \Phi_y \end{Bmatrix} d\Gamma = 0 \ldots\ldots 3.31$$

Using the relationship between deflection and elasticity into equation 3.31 to yield equation 3.32:

$$\int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & 0 & \frac{\partial w_1}{\partial y} \\ 0 & \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} [D] \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} d\Omega = \int_{\Omega} \begin{Bmatrix} w_1 f_x \\ w_2 f_y \end{Bmatrix} d\Omega + \int_{\Gamma_n} \begin{Bmatrix} w_1 \Phi_x \\ w_2 \Phi_y \end{Bmatrix} d\Gamma = 0 \ldots\ldots 3.32$$

Considering the assumption taken in Kirchhoff's theory for small deflection of plate given in equation 3.1 and 3.2 and inserting them in equation 3.32 one can obtain:

$$\int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & 0 & \frac{\partial w_1}{\partial y} \\ 0 & \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} [D] \begin{Bmatrix} -z \frac{\partial^2 w}{\partial x^2} \\ -z \frac{\partial^2 w}{\partial y^2} \\ -2z \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} d\Omega = \int_{\Omega} \begin{Bmatrix} w_1 f_x \\ w_2 f_y \end{Bmatrix} d\Omega + \int_{\Gamma_n} \begin{Bmatrix} w_1 \Phi_x \\ w_2 \Phi_y \end{Bmatrix} d\Gamma = 0 \ldots 3.33$$

Now one can use appropriate shape function to obtain stiffness matrix and force vector. Hence a general discritization function $w = Nd$ is used to obtain standard displacement equations

$$[K]\{d\} = \{F\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.34$$

Where {d} are appropriate parameters.

And the stiffness matrix can be expressed as

$$[K^e] = \int_{\Omega^e} [B]^T [D][B] d\,\Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.35$$

Where $B = \left[\frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^2}, 2\frac{\partial^2}{\partial x \partial y}\right]^T N$

N is the appropriate shape function

And the force vector can be expressed as

$$\{F\} = \int_{\Omega^e} [N]^T \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} + f_b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.36$$

f_b is the boundary contribution.

The basic idea of the finite element method is piece wise approximation. That is a solution of a complicated problem obtained by dividing the region of interest into small regions (finite elements) and approximating the solution over each sub-region by a simple function. In this section a polynomial function is used to obtain expression for triangular elements.

Consider a triangular element shown in fig 3.5. At each node displacements are introduced, the transverse displacement w and the slopes (rotations) about the x and y axes $[(\partial w/\partial y)\ and -(\partial w/\partial x)]$ are taken as degrees of freedom. The minus sign for the third degree of freedom indicates that if a positive displacement dw is taken at a distance dx from node one the rotation dw/dx about the y axis at node 1 will be opposite to the direction of the degree of freedom q_3 indicated in fig. 3.5

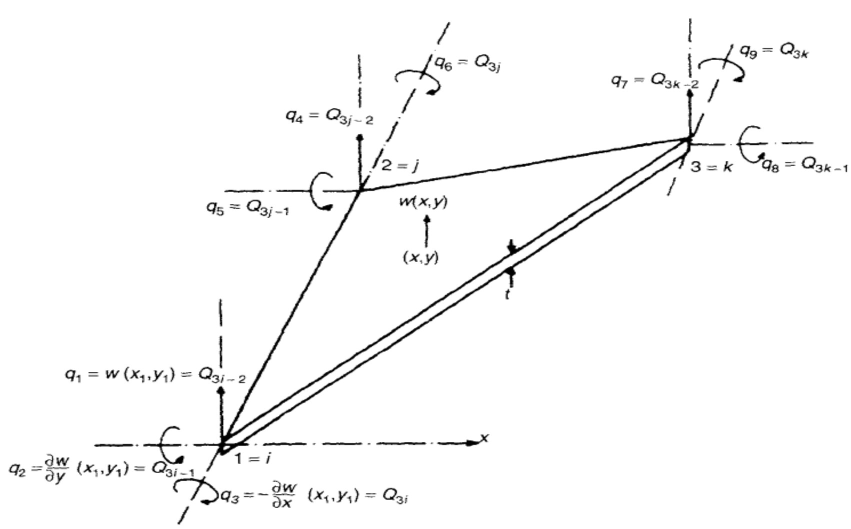

Fig. 3.5 Nodal degrees of freedom of a triangular plate [3]

Since there are nine displacement degrees of freedom in the element, the assumed polynomial $w(x, y)$ must also contain nine constant terms. To maintain geometric isotropy, the displacement model is taken as

$$w(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 (x^2 y + y^2 x) + \alpha_9 y^3$$
$$=[\eta]\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.37$$

Where $[\eta] = [\ 1\ \ x\ \ y\ \ \ x^2\ \ xy\ \ y^2\ \ x^3\ \ (x^2 y + xy^2)\ y^3\ ]$

$$\alpha = \begin{Bmatrix} \alpha_1 \\ \alpha_1 \\ \cdot \\ \cdot \\ \alpha_9 \end{Bmatrix}$$

The constants $\alpha_1, \alpha_2 \ldots \ldots \alpha_9$ have to be determined from the nodal condition, for instance from fig. 3.5 one can find the nodal values. The values for each node is given:

$$w(x, y) = q_1, \quad \frac{\partial w}{\partial y}(x, y) = q_{2,} \quad -\frac{\partial w}{\partial x}(x, y) = q_{3,} \quad at\ (x_1, y_1) = (0,0)$$
$$w(x, y) = q_4, \quad \frac{\partial w}{\partial y}(x, y) = q_{5,} \quad -\frac{\partial w}{\partial x}(x, y) = q_{6,} \quad at\ (x_2, y_2) = (0, y_2) \ldots . 3.38$$
$$w(x, y) = q_7, \quad \frac{\partial w}{\partial y}(x, y) = q_{8,} \quad -\frac{\partial w}{\partial x}(x, y) = q_{9,} \quad at\ (x_3, y_3) = (x_3, y_3)$$

Inserting the nodal values obtained in equation 3.37 in 3.36 the following matrix will be obtained,

$$q^e = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_9 \end{Bmatrix} = [\eta]\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . . 3.39$$

Where

$$[\eta] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & y_2 & 0 & 0 & y_2^2 & 0 & 0 & y_2^3 \\ 0 & 0 & 1 & 0 & 0 & 2y_2 & 0 & 0 & 3y_2^2 \\ 0 & -1 & 0 & 0 & -y_2 & 0 & 0 & -y_2^2 & 0 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 & x_3^3 & x_3^2 y_3 + x_3 y_3^2 & y_3^3 \\ 0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & 2x_3 y_3 + x_3^2 & 3y_3^2 \\ 0 & -1 & 0 & -2x_3 & -y_3 & 0 & -3x_3^2 & -y_3^2 + 2x_3 y_3 & 0 \end{bmatrix}$$

And the second derivative of the polynomial function is applied to obtain expression B.

$$\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_9 \end{Bmatrix}$$

$$B = L[\eta]^{-1} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 3.40$$

Inserting equation 3.38 into equation 3.33 to obtain expression for the stiffness matrix

$$[K^e] = [\eta]^{-T} \int_{\Omega^e}\int z^2 [L]^T [D][L] d\Omega [\eta]^{-T}$$

$$= [\eta]^{-T} \frac{h^3}{12} \int_{\Omega^e} [L]^T [D][L] d\Omega [\eta]^{-T} \dots\dots\dots 3.41$$

3.3.1 Global matrix

Before the element equations can be assembled, it is necessary to transform the element matrices and vectors derived in local coordinate systems so that all the elemental equations are referred to a common global coordinate system. Let a transformation matrix λ exist between the local and global coordinate systems. Triangular element shown in fig. 3.6 is used to illustrate how this transformation matrix is obtained.

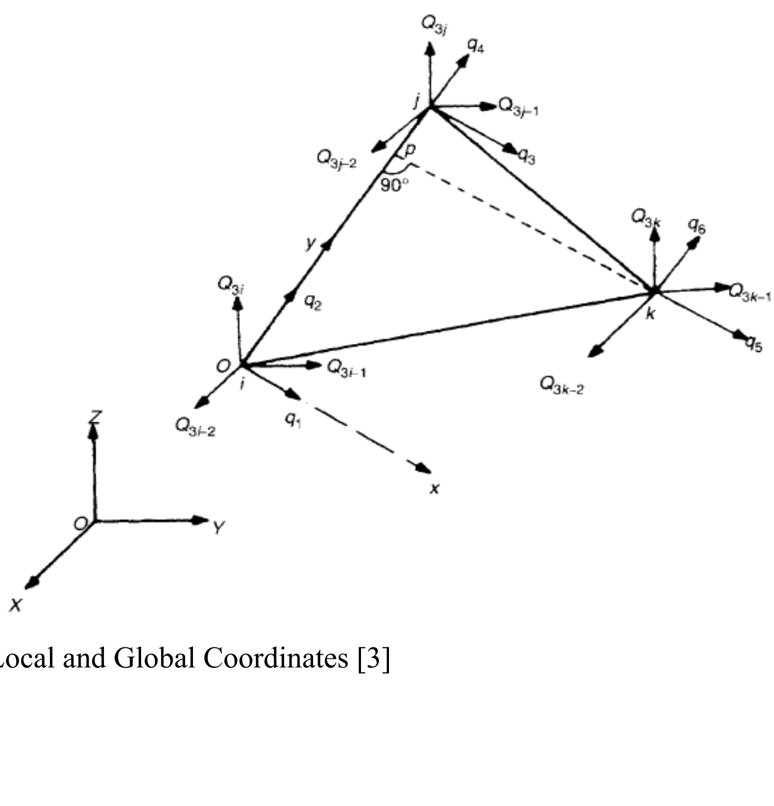

Fig. 3.6 Local and Global Coordinates [3]

Let the node number 1, 2 and 3 of the element correspond to the node numbers i, j and k respectively of the global system. The direction cosines of lines ox and oy with respect to the global x,y and z axes are required. Since the direction cosines of the line oy are the same as those of line ij. Hence the direction cosines of the line oy become

$$l_{ij} = \frac{x_j - x_i}{d_{ij}} \qquad m_{ij} = \frac{y_j - y_i}{d_{ij}} \qquad n_{ij} = \frac{z_j - z_i}{d_{ij}} \ldots\ldots\ldots\ldots\ldots\ldots 3.42$$

Where the distance between the points I and j is given by

$$d_{ij} = \left[\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2 + \left(z_j - z_i\right)^2\right]^{1/2}$$

Where (x_i, y_i, z_i) and (x_j, y_j, z_j) denote the (x, y, z) coordinates of points i and j respectively. To obtain the direction cosines of the line ox one should draw a perpendicular line kp from node k onto the line ij as shown in the fig. 3.6. Then direction cosines of the line ox will be the same as those of the line pk.

$$l_{pk} = \frac{x_k - x_p}{d_{pk}} \qquad m_{pk} = \frac{y_k - y_p}{d_{pk}} \qquad n_{pk} = \frac{z_k - z_p}{d_{pk}} \ldots\ldots\ldots\ldots\ldots\ldots 3.43$$

Where d_{pk} is the distance between the point p and k. The coordinate can be computed as

$$\begin{aligned} x_p &= x_i + l_{ij} d_{ip} \\ y_p &= y_i + m_{ij} d_{ip} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.44 \\ z_p &= z_i + n_{ij} d_{ip} \end{aligned}$$

Where d_{ip} is the distance between the points i and p. To find the distance d_{ip} the perpendicular condition of the lines ij and pk are considered

$$l_{ij} l_{pk} + m_{ij} m_{pk} + n_{ij} n_{pk} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.45$$

Using Eqs 3.43 and 3.44, eq 3.45 can be rewritten as

$$\frac{1}{d_{pk}}\left[l_{ij}\left(X_k - X_i - l_{ij} d_{ip}\right) + m_{ij}\left(Y_k - Y_i - m_{ij} d_{ip}\right) + n_{ij}\left(Z_k - Z_i - n_{ij} d_{ip}\right)\right] = 0 \ldots\ldots 3.46$$

Finally the distance d_{pk} can be found by considering the right-angle triangle ikp as

$$d_{pk} = \left({d_{ik}}^2 - {d_{ip}}^2\right)^{1/2} = \left[(X_k - X_i)^2 + (Y_k - Y_i)^2 + (Z_k - Z_i)^2 - {d_{ip}}^2\right]^{1/2} \ldots 3.47$$

The transformation matrix [λ] can now be constructed by using the direction cosine lines ij and pk as

$$[\lambda] = \begin{bmatrix} \lambda_{pk} & 0 & 0 \\ \lambda_{ij} & 0 & 0 \\ 0 & \lambda_{pk} & 0 \\ 0 & \lambda_{ij} & 0 \\ 0 & 0 & \lambda_{pk} \\ 0 & 0 & \lambda_{ij} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, 3.47$$

Where

$$\lambda_{pk}(1X3) = (l_{pk} \quad m_{pk} \quad n_{pk})$$

$$\lambda_{ij}(1X3) = (l_{ij} \quad m_{ij} \quad n_{ij})$$

$$0(1X3) = (0 \quad 0 \quad 0)$$

Considering the transformation matrix the stiffness matrix and force vector can be transformed to a global coordinate

$$\left[K^{(e)}\right] = [\lambda]^T[k^e][\lambda]$$

$$\left\{F^{(e)}\right\} = [\lambda]^T[f^e][\lambda] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, .3.48$$

Where K and F are in global coordinate and k and f are in local coordinate

3.3.2 Assembly of element matrices

Once the element characteristics, namely, the element matrices and element vectors are found in global coordinate system, the next step is to construct the overall system equations. The procedure for constructing the system equations from the element characteristics is the same regardless of the type of problem and the number and type of elements used.

The procedure of assembling the element matrices and vectors is based on the requirement of "compatibility" at the element nodes. This means that at the nodes where elements are connected, the values of the unknown nodal degrees of freedom are the same for all the elements joining at that node. In solid mechanics and structural problems, the nodal variables are usually generalized displacements, which can be translations, rotations, curvatures, or other spatial derivatives of translations. When the generalized displacements are matched at a common node, the nodal stiffnesses and nodal loads of each of the elements sharing the node are added to obtain the net stiffness and the net load at that node.

Let nel and sdof denote the total number of elements and nodal degree of freedom (including the boundary and restrained degrees of freedom) respectively. Let [K] the assembled system characteristics matrix of order of sdof x sdof and P is the characteristic load vector having sdof x 1 dimension. Thus, the global characteristic matrix and the global characteristic vector can be obtained by algebraic addition as

$$[K] = \sum_{e=1}^{nel} [k^e]$$

$$[P] = \sum_{e=1}^{nel} [p^e] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .3.49$$

One can use the flow chart shown in fig 3.7 to demonstrate how element matrix is assembled to system matrix. This flow chart can be used as a general procedure to assemble the element matrix.

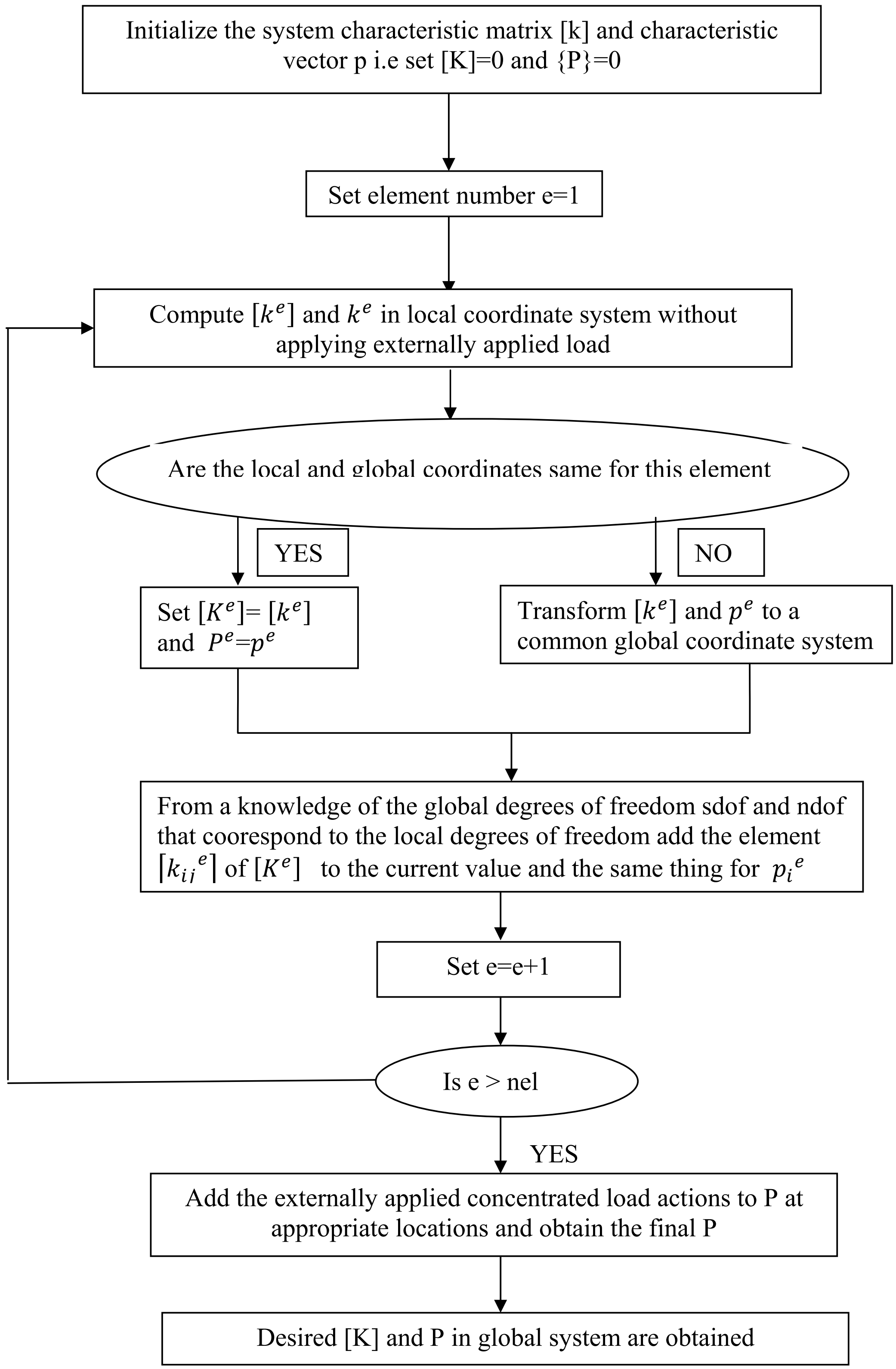

Fig. 3.7 shows the flow chart to assemble element matrix

Chapter 4-Result and Discussion

In the previous chapter analytical and finite element formulations of thin plates are considered. And in this section specific problems are considered to compare the result obtained from software developed based on Kirchhoff's theory for a linear triangular element with analytical solution. The comparisons are made by varying the thickness of the plate in order to obtain the optimum thickness where the thin plate theory works best. In addition, software called ANSYS is considered to verify the results obtained from the developed software.

4.1 Statement of the problem

In this section two specific problems are considered the first one is a concentrated load applied on a thin plate where as the other one is distributed load applied on the same plate. Clamped edge and simple supported plate are considered as boundary condition for this problem. The detailed description of the problem is described in the following section.

4.1.1 Concentrated load applied on thin plate

A concentrated load of 50,000 N is applied on an isotropic square plate as shown in the figure 4.1. This figure also shows the effect of the support by considering the central cross section of the plate. The plate has a dimension of 1m X 1m. Here mild steel with the following material property are considered:

Young's Modulus E = 2.0 e 11 Pa (200GPa)

Poisson's Ratio = 0.285

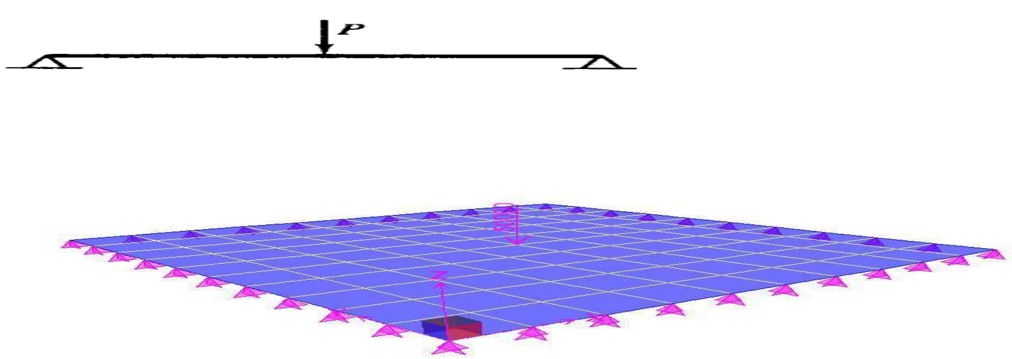

Fig. 4.1 Concentrated load applied on isotropic simple supported square plate

In this problem the objective is to obtain the total deflection in particular the maximum deflection using thin plate theory.

4.1.2 Distributed load applied on thin plate

A distributed load of 50,000 N is applied on isotropic square plate as shown in the figure 4.2. This figure also shows the effect of the support by considering the central cross section of the plate. The plate has a dimension of 1m X 1m. Here mild steel with the following material property are considered:

Young's Modulus E = 2.0 e 11 Pa (200 GPa)

Poisson's Ratio = 0.285

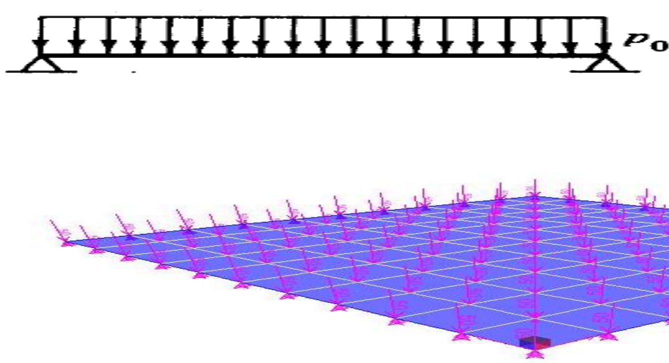

Fig. 4.2 distributed load applied on isotropic simple supported square plate

Similarly in this problem the maximum deflection for various thicknesses of a thin plate will be calculated and discussed.

4.2 Algorithm of the solution

Before going to the result, it is important to see the general solution procedures to solve finite element problems using computer program. These procedures can be broadly classified into three sections:

Preprocessor - In this part of the program essential information is collected from the user about the geometry and type of the problem. Some of the major input parameters are listed below

- ✓ Number of total nodes in the system
- ✓ Number of total elements in the system

- ✓ Coordinate values of every node in terms of the global coordinate system
- ✓ Boundary conditions
- ✓ Material property(Young's Modulus and Poisson's Ratio)
- ✓ Specifying applied force
- ✓ Size of the plate(thickness, length and width)

Processor – After taking user input all of the calculation will be conducted in this section of the program. Here major tasks of processor will be listed in this specific problem:

- ✓ Calculating stiffness matrices and force vectors for every element
- ✓ Assembling stiffness matrices into the system matrix and assembling force vector to a system vector
- ✓ Applying constraints (boundary conditions) to the system matrix and vector
- ✓ Calculating deflection of the plate

Postprocessor – Once the required calculation is performed in previous section one can display and plot the result in this part of the program.

- ✓ Displaying the result, in this specific problem a deflection for each node will be displayed.
- ✓ An open source software called gnuplot is used to plot the results

4.3 Result and discussion

The result obtained for the deflection of plate from software developed based on Kirchoff's thin plate theory as well as from ANSYS and analytical solution will be discussed for different load and boundary condition for plate having various thicknesses.

4.3.1 Concentrated load applied on simple supported plate

This section discusses about the result obtained when concentrated load applied on a simple supported square plate. Fig 4.3 shows the total deflection of the plate after the load is applied for mesh size of 0.1

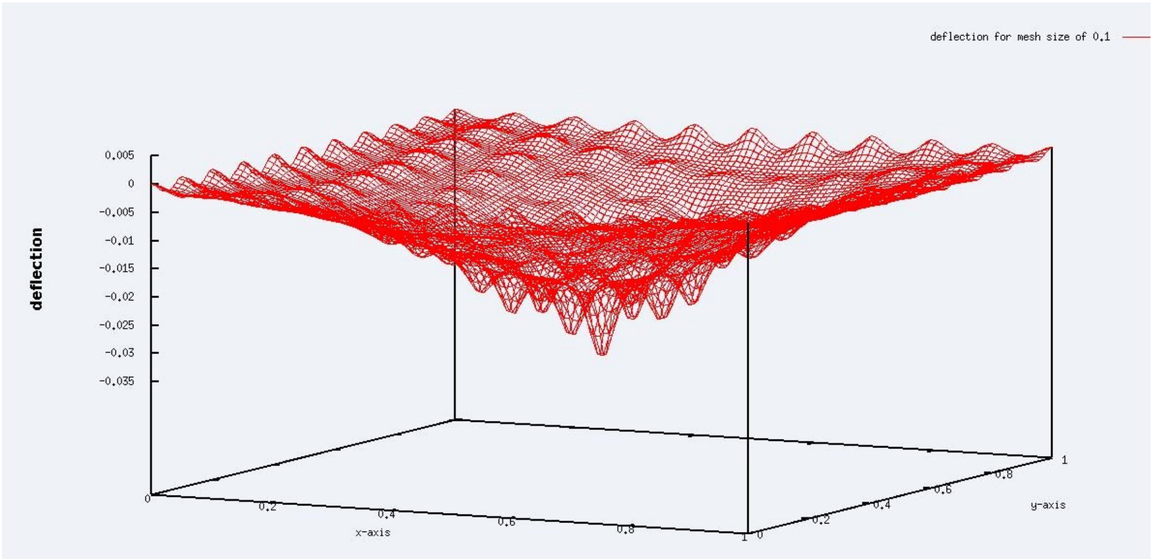

Fig 4.3 Deflection of the plate for concentrated load with mesh size of 0.1

The first analysis is among various mesh size of the computer program; in addition analytical solution is also included for comparison. Accordingly a solution is examined by varying the thickness of the plate is shown in table 4.1. The corresponding graph is depicted in fig 4.4

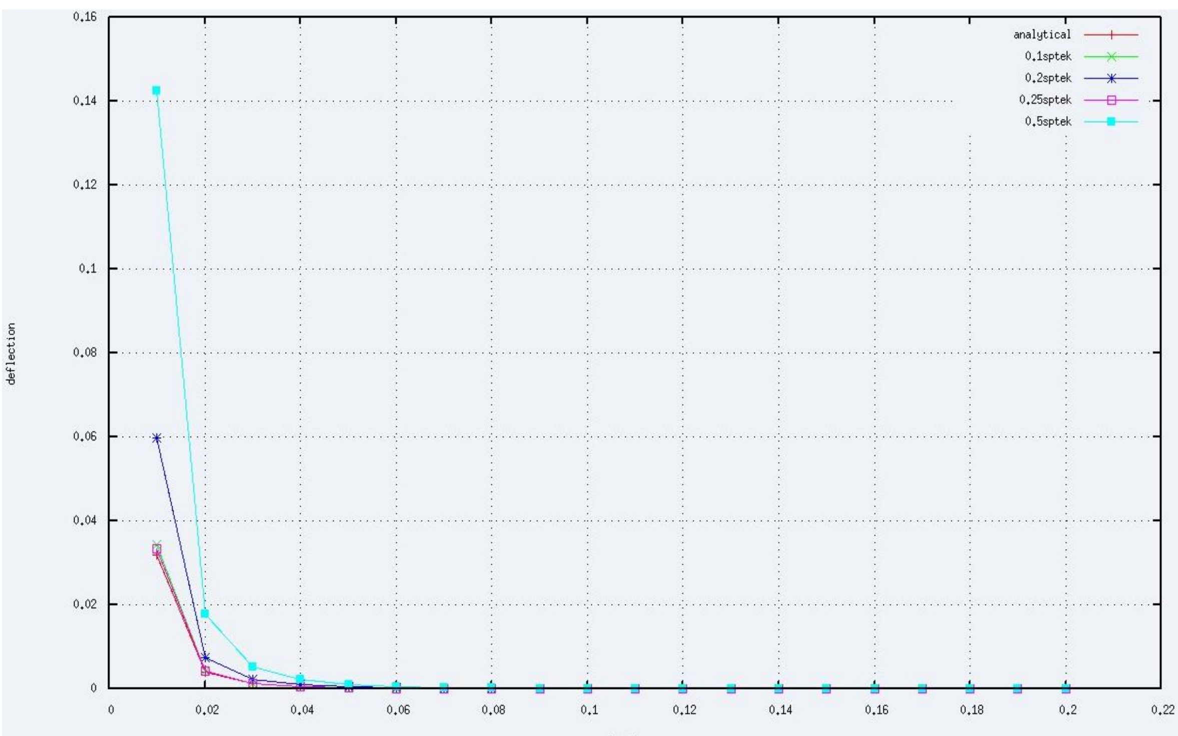

Fig 4.4 max deflection versus thickness of the plate for different mesh size

Table 4.1 maximum deflection for different mesh size

Thickness	Analytical	Sptek(0.1)	Sptek(0.2)	Sptek(0.25)	Sptek(0.5)
0.01	-0.0319734	-0.0344800	-0.0596800	-0.0333100	-0.1425000
0.02	-0.0039967	-0.0043100	-0.0074610	-0.0041640	-0.0178100
0.03	-0.0011842	-0.0012770	-0.0022110	-0.0012340	-0.0052780
0.04	-0.0004996	-0.0005388	-0.0009326	-0.0005205	-0.0022270
0.05	-0.0002558	-0.0002758	-0.0004775	-0.0002665	-0.0011400
0.06	-0.0001480	-0.0001596	-0.0002763	-0.0001542	-0.0006598
0.07	-0.0000932	-0.0001005	-0.0001740	-0.0000971	-0.0004155
0.08	-0.0000624	-0.0000673	-0.0001166	-0.0000651	-0.0002783
0.09	-0.0000439	-0.0000473	-0.0000819	-0.0000457	-0.0001955
0.1	-0.0000320	-0.0000345	-0.0000597	-0.0000333	-0.0001425
0.11	-0.0000240	-0.0000259	-0.0000448	-0.0000250	-0.0001071
0.12	-0.0000185	-0.0000199	-0.0000345	-0.0000193	-0.0000825
0.13	-0.0000146	-0.0000157	-0.0000272	-0.0000152	-0.0000649
0.14	-0.0000117	-0.0000126	-0.0000218	-0.0000121	-0.0000519
0.15	-0.0000095	-0.0000102	-0.0000177	-0.0000099	-0.0000422
0.16	-0.0000078	-0.0000084	-0.0000146	-0.0000081	-0.0000348
0.17	-0.0000065	-0.0000070	-0.0000122	-0.0000068	-0.0000290
0.18	-0.0000055	-0.0000059	-0.0000102	-0.0000057	-0.0000244
0.19	-0.0000047	-0.0000050	-0.0000087	-0.0000049	-0.0000208
0.2	-0.0000040	-0.0000043	-0.0000075	-0.0000042	-0.0000101

Furthermore it can be observed that the detail of information varies as one increase or decrease the mesh size. To appreciate this difference a three dimensional figures shown in fig 4.5 to fig. 4.8 are presented.

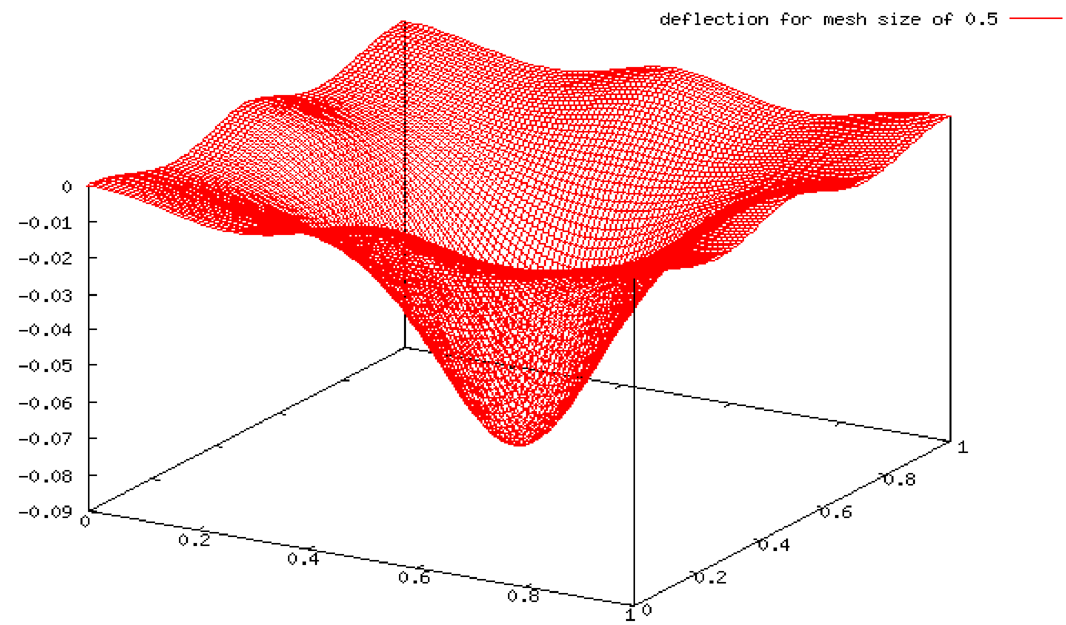

Fig 4.5 deflection of the plate for mesh size of 0.5

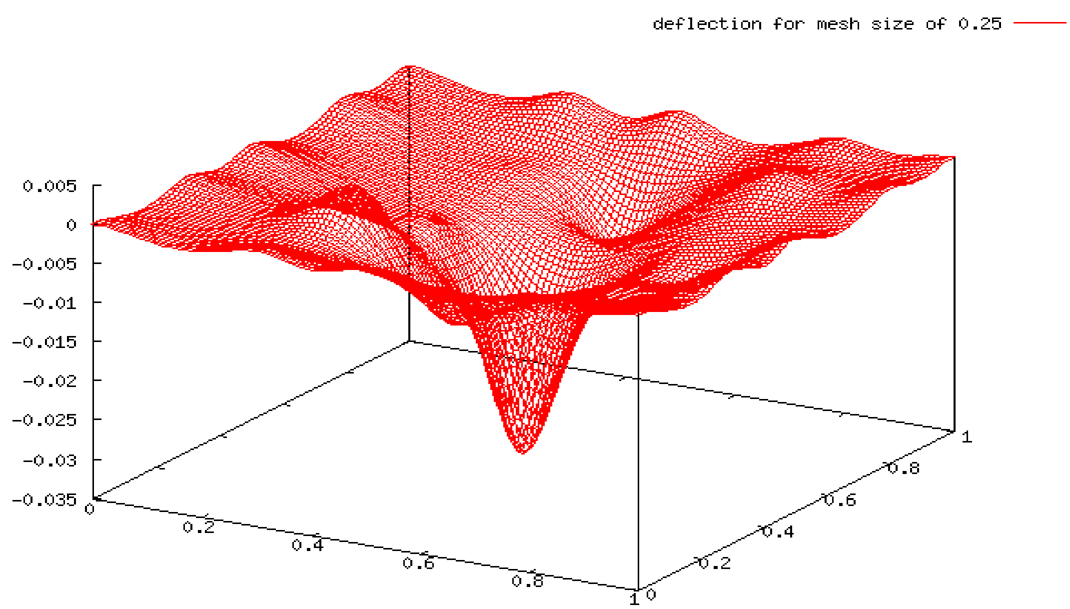

Fig 4.6 deflection of the plate for mesh size of 0.25

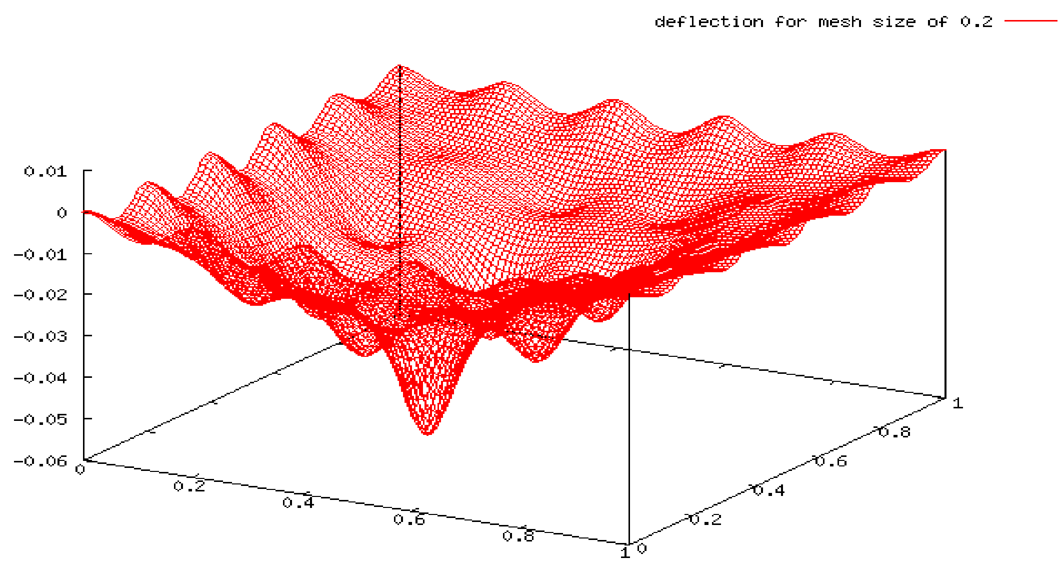

Fig 4.7 deflection of the plate for mesh size of 0.2

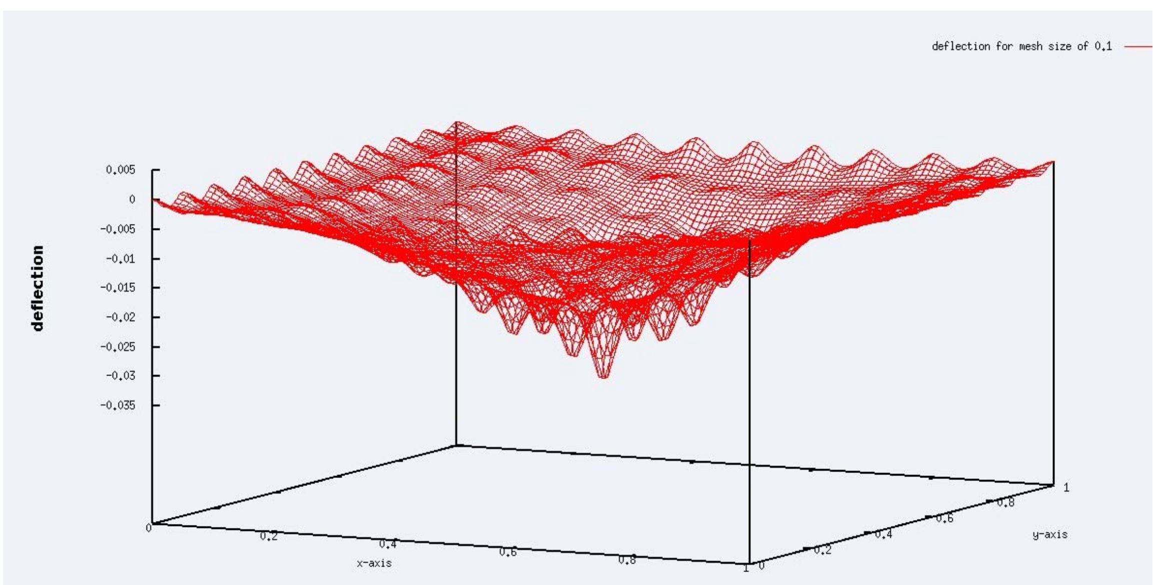

Fig 4.8 deflection of the plate for mesh size of 0.1

From the above figures and table 4.1 one can notice that the result approaches to the exact solution as the mesh size decrease. In addition the detail of information that can be obtained from the model will increase as the size of the mesh decrease. But one of the dicritization has even number of nodes and it was not possible to identify the exact position of the central node and hence the nearby node is selected to be the central node. This is the reason for getting a better result from 0.25 mesh size than 0.2 meshes for the central deflection. In the second analysis the best result of the software will be taken and compare it with the result obtained from ANSYS. Two Elements called Shell181 and Shell63 are found to be suitable to model the thin plate from ANSYS software. Hence these elements are considered and compared it with the software as shown in the following table and the corresponding graph is shown in fig 4.6

Table 4.2 maximum deflection for different element type

Thickness	Analytical	Sptek(0.1)	ansysis_shell181	Ansyis_shell63
0.01	-0.0319734	-0.0344800	-0.0320890	-0.0322440
0.02	-0.0039967	-0.0043100	-0.0040293	-0.0040310
0.03	-0.0011842	-0.0012770	-0.0012028	-0.0011940
0.04	-0.0004996	-0.0005388	-0.0005127	-0.0005040
0.05	-0.0002558	-0.0002758	-0.0002659	-0.0002580
0.06	-0.0001480	-0.0001596	-0.0001563	-0.0001492
0.07	-0.0000932	-0.0001005	-0.0001003	-0.0000940
0.08	-0.0000624	-0.0000673	-0.0000685	-0.0000629
0.09	-0.0000439	-0.0000473	-0.0000493	-0.0000442
0.1	-0.0000320	-0.0000345	-0.0000368	-0.0000322
0.11	-0.0000240	-0.0000259	-0.0000284	-0.0000242
0.12	-0.0000185	-0.0000199	-0.0000225	-0.0000187
0.13	-0.0000146	-0.0000157	-0.0000183	-0.0000147
0.14	-0.0000117	-0.0000126	-0.0000151	-0.0000117
0.15	-0.0000095	-0.0000102	-0.0000127	-0.0000095
0.16	-0.0000078	-0.0000084	-0.0000108	-0.0000078
0.17	-0.0000065	-0.0000070	-0.0000093	-0.0000066
0.18	-0.0000055	-0.0000059	-0.0000081	-0.0000055
0.19	-0.0000047	-0.0000050	-0.0000072	-0.0000047
0.2	-0.0000040	-0.0000043	-0.0000064	-0.0000040

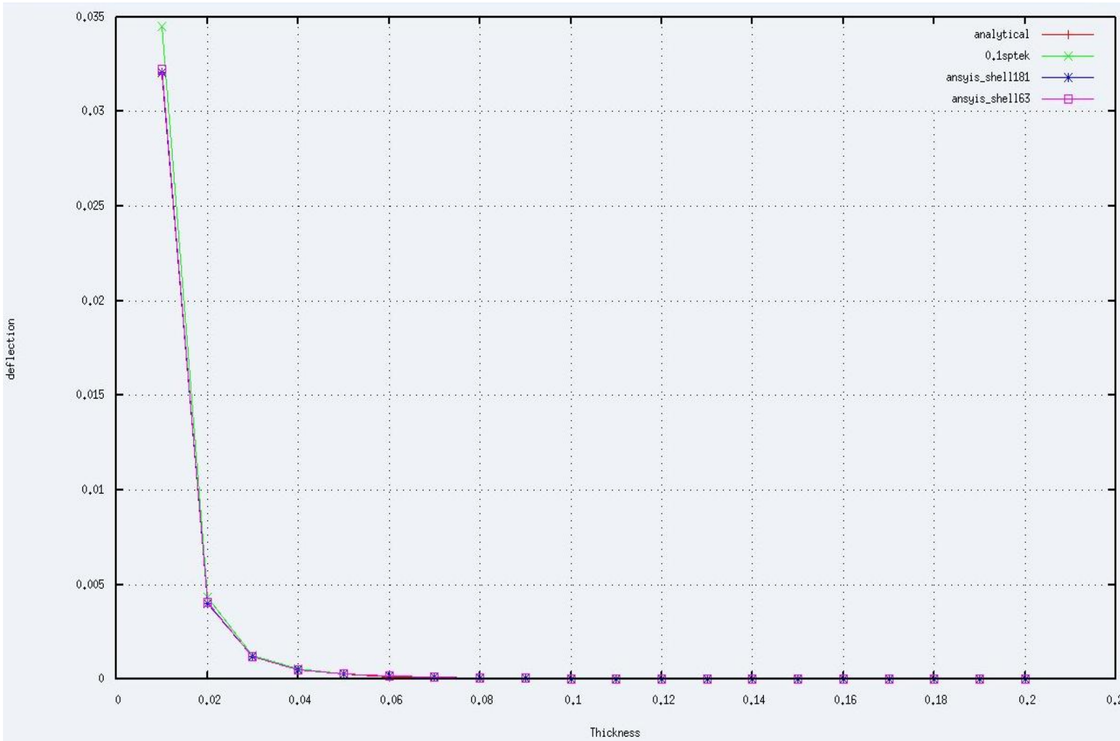

Fig 4.9 max deflection versus thickness of plate for different elements

From Table 4.2 and Fig. 4.9 one can notice the solution from the software to overlap with the analytical result as well as with two elements of Ansyis software. Small variation is observed when the thickness of the plate is 0.01 but other than this case the solution seems to overlap

4.3.2 Distributed load applied on simple supported plate

Similarly, a thin plate theory is used and applies a finite element procedure as it has been used for concentrated load to obtain the deflection of a plate. The following figure shows the deflection caused by distributed load applied on simple supported plate:

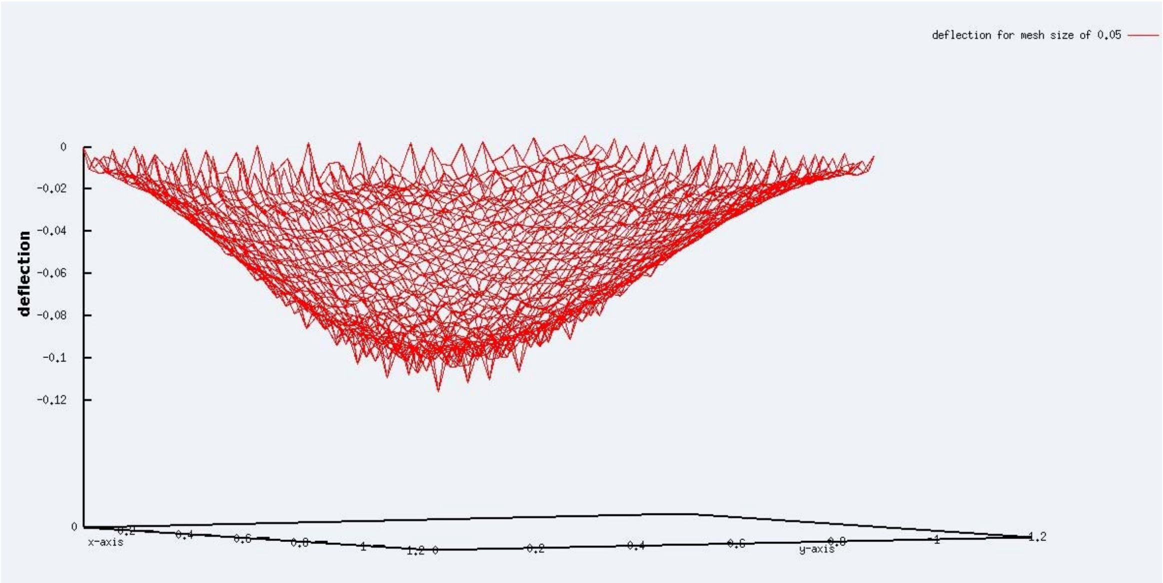

Fig 4.10 Deflection of the plate when distributed load is applied

The solution obtained from the finite element program when a distributed load applied on thin plate is compared with analytical solution and anysis's shell181 and shell63 elements. Table 4.3 lists the result obtained from different methods for various thickness sizes and also fig. 4.11 depicts the output obtained from the table.

Table 4.3 maximum deflection for different element type

Thickness	Analytical	Sptek(0.1)	Ansysis_shell181	Ansyis_shell63
0.01	-0.0112000	-0.0113239	-0.0144180	-0.0143170
0.02	-0.0014000	-0.0014155	-0.0018062	-0.0017890
0.03	-0.0004145	-0.0004194	-0.0005371	-0.0005303
0.04	-0.0001749	-0.0001769	-0.0002278	-0.0002237
0.05	-0.0000895	-0.0000875	-0.0001174	-0.0001145
0.06	-0.0000518	-0.0000524	-0.0000685	-0.0000663
0.07	-0.0000326	-0.0000330	-0.0000435	-0.0000417
0.08	-0.0000219	-0.0000221	-0.0000295	-0.0000280
0.09	-0.0000154	-0.0000155	-0.0000209	-0.0000196
0.1	-0.0000112	-0.0000113	-0.0000155	-0.0000143
0.11	-0.0000084	-0.0000085	-0.0000118	-0.0000108
0.12	-0.0000065	-0.0000063	-0.0000092	-0.0000083
0.13	-0.0000051	-0.0000050	-0.0000074	-0.0000065
0.14	-0.0000041	-0.0000041	-0.0000060	-0.0000052
0.15	-0.0000033	-0.0000034	-0.0000050	-0.0000042
0.16	-0.0000027	-0.0000028	-0.0000042	-0.0000035
0.17	-0.0000023	-0.0000023	-0.0000035	-0.0000031
0.18	-0.0000019	-0.0000019	-0.0000031	-0.0000028
0.19	0.0000016	-0.0000017	-0.0000027	-0.0000024
0.2	0.0000014	-0.0000014	-0.0000023	-0.0000021

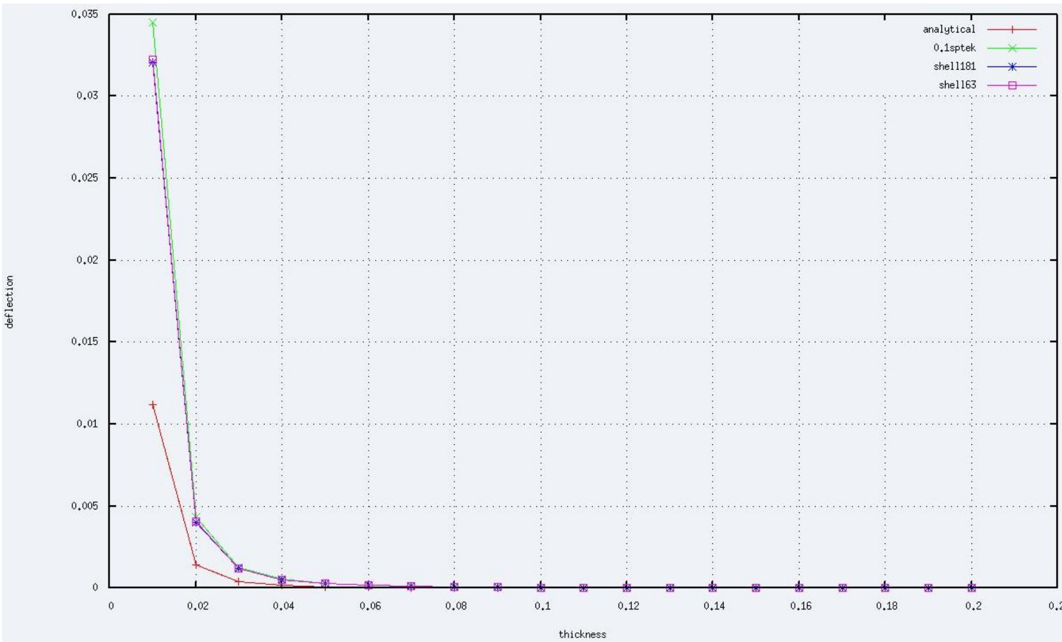

Fig 4.11 max deflection versus thickness of plate for different elements

For the distributed load as it can be noticed from table 4.3 and fig. 4.12 analytical solution shows some discrepancy for plate having thickness less than 0.04. Apart from that the finite element program overlaps with the analytical result as well as with two elements of ANSYS software.

4.3.3 Force applied on fixed supported plate

In addition to simple supported plates, the maximum deflection of clamped edge plates were evaluated under both concentrated and distributed loads as shown in fig 4.12 and fig 4.13. In this case a plate with 0.1 thicknesses is considered and the results were compared with those obtained from the ANSYS software as shown in Table 4.4

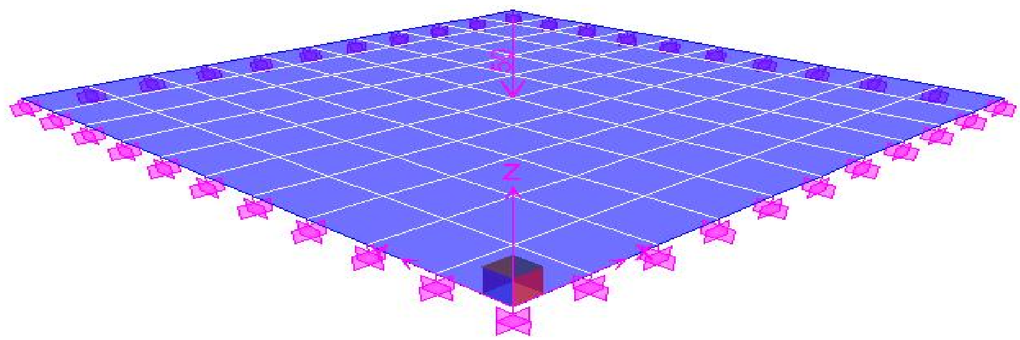

Fig 4.12 Concentrated load applied on fixed supported plate

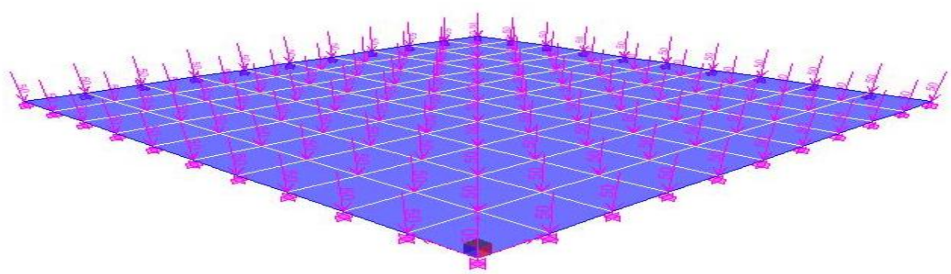

Fig 4.13 Distributed load applied on fixed supported plate

Table 4.4 maximum deflection for clamped edge plate

	Max. deflection Concentrated load	Max. deflection Distributed load
Analytical	-0.0000154	-0.0000035
Software(mesh size of 0.1)	-0.00002914	-0.00000481
Ansysis(Shell 63)	-0.00001572	-0.000001966
Ansysis(Shell 181)	-0.00006148	-0.00001608

Finally the total deflection of the fixed plate when a concentrated load is applied at the center is depicted in fig 4.14 for a plate with 0.1 thicknesses.

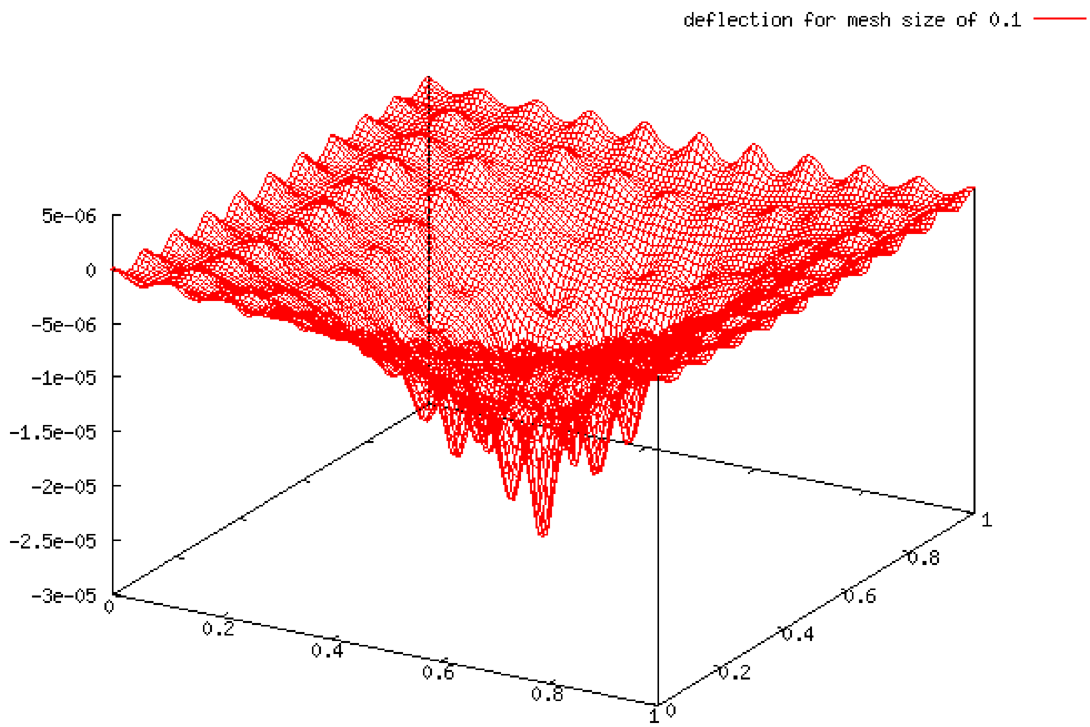

Fig 4.14 Deflection of clamped edge plate

Chapter 5-Conclusions and future directions

5.1 Conclusions

It is evident that the Finite Element Method is a pretty robust means of numerically solving for vast majority of structural problems. Of course the specimen used here was a simple flat square plate, but the observations made in this thesis can apply to even complex structures with minor changes like tailoring the right element and boundary condition. The numerical results provided by the software developed for this thesis or any other numerical code require careful scrutiny and interpretation to ensure that the known principles of applied mechanics have been properly and relevantly applied to the solution at hand. This is why starting with a simplified problem (like in this case a simply supported and clamped edge plate) before moving to more complex ones can be profitable.

In this thesis a FORTRAN program is written based on thin plate theory. And the result converges to the exact (analytical) solution as one decrease the mesh size. Meshing, is therefore an important aspect in running a numerical solution, and must always be treated with caution. A different mesh type can yield different answers depending on how the element code is set up.

Another important parameter in programming a finite element code is identifying appropriate element type. In this thesis a linear triangular element is used and the result is compared with shell63 and shell181 of ANSYS. And observation shows remarkably similar solution.

Hence, Kirchhoff's thin plate theory using a linear triangular element with mesh size of 0.1 is found to be suitable to model the static behavior of plate

5.2 Future directions

So far it has been discussed about the static behavior of a thin plate; one can extend this to dynamic analysis. The other thing which would be interesting for future work is to incorporate the idea of parallel programming. This paper can serve as starting point if someone is interested in dynamic analysis as well as parallel programming in finite element problems.

Appendex

Make file (maintain groups of programs)

```
F90 = gfortran
FFLAG = -c
FFLAGO = -O2
OBJ = main.o global.o solve.o decomp.o inverse.o eta.o  \

      stiffness.o transform.o
main:$(OBJ)
	$(F90) $(FFLAGO) -o main  $(OBJ)
main.o: main.f90 global.o stiffness.o mass.o input.in input1.in input2.in input3.in
	$(F90) $(FFLAG) main.f90
global.o: global.f90
	$(F90) $(FFLAG) global.f90
solve.o: solve.f90
	$(F90) $(FFLAG) solve.f90
decomp.o: decomp.f90
	$(F90) $(FFLAG) decomp.f90
inverse.o: inverse.f90
	$(F90) $(FFLAG) inverse.f90
eta.o: eta.f90
	$(F90) $(FFLAG) eta.f90
stiffness.o: stiffness.f90
	$(F90) $(FFLAG) stiffness.f90
transform.o: transform.f90
	$(F90) $(FFLAG) transform.f90
clean:
	rm -f *.o *.mod main
```

Global variables

```
MODULE global
Implicit none
 Integer :: nd,ne,nb,nn,nfix,i,j,mm,sdof,nmode,index
 Real :: anu,d12,dcl12,dcm12,d14,d43,dcl43,dcm43,T
 Real :: ep,a1,a2,a3,a4,a5,a6,BIG,E,rho
 Integer :: lm,ii,jj,jp,kp,lp,ix,kk,ll,ik,jk,in
 Real,allocatable, Dimension(:) :: cx,cy
 Real,allocatable, Dimension(:) :: area
 Real,allocatable, Dimension(:) :: bnd
 Real,allocatable, Dimension(:) :: omeg
 Real,allocatable, Dimension(:,:) :: k,gm,kz
```

```
 Real,allocatable, Dimension(:)  :: m
 Real,allocatable, Dimension(:) :: p
 Real,allocatable, Dimension(:,:) :: node
 Real,allocatable, Dimension(:,:) :: X



 Real, Dimension(9,9) :: et,bt,bte,btet,tk,slt,sltt,etinv,sm
 Real(8), Dimension(2) :: diff2
 Real, Dimension(9,1) :: b2
 Real, Dimension(9) :: lp2,s2,ni,indx
 Real, Dimension(9,2) :: lq2
 Real, Dimension(3) :: cxl,cyl
END MODULE global
```

Main program

```
Program main
   USE global
   use stiff
   use massmatrix
   INCLUDE"input.in"
   E=200.0E9
   anu=0.285
   rho=7861.093
   T=0.1
   index=2
   OPEN(UNIT=20,FILE='platedata11',ACTION='read',IOSTAT=l)
   read(20,*) bnd,cx,cy,node
   forall (i=1:nd) p(i)=0
   p(181)=-500000.0
   forall (i=1:nd,j=1:nb) k(i,j)=0.0
   do lm=1,ne
     !assigns the value of the node releative to the global value
     jp=node(lm,1)
     kp=node(lm,2)
     lp=node(lm,3)
     cxl(1)=0.0
     cyl(1)=0.0
     cxl(2)=0.0
     !(x2-x1)**2 + (y2-y1)**2
     d12=(cx(kp)-cx(jp))**2+(cy(kp)-cy(jp))**2
     ! [x2-x1)**2 + (y2-y1)**2]**1/2
     d12=sqrt(d12)  !x2-x1
     cyl(2)=d12
     cl12=(cx(kp)-cx(jp))/d12
     dcm12=(cy(kp)-cy(jp))/D12
     d14=dcl12*(cx(lp)-cx(jp))+dcm12*(cy(lp)-cy(jp)) !x3-x1
     ![(x3-x2)**2 + (y3-y2)**2]**1/2
     d43=(cx(lp)-cx(jp))**2+(cy(lp)-cy(jp))**2
     d43=d43-d14**2
     d43=sqrt(d43)  !x3
```

```
    cxl(3)=d43
    cyl(3)=d14
    dcl43=((cx(lp)-cx(jp))-(dcl12*d14))/d43
    dcm43=((cy(lp)-cy(jp))-(dcm12*d14))/d43

    !calculate the area of the triangle
    !1/2*x3*y2
    area(lm)=((cxl(3)-cxl(2))*(cyl(1)-cyl(2))-(cxl(2)-cxl(1))*(cyl(2)-cyl(3)))/2.0
    area(lm)=ABS(area(lm))
    ep=(E*T**3)/(12.0*(1.0-anu**2))
    forall(i=1:ND,j=1:NMODE) X(i,j)=0.0
    forall(i=1:9,j=1:9) et(i,j)=0.0
    call eta(d12,d14,d43,et)
    call stiffness(d12,d14,d43,ep,et,anu,sltt)
     jp=node(lm,1)
     kp=node(lm,2)
     lp=node(lm,3)
     ni(1)=jp*3-2
     ni(2)=jp*3-1
     ni(3)=jp*3
     ni(4)=kp*3-2
     ni(5)=kp*3-1
     ni(6)=kp*3
     ni(7)=lp*3-2
     ni(8)=lp*3-1
     ni(9)=lp*3
     DO i=1,9
          ik=ni(i)
       DO j=1,9
         jk=ni(j)
         in=jk-ik+1
         IF (in .LE. 0)  cycle
         k(ik,in)=k(ik,in)+sltt(i,j)
         gm(ik,in)=gm(ik,in)+sm(i,j)
       end do
     end do
 end do
 close(20)
     BIG=10.0E8
     DO I=1,NFIX
       IX=bnd(I)
       k(IX,1)=k(IX,1)*BIG
       gm(IX,1)=gm(IX,1)*BIG
     end do
 forall (i=1:nd,j=1:nb) kz(i,j)=k(i,j)
 CALL DECOMP (nd,nb,k)
 CALL SOLVE (nd,nb,mm,k,p,diff2)
OPEN(UniT=10,FILE='results',ACTION='write',STATUS='replace')
  j=1
  do i=1,nd,3
                  write(*,51) cx(j) , cy(j)
```

```
                write(10,51) cx(j) , cy(j)
            write(*,50) p(i)
            write(10,50) p(i)

                50 FORMAT(E16.4)
                        51 FORMAT(2f6.1,$)
            j=j+1
        end do
 CLOSE(10)
end program main
```

Subroutine that solves the matrix

```
SUBROUTINE SOLVE (n,nb,m,a,b,diff)
  implicit none
   real,dimension(n,nb)::A
   real,dimension(n,m)::B
   real(8),dimension(m)::diff
   integer :: j,i,k,irow,icol,ii,ik,n,nb,m
   do j=1,m
     b(1,j)=b(1,j)/a(1,1)
   end do
   do i=2,n
     do j=1,m
       diff(j)=b(i,j)
     end do
     do k=2,nb
       irow=i+1-k
       if (irow .LT. 1) cycle
         icol=i+1-irow
       if (icol .GT. nb) cycle
       do j=1,m
         diff(j)=diff(j)-a(irow,icol)*b(irow,j)
       end do
     end do
     do j=1,m
       b(i,j)=diff(j)/a(i,1)
     end do
   end do
   do  j=1,m
      b(n,j)=b(n,j)/a(n,1)
   end do
   do ii=2,n
      i=n+1-ii
      do j=1,m
        diff(j)=b(i,j)
      end do
      do k=2,nb
        ik=i-1+k
        if (ik .GT. n) cycle
        do j=1,m
          diff(j)=diff(j)-a(i,k)*b(ik,j)
```

```
      end do
    end do

    do J=1,m
      b(i,j)=diff(j)/a(i,1)
    end do
  end do
end subroutine solve
```

Subroutine that calculate the LU decomposition

```
 SUBROUTINE SOLVE (n,nb,m,a,b,diff)
  implicit none
  real,dimension(n,nb)::A
  real,dimension(n,m)::B
  real(8),dimension(m)::diff
  integer :: j,i,k,irow,icol,ii,ik,n,nb,m
  do j=1,m
    b(1,j)=b(1,j)/a(1,1)
  end do
  do i=2,n
    do j=1,m
      diff(j)=b(i,j)
    end do
    do k=2,nb
      irow=i+1-k
      if (irow .LT. 1) cycle
        icol=i+1-irow
      if (icol .GT. nb) cycle
      do j=1,m
        diff(j)=diff(j)-a(irow,icol)*b(irow,j)
      end do
    end do
    do j=1,m
      b(i,j)=diff(j)/a(i,1)
    end do
  end do
  do  j=1,m
     b(n,j)=b(n,j)/a(n,1)
  end do
  do ii=2,n
     i=n+1-ii
     do j=1,m
       diff(j)=b(i,j)
     end do
     do k=2,nb
       ik=i-1+k
       if (ik .GT. n) cycle
       do j=1,m
         diff(j)=diff(j)-a(i,k)*b(ik,j)
```

```
        end do
      end do

      do J=1,m
        b(i,j)=diff(j)/a(i,1)
      end do
   end do
 end subroutine solve
```

Subroutine that calculates the inverse of a matrix

```
SUBROUTINE MIGS (A,N,X,INDX)
  IMPLICIT NONE
  INTEGER, INTENT (IN) :: N
  INTEGER :: I,J,K
  INTEGER, INTENT (INOUT), DIMENSION (N) :: INDX
  REAL, INTENT (IN), DIMENSION (N,N):: A
  REAL, INTENT (INOUT), DIMENSION (N,N):: X
  REAL, DIMENSION (N,N) :: B
  DO I = 1, N
    DO J = 1, N
      B(I,J) = 0.0
    END DO
  END DO
  DO I = 1, N
    B(I,I) = 1.0
  END DO

  CALL ELGS (A,N,INDX)

  DO I = 1, N-1
    DO J = I+1, N
      DO K = 1, N
        B(INDX(J),K) = B(INDX(J),K)-A(INDX(J),I)*B(INDX(I),K)
      END DO
    END DO
  END DO

  DO I = 1, N
    X(N,I) = B(INDX(N),I)/A(INDX(N),N)
    DO J = N-1, 1, -1
      X(J,I) = B(INDX(J),I)
      DO K = J+1, N
        X(J,I) = X(J,I)-A(INDX(J),K)*X(K,I)
      END DO
      X(J,I) =  X(J,I)/A(INDX(J),J)
    END DO
  END DO
END SUBROUTINE MIGS
```

```
SUBROUTINE ELGS (A,N,INDX)

!

! Subroutine to perform the partial-pivoting Gaussian elimination.
! A(N,N) is the original matrix in the input and transformed matrix
! plus the pivoting element ratios below the diagonal in the output.
! INDX(N) records the pivoting order.  .
  IMPLICIT NONE
  INTEGER, INTENT (IN) :: N
  INTEGER :: I,J,K,ITMP
  INTEGER, INTENT (INOUT), DIMENSION (N) :: INDX
  REAL :: C1,PI,PI1,PJ
  REAL, INTENT (INOUT), DIMENSION (N,N) :: A
  REAL, DIMENSION (N) :: C
! Initialize the index
  DO I = 1, N
    INDX(I) = I
  END DO

! Find the rescaling factors, one from each row

  DO I = 1, N
    C1= 0.0
    DO J = 1, N
      C1 = AMAX1(C1,ABS(A(I,J)))
    END DO
    C(I) = C1
  END DO
! Search the pivoting (largest) element from each column

  DO J = 1, N-1
    PI1 = 0.0
    DO I = J, N
      PI = ABS(A(INDX(I),J))/C(INDX(I))
      IF (PI.GT.PI1) THEN
        PI1 = PI
        K   = I
      ENDIF
    END DO
! Interchange the rows via INDX(N) to record pivoting order

    ITMP    = INDX(J)
    INDX(J) = INDX(K)
    INDX(K) = ITMP
    DO I = J+1, N
      PJ  = A(INDX(I),J)/A(INDX(J),J)
! Record pivoting ratios below the diagonal
      A(INDX(I),J) = PJ
```

```
! Modify other elements accordingly

      DO K = J+1, N

        A(INDX(I),K) = A(INDX(I),K)-PJ*A(INDX(J),K)
      END DO
    END DO
  END DO
END SUBROUTINE ELGS
```

Calculate the value of eta

```
subroutine eta(y2,y3,x3,et)
   implicit none
   Real, intent(inout), Dimension(9,9) :: et
   Real,dimension(9,1) :: b
   Real, intent(in) :: y2,y3,x3
   integer :: i,j
   forall(i=1:9,j=1:9) et(i,j)=0.0
      et(1,1)=1.0
      et(2,3)=1.0
      et(3,2)=-1.0
      et(4,1)=1.0
      et(4,3)=y2
      et(4,6)=y2**2
      et(4,9)=y2**3
      et(5,3)=1.0
      et(5,6)=2.0*y2
      et(5,9)=3.0*y2**2
      et(6,2)=-1.0
      et(6,5)=-y2
      et(6,8)=-y2**2
      et(7,1)=1.0
      et(7,2)=x3
      et(7,3)=y3
      et(7,4)=x3**2
      et(7,5)=x3*y3
      et(7,6)=y3**2
      et(7,7)=x3**3
      et(7,8)=(x3*y3**2)+(x3**2)*y3
      et(7,9)=y3**3
      et(8,3)=1.0
      et(8,5)=x3
      et(8,6)=2.0*y3
      et(8,8)=2.0*x3*y3+x3**2
      et(8,9)=3.0*y3**2
      et(9,2)=-1.0
      et(9,4)=-2.0*x3
      et(9,5)=-y3
      et(9,7)=-3.0*x3**2
      et(9,8)=-(y3**2+2.0*x3*y3)
```

```
  end subroutine eta
```

Calculate the stiffness matrix

```
module stiff
contains
subroutine stiffness(y2,y3,x3,ep,et,anu,sltt)
 use global, only : dcl43,dcm43,dcl12,dcm12,tk, &
                   bt,indx,etinv,bte,btet,slt
 implicit none
 real, intent(inout),dimension(9,9) :: sltt,et
 integer :: i,j,k,l
 real :: y2,y3,x3,a1,a2,a3,a4,a5,a6,ep,anu
 forall (i=1:9,j=1:9) bt(i,j)=0.0
      a1=0.5*x3*y2*ep
      a2=(1.0/6.0)*(x3**2)*y2*ep
      a3=(1.0/12.0)*(x3**3)*y2*ep
      a4=(1.0/6.0)*x3*y2*(y2+y3)*ep
      a5=(1.0/12.0)*x3*y2*(y2**2+y2*y3+y3**2)*ep
      a6=(1.0/24.0)*(x3**2)*y2*(y2+2.0*y3)*ep
      bt(4,4)=4.0*a1
      bt(5,5)=2.0*(1.0-anu)*a1
      bt(6,4)=4.0*anu*a1
      bt(6,6)=4.0*a1
      bt(7,4)=12.0*a2
      bt(7,6)=12.0*anu*a2
      bt(7,7)=36.0*a3
      bt(8,4)=4.0*anu*a2+4.0*a4
      bt(8,5)=4.0*(1.0-anu)*(a2+a4)
      bt(8,6)=4.0*(a2+anu*a4)
      bt(8,7)=12.0*anu*a3+12.0*a6
      bt(8,8)=(12.0-8.0*anu)*(a3+a5+2.0*a6)-8.0*(1.0-anu)*a6
      bt(9,4)=12.0*anu*a4
      bt(9,6)=12.0*a4
      bt(9,7)=36.0*anu*a6
      bt(9,8)=12.0*a6+12.0*anu*a5
      bt(9,9)=36.0*a5
  forall (i=1:9,j=1:9) bt(i,j)= bt(j,i)

  call MIGS (et,9,etinv,indx)
  do i=1,9
    do j=1,9
      bte(i,j)=0.0
      do  k=1,9
          bte(i,j)=bte(i,j)+bt(i,k)*etinv(k,j)
      end do
    end do
  end do
  do j=1,9
```

```
    do l=1,9

       btet(j,l)=0.0
       do  k=1,9
           btet(j,l)=btet(j,l)+etinv(k,j)*bte(k,l)
       end do
    end do
  end do

  call transform(dcl43,dcm43,dcl12,dcm12,tk)
  do j=1,9
    do l=1,9
       slt(j,l)=0.0
       do k=1,9
          slt(j,l)=slt(j,l)+btet(j,k)*tk(k,l)
       end do
    end do
  end do
  do j=1,9
    do l=1,9
       sltt(j,l)=0.0
       do k=1,9
          sltt(j,l)=sltt(j,l) + tk(k,j)*slt(k,l)
       end do
    end do
  end do
 end subroutine  stiffness
end module stiff
```

Transform to global coordinates

```
subroutine transform(dcl43,dcm43,dcl12,dcm12,tk)
 implicit none
 real, intent(inout),dimension(9,9) :: tk
 real :: dcl43,dcm43,dcl12,dcm12
 integer :: i,j
  forall (i=1:9, j=1:9) tk(i,j)=0.0
    tk(1,1)=1.0
    tk(2,2)=dcl43
    tk(2,3)=dcm43
    tk(3,2)=dcl12
    tk(3,3)=dcm12
    tk(4,4)=1.0
    tk(5,5)=dcl43
    tk(5,6)=dcm43
    tk(6,5)=dcl12
    tk(6,6)=dcm12
    tk(7,7)=1
    tk(8,8)=dcl43
    tk(8,9)=dcm43
    tk(9,8)=dcl12
```

```
  tk(9,9)=dcm12
end subroutine transform
```

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